ABSTRACT

A discussion is given of a general probabilistic approach to the derivation of the failure probability conditioned by nondestructive (ND) measurements and of an optimal accept/reject procedure. This approach involves the use of explicit stochastic models of both the ND measurement process and the failure process (including a postulated stress environment). The overall decision logic involves a number of online and off-line inputs and outputs which will be described in detail with some indications of the kinds that are of interest to various categories of users. Particular emphasis will be placed upon the operating characteristic curve (i.e., the false-rejection probability vs. the false-acceptance probability representing a broad spectrum of optimal decision procedures) and its significance as a measure of the performance and cost-effectiveness of NDE systems. Explicit results will be given for the case of ceramic NDE with acoustical scattering measurements and two alternative failure models. The first is one in which the fracture process originates at a void surrounded by peripheral microcracks and the second involves fracture originating in a subcritical inclusion. Particular attention will be devoted to limiting situations in which the unconditional failure probability is small and/or in which the ND measurements are accurate and sufficiently diverse.

INTRODUCTION

The purpose of this paper is to present a description of progress made since the last review of quantitative NDE on the subject of probabilistic failure prediction and optimization of accept/reject criteria. This work goes beyond other work on reliability theory by making use of explicit physical models of both the failure and measurement processes. The resultant formalism enables one to bridge the gap between the ND measurements and the concerns of the ultimate user.

Although our methodology applies in principle to any material, our explicit applications will be made to structural ceramics. We will make the following simplifying assumptions:

1. The ND measurement, or set of such measurements, will be performed at a single time and a single accept/reject decision will be made on the basis of the result.
2. Only the most significant (e.g., the largest defect in a first-piece causes failure, if any, and the less significant defects in aggregate have a negligible probability of causing failure.
3. The applied stress is spatially uniform. The probability of failure depends only on the maximum stress and is independent of the stress history up to that time.

In the following sections we discuss first the general theory of failure prediction and accept/reject decisions. In later sections, we discuss the applications to failure in ceramics due to voids and subcritical inclusions, respectively. Various output properties are considered with particular emphasis on plots of false-reject vs false-accept probabilities. Although the failure models have been validated with real data on fracture and the associated causative defect in each of a number of test pieces, the assessments of the overall decision formalisms have been carried out only with artificial theoretical data based upon the relevant models of the measurement and failure processes.
that are connected with the physical model of the failure process and with the model of the non-destructive measurements.

![Diagram](https://via.placeholder.com/150)

**Fig. 2** Analysis and optimization of NDE decision process.

In this discussion we limit our attention to the case in which each test piece contains only one defect that is significant contribution to failure. We can assume, for example, that the largest defect in a given test piece has a much larger probability of causing failure at a given applied stress than the probability due to the combined effect of all of the smaller defects. With this assumption, we can formulate our problem as though only one defect were present.

Further discussion is necessary to clarify the meaning of failure, in particular what kind of event constitutes failure and under what conditions. In all cases in this paper, failure means fracture (e.g., the propagation of a crack across the component with resultant division into two or more pieces) and not some relatively benign occurrence such as a small amount of irreversible bending. The conditions under which failure occurs must be specified and the nature of this specification depends on the kind of material involved. For example, in the case of a brittle material one need only specify the stress environment (e.g., a static applied stress in the simplest case) since the elapsed time is irrelevant (within wide limits) since fatigue and aging processes do not occur to a significant extent. In the case of fatigue of metals under a cyclic stress of constant amplitude, it is necessary to specify this amplitude and the number of cycles. However, here the elapsed time (for a given number of cycles) is not important since the fatigue process is almost independent (within wide limits) of the rate with which the stress cycles are executed. There is also the problem of stochastic stress environments which we will not discuss here. Failure depends not only on the stress environment but depends on the thermal and chemical environments (e.g., the temperature and humidity); however, in the ensuing discussion we will assume a standard temperature and a chemically inert environment.

The result of inspection measurements are a given test piece will be represented by the n-dimensional vector \( y \). In the specific examples to be considered later, the inspection consists of a set of measurements of longitudinal-to-longitudinal scattering of elastic waves in the Rayleigh regime (i.e., low frequency or, equivalently, long wavelength) with various combinations of transmitter and receiver positions. Then, one scalar quantity represents the final result of measurement for each combination.

We next introduce a binary variable \( c \) describing the structural performance of the test piece. The variable \( c \) takes the value 0 if the test piece fails under a specified applied stress and the value 1 if it survives. Although the variable \( c \) is deterministically defined, it is only probabilistically related to the variable \( y \). In simple cases, the binary variable \( c \) can be related to more conventional variables (e.g., the failure stress \( \sigma_F \) or the time to failure \( t_F \)). The variable \( c \) can still be given a precise meaning when one considers cases with more subtle definitions of failure and/or with random stress environments.

