On the nonlinear regime of electrically induced viscous jets with finite conductivity

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On the nonlinear regime of electrically induced viscous jets with finite conductivity

by

Saulo Orizaga

A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:
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Iowa State University
Ames, Iowa
2014

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I would like to dedicate my thesis work to my parents, brothers and sisters, and give particular credit to my wife, Monique, and my kids: Isaiah, Saulo and Ahava. Monique and my kids have been able to deal in a very caring way with the demanding work schedule that I have had for the past four years at Iowa State University. I deeply appreciate all their love and support. I also would like to thank my sisters who have always been a tremendous support during difficult times. I definitely enjoyed the holidays because I could spend more time with the best sisters in the world. We always have a good time and sometimes we even have adventures (I remember that crazy-nice road trip from Texas to California). To my brother (mi hermano) I would like to dedicate part of this work as well. He has always been a good supportive brother and always motivated me towards higher education. We have both been undertaking challenges in life. I am finishing my current project and I wish him the best on his current project. I expect to hear the good news for his project really soon. Te quiero un chingo carnal. I would like to thank my parents for all their efforts with me. They have given me wise and loving guidance during my early years and again in recent years as I pursue my current life goals. I have the most sincere appreciation and gratitude to them. (Como quisiera que el tiempo no pasara mis viejos, grandes son ustedes en mi vida y mi alma.)
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CHAPTER 1. GENERAL INTRODUCTION

We focus our investigation to electrically driven charged viscous jets with finite electrical conductivity in the presence of either a uniform or non-uniform externally applied electric field. Our efforts go towards the nonlinear local regime of the problem in which practical applications can benefit from our studies.

1.1 Introduction

The investigations of electrically driven jets are important due to the practical applications to electrospraying [12] and electrospinning [10,11]. Electrospinning is a process for manufacturing high volumes of very thin fibers that typically range from 100 nm to 1 micron, with lengths up to several hundred of meters depending on the application, from a vast variety of materials, including polymers, composites and ceramics [3,5,21]. In this process, nanofibers are produced by solidification of a polymer solution stretched by an electric field. The unique properties of nanofibers are extraordinarily high surface area per unit mass, very high porosity, tunable pore size and surface properties, layer thinness, high permeability, low basic weight, ability to retain electrostatic charges and cost effectiveness. These electrospun nanofibers have many practical applications to different areas including wound dressing, drug or gene delivery vehicles, high quality filters, biosensors, fuel cell membranes and electronics, tissue-engineering processes.

The governing electro-hydrodynamic equations for electrical driven jets [7,8] include conservation of mass, momentum, conservation of charge and electric potential. These partial differential equations (PDEs) are complicated due to the nonlinearities and even though they were published in a series of wonderful investigations by J.R Melcher and G.I Taylor [7,8] back
1969, no analytical results are currently available. The available investigations of this nonlinear problem are limited to cases of linear stability [10,20] or temporal instability alone of some modes with dyad resonance conditions [22].

The motivation for this investigation is to contribute in the understanding of electrically driven jets at the nonlinear stage, which at the application level this problem is still performed on a trial and error basis due to the the complexity of the mathematical formulation.

1.2 Dissertation Organization

This dissertation is organized in the format of a dissertation containing journal papers. In the general introduction, the research problem and related pertinent background information are presented. Additionally, a literature review is included.

This dissertation is organized as follows :

- Chapter 2: Contains the paper "Resonant Spatial Instability and Nonlinear Wave Interactions in Electrically Driven Jets", submitted to *Journal of Applied Mathematics and Informatics*. The paper includes the formulation of the jet flow system using the Weakly Nonlinear Wave Theory [15,25] and results about the dispersion relation that governs the jet flow system and the nonlinear wave interactions that exist due to disturbances that evolve in space. Results related to dyad (two-wave) interactions are presented and compared to those obtained from classical linear theory. Resonant Spatial instability provides an amplifying effect on the instabilities for most of the cases but this in turn provides a significant reduction in the jet radius.

- Chapter 3: Contains the paper "On a non-linear investigation of an electrospinning model under combined space and time evolving instabilities", submitted to *International Journal of Non-Linear Mechanics*. The paper includes results about the combined space and time evolving instabilities and the numerical solution to the dispersion relation for the jet flow system. The linear modes of instability for both time and space are studied in the nonlinear regime by the Weakly Nonlinear Wave Theory [15,25] with two waves satisfying
the dyad condition in the dispersion relation. The PDE system governing the amplitude of disturbances is solved and the perturbation quantities are reconstructed, presented and compared with cases in the absence of nonlinear wave interactions. This study provides a more realistic study since both instabilities for space and time were considered. Combined space and time resonant instability provides a very distinct evolution of the perturbation quantities that allows for the jet radius to reduce at earlier time and space when compared to the cases of linear theory and these modified results are of high interest in practical applications.

- Chapter 4: Contains the paper "Weakly Nonlinear Wave Interactions in Electrically Driven Jets via Resonant Triad Modes", submitted to *Journal of Engineering Mathematics*. The paper includes results of the study of electrically driven jets under combined spatial and temporal resonant instability of modes that satisfy the triad (three-wave) condition. Considering the jet flow system under the Weakly Nonlinear Wave Theory with triads [15,25], the results containing single and multiple triad modes for each fluid are presented. In this study different fluid phase velocities are considered for the perturbation quantities and results for the electric field and jet radius are presented. The study of electrically driven jets under the triad resonant instability provides new operating modes in which electric field in the jet increases while jet radius reduces at higher rates when compared to the dyad (two-wave) resonant instability studied in Chapter 3.

- Chapter 5: We conclude with a discussion and direction for future work.

1.3 Literature Review

Several investigations have been done in order to contribute in the understanding of electrically driven jets and their applications. Authors have broken down into different modeling cases in order to understand smaller pieces of the bigger complicated problem. This problem was studied in two cases, one under the influence of electrical effects and the other for absence of such effects. In the absence of electrical effects, it is known that, as far as physical application is concerned, spatially growing disturbances are more appropriate and realized than the
temporally growing counterpart for the free shear flows, which includes jet flows [14]. In particular, the earlier spatial instability studies [18] showed better agreement with the experimental results. Further investigations of spatial instability of free shear flows and jets were reported by a number of authors including those by Tam [15], Soderberg [2] and Healey [16]. Soderberg [2] showed, in particular, agreement between the results of the linear spatial instability and the experiments.

In the presence of electrical effects, linear temporal instability of the electrically forced jets has been studied theoretically by several authors [3,6,10,11]. Hohman et al. [10] studied linear temporal instability of an electrically forced jet with a uniform applied field. The equations for the dependent variables of the superposed disturbances were based on the long wavelength and asymptotic approximations of the governing electro-hydrodynamic equations. For the axisymmetric case, the authors detected, in particular, two temporal instability modes, and they discussed the properties of such instability modes in various possible limits. Other investigations of electrically driven jets with applications in electrospinning of nanofiber are reported in the papers by Li and Xia [5] and Yu et al. [21]. Riahi [20] studied linear spatial instability of the electrically forced jets with either a uniform or a variable applied field for ideal cases of zero or infinite conductivity. He derived the relevant equations for the perturbations under approximations of the types used in Hohman et al. [10], but he also determined the condition under which the basic state surface charge can be non-zero. For neutral temporal stability boundary, he detected two new spatial modes of instability under certain conditions. He found, in particular, that both of these modes can be enhanced with increasing the strength of the field. Orizaga and Riahi [24] extended the idealized work due to Riahi [20] to the realistic cases of water and glycerol jets. They detected two modes of instability one of which was enhanced with increasing the strength of the electric field, while the growth rate of the other modes decreases with increasing the strength of the field. However, the growth rates of both mode increased mostly with decreasing the axial wavelength of the disturbance.

In the early 1970s, D.A Saville conducted investigations to contribute on the understanding of electrically driven jets. His investigation in [9] considered fluid and modeling properties that included jets with infinity conductivity and zero viscosity. His work contributed in the
understanding at the linear stage for temporal instability of theoretical and idealistic electrically
driven jets [9]. More recently, on a series of investigations [10,11] the work of D.A Saville was
extended to consider more realistic cases of the jet flow in which the theoretical frame work
could be confirmed at the experimental level. The work on [9,10,11] concerns with the linear
temporal instability of electrically driven jets.

One way to broaden the understanding of electrically driven jets is to extend to its nonlinear
regime which is often not studied due to the limitations in linear theory and the complexity of
the nonlinearities in the modeling problem. In theoretical and computational fluid mechanics
several authors have considered the nonlinearities in a fluid flow problem by considering non-
linear wave interactions [15,25]. One way to considering nonlinear wave interaction in a flow
system is by resonant wave interactions [15].

Relevant investigations related to resonant wave dyads and triads in various flow systems
have been investigated by a number of authors in the past [15,25]. Criak [18,25] showed the
existence of resonant instability mode triads in flow over a flat plate. El-Hady [13] presented
computational results for some resonant instability modes in a three-dimensional incompressible
boundary layer flow. Vonderwell and Riahi [23] studied theoretically and computationally
instability modes that satisfy the triad resonance conditions in three-dimensional compressible
(subsonic) boundary layer flow over a swept wing. Detuning parameters were used for the
wave numbers and frequencies for the resonance conditions in both space and time. Different
resonant instability and wave interaction due to cross flow and Tollmien-Schlichting modes [4]
were studied for both the incompressible and compressible cases. The authors determined the
effects of compressibility on flow features, instability modes, and wave interactions.

In regards to the resonant wave interactions, no investigation had been carried out for
the electrically driven jets prior to the work of Orizaga and Riahi [22]. They conducted the
investigation of temporal linear instabilities and their nonlinear wave interactions. They found
that the instabilities of resonant type evolving in time were able to produce both favorable and
unfavorable results. The unfavorable condition was in the sense that the instabilities detected
on the linear case were actually modified by stronger types of instabilities for most cases but at
the same time this in turn provided a significant reduction on the jet radius, which is of high
interest for practical applications.

This dissertation is aimed to extend the work done in [9,10,11,20,22,24] and to contribute in the understanding of electrically driven jets at the nonlinear stage for the realistic cases of spatial and temporal resonant instabilities. The studies are completed on the following order, nonlinear resonant spatial instability, combined spatial and temporal dyad resonant instability and combined spatial and temporal triad resonant instability of electrically driven jets with finite electrical conductivity in the presence of a uniform or non-uniform applied electric field.
Bibliography


CHAPTER 2. RESONANT SPATIAL INSTABILITY AND NONLINEAR WAVE INTERACTIONS IN ELECTRICALLY DRIVEN JETS

A paper submitted to Journal of Applied Mathematics and Informatics
Saulo Orizaga, Daniel N. Riahi and L. Steven Hou

Abstract

Spatial instability and the nonlinear wave interactions of the modes that satisfy the dyad resonance conditions are investigated for both cases of constant and variable externally imposed electric fields. We apply mathematical model based on the governing electro-hydrodynamic equations, which is developed and used for the spatially growing modes with resonance and their nonlinear wave interactions in electrically driven jet flows, leads to equations for the unknown amplitudes of such waves. These equations are solved for both water-glycerol mixture and glycerol jet cases, and the expressions for the dependent variables of the corresponding modes are determined. The energy exchanged during the interaction is very strong and for most cases the domain of the dependent variables was significantly reduced. The amplified instability was also found to provide a significant reduction in the jet radius, which is a favorable result for practical applications.

2.1 Introduction

In this paper we consider spatial instability modes that satisfy dyad resonance conditions and the associated nonlinear interactions in a cylindrical jet of viscous fluid with finite electrical conductivity and a static charge density and in the presence of an external constant or variable electric field. The investigations of electrically forced jets are important particularly
in applications such as those to electrospraying [11] and electrospinning [9,10]. For a more
detailed introduction we will refer the reader to [21].

The importance of resonant wave interaction relies on the behavior that a system can
undergo with regards of large amplitude oscillations at certain frequencies. The frequencies
that generate larger oscillation amplitudes compared to all other frequencies are called resonant
frequencies. These frequencies are very critical because a system behavior could be altered
even by small periodic driving forces. In general, the system could be stabilized or destabilized
by such driving forces, which for our the problem investigation these take the form of the
nonlinearities in the original system.

For linear spatial instability, we implemented a search and detected several modes that
satisfy the dyad resonance conditions and use them in the system of nonlinear equations to
determine the spatial dependent amplitude functions. We found interesting results about the
resulting instability due to the nonlinear interactions of such modes. In particular, the instabil-
ity that is generated by the nonlinear wave interactions of such modes is mostly of amplifying
effect. The energy exchanged during the interaction is very strong and for most cases the
domain of the dependent variables was significantly reduced. The amplified instability was
also found to provide a significant reduction in the jet radius, which is a favorable result for
practical applications.

2.2 Mathematical Formulation and Analysis

Our mathematical modeling of the electrically driven jets is based on the governing elec-
trohydrodynamic equations [8] for the mass conservation, momentum, charge conservation and
for the electric potential, which are described in [9,10,19,21]. The work of resonant spatial
instability that we consider in this paper is a research continuation of the work done in [21].
In contrast to the time evolving instability that was studied in [21], here we study particular
realistic features of spatial instabilities of the modes that enhance significantly in space as ob-
served in experiments [10]. We find that such spatial instabilities subjected to dyad resonance
conditions can be significantly stronger that corresponding temporal instability of such modes
and occur at a very short distance after the jet is emitted \[7,8\].

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \quad (2.1.1)
\]

\[
\rho \frac{D\vec{u}}{Dt} = -\nabla P + \nabla \cdot (\mu \vec{u}) + q \vec{E} \quad (2.1.2)
\]

\[
\frac{Dq}{Dt} + \nabla \cdot (K \vec{E}) = 0 \quad (2.1.3)
\]

\[
\vec{E} = -\nabla \Phi \quad (2.1.4)
\]

where \(\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla\) is the total derivative, \(\vec{u}\) is the velocity vector, \(P\) is the pressure, \(\vec{E}\) is the electric field vector, \(\Phi\) is the electric potential, \(q\) is the free charge density, \(\rho\) is the fluid density, \(\mu\) is dynamic viscosity, \(K\) is electric conductivity and \(t\) is the time variable.

The internal pressure in the jet was found by taking into account the balances across the free boundary of the jet between the pressure, viscous forces, capillary forces and the electric energy density plus the radial self-repulsion of the free charges on the free boundary \[8\], and it leads to the following expression for the pressure \(P\) in the jet

\[
P = \gamma \kappa - \left[\left(\frac{\epsilon - \tilde{\epsilon}}{8\pi}\right)E^2 - \frac{4\pi}{\tilde{\epsilon}}\sigma^2\right] \quad (2.2)
\]

where \(\gamma\) is the surface tension, \(\kappa\) is twice the mean curvature of the interface, \(E\) is the magnitude of the electric field, \(\epsilon/(4\pi)\) is the permittivity constant in the jet, \(\tilde{\epsilon}/(4\pi)\) is the permittivity constant in the air and \(\sigma\) is the surface free charge.

Following the previous investigation \[9\], we consider a cylindrical fluid jet moving axially. The fluid of air is considered as the external fluid, and the internal fluid of jet is assumed to be Newtonian and incompressible. We use the governing Equations (2.1)-(2.2) in the cylindrical coordinate system with the origin at the center of nozzle exit section, where the jet flow is emitted with axial z-axis along the axis of the jet. We consider the axisymmetric form of the dependent variables in the sense that the azimuthal velocity is zero and there are no variations of the dependent variables with respect to the azimuthal variable.

