1969

Operator and dual operator bases in linear topological spaces

William Buhmann Johnson
Iowa State University

JOHNSON, William Buhmann, 1944-
OPERATOR AND DUAL OPERATOR BASES
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Iowa State University, Ph.D., 1969
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

OPERATOR AND DUAL OPERATOR BASES
IN LINEAR TOPOLOGICAL SPACES

by

William Buhmann Johnson

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
1969

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I. INTRODUCTION

Let V be a Hausdorff linear topological space. A basis of subspaces for V is a sequence $\{M_n\}_{n=1}^{\infty}$ of subspaces of V such that for every x in V, there is a unique sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in M_n$ such that

$$x = \sum_{n=1}^{\infty} x_n .$$

Bases of subspaces were introduced by Grinblyum in [6] and have been extensively studied by McArthur and his students. Note that a summation basis can be considered to be a basis of subspaces $\{M_n\}_{n=1}^{\infty}$ where each M_n is one dimensional.

It is known that many of the theorems concerning summation bases have analogues in the theory of bases of subspaces. For example, the weak basis theorems of [1], [2], and [3] hold true for bases of subspaces if each M_n is closed [11]; the characterization of reflexivity of a Banach space in terms of a boundedly complete and shrinking basis given by James in [7] is true for a basis of subspaces if each M_n is reflexive [15].

Surprisingly enough, most of the proofs of theorems involving a basis of subspaces do not use the M_n's but consider instead a sequence of linear projections defined in terms of the M_n's. Let $\{M_n\}_{n=1}^{\infty}$ be a basis of subspaces for V. For each n, define P_n by $P_n(x) = x_n$, where

$$\sum_{i=1}^{\infty} x_i$$

is the expansion of x in terms of the elements of the M_i's. The uniqueness of the expansion guarantees that each P_n is a linear projection of V onto M_n and that P_nP_m is the zero operator if $n \neq m$. The sequence of partial sum operators associated with the basis $\{M_n\}_{n=1}^{\infty}$ is the sequence $\{S_n\}_{n=1}^{\infty}$ defined by

$$S_n = \sum_{i=1}^{n} P_i .$$

Each S_n is a linear projection of V onto the span of

$$\bigcup_{i=1}^{n} M_i .$$

Further, for each n and m, $S_nS_m = S_{\min(n,m)}$; and for each x in V,

$$\lim_{n\to\infty} S_n(x) = x.$$

If each P_n is continuous (equivalently, if each S_n is continuous), $\{M_n\}_{n=1}^{\infty}$ is called a Schauder basis of subspaces for V or a Schauder decomposition for V.

The theorems concerning bases of subspaces usually utilize only the properties of the partial sum operators associated with a basis of subspaces. In this dissertation we abstract the properties of these operators to get the following definition:

Definition I.1. Let V be a Hausdorff linear topological space and let $\{S_d : d \in D\}$ be a net of linear projections of finite range on V. V is said to be an operator basis--O.B.--(resp. dual operator basis--D.O.B.) for V provided

(1) for each x in V, the net $\{S_d(x) : d \in D\}$ is bounded and converges to x;

(2) $S_e S_d = S_d$ whenever $e \geq d$ (resp. $S_d S_e = S_d$ whenever $e \geq d$);

(3) if $\{x_j\}$ is a net in V which converges to x, and if

$$\lim_j S_d(x_j) = y_d$$

uniformly on d in D, and if

$$\lim_{d} y_d = x,$$

then $S_d(x) = y_d$ for all d in D.

If each S_d is continuous, $\{S_d\}$ is said to be a Schauder O.B.--S.O.B.--(resp. Schauder D.O.B.--S.D.O.B.). If, in addition, $\{S_d\}$ is uniformly bounded on bounded sets, $\{S_d\}$ is said to be an u.b.S.O.B. (resp. u.b.S.D.O.B.). If $\{S_d\}$ is equicontinuous, $\{S_d\}$ is said to be an e-S.O.B. (resp. e-S.D.O.B.).

Some comments are in order about the various parts of Definition I.1. In most of the work that follows, the assumption that each S_d has finite dimensional range is not essential. In particular, this condition could be replaced in Chapter II by the assumption of closed range and in Chapter III by the assumption of reflexive range. However, it seems doubtful that such a general theory has any useful applications and the assumption of finite range considerably simplifies the proofs of several theorems. Note that $\{I\}$, where I is the identity operator on V, satisfies all the conditions of Definition I.1 other than the finite range condition.

The pointwise boundedness condition in (1) does not arise in the definition of basis of subspaces because a convergent sequence is automatically bounded--in fact, it is totally bounded. As we shall see, this mild-seeming condition has powerful implications. Roughly speaking, one can say that the structure of a space with a S.O.B. or S.D.O.B. is determined by the structure of its bounded sets.

Condition (2) for O.B.'s says that the ranges of the S_d's are directed by inclusion. Condition (2) for D.O.B.'s says that the null spaces of the S_d's are directed by containment. Note that the sequence of partial sum operators associated with a basis of subspaces satisfies both of these conditions.

Condition (3) replaces the uniqueness of expansion condition for a basis of subspaces. The relationship between (3) and the uniqueness of expansion condition is made precise in Theorem II.1.

In Chapter II, we consider conditions on V which will guarantee that an O.B. or D.O.B. for V is a S.O.B. or S.D.O.B. for V. The corresponding results for summation bases are in [11].

In Chapter III, we deal with applications of S.O.B.'s and S.D.O.B.'s to the duality theory of locally convex spaces. Note that if $\{S_d : d \in D\}$ is a S.O.B. (resp. S.D.O.B.) for V, then $\{S_d^* : d \in D\}$ is a S.D.O.B. (resp. S.O.B.) for V* if V* is endowed with the weak* topology. (This relationship justifies the use of "dual" in "dual operator basis".)

The main result of Chapter III is that a locally convex space with a S.D.O.B. is semi-reflexive if and only if the S.D.O.B. is shrinking (Definition III.7) and boundedly complete (Definition III.4). The corresponding theorem for Schauder bases in Banach spaces was proved by James in [7]. Retherford [14] established the "if" part of the theorem for Schauder bases in locally convex spaces, and he proved the "only if" part for Schauder bases in complete, reflexive spaces.

Theorems III.11, III.12, and III.14 discuss the duality between shrinking S.O.B.'s and boundedly complete S.D.O.B.'s. In [4], Dubinsky and Retherford prove III.11 and III.12 and a restricted variant of III.14 for Schauder bases in locally convex spaces.

The other results of Chapter III are structure theorems for locally convex spaces with a S.O.B. or S.D.O.B. These theorems generalize the best known results for Schauder bases. A typical example is Theorem III.5, which shows that a space with a boundedly complete e-S.D.O.B. is complete. In [4], Dubinsky and Retherford show that a space with a boundedly complete e-Schauder basis is sequentially complete.