The final stage of the decision process requires a knowledge of the joint probability function \( P(y, c) \). It is also of interest to consider two derived probability functions, namely

\[
P(c) = \int_{y} P(y, c)
\]

and

\[
P(y|c) = \frac{P(y, c)}{P(c)}
\]

The first function, \( P(c) \), is the unconditional probability of performance (failure or survival). The second function, \( P(y|c) \), is the probability density of \( y \) given the performance \( c \). It describes the normalized populations of test pieces in \( y \)-space of objects that are going to survive \( (c = 1) \) and fail \((c = 0)\), respectively. The nature of \( P(y|c) \) is illustrated in Fig. 3 for the case in which \( y \) is scalar.

![Diagram](https://via.placeholder.com/150)

**Fig. 3** The nature of failing and surviving populations and classification errors.

Two additional probability functions are also of interest, i.e.,

\[
P(y) = \int_{c} P(y, c)
\]

and

\[
P(c|y) = \frac{P(y, c)}{P(y)}
\]
The function $P(y)$ is the unconditional probability density of $y$ and it represents the total population of test pieces, i.e., the combination of the surviving and failing sub-populations. The function $P(c|y)$ is the probability of performance $c$ given the measurements $y$.

To carry out an optimization of the accept/reject decision in terms of minimum cost, we need two kinds of inputs: (a) the probability function $P(y,c)$ discussed above and (b) an optimality criterion that assigns an average cost to each candidate decision procedure. To formulate the optimality criterion we start with the introduction of the loss (or cost) function $L(c,c)$, which is the loss incurred if we decide the performance is $c(y)$ when it is actually $c$. For a given $y$ the decision $c = 1$ (survival) leads to acceptance and conversely $c = 0$ (future failure) leads to rejection. Thus the losses $L(0,1)$ and $L(1,0)$ are associated with false rejection and false acceptance, respectively. In the present analysis, the nature of the NDE measurement is assumed to be given and hence its cost is not explicitly considered. Typically the cost of false rejection is the cost of the test piece. On the other hand, the cost of false acceptance can be very large and clearly involves product-liability considerations. The optimality criterion to be considered here is the average loss (which is called risk, $R$, in the decision theory literature) given by

$$R = \sum_c \int dy \, L(c(y),c) \, P(y,c)$$

(5)

The parameters $w_0$, $w_1$ and $b$ are dependent solely on the loss function $L(c,c)$ and the unconditional performance probability $P(c)$. The quantities $e_0$ and $e_1$ are the two types of misclassification probabilities (or rates). Specifically, $e_0$ is the probability that we decide $c = 1$ (survival) when actually $c = 0$ (failure) and will call it the false-acceptance probability. Similarly, $e_1$ is the probability that we decide $c = 0$ (failure) when actually $c = 1$ (survival) and correspondingly we will call this the false-rejection probability. The nature of $e_0$ and $e_1$ is illustrated in Fig. 3.

With a given loss function a short calculation (2) leads to the result that the optimal decision rule is given by the scheme:

$$A(y) > \lambda \implies \hat{c} = 1 \text{ (i.e., accept);}$$

$$A(y) < \lambda \implies \hat{c} = 1 \text{ (i.e., accept);}$$

(6)

where

$$A(y) = P(y|1)P(y|0),$$

(7)

and where

$$\lambda = \frac{w_0}{w_1}.$$

(8)

The senses of the inequalities in (6) are based upon the assumption that the coefficients $w_0$ and $w_1$ are positive.

In order to deal separately with the models of failure and measurement, it is necessary to introduce a state vector $x$ having the property

$$P(y,c|x) = P(y|x)P(c|x),$$

(9)

i.e., when $x$ is specified $y$ and $c$ become statistically independent. From (1) we infer the relations

$$P(y,c) = P(y|x)P(c|x)$$

(10)

$$P(c|x,y) = P(c|x).$$

(11)

In words, (10) means that the probability density of $y$ given $x$ and $c$ does not depend on $c$ because the specification of $x$ represents sufficiently comprehensive knowledge that the additional specification of $c$ is irrelevant. A similar statement can be applied to (11).

Up to now we have not discussed how the state vector is to be related to the underlying physical realities. It is important to emphasize at this point that, although the physicists concept of state (at least in the classical case) is a possible realization of our concept, our likely choice is far cruder than that of the physicist. For example, in the case of ceramics, the state of a spherical voids with peripheral microcracks would be given by its radius. The nature of the microcracks would not be included in the state vector because there is no ND measurement procedure presently available for detecting them.