Following the approximations carried out in \[9\] for a long and slender jet in the axial direction, we consider the length scale in the axial direction to be large in comparison to that in the radial direction and use a perturbation expansion in the small jet aspect ratio. We expand the dependent variables in a Taylor series in the radial variable \(r\). Then such expansions are
used in the full axisymmetric system and keep only the leading terms. These lead to relatively simple equations for the dependent variables as functions of \( t \) and \( z \) only. Following the method of approach in [9], we employ (1d) and Coulombs integral equation to arrive at an equation for the electric field, which is essentially the same as the one derived in [9] and will not be repeated here. We non-dimensionalize these equations using \( r_0 \) (radius of the cross-sectional area of the nozzle exit at \( z = 0 \)), \( E_0 = \gamma/[(\tilde{\epsilon})r_0]^{1/2}, t_0 = (\rho r_0/\gamma)^{1/2}, (r_0/t_0) \) and \((\gamma\tilde{\epsilon}/r_0)^{1/2}\) as scales for length, electric field, time, velocity and surface charge, respectively. The resulting non-dimensional equations have the following form [19]

\[
\frac{\partial}{\partial t}(h^2) + \frac{\partial}{\partial z}(h^2 v) = 0 \tag{2.3.1}
\]

\[
\frac{\partial}{\partial t}(h\sigma) + \frac{\partial}{\partial z}(hv\sigma) + \frac{1}{2} \frac{\partial}{\partial z}(h^2 E K^* \tilde{K}(z)) = 0 \tag{2.3.2}
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = - \frac{\partial}{\partial z} \left[ h \left[ 1 + \left( \frac{\partial h}{\partial z} \right)^2 \right]^{-\frac{3}{2}} - \frac{\partial^2 h}{\partial z^2} \left[ 1 + \left( \frac{\partial h}{\partial z} \right)^2 \right]^{-\frac{3}{2}} - \frac{E^2}{8\pi} - 4\pi\sigma^2 \right] + \frac{2E\sigma}{h\sqrt{\beta}} + \frac{3v^*}{h^2} \frac{\partial^2 \sigma}{\partial z^2} \tag{2.3.3}
\]

\[
E_b(z) = E - \ln(\chi) \left[ \frac{\beta}{2} \frac{\partial^2}{\partial z^2}(h^2 E) - 4\pi\sqrt{\beta} \frac{\partial}{\partial z}(h\sigma) \right] \tag{2.3.4}
\]

where \( v(z,t) \) is the axial velocity, \( h(z,t) \) is the radius of the jet cross-section at the axial location \( z \), \( \sigma(z,t) \) is the surface charge, the conductivity \( K \) is assumed to be a function of \( z \) in the form \( K = K_0\tilde{K}(z) \), where \( K_0 \) is a constant of dimensional conductivity and \( \tilde{K}(z) \) is a non-dimensional variable function, \( K^* = K_0 \left\{ \rho r_0^3/[(\gamma\tilde{\epsilon})(\tilde{\epsilon})^2] \right\}^{0.5} \) is the non-dimensional conductivity parameter, \( \beta = \epsilon/\tilde{\epsilon} - 1 \), \( v^* = [\nu^2 r_0^2/(\gamma\tilde{\epsilon})]^{0.5} \) is the non-dimensional viscosity parameter, \( E(z,t) \) is the electric field, \( E_b(z) \) is an applied electric field and \( 1/\chi \) is the local aspect ratio, which is assumed to be small.

Next, we determine the electrostatic equilibrium solution, which is referred to here as the basic state solution, to the Eqs. (2.3.1)-(2.3.4). The basic state solutions for the dependent variables, which are designated with a subscript b, are those derived in [19] and are given below.
where both $\Omega$ and $\sigma_0$ are constant quantities. Here $\sigma_0$ is the background free charge density. We set $\delta = 8\sigma_0\pi/(\Omega\sqrt{\beta})$ to be a small parameter ($\delta << 1$) and consider a series expansion in powers of $\delta$ for all the dependent variables for the case of variable applied field.

In this paper we investigate the cases where applied electric field can be either uniform ($\delta = 0$) or non-uniform ($\delta \neq 0$). This is related the electric field that is generated between the high voltage, which is applied at nozzle, and the distance from the nozzle to the grounded collector plate. This allows for perfect or imperfect alignment on the collector plate with respect to the nozzle orifice. Here we begin with our perturbation analysis of the system. We consider each dependent variable as sum of its basic state solution plus a small perturbation. Thus, we write

$$h, v, \sigma, E = (h_b, v_b, \sigma_b, E_b) + (\bar{h}, \bar{v}, \bar{\sigma}, \bar{E})$$

where $h, v, \sigma$ and $E$ are the dependent variables for the perturbation quantities. We then use Eqs. (2.4)-(2.5) in (2.3). Since our present nonlinear investigation is to take into account only nonlinear interactions due to resonant dyads, we keep only quadratic nonlinearities and products of up to two dependent variables in the resulting equations, which can be written in the vector form

$$Lq = N$$

where $q = (h, v, \sigma, E)^T$ is the perturbation vector, and the linear operator $L$ and the nonlinear operator $N$, which are lengthy and will not be given in this paper. We now assume that the perturbation quantities are small and have the following form

$$(\bar{h}, \bar{v}, \bar{\sigma}, \bar{E}) = \epsilon(h_1, v_1, \sigma_1, E_1) + \epsilon^2(h_2, v_2, \sigma_2, E_2)$$

(2.7)
where the small parameter \( \epsilon (\epsilon << 1) \) characterizes the magnitude of the disturbance quantity that causes perturbation. This parameter will play a very important role for the proper arrangement of the equations as well as the detection of the linear spatial instability modes that satisfy the resonance conditions.

### 2.2.1 Linear Problem

For the linear case we keep only leading order terms in Eqn. (2.7). We consider the following form for the perturbation quantities

\[
(h_1, v_1, \sigma_1, E_1) = (h', v', \sigma', E') \exp[i\omega t + (i + ks)z] 
\]  

(2.8)

which is constructed from plane waves that oscillate as well as grow or decay in the spatial direction. Using Eqn. (2.8) in Eqn. (2.6), we linearize with respect to the amplitude of perturbation and following [21] we arrive at the dispersion relation, which has the following form

\[
\frac{1}{4\pi \sqrt{\beta}} (4\pi (\sqrt{\beta} \omega) \left( -i(k - is)^2 \right) \left( 1 + (k - is)^2 - 8\pi \sigma_0^2 \right) + 6(k - is)^2 v^* \omega + 2i\omega^2) - \\
8iK^* \pi(k - is)^3 \sqrt{\beta} \sigma_0 \Omega - 2K^* (k - is)^2 (\Omega^2) + (k - is)^2 \sqrt{\beta} (4k \pi s \beta \omega (-1 + 2s^2) + 8\pi \sigma_0^2 - 6iv^* \omega) - 8k^2 K^* \pi^2 \sqrt{\beta} (1 - 6s^2) + 8\pi \sigma_0^2 + 6iv^* \omega + 16ikK^* \pi^2 s \sqrt{\beta} (1 - 2s^2 + 8\pi \sigma_0^2 + 6iv^* \omega) - 2k^4 (4K^* \pi^2 + 4i^{\pi} \omega^2) - \\
i \pi \beta \omega + 8iK^* \pi^2 s \sqrt{\beta} + i \pi s \beta \omega + 8K^* \pi^2 \sqrt{\beta} (-s^4 + 2\omega^2 + s^2 (1 + 8\pi \sigma_0^2 + 6iv^* \omega)) - 2i \pi (32\pi \sigma_0 + \beta s^4 - 2\omega^2 + s^2 (-1 + 8\pi \sigma_0^2 - 6iv^* \omega)) + 64K^* \pi^2 \\
(ik + s) \sigma_0 \Omega - (k - is)^2 \sqrt{\beta} \left( 4K^* \pi + i \sqrt{\beta} \omega \right) \Omega^2 \log [\lambda] = 0
\]  

(2.9)

We solved (2.9) for \( s \) and \( \omega \) numerically using Newton method [21] for given values of the parameters and we used initial guess from suggested values provided by different solution branches obtained in [19,9]. For either water-glycerol mixture jet or glycerol jet, we were able to search and detected several modes that satisfy dyad resonance conditions. We then used such modes to investigate their nonlinear wave interactions.
2.2.2 Nonlinear Problem

Here we investigate the effects of the nonlinear interactions of the modes that can satisfy dyad resonance conditions on the nonlinear spatial instabilities of the jet. Introducing a slowly varying space variable \( z_s = \epsilon z \), we write the solution to the linear version of equation (2.6)

\[
L q_1 = 0 \quad (2.10.1)
\]

\[
q_1 \equiv (h_1, v_1, \sigma_1, E_1)^\top \quad (2.10.2)
\]

in the following form

\[
q_1 = \sum_{n=1}^{2} A_n(z_s) q_{1n} \exp[i(k_n z + \omega_n t) + s_n z] + c.c \quad (2.10.3)
\]

\[
q_{1n} \equiv (h_{1n}, v_{1n}, \sigma_{1n}, E_{1n})^\top \quad (2.10.4)
\]

where \( q_{1n} \) is a vector with constant elements and c.c. denotes the complex conjugate of the preceding expression. We included in (2.10.3) terms due to two modes labeled as the mode 1 and the mode 2 with the corresponding amplitude functions \( A_n(z_s)(n=1,2) \), wave numbers \( k_n(n=1,2) \), frequencies \( \omega_n(n=1,2) \) and small growth rates \( s_n \) (\( s_n = \epsilon \tilde{s}_n \) with \( \tilde{s}_n \) of order one value or less) that satisfy the dyad resonance conditions in the sense that \( k_2 = 2k_1 \) and \( \omega_2 \approx 2\omega_1 \). We refer to the resonance as a perfect resonance [1,14] if \( \omega_2 = 2\omega_1 \). For this investigation we have the wave number \( k \), coming from a non-discrete set of values under the modeling assumptions hence, we consider the more general case of near resonance [1] when

\[
(k_2, \omega_2) = 2(k_1, \omega_1) + \epsilon(0, \mu) \quad (2.10.5)
\]

where \( \mu \) is an order one quantity and the so-called detuning parameter [12] \( \epsilon \mu \) represents a small deviation from perfect resonance. Using (2.10,3) in (2.10.1), we find

\[
L_n q_{1n} = 0 \quad (n = 1, 2) \quad (2.11)
\]

where \( L_n \) has the same form as \( L \), provided \( (\partial/\partial z) \) and \( (\partial/\partial t) \) are replaced by \( (i k_n + S_n) \) and \( i \omega_n \), respectively. Using Eqn. (2,7) and (2.10) in Eqn. (2.6), we find that in the order \( \epsilon^2 \) the
following nonlinear equation becomes

\[ Lq_2 = N_1 \]  \hspace{1cm} (2.12.1)
\[ q_2 \equiv (h_2, v_2, \sigma_2, E_2)^\top \]  \hspace{1cm} (2.12.1)

where the expression for the nonlinear operator \( N_1 \), which is lengthy and will not be given. We impose solution to (2.12) in the following form

\[ q_2 = \sum_{n=1}^{2} q_{2n} \exp[i(k_n z + \omega_n t) + s_n z] + c.c \]  \hspace{1cm} (2.13.1)
\[ q_{2n} \equiv (h_{2n}, v_{2n}, \sigma_{2n}, E_{2n})^\top \]  \hspace{1cm} (2.13.2)

where the vector \( q_{2n} \) is a function of \( z_s \). Using Eqn. (2.13.1) in (2.12.1), we find

\[ L_n q_{2n} = N_{1n} \quad (n = 1, 2) \]  \hspace{1cm} (2.14)

where the expressions for the vectors \( N_{1n} (n = 1, 2) \) are lengthy and will not be given. To determine the nonlinear differential equations for the amplitude functions \( A_n(z_s) \), we need to apply the so-called solvability conditions or also known as the Fredholm alternative [14,22] for the Eq. (2.14). Here we briefly provide the general idea of the solvability condition.

**Theorem 2.2.1**  Let \( L_n \) be a linear operator on the domain \( DL_n \) dense in Hilbert space \( \mathcal{H} \) and consider the problem from Equation (2.14) and its corresponding homogenous adjoint problem

\[ L_n q_{2n} = N_{1n} \quad (n = 1, 2) \]  \hspace{1cm} (2.14.1)
\[ L_n^{(a)} q_n^{(a)} = 0 \]  \hspace{1cm} (2.14.2)
\[ q_n^{(a)} = (h_n^{(a)}, v_n^{(a)}, \sigma_n^{(a)}, E_n^{(a)})^\top \]  \hspace{1cm} (2.14.3)

Then the orthogonal complement of the range of \( L_n \) is the null space of the adjoint.

\[ \text{Range}[L_n]^\perp = \text{Nullspace}[L_n^{(a)}] \]  \hspace{1cm} (2.14.4)

The above provides a sufficient condition for the solvability to our problem. Here we briefly address components on Theorem 2.2.1, \( L_n \) after the perturbation approach takes the form of
a matrix with entries that satisfied the resonant condition, therefore these entries are real and imaginary. We will use the usual inner product for vectors with four entries that can contain complex elements, which will induce the norm so that our underlying space becomes a Hilbert Space. In our problem the adjoint will take place as the conjugate transpose of $L_n$. Making use of Theorem 2.2.1, we now proceed with our problem. For the proof of Theorem 2.2.1 and more details, we will direct the reader to [13].

Since forming the solvability condition for a nonlinear system requires knowing the solution to the so-called the related linear adjoint system [14], we first need to determine the linear adjoint system of (2.11) and then find its solution. The adjoint solution $q_{1n}^{(a)}$ is defined by the property [13]

$$ (L_n q_{1n}, q_{1n}^{(a)}) = (q_{1n}, L_n^{(a)} q_{1n}^{(a)}), \quad (n = 1, 2) \tag{2.15.1} $$

$$ q_{1n}^{(a)} = (h_{1n}^{(a)}, v_{1n}^{(a)}, \sigma_{1n}^{(a)}, E_{1n}^{(a)})^\top \tag{2.15.2} $$

Here we denote the usual inner product $(x, y) = x^\top y^*$, where $^*$ is used for the complex conjugate. Eqs. (2.15.1)-(2.15.2) follow since (2.11) is the related linear problem of (2.14). Here $L_n^{(a)}$ is the linear adjoint operator, and $q_{1n}^{(a)}$ is the solution vector to the homogenous adjoint problem, which represents the null space of the adjoint operator $L_n^{(a)}$. Taking inner product of (2.11) with $q_{1n}^{(a)}$ and using (2.15), we look for non-trivial solution of the adjoint system

$$ L_n^{(a)} q_{1n}^{(a)} = 0, \quad (n = 1, 2) \tag{2.16} $$

where $L_n^{(a)}$ is a 4 by 4 matrix differential operator, which is lengthy and is not given in this paper. Taking inner product of (2.14) with the adjoint solution $q_{1n}^{(a)}$ found from Eq. (16) and making use of the property (2.15.1), we arrive at the solvability conditions for Eq. (2.14) that are in agreement with (2.14.4) for both $n = 1, 2$.

$$ (N_{1n}^{(a)} q_{1n}^{(a)}) = 0, \quad (n = 1, 2) \tag{2.17} $$

Each of this $n$ values will produce a lengthy second order ordinary differential equation. The system of nonlinear differential equations contains complex conjugate of the amplitude
functions, complex coefficients and exponential components that have the corresponding spatial growth rates of the resonant modes we found earlier. The two differential equations that govern the slowly varying amplitudes functions $A_1(z_s)$ and $A_2(z_s)$, which are modulated by the nonlinear wave interactions, have the following form

$$c_1 \frac{d^2 A_1(z_s)}{dz_s^2} + c_2 \frac{dA_1(z_s)}{dz_s} + c_3 A_1(z_s) A_2(z_s)^* \exp[i \epsilon \mu t + s_2 z] = 0 \quad (2.18.1)$$

$$d_1 \frac{d^2 A_2(z_s)}{dz_s^2} + d_2 \frac{dA_2(z_s)}{dz_s} + d_3 A_2(z_s) + d_4 A_1(z_s)^2 \exp[(2s_1 - s_2)z - i \epsilon \mu t] = 0 \quad (2.18.2)$$

where * indicates complex conjugate and the expressions for the constant complex coefficients $c_n (n = 1, 2, 3)$, and $d_m (m = 1, 2, 3, 4)$ are given in the Appendix A. The solutions to (2.18) for $A_1$ and $A_2$ versus spatial variable are found numerically using a Runge-Kutta fourth-order scheme for given values of the parameters and prescribed initial conditions on the amplitudes functions. The approach was to allow the solutions to evolve along the spatial direction and this was done by providing the values of the function and the derivative of the function at spatial location zero. We then used these slowly varying space dependent amplitudes and use them on the imposed solution for the nonlinear problem from equation (2.6).

