Wave propagation in a fully ionized plasma with uniform velocity

Leverne Kenneth Seversike

Iowa State University
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Leverne Kenneth Seversike

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I. INTRODUCTION

The propagation of waves or disturbances in a plasma has been of considerable interest since the existence of a plasma was first established. Many problems, such as, solar flares, sun spots, geomagnetic storms, and communications, are associated with the plasma wave phenomenon. An area of great concern in the aerospace field is that of communication with a vehicle that may be surrounded by a plasma sheath, such as a space probe or a body entering a planetary atmosphere at high speeds. The ionized layer of gas around the body tends to severely attenuate the electromagnetic signal thereby causing a communication black-out. As a result a better understanding of wave propagation in an ionized media is desired.

A number of investigations of plasma wave propagation have been conducted beginning with the work of Tonks and Langmuir (25) in 1929. This study examined the longitudinal electron oscillations in a plasma by considering only the motion of the electrons and neglecting the effects of energy interchange between the fluid particles. Bohm and Gross (7, 8) extended this work by considering the effect on the plasma in the presence of an externally applied electric field in one case and an externally applied magnetic field as another case. Bernstein (6) also studied longitudinal electron oscillations but neglected the effect due to collisions of the electrons and the heavier particles, ions and neutrals.

The above mentioned analyses all deal with high frequency oscillations so that neglecting the motion of the heavier particles with respect to the motion of the electrons and neglecting the collision effects are both
fairly good assumptions. In the investigation of low frequency oscillations, the motion of the electrons is neglected in comparison with the motion of the heavier particles. Alfvén (1) found that in the presence of a magnetic field this leads to transverse waves similar to electromagnetic waves. These waves are in contrast to the conventional longitudinal pressure waves associated with fluids. The transverse waves, called Alfvén waves, are due to the coupling of the fluid dynamic and electromagnetic forces acting on the plasma.

Piddington's (18) account, in which the motions of both the ions and electrons are considered, states that only four basic types of waves can propagate through an electrically conducting medium in the presence of an externally applied magnetic field. Two of the wave modes are longitudinal or pressure waves. One of these corresponds to the usual sound wave occurring in fluids and the other longitudinal waves appear as an electron pressure wave. The remaining two waves are transverse in character and are essentially independent of pressure. At low frequencies, they are the hydromagnetic waves first described by Alfvén (1) and at high frequencies they correspond to electromagnetic or radio waves.

All of the preceding analyses have been carried out from the microscopic point of view using the Boltzmann equation to describe the plasma. Another approach in the study of plasma dynamics is the macroscopic point of view in which the plasma is treated as a continuum and the conservation equations are used to describe the plasma. In the macroscopic treatment the plasma can either be considered from single-fluid theory as done by Herlofson (12) and Ludford (13) or from the multi-fluid theory as done by
The use of the multi-fluid theory permits the investigation of the interaction between electrons, ions, and neutral particles, or in other words, the interaction between the various wave modes.

Bailey's (2, 3) analyses are two-fluid treatments, i.e., the plasma is assumed to be fully ionized so that the only species present are ions and electrons. The complete set of equations were not employed since the pressure and temperature of the ions and electrons were not considered as independent variables. As a result, the energy equation and the equation of state were not included in the analyses. These accounts did allow for the possibility that the plasma could have a uniform flow velocity, however.

The works of both Pai (15, 16) and Tanenbaum (21) are also two-fluid treatments but the complete set of equations was employed involving all of the fluid dynamic variables, i.e., pressure, density, temperature, and the velocity vector for both the ions and electrons as well as the electric and magnetic field vectors. Their result that only four types of waves can propagate in a fully ionized plasma is in agreement with that of Piddington. They also found that all of the waves do not propagate for all frequencies and that some of the waves are highly damped. In contrast, Bailey found that it is also possible for some of the waves to grow in magnitude for certain frequency ranges due to the uniform plasma flow velocity. This condition for growing waves is also mentioned in the limited investigations of Epstein (11) and Unz (24).

Recently, Tanenbaum and Mintzer (22) have investigated wave propagation in a partly ionized plasma using a three-fluid theory to describe
a plasma consisting of three species, electrons, ions, and neutrals. They found that three longitudinal wave modes exist, one associated with each specie, but that only the two basic transverse waves are present with the neutral particles acting primarily as a damping influence.

In the present analysis, wave propagation in a fully ionized plasma is investigated from a two-fluid theory using the complete set of fluid dynamic equations assuming that both the electrons and ions have the same uniform initial flow velocity. The plasma consists of a mixture of singly charged ions and electrons and the initial flow velocity is assumed to be in the same direction as the wave propagation. The degree of ionization is constant since the plasma is assumed to be in thermal and chemical equilibrium at all times. This last restriction is not so severe as it first seems since the temperature variation of the plasma is relatively small. The plasma is permeated by an externally applied magnetic field of uniform strength and is assumed to be unbounded so that no limitations or boundary conditions must be satisfied. A full dispersion equation is obtained and studied to see if only four wave types are present and if these waves are damped, undamped, or growing waves when the plasma has a uniform velocity.
II. LIST OF SYMBOLS

a  Sound speed
B  Magnetic flux density vector
\( c \)  Speed of light
\( c_p \)  Specific heat at constant pressure
\( \mathbf{D} \)  Dielectric displacement vector
\( \mathbf{E} \)  Electric field strength vector
e  Electron charge
\( \mathbf{F}_{ij} \)  Fluid interaction force between different species
\( \mathbf{H} \)  Magnetic field strength vector
h  Enthalpy
\( \mathbf{j} \)  Electric current density vector
k  Boltzmann constant
m  Particle mass
N  Particle number density
p  Fluid pressure
T  Fluid temperature
t  Time
\( \mathbf{U} \)  Fluid velocity vector
\((u,v,w)\)  \( x,y, \) and \( z \) velocity components, respectively
\( V \)  Wave propagation speed or phase velocity
\( V_{x_1} \)  \( x \)-wise Alfvén speed
\((x,y,z)\)  Rectangular coordinates
Z  Modified frequency, \( \omega-\lambda \) \( U_{cx} \)
\( \alpha_{12} \) Friction coefficient between species

\( \alpha_{12}^* \) See Equation 5

\( \gamma \) Ratio of specific heats

\( \lambda \) Complex wave number

\( \mu_e \) Magnetic permeability

\( \rho \) Fluid density

\( \rho_e \) Electric charge density

\( \sigma \) Electrical conductivity

\( \omega \) Plasma frequency

\( \omega_i \) Ion plasma frequency

\( \omega_e \) Electron plasma frequency

\( \omega_{ci} \) Ion cyclotron frequency

\( \omega_{ce} \) Electron cyclotron frequency

Subscripts:

i Refers to species, 1-ions, 2-electrons

o Refers to initial condition

Superscript:

( )' Refers to perturbation value
III. DERIVATION OF EQUATIONS

In this chapter the linearized small perturbation equations are obtained for a fully ionized plasma with a uniform externally applied magnetic field. The plasma has a uniform initial flow velocity in the direction of the wave propagation such that the velocity of both the electrons and ions is the same.

A. Plasma Description

Before writing the equations describing the relationships for the flow variables it is necessary to specify the type of media being considered. For this analysis the plasma is assumed to be a mixture of singly charged ions and electrons so that the plasma is said to be fully ionized. The degree of ionization remains constant at all times since thermal and chemical equilibrium are assumed. The plasma is taken to be electrically neutral and a homogeneous continuum. As a result of the last assumption the conventional fluid dynamic equations are used. In addition, the ion and electron gases are assumed to be inviscid and nonheat-conducting within their own species but a viscous interaction between the two species is accounted for in the governing equations. The perfect gas law is considered to apply to each of the gases. As far as electrical and magnetic properties are concerned the plasma is assumed to be homogeneous and isotropic, i.e., the magnetic permeability $\mu_e$ and the inductive capacity $\epsilon$ are constant.
B. Governing Equations

The equations governing the behavior of the plasma are based on certain physical laws. The fluid dynamic equations are as follows: (9, 14, 15, 16, 17, 19, 20). The conservation of mass equation

\[ \frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{U}_i) = 0 \]  

(1)

where we have assumed that there is no mass production due to ionization processes or chemical reactions.

The conservation of momentum equation

\[ \rho_i \frac{D\mathbf{U}_i}{Dt} + \nabla \mathbf{p}_i = \rho_e \left[ \mathbf{E} + \frac{\mu_e}{\mu_{i\ell}} (\mathbf{U}_i \times \mathbf{H}) \right] + \mathbf{F}_{ij} \]  

(2)

neglecting viscous forces within a species and gravitational forces.

The energy equation

\[ \rho_i \frac{Dh_i}{Dt} = \frac{Dp_i}{Dt} + \frac{j^2}{\sigma} \]  

(3)

neglecting viscous, heat-conduction, and radiation effects.