In any case, with the introduction of the state variable $x$, defined in terms of the decorrelation of failure and measurement processes (i.e., by Eq. (1)), we can write the joint probability of measurement and performance in the form

$$P(g,c) = \int dx \, P(y,c|x) \, P(x)$$

(12)

where (9) was used in obtaining the last line of (12). The integration on $x$ is assumed to span the entire domain of definition of state space, unless otherwise specified.

The schematic illustration in Fig. 2 of the analysis and optimization of the NDE decision process reflects the advantages of the separate modelling of measurement and failure achievable through the introduction of the state vector $x$.

**FAILU RE DUE TO A VOID WITH PERIPHERAL MICROCRACKS**

It is known that voids, which are almost always present in ceramics, are frequently the sites for the initiation of catastrophic crack growth under sufficiently large applied stresses. We consider a model of this process involving a random set of microcracks the periphery of the void with each crack having an independent probability of propagating to failure. We present a detailed treatment of the perhaps over-simplified case in which it is assumed that the propagation probability depends only on the local stress* at the void surface. Later, a brief analysis will be given of

*Here the local stress is defined as the stress that would be induced at the point in question by the applied stress if no microcracks were present.
the use in which it is assumed that the stress gradient also influences the propagation probability. For the sake of simplicity, the voids are assumed to be spherical.

Three independent considerations are involved in the assembly of a decision framework: the estimate of the pertinent defect dimensions from the inspection measurement y given the defect state x; the probability of performance c at a specified stress level \( \sigma_{eq} \) given the defect state x; and the a priori probability density of the state x corresponding to the distribution of defects. Each of these inputs is examined separately and then combined to provide the optimal accept/reject decision rule and associated decision performance measures.

**Measurement Process.** The relevant conditional probability density \( P(y|x) \) is implied by the stochastic measurement process

\[
y = n_0^3 + r
\]

where \( y \) is a possible measured value of \( A(w)/w^2 \), i.e., the scattering amplitude for longitudinal-to-longitudinal backscatter divided by the square of the frequency \( w \), evaluated at a sufficiently small value of \( \omega \). The quantity \( n_0^3 \) is the theoretical value of the above quantity when the state \( x = a \) (the void radius) is assumed to be known. The coefficient \( n \) depends only on the known properties of the host material and is given by the expression

\[
n = \frac{1}{3} \left( 1 + \frac{1}{2} \right) \left( 1 - 2v \right) \frac{10(1 - 2v)}{1 - 5v} \tag{1a}
\]

where \( c_0 \) is the propagation velocity of longitudinal elastic waves and \( v \) is Poisson's ratio. The experimental error \( r \) is assumed to be a Gaussian random variable with zero mean and covariance \( Cr \).

**Failure Process.** This subsection deals with the calculation of \( P(c|x) \), the probability of the performance \( c \) given the state \( x = a \) of the significant defect. In the present model the only type of defect that is significant in the context of structural failure is a spherical void. As illustrated in Fig. 4 this void has randomly positioned cracks distributed at its surface. With a specified applied stress, each crack has the potential of propagating into a large crack, subsequently causing structural failure. The probability of this happening is a function of the local stress \( \sigma_{eq}(r) \) in the neighborhood of the crack. The cracks are, in this instance, considered to be much smaller than the void diameter, so that the effects of stress gradients into the host can be neglected. The modifications that pertained when this condition is not satisfied will be discussed later.

Based upon this model, the probability of survival, given that the state \( x = a \) is specified, is

\[
P(1|a) = 1 - P(0|a) = \exp \left( -4n_0^3 a^2 \langle Q_A \rangle \right) \tag{2}
\]

where \( n_0 \) is the average surface density of cracks on the surface of the spherical void and \( \langle Q \rangle = \langle Q(\sigma_{eq}) \rangle \) is the probability that a crack at the position \( r \) on the surface will propagate to failure. The symbol \( \langle Q_A \rangle \) denotes the area average of \( Q \) over the surface of the void.

The A Priori Probability Density of Defects. Studies of defect densities in ceramics indicate that the large value extreme, of interest to fracture problems, can frequently be characterized by the cumulative distribution

\[
F(a) \equiv \int_0^a P(a)da = 1 - \exp \left[ - \left( \frac{a}{a_c} \right)^k \right] \tag{3}
\]

where \( a_c \) is a characteristic radius and \( k \) is a constant exponent.

**Results.** Here we combine the outputs of the last three subsections to yield \( P(y,c) \) from which we deduce \( P(y|c) \) and the classification errors \( e_0 \) and \( e_1 \).