### 2.3 Results and Discussion

In the present study we provide spatial instability results that are based on the already obtained linear and nonlinear solutions in the previous section which include (2.9), (2.10.3-2.10.4) and (2.18). We consider two types of fluids for the jet, which can be representative as those that can be used in the experimental investigation for the problem such as water-glycerol mixture and glycerol. For such fluids, we set representative values of the parameters to be $K^* = 19.60$, $\nu^*$ (glycerol) = 9.05384, $\nu^*$ (water-glycerol) = 0.60764, and $\beta = 77$. The main typical results for some of the detected modes that satisfied the dyad resonance conditions are briefly presented respectively for water-glycerol mixture and glycerol jets in the following two sub-sections.
2.3.1 The Case of Water-glycerol Mixture Jet

Figs. 2.1 and 2.2 represent the growth rate $s$ versus the axial wave number $k$ for the non-uniform applied field ($\sigma_0 = 0.1$) and the uniform applied field ($\sigma_0 = 0$), respectively, and for different values of $\Omega$. It can be seen from the Fig. 1 that the growth rate $s$ are divided into two main branches and in the case where $\Omega = 4$ there is even a third branch for the growth rates $s$, in particular, when we consider the uniform applied field. For the branch containing the lower range wave numbers we can detect that the instabilities grow in moderate magnitude as the wave number is increased, but we can see significant instability growth by incrementing the magnitude of the $\Omega$. We can also observe that the range of values for the wave number increases for larger values of $\Omega$. From the higher wave number values on Fig. 2.1, which is the other solution branch, we detect the similar effect as for the low wave numbers in the sense that $s$ increases as $k$ increases, but now at a higher rate. This solution branch for the spatial growth rate becomes more stable for higher values of $\Omega$ and at the same time the wave number range where these modes operate gets reduced. In Fig. 2.2, we have the same representation as Fig. 2.1, but now for ($\sigma_0 = 0$). The results obtained from this figure indicate that uniform applied field provides stabilization since all modes of instability were reduced compared to non-uniform variable applied field ($\sigma_0 = 0.1$). This result is in agreement with [23].

For $\Omega = 3$, we found the modes of instability that satisfy the dyad resonance conditions for non-uniform applied field where $\sigma_0 = 0.1$. Figs. 2.3 and 2.4 present results for the dependent variables of the perturbation versus $z$ for $t = 1, k_1 = 0.1, k_2 = 0.2, \omega_1 = 0.09488, \omega_2 = 0.19131, s_1 = 1.68026$ and $s_2 = 1.68949$. The initial conditions chosen for the amplitude functions were $A_{10} = 0.1 + 0.1i$ and $A_{20} = -0.1 + 0.1i$. It can be seen from the results presented in the Fig. 2.3 that the nonlinear instability is enhanced significantly with axial direction for $z > 2.5$. Such instability is due to the nonlinear resonant mode interactions between to oscillatory modes that form a dyad. The results presented in the Fig. 2.4 indicate that the instability is much weaker when nonlinear resonant mode interactions were not included. We detect a reduction on the domain of the perturbations of about a factor of two due to the energy exchanged during wave interactions. The nonlinear effects modify the linear spatial instability
perturbation variables in an amplifying way except for the velocity perturbation plot which is also modified in direction of acceleration. The overall description aside from the mentioned points above for Fig. 4 was a weaker type of instability compared to Fig 2.3.

2.3.2 The Case of Glycerol Mixture Jet

Figs. 2.5 and 2.6 present the growth rate $s$ versus the axial wave number for the non-uniform applied field, where $\sigma_0 = 0.1$, and the uniform applied field, where $\sigma_0 = 0$, respectively, and for different values of $\Omega$. It can be seen from these figures that growth rate $s$ undergoes a similar branching process, but for these cases under this higher viscosity from glycerol these two modes seem to carry similar properties. The properties include in either case of 1 mode or 2 modes depending on $\Omega$, the modes increase as $k$ increases and all these modes are enhanced as the magnitude of the electric field is intensified. Here $\Omega$ is strictly destabilizing for the spatial growth rates $s$. The uniform applied field provides a slightly advantage in the sense of stability compared to the non-uniform applied field. For the case for $\sigma_0 = 0.1$, the simulations detected stronger type of instabilities.

For $\Omega = 1$, we found modes that satisfy the dyad resonance conditions for non-uniform applied field where $\sigma_0 = 0.1$. Figs. 2.7 and 2.8 present results for the dependent variables of the perturbation versus $t$ in the presence of the nonlinear mode interactions (Figure 2.7) and in the absence of such interactions (Figure 2.8) and for $t = 1, k_1 = 0.07, k_2 = 0.14, \omega_1 = 0.00319, \omega_2 = 0.00639, s_1 = 1.21478$ and $s_2 = 1.21859$. The initial conditions for the amplitude functions were $A_{10} = A_{20} = 0.1 + 0.1i$. It can be seen from the results presented in these figures that the amplitudes of the perturbation quantities are modified from having a larger domain $z = 10$, and then being reduced to about $z = 1.5$. Here we can see that the wave interaction was such that in the presence of nonlinear effects the perturbation plots exhibit very large steep solutions, which provides a significant jet radius reduction and jet acceleration.
2.4 Conclusion

We carried out mathematical modeling and numerical investigation of linear spatial instability as well as the nonlinear interactions of those detected modes that satisfy the dyad resonance conditions in electro-hydrodynamic system for electrically forced slender water-glycerol and glycerol jet flows with externally imposed either uniform or non-uniform applied field. Linear instability results indicate that non-uniform applied field and water-glycerol jet exhibited larger instabilities compared to the uniform applied field and glycerol jet cases, respectively. We were able to uncover different types of modes that can satisfy the dyad resonance conditions, and we calculated the effects of the nonlinear interactions due to these modes on the dependent variables of the jet flows. For either water-glycerol mixture jet or glycerol jet, we found, in particular, that the main effects of the nonlinear interactions of these modes are mostly to increase the amplitudes of the dependent variables of the perturbation superimposed on the jet flow but also helped to decrease the jet radius. These results, which indicate both undesirable effects of such modes for enhancing the jet instability as well as the beneficial effects of these modes for reducing the jet radius, can be of interest in applications of electrospinning to determine necessary means to control instabilities in jets in order to produce higher quality fibers. The significant reduction of jet radius is important in such applications for production of high quality fibers of nano-scale size in radius.

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Appendix A

The expressions for the coefficients $c_n (n = 1, 2, 3)$ and $d_m (m = 1, 2, 3, 4)$ are given below

\[
c_1 = \left( \frac{1}{2} \beta (E_{11} + 2h_{11} \Omega) \ln(\mathcal{X}) \right) E_{11}^{(a)} \tag{A.1}
\]

\[
c_2 = -v_{11} h_{11}^{(a)} + \left( -\frac{K^*}{2} E_{11} - v_{11} \sigma_0 - h_{11} K^* \Omega \right) v_{11}^{(a)} + (3(i k_1 + s_1)^2 - 1)h_{11} + 8\pi \sigma_0 \sigma_{11} + 6\nu^\ast (i k_1 + s_1) v_{11} + \left( \frac{\Omega}{4\pi} E_{11} \right) \sigma_{11}^{(a)} - \left( 4\pi \sqrt{\beta} (\sigma_0 h_{11} + \sigma_{11}) \ln(\mathcal{X}) \right) E_{11}^{(a)} \tag{A.2}
\]

\[
c_3 = \left( (i k_1 + s_1 + s_2) v_{12} + 6\nu^\ast (i k_1 + s_1) \right) h_{11}^{(a)} + \left( h_{12} K^*(i k_1 + s_1 + s_2) E_{11}^{(a)} + (i k_1 + s_1 + s_2) \left( E_{12} K + v_{12} \sigma_0 + h_{12} K^* \Omega \right) + i \sigma_{12} (\epsilon \mu + \omega_1) \right) \sigma_{11}^{(a)} v_{11}^{(a)} + \left( \frac{1}{4\pi \sqrt{\beta}} \right) \left( \left( E_{12} (i k_1 + s_1 + s_2) \right) \sqrt{\beta} + 8\pi (\sigma_{12} - \sigma_0 h_{12}) E_{11}^{(a)} 4\pi (k_1 + is_1) (2k_1 + is_2) (h_{12} (i k_1 + s_1 + s_2) + 6\nu^\ast v_{12}) \sqrt{\beta} - 2E_{12} \sigma_0 + 4h_{12} \sigma_0 \Omega - 2\sigma_{12} \Omega) + (i k_1 + s_1 + s_2) v_{12} \sqrt{\beta} v_{11}^{(a)} + 2(E_{12} + 4\pi (i k_1 + s_1 + s_2) v_{12} + \sigma_0) \sqrt{\beta} v_{11}^{(a)} = \sqrt{\beta} E_{11}^{(a)} + (4i \pi \sigma_{12} + (k_1 + i (s_1 + 2)) \sqrt{\beta} (E_{12} + h_{12} \Omega)) h_{11}^{(a)} + 4i \pi h_{12} \sigma_{11}^{(a)} E_{11}^{(a)} \tag{A.3}
\]

\[
d_1 = \left( \frac{1}{2} \beta (E_{12} + 2h_{12} \Omega) \ln(\mathcal{X}) \right) E_{12}^{(a)} \tag{A.4}
\]

\[
d_2 = -v_{12} h_{12}^{(a)} - \left( \frac{K^*}{2} E_{12} + \sigma_0 v_{12} + h_{12} K^* \Omega \right) v_{12}^{(a)} + \frac{h_{12}}{4\pi} (4\pi - (1 + 3(2k_1 is_1)^2) + 6(8\pi \sigma_0 \sigma_{12} + \nu^\ast (2i k_1 + s_2) v_{12}) + \Omega E_{12} \sigma_{12}^{(a)} - (4\pi \sqrt{\beta} (\sigma_0 h_{12} + \sigma_{12}) \ln(\mathcal{X})) E_{12}^{(a)} \tag{A.5}
\]

\[
d_3 = -2i \mu h_{12} h_{12}^{(a)} - (i \mu \sigma_{12} + i \mu \sigma_0 h_{12}) v_{12}^{(a)} + i \mu v_{12} \sigma_{12}^{(a)} \tag{A.6}
\]

\[
d_4 = -2i [h_{11} (2k_1 v_{11} - 2i s_1 v_{11} + h_{11} \omega_1)] h_{11}^{(a)} + -[(i k_1 + s_1) (K^* h_{11} E_{11} + 2v_{11} (\sigma_0 h_{11} + \sigma_{11} + 2i h_{11} \sigma_{11} \omega_1))] v_{12}^{(a)} + \frac{1}{4\pi \sqrt{\beta}} \left( (E_{12} (i k_1 + s_1) \sqrt{\beta} + 8\pi (\sigma_{12} - \sigma_0 h_{11})) E_{12}^{(a)} + 4\pi (i k_1 + s_1) \sqrt{\beta} (-h_{11} (k_1 - is_1)^2 + 6h_{11} \nu^\ast (i k_1 + s_1) v_{11} - v_{11}^2 + 8 \sigma_{11}^2) + \pi h_{11} \Omega (\sigma_0 h_{11} - \sigma_{11})) \sigma_{12}^{(a)} + [2h_{11} \sqrt{\beta} (i k_1 + s_1) \ln(\mathcal{X}) (-4\pi \sigma_{12} + (i k_1 + s_1) \sqrt{\beta} (2E_{11} + h_{11} \Omega))] E_{12}^{(a)} \tag{A.7}
\]
Fig 2.1. The growth rate $s$ versus the axial wave number $k$ and for water-glycerol mixture jet with $\Omega = 1$ (thin solid line), 2 (dotted line), 3 (dashed line) and 4 (thick solid line) and for variable applied field ($\sigma_0 = 0.1$).

Fig 2.2. The same as in the figure 1 but for constant applied field ($\sigma_0 = 0$).

Fig 2.3. Perturbation quantities $h_1$ (thin solid line), $v_1$ (dotted line), $\sigma_1$ (dashed line) and $E_1$ (thick solid line) versus the time variable $t$ for water-glycerol jet and for the two modes 1 and 2 that satisfy the dyad resonant conditions. Here $\sigma_0 = 0.1, t = 1, k_1 = 0.1, k_2 = 0.2, \omega_1 = 0.09488, \omega_2 = 0.19131, s_1 = 1.68026$ and $s_2 = 1.68949$ and $\Omega = 3$ and nonlinear mode interactions are fully taken into account.

Fig 2.4. The same as in the figure 3 but in the absence of nonlinear interactions.
Fig 2.5. The same as in the figure 1 but for glycerol jet.

Fig 2.6. The same as in the figure 2 but for glycerol jet.

Fig 2.7. The same as in the figure 3 but for $\sigma_0 = 0.1, \Omega = 1, t = 1, k_1 = 0.07, k_2 = 0.14, \omega_1 = 0.00319, \omega_2 = 0.00639, s_1 = 1.21478$ and $s_2 = 1.21859$.

Fig 2.8. The same as in the figure 7 but in the absence of nonlinear interactions.
Bibliography


CHAPTER 3. ON A NON-LINEAR INVESTIGATION OF AN ELECTRO-SPINNING MODEL UNDER COMBINED SPACE AND TIME EVOLVING INSTABILITIES

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Abstract

We study the nonlinear problem of axisymmetric electrically driven jets with applications to electrospinning. In our investigation we consider the model based on the governing electrohydrodynamic equations that unifies the two types of instabilities that occur during the fine fiber production due to spatial and temporal growing disturbances. The model is approached from a classical stability point of view in the early stage and then it is treated with a weakly nonlinear wave theory of certain dyad resonance modes that later involve the use of Newton’s Method to solve a dispersion relation and finally the Method of Lines (MOL) to solve a system of PDEs that governs the combined time and space evolving amplitude instability functions. We found, in particular, that for certain parameter values of the jet flow system, there are some resonance modes that can dominate the jet for its temporal and spatial evolution. We were able to detect nonlinear properties in our investigation that allowed a favorable change in the dynamics of the jet flow from a thickening to a thinning jet. Our model was able to uncover new parameter regimes for both spatial and time modes in which instabilities were significantly enhanced and jet radius was reduced approaching to nano-scale size, which is the desired mechanism in the electrospinning process.
3.1 Introduction

3.1.1 Electrospinning Background and Relevant Investigations

Electrospinning is a process for manufacturing high volumes of very thin fibers that typically range from 100 nm to 1 micron, with lengths up to several hundred of meters depending on the application, from a vast variety of materials, including polymers, composites and ceramics [3,5,20]. In this process, nanofibers are produced by solidification of a polymer solution stretched by an electric field. The unique properties of nanofibers are extraordinarily high surface area per unit mass, very high porosity, tunable pore size and surface properties, layer thinness, high permeability, low basic weight, ability to retain electrostatic charges and cost effectiveness. These electrospun nanofibers have many practical applications to different areas including wound dressing, drug or gene delivery vehicles, high quality filters, biosensors, fuel cell membranes and electronics, tissue-engineering processes.

In this paper we consider the nonlinear problem of the axisymmetric electrically driven jets with finite electrical conductivity and under the presence of a uniform and non-uniform applied electric field [9,10,19], but now we include realistic features that are found in the manufacturing process of electrospinning including time and space evolving instabilities. We approach the nonlinear problem by considering combined space and time resonant instability modes that can extract and quantify the nonlinear wave interaction of such growing disturbances. Resonant wave interactions play a very important role in the theory of weakly nonlinear instability of weakly nonlinear waves. In general terms, understanding the small amplitude solutions of various physical problems requires a first natural step, which is the linearization of the equation or system of equations. Then by the use of harmonic analysis one can apply the principle of superposition to provide a solution representation of the linearized problem in the form of plane waves.

\[ A \exp [i(k \cdot x - \omega t)] \]

where \( \omega \) is the frequency of the wave, \( k \) is a wave number vector, whose magnitude is referred to as wave number, and \( A \) is the amplitude of the wave. The wave number and the frequency are then involved mostly in the resulting dispersion relation from the linear system derived.
from particular physical problem. In the study of weakly nonlinear wave interactions solutions we look for the solutions in the form of plane waves as we described above by working with the linearized system, but the amplitudes of the plane waves will no longer be constant. Instead, the amplitudes are slowly evolving or modulated by nonlinear wave interactions, which occur due to the nonlinearities that exist on the original equations and boundary conditions of the system.

The importance of resonant wave interaction relies on the behavior that a system can undergo with regards of large amplitude oscillations at certain frequencies. The frequencies that generate larger oscillation amplitudes compared to all other frequencies are called resonant frequencies. These frequencies are very critical because a system behavior could be altered even by small periodic driving forces [1,14]. In general, the system could be stabilized or destabilized by such driving forces, which for our the problem investigation these take the form of the nonlinearities in the original system.