The equation of state for a perfect gas

\[ p_i = \rho_i \frac{k}{\mu_i} T_i \]  

(4)

The electromagnetic or Maxwell's equations in the rationalized MKS units system are as follows:

The statement of Ampere's Law

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \]  

(5)

The statement of Faraday's Law

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  

(6)

In the above equations the subscript \( i = 1 \) refers to the ions and \( i = 2 \) refers to the electrons.
The interaction between the ions and electrons is represented by the force $\mathbf{F}_{ij}$ appearing in Equation 2. It is assumed that this force is proportional to the difference between the mean flow velocities of the ions and electrons (17)

$$\mathbf{F}_{12} = \alpha_{12} (\mathbf{U}_1 - \mathbf{U}_2) = -\mathbf{F}_{21}$$

(7)

where $\alpha_{12} = \alpha_{21}$ is the friction coefficient and as a first approximation (17)

$$\sigma = -\frac{e^2 N_e^2}{\alpha_{12}}.$$  

(8)

This gives a relationship between the friction coefficient and the scalar electrical conductivity. The fluid density $\rho_i$ is equal to $m_i N_i$ where $m_i$ is the mass of the individual particles. The charge density for each species $\rho_e$ is equal to $\pm e N_i$ where $e$ is the absolute electron charge (the charge of an ion is $+e$ and of an electron is $-e$). One of the advantages of using two-fluid theory is the simple expression for electric current

$$\mathbf{J} = e \left[ N_1 \mathbf{U}_1 - N_2 \mathbf{U}_2 \right].$$

(9)

This expression is exact thus having an advantage over using the generalized Ohm's Law as is done in single-fluid theory. In Equation 3 $h_1$ is the enthalpy and is equal to $c_p T_1$ where $c_p$ is the specific heat at constant pressure and is assumed constant.

C. Initial Conditions

Initially the plasma has a uniform flow velocity $U_{ox}$ with a pressure $p = 2p_o$, temperature $T_o$, and number density $N = 2N_o$. The partial pressure and number density for both the ions and electrons are $p_o$ and $N_o$, respectively. The externally applied magnetic field is oriented such that the z-component is zero. Hence, $\mathbf{H}_o = i\mathbf{H}_{ox} + j\mathbf{H}_{oy}$ where $\mathbf{H}_{ox}$ and $\mathbf{H}_{oy}$ are
constant. Since the plasma is of infinite extent, there are no boundary conditions that must be satisfied.

All of the specified initial conditions must also satisfy Equations 1 through 6. From Equation 2 we have the condition

$$[\mathbf{E}_o + \mu e (\mathbf{U}_o \times \mathbf{B}_o)] = 0 \quad (10)$$

Hence, we see that

$$E_{ox} = 0$$
$$E_{oy} = 0$$
$$E_{oz} = - \mu e U_{ox} B_{oy} \quad (11)$$

The subscript $o$ refers to initial conditions. This is a rather interesting result since it means that an initial electric field with components given by Equation 11 must be applied to the plasma to satisfy Equation 10. If only a longitudinal magnetic field is considered, i.e., $H_{oy} = H_{oz} = 0$, then $E_{oz} = 0$ also, so that there will be no initial electric field.

Another interesting result can be observed by examining Equation 5. We see that $J_o = 0$ since all of the initial conditions are constant. Comparing this with Equation 9

$$\mathbf{J}_o = e [N_{o1} \mathbf{U}_{o1} - N_{o2} \mathbf{U}_{o2}] = 0 \quad (12)$$

Thus, we find that the initial flow velocities of the ions and electrons must be identified as was initially assumed, i.e., the plasma velocity is $U_{ox}$.

D. Small Perturbation Equations

The plasma is perturbed from equilibrium by a small disturbance such that the subsequent flow variables are
\[ \bar{U}_1 = U_0 + \bar{U}_1(x,t) = i[U_o^{\prime} + u_1^{\prime}(x,t)] + j v_1^{\prime}(x,t) + k w_1^{\prime}(x,t) \]
\[ p_1 = p_0 + p_1(x,t) \]
\[ T_1 = T_0 + T_1(x,t) \]  
(13)
\[ N_1 = N_0 + N_1(x,t) \]
\[ \bar{E} = \bar{E}_0 + \bar{E}_1(x,t) = i E_x^{\prime}(x,t) + j E_y^{\prime}(x,t) + k[E_{oz} + E_z^{\prime}(x,t)] \]
\[ \bar{H} = \bar{H}_0 + \bar{H}_1(x,t) = i [H_{ox} + h_x^{\prime}(x,t)] + j[H_{oy} + h_y^{\prime}(x,t)] + k h_z^{\prime}(x,t) \]

assuming that the perturbed quantities are functions of one space coordinate \( x \) and time \( t \) only. Therefore, we are considering wave propagation along the \( x \)-axis, the same direction as the uniform flow velocity. The perturbed quantities are denoted by \( (\cdot)^\prime \) and are assumed small as compared to the initial values so that second-order and higher-order terms in these quantities can be neglected. As a result the equations of motion can be linearized.

The linearized small perturbation equations are obtained by substituting Equation 13 into Equations 1 through 6 making use of Equations 7, 9, and 11 and neglecting the second-order and higher-order terms of the perturbed quantities. We therefore have the following set of linearized equations.

Maxwell's equations give
\[ \nabla \times \bar{H}^\prime = e[N_o^\prime(\bar{U}_1^\prime - \bar{U}_2^\prime) + \bar{U}_0^\prime(N_1^\prime - N_2^\prime)] + \frac{\partial \bar{E}^\prime}{\partial t} \]  
(14)
\[ \nabla \times \bar{E}^\prime = - \mu e \frac{\partial \bar{H}^\prime}{\partial t} \]  
(15)

The mass conservation equation yields
\[ \frac{\partial N_1^\prime}{\partial t} + N_o \nabla \cdot \bar{U}_1^\prime + \bar{U}_0 \nabla N_1^\prime = 0 \]  
(16)
\[ \frac{\partial N_2'}{\partial t} + N_0 \nabla \cdot \vec{U}_2' + \vec{U}_0 \cdot \nabla N_2' = 0 \]  \hspace{1cm} (17)

The energy equation gives
\[ c_{p_1} \mu_{10} \left( \frac{\partial T_1'}{\partial t} + U_{10x} \frac{\partial T_1'}{\partial x} \right) = \frac{\partial p_1'}{\partial t} + U_{10x} \frac{\partial p_1'}{\partial x} \]  \hspace{1cm} (18)
\[ c_{p_2} \mu_{20} \left( \frac{\partial T_2'}{\partial t} + U_{20x} \frac{\partial T_2'}{\partial x} \right) = \frac{\partial p_2'}{\partial t} + U_{20x} \frac{\partial p_2'}{\partial x} \]  \hspace{1cm} (19)

The conservation of momentum equation gives
\[ \frac{\partial \vec{U}_1'}{\partial t} + U_{10x} \frac{\partial \vec{U}_1'}{\partial x} + \frac{1}{m_1 N_0} \nabla p_1' = \frac{\alpha_{12}}{m_1 N_0} (\vec{U}_1' - \vec{U}_2') \]  \hspace{1cm} (20)
\[ + \frac{e}{m_1} [\vec{E}' + \mu_e (\vec{U}_1' \times \vec{B}_0 + \vec{U}_0 \times \vec{B}')] \]
\[ \frac{\partial \vec{U}_2'}{\partial t} + U_{20x} \frac{\partial \vec{U}_2'}{\partial x} + \frac{1}{m_2 N_0} \nabla p_2' = \frac{\alpha_{12}}{m_2 N_0} (\vec{U}_2' - \vec{U}_1') \]  \hspace{1cm} (21)
\[ - \frac{e}{m_2} [\vec{E}' + \mu_e (\vec{U}_2' \times \vec{B}_0 + \vec{U}_0 \times \vec{B}')] \]

The equation of state, becomes
\[ \frac{p_1'}{p_0} = \frac{T_1'}{T_0} + \frac{N_1'}{N_0} \]  \hspace{1cm} (22)
\[ \frac{p_2'}{p_0} = \frac{T_2'}{T_0} + \frac{N_2'}{N_0} \]  \hspace{1cm} (23)

As the problem is initially posed we have from Equations 14 through 23 eighteen fundamental scalar equations describing the plasma and from Equation 13 eighteen perturbed quantities as variables so that the problem is well defined.

In wave phenomena we are looking for periodic solutions to the perturbation equations. Hence, we assume that the perturbation quantities
are proportional to
\[ e^{i(\omega t - \lambda x)} \]  
(24)

where \( \omega \) is a given real frequency, \( \lambda \) is the wave number which may be complex, i.e., \( \lambda = \lambda_R^\text{i} + \lambda_I^\text{i} \) with \( \lambda_R^\text{i} \) and \( \lambda_I^\text{i} \) real numbers. The wave phase velocity is \( V = \frac{\omega}{\lambda_R^\text{i}} \). The wave number is primarily a function of \( \omega \) so that the problem becomes one of determining an equation for \( \lambda \) in terms of \( \omega \). Considering propagation in the positive x-direction, a negative value for \( \lambda_I^\text{i} \) indicates that the wave is damped, while a positive value means that the wave amplitude is increasing as the disturbance moves away from the disturbance center \( (x = 0) \) or the wave is said to be growing. If \( \lambda \) is a pure real number, the wave propagates without attenuation. If \( \lambda \) is a pure imaginary number the wave is of exponential type and does not propagate.