In Chapter IV we restrict our attention to Banach spaces. The most important results of this chapter are Theorem IV.2 and Corollary IV.6. Theorem IV.2 shows that every separable Banach space which has a D.O.B. admits a Schauder decomposition into finite dimensional subspaces. Corollary IV.6 shows that a separable Banach space which admits an O.B. $\{S_d : d \in D\}$ with

$$\lim_d \|S_d\| = 1$$

must also admit a Schauder decomposition into finite dimensional subspaces. We think that the technique of infinite compositions of linear operators used in Theorem IV.2 (and in Theorem IV.5) is of special interest, although it does not seem to have been exploited previously.

As an application of Theorem IV.2, we get an easy proof of the Michael and Pełczynski result [13] that if K is a compact metric space, $C(K)$ is a π_1^∞ space. The proof utilizes the fact that if K is a compact Hausdorff space, then $C(K)$ admits a very special kind of S.D.O.B. (Theorem IV.8).

Throughout this work we use the notation and terminology of [9]. Particularly in Chapter III, we continually reference the results of [9], so the reader will probably find it convenient to have this book on hand when reading Chapter III. We also use the following notation throughout this dissertation. V represents a Hausdorff linear topological space over the real or complex field. If S is a linear operator on V, $R(S)$ denotes the range of S and $\ker(S)$ denotes the null space of S. "0" is used for the null vector of a linear space. "I" denotes the identity operator. If A is a subset of a linear space, $\mathrm{sp}\, A$ denotes the linear span of A. If A is a subset of a topological space, X, $\mathrm{cl}\, A$ denotes the closure of A in X. "If and only if" is ordinarily abbreviated to "iff".

II. WEAK BASIS THEOREMS

Theorem II.1 justifies the assertion that O.B.'s and D.O.B.'s generalize the concept of finite dimensional bases of subspaces.

<u>Theorem II.1</u>. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of linear operators on a linear topological space V. Then $\{S_n\}_{n=1}^{\infty}$ is both an O.B. and D.O.B. for V iff $\{S_n\}_{n=1}^{\infty}$ are the partial sum operators associated with a finite dimensional basis of subspaces for V.

<u>Proof</u>. To go one way, suppose that $\{S_n\}_{n=1}^{\infty}$ is both an O.B. and D.O.B. for V. Let $M_1 = R(S_1)$, $M_n = R(S_n - S_{n-1})$ $(n > 1)$. Note that each M_n is finite dimensional. $\{M_n\}_{n=1}^{\infty}$ is representing, because if $x \in V$,

$$x = \lim_n S_n(x) = S_1(x) + \sum_{n=2}^{\infty} (S_n - S_{n-1})(x).$$

We show that this representation of x is unique. Suppose that

$$x = \sum_{n=1}^{\infty} x_n ,$$

where $x_n \in M_n$. Let U be a neighborhood of 0 in V.

Since the series

$$\sum_{n=1}^{\infty} x_n$$

converges, there is N such that if $N \leq n < m$,

$$\sum_{k=n+1}^{m} x_k \in U .$$

It follows that

$$\lim_{j} S_n\left(\sum_{i=1}^{j} x_i\right) = \sum_{i=1}^{n} x_i ,$$

uniformly on n. We conclude from I.1 (3) that

$$S_n(x) = \sum_{i=1}^{n} x_i ,$$

for all n. Thus $S_1(x) = x_1$, and $(S_n - S_{n-1})(x) = x_n$, for $n > 1$.

To go the other way, suppose that $\{S_n\}_{n=1}^{\infty}$ are the partial sum operators associated with a finite dimensional basis of subspaces for V. $\{S_n\}_{n=1}^{\infty}$ obviously satisfies conditions (1) and (2) of I.1. Let $\{x_j\}$, x, and $\{y_n\}_{n=1}^{\infty}$ be as in (3) of I.1. Since $R(S_n)$ is finite dimensional for each n, $R(S_n - S_{n-1})$ is finite dimensional for each $n > 1$. In particular, these subspaces are closed. Thus $y_1 \in R(S_1)$ and for $n > 1$,

$$y_n - y_{n-1} = \lim_j (S_n - S_{n-1})(x_j) \in R(S_n - S_{n-1}).$$

But

$$x = \lim_n y_n = y_1 + \sum_{n=2}^{\infty} y_n - y_{n-1},$$

so that $y_1 = S_1(x)$ and $y_n - y_{n-1} = (S_n - S_{n-1})(x)$ for $n > 1$. From this it follows that $y_n = S_n(x)$, for all n.

QED

The following lemma provides most of the machinery for deriving the so-called "weak basis" theorems for O.B.'s and D.O.B.'s. The lemma is a straightforward generalization of Lemma 2 in [11].

<u>Lemma II.2</u>. Let (V,T) be a [locally convex] linear topological space. Let $\{S_d : d \in D\}$ be a [weak] O.B. or a [weak] D.O.B. for V. Then there is a linear topology T' for V such that

(1) $T \subset T'$;

(2) $\{S_d : (V,T') \to (V,T) : d \in D\}$ is equicontinuous;

(3) if (V,T) is locally convex, so is (V,T');

(4) if (V,T) is metrizable, so is (V,T');

(5) if (V,T) is complete or quasi-complete (bounded Cauchy nets are convergent) or sequentially complete, then so is (V,T').

Proof. Let L be a local base of closed, circled T neighborhoods of 0. If T is locally convex, let each member of L be convex. Let $L' = \{U' : U \in L\}$, where

$$U' = \bigcap_{d \in D} S_d^{-1}(U) .$$

McArthur's proof of Lemma 2 in [11] shows that L' is a local base for a linear topology, T', on V, and that (1), (2), and (3) are satisfied. (4) is satisfied because if T is metrizable, L can be chosen to be countable, so that L' is countable. We show that (5) holds. Suppose that (V,T) is complete (resp. quasi-complete; resp. sequentially complete). Let $\{x_a : a \in A\}$ be a T'-Cauchy net (resp. bounded T'-Cauchy net; resp. T'-Cauchy sequence). By (1), $\{x_a\}$ is T-Cauchy (and T-bounded if $\{x_a\}$ is T'-bounded), and thus T-converges to, say, x. Since $\{x_a\}$ is T'-Cauchy, it follows from the definition of T' that $\{S_d(x_a) : a \in A\}$ is T-Cauchy, uniformly on $d \in D$. Thus

$$T - \lim_{a} S_d(x_a)$$

exists for each $d \in D$, and in fact uniformly on $d \in D$ [hence weakly uniformly on $d \in D$]. Let

$$y_d = T - \lim_{a} S_d(x_a).$$

We show that $\{y_d : d \in D\}$ [weakly] converges to x.