Resonance wave interactions in an ongoing study in many areas for example nonlinear optics, plasma physics, materials and mechanical engineering and fluid mechanics, etc. The general ideas of resonance wave interactions have been attributed by several authors such as H.J. Beth and Diederik Korteweg [13], but more recent work has been done by Rott [1]. He investigated the internal resonance for the double pendulum problem, which was modeled by a system of ordinary differential equations that described the dynamics of the angles of rotation $\theta_1$ and $\theta_2$ in the system. He provided the results, which are now implemented in several areas of research, relating the small oscillations normal modes of the system and the corresponding modifications of such modes due to the nonlinearities of $\theta_1$ and $\theta_2$. Rott [1] showed that these nonlinearities treated as forcing terms in the system are the weak interactions of the normal modes. He found that there is a slow periodic interchange of energy between the two normal modes. The resonance he investigated was of dyad type, which is the same type of resonance we implement for this investigation by mathematically adapting those ideas to a larger system.

In regards to the resonant wave interactions in electrospinning applications, no investigation had been carried out for the electrically driven jets prior to the work of Orizaga and Riahi [21]. They conducted the investigation of temporal linear instabilities and their nonlinear wave
interactions. They found that the instabilities of resonant type evolving in time were able to produce both favorable and unfavorable results. The unfavorable condition was in the sense that the instabilities detected on the linear case were actually modified by stronger types of instabilities for most cases but at the same time this in turn provided a significant reduction on the jet radius, which is of high interest for practical applications.

In this investigation in contrast with [21], we considered a more realistic study, which included two types of instabilities in a combined form for space and time. We modeled the combined spatial and temporal evolving instabilities for axysymmetric electrically driven jets, which are known to exist by the experiments done in [7,8,10]. We found, in particular, that for certain parameter values of the jet flow system, there are some resonance modes that can dominate the jet for its temporal and spatial evolution. We were able to detect nonlinear properties in our investigation that allowed a favorable change in the dynamics of the jet flow from a thickening to a thinning jet. Our model was able to uncover new parameter regimes for both spatial and time modes in which instabilities were significantly enhanced and jet radius was reduced approaching to nano-scale size, which is the desired mechanism in the electrospinning process.

3.1.2 Paper Organization

This brief section is used to describe the paper organization. In section 2, we will talk about the particular model that we use for the mathematical modeling of the axisymmetric electrically driven jets. We will also talk about the classical stability approach for nonlinear model, the nonlinear approach in the dyad resonance sense, the complications in finding the numerical solution to the dispersion relation arising from the stability analysis of the problem and the numerical treatment done to the system of nonlinear partial differential equations (PDEs) governing the amplitude functions carrying the nonlinear wave interactions with instability evolving along spatial and temporal direction. In Section 3, we will provide the main results, simulations and discussions for this investigation, and in section 4 we provide some concluding remarks.
3.2 Electrospinning Model and Nonlinear Wave Theory

3.2.1 Mathematical Formulation

Our mathematical modeling of the electrically driven jets is based on the governing electrohydrodynamic equations [8] for the mass conservation, momentum, charge conservation and for the electric potential, which are described in detail in [9,10,19,21]. The work of combined resonant spatial and temporal instability that we consider in this paper is a research continuation of the work done in [21]. In contrast to the time evolving instability that was studied in [21], here we study particular realistic features of combined space and time evolving disturbances of the modes that enhance significantly in space and time as observed in experiments [10]. We use the non-dimensional form of the modeling equations for long slender axisymmetric electrically driven jets with finite conductivity under a uniform or nonuniform applied electric field [19,21].

\[
\frac{\partial}{\partial t} (h^2) + \frac{\partial}{\partial z}(h^2v) = 0 \tag{3.2}
\]

\[
\frac{\partial}{\partial t}(h\sigma) + \frac{\partial}{\partial z}(hv\sigma) + \frac{1}{2}\frac{\partial}{\partial z}(h^2EK^*\tilde{K}(z)) = 0 \tag{3.3}
\]

\[
\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{\partial}{\partial z}\left[ h \left( 1 + \left( \frac{\partial h}{\partial z} \right)^2 \right)^{-\frac{3}{2}} - \frac{\partial^2 h}{\partial z^2} \left[ 1 + \left( \frac{\partial h}{\partial z} \right)^2 \right]^{-\frac{3}{2}} - \frac{E^2}{8\pi} - 4\pi\sigma^2 \right] + \frac{2E\sigma}{h\sqrt{\beta}} + \frac{3v^*}{h^2 \frac{\partial}{\partial z}(h^2v)} \tag{3.4}
\]

\[
E_b(z) = E - \ln(\lambda) \left[ \frac{\beta}{2} \frac{\partial^2}{\partial z^2}(h^2E) - 4\pi\sqrt{\beta} \frac{\partial}{\partial z}(h\sigma) \right] \tag{3.5}
\]

where the dependent variables are \( h(z,t) \) the radius of the jet cross-section at the axial location \( z \), \( v(z,t) \) the axial velocity, \( \sigma(z,t) \) the surface charge, \( E(z,t) \) the electric field, and the parameters are \( K^* \) the non-dimensional conductivity, \( \tilde{K}(z) \) is a non-dimensional variable function, \( \beta \) is the permittivity ratio constant of fluids, \( \nu^* \) is the non-dimensional viscosity, \( E_b(z) \) is an applied electric field and \( 1/\lambda \) is the local aspect ratio, which is assumed to be small.

3.2.2 Stability Analysis

In this section, we address the problem of local linear stability of axisymmetric perturbations of an electrified fluid flow due to combined spatial and temporal instabilities. To determine the
linear stability we express the solution to Equations (3.2-3.5) as a sum of its equilibrium state plus a small perturbation. The equilibrium point often referred as an electro-static equilibrium point was obtained from the work of Riahi [19]. This Equilibrium point provides the desired mechanisms related to electrospinning process (jet can be reduced by the use of this point). To determine the linear stability, we solve for the dynamics of small disturbances to the constant jet radius, velocity, surface charge and electric field with the following solution

\[ h = 1 + h_1 e^{(f+i\omega)t+(s+ik)z} \]  
\[ v = 0 + v_1 e^{(f+i\omega)t+(s+ik)z} \]  
\[ \sigma = \sigma_0 + \sigma_1 e^{(f+i\omega)t+(s+ik)z} \]  
\[ E = \Omega(1 - \delta z) + E_1 e^{(f+i\omega)t+(s+ik)z} \]

where \( h_1, v_1, \sigma_1 \) and \( E_1 \) are assumed to be small. The growth rates are \( s \) and \( f \) for spatial and temporal cases respectively, \( k \) is the wave number, \( \omega \) is the real frequency. Here both \( \Omega \) and \( \sigma_0 \) are constant quantities. \( \sigma_0 \) is the background free charge density and \( \Omega \) the magnitude of the applied electric field. We set \( \delta = 8\sigma_0\pi/(\Omega \sqrt{\beta}) \) [19] to be a small parameter (\( \delta << 1 \)) and consider a series expansion in powers of \( \delta \) for all the dependent variables for the case of non-uniform applied field.

In this paper we investigate the cases where applied electric field can be either uniform (\( \delta = 0 \)) or non-uniform (\( \delta \neq 0 \)). This is related the electric field that is generated between the high voltage, which is applied at nozzle, and the distance from the nozzle to the grounded collector plate. This allows for perfect or imperfect alignment on the collector plate with respect to the nozzle orifice [9,10]. Substituting Eqs.(3.6)-(3.9) into Eqs.(3.2)-(3.5) and dividing by exponential part gives
(ik + s)v_1 + 2h_1(f + i\omega) = 0 \quad (3.10)

E_1K^*(ik + s) + 2v_1(ik\sigma_0 + s\sigma_0) + 2\sigma_1(f + i\omega) + 2h_1(f\sigma_0 + i\sigma_0\omega + ikK^*\Omega + K^*s\Omega) = 0 \quad (3.11)

v_1(f + 3k^2v^* - 6iksv^* - 3s^2v^* + i\omega) + E_1\left(-\frac{2\sigma_0}{\sqrt{\beta}} \frac{ik\Omega}{4\pi} - \frac{s\Omega}{4\pi}\right) + \sigma_1(-8ik\pi\sigma_0 - 8\pi s\sigma_0 - \frac{2\Omega}{\sqrt{\beta}}) + h_1(ik^3 + s + 3k^2s - s^3 - ik(-1 + 3s^2) + \frac{2\sigma_0\Omega}{\sqrt{\beta}}) = 0 \quad (3.12)

4i\pi(k - is)\sqrt{\beta}\sigma_1\text{Ln}\left[\frac{100}{89k}\right] + h_1(k - is)\sqrt{\beta}\left(4i\pi\sigma_0 + (k - is)\sqrt{\beta}\Omega\right)\text{Ln}\left[\frac{100}{89k}\right] + \frac{1}{2}E_1(2 + (k - is)^2\beta\text{Ln}\left[\frac{100}{89k}\right]) = 0 \quad (3.13)

The above Eqs. (3.10)-(3.13) are algebraic equations that can generate a nontrivial solution only if the determinant of the coefficient matrix is zero. This gives rise to the dispersion relation to our modeling problem

\[-\frac{1}{4}(ik + s)(\frac{1}{\sqrt{\beta}}(4(k - is)\sqrt{\beta}(1 + (k - is)^2 + 8\pi\sigma_0^2) - if + \omega) - 16\sigma_0(f - 2K^*\pi(k - is)^2\sqrt{\beta} + \omega)\Omega - 8iK^*(k - is)\Omega^2) + \frac{1}{\pi}(k - is)^2(4K^*\pi + \sqrt{\beta}(f + i\omega))(-2i\pi(k - is)\sqrt{\beta}(1 + (k - is)^2 + 8\pi\sigma_0^2) - 16\pi\sigma_0\Omega + (-ik - s)\sqrt{\beta}\Omega^2)\log\left[\frac{100}{89k}\right] + (f + i\omega)(2(k - is)^2(2\pi(4\sigma_0^2 + K^*\sqrt{\beta}(f + 3(k - is)^2v^* + i\omega)) + (ik + s)\sqrt{\beta}\sigma_0\Omega)\log\left[\frac{100}{89k}\right] + \frac{1}{\sqrt{\beta}}(\sqrt{\beta}(f^2 + 3f(k - is)^2v^* - 8\pi(k - is)^2\sigma_0^2 + 2if\omega + 3i(k - is)^2v^*\omega - \omega^2) + 2(ik + s)\sigma_0\Omega)(-2 - (k - is)^2\beta\log\left[\frac{100}{89k}\right]))) = 0 \quad (3.14)\]

The above dispersion relation is written in the compact form after prescribing parameter setting values and fluid characteristic values

\[D(s, f, w; k) = 0 \quad (3.15)\]

### 3.2.3 Numerical Solution to the Dispersion Relation

In this section we address the methods implemented as well the complications encountered in solving Eq.(3.15). To solve the dispersion relation we first make use the property for complex value functions and we separate real and imaginary components with the tools find in
In order to implement a numerical scheme on the dispersion relation in Eq.(3.15), we make use of Eq.(3.16) to arrive at the system of nonlinear equations

\[ f(s, f, \omega; k) = 0 \]  \hspace{1cm} (3.17)

\[ g(s, f, \omega; k) = 0 \]  \hspace{1cm} (3.18)

The nonlinear system in Eqs. (3.17)-(3.18) is suitable for the Newton’s Method provided that two major obstacles are overcome. First we need to reduce the independent variables so that the root finding algorithm can take place (i.e. we need 2 equations and 2 unknowns) and then we need a good initial guess for the method to converge. The first attempt was to run the Newton’s Method twice to solve the nonlinear system first solve the system for neutral temporal instability and obtain \((s, \omega)\) and then use this information on \(\omega\) to prescribe values for the nonlinear system and again solve the system to obtain \((s^*, f^*)\), where the “*” denotes the solution to the dispersion relation. The complications with this was that the initial guess was not chosen from previous research papers as it was done in [19,21] since no investigations that model combined spatial and temporal instabilities were available. The grow rates found by the Newton’s Method \((s^*, f^*)\) were not vanishing Eq.(3.15) and the system showed high sensitivity to the initial guess and to the small variations in wavenumber as well as to the to not so small variations on the non-dimensional viscosity fluid parameters. The second obvious approach was to reverse the order of the previous mentioned approach first solve nonlinear system with neutral spatial instability and solve for \((\omega, f)\) then prescribe data about \(\omega\) on the nonlinear system to once again solve the system by Newton’s Method to arrive at \((s^*, f^*)\). Reversing the approach produce little to no improve results since we still experience complications with regards the convergence of the Newton’s Method.

On a more positive note we were able to overcome the problem related to the dispersion relation on Eq.(3.15) and this was done by running long simulations in Mathematica that allowed us to study a very broad range of values for each parameter in order to properly reduce
the independent variables and to choose the appropriate initial guess values. We were able to find particular functions for $\omega$ in terms of wave number $k$ so that the only dependent variables in the nonlinear system become $s$ and $f$. We were able to treat the high sensitivity in the system with respect to changes in parameter values by adapting a parameter continuation approach to our algorithm for Newton's Method. Parameter continuation was first implemented in our low viscous fluid by updating each initial guess as we ran the Newton’s Method along all possible wavenumber values and we were able to solve for dispersion relation for the low viscous fluid. We then use the information on the low viscous fluid to implement a parameter continuation approach that allowed us to run Newton’s Method by now slowly incrementing the viscosity values and update each initial guess until we were able to arrive at the viscosity value that was needed for the investigation. Once we arrived at the objected viscosity parameter again we ran the Newton’s Method with parameter continuation approach in order properly solve Eq.(3.15) for higher viscosity fluid.

Next, we would like to provide the idea of the algorithm and present some basic components in the strategy for solving the dispersion relation. Using the vector notation for Eqs.(3.17)-(3.18) and representing that system as $F(X) = 0$ with $X := (s,f)$ allows us to implement Newton’s Method by properly taking into account the several parameters that variate along the simulation process. Parameters that variate include the fluid viscosity $\nu^*$, uniform and non-uniform applied field $\delta$, intensity of the electric field $\Omega$, and the wave number $k$. Here we present the primary structure of the Newton’s Method with the parameter continuation approach implemented in Mathematica Software

**Algorithm/Pseudocode:**

For loop structure with parameter continuation approach for $\nu^*$

Nested For loop structure for each $\delta$, $\Omega$ and $k$

Initial guess $X$

For $i = 1$ to Max. number of iterations.

$X_{new} = X - InverseJacobian[F(X)]F(X)$

If $\|X_{new} - X\|_2 < Tol$, Break
\[ X = X_{\text{new}} \]

For the low viscous fluid we used (water glycerol mixture) \( \nu^* = 0.60764 \) and higher viscous fluid (glycerol) \( \nu^* = 9.05384 \). We were able to solve the problem of linear combined spatial and temporal instability for different magnitudes of the applied electric field under a uniform and non-uniform applied electric field (See Figures 3.1-3.2). We stored the data for the linear problem in matrices that were later used for search and detection of certain modes of instability that satisfy the triad resonance conditions. We then use such modes to investigate the nonlinear problem associated with the nonlinear wave interactions.