Substituting Equation 24 into Equations 14 through 23 yields the fundamental linearized algebraic equations in terms of the frequency \( \omega \) and the wave number \( \lambda \). At this point we also make the assumption that the external magnetic field is oriented such that it is parallel to the x-axis and the initial flow velocity, i.e., \( \vec{H}_0 = i \vec{H}_{ox} \). Referring to Equation 11, we see that \( E_{ox} = E_{oy} = E_{oz} = 0 \). We therefore have the linearized small perturbation equations for a fully ionized plasma with the external magnetic field aligned with the initial flow velocity.

\[ i\omega E_x^i + e [N_0(u_1 - u_2)] + U_{ox} (N_1 - N_2) = 0 \]  
(25)
\[ i\omega E_y^i + e [N_0(v_1 - v_2)] = i \lambda h_z^i \]  
(26)
\[ i\omega E_z^i + e [N_0(w_1 - w_2)] = -i \lambda h_y^i \]  
(27)
\[ \omega \mu_e h_x^i = 0 \]  
(28)
\[ \omega \mu_e \frac{h_y}{y} = -\lambda E_z^f \]  
\[ \omega \mu_e \frac{h_z}{z} = \lambda E_y^f \]  
\[ (\omega - \lambda U_{ox}) N_1^f - \lambda N_0 u_1^f = 0 \]  
\[ (\omega - \lambda U_{ox}) N_2^f - \lambda N_0 u_2^f = 0 \]  
\[ m_1 c \rho_1^0 N_1^f = b_1^f \]  
\[ m_2 c \rho_2^0 N_2^f = b_2^f \]  
\[ i m N_0 (\omega - \lambda U_{ox}) u_1^f - i \lambda \rho_1^f = \alpha_{12} (u_1^f - u_2^f) + e N_0 E_x^f \]  
\[ i m N_0 (\omega - \lambda U_{ox}) v_1^f = \alpha_{12} (v_1^f - v_2^f) + e N_0 E_y^f + \mu e N_0 (H_{ox} w_1^f - U_{ox} h_1^f) \]  
\[ i m N_0 (\omega - \lambda U_{ox}) w_1^f = \alpha_{12} (w_1^f - w_2^f) + e N_0 E_z^f + \mu e N_0 (U_{ox} h_1^f - H_{ox} v_1^f) \]  
\[ i m N_0 (\omega - \lambda U_{ox}) u_2^f - i \lambda \rho_2^f = \alpha_{12} (u_2^f - u_1^f) - e N_0 E_x^f \]  
\[ i m N_0 (\omega - \lambda U_{ox}) v_2^f = \alpha_{12} (v_2^f - v_1^f) - e N_0 E_y^f - \mu e N_0 (H_{ox} w_1^f - U_{ox} h_1^f) \]  
\[ i m N_0 (\omega - \lambda U_{ox}) w_2^f = \alpha_{12} (w_2^f - w_1^f) - e N_0 E_z^f - \mu e N_0 (U_{ox} h_1^f - H_{ox} v_1^f) \]  
\[ \frac{\rho_1}{\rho_0} = \frac{T_1}{T_0} + \frac{N_1}{N_0} \]  
\[ \frac{\rho_2}{\rho_0} = \frac{T_2}{T_0} + \frac{N_2}{N_0} \]  

Upon examining the fundamental equations we see that they can be separated into three independent groups as follows:

1. Equation 28 expresses the condition on \( h_x^f \), indicating no perturbation in the magnetic field in the x-direction, the direction of wave propagation.
2. The second group consists of the variables \( E_x^f, u_1^f, u_2^f, N_1^f, N_2^f, T_1^f, T_2^f, \rho_1^f, \) and \( \rho_2^f \) which are governed by Equations 25, 31, 32, 33, 34,
35, 38, 41, and 42. This group characterizes the longitudinal wave motion. Note that the external magnetic field does not appear in these equations so that the resulting longitudinal wave motion behaves the same if a magnetic field is present or is not present. The longitudinal waves will be studied in detail in Chapter IV.

(3) The third group consists of the variables $E'_y$, $E'_z$, $v'_1$, $v'_2$, $w'_1$, $w'_2$, $h'_y$, and $h'_z$ which are governed by Equations 26, 27, 29, 30, 36, 37, 39, and 40. Since the variables present are perpendicular to the direction of propagation, this group is said to describe the transverse waves. These will be examined in Chapters V and VI. Chapter V will consider the case without an external magnetic field and Chapter VI will deal with the magnetic field present.
IV. LONGITUDINAL WAVES

In this chapter we will study the second group of variables and equations which characterize the longitudinal waves. The eigenvalues $\lambda$ for the modes of wave propagation are found from the determinantal equation obtained by setting the determinant of the coefficients for the governing equations equal to zero.

\[
\begin{bmatrix}
\omega & eN_o & -eN_o & eU_{ox} & -eU_{ox} & 0 & 0 & 0 & 0 \\
0 & -\lambda N_o & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\lambda N_o & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & m_1 p_1 N_o & 0 & -l & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_2 p_2 N_o & 0 & -l \\
-eN_o & (im_1 N_o z - \alpha_{12}) & \alpha_{12} & 0 & 0 & 0 & 0 & -i\lambda & 0 \\
eN_o & \alpha_{12} & (im_2 N_o z - \alpha_{12}) & 0 & 0 & 0 & 0 & -i\lambda & 0 \\
0 & 0 & 0 & p_0 T_0 & 0 & p_0 N_0 & 0 & -N_o T_0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_0 T_0 & 0 & p_0 N_0 & 0 & -N_o T_0 \\
\end{bmatrix} = 0
\]

where $z = \omega - \lambda U_{ox}$. The resulting determinantal equation is

\[A_4 \lambda^4 + A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 = 0\]  \hspace{1cm} (44)

where

\[A_4 = \left[1 - \frac{U_{ox}^2}{a_1^2} + \frac{U_{ox}^4}{a_1^4 a_2^2}\right] \]  \hspace{1cm} (45)

\[A_3 = \frac{\omega U_{ox}}{a_1^2} \left[2 - \frac{4U_{ox}^2}{a_1^2 a_2^2} + i \alpha_{12} \left(2 \frac{m_2}{m_1} - \frac{U_{ox}^2}{a_1^2 a_2^2}\right)\right] \]  \hspace{1cm} (46)

\[A_2 = \frac{a_2^2}{a_1^2} \left[-1 + 2 \frac{U_{ox}^2}{a_1^2 a_2^2} + \frac{U_{ox}^4}{a_2^4} (6 - \frac{a_2^2}{a_1^2}) + i \alpha_{12} \left(-2 \frac{m_2}{m_1} + 3 \frac{U_{ox}^2}{a_2^2}\right)\right] \]  \hspace{1cm} (47)
\[ A_1 = \frac{\omega^2 \alpha_1}{\omega_1^2 \omega_2^2} \left[ -1 + 2 \frac{\omega_1^2}{\omega^2} - 3 \omega_1^2 \right] \]  
\[ A_0 = \frac{\omega_1}{\omega_1^2 \omega_2^2} \left[ 1 - \frac{\omega_1^2}{\omega^2} + i \omega_1^2 \right] \]

with the ion plasma frequency
\[ \omega_1 = \left( \frac{e^2 N_0}{m_1} \right)^{1/2} \]  
the electron plasma frequency
\[ \omega_2 = \left( \frac{e^2 N_0}{m_2} \right)^{1/2} \]  
the ion sound speed
\[ a_1 = \left( \frac{\gamma P_0}{m_1 N_0} \right)^{1/2} \]  
the electron sound speed
\[ a_2 = \left( \frac{\gamma P_0}{m_2 N_0} \right)^{1/2} \]  
and
\[ \alpha_{12} = \frac{a_1}{a_2} \]

In these equations \( m_2 \) was neglected in comparison with \( m_1 \) in terms involving the quantity \((m_1 + m_2)\) and recognizing that \( m_1 c_{pl} = m_2 c_{p2} = \frac{5}{2} \frac{P_0}{N_0 T_0} \).

The roots of Equation 44 are the wave numbers for the corresponding modes of wave propagation and are primarily functions of the frequency. One important result can be noted immediately. Since the externally applied magnetic field does not appear in the characteristic equation, the propagation of longitudinal waves appears the same whether or not the magnetic field is present. Hence, the magnetic field has no influence on the longitudinal waves.
It is of interest to examine several cases for the longitudinal wave equation.

A. Stationary Plasma

For this case $U_{\text{ox}} = 0$ and Equation 44 reduces to

$$\lambda^4 + \frac{\alpha^2}{a_1^2} [ -1 + 2 \frac{\alpha^2}{\omega^2} - 2 i \alpha_{12} \frac{m_2}{m_1} ] \lambda^2 + \frac{\omega^4}{a_1^2 a_2^2} [ 1 - \frac{\omega^2}{\alpha_{12}^2} + i \alpha_{12} ] = 0 .$$

(55)

This is the same result as presented by Pai (15). In general the solutions for the wave modes are

$$\lambda_{1}^2 = \frac{\alpha^2}{a_1^2,} \frac{1}{2} [ 1 - 2 \frac{\alpha^2}{\omega^2} + 2 i \alpha_{12} \frac{m_2}{m_1} + [(1 - 2 \frac{\alpha^2}{\omega^2} + 2 i \alpha_{12} \frac{m_2}{m_1})^2]
- 4 \frac{a_1^2}{a_2^2} \frac{\alpha_{12}}{\omega^2} (1 - \frac{\alpha_{12}^2}{\omega^2} + i \alpha_{12}) \frac{1}{2} \right]$$

(56)

$$\lambda_{2}^2 = \frac{\alpha^2}{a_1^2} \frac{1}{2} [ 1 - 2 \frac{\alpha^2}{\omega^2} + 2 i \alpha_{12} \frac{m_2}{m_1} + [(1 - 2 \frac{\alpha^2}{\omega^2} + 2 i \alpha_{12} \frac{m_2}{m_1})^2]
- 4 \frac{a_1^2}{a_2^2} \frac{\alpha_{12}}{\omega^2} (1 - \frac{\alpha_{12}^2}{\omega^2} + i \alpha_{12}) \frac{1}{2} \right]$$

(57)

Note that for each mode there is a positive value for $\lambda$ and a negative value. The positive value indicates propagation in the positive $x$-direction and the negative value propagation in the negative $x$-direction.