It is desirable to introduce the dimensionless variables \( z = y/(c_{eq})^2 \), \( \xi = a/a_c \), and an additional dimensionless parameter \( \kappa = n_0^3 / C_{eq} \), which is signal-to-noise ratio characterizing the observation of elastic waves scattered from a spherical void of radius \( a_c \). Another useful quantity is the dimensionless failure parameter

\[
\zeta = 4n_0^3 a_c^2 \langle Q_A \rangle \tag{4}
\]

whose significance is given by \( P(1|x) = P(1|a) = \exp(-\xi) \) when \( a = a_c \) (i.e., the void has the critical radius defined by (9)). We actually compute \( P(z|c) \) instead of \( P(y|c) \) with a scale factor introduced into the normalization.

In Fig. 5 we present plots of \( P(z|0) \) and \( P(z|1) \) vs. \( z \) for \( k = 3 \), \( k = 10 \) and \( \zeta = 0.01 \). These figures show the structure of the \( c = 0 \) class (i.e., the normalized population of objects that are going to fail) and \( c = 1 \) class (i.e., the normalized complementary population of objects that are going to survive). Moreover, they show the nature of the overlap of the two classes.

In Fig. 6 we also give a plot of \( e_1 \) vs. \( e_0 \) for the same parameter values. This is the so-called "operating characteristic" of the system. It is to be emphasized again that \( e_0 \) is the falsely accepted fraction of objects that are actually going to fail. Conversely, \( e_1 \) is the falsely rejected fraction of objects that are actually going to survive.
Effect of Stress Gradients. A preliminary investigation has been made of the influence of the stress gradient effect on NDE performance. We take this effect into account by assuming that $\langle Q \rangle_A$ depends upon the spherical void radius $a$. Proceeding on a phenomenological level, let us assume that

$$\langle Q \rangle_A = f(\sigma_{ab}) a^m$$

(5)

where $\zeta$ is defined now by

$$\zeta = 4\pi a^m f(\sigma_{ab})$$

(7)

Clearly, many other modifications of $\langle Q \rangle_A$ may have greater physical justification, i.e., the case discussed by Evans, Biswas and Fulrath. However, the above modification enables us to obtain relatively simple results without difficulty.

We now obtain with $c >> 1$ and $\zeta << 1$ (i.e., large signal-to-noise ratio and low a priori failure probability) the results

$$p(c = 0) = \zeta(1 + \frac{m}{k})$$

(8)

$$p(c = 1) = 1 - \zeta(1 + \frac{m}{k})$$

(9)

$$e_1 = \exp (-u^*)$$

(10)

$$e_0 = \frac{\gamma(1 + \frac{m}{k} + u^*)}{\Gamma(1 + \frac{m}{k})}$$

(11)

where

$$\Gamma(a) = \int_0^\infty dt \, t^{a-1} \exp (-t)$$

(12)

is the gamma function and

$$\gamma(a, x) = \int_x^\infty dt \, t^{a-1} \exp (-t)$$

(13)

is the incomplete gamma function. The variable $u^*$ is defined by

$$u^* = (\gamma / \pi a C)^3 k / 3$$

(14)

We now consider the quantitative determination of the dependence of the operating characteristic upon the ratio $m/k$. In Fig. 7 we present a plot of the false acceptance probability $e_0$ vs. $m/k$ for a fixed false-rejection probability $e_1 = \exp (-1) = 0.368$ corresponding to $u = 1$. This result strongly suggests that this $e_1$ vs. $e_0$ curve moves closer to the horizontal and vertical axes as $m/k$ increases, i.e., the performance of the NDE system improves as this ratio increases. It is clear that with $m = 2$ the previous case with no stress.
gradient effect is obtained. In this case the improvement can be due only to the decrease of \( k \). This corresponds to an increase of the width of the combined populations of surviving and failing components relative to the width of the overlap region. On the other hand, with \( k \) fixed, the improvement can be brought about only by the increase of \( m \). This corresponds to the converse of the situation just discussed, i.e., the decrease of the overlap region relative to the width of the combined populations.

**Failure Due to a Subcritical Inclusion**

We turn now to the consideration of failure due to a subcritical inclusion. An example of such a system is an Si inclusion in Si3N4. The work reported here is largely due to Fertig and Meyer. Here, it is assumed that a crack is first nucleated in the interior of an inclusion of lower toughness than the host. The "bottleneck" in the failure process is the propagation of the crack through the inclusion boundary into the host, a process requiring a substantially higher level of applied stress than that required to produce the earlier stages of crack development within the inclusion.