### 3.2.4 Nonlinear Wave Theory and Dyad Resonant Modes

In this section, we apply the theory of weakly nonlinear wave theory under dyad or triad resonant modes [1,14,21,22]. We start by considering the following

\[
\begin{align*}
 h &= 1 + h_1, \\
 v &= 0 + v_1, \\
 \sigma &= 1 + \sigma_1 \\
 E &= E_0 + E_1
\end{align*}
\]

and substitute these in Eqs.(3.2)-(3.5) and we keep linear terms on left hand side and products of up to two perturbation quantities on the right hand side. For simplicity and proper arrangement of equations we drop the subscript labels and obtain

\[
\frac{2}{\partial t} \frac{\partial h}{\partial t} + \frac{\partial v}{\partial z} = -2h \frac{\partial h}{\partial t} - 2h \frac{\partial v}{\partial z} - 2v \frac{\partial h}{\partial z} = -\frac{\partial (h \sigma)}{\partial t} - \sigma_0 h \frac{\partial h}{\partial z} - \sigma \frac{\partial v}{\partial z} - \sigma \frac{\partial v}{\partial z},
\]

\[
\frac{1}{2} K^* \left[ 2 \frac{\partial (h \sigma)}{\partial z} + \Omega \frac{\partial (h^2)}{\partial z} \right] = -\frac{\partial (h \sigma)}{\partial t} - \sigma_0 h \frac{\partial h}{\partial z} - \sigma \frac{\partial v}{\partial z} - \sigma \frac{\partial v}{\partial z} - \sigma \frac{\partial v}{\partial z} - \sigma \frac{\partial v}{\partial z},
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial h}{\partial z} - \frac{\partial^2 h}{\partial z^2} - \frac{\Omega}{4 \pi} \frac{\partial E}{\partial z} - 8\pi \sigma_0 \frac{\partial \sigma}{\partial z} - \frac{2}{\sqrt{\beta}} \left[ \Omega \sigma + \sigma_0 (E + \Omega h) \right] - 3\nu \frac{\partial^2 v}{\partial z^2} = -v \frac{\partial h}{\partial z} + \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial h}{\partial z} \right)^2
\]

\[
+ \frac{1}{4 \pi} \rho E \frac{\partial E}{\partial z} + 8\pi \sigma \frac{\partial \sigma}{\partial z} + \frac{2}{\sqrt{\beta}} \left[ \Omega \sigma_0 h^2 - \Omega \sigma_0 h - \sigma_0 Eh + E \sigma \right] + 6\nu \frac{\partial}{\partial z} \left( \frac{h \sigma}{\partial z} \right) - 6\nu h \frac{\partial^2 v}{\partial z^2}
\]

\[
E + \ln \left[ \frac{100}{89k} \right] \left( -\frac{\beta}{2} \left( \frac{\partial^2h}{\partial z^2} + \frac{\partial^2E}{\partial z^2} \right) + 4\pi \sqrt{\beta} \left( \frac{\partial \sigma}{\partial z} + \sigma_0 \frac{\partial h}{\partial z} \right) \right) = \ln \left[ \frac{100}{89k} \right] \left( \frac{\beta}{2} \left( \frac{\partial^2(hE)}{\partial z^2} + \Omega \frac{\partial^2h^2}{\partial z^2} \right) 
\right)
\]

\[
- 4\pi \frac{\partial (\sigma h)}{\partial z}
\]

\[
(3.21)
\]

\[
(3.22)
\]
Next we consider the solution form for Eqs.(3.19)-(3.22) in the following form \((h, v, \sigma, E) = \epsilon(h_1, v_1, \sigma_1, E_1) + \epsilon^2(h_2, v_2, \sigma_2, E_2)\), which provides different order of perturbation magnitudes that are required to treat the nonlinear problem in a dyad resonant setting. We also introduce the slowly varying variables for both space and time that are required to capture the nonlinear wave interactions[1,14]. Letting the different order perturbations have the following form

\[
(h_1, v_1, \sigma_1, E_1) = \sum_{n=1}^{2} (h_{1n}, v_{1n}, \sigma_{1n}, E_{1n}) A_n(z_s, t_s) e^{[(f_n+i\omega_n)t+(s_n+i\kappa_n)z]} + c.c \quad (3.23)
\]

\[
(h_2, v_2, \sigma_2, E_2) = \sum_{n=1}^{2} (h_{2n}(z_s, t_s), v_{2n}(z_s, t_s), \sigma_{2n}(z_s, t_s), E_{2n}(z_s, t_s)) e^{[(f_n+i\omega_n)t+(s_n+i\kappa_n)z]} + c.c \quad (3.24)
\]

where \((h_{1n}, v_{1n}, \sigma_{1n}, E_{1n})\) are constants, \((h_{2n}(z_s, t_s), v_{2n}(z_s, t_s), \sigma_{2n}(z_s, t_s), E_{2n}(z_s, t_s))\) are functions of slowly varying time and space, \(c.c\) represents the complex conjugate of the preceding expressions and \(A_n(z_s, t_s)\) are the amplitude functions that carry the information on the nonlinear wave interactions of the triad resonant modes. Making use of Eqs.(3.23)-(3.24) along with \(\frac{\partial}{\partial z} := \frac{\partial}{\partial z} + \epsilon \frac{\partial}{\partial z_s}, \frac{\partial^2}{\partial z^2} := \frac{\partial^2}{\partial z^2} + 2\epsilon \frac{\partial^2}{\partial z \partial z_s} + \epsilon^2 \frac{\partial^2}{\partial z_s^2}\) and \(\frac{\partial}{\partial t} := \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial t_s}\), where \(z_s := \epsilon z\) and \(t_s := \epsilon t\), in Eqs.(3.19)-(3.22) gives the following system.
2 \frac{\partial h_2}{\partial t} + \frac{\partial v_2}{\partial z} = -2h_1 \frac{\partial h_1}{\partial t} - 2h_1 \frac{\partial v_1}{\partial z} - 2v_1 \frac{\partial h_1}{\partial z} - 2v_1 \frac{\partial v_1}{\partial z} - 2 \frac{\partial h_1}{\partial t_s} - 2 \frac{\partial v_1}{\partial t_s} \tag{3.25}

\frac{\partial \sigma_2}{\partial t} + \sigma_0 \frac{\partial h_2}{\partial t} + \sigma_0 \frac{\partial v_2}{\partial z} + \frac{1}{2} K^* [2 \Omega \frac{\partial h_2}{\partial z} + \frac{\partial E_2}{\partial z}] = - \frac{\partial (h_1 \sigma_1)}{\partial t} - \sigma_0 v_1 \frac{\partial h_1}{\partial z} - \sigma_1 \frac{\partial v_1}{\partial z} - \sigma_0 h_1 \frac{\partial v_1}{\partial z} - \sigma_0 \frac{\partial \sigma_1}{\partial t_s} - \frac{1}{2} K^* [2 \Omega \frac{\partial (h_1 \sigma_1)}{\partial z} + \frac{\partial (E_1)}{\partial z}] - \sigma_0 \frac{\partial \sigma_1}{\partial t_s} - \frac{\partial \sigma_1}{\partial t_s} \tag{3.26}

\frac{\partial v_2}{\partial t} + \frac{\partial h_2}{\partial z} - \frac{\partial^2 h_2}{\partial z^2} - \frac{\Omega \partial E_2}{4 \pi} \frac{\partial \sigma_2}{\partial z} - \frac{2}{\sqrt{\beta}} [\Omega \sigma_2 + \sigma_0 E_2 + \Omega \sigma_0 h_2] - 3 \nu^* \frac{\partial^2 v_2}{\partial z^2} = -v_1 \frac{\partial v_1}{\partial z} \nonumber

+ \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial h_1}{\partial z} \right)^2 + \frac{1}{4 \pi} \frac{E_1 \partial E_1}{\partial z} + 8 \pi \sigma_1 \frac{\partial \sigma_1}{\partial z} + \frac{2}{\sqrt{\beta}} [\Omega \sigma_0 h_1^2 - \Omega h_1 - \sigma_0 E_1 h_1 + E_1 \sigma_1] + 6 \nu^* \frac{\partial}{\partial z} \left( h_1 \frac{\partial v_1}{\partial z} \right) - 6 \nu^* h_1 \frac{\partial^2 v_1}{\partial z^2} - h_1 \frac{\partial h_1}{\partial z_s} + 3 \left( \frac{\partial^2}{\partial z^2} \frac{\partial \sigma_1}{\partial z} \right) h_1 + \frac{\Omega}{4 \pi} \frac{\partial E_1}{\partial z_s} + 8 \pi \sigma_0 \frac{\partial \sigma_1}{\partial z_s} + 6 \nu^* \left( \frac{\partial}{\partial z} \frac{\partial}{\partial z_s} \right) v_1 - \frac{\partial v_1}{\partial t_s} \tag{3.27}

E_2 + \ln \left[ \frac{100}{89 k} \right] \left( -\frac{\beta}{2} \left( \frac{\partial^2 h_2}{\partial z^2} + \frac{\partial^2 E_2}{\partial z^2} \right) + 4 \pi \sqrt{\beta} \left( \frac{\partial \sigma_2}{\partial z} + \sigma_0 \frac{\partial h_2}{\partial z} \right) \right) = \ln \left[ \frac{100}{89 k} \right] \left( \frac{\beta}{2} \left( \frac{\partial^2 (2 h_1 E_1)}{\partial z^2} \right) + \frac{\Omega}{89 k} \right) \left( \frac{\beta}{2} \left( \frac{\partial^2 (2 h_1 E_1)}{\partial z^2} \right) - 4 \pi \sqrt{\beta} \frac{\partial (\sigma_1 + \sigma_0 h_1)}{\partial z} \right) \tag{3.28}

In order to treat Eqs.(3.25)-(3.28) we require the solution from section 2.1 Eq.(3.15) and seek for solutions that satisfy the dyad resonance conditions under perfect resonance \((k_2, \omega_2) = 2(k_1, \omega_1) [1, 14]\). Each solution associated with each particular fluid is in the form of \((k_n^*, \omega_n^*, s_n^*, f_n^*)\) for the wavenumber, real frequency, spatial growth rate and temporal growth rate along with the parameters used in Eq.(3.15) for \(n = 1, 2\). We implemented a code in Mathematica to search for resonance modes in the matrices solutions we stored for Eq.(3.15) in section 2.1 and we were able to detect several modes that satisfy dyad resonance conditions. We then used such modes to investigate their nonlinear wave interactions.

We now continue with the nonlinear problem formulation and for this we will require the solutions of the constant amplitude values in Eq.(3.23) \((h_{1n}, v_{1n}, \sigma_{1n}, E_{1n})\) for \(n = 1, 2\). We formulate the problem by making use of Eqs.(3.19)-(3.22) and linearizing with respect to the amplitude of the perturbation along with the amplitude function to produce the following linear systems.
\[(ik_n + s_n)v_{1n} + 2h_{1n}(f_n + i\omega_n) = 0 \quad (3.29)\]

\[E_{1n}K^*(ik_n + s_n) + 2v_{1n}(ik_n\sigma_0 + s_n\sigma_0) + 2\sigma_{1n}(f_n + i\omega_n) + 2h_{1n}(f_n\sigma_0 + i\sigma_0\omega_n + ik_nK^*\Omega + K^*s_n\Omega) = 0 \quad (3.30)\]

\[v_{1n}(f_n + 3k_n^2v^* - 6ik_n s_n v^* - 3s_n^2 v^* + i\omega_n) + E_{1n} \left(-\frac{2\sigma_0}{\sqrt{\beta}} - \frac{ik_n\Omega}{4\pi} - \frac{s_n\Omega}{4\pi}\right) + \sigma_{1n}(-8ik_n\pi\sigma_0 - 8\pi s_n\sigma_0 - \frac{2\Omega}{\sqrt{\beta}}) + h_{1n} \left(\frac{2\sigma_0\Omega}{\sqrt{\beta}}\right) = 0 \quad (3.31)\]

\[4i\pi(k_n - is_n)\sqrt{\beta}\sigma_{1n}\ln \left[\frac{100}{89k_n}\right] + h_{1n}(k_n - is_n)\sqrt{\beta} \left(4i\pi\sigma_0 + (k_n - is_n)\sqrt{\beta}\right) \ln \left[\frac{100}{89k_n}\right] + \frac{1}{2}E_{1n}(2 + (k_n - is_n)^2)\beta\ln \left[\frac{100}{89k_n}\right] = 0 \quad (3.32)\]

Using the dyad resonant modes we detected and prescribing suitable values for \((h_{1n})\) for \(n = 1, 2\) according to the weakly nonlinear theory[1,14] we can solve the above linear systems. Now we set up the problem for the adjoint variables \((h_{1n}^{(a)}, v_{1n}^{(a)}, \sigma_{1n}^{(a)}, E_{1n}^{(a)})\) that are required by the related adjoint problem for the nonlinear problem. The strategy for solving for the adjoint variables is to use the inner product, \((x, y) = x^\top y\), where the overline denotes the complex conjugate. Using the inner product of the linear system formed by Eqs.(3.29)-(3.32) against \((h_{1n}^{(a)}, v_{1n}^{(a)}, \sigma_{1n}^{(a)}, E_{1n}^{(a)})\) gives the single equation (by using the left hand sides of Eqs.(3.29)-(3.32))

\[(L.H.S_{Eq.(3.29)})h_{1n}^{(\pi)} + (L.H.S_{Eq.(3.30)})v_{1n}^{(\pi)} + (L.H.S_{Eq.(3.31)})\sigma_{1n}^{(\pi)} + (L.H.S_{Eq.(3.32)})E_{1n}^{(\pi)} = 0 \quad (3.33)\]

we factor Eq.(3.33) with the help of Mathematica in the following form \((adj.eq1)h_{1n} + (adj.eq2)v_{1n} + (adj.eq3)\sigma_{1n} + (adj.eq4)E_{1n} = 0\) and we simultaneously solve the four equations \((adj.eq1) - (adj.eq4)\), which are lengthy and will not be given, to zero to obtain the solution to the adjoint variables. We now have both the regular amplitude variables and the adjoint amplitude variables for \(n = 1, 2\).

We now proceed with the nonlinear system of PDEs from Eqs.(3.19)-(3.22) and due to the size of the system we will work each case separately for \(n = 1, 2\). Using Eqs.(3.23)-(3.24) in the nonlinear Eqs.(3.19)-(3.22) and keeping on the right hand sides only terms that resonate[14] for the case \(n=1\), we obtain
\[ L.H.S_{Eq.(3.29)} = (b_1 \frac{\partial A_1}{\partial z_s} + (c_1) \frac{\partial A_1}{\partial t_s} + (d_1)A_1A_2e^{[f_2t+s_2z]} \] (3.34)

\[ L.H.S_{Eq.(3.30)} = (b_2 \frac{\partial A_1}{\partial z_s} + (c_2) \frac{\partial A_1}{\partial t_s} + (d_2)A_1A_2e^{[f_2t+s_2z]} \] (3.35)

\[ L.H.S_{Eq.(3.31)} = (b_3 \frac{\partial A_1}{\partial z_s} + (c_3) \frac{\partial A_1}{\partial t_s} + (d_3)A_1A_2e^{[f_2t+s_2z]} \] (3.36)

\[ L.H.S_{Eq.(3.32)} = (a_4)\frac{\partial^2 A_1}{\partial z_s^2} + (b_4)\frac{\partial A_1}{\partial z_s} + (c_4)\frac{\partial A_1}{\partial t_s} + (d_4)\overline{A_1}A_2e^{[f_2t+s_2z]} \] (3.37)

where the overline in the amplitude function \( A_1(z_s, t_s) \) denotes the complex conjugate, coefficients \((b_i, c_i, d_i)\) for \( i = 1 \) to \( 4 \) and \( a_4 \) will not be given for the above equations at this stage, but will be given once we have arrived at final set of equations for \( n = 1, 2 \). We now consider the Eqs.(3.34)-(3.37) for the case \( n=1 \) in vector-matrix form which can be written as \( Lx = N \) with its related homogeneous adjoint problem \( L^a x^{(a)} = 0 \). Here \( x \) represents the vector containing the constant amplitude variables \((h_{1n}, v_{1n}, \sigma_{1n}, E_{1n})\), \( L \) represents the coefficient matrix and \( N \) represents the right hand side vector. We then then can apply the solvability conditions[13] for the above system. Pairing \( N \) against \( x^{(a)} \) gives the following \((N, x^{(a)}) = (Lx, x^{(a)}) = (x, L^a x^{(a)}) = (x, 0) = 0\), hence we obtain the solvability condition for Eqs.(3.34)-(3.37) for \( n=1 \) as \((N, x^{(a)}) = 0\) which gives the following nonlinear PDE

\[ (a_4 * E_{11}^{(a)}) \frac{\partial^2 A_1}{\partial z_s^2} + (b_1 * h_{1n}^{(a)} + b_2 * v_{1n}^{(a)} + b_3 * \sigma_{1n}^{(a)} + b_4 * E_{1n}^{(a)}) \frac{\partial A_1}{\partial z_s} + (c_1 * h_{1n}^{(a)} + c_2 * v_{1n}^{(a)} + c_3 * \sigma_{1n}^{(a)} + c_4 * E_{1n}^{(a)}) \frac{\partial A_1}{\partial t_s} + (d_1 * h_{1n}^{(a)} + d_2 * v_{1n}^{(a)} + d_3 * \sigma_{1n}^{(a)} + d_4 * E_{1n}^{(a)}) \overline{A_1}A_2e^{[f_2t+s_2z]} = 0 \] (3.38)

We follow an analogous approach when \( n=2 \) to find the solvability conditions for Eqs.(3.34)-(3.37). We also write Eq.(3.38) in compact form to express the system of nonlinear system of PDEs that govern the evolution of the amplitude functions \( A_1(z_s, t_s) \) and \( A_2(z_s, t_s) \)

\[ (C_1) \frac{\partial^2 A_1}{\partial z_s^2} + (C_2) \frac{\partial A_1}{\partial z_s} + (C_3) \frac{\partial A_1}{\partial t_s} + (C_4) \overline{A_1}A_2 \exp[f_2t+s_2z] = 0 \] (3.39)

\[ (D_1) \frac{\partial^2 A_2}{\partial z_s^2} + (D_2) \frac{\partial A_2}{\partial z_s} + (D_3) \frac{\partial A_2}{\partial t_s} + (D_4)A_1^2 \exp[(2f_1 - f_2)t + (2s_1 - s_2)z] = 0 \] (3.40)

where the coefficients \((C_i, D_i)\) for \( i = 1 \) to \( 4 \) are given on Appendix B .Eqs(3.39)-(3.40) govern the dynamics of the nonlinear wave interactions for instabilities that evolve in time and
space. This concludes the weakly nonlinear wave theory for the dyad resonant modes. In the next section we consider numerical approach to solve the above system and present the typical results obtained from the dyad resonant problem.