1. Ideal Plasma

For an ideal plasma the electrical conductivity is infinite, i.e., $\alpha_{12} = 0$. Equations 56 and 57 then become for the two wave modes

$$\lambda_{1}^2 = \frac{\alpha^2}{a_1^2} \frac{1}{2} [ 1 - 2 \frac{\alpha^2}{\omega^2} + [(1 - 2 \frac{\alpha^2}{\omega^2})^2 - 4 \frac{a_1^2}{a_2^2} (1 - \frac{\alpha^2}{\omega^2}) \frac{1}{2} \right]$$

(58)

$$\lambda_{2}^2 = \frac{\alpha^2}{a_1^2} \frac{1}{2} [ 1 - 2 \frac{\alpha^2}{\omega^2} - [(1 - 2 \frac{\alpha^2}{\omega^2})^2 - 4 \frac{a_1^2}{a_2^2} (1 - \frac{\alpha^2}{\omega^2}) \frac{1}{2} \right]$$

(59)
These modes can now be examined for various frequency ranges.

For very low frequencies, $\omega \ll \omega_1, \omega_2$, the two solutions are

$$\lambda_{1}^2 = \frac{\omega^2}{2a_{1}^2} = \frac{\omega^2}{a_p^2}$$  \hfill (60)

$$\lambda_{2}^2 = - \left( \frac{-2a_{1}^2}{a_p^2} \right)$$  \hfill (61)

where the plasma sound speed is

$$a_p = \left( \frac{2\gamma p_0}{m_1 N_0} \right)^{\frac{1}{2}}$$  \hfill (62)

The first mode is therefore an undamped wave traveling at the speed of sound of the plasma. This corresponds to an ordinary pressure wave in fluid dynamics. The second mode is an exponential wave which does not propagate. This type of wave is called an evanescent wave.

At slightly higher frequencies such that $\omega < \omega_2, \lambda_{1}^2$ is always positive and $\lambda_{2}^2$ is always negative. Hence, the wave for $\lambda_{1}$ still propagates as an undamped wave but the speed decreases with increasing frequency. The second wave remains as a damped wave but the damping factor decreases with increasing frequency.

When the frequency is equal to the electron plasma frequency, $\omega = \omega_2$, the two solutions are

$$\lambda_{1}^2 = \frac{\omega^2}{a_{1}^2} \left[1 - \frac{a_{1}^2}{\omega^2} \right]$$  \hfill (63)

$$\lambda_{2}^2 = 0$$  \hfill (64)

The first wave mode remains undamped and propagates with a finite velocity while the second wave mode is changing from an evanescent wave to a
propagating wave. The propagation velocity is said to be infinite.

For \( \omega > \alpha_2 \), both \( \lambda_1^2 \) and \( \lambda_2^2 \) are positive so that both waves are undamped and propagate at finite speeds.

Finally for very large frequencies, \( \omega \gg \alpha_2 \), the limiting case, Equations 58 and 59 are

\[
\lambda_1^2 = \frac{\omega^2}{\alpha_1^2} \quad (65)
\]

\[
\lambda_2^2 = \frac{\omega^2}{\alpha_2^2} \quad (66)
\]

The first represents an undamped wave traveling at the speed of sound for the ion gas \( \alpha_1 \) and the second is an undamped wave traveling at the speed of sound for the electron gas \( \alpha_2 \). As a result of this last case, the wave associated with \( \lambda_1 \) can be thought of as due to the ions and the wave for \( \lambda_2 \) as due to the electrons.

A plot of the phase velocity or propagation velocity as a function of frequency for the two wave modes is presented in Figure 1.

2. Real plasma

In a real plasma the electrical conductivity is finite, although for most physical situations the conductivity is quite large due to the high plasma temperatures. For this case we must return to the general solutions given in Equations 56 and 57. Pai (15) and Tanenbaum (21) have shown that these two solutions indicate that the two modes of wave propagation are damped waves.

If we consider the electrical conductivity as finite but very large, such that we may neglect second-order and higher-order terms of \( \alpha_{12} \), the solutions to the determinantal equation are
\[ \lambda_{1}^{2} = \lambda_{01}^{2} + \frac{a_{1}}{a_{1}} \frac{\omega_{1}^{2}}{\alpha_{12}^{2}} \frac{m_{2}}{m_{1}} \left( 1 - 2 \frac{\omega_{1}^{2}}{\omega_{2}^{2}} \frac{1}{\alpha_{12}^{2}} \right) \quad (67) \]

\[ \lambda_{2}^{2} = \lambda_{02}^{2} + \frac{a_{1}}{a_{1}} \frac{\omega_{2}^{2}}{\alpha_{12}^{2}} \frac{m_{2}}{m_{1}} \left( 1 + 2 \frac{\omega_{1}^{2}}{\omega_{2}^{2}} \frac{1}{\alpha_{12}^{2}} \right) \quad (68) \]

where \( \lambda_{01}^{2} \) and \( \lambda_{02}^{2} \) correspond to the ideal plasma solutions given by Equations 58 and 59, respectively, and

\[ B = \left[ 1 - 2 \frac{\omega_{1}^{2}}{\omega_{2}^{2}} - 4 \frac{a_{1}}{a_{2}} \frac{\omega_{1}^{2}}{\omega_{2}^{2}} \left( 1 - \frac{\omega_{2}^{2}}{\omega_{1}^{2}} \right) \right] . \quad (69) \]

We can write the wave number as

\[ \lambda = \lambda_{R} + i \lambda_{I} \quad (70) \]

with the wave phase velocity \( V = \frac{\omega}{\lambda_{R}} \) and the damping factor \( \lambda_{I} \).

For the undamped propagating waves in the ideal plasma, i.e., \( \lambda_{01}^{2}, \lambda_{02}^{2}, > 0 \), we can show from Equations 67, 68, and 70 that \( \lambda_{R1}^{2} \approx \lambda_{01}^{2} \) and \( \lambda_{R2}^{2} \approx \lambda_{02}^{2} \). This means that for the first order approximation the wave phase velocity of the two waves for a real plasma are the same as for an ideal plasma. Also we find that

\[ \lambda_{I1}^{2} = \frac{G_{1}^{2}}{\alpha_{12}^{2}} \frac{m_{2}}{m_{1}} \lambda_{R1}^{2} \quad (71) \]

\[ \lambda_{I2}^{2} = \frac{G_{1}^{2}}{\alpha_{12}^{2}} \frac{m_{2}}{m_{1}} \lambda_{R2}^{2} \quad (72) \]

where

\[ G_{1} = \frac{\omega_{2}^{2}}{2a_{1}^{2}} \frac{m_{2}}{m_{1}} \left( 1 + 2 \frac{\omega_{1}^{2}}{\omega_{2}^{2}} \frac{1}{\alpha_{12}^{2}} \right) \quad (73) \]

with the negative sign taken in \( G_{1} \) for \( \lambda_{I1}^{2} \) and the positive sign used for \( \lambda_{I2}^{2} \). Therefore, the waves are damped and the damping factor increases
with decreasing electrical conductivity and decreases with increasing frequency.

For the evanescent waves in the ideal plasma, i.e., $\lambda_{o2}^2 < 0$, we find that

$$\lambda_{R2}^2 = \frac{\omega_{o2}^2}{(-\lambda_{o2}^2)} \left[ \frac{m_\perp^2}{m_\parallel} \left( 1 + \frac{2}{m_\perp} \frac{1}{\omega_B^2} \right) \right]^2$$

(74)

and

$$\lambda_{12}^2 = (-\lambda_{o2}^2).$$

(75)

As a result the waves propagate at a velocity which is proportional to the conductivity which is very large and the damping factor is the same as for the ideal plasma.

B. Plasma with Uniform Velocity

We return to the general equation for longitudinal waves, Equation 44, for this case. This is a fourth order polynomial equation in $\lambda$ so that there are four solutions. To solve this it is necessary to use a numerical technique involving a digital computer to facilitate the computations. Numerical values for a plasma were chosen and are described in the Appendix. The roots were then extracted using the Cyclone digital computer of the Iowa State University Computation Center. Since $\lambda$ is primarily a function of frequency, solutions were obtained for frequencies ranging from 1.0 rad/sec to $1 \times 10^{20}$ rad/sec. This range covers the frequencies normally observed in physical phenomena (9). As was observed for the stationary plasma the four roots always occur in pairs, one pair corresponding to propagation of an ion wave and the other pair to propagation of an electron wave. Hence we see that the two basic
longitudinal wave modes which appeared in the stationary plasma are present in a moving plasma but the characteristics are somewhat different as can be seen by examining several cases.

1. Ideal plasma

First we will look at the situation for $\omega_0 = 0$, which means that all of the coefficients in Equation 44 are real numbers. The variation of the phase velocity as a function of frequency for the two basic longitudinal wave modes is shown in Figure 2.

The ion wave phase velocity appears exactly as it does for the stationary plasma. Hence, the wave starts at low frequency with a phase velocity equal to the plasma sound speed and as the frequency increases the speed decreases slightly until the frequency approaches the electron plasma frequency. At this point the phase velocity decreases more rapidly and approaches the ion sound speed as the frequency becomes quite large. In contrast to the stationary plasma case, a damping factor is present for the ion wave so that the wave is very lightly damped. The damping factor decreases with increasing frequency. It can therefore be concluded that the plasma velocity has no effect on the ion wave phase velocity but it does produce a damping effect on the wave.