Let K be a [weak] neighborhood of 0 in (V,T). [Let J be a weak neighborhood of 0 such that $J + J \subset K$.] Let $U \in L$ such that $U + U + U \subset K$ [such that $U + U \subset J$.] Let $N \in A$ such that if $a \geq N$, $x - x_a \in U$. Choose $N' \in A$ such that if $a \geq N'$, $S_d(x_a) - y_d \in U$, for all $d \in D$. Fix $a \in A$ such that a follows both N and N'. Choose $M \in D$ such that for $d \geq M$, $x_a - S_d(x_a) \in U$ [such that $x_a - S_d(x_a) \in J$]. Then if $d \geq M$, $x - y_d = (x - x_a) + (x_a - S_d(x_a)) + (S_d(x_a) - y_d) \in K$. Thus $\{y_d\}$ [weakly] converges to x. It follows from I.1, (3) that $S_d(x) = y_d$, for all $d \in D$. But $\{S_d(x_a) : a \in A\}$ T-converges to $y_d(= S_d(x))$ uniformly on $d \in D$, so it follows from the definition of T' that $\{x_a\}$ T'-converges to x. QED

<u>Theorem II.3</u>. Let $\{S_d : d \in D\}$ be a weak S.O.B. for a locally convex space V. Suppose either that $\{S_d\}$ is u.b. and V is evaluable, or that V is barrelled. Then $\{S_d\}$ is an e-S.O.B. for V.

<u>Proof</u>. Either hypothesis guarantees that $\{S_d\}$ is equicontinuous. Now $\{S_d(x) : d \in D\}$ is eventually x if

$$x \in \bigcup_{d \in D} R(S_d) ,$$

so $\{S_d : d \in D\}$ converges pointwise to the identity operator, I, on the subspace

$$\bigcup_{d \in D} R(S_d) .$$

Since $\{S_d\}$ is equicontinuous, it converges pointwise to I on

$$\mathrm{cl}\Big[\bigcup_{d \in D} R(S_d)\Big] = \text{weak-cl}\Big[\bigcup_{d \in D} R(S_d)\Big] = V. \qquad \text{QED}$$

<u>Theorem II.4</u>. An O.B. (resp. a D.O.B.) in a complete linear metric space is an e-S.O.B. (resp. an e-S.D.O.B.).

<u>Proof</u>. Immediate from Lemma II.2 and the open mapping theorem. QED

Theorem II.5. A weak O.B. for a Frechet space is an e-S.O.B.

Proof. Lemma II.2 and the open mapping theorem imply that the elements of the O.B. are continuous. The desired conclusion then follows from Theorem II.3. QED

III. APPLICATIONS TO DUALITY THEORY

In this chapter we let V be a Hausdorff, locally convex, linear topological space. Endow V^*, the dual of V, with the strong topology $s(V^*,V)$. Recall that a local base for V^* is $\{B^{\circ} : B$ is a bounded subset of $V\}$, where $B^{\circ} = \{f \in V^* : |f(b)| \leq 1,$ for all $b \in B\}$. Alternatively, V^* has the topology of uniform convergence on bounded subsets of V.

If P is a continuous linear operator on V into V, define $P^* : V^* \to V^*$ by $P^*(f) = f \circ P$. P^* is necessarily continuous on V^* [9, p.204, 21.6]. If $\{S_d : d \in D\}$ is a S.D.O.B. for V, let

$$Y = \bigcup_{d \in D} R(S_d^*) ,$$

and let $\bar{Y}$ be the strong closure of Y in V^*. Note that Y, and hence $\bar{Y}$, is a linear subspace of V^*.

One of the most interesting duality-theory results concerning Schauder bases in Banach spaces is the theorem of James [7] which shows that a Banach space with a Schauder basis is reflexive iff the basis is both shrinking (Definition III.7) and boundedly complete (Definition III.4). More recently, Retherford [14] has related the semi-reflexivity of a locally convex space to the existence of a shrinking and boundedly complete Schauder basis for the space. The main results of this section, Theorems III.15 and III.17, show that a locally convex space with a S.D.O.B. is semi-reflexive iff the S.D.O.B. is both shrinking and boundedly complete. Theorem III.15, when applied to Schauder bases, generalizes the best known results of this kind (see [4] and [14]).

Singer [16] noted a duality between shrinking and boundedly complete Schauder bases. If $\{S_n\}_{n=1}^{\infty}$ are the partial sum operators associated with a Schauder basis for a Banach space V, then $\{S_n\}_{n=1}^{\infty}$ is shrinking iff $\{S_n^*\}_{n=1}^{\infty}$ is a boundedly complete basis for V^*; $\{S_n\}_{n=1}^{\infty}$ is boundedly complete iff $\{S_n^*\}_{n=1}^{\infty}$ is a shrinking basis for $\bar{Y}$. Dubinsky and Retherford [4] extended this result to Schauder bases in certain kinds of locally convex spaces. Theorems III.11, III.12, and III.14 verify that under reasonable conditions on V (which are satisfied whenever V is quasi-complete and evaluable), there is a duality between shrinking and boundedly complete S.O.B.'s and S.D.O.B.'s.

The following three known lemmas are useful for obtaining the results of this chapter.

<u>Lemma III.1</u>. Let $\{P_i : i \in J\}$ be a uniformly bounded family of continuous linear operators on V into V. Then

$\{P_i^* : i \in J\}$ is equicontinuous.

<u>Proof</u>. Let B^o be a basic neighborhood of 0 in V^*. Since B is bounded and $\{P_i : i \in J\}$ is uniformly bounded, $C = \{P_i(b) : i \in J, b \in B\}$ is bounded in V. Hence C^o is a neighborhood of 0 in V^*. We assert that $P_i^*[C^o] \subset B^o$, for all $i \in J$. Suppose that $f \in C^o$, $i \in J$, and $b \in B$. Then $|P_i^*(f)(b)| = |f(P_i(b))| \leq 1$, since $P_i(b) \in C$. Hence $P_i^*(f) \in B^o$, and thus $\{P_i^* : i \in J\}$ is equicontinuous. QED

<u>Lemma III.2</u>. Let V be sequentially complete and let $\{P_i : i \in J\}$ be a pointwise bounded family of continuous linear operators on V. Then $\{P_i : i \in J\}$ is uniformly bounded.

<u>Proof</u>. Immediate from [9, p.105, 12.4]. QED

<u>Lemma III.3</u>. A semi-reflexive space is sequentially complete.

<u>Proof</u>. Immediate from [9, p.190, 20.2]. QED

<u>Definition III.4</u>. Let $\{S_d : d \in D\}$ be a S.D.O.B. for V. $\{S_d\}$ is boundedly complete iff for each bounded net $\{x_d : d \in D\}$ in V satisfying $S_e(x_d) = x_e$, for all $e \leq d$, there is $x \in V$ such that $S_d(x) = x_d$, for all $d \in D$.

Of course, the statement in the above definition that $S_d(x) = x_d$, for all $d \in D$ is equivalent to the statement that $\{x_d\}$ is convergent.

<u>Theorem III.5</u>. Let $\{S_d : d \in D\}$ be a boundedly complete e-S.D.O.B. for V. Then V is complete.