The geometrical nature of the model of the defect and its observation by elastic wave scattering is depicted in Fig. 8. We assume a semi-infinite specimen with known host material. With Cartesian coordinate system partially shown, the boundary of the specimen is parallel to the xy-plane and the outward pointing normal lies in the positive z-direction. We assume that the defect is an ellipsoidal inclusion (although the subsequent analysis is limited for the sake of brevity to the spheroidal case) with a known included material. We explicitly show a pulse-echo (i.e., backscatter) measurement with the incident wave pointed in the negative z-direction. However, additional transducer configurations will be considered later.

\[ x = \begin{pmatrix} a \\ c \\ 0 \\ y_z \end{pmatrix} \]  

where \( \theta \) is the azimuthal angle (in the xy-plane), \( \gamma_x \) the direction cosine of \( \mathbf{w} \) relative to the z-axis. The vector \( \mathbf{w} \) can be expressed in terms of \( \theta \) and \( \gamma_z \) as follows

\[ \mathbf{w} = (1 - \gamma_z^2)^{1/2} (\mathbf{e}_x \cos \theta + \mathbf{e}_y \sin \theta) + \gamma_z \mathbf{e}_z \]

where \( \mathbf{e}_x \), \( \mathbf{e}_y \), and \( \mathbf{e}_z \) are the unit vectors in the Cartesian coordinate directions.

**Measurement Process.** We assume that the measurements consist of an arbitrary number of low-frequency longitudinal-to-longitudinal backscatter processes. These are collectively represented by a standard stochastic model of the generic form

\[ y = f(x) + r \]

where \( y \), \( f(x) \), and \( r \) are \( N \)-dimensional vectors (but considerable attention will be devoted to the case \( N = 1 \)). The exact theoretical measurement \( f(x) \) contains an \( n \)th component given by

\[ f_n(x) = \lambda_n L (\mathbf{e}_n - \mathbf{e}_n; x) \]

where it is assumed that \( \mathbf{e}_n - \mathbf{e}_n = \mathbf{e}_n \).

The conditional probability density \( P(y|x) \) is then given by

\[ P(y|x) = G(y - f(x), C_r) \]

where \( G(., .) \) is the \( N \)-dimensional Gaussian probability density given by

\[ G(u,C) = (2\pi)^{-N/2} (\text{Det } C)^{-1/2} \exp\left(-\frac{1}{2} u^T C^{-1} u\right) \]

**Failure Process.** It is assumed that a uniaxial stress is applied in the x-direction. At sufficiently high levels the stress causes the initiation of processes described earlier. An interior defect thus causes a crack to propagate through the inclusion. We make the rather crude assumption that this crack forms, as represented by the dashed line AA in Fig. 8, a plane intersecting the geometrical center of the spheroid and having an orientation perpendicular to the axis of the applied stress, i.e., the x-axis. At a sufficiently higher value of the applied stress the crack will propagate from the lower toughness inclusion (e.g., Si) into the higher toughness host material (e.g., Si3N4). We assume that the condition for this event can be adequately represented by an empirically recalibrated version of simple fracture mechanics with a Gaussian random additive variable representing the inherent variability in the fracture process.
In explicit mathematical terms, we assume that the performance variable \( c \) is given by

\[
c = H(c_T - \sigma_w)
\]  

(5)

where \( H(\cdot) \) is the Heaviside unit step function, \( \sigma_w \) is the applied stress, and \( c_T \) is the failure stress. The latter quantity is a random variable by the random process

\[
\sigma_F = \alpha + \beta \sigma_p + s
\]  

(6)

where \( \sigma_p \) is the failure stress predicted according to simple fracture mechanics, \( \alpha \) and \( \beta \) are empirical recalibration constants, and \( s \) is a Gaussian random variable with zero mean and variance \( \sigma_0^2 \).

The application of simple fracture mechanics, i.e., the computation of yield stress under the assumption that the ellipsoidal crack is surrounded solely by host material, gives

\[
\sigma_p = \frac{K_c}{Z(c'/a')} (7)
\]

where \( K_c \) is the fracture toughness, \( a' \) and \( c' \) are the major and minor semi-axis lengths of the fully developed inclusion crack, and \( Z(\cdot) \) is a function defined by

\[
Z(u) = \int_0^{\infty} du (1 - \sin^2 u)^{1/2} \exp \left( - \frac{1}{2} t^2 \right) (8)
\]

As stated earlier, we assume that the fully developed crack inside the inclusion is represented by the cross section formed by a plane, perpendicular to the x-axis, passing through the center of the spheroid. A straightforward geometrical analysis yields the result

\[
a' = a \]  

(9)

\[
c' = (a^2 + (c^2 - a^2)w^2)^{1/2} \]  

(10)

where \( w \) is the length of the projection of \( w \) (the unit vector defining the axis of symmetry of the spheroid) onto the crack plane. Using (2) we obtain

\[
w^2 = 1 - (1 - a^2) \cos^2 \theta . \]  

(11)

Equations (7)-(11) thus give \( \sigma_p \) as a function of the state vector \( x \) defined by (1).