The character of the longitudinal wave associated with the electron motion is quite different from the stationary plasma case except in the high frequency region. For $\omega > \omega_p$, the phase velocity approaches the electron sound speed as the frequency becomes large. This wave is also damped in this region when the plasma has an initial flow velocity. The
damping factor decreases with increasing frequency. For frequencies less than the electron plasma frequency, the electron wave propagates at approximately the plasma sound speed with the speed increasing with increasing frequency. Near \( \omega = \omega_2 \) the propagation speed increases sharply approaching an infinite value at the electron plasma frequency. Unlike any waves encountered for a stationary plasma, this wave grows as it moves away from the disturbance center. The amplification factor \( \lambda_I \) is quite small so that the growth rate is also small and decreases with increasing frequency. This type of disturbance growth has been observed experimentally but at present there is no satisfactory explanation for this phenomenon \( (19) \). Perhaps one reason that the electrons are affected and the ions are not is due to the large difference in the mass of the particles.

2. **Real plasma**

For the numerical example, the electrical conductivity is relatively large so that the results will correspond with the approximate situation examined for the stationary fluid. In this case the coefficients appearing in Equation 44 are complex so that, in general, the solutions will be complex also. As was shown for the stationary plasma, the wave propagation speeds are essentially unaffected by the electrical conductivity but the damping factors do vary slightly.

In the case of the ion wave, the damping factor is increased slightly over the ideal plasma case but the wave is still lightly damped. The damping factor for the electron wave is also increased slightly, while the amplification factor is reduced. Hence, the growth rate is lessened in a real plasma as compared with an ideal plasma.
Figure 1. Longitudinal waves for a stationary plasma.
Data calculated for $p_o = 1$ atmos. and $T_o = 15,000^\circ K$. 

**PHASE VELOCITY-LOG$_{10} V$ (meters/sec.)**

**FREQUENCY-LOG$_{10} \omega$ ($\text{rad./sec.}$)**

- Electron wave
- Ion wave
Figure 2. Longitudinal waves for a drifting plasma.
Data calculated for $p_o = 1$ atmos., $T_o = 15,000$ K, and $U_{ox} = 3000$ meters/sec.
V. TRANSVERSE WAVES WITHOUT AN EXTERNAL MAGNETIC FIELD

Unlike the longitudinal waves the presence of an external magnetic field affects the transverse waves propagating in a plasma. In the present chapter we will examine the transverse waves without an external magnetic field and in the following chapter the effect of an external magnetic field on the transverse waves.

The equations governing the propagation of transverse waves are characterized by the variables which are perpendicular to the direction of propagation, i.e., $E_y^t$, $E_z^t$, $v_1^t$, $v_2^t$, $w_1^t$, $w_2^t$, $h_1^t$, and $h_2^t$ are Equations 26, 27, 29, 30, 36, 37, 39, and 40. If we set $H_{ox} = 0$ these equations become

\begin{align}
&i \omega E_y^t + e [N_o (v_1^t - v_2^t)] = i \lambda h_z^t \quad (76) \\
&i \omega E_z^t + e [N_o (w_1^t - w_2^t)] = -i \lambda h_y^t \quad (77) \\
&\omega \mu_e h_y^t = -\lambda E_z^t \quad (78) \\
&\omega \mu_e h_z^t = \lambda E_y^t \quad (79) \\
&\text{im} N_o (\omega - \lambda U_{ox}) v_1^t = \alpha_{12} (v_1^t - v_2^t) + e N_o E_y^t - \epsilon_0 \epsilon_0 U_{ox} h_z^t \quad (80) \\
&\text{im} N_o (\omega - \lambda U_{ox}) w_1^t = \alpha_{12} (w_1^t - w_2^t) + e N_o E_z^t + \epsilon_0 \epsilon_0 U_{ox} h_y^t \quad (81) \\
&\text{im} N_o (\omega - \lambda U_{ox}) v_2^t = \alpha_{12} (v_2^t - v_1^t) - e N_o E_y^t + \epsilon_0 \epsilon_0 U_{ox} h_z^t \quad (82) \\
&\text{im} N_o (\omega - \lambda U_{ox}) w_2^t = \alpha_{12} (w_2^t - w_1^t) - e N_o E_z^t - \epsilon_0 \epsilon_0 U_{ox} h_y^t \quad (83)
\end{align}

Note that this set of equations can again be separated into the following two groups:

(1) Equations 76, 79, 80, and 82 involving the variables $E_y^t$, $v_1^t$, $v_2^t$, and $h_z^t$
(2) Equations 77, 78, 81, and 83 involving the variables $E_z$, $v_1^i$, $v_2^i$, and $h_y^i$.

Each of these groups characterize a transverse wave so that the characteristics of the wave can be obtained by examining only one of the above groups. The other group describes a wave of identical appearance except that it has a 90° rotation about the x-axis. Considering the first group, the determinantal equation is obtained from

$$
\begin{vmatrix}
 i\omega & -i\lambda & eN_0 & -eN_0 \\
 -\lambda & a_{11} & 0 & 0 \\
 -eN_0 & e_{12}^N U_{ox} & (imN_0 z - \alpha_{12}) & \alpha_{12} \\
 eN_0 & -e_{12}^N U_{ox} & \alpha_{12} & (imN_0 z - \alpha_{12}) \\
\end{vmatrix} = 0
$$

(84)

where $z = \omega - \lambda U_{ox}$. Again neglecting $m_2$ with respect to $m_1$ and introducing the speed of light

$$
c = \left(\frac{1}{\mu_e^0\epsilon^0}\right)^{\frac{1}{2}}
$$

(85)

the resulting polynomial equation in $\lambda$ is

$$
\left(\frac{U_{ox}}{\omega}\right) \lambda^3 + \left[1 + i \alpha_{12}^N\right] \lambda^2 + \left(-\frac{U_{ox}}{c^2}\right) \left(1 - \frac{\omega^2}{\omega^2}\right) \lambda \\
+ \frac{\omega^2}{c^2} \left[\frac{\omega^2}{\omega^2} - (1 + i \alpha_{12}^N)\right] = 0
$$

(86)

The wave modes can be obtained by solving Equation 86 for the wave number $\lambda$, which may be complex, as a function of the frequency $\omega$.

**A. Stationary Plasma**

For this case, $U_{ox} = 0$, Equation 86 becomes

$$
\left[1 + i \alpha_{12}^N\right] \lambda^2 + \frac{\omega^2}{c^2} \left[\frac{\omega^2}{\omega^2} - (1 + i \alpha_{12}^N)\right] = 0
$$

(87)

or solving for $\lambda^2$ we have
\[ \lambda^2 = \frac{\omega_p^2}{c^2} \left[ 1 - \frac{\omega_p^2}{\omega^2} \frac{(1 - i \alpha_i^2)}{\alpha_i^2 (1 + \alpha_i^2)} \right] \]  

(88)

which is the same result as presented by Pai (15). Hence we have a wave propagating in the positive x-direction and a corresponding wave propagating in the negative x-direction for the first group of variables \( E'_x, v'_1, v'_2 \) and \( h'_z \). In addition we have a similar wave mode associated with the second group of variables \( E'_z, w'_1, w'_2 \) and \( h'_y \). Therefore, in the plasma there are two basic transverse wave modes present.

1. **Ideal plasma**

   Equation 88 now becomes
   \[ \lambda^2 = \frac{\omega_p^2}{c^2} \left[ 1 - \frac{\omega_p^2}{\omega^2} \right] \]  

   (89)

   For \( \omega < \omega_p \), \( \lambda \) is a pure imaginary number so that the wave is a non-propagating or evanescent wave.

   At the electron plasma frequency, \( \omega_p \), the wave is changing from an evanescent wave to a propagating wave and the phase velocity is said to be infinite.

   When \( \omega > \omega_p \), the wave is an undamped propagating wave with the phase velocity given by
   \[ v = \frac{\omega}{\lambda_R} = \frac{c}{\left[ 1 - \frac{\omega_p^2}{\omega^2} \right]^{\frac{1}{2}}} \]  

   (90)

   As a result the phase velocity decreases with increasing frequency and approaches the limiting case where the wave propagates at the speed of light. This is shown in Figure 3.

2. **Real plasma**

   For this case we return to the general expression for the wave
number, Equation 88, written as follows

\[ \lambda^2 = (\lambda_R + i\lambda_I)^2 = \frac{\omega_p^2}{c^2} \left[ 1 - \frac{\omega_p^2}{\omega^2 (1 + \alpha_{12}^2)} \right] + i \frac{\omega_p^2}{c^2} \frac{\alpha_{12}^2}{(1 + \alpha_{12}^2)} \]

\[ = k_1 + i k_2 . \]  

(91)

Solving for \( \lambda_R \) and \( \lambda_I \) in terms of \( k_1 \) and \( k_2 \) we have

\[ \lambda^2_R = \frac{1}{2} \left[ k_1 + (k_1^2 + k_2^2)^{1/2} \right] \]

(92)

\[ \lambda^2_I = \frac{1}{2} \left[ -k_1 + (k_1^2 + k_2^2)^{1/2} \right] . \]

(93)

As was done for the longitudinal waves, we consider the electrical conductivity as large but still finite. Hence, we find that for \( \omega < \omega_e \), \( k_1 < 0 \), with the result that

\[ \lambda^2_R = \frac{1}{4} \frac{k_2^2}{|k_1|} , \quad \lambda^2_I = \left| k_1 \right| . \]

(94)

Therefore, the phase velocity is very large as is the damping factor with the result that the wave is highly damped and essentially does not propagate. The wave is very similar to the ideal plasma case.

If \( \omega > \omega_e \), \( k_1 > 0 \), and it can be shown that

\[ \lambda^2_R = k_1 , \quad \lambda^2_I = \frac{1}{4} \frac{k_2^2}{k_1} . \]

(95)

This yields a damped wave with phase velocity nearly identical with that for the undamped wave in an ideal plasma, i.e., \( V = \frac{\omega}{k_1^{1/2}} \). The damping factor increases with decreasing electrical conductivity and decreases with increasing frequency.