3.2.5 Numerical Treatment for Governing PDE System

This section we discuss the numerical algorithm that we used to solve Eqs.(3.39)-(3.40). We implemented the Method of Lines (MOL) to solve the above system. We used a backward in time finite difference discretization for the time derivative appearing in the system and then we prescribed the initial condition along the axial direction which in turn transformed the PDE system into a system of ordinary differential equations (ODEs). We then iterate along time and allow at each time step for the solutions to evolve along the z axial direction by clamping the amplitude functions and their derivatives at the initial z axial direction. Using the appropriate slowly varying dependent variables throughout the PDE system and discretizing time in Eqs.(3.39)-(3.40) we obtain

\[
(C_1) \frac{\partial^2 A_1}{\partial z_s^2} + (C_2) \frac{\partial A_1}{\partial z_s} + (C_4) A_1 A_2 \exp\left(\frac{f_2 t_s}{\epsilon} + \frac{s_2 z_s}{\epsilon}\right) + (C_3) \frac{A_1 - A_1^{b0}}{\Delta t_s / \epsilon} = 0 \tag{3.41}
\]

\[
(D_1) \frac{\partial^2 A_2}{\partial z_s^2} + (D_2) \frac{\partial A_2}{\partial z_s} + (D_4) A_1^2 \exp\left((2f_1 - f_2) \frac{t_s}{\epsilon} + (2s_1 - s_2) \frac{z_s}{\epsilon}\right) + (D_3) \frac{A_2 - A_2^{b0}}{\Delta t_s / \epsilon} = 0 \tag{3.42}
\]

The Eqs.(3.41)-(3.42) are can now be treated ODEs in terms of the dependent variables \((z_s, t_s)\). Here \(t_s\) will be fixed depending on the iteration step along the time variable. In the notation above for the time discretization we label \(A_1^{t0}\) as the previous known stage with respect to time and \(A_1^t\) will represent the amplitude function \(A_1(z_s, t_s)\) for a given \(t_s\) according to the time step selected, which complies with the C.F.L condition, and the iteration process. The same is applied for the second amplitude function \(A_2(z_s, t_s)\). Using the notation described about the above equations, we now proceed to present the algorithm used to solve the Eqs.(3.39)-(3.40).

**Algorithm/Pseudo – code** :

Discretize time and space according to C.F.L condition \(\Delta t / (\Delta x)^2 \leq 1/2\)
Initialize: $A_{t0}^1(z_s,t_s), A_{t0}^2(z_s,t_s), A_1(0,t_s), A_2(0,t_s), A_1'(0,t_s)$ and $A_2'(0,t_s)$

For $i = 1$ to $n$

Solve ODE system from Eqs.(3.41)-(3.42)

Assign $A_{t0}^1(z_s,t_s):= A_1(z_s,t_s)$ and $A_{t0}^2(z_s,t_s):= A_2(z_s,t_s)$

End

The algorithm was implemented in Mathematica Software and the solutions obtained were transformed back to the regular space and time variables. These solutions were then used to construct perturbation quantities in the form of Eq.(3.23) to provide a study and graphical representation of the nonlinear wave interactions of the dyad resonant modes studied for this investigation.

### 3.3 Simulation Results and Discussion

In the present study we considered the nonlinear investigation of the combined spatial and temporal instability of electrically driven jets with finite electrical conductivity and under the presence of both uniform and non-uniform applied electric field. We consider two types of fluids for the jet, which can be representative as those that can be used in the experimental investigation for the problem such as water-glycerol mixture and glycerol [9,10]. For such fluids, we set representative parameter values to be $K^* = 19.60$, $\nu^*$ (glycerol) = 9.05384, $\nu^*$ (water-glycerol) = 0.60764, and $\beta = 77$. We also include experimental and simulations values $\sigma_0 = 0$ for the constant applied field and $\sigma_0 = 0.10$ for the variable applied field. We also set $\epsilon = 0.01$ which is required to change the regular dependent variables for space and time to slowly varying and to capture the dynamics of the nonlinear wave interactions[1,14]. For this investigation we consider several different magnitudes of the applied electric field $\Omega = 0.5, 1.0, 1.5$ and 2.3. In regards to provide solution representations for this investigation, we first consider the dispersion relation in Eq.(3.15) and solve this using the Newton’s Method by carrying the appropriate parameter continuation settings. The solutions of Eq.(3.15) are investigated for both uniform and non-uniform applied electric field. Only those solutions that resonate according to the dyad settings will be used to perform the nonlinear study. Graphical representations are provided
for numerical solution of the dispersion relation. In order to provide a qualitative study of
the quality of the fiber production we solve for Eqs.(3.41)-(3.42) via Method of Lines(MOL)
and use these amplitude functions $A_1(z,t)$ and $A_2(z,t)$ to reconstruct the solution in the form
of Eq.(3.23) which solves Eqs.(3.19)-(3.22). This reconstruction for the solution provides a
simulation of the time and space evolution process for the fiber fabrication with respect to
axisymmetric type of disturbances. We will refer to these simulation plots as the perturbation
plots. The results for the solution of the dispersion relation and the main typical results for the
perturbation plots and their time and space evolution for water-glycerol mixture and glycerol
jets are briefly presented in the following two sub-sections.

3.3.1 The Case of Water-glycerol Mixture Jet

Figs. 3.1 and 3.2 represent the temporal and spatial growth rate $f$ and $s$ versus the axial
wave number $k$ for the non-uniform applied electric field ($\sigma_0 = 0.1$), respectively, and for
different values of $\Omega$. It can be seen from the Fig. 3.1 that the temporal growth rate $f$
undergoes to a stabilization process as the values of $\Omega$ are increased which gives rise to a
classical Rayleigh instability. For the largest value of the applied electric field the temporal
instability is neutralized and for the wave number values of $k > 0.7$ the temporal growth
rate sets at a stable mode. Temporal growth rate $f$ mostly decreases for increased wavenumber
values. For the case of constant applied electric field, we detected a small decrease of instability
for the temporal growth rate as compared to case where ($\sigma_0 = 0.1$). For the spatial growth
rate $s$, we can observe from Fig. 3.2 that by increasing the magnitude of the electric field $\Omega$
the instabilities are enhanced and thus producing a conducting type of instability[9,10]. The
growth rates $s$ increase for larger wavenumber $k$, specially near $k = 1.0$. For the case of constant
applied field, the spatial growth rates are slightly increased as compared to the non-uniform
applied field.

For uniform applied electric field, we found some modes of combined spatial and temporal
instability that satisfy the dyad resonant conditions. These modes of instability were present
for both cases of constant and variable applied field. The nonlinear resonant mode interaction
due to modes that are present in the case of uniform applied field, was found to be weaker
the majority of the time in the sense that magnitudes of the perturbations due to such modes grow with in space and time at moderate lower values than those due to a non-uniform applied field. For this study we will do a comparative approach between the nonlinear problem and the linear problem. In Fig. 3.3(a)-(b), we present perturbation plots with nonlinear wave interactions present and in the absence of such interactions, respectively. We can observe important components that directly impact the quality production of the fiber. First we discuss the space evolution, Fig. 3.3(b) shows that the jet radius (thin-line) grows for \( z > 4.5 \) and at the same time the velocity drops (dotted-line) after \( k > 3 \). Second, Fig. 3.3(d) shows similar fiber thickening for time values greater than \( t = 42 \) and jet velocity decreased for \( t > 40 \). These results are obtained from the linear problem and since the goal of electrospinning process is to reduce the jet radius these results obtained from the linear theory are not of great interest [9,10]. However, performing the nonlinear investigation of the problem in a dyad resonance setting we obtain very different and favorable results. In Fig. 3.3(a), we observe a strong wave interaction of the modes that satisfy the resonance conditions and this is quantified by the considerable reduction of the domain of the perturbation plots. The jet radius path is modified from a thickening jet to a thinning jet by the nonlinear problem. In Fig.3(c), we obtain another modified perturbation plot that now provides jet velocity increased and jet reduction increased as time evolves which is a mechanism that makes sense as a physical problem and is a desired process at the manufacturing of the small fibers. For Fig. 3(e)-(f), we have provided the reconstructed solution in Eq.(3.23), but only for \( h_1(z,t) \). Here we show the different contour lines and several levels of fast transitions in the solution as well as a three dimensional plot. We make use of such information to first be able to detect solutions paths which lead to jet reduction and then with the contour plot we are able to select the steepest descent path in terms of jet reduction. For this investigation, the contour plots will always be used for choosing the path of highest rate of jet reduction. Darker regions in the contour plots indicate fast transitions when \( h_1(z,t) \) decreases and lighter regions indicate change in transitions where the jet radius grows. The experimentally relevant solution found are presented in Fig. 3.3(a) and Fig. 3.3(c) in which the solution paths in either space or time are able to reduce the jet radius to nano-scale size.
For non-uniform applied electric field, we consider another mode that satisfied the dyad resonant conditions and considered the quantitative changes taking place in the perturbation quantities for the jet flow. For Figs. 3.4(a)-(b), Linear theory shows a jet thinning at significant rate for \( z > 5 \) and a jet with reduced speed at about \( z > 6 \). These results are modified the nonlinear investigation as shows in Fig. 3.4(a) in which the jet radius is reduced at a much higher rate after about \( z = 4.2 \) and this is while jet radius is gaining considerable amount of speed (thin-line) growing at at large rate. For the time evolving instabilities, we have in Fig. 3.4(d) what seems to be real good perturbation plots since jet is gaining speed and thinning for \( t > 20 \). Unfortunately, this case is not good for practical applications since this study is combined spatial and temporal instability we must account for both instabilities at the same time. In other words, jet losing velocity in Fig. 3.4(b) is not of interest. For Figure 3.4(c), jet is reduced for all values of \( t \) and jet velocity increases for values larger than \( t = 15 \). This translates translates into an improvement in jet dynamics with respect to time evolving instabilities. In Figs. 3.4(e)-(f), we provide the solution for \( h_1(z,t) \) and its contour level plot accounting for fast transitions in jet radius. We make use of both plots to find new solution paths that we were able to detect under the nonlinear investigation. These solutions paths exhibited a considerable level of freedom with regards to selection of time and space cross-section. We used Fig. 3.4(a) and Fig. 3.4(c) to present solution paths along space and time in which jet radius undergoes to a significant reduction needed to produce the small fibers.

### 3.3.2 The Case of Glycerol Jet

We now provide results for the higher viscosity glycerol fluid and we begin by solving the dispersion relation. Figures 3.5 and 3.6 present the temporal growth rate \( f \) and the spatial growth rate \( s \) versus the axial wave number for the uniform applied field \((\sigma_0 = 0.0)\) for different values of \( \Omega \). The temporal growth rate that is shown in Fig. 3.5 shows no variations with respect to changes in the applied electric field and this is due to the small order changes mainly in the \( 10^{-3} \) for \( f \). The temporal growth rate decreases as wavenumber \( k \) increases and there is a significant region in which \( f \) is found to be stable for \( k > .45 \). For the spatial growth rate \( s \), we detected two branches of instabilities and each branch evolved different with respect to changes
in $\Omega$. In Fig. 3.6, we detected the classical Rayleigh [10] type of instability which suppresses the instability for increased values in the applied electric field for $k < 0.58$. For $k > 0.58$, we detected another spatial instability branch and for $k < 0.07$ the instability is of Rayleigh type but for $k > 0.7$ it is of the conducting type, which is enhanced by increasing the strength of $\Omega$. The second branch of instability for $s$ becomes more dominant as the wavenumber increases and for larger magnitudes of the applied electric field.

For uniform applied field, we found modes that satisfy the dyad resonance conditions for the combined spatial and temporal instabilities of glycerol jets. Fig. 3.7 presents the perturbation plots under nonlinear wave interactions and in the absence of such interactions for both time and space evolutions. Fig. 3.7(a)-(b) provide visually almost identical plots that include favorable results in jet speed increase and jet radius reduction as space $z$ evolves but nonlinear interactions case had a very small advantage over the linear case in the order of $10^{-2}$. For the case of time evolving instabilities, we observe in Fig. 3.7(d) that jet radius (thin-line) is reduce for $t > 1$ and jet velocity (dotted-line) is increasing for all $t$ values. The nonlinear treatment of the problem modifies those perturbation plots as obtained in Fig. 3.7(d) by the ones in Fig. 3.7(c) and the major notable changes occur with the jet electric field (thick-line), and jet velocity (dotted-line) and the small change of order $10^{-2}$ on the jet radius (thin-line). Here we attribute the type modifications found from the linear to the nonlinear problem to the higher viscosity value of glycerol jet and to the case of $(\sigma_0 = 0.0)$. The solution for $h_1(z,t)$ for the nonlinear problem is found to have small restriction in which different solution paths could provide a significant jet reduction. In Fig. 3.7(e)-(f), we can detect the different transition levels in the contour plot and evolution of the jet radius along space and time in the three dimensional plot. Making appropriate selection and avoiding undesirable solution paths is of high importance for this investigation. For example, spatial evolving trajectories from $t = 0.3$ to $t = 1.2$ are of no interest since these trajectories land on the lighter colors of the contour plots, which means jet will not be reduced. Therefore, the solution paths of interest are shown in Fig. 3.7(a) and Fig. 3.7(c), which provide the required mechanism of interest in the electrospinning process.

For non-uniform applied field, we considered our last mode to presented in this paper. This mode is another mode that resonates in a dyad sense for the combined space and time evolving
disturbances. Fig. 3.8(a)-(b) presents how the nonlinear wave interaction modifies the perturbation plots obtained from a classical linear instability point of view. In Fig. 3.8(b), the jet radius (thin-line) increases at a high rate for $z > 4$ and this is while jet velocity (dotted-line) is decreasing. This mechanism which has no practical use in the fiber production is positively modified by the nonlinear problem. The dyad wave interaction modifies this mechanism in a favorable way as shown in Fig. 3.8(a) in which jet radius is reduced for $z > 0.5$ and jet accelerates for $z > 0.31$. For time evolutions, the perturbations plots also undergo to a change in dynamics from thickening to thinning jet when nonlinear wave interactions are taken into account. Fig. 3.8(c)-(d) provides an illustration of favorable change in mechanism that is obtained for the perturbation plots with respect to temporal instabilities. For both temporal and spatial instability the nonlinear investigation was able to uncover favorable modes of operation that linear theory was not able to detect. The solution $h_1(z,t)$ for Eq. (3.23) uncovers new solutions paths that are in a sense more restricted than those seen in Fig. 3.8(e)-(f), but if chosen in a proper way can provide a significant jet radius reduction as shown in Fig. 3.8(a) and Fig. 3.8(c). Having a reduced axial direction $z$ values and attaining a jet reduction is of high importance in the manufacturing process of the small fibers. This is because small $z$ values translates to less material to be required to produce the high quality fibers.

### 3.4 Concluding Remarks and Future Research

The main task of this paper was to consider combined space and time evolving instabilities due to disturbances of axysymmetric type in electrically driven jets, which could have a direct impact on the manufacturing process of electrospinning. We carried out mathematical modeling, theoretical and numerical investigation of linear combined spatial and temporal instability as well as the nonlinear interactions of those detected modes that satisfy the dyad resonance conditions in electro-hydrodynamic system for axisymmetric electrically forced slender water-glycerol and glycerol jet flows with externally imposed either uniform or non-uniform applied field. We found new parameter regimes for the electrospinning model in which the electrospinning process can benefit by successfully approaching desired properties in the production fiber at earlier $z$-axial and $t$-axis locations. This results in shortening the electrospinning process
by reducing the amount of solution needed and the time involved in the process, which can improve the productivity of the small fibers. We were able to detect nonlinear properties in our investigation that allowed a favorable change in the dynamics of the jet flow that changed from a thickening to a thinning jet. Considering the studied electrospinning model in a nonlinear wave theory approach under the dyad resonant settings provided a very a modified mechanism as the one obtained from a classical linear stability theory point of view. This mechanism can be of high interest in practical applications of electrospinning to determine necessary means to control and predict instabilities in jets in order to produce higher quality fibers.