We turn finally to the calculation of \( P(c|x) \). First we observe that, according to the stochastic model (b), the conditional probability density of \( \sigma_p \) is given by

\[
P(\sigma_p|x) = G(\sigma_p - \alpha - \beta \sigma_p(x), \sigma_0^2) \]  

(12)

where \( G(\cdot, \cdot) \) is the Gaussian function defined by (4b) which in the present case is specialized to a case of a scalar variable, i.e.,

\[
G(u, c) = (2\pi c)^{-1/2} \exp(-\frac{1}{2} c^{-1} u^2) \]  

(13)

Using (5) we obtain

\[
P(c = 0|x) = \int_{-\infty}^{c} d\sigma_p P(\sigma_p|x) \]  

(14)

where the function \( \phi(u) \) is the error integral

\[
\phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} dt \exp \left( - \frac{1}{2} t^2 \right) . \]  

(15)

Clearly, the conditional probability \( P(\sigma = 1|x) \) is given by

\[
P(c = 0|x) + P(c = 1|x) = 1, \]  

(16)

the normalization condition.

A Priori Probability Density. In the next several paragraphs, we discuss the a priori probability density \( P(x) \), which is more complicated than that assumed in the previous section because the state vector \( x \) is now four-dimensional in order to characterize the spheroidal geometry.

We assume that the semi-axis lengths are independent of the angle variables. Furthermore, we assume that the latter are distributed with axial symmetry about the z-axis. These assumptions imply that

\[
P(x) = P(a, c, \theta, \gamma_z) = P(a, c) P(\theta) P(\gamma_z) \]  

(17)

with

\[
P(\theta) = \frac{1}{2\pi} \]  

(18)

If the axis of symmetry (represented by the unit vector \( \theta \)) is completely isotropically distributed, then we obtain

\[
P(\gamma_z) = 1 \]  

(19)

under the assumption that \( \gamma_z \) is constrained to be positive. If the axis of symmetry is preferentially nearly parallel to the z-axis, then we could assume a probability density of the form

\[
P(\gamma_z) = (1 + v) \gamma_z^v \]  

(20)

with \( v > 0 \).

Here the distribution of semi-axis lengths is constrained by the inequality \( a > c \) in order to limit the class of spheroids to the oblate case. In order to imitate the a priori probability density used in the last section, we assume

\[
P(a, c) = B g(a) h(c) \]  

(21)

where

\[
g(a) = \left( \frac{a}{a_c} \right)^{k-1} \frac{K_c}{\sigma_c} \exp(-\frac{a}{\sigma_c} k) \]  

(22)

\[
h(c) = \left( \frac{c}{c_c} \right)^{k-1} \frac{K_c}{\sigma_c} \exp(-\frac{c}{\sigma_c} k) \]  

(23)
The exponent k determines the sharpness of the decline of $g(a)$ or $h(c)$ when $a$ or $c$ exceeds the characteristic values $a_c$ or $c_c$, respectively.

Combination of Probabilities. In order to compute the probability functions $P(x,c)$, $P(c|y)$, etc., we must first combine the various results of the previous paragraphs according to the relation

$$ P(c,y) = \int dx \, P(c|x) \, P(y|x) \, P(x) $$

The calculation of $P(c|y)$, $P(y)$, and $P(c)$ have been discussed earlier.

In terms of the functions pertaining explicitly to the present case of failure due to subcritical inclusions, we obtain

$$ P(y) = \int dx \, G(y-f(x), c_r) $$

where $G(\cdot,\cdot)$ is the Gaussian probability density defined by (4b), and $f(x)$ is given by (4). In the case of $P(c,y)$ we need consider only $P(c=0,y)$ now given by

$$ P(c=0,y) = \int dx \, \phi(C^{-1/2}(a-a_0-s_{np}(x))) \, G(y-f(x), c_r) P(x) $$

where $\phi(\cdot)$ is defined by (15). We can use the relation

$$ P(c=0,y) + P(c=1,y) = P(y) $$

(28)

to obtain other functions of interest.