The effect of the electrical conductivity is to produce a damping of the propagating wave without altering the phase velocity.
B. Plasma with Uniform Velocity

The behavior of the transverse waves with an initial plasma flow velocity is examined by studying the roots of the general cubic Equation 86. We again use the numerical example described in the Appendix and solve for the values of $\lambda$ as a function of frequency using a numerical technique and the Cyclone digital computer. The same frequency spectrum is considered as was used for the longitudinal waves.

1. Ideal plasma

For this case the coefficients in Equation 86 are real numbers. Hence, one of the solutions $\lambda$ of the cubic equation is always a real number and the remaining two solutions may be complex numbers, but we find for this case that all of the roots are real numbers. Two of the roots occur as a pair, representing wave modes of identical character, one wave propagating in the positive $x$-direction and the other wave propagating in the negative $x$-direction. The other real solution corresponds to a disturbance which propagates at the same speed as the uniform plasma flow velocity. These waves propagate unattenuated.

The behavior of the phase velocity as a function of frequency for the waves is shown in Figure 4.

At low frequencies the wave phase velocity is nearly constant at approximately the speed of light $c$ until the frequency nears the ion plasma frequency where the phase velocity begins to increase rapidly. At $\omega = \omega_2$ the phase velocity is infinite.

Beyond $\omega > \omega_2$ the phase velocity appears the same as it does for the stationary plasma case, i.e., as the frequency increases it rapidly
decreases and approaches the speed of light as the frequency becomes very large.

2. Real plasma

In this case the coefficients in the determinantal equation are complex numbers so that the wave numbers are also expected to be complex. Again one of the solutions $\lambda$ corresponds to a wave propagating at the same speed as the initial plasma flow velocity, but in this case the wave is lightly damped with the damping factor decreasing with increasing frequency. The wave phase velocity for the other wave, the transverse wave, appears exactly as it did for the ideal plasma case.

When $\omega < \omega_p$, the transverse wave is highly damped with the damping factor decreasing with increasing frequency. This is in contrast to no damping for the ideal plasma.

If $\omega > \omega_p$, the wave is slightly damped. Again the damping factor decreases with increasing frequency.
Figure 5. Transverse waves for a stationary plasma without external magnetic field. Data calculated for $p = 1$ atmos., and $T_e = 15,000$ K.
Figure 4. Transverse waves for a drifting plasma without external magnetic field. Data calculated for \( p_0 = 1 \) atmos., \( T_0 = 15,000^\circ \text{K} \), and \( U_{ox} = 3000 \) meters/sec.
VI. TRANSVERSE WAVES WITH A LONGITUDINAL EXTERNAL MAGNETIC FIELD

When an external magnetic field aligned with the plasma flow velocity is present the transverse waves are governed by Equations 26, 27, 29, 30, 36, 37, 39, and 40 involving the variables $E_y', E_z', v_1', v_2', w_1', w_2', h'_1$, and $h'_2$. The determinantal equation is obtained from

\[
\begin{vmatrix}
\omega & 0 & -eN_0 & 0 & 0 & 0 & -i\lambda \\
0 & \omega & 0 & 0 & eN_0 & -eN_0 & i\lambda & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & \omega \mu_e & 0 \\
-\lambda & 0 & 0 & 0 & 0 & 0 & 0 & \omega \mu_e \\
eN_0 & 0 & (im_1N_0z-\alpha_{12}) & \alpha_{12} & -e\mu_eN_0H_{ox} & 0 & 0 & e\mu_eN_0U_{ox} \\
0 & -eN_0 & e\mu_eN_0H_{ox} & 0 & (im_1N_0z-\alpha_{12}) & \alpha_{12} & -e\mu_eN_0U_{ox} & 0 \\
eN_0 & 0 & \alpha_{12} & (im_2N_0z-\alpha_{12}) & 0 & e\mu_eN_0H_{ox} & 0 & -e\mu_eN_0U_{ox} \\
0 & eN_0 & 0 & -e\mu_eN_0H_{ox} & \alpha_{12} & (im_2N_0z-\alpha_{12})e\mu_eN_0U_{ox} & 0 & 0 \\
\end{vmatrix} = 0
\]

where $z = \omega - \lambda U_{ox}$.

As was done in the preceding cases, we use the fact that $\frac{m_2}{m_1} << 1$ in expanding the above determinant. The equations describing the plasma and its associated phenomena have been written in non-relativistic terms, i.e., we have assumed that the fluid velocities are much less than the speed of light. As a result $\frac{U_{ox}}{c} << 1$ so that higher-order terms can be neglected in the expansion. Hence, the resulting polynomial determinantal equation in $\lambda$ is

\[
C_8\lambda^8 + C_7\lambda^7 + C_6\lambda^6 + C_5\lambda^5 + C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 = 0
\]

where
\[ c_6 = \frac{\omega_0^4}{\omega_a^4} \left[ 6 - \frac{\omega_x^2}{\omega_a^2} + 2 \frac{\omega_0^2}{\omega_a^2} - \alpha_{12}^2 + 6 \alpha_{12}^2 \right] \]

\[ c_5 = \frac{U_{ox}}{c^2} \left[ -4 + 2 \frac{\omega_x^2}{\omega_a^2} - 8 \frac{\omega_0^2}{\omega_a^2} + 2 \alpha_{12}^2 + 21 \alpha_{12}^2 \left( -3 + \frac{\omega_x^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} - \frac{\omega_0^2}{\omega_a^2} \right) \right] \]

\[ c_4 = 1 - \frac{\omega_x^2}{\omega_a^2} \left( 1 - \frac{\omega_0^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} \right) - \frac{\omega_0^2}{\omega_a^2} + \frac{\omega_x^2}{\omega_a^2} \left( 12 + \frac{\omega_0^2}{\omega_a^2} \right) \]

\[ c_3 = \frac{\omega U_{ox}}{c^2} \left[ 8 + 4 \frac{\omega_x^2}{\omega_a^2} \left( \frac{\omega_x^2}{\omega_a^2} - 1 \right) - 4 \frac{\omega_0^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} + 2 \frac{\omega_0^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} \right] \]

\[ c_2 = \frac{\omega^2}{c^2} \left[ -2 + 2 \frac{\omega_x^2}{\omega_a^2} \left( 1 - \frac{\omega_x^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} \right) + 2 \frac{\omega_x^2}{\omega_a^2} + 2 \frac{\omega_0^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} \left( 1 + 3 \frac{\omega_0^2}{\omega_a^2} \right) \right] \]

\[ c_1 = \frac{\omega^3 U_{ox}}{c^4} \left[ -4 - \frac{\omega_0^2}{\omega_a^2} \left( \frac{\omega_x^2}{\omega_a^2} - 2 \right) + 2 \frac{\omega_x^2}{\omega_a^2} \left( 1 - \frac{1}{\omega_a^2} \right) + 2 \alpha_{12}^2 \right. \]

\[ + 21 \alpha_{12}^2 \left( -3 + \frac{\omega_x^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} \right) \]

\[ c_0 = \frac{\omega^4}{c^4} \left[ 1 - \frac{\omega_x^2}{\omega_a^2} \left( 1 - \frac{\omega_0^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} \right) - 2 \frac{\omega_0^2}{\omega_a^2} - \frac{\omega_x^2}{\omega_a^2} \left( 2 - \frac{\omega_0^2}{\omega_a^2} \right) - \alpha_{12}^2 \right. \]

\[ + 2 \alpha_{12}^2 \left( 1 - \frac{\omega_0^2}{\omega_a^2} \frac{\omega_x^2}{\omega_a^2} \right) \]
and the ion cyclotron frequency

\[ \omega_{\text{xl}} = \frac{e\mu_e H}{m_i} \]  \hspace{1cm} (107)

denotes the electron cyclotron frequency

\[ \omega_{\text{x2}} = \frac{e\mu_e H}{m_2} \]  \hspace{1cm} (108)

the ion plasma frequency \( \omega_i \), the electron plasma frequency \( \omega_e \), \( \alpha_{12}^2 \), and the speed of light \( c \) are as previously defined.

Equation 97 must then be solved for the wave number \( \lambda \) for each of the wave modes as a function of the frequency \( \omega \). Since the determinantal equation is in general an eighth order polynomial it must be solved by numerical techniques with the aid of a digital computer. Several cases were examined as follows.

A. Stationary Plasma

Setting \( U_{\text{ox}} = 0 \) in Equations 97 through 106 we obtain

\[ D_4 \lambda^4 + D_2 \lambda^2 + D_0 = 0 \]  \hspace{1cm} (109)

where

\[ D_4 = \left[ 1 - \frac{\omega_i^2}{\omega^2} (1 - \frac{\omega_{\text{x1}}^2}{\omega^2}) - \frac{\omega_{\text{x1}}^2}{\omega^2} - \omega_{\text{x2}}^2\omega_e^2 \alpha_{12}^2 (1 - \frac{\omega_{\text{x2}}^2}{\omega^2} \frac{m_2}{m_1}) \right] \]  \hspace{1cm} (110)

\[ D_2 = \frac{\omega_i^2}{c^2} \left[ -2 + 2 \frac{\omega_{\text{x2}}^2}{\omega^2} (1 - \frac{\omega_{\text{x1}}^2}{\omega^2} - \frac{\omega_{\text{x2}}^2}{\omega^2}) + 2 \frac{\omega_{\text{x1}}^2}{\omega^2} + 2 \frac{\omega_{\text{x2}}^2}{\omega^2} + 2 \alpha_{12}^2 \right] \]  \hspace{1cm} (111)

\[ D_0 = \frac{\omega_i^4}{c^4} \left[ 1 - \frac{\omega_{\text{x1}}^2}{\omega^2} (1 - \frac{\omega_{\text{x2}}^2}{\omega^2}) - 2 \frac{\omega_{\text{x1}}^2}{\omega^2} \frac{\omega_{\text{x2}}^2}{\omega^2} - \frac{\omega_{\text{x2}}^2}{\omega^2} - \frac{\omega_{\text{x2}}^2}{\omega^2} (2 - \omega_{\text{x1}}^2 \frac{m_2}{m_1}) \right] \]  \hspace{1cm} (112)

which is the more familiar dispersion equation, a quadratic in \( \lambda^2 \), for
transverse waves as presented by Pai (16). This equation can readily be solved for the two roots for \( \lambda^2 \) which represent the two basic transverse waves. One of these modes is primarily associated with the electrons and the other one with the ions of the plasma. These will be examined in more detail.