<u>Proof</u>. Let $V^{\sim}$ be the completion of V. For each d in D, let $S_d^{\sim}$ be the continuous extension of S_d to $V^{\sim}$. Note that $\{S_d^{\sim} : d \in D\}$ is equicontinuous (see [9, p.38, 5.5 ff.]), so since it converges to I pointwise on V, it converges to I pointwise on $cl(V) = V^{\sim}$. Now $R(S_d^{\sim}) = R(S_d)$, because the latter is finite dimensional and hence complete. Thus if y is in $V^{\sim}$, $\{S_d^{\sim}(y) : d \in D\}$ is a bounded net in V which converges to y. Obviously if $e \leq d$, $S_e(S_d^{\sim}(y)) = S_e^{\sim}(y)$. But since $\{S_d : d \in D\}$ is boundedly complete, $\{S_d^{\sim}(y) : d \in D\}$ must converge in V. In other words, y is in V. QED

<u>Remark III.6</u>. It follows from Theorem III.5 and [9, p.192, 20.4] that if V is evaluable and has a boundedly complete u.b.S.D.O.B., then V is barrelled.

<u>Definition III.7</u>. Let $\{S_d : d \in D\}$ be a S.O.B. (resp. a S.D.O.B.) for V. $\{S_d\}$ is shrinking iff $\{S_d^* : d \in D\}$ is a S.D.O.B. (resp. a S.O.B.) for V^*.

<u>Theorem III.8</u>. Let $\{S_d : d \in D\}$ be a shrinking S.O.B. or S.D.O.B. for V. Then $\{S_d\}$ is uniformly bounded.

<u>Proof</u>. For each $f \in V^*$, $\{S_d^*(f) : d \in D\}$ is bounded, hence for each $f \in V^*$, $\{f \circ S_d : d \in D\}$ is uniformly bounded, hence $\{S_d : d \in D\}$ is uniformly bounded. QED

<u>Remark III.9</u>. Theorem III.8 and Lemma III.1 show that if $\{S_d : d \in D\}$ is shrinking, then $\{S_d^* : d \in D\}$ is an equicontinuous basis.

Let $\{S_d : d \in D\}$ be a shrinking, weak S.O.B. for V, and suppose that V is evaluable. Theorem III.8 shows that $\{S_d\}$ is uniformly bounded, so that it follows from Theorem II.3 that $\{S_d\}$ is an e-S.O.B. for V. Thus we have:

<u>Corollary III.10</u>. A weak, shrinking S.O.B. for an evaluable space is an e-S.O.B.

<u>Theorem III.11</u>. Let $\{S_d : d \in D\}$ be a shrinking S.O. B. for V, and suppose that V is evaluable. Then $\{S_d^* : d \in D\}$ is a boundedly complete e-S.D.O.B. for V*.

<u>Proof</u>. In view of Theorem III.8 and Lemma III.1, we need prove only that $\{S_d^* : d \in D\}$ is boundedly complete. Let $\{x_d : d \in D\}$ be a strongly bounded net in V* such that $S_e^*(x_d) = x_e$, for all $e \leq d$. Since V is evaluable,

$\{x_d : d \in D\}$ is equicontinuous [9, p.192, 20.4]. Let f be in $R(S_e)$. Then

$$\lim_d x_d(f) = \lim_d x_d(S_e(f)) = \lim_d S_e^*(x_d)(f) = x_e(f),$$

so that $\{x_d : d \in D\}$ converges pointwise on

$$\bigcup_{e \in D} R(S_e).$$

Since $\{x_d : d \in D\}$ is equicontinuous and

$$\bigcup_{e \in D} R(S_e)$$

is dense in V,

$$\lim_d x_d(f)$$

exists for each f in V. Let x be defined by

$$x(f) = \lim_d x_d(f),$$

for all f in V. x is in V^* (i.e., x is continuous on V) because it is the pointwise limit of an equicontinuous net. Clearly $S_d^*(x) = x_d$, for all d in D.

QED

Theorem III.12. Let $\{S_d : d \in D\}$ be a boundedly complete S.D.O.B. for a barrelled space V. Then $\{S_d^* : d \in D\}$ is a shrinking e-S.O.B. for $\bar{Y}$.

Proof. Note that $\{S_d\}$ is equicontinuous, since V is barrelled, so that by Lemma III.1, $\{S_d^*\}$ is equicontinuous. A standard argument (e.g., that used in Theorem II.3) shows that $\{S_d^* : d \in D\}$ is an e-S.O.B. for $\bar{Y}$. Let f be in $\bar{Y}^*$. Since S_d is a projection, $R(S_d)^*$ and $R(S_d^*)$ are naturally isomorphic. Since $R(S_d)$ is finite dimensional and thus reflexive, we can find x_d in $R(S_d)$ such that for all y in $R(S_d^*)$, $y(x_d) = f(y)$. Now if $e \leq d$, $y(x_d) = f(y) = y(x_e)$, for all y in $R(S_e^*)$. Thus the totality of $R(S_e^*)$ over $R(S_e)$ implies that $S_e(x_d) = S_e(x_e) = x_e$, for all $e \leq d$. We show that $\{x_d : d \in D\}$ is bounded. Let y be in V^*. Since $\{S_d^*\}$ is equicontinuous, $\{S_d^*(y) : d \in D\}$ is bounded and hence $\{f(S_d^*(y)) : d \in D\}$ is bounded. But

$$\{f(S_d^*(y)) : d \in D\} = \{S_d^*(y)(x_d) : d \in D\} = \{y(x_d) : d \in D\}.$$

Thus $\{x_d : d \in D\}$ is weakly bounded, hence bounded. Since $\{S_d : d \in D\}$ is boundedly complete, there is x in V such that $S_d(x) = x_d$, for all d in D. Clearly $y(x) = f(y)$, for all y in Y, so $y(x) = f(y)$, for all y in $\bar{Y}$. This argument and the totality of Y over V show that V is canonically (algebraically) isomorphic to $\bar{Y}^*$. But since the topology of $\bar{Y}^*$ is the topology of uniform convergence on bounded subsets of $\bar{Y}$ and the topology of V is the stronger topology of uniform convergence on weak* bounded subsets of V^* [9, p.171, 18.7 and p.156, 17.7], $\{S_d^{**} : d \in D\}$ is a S.D.O.B. for $\bar{Y}^*$. QED

Remark III.13. Under the hypotheses of Theorem III.12, V and $\bar{Y}^*$ are isomorphic. Let T be the barrelled topology on V and let T' be the topology on V of uniform convergence on bounded subsets of $\bar{Y}$. We are asserting that $T = T'$. Theorem III.12 shows that T' is weaker then T. Let $\{x_j\}$ be a net in V which T'-converges to 0. Let C be a weak*-bounded subset of V^*. C is strongly bounded [9, p.171, 18.7], so that $J = \cup \{S_d^*[C] : d \in D\}$ is a strongly bounded subset of $\bar{Y}$, and thus $\{f(x_j)\}$ converges to 0 uniformly on f in J. Let $\bar{J}$ be the weak*-closure of J. Clearly C is a subset of $\bar{J}$. Since $\{x_j\} \cup \{0\}$ can be considered as a set of weak* continuous functions on V^* and $\{f(x_j)\}$ converges to 0 uniformly for f in the weak* dense subset J of $\bar{J}$, $\{f(x_j)\}$ converges to 0 uniformly for f

in $\bar{J}$. Thus $T \subset T'$. QED

One obvious, but interesting, application of Theorem III.12 and Remark III.13 is that a Banach space with a boundedly complete S.D.O.B. is isomorphic to a conjugate Banach space.