Computations. In the numerical computations, we have used a Monte Carlo technique in which quantities of the type $dx(\cdot)P(\cdot)$ are replaced by $\sum \sum \sum$ where the samples of the state vector in the set $S$ have been drawn at random in accordance with the probability density $P(x)$.

Throughout the computations we have employed the centimeter-gram-microsecond (c-g-µs) system of physical units except in the case of stress or pressure which is expressed in pascals (Pa). This exception entails no difficulty because stresses will always be divided by other quantities involving the same units.

In all computations we will uniformly use the following assumptions and parameter values.

1. A priori statistics are partly defined by the assumption that the angular distribution of the axis of symmetry, defined by $w$, is isotropic, i.e., (18) and (19) hold. It is further defined by assuming the distribution of semi-axis lengths is given by (21)-(24) with $k=1$, $a_0=0.0325$, and $c_c=0.0075$.

2. The material properties of the host $(S_1(M))$ are assumed to take the values $\rho=3.200$, $\lambda=1.586$, and $\mu=1.250$ (\(\rho\), \(\lambda\), and \(\mu\) are the density and the two Lamé constants). The corresponding properties of the inclusion $(S_i)$ are assumed to take the values $\rho'=2.340$, $\lambda'=0.527$, and $\mu'=0.580$.

3. In the failure process the empirical recalibration constants take the values $\alpha=0.997 \times 10^3$ Pa and $\beta=0.541$. For the fracture toughness constant we assume $K_c=0.500 \times 10^3$ Pa cm$^{1/2}$.

The purely statistical parameters $c_a$ and $c_c$ and the applied stress $\sigma_a$ will be discussed later.

In the following paragraphs we present numerical results for three cases to illustrate the separate effects of randomness and completeness in the measurement process and randomness in the failure process.

Case 1. One Measurement. Random Measurement and Failure Processes. In this case we consider a single ND measurement, i.e., a pulse-echo, longitudinal-to-longitudinal scattering of elastic waves with the incident propagation in the negative $z$-direction. The random experimental error is represented by the standard deviation $C_{22}^e=10^6$ corresponding to a signal-to-noise ratio of about 10, using as a signal standard the return from a scattered wave with $a=a_0$, $c=c_c$, $\gamma_z=1$ and $\theta$ irrelevant. It is to be stressed that the single measurement assumed here represents a decidedly incomplete set of measurements since the state vector is four-dimensional. In the failure process, we assume the applied stress $\sigma_a=2.5 \times 10^6$ Pa and a standard deviation of $\sigma_F$ given by $C_{F/2}^e=0.367 \times 10^6$ Pa.

In Fig. 9 we show the computed curves of $P(y|c=0)$ and $P(y|c=1)$ representing the failing and surviving populations. It is clear that the severe overlap will yield a rather poor NDE performance as indicated by the plot of false rejection probability $\epsilon_1$ vs. false acceptance probability $\epsilon_0$ shown in Fig. 10. The poor performance is associated with three factors:

![Fig. 9 Probability densities of falling (c=0) and surviving (c=1) populations vs. ND measurement y.](image-url)
incompleteness in the set of measurements, randomness in the measurement process, and randomness in the failure process. The two remaining cases will throw some light on this matter.

Case 2. One Measurement. Deterministic Measurement and Failure Processes. Here we consider again a single measurement of the same kind as in the last case. However, for the sake of understanding we eliminate the randomness from the measurement and failure processes by setting the variances $C_s = C_0 = 0$. The resultant NDE performance (hypothetical) is given by the $e_1$ vs. $e_0$ plot in Fig. 11. Although there is a marked improvement in the performance, i.e., the curve has moved closer to the horizontal and vertical axes, the performance is hardly what one would expect from a perfect system. This is due, as one might expect, to the serious incompleteness of the measurement set. Incidentally, the lack of smoothness of the curve is due to the relatively small fraction of Monte Carlo samples that actually affect the final answer.

It is desirable to add a few clarifying remarks concerning the effect of an incomplete measurement set. A deterministic measurement model implies that the relation

$$P(y|x) = \delta(y-f(x))$$

(29)

holds, i.e., a given value of the vector $x$ implies a unique value of $y$. This result holds regardless of the dimensionality of $y$ relative to the dimensionality of $x$ and it is obviously valid when $y$ is scalar. However, in the present case of an incomplete measurement set one cannot make analogous statements concerning $P(x|y)$. Here a given value of the scalar $y$ does not imply a unique value of the four-dimensional vector $x$, but only a unique three-dimensional subspace. Thus here $P(x|y)$ is approximately proportional to $P(x)$ with a constraint that $x$ lies in this subspace. More precisely $P(x|y)$ is given by

$$P(x|y) = P(y|x) P(x)/P(y)$$

$$= \delta(y-f(x)) P(x)/P(y)$$

(30)

with $P(y)$ playing the role of a normalization constant. It is obvious that if $P(x)$ is sharply peaked in this subspace (defined by $f(x) = y$) then the lack of completeness in the measurement set does not lead to a serious degradation of NDE performance.