1. **Ideal plasma**

With \( \alpha^2 = 0 \), the two solutions for \( \lambda^2 \) which represent the wave modes are

\[
\lambda_1^2 = \frac{-R_2 + (R_2^2 - R_0 R_4)^{1/2}}{R_0} \quad (113)
\]

\[
\lambda_2^2 = \frac{-R_2 - (R_2^2 - R_0 R_4)^{1/2}}{R_0} \quad (114)
\]

where

\[
R_0 = \frac{c^2}{\omega^2} [1 - \frac{\omega^2_{\perp} (1 - \frac{\omega^2_1}{\omega^2} - 2 \frac{\omega^2_1}{\omega^2} - \frac{\omega^2_{11}}{\omega^2} - \frac{\omega^2_{12}}{\omega^2} - 2 \frac{\omega^2_{22}}{\omega^2})]} \quad (115)
\]

\[
R_2 = \frac{\omega^2}{c^2} [-2 + \frac{\omega^2_{\perp}}{\omega^2} (1 - \frac{\omega^2_1}{\omega^2} - \frac{\omega^2_1}{\omega^2} + 2 \frac{\omega^2_{11}}{\omega^2} + 2 \frac{\omega^2_{12}}{\omega^2} - \frac{\omega^2_{12}}{\omega^2})] \quad (116)
\]

\[
R_4 = [1 - \frac{\omega^2_{\perp}}{\omega^2} (1 - \frac{\omega^2_1}{\omega^2} - \frac{\omega^2_{11}}{\omega^2})] \quad (117)
\]

For very low frequencies the two waves are always undamped and propagate at the same speed

\[
V = V_{x1} \left(1 + \frac{V_{x1}^2}{c^2}\right)^{1/2} \quad (118)
\]

where the x-wise ion Alfvén wave speed is

\[
V_{x1} = \frac{\mu}{m_1 N_0} \quad (119)
\]

which is essentially the x-wise plasma Alfvén wave speed. For the numerical example \( \frac{V_{x1}}{c} \ll 1 \), hence, \( V = V_{x1} \), so that the two waves propagate at the plasma Alfvén wave speed. This agrees with the results
previously obtained.

So long as \( \omega < \omega_{x1} \), the two waves are still undamped but propagate at different speeds both decreasing with increasing frequency. The wave described by \( \lambda^2_1 \) is associated with the ions and has a larger phase velocity than the wave associated with the electrons given by \( \lambda^2_2 \).

When \( \omega = \omega_{x1} \),

\[
\lambda^2_1 \approx \frac{\omega_{x2}^2}{2c^2} \left( \frac{\omega_{x2}}{\omega_{x1}} + \frac{\omega_{x1}}{\omega_e} \right) ; \quad \lambda^2_2 = \infty . \tag{120}
\]

This, the first mode is still present but the second mode, electron wave, has a propagation speed of zero and does not propagate.

In the region \( \omega_{x1} < \omega < \omega_{x2} \), the nature of the solutions for \( \lambda^2 \) depends to some extent on the cyclotron frequencies and the plasma frequencies. The solution \( \lambda^2_1 \) indicates that the ion wave is always undamped and propagates with a finite speed. The electron wave, or \( \lambda^2_2 \), starts as an evanescent wave and may become a propagating undamped wave, depending upon the magnitude of the external magnetic field. For the propagating waves, the propagation speed of the electron wave is larger than for the ion wave, both decreasing with increasing frequency.

If \( \omega = \omega_{x2} \), we have

\[
\lambda^2_1 = \infty , \quad \lambda^2_2 \approx \frac{\omega^2}{c^2} \left[ 1 - \frac{\omega_{x2}^2}{2\omega^2} \right] . \tag{121}
\]

As a result, the ion wave no longer propagates since \( V = 0 \). The electron wave still exists, however.

For \( \omega > \omega_{x2} \), the ion wave starts as an evanescent wave but as the frequency increases it changes to an undamped propagating wave. The
electron wave may start as a non-propagating wave or a propagating wave depending upon the strength of the external magnetic field. The change-over frequency where the wave changes from a non-propagating to a propagating wave is higher for the electron wave than for the ion waves.

In the high-frequency regime, $\omega \gg \omega_2$, $\omega_2$, the two waves propagate as undamped waves at the speed of light. Hence, the two waves appear as the conventional electromagnetic waves.

Figure 5 shows the variation of phase velocity with frequency for the two basic transverse waves.

2. Real plasma

For this case the electrical conductivity is a finite quantity so that the waves must be examined from Equation 109. The coefficients are complex numbers so that the solutions for $\lambda$ are also expected to be complex. It has been shown by Pai (16) and Tanenbaum (21) that this leads to damped waves.

As we have done before we consider the electrical conductivity as very large but still finite. We then write the complex wave number as $\lambda = \lambda_R + i\lambda_I$.

Denoting $\lambda_o^2$ as the solution for the ideal plasma, we then compare the real plasma solutions with the ideal plasma. If $\lambda_o^2 > 0$, i.e., undamped propagating waves, we find for the real plasma that $\lambda_R^2 = \lambda_o^2$ and $\lambda_I$ is proportional to $\alpha_12$. Hence, the electrical conductivity does not affect the speed of propagation but the damping factor increases with decreasing electrical conductivity. Thus, the propagating waves are damped in a real plasma.
For $\lambda^2_0 < 0$, non-propagating waves in an ideal plasma, we find that the wave will propagate at a very high speed since $V$ is inversely proportional to $\alpha_{12}$ but that the damping factor is essentially unaffected, $\lambda_I \equiv (-\lambda^2_0)$. As a result the wave propagates at a very high speed but it is highly damped.

B. Plasma with Uniform Velocity

We must now return to the general expression for $\lambda$ given by Equation 97. This is an eighth order polynomial so that it would appear to have eight roots which must be examined as possible wave modes. Using the numerical example described in the Appendix the roots were extracted by a numerical procedure involving a digital computer. The frequency range used is the same as previously taken, namely, $\omega = 1 \times 10^4$ rad/sec to $1 \times 10^{20}$ rad/sec.

1. Ideal plasma

Under this condition the coefficients appearing in Equation 97 are real numbers. One of the solutions $\lambda$ corresponds to a disturbance which propagates at the same speed as the initial plasma flow velocity. This wave is lightly damped with the damping decreasing as the frequency increases.

Although the determinantal equation is an eighth order polynomial, only the two basic transverse waves, one associated with the ions and the other with the electrons, plus the above mentioned wave are found to be present. The higher order of the polynomial merely acts to alter the characteristics of the basic wave modes.

The behavior of the phase velocity as a function of the frequency for the waves is shown in Figure 6.
At very low frequencies the electron wave propagates at approximately the x-wise Alfvén velocity with the phase velocity decreasing slightly with increasing frequency. Near the ion cyclotron frequency the phase velocity decreases rapidly approaching zero. In this frequency region, the electron wave is very lightly damped with the damping factor decreasing with increasing frequency.

The ion wave also starts at approximately the x-wise Alfvén velocity for very low frequencies and as the frequency increases the phase velocity increases slightly until at $\omega = \omega_{x1}$ the phase velocity is equal to the ion sound speed. As the frequency continues to increase the speed now decreases slightly until the frequency nears the electron cyclotron frequency where it decreases rapidly approaching zero as $\omega$ approaches $\omega_{x2}$. For this frequency spectrum, the ion wave is lightly damped with the damping factor decreasing with increasing frequency.

In the frequency region $\omega_{x1} < \omega < \omega_{x2}$, an electron wave is also present in addition to the ion wave discussed above. The phase velocity is very large for $\omega$ near $\omega_{x1}$, and decreases quite rapidly, approaching the speed of light. The electron wave is highly damped in this region and the damping factor decreases slightly with increasing frequency.

For $\omega_{x2} < \omega < \omega_2$, the electron wave propagates at essentially the speed of light and is lightly damped. In addition the ion wave is also present. Near $\omega_{x2}$ the phase velocity for the ion wave is very large and decreases rapidly with increasing frequency approaching the speed of light for $\omega$ slightly larger than the electron plasma frequency.
The ion wave is also highly damped in the region near $\omega_2$, but the damping factor decreases with increasing frequency.

For $\omega > \omega_2$, both the ion and electron waves propagate at the speed of light and both are very lightly damped. With increasing frequency the damping factor shows a slight decrease in magnitude.

2. **Real plasma**

The solutions for the wave number $\lambda$ as a function of frequency were again obtained from the general transverse wave expression, Equation 97, this time with complex coefficients. A wave propagating at the same speed as the initial flow velocity is again found to be present. As was the case for the ideal plasma, this wave is lightly damped but more so than for the ideal plasma. The damping factor decreases with increasing frequency.