Further investigation of nonlinear wave interaction and resonances for the corresponding modes due to the non-axisymmetric disturbances will be good a problem to consider in near future. As is evident from the well-known experiments by Taylor [7,8] that the electrically forced jets can be non-axisymmetric and whip for sufficiently large values of the strength of the electric field and further down-stream away from the jet orifice, we can expect that spatial and temporal instabilities and nonlinear interactions of the modes due to the non-axisymmetric disturbances can dominate over the axisymmetric ones in some domain in the axial direction if Ω is sufficiently large.

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Appendix B

The expressions for the coefficients $C_n(n = 1, 2, 3, 4)$ and $D_n(m = 1, 2, 3, 4)$ for Eqs.(3.39)-(3.40) are given below

\[ C_1 = \left(\frac{1}{2}E_{11}\beta\log \left[ \frac{100}{89k_1} \right] + h_{11}\Lambda\log \left[ \frac{100}{89k_1} \right] \right)E_{11}^{(\sigma)} \quad (B.1) \]

\[ C_2 = (-v_{11})h_{11}^{(\sigma)} + (-\frac{E_{11}K^*}{2} - v_{11}\sigma_0 - h_{11}K^*\Lambda)v_{11}^{(\sigma)} + (h_{11}(-1 - (k_1 - is_1)^2) + 6(ik_1 + s_1)v_0 v_{11} + 8\pi\sigma_0\sigma_1 + \frac{E_{11}\Omega}{4\pi})\sigma_{11}^{(\sigma)} + (-4\pi\sqrt{\beta} (h_{11}\sigma_0 + \sigma_{11})\log \left[ \frac{100}{89k_1} \right] )E_{11}^{(\sigma)} \quad (B.2) \]

\[ C_3 = (-h_{11})h_{11}^{(\sigma)} + (-h_{11}\sigma_0 - \sigma_{11})v_{11}^{(\sigma)} + (-v_{11})\sigma_{11}^{(\sigma)} + (0)E_{11}^{(\sigma)} \quad (B.3) \]

\[ C_4 = (-2\left((ik_1 + s_1 + s_2)v_{12} + h_{12}(f_1 + f_2 + i\omega_1)\right)h_{11}^{(\sigma)} + (h_{12}K^* (ik_1 + s_1 + s_2))E_{11}^{(\sigma)} + ((ik_1 + s_1 + s_2)(E_{12}K^* + v_{12}\sigma_0 + h_{12}K^*\Lambda) + \sigma_{12}(f_1 + f_2 + i\omega_1))h_{11}^{(\sigma)} + (i k_1 + s_1 + s_2)(h_{12}\sigma_0 + \sigma_{12})v_{11}^{(\sigma)} + ((ik_1 + s_1 + s_2)v_{12} + h_{12}(f_1 + f_2 + i\omega_1))\sigma_{11}^{(\sigma)} + \frac{1}{4\pi\sqrt{\beta}}((E_{12}(ik_1 + s_1 + s_2))\sqrt{\beta} + 8\pi(-h_{12}\sigma_0 + \sigma_{12}))E_{11}^{(\sigma)} + 4\pi(((k_1 + is_1)(2k_1 - is_2)(h_{12}(ik_1 + s_1 + s_2) + 6v_0v_{12})\sqrt{\beta} - 2E_{12}\sigma_0 + 4h_{12}\Omega - 2s_1\sigma_1\Omega))h_{11}^{(\sigma)} + (6h_{12}(k_1 + is_1)(2k_1 - is_2)v_0 + (-ik_1 - s_1 - s_2)v_{12})\sqrt{\beta} + 2(E_{12} + 4\pi(ik_1 + s_1 + s_2)\sqrt{\beta}\sigma_{12} - h_{12}\Omega)\sigma_{11}^{(\sigma)} + ((k_1 - i(s_1 + s_2))\sqrt{\beta} \left(h_{12}(k_1 - i s_1 + s_2))\sqrt{\beta}E_{11}^{(\sigma)} + (4i\pi\sigma_{12} + (k_1 + i(s_1 + s_2))\sqrt{\beta}(E_{12} + h_{12}\Omega)h_{11}^{(\sigma)} + 4ih_{12}\pi\sigma_{11}^{(\sigma)} \right) \log \left[ \frac{100}{89k_1} \right] )E_{11}^{(\sigma)} \quad (B.4) \]

\[ D_1 = \left(\frac{1}{2}\beta(E_{12} + 2h_{12}\Omega)\log \left[ \frac{50}{89k_1} \right] \right)E_{12}^{(\sigma)} \quad (B.5) \]

\[ D_2 = (-v_{12})h_{12}^{(\sigma)} + (-\frac{E_{12}K^*}{2} - v_{12}\sigma_0 - h_{12}K^*\Lambda)v_{12}^{(\sigma)} + (h_{12}(1 - 12k_1^2 + 12i k_1 s_2 + 3s_2^2) + 12i k_1 v_0 v_{12} + 6s_2 v_0 v_{12} + 8\pi\sigma_0\sigma_{12} + \frac{E_{12}\Omega}{4\pi})\sigma_{12}^{(\sigma)} + (-4\pi\sqrt{\beta} (h_{12}\sigma_0 + \sigma_{12})\log \left[ \frac{50}{89k_1} \right] )E_{12}^{(\sigma)} \quad (B.6) \]

\[ D_3 = (-h_{12})h_{12}^{(\sigma)} + (-h_{12}\sigma_0 - \sigma_{12})v_{12}^{(\sigma)} + (-v_{12})\sigma_{12}^{(\sigma)} + (0)E_{12}^{(\sigma)} \quad (B.7) \]

\[ D_4 = -2(h_{11}(f_1 h_{11} + 2ik_1 v_{11} + 2s_1 v_{11} + i\omega_1))h_{12}^{(\sigma)} - 2((2E_{11}h_{11}K^*(ik_1 + s_1) + 2(ik_1 + s_1) v_{11}\sigma_{11} + h_{11}^2 K(k_1 + s_1)\Omega + 2h_{11}((ik_1 + s_1)v_{11}\sigma_0 + \sigma_{11}(f_1 + i\omega_1)))v_{12}^{(\sigma)} + \frac{1}{4\pi\sqrt{\beta}}((E_{11}(ik_1 + s_1)\sqrt{\beta} + 8E_{11}\pi(-h_{11}\sigma_0 - \sigma_{11}) + 4\pi(ik_1 + s_1)\sqrt{\beta}(h_{11}^2 (k_1 - is_1)^2 + 6h_{11}(ik_1 + s_1)v_{0}v_{11} - v_{11}^2 + 8\pi\sigma_1^2) + 8h_{11}\pi(h_{11}\sigma_0 - \sigma_{11})\Omega))\sigma_{12}^{(\sigma)} - 2(h_{11}(k_1 - is_1)\sqrt{\beta}(4i\pi\sigma_{11} + (k_1 - is_1)\sqrt{\beta}(2 E_{11} + h_{11}\Omega)))E_{12}^{(\sigma)} \quad (B.8) \]
Figure Captions

Fig. 3.1. The temporal growth rate $f$ versus the axial wave number $k$ and for water-glycerol mixture jet with $\Omega = 0.5$ (thin solid line), 1 (dotted line), 1.5 (dashed line) and 2.3 (thick solid line) and for non-uniform applied electric field ($\sigma_0 = 0.1$).

Fig. 3.2. The same as in the Fig.1 but for spatial growth rate $s$ versus wavenumber $k$.

Fig. 3.3. Perturbation quantities $h_1$ (thin solid line), $v_1$ (dotted line), $\sigma_1$(dashed line) and $E_1$ (thick solid line) versus the time variable $t$ and for space variable $z$ for water-glycerol mixture jet and for the two modes 1 and 2 that satisfy the dyad resonant conditions. Subfigures Fig.1a and Fig1.b represent the perturbation plots when nonlinear resonant wave interactions are considered and in the absence of such nonlinear modes evolving in space. Similarly for Subfigures Fig1.c and Fig1.d represent the same as in Fig1.a and Fig1.b but for time evolving instabilities. Subfigures Fig.1e and Fig.1f represents the numerical solution of the nonlinear problem for jet radius $h_1(z,t)$ with a contour plot and a 3-d plot. For the contour plot, the vertical axis is set for time $t$ and horizontal axis for space $z$. Darker regions in the contour plots indicate radius thinning and lighter region indicates thickening of the jet. Here $t = 6$ and $z = 0.2$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings included $\epsilon = 0.01, \nu^* = 0.60764, \beta = 77.0, K^* = 19.6, \sigma_0 = 0.0, k_1 = \omega_1 = 0.39, k_2 = \omega_2 = 0.78, s_1 = 1.6580, s_2 = 1.6455, f_1 = 0.2034, f_2 = 0.01231$ and $\Omega = 2.3$.

Fig. 3.4. Same as in Fig. 5 but for $t = 5$ and $z = 4.2$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig.5 except for another mode found with resonant properties $\sigma_0 = 0.1, k_1 = \omega_1 = 0.1, k_2 = \omega_2 = 0.2, s_1 = 1.67284, s_2 = 1.67315, f_1 = 0.169813$ and $f_2 = 0.161083$.

Fig. 3.5. The same as Fig. 1 but for the case of glycerol fluid.

Fig. 3.6. The same as Fig. 2 but for the case of glycerol fluid.
Fig. 3.7. Same as in Fig. 5 but for the case of glycerol jet with $t = 1.5$ and $z = 11.2$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig.5 except for the viscosity $\nu^* = 9.05384$ along with $\sigma_0 = 0.0, k_1 = .05, \omega_1 = 1.25, k_2 = 0.1, \omega_2 = 2.5, s_1 = 0.460463, s_2 = 0.460455, f_1 = 5.72056$ and $f_2 = 5.51678$

Fig. 3.8. Same as in Fig. 7 but with $t = 1.55$ and $z = 0.31$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig.7 except for $\sigma_0 = 0.1, k_1 = 0.1, \omega_1 = 2.5, k_2 = 0.2, \omega_2 = 5.0, s_1 = 0.460679, s_2 = 0.460673, f_1 = 5.51216$ and $f_2 = 4.69756$. 
Figure 3.1. The temporal growth rate $f$ versus the axial wave number $k$ and for water-glycerol mixture jet with $\Omega = 0.5$ (thin solid line), 1 (dotted line), 1.5 (dashed line) and 2.3 (thick solid line) and for non-uniform applied electric field ($\sigma_0 = 0.1$).

Figure 3.2. The same as in the Fig.1 but for spatial growth rate $s$ versus wavenumber $k$. 
Figure 3.3. Perturbation quantities $h_1$ (thin solid line), $v_1$ (dotted line), $\sigma_1$ (dashed line) and $E_1$ (thick solid line) versus the time variable $t$ and for space variable $z$ for water-glycerol mixture jet and for the two modes 1 and 2 that satisfy the dyad resonant conditions. Subfigures Fig.1a and Fig1.b represent the perturbation plots when nonlinear resonant wave interactions are considered and in the absence of such nonlinear modes evolving in space. Similarly for Subfigures Fig1.c and Fig1.d represent the same as in Fig1.a and Fig1.b but for time evolving instabilities. Subfigures Fig.3.1e and Fig.3.1f represents the numerical solution of the nonlinear problem for jet radius $h_1(z,t)$ with a contour plot and a 3-d plot. For the contour plot, the vertical axis is set for time $t$ and horizontal axis for space $z$. Darker regions in the contour plots indicate radius thinning and lighter region indicates thickning of the jet. Here $t = 6$ and $z = 0.2$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings included $\epsilon = 0.01, \nu^* = 0.60764, \beta = 77.0, K^* = 19.6, \sigma_0 = 0.0, k_1 = \omega_1 = 0.39, k_2 = \omega_2 = 0.78, s_1 = 1.6580, s_2 = 1.6455, f_1 = 0.2034, f_2 = 0.01231$ and $\Omega = 2.3$. 
Figure 3.4. Same as in Fig. 3.5 but for $t = 5$ and $z = 4.2$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig.5 except for another mode found with resonant properties $\sigma_0 = 0.1, k_1 = \omega_1 = 0.1, k_2 = \omega_2 = 0.2, s_1 = 1.67284, s_2 = 1.67315, f_1 = 0.169813$ and $f_2 = 0.161083$.
Figure 3.5. The same as Fig. 3.1 but for the case of glycerol fluid.

Figure 3.6. The same as Fig. 3.2 but for the case of glycerol fluid.
Figure 3.7. Same as in Fig. 3.5 but for the case of glycerol jet with $t = 1.5$ and $z = 11.2$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig.3.5 except for the viscosity $\nu^* = 9.05384$ along with $\sigma_0 = 0.0, k_1 = .05, \omega_1 = 1.25, k_2 = 0.1, \omega_2 = 2.5, s_1 = 0.460463, s_2 = 0.460455, f_1 = 5.72056$ and $f_2 = 5.51678$.
Figure 3.8. Same as in Fig. 3.7 but with $t = 1.55$ and $z = 0.31$ for first and second pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig.3.7 except for $\sigma_0 = 0.1, k_1 = 0.1, \omega_1 = 2.5, k_2 = 0.2, \omega_2 = 5.0, s_1 = 0.460679, s_2 = 0.460673, f_1 = 5.51216$ and $f_2 = 4.69756$
Bibliography


CHAPTER 4. WEAKLY NONLINEAR WAVE INTERACTIONS IN ELECTRICALLY DRIVEN JETS VIA RESONANT TRIAD MODES

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Abstract

We study the nonlinear problem of axisymmetric electrically driven jets with applications to electrospinning. In our investigation we consider the model based on the governing electrohydrodynamic equations. The system of equations is studied under the case of both space and time evolving instabilities, in the presence of either a uniform or non-uniform electric field, and viscous jets with finite conductivity. At the linear stage we were able to detect different fluid flow instabilities including Rayleigh and Conducting instabilities in the jet by solving the dispersion relation for the governing system. We then study the nonlinear regime of this problem by carrying over solutions of the dispersion relation to the nonlinear problem using the weakly nonlinear wave theory and resonance triad modes. We found, in particular, that for certain parameter values of the jet flow system, there are some triad resonance modes that can dominate the jet for its temporal and spatial evolution. We were able to detect wave interactions that allowed a favorable change in the dynamics of the jet flow from a thickening to a thinning jet. Parameter settings that could not be of any practical use under classical linear investigation turn out to be of interest under our investigation. Our investigation was able to uncover new parameter regimes for both spatial and time modes in which instabilities as well as electric field in the jet were significantly enhanced and jet radius was reduced, which is the desired mechanism in the applications of electrically driven jets.
4.1 Introduction

4.1.1 Relevant Investigations and Practical Applications

The investigations of electrically driven jets are important due to the practical applications to electrospraying [11] and electrospinning [9,10]. Electrospinning is a process for manufacturing high volumes of very thin fibers that typically range from 100 nm to 1 micron, with lengths up to several hundred of meters depending on the application, from a vast variety of materials, including polymers, composites and ceramics [3,5,20]. In this process, nanofibers are produced by solidification of a polymer solution stretched by an electric field. The unique properties of nanofibers are extraordinarily high surface area per unit mass, very high porosity, tunable pore size and surface properties, layer thinness, high permeability, low basic weight, ability to retain electrostatic charges and cost effectiveness. These electrospun nanofibers have many practical applications to different areas including wound dressing, drug or gene delivery vehicles, high quality filters, biosensors, fuel cell membranes and electronics, tissue-engineering processes. The motivation for this investigation is to contribute in the understanding of electrically driven jets, which at the application level this problem is still performed on a trial and error basis due to the complexity of the problem.

In this paper we consider the nonlinear problem of the axisymmetric electrically driven jets with finite electrical conductivity and under the presence of a uniform and non-uniform applied electric field [9,10,19], but now we include realistic features that are found in the manufacturing process of electrospinning including time and space evolving instabilities that are known to exist [8]. We approach the nonlinear problem by considering combined space and time triad resonant instability modes that can extract and quantify the nonlinear wave interaction of such growing disturbances. Resonant wave interactions play a very important role in the theory of weakly nonlinear instability of weakly nonlinear waves. In general terms, understanding the small amplitude solutions of various physical problems requires a first natural step, which is the linearization of the equation or system of equations. Then by the use of harmonic analysis one can apply the principle of superposition to provide a solution representation of the linearized problem in the form of plane waves.
$A \exp \left[ i (k \cdot x - \omega t) \right]$ \hspace{1cm} (4.1)

where $\omega$ is the frequency of the wave, $k$ is a wave number vector, whose magnitude is referred to as wave number, and $A$ is the amplitude of the wave. The wave number and the frequency are then involved mostly in the resulting dispersion relation from the linear system derived from particular physical problem. In the study of weakly nonlinear wave interactions solutions we look for the solutions in the form of plane waves as we described above by working with the linearized system, but the amplitudes of the plane waves will no longer be constant. Instead, the amplitudes are slowly evolving or modulated by nonlinear wave interactions, which occur due to the nonlinearities that exist on the original equations and boundary conditions of the system.