Theorem III.14. Let $\{S_d : d \in D\}$ be a S.D.O.B. for a quasi-complete space (V,T) (i.e., bounded T-Cauchy nets are T-convergent). Suppose that $\{S_d^* : d \in D\}$ is a shrinking S.O.B. for $\bar{Y}$. Then $\{S_d : d \in D\}$ is boundedly complete.

Proof. Note that by Lemma III.2, $\{S_d : d \in D\}$ is uniformly bounded, so that by Lemma III.1, $\{S_d^* : d \in D\}$ is equicontinuous. Since $\bar{Y}$ is total over V, we can identify V with a subset of $\bar{Y}^*$. The relativised topology, T', induced on V by the strong topology on $\bar{Y}^*$ is the topology of uniform convergence on (strongly) bounded subsets of $\bar{Y}$. Now the weak* -bounded and strongly -bounded subsets of V^* agree because V is quasi-complete [9, p.170, 18.5], so that the proof of Remark III.13 shows that T is weaker than T'. Let $\{x_d : d \in D\}$ be a bounded net in V such that $S_e(x_d) = x_e$, for all $e \leq d$. To show that $\{x_d : d \in D\}$ is T-convergent, it is sufficient to show that $\{x_d : d \in D\}$ is T-Cauchy. Now $\{x_d : d \in D\}$ is T-bounded, hence is equicontinuous on V^*. If f is in

$R(S_e^*)$,

$$\lim_d f(x_d) = \lim_d S_e^*(f)(x_d) = \lim_d f(S_e(x_d)) = f(x_e).$$

Since $\{x_d : d \in D\}$ is equicontinuous on V^*,

$$\lim_d f(x_d)$$

exists for all f in $\bar{Y}$. Define F on $\bar{Y}$ by

$$F(f) = \lim_d f(x_d).$$

F is continuous on $\bar{Y}$, because it is the pointwise limit of the equicontinuous net $\{x_d : d \in D\}$. Since $\{S_d^* : d \in D\}$ is shrinking, it follows that if B is a bounded subset of $\bar{Y}$, then

$$\lim_d f(x_d) = F(f),$$

uniformly on f in B. Thus $\{x_d : d \in D\}$ is T'-Cauchy, hence T-Cauchy. QED

<u>Theorem III.15</u>. Let $\{S_d : d \in D\}$ be a S.D.O.B. for a semi-reflexive space V. Then $\{S_d\}$ is both shrinking and

boundedly complete.

Proof. Since V is semi-reflexive it follows from Lemmas III.3 and III.2 that $\{S_d\}$ is uniformly bounded, and thus from Lemma III.1 that $\{S_d^* : d \in D\}$ is equicontinuous. Since V is semi-reflexive, the weak* and weak topologies on V^* agree, so that our usual argument shows that $\{S_d^* : d \in D\}$ is a S.O.B. for V^*. Thus $\{S_d : d \in D\}$ is shrinking.

To show that $\{S_d\}$ is boundedly complete, we let $\{x_d : d \in D\}$ be a bounded net in V such that $S_e(x_d) = x_e$ whenever $e \leq d$. We show that $\{x_d : d \in D\}$ is weakly Cauchy (hence weakly convergent by [9, p.190, 20.2]). Let f be in V^* and let $\varepsilon > 0$. Since $\{S_d : d \in D\}$ is shrinking and $\{x_d : d \in D\}$ is bounded, there is $\bar{d}$ in D such that if $d \geq \bar{d}$, $|S_d^*(f)(x_i) - f(x_i)| < \varepsilon/2$, for all i in D. Now suppose that d and e both follow $\bar{d}$. Pick j in D so that j follows both d and e. Then

$$|f(x_d) - f(x_e)| \leq |f(x_d) - f(x_j)| + |f(x_j) - f(x_e)| =$$

$$= |f(S_d(x_j)) - f(x_j)| + |f(x_j) - f(S_e(x_j))| =$$

$$= |S_d^*(f)(x_j) - f(x_j)| + |f(x_j) - S_e^*(f)(x_j)| < \varepsilon.$$

Thus $\{x_d : d \in D\}$ is weakly Cauchy and thus weakly converges to, say, x. Clearly $S_d(x) = x_d$, for all d in D, so $\{S_d : d \in D\}$ is boundedly complete. QED

An easy modification of the above proof yields the following corollary.

Corollary III.16. If $\{S_n\}_{n=1}^{\infty}$ is a shrinking S.D.O.B. for a weakly sequentially complete space, then $\{S_n\}_{n=1}^{\infty}$ is boundedly complete. In particular, a shrinking finite dimensional Schauder decomposition for a weakly sequentially complete space is boundedly complete.

Theorem III.17. Let $\{S_d : d \in D\}$ be a boundedly complete, shrinking S.D.O.B. for V. Then V is semi-reflexive.

Proof. Let F be in V^{**}. As in the proof of Theorem III.12, for each d in D there is x_d in $R(S_d)$ such that for all f in $R(S_d^*)$, $f(x_d) = F(f)$, and $S_e(x_d) = x_e$ whenever $e \leq d$. We show that $\{x_d : d \in D\}$ is bounded. It is sufficient to show that $\{x_d\}$ is weakly bounded. Let f be in V^*. Then

$$f(x_d) = f(S_d(x_d)) = F(S_d^*(f)).$$

But $\{S_d^*(f) : d \in D\}$ is bounded and F is continuous, so that $\{F(S_d^*(f)) : d \in D\}$ is bounded. This shows that $\{x_d : d \in D\}$ is bounded. Thus there is x in V such that $S_d(x) = x_d$, for all d in D. Clearly $f(x) = F(f)$, for all f in Y. Since x and F are both continuous, $f(x) = F(f)$, for all f in $\bar{Y}$. But $\bar{Y} = V^*$, since $\{S_d : d \in D\}$ is shrinking. Thus the canonical embedding of V into V^{**} is onto, and V is semi-reflexive. QED

A semi-reflexive space is reflexive iff it is evaluable (or, equivalently, barrelled) [9, p.194, 20.6 and 20.7]. Thus to characterize reflexive spaces which admit a S.D.O.B. it is natural to ask what properties a S.D.O.B. in an evaluable or barrelled space must have.

Suppose that V is barrelled (resp. evaluable) and $\{S_d : d \in D\}$ is a S.O.B. or S.D.O.B. for V (resp. a u.b.S.O.B. or u.b.S.D.O.B. for V). Let f be a bounded linear functional on V satisfying the condition that

$$\lim_d f(S_d(x)) = f(x)$$

for all x in V. For each d in D, $R(S_d)$ is finite dimensional, so that f is continuous on $R(S_d)$. Thus $f \circ S_d$ is continuous, for each d in D. But $\{S_d : d \in D\}$ is uniformly bounded and f is bounded, so that $\{f \circ S_d : d \in D\}$ is a uniformly bounded net of continuous linear functionals on V. It follows from [9, p.191, 20.3] that $\{f \circ S_d : d \in D\}$ is equicontinuous. Thus f, the pointwise limit of $\{f \circ S_d : d \in D\}$, is continuous.