The two basic transverse wave modes are also present and their propagation velocities are found to behave in exactly the same fashion as for an ideal plasma. This is shown in Figure 6. The damping factors for the waves are larger for the real plasma as compared with the ideal plasma. The increase is not very great, but there is a slight increase.
Figure 5. Transverse waves for a stationary plasma with a longitudinal external magnetic field. Data calculated for $p_0 = 1$ atmos., $T_0 = 15,000^\circ$K, and $B_{ox} = 1000$ gauss.
Figure 6. Transverse waves for a drifting plasma with a longitudinal external magnetic field. Data calculated for $p_0 = 1$ atmos., $T_0 = 15,000^\circ K$, $U_{ox} = 3000$ meters/sec, and $B_{ox} = 1000$ gauss.
VII. CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDIES

Following is a summary of the results obtain in the study of wave propagation in a fully ionized plasma. As a result of this study several additional areas for continued study became apparent and are discussed in this section also.

A. Conclusions

1. Without an external magnetic field

For a stationary plasma there are four basic wave modes present, two longitudinal waves and two transverse waves. At all times the longitudinal and transverse waves are entirely independent of one another. The waves do not always appear as propagating waves throughout the entire frequency range. For frequencies below the electron plasma frequency $\omega_p$, only the longitudinal wave associated with the ions is present and propagates at a speed essentially equal to the plasma sound speed. The other three wave modes are evanescent waves. For frequencies larger than $\omega_p$, all four waves are present. One of the longitudinal waves, the ion wave, propagates at a speed equal to the ion sound speed while the other longitudinal wave, the electron wave, propagates at a speed essentially equal to the electron sound speed. Both of the transverse waves, which are independent of one another, propagate at the speed of light.

When the plasma has a uniform flow velocity, the two longitudinal waves and the two transverse waves are present along with a disturbance which propagates at a speed equal to the initial flow velocity. The propagation speed for this wave is independent of the frequency but the
The characteristics of the four basic wave modes are also altered. In the frequency region below $\omega_2$, all four of the basic waves are now present. The two transverse waves, still independent, propagate at essentially the speed of light and are undamped. The longitudinal ion wave appears exactly as it did for the stationary plasma except that it is lightly damped. The longitudinal electron wave propagates at approximately the plasma sound speed and the amplitude of the wave increases slowly so that the disturbance tends to grow. For frequencies larger than $\omega_2$, the propagation speeds for all four basic waves are the same as in the stationary plasma case, but the waves are lightly damped.

2. With a longitudinal external magnetic field

The longitudinal waves are unaffected by the introduction of an external magnetic field with or without an initial plasma flow velocity so they will not be discussed any further.

For the stationary plasma the two basic transverse waves do not appear as propagating waves for all frequencies. In the very low frequency region, both the ion and electron transverse waves are present and propagate at approximately the Alfven speed. In the high frequency region, or $\omega > \omega_2$, both waves are also present and propagate at the speed of light. In the intermediate frequency region, the waves may be propagating or evanescent waves depending to a large extent upon the magnitude of the external magnetic field applied.

With an initial plasma flow velocity, the two basic transverse waves are found to be present for all frequencies but their characteristics are
altered from the stationary plasma case. In addition, a disturbance which propagates at the same speed as the initial flow velocity is also present. This wave is lightly damped but the phase velocity is insensitive to the frequency. In the very low and very high frequencies ranges the phase velocities for the two transverse waves are essentially the same as for the stationary plasma, but the waves are lightly damped now. In the intermediate frequency region the two waves are always found to be present as propagating waves but their exact behavior depends to some extent on the magnitude of the external magnetic field. When the phase velocity is less than the speed of light, these waves are lightly damped. For the phase velocities larger than the speed of light the waves are heavily damped.

The primary effect of considering a finite electrical conductivity is to increase the damping factor for all of the wave modes. For the one longitudinal wave which grows, finite conductivity tends to reduce the amplification factor.

The principal effect of the initial plasma flow velocity is that of producing an additional force due to the interaction of the flow velocity and the induced magnetic field which then acts on the fluid particles. As a result all of the wave modes appear throughout the entire frequency spectrum and with the exception of the one longitudinal wave which grows, the waves are always damped.

B. Suggestions for Further Studies

The present analysis considers only the case for a longitudinal external magnetic field so that it would be of interest to study the
case for a transverse external magnetic field. As was indicated in Chapter 3 this would necessitate the inclusion of an external electric field to satisfy the initial condition of no electric current.

An arbitrarily oriented external magnetic field is another case that would be of interest. In each of these studies, the longitudinal and transverse waves would probably interact and not be independent of one another.

Instead of considering the uniform plasma flow velocity in the direction of wave propagation as was done in this study, the uniform velocity could also be transverse. This would lead to a different set of governing equations and quite likely different wave propagation characteristics.

Additional studies, extensions of the above, could be undertaken using a partly ionized plasma composed of ions, electrons, and neutral particles using a three-fluid theory to describe the plasma. The resulting characteristic equations for the wave numbers would be quite complex and difficult to analyze, however.
VIII. REFERENCES


IX. ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. C. T. Hsu for his guidance, suggestions, and encouragement offered during the course of this investigation; to Dr. E. W. Anderson for his assistance; to the Iowa State Computation Center; and to the Iowa Engineering Experiment Station for their financial assistance.
X. APPENDIX

Some of the characteristics of the plasma chosen for the numerical analysis along with the calculated constants are presented herein.

Electron charge - \( e = 1.602 \times 10^{-19} \) coulombs

Permeability - \( \mu_e = 4\pi \times 10^{-7} \) henries/meter

Dielectric constant - \( \epsilon = 8.854 \times 10^{-12} \frac{\text{coul}^2}{\text{kg \cdot meter}^s} \)

Electron mass - \( m_e = 9.108 \times 10^{-31} \) kg

Ion mass - \( m_i = 2.658 \times 10^{-26} \) kg (oxygen gas)

Initial number density - \( N_0 = 1.54 \times 10^{17} / \text{cm}^3 \)

(based on oxygen gas at \( T_0 = 15,000^\circ \text{K} \)
\( P_0 = 1 \) atmos.)

Electrical conductivity - \( \sigma = 66 \) mhos/cm

Initial magnetic flux density - \( B_{ox} = 1 \times 10^{-1} \) webers/meter\(^2\)

Initial plasma flow velocity - 3000 meter/sec

Speed of light - \( c = 2.998 \times 10^8 \) meter/sec

Ion sound speed - \( a_i = 3.60 \times 10^3 \) meter/sec

Electron sound speed - \( a_e = 6.15 \times 10^5 \) meter/sec

Plasma sound speed - \( a_p = 5.31 \times 10^8 \) meter/sec

Alfvén wave speed - \( V_{xl} = 1.392 \times 10^9 \) meter/sec

Ion plasma frequency - \( \omega_i = 1.294 \times 10^{11} \) rad/sec

Electron plasma frequency - \( \omega_e = 2.21 \times 10^{13} \) rad/sec

Ion cyclotron frequency - \( \omega_{i\lambda} = 6.02 \times 10^5 \) rad/sec

Electron cyclotron frequency - \( \omega_{e\lambda} = 1.759 \times 10^{10} \) rad/sec

Friction frequency - \( \omega_f = 6.57 \times 10^6 \) rad/sec

where \( \omega^{*}_{\lambda} = \frac{\omega_{i\lambda}}{\omega} \)
Based upon the above values the coefficients appearing on the determinantal equations for the longitudinal and transverse waves were calculated using the Cyclone digital computer of the Iowa State Computational Center. The roots of these equations were then extracted using a polynomial root solver program which is maintained by the Center for use with the Cyclone.

In the following tables an indication of the order of magnitude for the damping or amplification factor for the wave modes is given. The values correspond to the lower end of the frequency range indicated and decrease with increasing frequency.

Table 1. Longitudinal waves

<table>
<thead>
<tr>
<th>( U_{ox} ) meters/sec</th>
<th>( \alpha_{12} ) /meter</th>
<th>( \lambda_1 ) /meter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \omega &lt; \omega_2 )</td>
<td>( \omega &gt; \omega_2 )</td>
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<table>
<thead>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
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</tr>
<tr>
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<td>0</td>
<td>-10^{-4}</td>
</tr>
<tr>
<td>3000</td>
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<table>
<thead>
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</tr>
<tr>
<td>0</td>
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<td>-10^{-7}</td>
</tr>
<tr>
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<td>+10^{-7}</td>
</tr>
<tr>
<td>3000</td>
<td>non-zero</td>
<td>+10^{-7}</td>
</tr>
</tbody>
</table>
Table 2. Transverse waves without an external magnetic field

<table>
<thead>
<tr>
<th>$U_{oX}$ meters/sec</th>
<th>$\alpha_{12}$</th>
<th>$\lambda_1$/meter</th>
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</thead>
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<td>$\omega &lt; \omega_2$</td>
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</tr>
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<td>Electron wave</td>
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<td></td>
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</tr>
<tr>
<td></td>
<td>3000</td>
<td>non-zero</td>
</tr>
<tr>
<td>Drifting wave</td>
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</tr>
<tr>
<td></td>
<td>3000</td>
<td>non-zero</td>
</tr>
</tbody>
</table>

Table 3. Transverse waves with a longitudinal external magnetic field

<table>
<thead>
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<th>$U_{oX}$ meters-sec</th>
<th>$\alpha_{12}$</th>
<th>$\lambda_1$/meter</th>
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<td>Right-circularly polarized</td>
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