Case 3. Complete Measurement Set. Deterministic Measurement Process but a Random Failure Process. In this case, we assume a significantly large diversity of very accurate measurements that the measurement vector $y$ implies a unique estimate of $x$, namely $\hat{x}(y)$, with a negligible a posteriori variance (more precisely, a covariance matrix $\text{Cov}(x|y)$, whose eigenvalues are sufficiently small in an appropriate sense). This means that we can write

$$P(x|y) = P(x = \hat{x}(y))$$

(31)

or, in more explicit terms

$$P(x|y) = \phi(C_x^{1/2}(x_{\text{ref}} - a \Psi_p(x(y))))$$

(32)

where $\phi(\cdot)$ is defined by (15). In the damage process, we assume the same standard deviation as in Case 1, i.e., $C_s = 0.367 \times 10^5$ Pa, but with a somewhat higher applied stress, i.e., $\sigma_0 = 3.5 \times 10^5$ Pa.

The resultant plot of $e_1$ vs. $e_0$, the NDE operating characteristic, is presented in Fig. 12. This highly satisfactory result demonstrates clearly that randomness in the present failure process (failure initiated in subcritical inclusions) is not a significant contributor to the degradation of NDE performance. In order to understand the relative contributions of incompleteness and randomness in the measurement process, it would be interesting to investigate the case in which the measurement set is complete but randomness in the form of measurement error remains. Because of excessive computational labor, this has not yet been done.
We have set up a complete formalism for the calculation of \( P(c|y) \), the probability of performance (failure or survival) of a structural component given the results of ND measurements. With the definition of a suitable loss function giving the costs of wrong decisions, an optimal accept/reject decision procedure was derived. With the inclusion of \( P(y) \), the probability density of ND measurement results on the entire population of failing and surviving components, it was possible to calculate the so-called operating characteristic, the plot of the probability of false-rejection vs. the probability of false-acceptance for all possible loss functions. This curve provides a unique characterization of the behavior of the NDE system independently of the loss function and a priori component performance probability.

The discussion here involves a basic approximation, namely that the most significant (from the standpoint of the probability of causing failure) defect is considerably more significant than the combined effect of all of the remaining defects. The specific application of this formalism was to the case of brittle fracture in ceramics. We considered two kinds of defects: voids and subcritical inclusions. In the first case, failure is associated with peripheral microcracks, any one of which may propagate to failure. In the second case the failure is connected with the possibility of a crack propagating from a lower-toughness inclusion into a higher-toughness host. In each case we assumed that the ND measurements consisted of a set of low-frequency, L-to-L, pulse-echo scattering measurements (it is understood that a set composed of a single measurement is an admissible special case). The analysis of failure in ceramics is especially simple because to a high degree of approximation there is no slow evolution of failure (e.g., like fatigue in metals) before rapid catastrophic failure occurs. Thus, here the probability of failure depends, in the case of a uniaxial applied stress, only upon the maximum positive (i.e., tensile) stress applied during an appropriate time interval. In present treatment, we regarded this maximum stress as a parameter with an arbitrarily specified value.

The problem of estimating the conditional probability of failure for the two kinds of failure mechanisms in ceramics has been investigated with the aid of synthetic (i.e., theoretical) ND measurement data. The operating characteristic (i.e., the plot of false rejection probability vs false-acceptance probability) was determined for various combinations of parameter values. It is noteworthy that the unconditional failure probability (i.e., the fraction of the total population that would fail under the assumed applied stress) had very little influence on the operating characteristics, thereby reinforcing the notion that these curves reflect the incremental value of NDE. Calculations conducted for the case of voids with peripheral microcracks, without stress gradient effects taken into account, yielded operating characteristics that were rather disappointing, a feature that was due in most cases almost entirely to randomness inherent in the failure model. However, the calculations carried out for the case of subcritical inclusions yield very different results. We obtained poor and good operating characteristics depending on the degree of completeness of the set of ND measurements. In all cases, the degradation due to randomness in the failure model was very minor.

ACKNOWLEDGEMENT

This research was sponsored by the Center for Advanced NDE operated by the Science Center, Rockwell International, for the Advanced Research Projects Agency and the Air Force Materials Laboratory under Contract No. F33615-74-C-5180.

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