The preceding observation motivates Definition III.18.

Definition III.18. Let $\{S_d : d \in D\}$ be a S.O.B. or S.D.O.B. for V. $\{S_d : d \in D\}$ is full iff every bounded linear functional, f, satisfying

$$\lim_d f(S_d(x)) = f(x)$$

for all x in V, is continuous.

Obviously every S.D.O.B. or S.O.B. for a bound space is full. The remarks preceding Definition III.18 prove the following theorem.

Theorem III.19. A S.O.B. or S.D.O.B. for a barrelled space is full. An u.b.S.O.B. or u.b.S.D.O.B. for an evaluable space is full.

The definition of full bases is similar to Jones' definition in [8] of A' Schauder bases. One of Jones' results is that if V admits an A' Schauder basis, then V* is complete. Theorem III.20 extends this result to full S.O.B.'s and full S.D.O.B's.

Theorem III.20. Let $\{S_d : d \in D\}$ be a full S.O.B. or full S.D.O.B. for V. Then V* is complete.

Proof. In view of [9, p.169, 18.4], it is sufficient to show that every linear functional which is continuous on bounded sets is continuous. But this follows immediately from the definition of full. QED

A similar theorem is the following.

Theorem III.21. Let $\{S_d : d \in D\}$ be a full S.O.B. or full S.D.O.B. for a Mackey space V. Let X be a complete locally convex space, and let $L(V,X)$ be the space of all continuous linear maps on V into X endowed with the topology of uniform convergence on bounded subsets of V. Then $L(V,X)$ is complete.

Proof. As in Theorem III.20, it is sufficient to show that every linear map K on V into X which is continuous on bounded sets is continuous. Theorem III.20 guarantees that such a K is continuous considered as a mapping from V into $(X,w(X,X^*))$. But then by [9, p.203-4, 21.5 and 21.4], K is Mackey continuous, hence continuous. QED

Remark III.22. We have seen that a barrelled space with a S.O.B. or S.D.O.B. has a complete dual space. Since it is well known that there is a barrelled space which does not have a complete dual space, there is a barrelled space which does not admit either a S.O.B. or S.D.O.B.

Let V be reflexive and let $\{S_d : d \in D\}$ be a S.D.O.B. for V. $\{S_d : d \in D\}$ is shrinking and boundedly complete by Theorem III.15. V is barrelled [9, p.194, 20.6], so that $\{S_d : d \in D\}$ is full by Theorem III.19. Since V is evaluable, every bound absorbing barrel is a neighborhood of 0. These observations lead us to consider the following theorem.

Theorem III.23. Let (V,T) be a Mackey space and let $\{S_d : d \in D\}$ be a S.D.O.B. for (V,T). Let T' be the topology on V which has for a local base the collection of all bound absorbing barrels in (V,T). Then (V,T) is reflexive iff $\{S_d : d \in D\}$ is shrinking, boundedly complete, and full, and $\{S_d : d \in D\}$ is a S.D.O.B. for (V,T').

Proof. The "only if" part follows from the remarks preceding the theorem. To go the other way, note that (V,T) is semi-reflexive by Theorem III.17. Now suppose f is a T'-continuous linear functional on V. Note that (V,T) and (V,T') have the same bounded sets, so that f is a T-bounded linear functional. Since $\{S_d : d \in D\}$ is a S.D.O.B. for (V,T') and f is T'-continuous,

$$\lim_d f(S_d(x)) = f(x)$$

for all x in V. It follows from the fullness of $\{S_d : d \in D\}$ that f is T-continuous. Thus T' is compatible with the duality (V, V^*). Since T is Mackey, $T = T'$, so that T is evaluable. QED

IV. APPLICATIONS TO BANACH SPACE THEORY

In this chapter we let X be a Banach space. If $\{S_d : d \in D\}$ is an O.B. (resp. a D.O.B.) for X, it follows from Theorem II.4 that $\{S_d : d \in D\}$ is uniformly bounded. If $\lambda \geq \sup\{\|S_d\| : d \in D\}$, we say that $\{S_d : d \in D\}$ is a π_λ (resp. a dual π_λ) decomposition for X, and X is called a π_λ space (resp. a dual π_λ space).

Lindenstrauss introduced the concept of π_λ spaces in order to study Hahn-Banach theorems for compact operators. His proof of Lemma 3.1 in [10] shows that a dual π_λ space is a π_β space for every $\beta > \lambda$, so that the results of [10] apply to dual π_λ spaces as well.

More recently, π_1^∞ spaces (see Definition IV.7) have been studied by Michael and Pełczynski, [12] and [13]. One of their main results is that if K is a compact metric space, then $C(K)$ is a π_1^∞ space.

In this section we relate the concepts of π_λ and dual π_λ spaces to the theory of finite dimensional bases of subspaces. In particular, Theorem IV.2 shows that every separable dual π_λ space admits a finite dimensional basis of subspaces; Corollary IV.6 shows that if X is separable and X is a π_λ space for every $\lambda > 1$, then X admits a finite dimensional basis of subspaces.

The following notation is used in this chapter: If

$a > 0$, $B(a) = \{x : \|x\| \leq a\}$. As usual, I denotes the identity operator. $C(K)$ is the Banach space of scalar (i.e., real or complex) valued continuous functions on the compact Hausdorff space K, endowed with the sup norm. ℓ_n^∞ is $C(\{i\}_{i=1}^n)$, where $\{i\}_{i=1}^n$ has the discrete topology.

<u>Lemma IV.1</u>. Let X be a normed space and let Y be a separable subspace of X. Suppose that $\{S_d : d \in D; \leq\}$ is an equicontinuous net of linear operators of finite range on X which converges pointwise to I. Let M and a be positive numbers. Then there is $\{d_1 \leq d_2 \leq d_3 \leq \cdots\} \subset D$ such that

$$\lim_{n\to\infty} S_{d_n}(x) = x,$$

for each x in Y, and $S_{d_{n+1}}$ moves each point of

$$B(M) \cap \operatorname{sp} \bigcup_{i=1}^{n} R(S_{d_i})$$

a distance less than $a/2^n$.

<u>Proof</u>. Let $\{x_i\}_{i=1}^\infty$ be dense in Y. Choose d_1 in D such that $\|x_1 - S_{d_1}(x_1)\| < a$. Suppose that

$d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n$ have been chosen. Choose d_{n+1} in D such that $d_n \leq d_{n+1}$ and for each x in

$$A = \{x_i\}_{i=1}^{n+1} \cup [B(M) \cap \operatorname{sp} \bigcup_{i=1}^{n} R(S_{d_i})],$$

$$\|x - S_{d_{n+1}}(x)\| < a/2^n.$$

This choice is possible because $\{S_d : d \in D\}$ converges pointwise to I and is equicontinuous, so that the convergence is uniform on compact sets. A is closed, bounded, and finite dimensional, hence is compact. Now for each i,

$$\lim_{n\to\infty} S_{d_n}(x_i) = x_i.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in Y and $\{S_{d_n}\}_{n=1}^{\infty}$ is equicontinuous, $\{S_{d_n}\}_{n=1}^{\infty}$ converges pointwise on Y to I.

QED

Before proceeding to the main results of this chapter, we note that the results of Chapter II show that a sequence $\{S_n\}_{n=1}^{\infty}$ of operators of finite range on X are the partial sum operators associated with a finite dimensional basis of subspaces of X iff $\{S_n\}_{n=1}^{\infty}$ is both a π_λ decomposition and dual π_λ decomposition for X. $G(\{S_n\})$ is then defined to be $\sup\{\|S_n\|\}_{n=1}^{\infty}$, and $G(\{S_n\})$ is called the Grinblyum constant of the basis. If $G(\{S_n\}) = 1$, the basis is said to be monotone.

Theorem IV.2. Let X be a separable dual π_λ space and let $M > \lambda$. Then X admits a finite dimensional basis of subspaces with Grinblyum constant no larger than M.

Proof. By Lemma IV.1, we can assume that X has a dual π_λ decomposition $\{S_n\}_{n=1}^{\infty}$ such that for each n and each

$$x \in B(M) \cap \operatorname{sp} \bigcup_{i=1}^{n} R(S_i),$$

$$\|x - S_{n+1}(x)\| < (M - \lambda)/2^n. \tag{1}$$

For $j \geq n$, let $T_n^j = S_j S_{j-1} \cdots S_n$. Now if $j > n$,

$$\|T_n^j\| \leq \left[\sum_{i=n}^{j-1} (M - \lambda)/2^i\right] + \lambda < M. \tag{2}$$

If $j = n + 1$, (2) follows from the fact that $\|S_n\| \leq \lambda$, (1), and the inequality

$$\|T_n^{n+1}(x)\| \leq \|S_{n+1}S_n(x) - S_n(x)\| + \|S_n(x)\|.$$

In general, if (2) holds for j, then for $x \in B(1)$

$$\begin{aligned}\|T_n^{j+1}(x)\| &\leq \|S_{j+1}T_n^j(x)\| + \|T_n^j(x)\| \\ &\leq (M - \lambda)/2^j + \left[\sum_{i=n}^{j-1} (M - \lambda)/2^i\right] + \lambda,\end{aligned}$$

so that (2) also holds if $j + 1$ is substituted for j. Note that this argument also shows that for $x \in B(1)$ and $j > i \geq n$,

$$\|T_n^j(x) - T_n^i(x)\| \leq \sum_{k=i}^{j-1} (M - \lambda)/2^k. \tag{3}$$

Thus the Cauchy criterion guarantees that

$$\lim_{j\to\infty} T_n^j(x)$$

exists for each $x \in X$ and $n = 1,2,3,\cdots$. Let

$$T_n = \lim_{j\to\infty} T_n^j .$$

Clearly each T_n is linear and $\|T_n\| \leq M$.

Now for $n \geq m$ and $j \geq m$,

$$T_m^j T_n = S_j \cdots S_m \lim_{i\to\infty} S_i \cdots S_n$$

$$= \lim_{i\to\infty} S_j \cdots S_m S_i \cdots S_n$$

$$= \lim_{i\to\infty} S_j \cdots S_m = T_m^j.$$

Thus for $n \geq m$, $T_m T_n = T_m$. Similarly, for $j \geq m \geq n$,

$$T_m^j T_n = S_j \cdots\cdot S_m \lim_{i\to\infty} S_i \cdots\cdot S_n$$

$$= \lim_{i\to\infty} S_j \cdots\cdot S_m S_i \cdots\cdot S_n$$

$$= S_j \cdots\cdot S_m S_{m-1} \cdots\cdot S_n = T_n^j.$$

Thus for $m \geq n$, $T_m T_n = T_n$. That is, $T_m T_n = T_{\min(n,m)}$.

We next show that $\{T_n\}_{n=1}^{\infty}$ pointwise converges to I. Since

$$\bigcup_{n=1}^{\infty} R(S_n)$$

is dense in X and $\{T_n\}_{n=1}^{\infty}$ is equicontinuous, it is sufficient to show that for each

$$x \in \bigcup_{n=1}^{\infty} R(S_n), \quad \lim_{n\to\infty} T_n(x) = x.$$

Let

$$x \in \bigcup_{n=1}^{\infty} R(S_n),$$

say $x \in R(S_i)$, and without loss of generality assume that $x \in B(1)$. If $j > n > i$, we have from (3) and (1) that

$$\begin{aligned} \|T_n^j(x) - x\| &\le \|T_n^j(x) - S_n(x)\| + \|S_n(x) - x\| \\ &\le \Big[\sum_{k=n}^{j-1} (M - \lambda)/2^k\Big] + (M - \lambda)/2^{n-1}. \end{aligned}$$

Passing to the limit on j, we get that for $n > i$,

$$\|T_n(x) - x\| \le \sum_{k=n-1}^{\infty} (M - \lambda)/2^k.$$

Passing to the limit on n, we have that

$$\lim_{n\to\infty} \|T_n(x) - x\| = 0.$$

Now for each n,

$$S_nT_n = S_n \quad \text{and} \quad T_nS_n = T_n, \tag{4}$$

so that $\ker(T_n) = \ker(S_n)$, and thus $R(T_n)$ and $R(S_n)$ have the same dimension. Therefore $\{T_n\}_{n=1}^{\infty}$ is a π_M-dual π_M decomposition for X, and the remarks preceding the theorem complete the proof. QED

<u>Remark IV.3</u>. Using the notation of Theorem IV.2, we have from (4) that T_n is an isomorphism from $R(S_n)$ onto $R(T_n)$ with inverse S_n. Thus for each n, $d(R(T_n), R(S_n)) \leq M\lambda$, where $d(A,B) = \inf\{\|T\| \cdot \|T^{-1}\| \ : T$ is an isomorphism from A onto $B\}$. If each S_n is of norm 1, then each T_n is of norm 1, so that $R(T_n)$ and $R(S_n)$ are isometric. Of course, in this case the generated Schauder decomposition is monotone.

<u>Corollary IV.4</u>. Let X be a dual π_1 space and let Y be a separable subspace of X. Then there is a separable subspace Z of X such that $Y \subset Z$ and Z has a π_1-dual

π_1 decomposition.

<u>Proof</u>. Let $\{S_d : d \in D\}$ be a dual π_1 decomposition for X. Using Lemma IV.1, we can find $\{d_1 \leq d_2 \leq d_3 \leq \cdots\} \subset D$ such that

$$\lim_{n\to\infty} S_{d_n}(x) = x,$$

for each $x \in Y$, and $S_{d_{n+1}}$ moves each point of

$$\mathrm{sp} \bigcup_{i=1}^{n} R(S_{d_i}) \cap B(1)$$

a distance less than $1/2^n$. Let

$$Z = \{x \in X : \lim_{n\to\infty} S_{d_n}(x) = x\}.$$

Clearly Z is a separable (closed) subspace of X and $Y \subset Z$. Now $\{S_{d_n}\}_{n=1}^{\infty}$ is a dual π_1 decomposition for Z because each $R(S_{d_n})$ is a subset of Z. Thus by Theorem IV.2 and Remark IV.3 Z has a π_1-dual π_1 decomposition.

The proof of the following theorem is very similar to the proof of Theorem IV.2, so we omit the details of the proof.

<u>Theorem IV.5</u>. Let X be a Banach space and let $\{S_n\}_{n=1}^{\infty}$ be a π_λ decomposition for X. Suppose that there is a sequence $\{P_n\}_{n=1}^{\infty}$ such that for each n, P_n is a linear projection from $R(S_{n+1})$ onto $R(S_n)$, and that

$$\prod_{n=1}^{\infty} \|P_n\| = K < \infty.$$

Then X has a finite dimensional basis of subspaces with Grinblyum constant no larger than λK.

<u>Sketch of proof</u>. For $n > j$, let $T_j^n = P_j P_{j+1} \cdots P_{n-1} S_n$. For each j, let

$$T_j = \lim_{n\to\infty} T_j^n.$$

(This pointwise limit exists because $\|T_j^n\| \le \lambda K$, and for each m, $\{T_j^n\}_{n=j+1}^{\infty}$ is eventually constant on $R(S_m)$.) It follows by an argument similar to the used in Theorem IV.2 that $\{T_n\}_{n=1}^{\infty}$ is a $\pi_{\lambda k}$-dual $\pi_{\lambda k}$ decomposition for X. QED

The following corollary is an immediate consequence of Theorem IV.5 and the proof of Lemma 2.1 in [10].

Corollary IV.6. Let X be a separable Banach space and suppose that X is a π_λ space for every $\lambda > 1$. Then X admits a finite dimensional basis of subspaces.

Definition IV.7. A π_1 (resp. dual π_1) decomposition $\{S_d : d \in D\}$ is a π_1^∞ (resp. dual π_1^∞) decomposition iff each $R(S_d)$ is isometric to an $\ell^\infty_{n(d)}$ space.

It is known [10] that every $C(K)$ space "almost" has a π_1^∞ decomposition, $\{S_d : d \in D\}$, in the sense that each $R(S_d)$ is almost isometric to $\ell^\infty_{n(d)}$, and that if K is compact metric, $C(K)$ is a π_1^∞ space [12]. It is not known whether every $C(K)$ space is a π_1^∞ space. However, Theorem IV.8 shows that every $C(K)$ space is a dual π_1^∞ space.

Recall that $\{f_i\}_{i=1}^n \subset C(K)$ is a peaked partition of unity iff each f_i is non-negatively real-valued,

$$\sum_{i=1}^n f_i$$

is the constant 1 function, and $\|f_i\| = 1$. $\mathrm{Sp}(\{f_i\}_{i=1}^n)$ is then called a peaked partition subspace, and is isometric to ℓ_n^∞ (cf., e.g., [12]).

Theorem IV.8. Let K be compact Hausdorff. Then $C(K)$ has a dual π_1^∞ decomposition $\{S_d : d \in D\}$ such that each $R(S_d)$ is a peaked partition subspace.

Proof. Let D be the collection of all ordered pairs $(\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n)$ such that $\{U_i\}_{i=1}^n$ is a minimal open cover of K and

$$x_i \in U_i - \bigcup_{j \neq i} U_j.$$

Partially order D by

$$(\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n) \leq (\{V_j\}_{j=1}^m, \{y_j\}_{j=1}^m) \quad \text{iff} \quad \{V_j\}_{j=1}^m$$

refines $\{U_i\}_{i=1}^n$ and $\{x_i\}_{i=1}^n \subset \{y_j\}_{j=1}^m$. It is straightforward to verify that D is directed by $\leq$. For each $(\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n) \in D$, pick a peaked partition of unity $\{f_i\}_{i=1}^n$ such that f_i vanishes outside U_i (hence $f_i(x_j) = \delta_{ij}$). For each $d = (\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n)$ in D, define the projection S_d by

$$S_d(f) = \sum_{i=1}^n f(x_i)f_i,$$

where $\{f_i\}_{i=1}^n$ is the peaked partition of unity associated with d. If $d = (\{U_i\}_{i=1}^n, \{x_i\}_{i=1}^n)$ is in D, then clearly

$$\ker(S_d) = \{f \in C(K) : f(x_1) = f(x_2) = \cdots = f(x_n) = 0\}.$$

Thus if $d \leq e$, $\ker(S_e) \subset \ker(S_d)$, and hence $S_d S_e = S_d$. Obviously $\|S_d\| = 1$, for all $d \in D$. To complete the proof we must show that the net $\{S_d : d \in D; \leq\}$ pointwise converges to I. Let $f \in C(K)$ and let $\varepsilon > 0$. Choose a minimal open cover $\{V_j\}_{j=1}^n$ of K such that if $\{x,y\} \subset V_j$, then $|f(x) - f(y)| < \varepsilon$. Suppose $d = (\{U_i\}_{i=1}^m, \{x_i\}_{i=1}^m)$ is in D such that $\{U_i\}_{i=1}^m$ refines $\{V_j\}_{j=1}^n$. Then for all $x \in K$,

$$|f(x) - S_d(f)(x)| = |f(x) - \sum_{i=1}^m f(x_i)f_i(x)| =$$

$$= |\sum_{i=1}^m f_i(x)(f(x) - f(x_i))| \leq \sum_{i=1}^m f_i(x)\,|f(x) - f(x_i)| = k,$$

where $\{f_i\}_{i=1}^m$ is the peaked partition of unity associated with d. Now if $x \in U_i$, $|f(x) - f(x_i)| < \varepsilon$, since $\{U_i\}_{i=1}^m$ refines $\{V_j\}_{j=1}^n$. If $x \notin U_i$, then $f_i(x) = 0$. Hence

$$k < \sum_{i=1}^{m} f_i(x)\varepsilon = \varepsilon.$$

This completes the proof. QED

Remark IV.9. The proof of Corollary IV.4 shows that a separable subspace of a dual π_1^∞ space, X, is contained in a separable π_1^∞ - dual π_1^∞ subspace of X. Thus by Theorem IV.8, every separable subspace of $C(K)$ is contained in a separable π_1^∞ subspace of $C(K)$. In particular, when K is compact metric, we have the result of Michael and Pełczynski [13], that $C(K)$ is a π_1^∞ space.

Recall that a Hausdorff space K is a Boolean space iff the compact-open subsets of K form a base for the topology. In [5], Dyer notes that Theorem IV.8 can be improved for Boolean spaces:

Theorem IV.10. If K is a compact Boolean space, then $C(K)$ has a π_1^∞ - dual π_1^∞ decomposition $\{S_d : d \in D\}$ such that for each $d \in D$, $R(S_d)$ is spanned by the characteristic functions of the elements of a pairwise disjoint compact-open cover of K.

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VI. ACKNOWLEDGMENTS

I wish to thank Professor James A. Dyer, my major professor, for suggesting the problem considered in this thesis, for directing the research which led to this thesis, and, most of all, for arousing my interest in mathematics.

I would also like to thank the National Aeronautics and Space Administration for financial assistance.