The importance of resonant wave interaction relies on the behavior that a system can undergo with regards of large amplitude oscillations at certain frequencies. The frequencies that generate larger oscillation amplitudes compared to all other frequencies are called resonant frequencies. These frequencies are very critical because a system behavior could be altered even by small periodic driving forces [1,14]. In general, the system could be stabilized or destabilized by such driving forces, which for our the problem investigation these take the form of the nonlinearities in the original system.

Resonance wave interactions is an ongoing study in many areas for example nonlinear optics, plasma physics, materials and mechanical engineering and fluid mechanics, etc. The general ideas of resonance wave interactions have been attributed by several authors such as H.J. Beth and Diederik Korteweg [13], but more recent work has been done by Rott [1]. He investigated the internal resonance for the double pendulum problem, which was modeled by a system of ordinary differential equations that described the dynamics of the angles of rotation $\theta_1$ and $\theta_2$ in the system. He provided the results, which are now implemented in several areas of research, relating the small oscillations normal modes of the system and the corresponding modifications of such modes due to the nonlinearities of $\theta_1$ and $\theta_2$. Rott [1] showed that these nonlinearities treated as forcing terms in the system are the weak interactions of the normal modes. He
found that there is a slow periodic interchange of energy between the two normal modes. The resonance he investigated was of dyad (two-wave) type, which is the similar type of resonance (triads) we implement for this investigation by mathematically adapting those ideas to a larger system and far more complicated system.

In regards to the resonant wave interactions in electrospinning applications, no investigation had been carried out for the electrically driven jets prior to the work of Orizaga and Riahi [21]. They conducted the investigation of temporal linear instability of the modes that satisfy the dyad resonance conditions. They found that the instabilities of resonant type evolving in time were able to produce both favorable and unfavorable results. The unfavorable condition was in the sense that the instabilities detected on the linear case were actually modified by stronger types of instabilities for most cases but at the same time this in turn provided a significant reduction on the jet radius, which is of high interest for practical applications.

In this investigation in contrast with [21], we considered modes that satisfy the triad resonance conditions with higher instabilities of single and multiple triad forms and a more realistic study, which included two types of instabilities in a combined form for space and time. We modeled the combined spatial and temporal evolving instabilities for axisymmetric electrically driven jets, which are known to exist by the experiments done in [7,8,10]. We found, in particular, that for certain parameter values of the jet flow system, there are some triad resonance modes that can dominate the jet for its temporal and spatial evolution. We were able to detect nonlinear properties in our investigation that allowed a favorable change in the dynamics of the jet flow from a thickening to a thinning jet. Our investigation was able to uncover new parameter regimes for both spatial and time modes in which instabilities were significantly enhanced and jet radius was reduced along with an increasing electric field, which is the desired mechanism in the electrospinning process.

4.1.2 Paper Organization

This brief section is used to describe the paper organization. In section 2, we will present the mathematical modeling of the axisymmetric electrically driven jets. We will also talk about the classical stability approach for the nonlinear problem, the nonlinear approach in
the triad resonance sense, the numerical solution to the dispersion relation arising from the
stability analysis of the problem and the numerical solution to the nonlinear partial differential
equations (PDEs) governing the amplitude functions carrying the nonlinear wave interactions
that evolve along time and space. In Section 3, we will provide the main results, simulations
and discussions for this investigation, and in section 4 we provide some concluding remarks.

4.2 Electrically Driven Jets Modeling and Nonlinear Wave Theory

4.2.1 Mathematical Formulation

Our mathematical modeling of the electrically driven jets is based on the governing electro-
hydrodynamic equations [8] for the mass conservation, momentum, charge conservation and for
the electric potential, which are described in [9,10,19,21]. The work of triad resonant combined
time and spatial instability that we consider in this paper is a research continuation of the work
done in [21]. In contrast to the time evolving instability that was studied in [21], here we study
particular realistic features of spatial and temporal instabilities of the modes that enhance sig-
nificantly in space and time as observed in experiments [10]. We find that combined space and
time evolving instabilities subjected to triad resonance conditions can be significantly larger
in magnitude that corresponding temporal instability of such modes and can occur at a very
short distance after the jet is emitted [7,8].

\[
\begin{align*}
\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} &= 0 \\
\rho \frac{D\vec{u}}{Dt} &= -\nabla P + \nabla \cdot (\mu \vec{u}) + q\vec{E} \\
\frac{Dq}{Dt} + \nabla \cdot (K\vec{E}) &= 0 \\
\vec{E} &= -\nabla \Phi
\end{align*}
\] (4.1.1)

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \) is the total derivative, \( \vec{u} \) is the velocity vector, \( P \) is the pressure, \( \vec{E} \) is
the electric field vector, \( \Phi \) is the electric potential, \( q \) is the free charge density, \( \rho \) is the fluid
density, \( \mu \) is dynamic viscosity, \( K \) is electric conductivity and \( t \) is the time variable.
The internal pressure in the jet was found by taking into account the balances across the free boundary of the jet between the pressure, viscous forces, capillary forces and the electric energy density plus the radial self-repulsion of the free charges on the free boundary [8], and it leads to the following expression for the pressure $P$ in the jet

\[ P = \gamma \kappa - \frac{[(\epsilon - \bar{\epsilon})/(8\pi)]E^2 - (4\pi/\bar{\epsilon})\sigma^2}{8} \] (4.1.5)

where $\gamma$ is the surface tension, $\kappa$ is twice the mean curvature of the interface, $E$ is the magnitude of the electric field, $\epsilon/(4\pi)$ is the permittivity constant in the jet, $\bar{\epsilon}/(4\pi)$ is the permittivity constant in the air and $\sigma$ is the surface free charge.

Following the previous investigation [9], we consider a cylindrical fluid jet moving axially. The fluid of air is considered as the external fluid, and the internal fluid of jet is assumed to be Newtonian and incompressible. We use the governing Equations 4.1 in the cylindrical coordinate system with the origin at the center of nozzle exit section, where the jet flow is emitted with axial $z$-axis along the axis of the jet. We consider the axisymmetric form of the dependent variables in the sense that the azimuthal velocity is zero and there are no variations of the dependent variables with respect to the azimuthal variable. Following the approximations carried out in [9,19] for a long and slender jet in the axial direction, we consider the length scale in the axial direction to be large in comparison to that in the radial direction and use a perturbation expansion in the small jet aspect ratio. We expand the dependent variables in a Taylor series in the radial variable $r$. Then such expansions are used in the full axisymmetric system and keep only the leading terms. These lead to relatively simple equations for the dependent variables as functions of $t$ and $z$ only. Following the method of approach in [9], we employ (1d) and Coulombs integral equation to arrive at an equation for the electric field, which is essentially the same as the one derived in [9] and will not be repeated here. We nondimensionalize these equations using $r_0$ (radius of the cross-sectional area of the nozzle exit at $z = 0$), $E_0 = \gamma/[(\bar{\epsilon})r_0]^{1/2}$, $t_0 = (\rho r_0/\gamma)^{1/2}$, $(r_0/t_0)$ and $(\gamma \bar{\epsilon}/r_0)^{1/2}$ as scales for length, electric field, time, velocity and surface charge, respectively. The resulting non-dimensional equations have the following form [19]
\[ \frac{\partial}{\partial t} (h^2) + \frac{\partial}{\partial z} (h^2v) = 0 \] (4.2)

\[ \frac{\partial}{\partial t} (h\sigma) + \frac{\partial}{\partial z} (hv\sigma) + \frac{1}{2} \frac{\partial}{\partial z} (h^2EK^*\tilde{K}(z)) = 0 \] (4.3)

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = - \frac{\partial}{\partial z} \left[ h \left[ 1 + \left( \frac{\partial h}{\partial z} \right)^2 \right]^{-\frac{3}{2}} - \frac{\partial^2 h}{\partial z^2} \left[ 1 + \left( \frac{\partial h}{\partial z} \right)^2 \right]^{-\frac{3}{2}} - \frac{E^2}{8\pi} - 4\pi\sigma^2 \right] + \frac{2E\sigma}{h\sqrt{\beta}} + \frac{3v^*}{h^2} \frac{\partial}{\partial z} (h^2\frac{\partial v}{\partial z}) \] (4.4)

\[ E_b(z) = E - \ln(\mathcal{X}) \left[ \frac{\beta}{2} \frac{\partial^2}{\partial z^2} (h^2E) - 4\pi\sqrt{\beta} \frac{\partial}{\partial z} (h\sigma) \right] \] (4.5)

where the dependent variables are \( h(z, t) \) the radius of the jet cross-section at the axial location \( z \), \( v(z, t) \) the axial velocity, \( \sigma(z, t) \) the surface charge, \( E(z, t) \) the electric field, and the parameters are \( K^* \) the non-dimensional conductivity, \( \tilde{K}(z) \) is a non-dimensional variable function, \( \beta \) is the permittivity ratio constant of fluids, \( \nu^* \) is the non-dimensional viscosity, \( E_b(z) \) is an applied electric field and \( 1/\mathcal{X} \) is the local aspect ratio, which is assumed to be small.

### 4.2.2 Stability Analysis

In this section, we address the problem of local linear stability of axisymmetric perturbations of an electrified fluid flow due to combined spatial and temporal instabilities. To determine the linear stability we express the solution to Equations (4.2-4.5) as a sum of its equilibrium state plus a small perturbation. The equilibrium point often referred as an electro-static equilibrium point was obtained from the work of Riahi [19]. This Equilibrium point provides the desired mechanisms related to electrospinning process (jet can be reduced by the use of this point). To determine the linear stability, we solve for the dynamics of small disturbances to the constant jet radius, velocity, surface charge and electric field with the following solution

\[ h = 1 + h_1 e^{(f+i\omega)t+(s+ik)z} \] (4.6)

\[ v = 0 + v_1 e^{(f+i\omega)t+(s+ik)z} \] (4.7)

\[ \sigma = \sigma_0 + \sigma_1 e^{(f+i\omega)t+(s+ik)z} \] (4.8)

\[ E = \Omega (1 - \delta z) + E_1 e^{(f+i\omega)t+(s+ik)z} \] (4.9)
where $h_1$, $v_1$, $\sigma_1$ and $E_1$ are assumed to be small. The growth rates are $s$ and $f$ for spatial and temporal cases respectively, $k$ is the wave number, $\omega$ is the real frequency. Here both $\Omega$ and $\sigma_0$ are constant quantities. $\sigma_0$ is the background free charge density and $\Omega$ the magnitude of the applied electric field. We set $\delta = 8\sigma_0\pi/(\Omega\sqrt{\beta})$ [19] to be a small parameter ($\delta \ll 1$) and consider a series expansion in powers of $\delta$ for all the dependent variables for the case of non-uniform applied field.

In this paper we investigate the cases where applied electric field can be either uniform ($\delta = 0$) or non-uniform ($\delta \neq 0$). This is related the electric field that is generated between the high voltage, which is applied at nozzle, and the distance from the nozzle to the grounded collector plate. This allows for perfect or imperfect alignment on the collector plate with respect to the nozzle orifice [9,10]. Substituting Eqs.(4.6)-(4.9) into Eqs.(4.2)-(4.5) and dividing by exponential part gives

\[(ik + s)v_1 + 2h_1(f + i\omega) = 0 \quad (4.10)\]
\[E_1K^*(ik + s) + 2v_1(ik\sigma_0 + s\sigma_0) + 2\sigma_1(f + i\omega) + 2h_1(f\sigma_0 + i\sigma_0\omega + ikK^*\Omega + K^*s\Omega) = 0 \quad (4.11)\]
\[v_1(f + 3k^2v^* - 6iksv^* - 3s^2v^* + i\omega) + E_1\left(-\frac{2\sigma_0}{\sqrt{\beta}} - \frac{ik\Omega}{4\pi} - s\frac{\Omega}{4\pi}\right) + \sigma_1(-8ik\pi\sigma_0 - 8\pi s\sigma_0 - \frac{2\sigma_0\Omega}{\sqrt{\beta}}) = 0 \quad (4.12)\]
\[4i\pi(k - is)\sqrt{\beta}\sigma_1\ln\left[\frac{100}{89k}\right] + h_1(k - is)\sqrt{\beta}\left(4i\pi\sigma_0 + (k - is)\sqrt{\beta}\Omega\right)\ln\left[\frac{100}{89k}\right] + \frac{1}{2}E_1(2+ \frac{100}{89k}) = 0 \quad (4.13)\]

The above Eqs. (4.10)-(4.13) are algebraic equations that can generate a nontrivial solution only if the determinant of the coefficient matrix is zero. This gives rise to the dispersion relation to our modeling problem.
\[
- \frac{1}{4}(ik + s)(\frac{1}{\sqrt{\beta}}(4(k - is)\sqrt{\beta}(1 + (k - is)^2 + 8\pi \sigma_0^2))(-if + \omega) - 16\sigma_0(f - 2K^*\pi(k - is)^2\sqrt{\beta}
+ i\omega)\Omega - 8iK^*(k - is)\Omega^2) + \frac{1}{\pi}(k - is)^2(4K^*\pi + \sqrt{\beta}(f + i\omega))(-2i\pi(k - is)\sqrt{\beta}(1 + (k - is)^2
+ 8\pi \sigma_0^2) - 16\pi \sigma_0 \Omega + (-ik - s)\sqrt{\beta}\Omega^2)\text{Log}\left[\frac{100}{89k}\right] + (f + i\omega)(-2(k - is)^2(2\pi(4\sigma_0^2 + K^*\sqrt{\beta}(f
+ 3(k - is)^2v^* + i\omega)) + (ik + s)\sqrt{\beta}\sigma_0 \Omega)\text{Log}\left[\frac{100}{89k}\right] + \frac{1}{\sqrt{\beta}}(\sqrt{\beta}(f^2 + 3f(k - is)^2v^* - 8\pi(k - is)^2
\sigma_0^2 + 2if\omega + 3i(k - is)^2v^*\omega - \omega^2) + 2(ik + s)\sigma_0 \Omega)(-2 - (k - is)^2\beta\text{Log}\left[\frac{100}{89k}\right]) = 0
\] (4.14)

The above dispersion relation is written in the abstract form
\[
D(s, f, w; k; v^*; K^*; \sigma_0; \Omega; \beta) = 0
\] (4.14.1)

We can express a more compact dispersion relation after prescribing parameter settings and fluid characteristic values
\[
D(s, f, w; k) = 0
\] (4.15)

4.2.3 Dispersion Relation for the Jet Flow System

In this section we address the methods implemented in solving Eq.(4.15). To solve the dispersion relation we first make use the property for complex value functions and we separate real and imaginary components with the tools found in Mathematica Software
\[
D(s, f, \omega; k) = f(s, f, \omega; k) + ig(s, f, \omega; k) = 0
\] (4.16)

In order to implement a numerical scheme on the dispersion relation in Eq.(4.15), we make use of Eq.(4.16) to arrive at the system of nonlinear equations
\[
f(s, f, \omega; k) = 0
\] (4.17)
\[
g(s, f, \omega; k) = 0
\] (4.18)

The nonlinear system in Eqs. (4.17)-(4.18) is suitable for the Newton’s Method provided that two major obstacles are overcome. First, we need to reduce the independent variables so
that the root finding algorithm can take place (i.e. we need 2 equations and 2 unknowns) and then we need a good initial guess for the method to converge (this solution must also be of interest in the combined temporal and spatial instability sense).

The dispersion relation on Eq. (4.15) was solved numerically and this was done by running long simulations in Mathematica that allowed us to study a very broad range of values for each parameter in order to properly reduce the independent variables and to choose the appropriate initial guess values. We were able to find particular functions for $\omega$ in terms of wave number $k$ so that the only dependent variables in the nonlinear system become $s$ and $f$. We were able to treat the high sensitivity in the system with respect to changes in parameter values by adapting a parameter continuation approach to our algorithm for Newton’s Method. Parameter continuation was first implemented in our low viscous fluid by updating each initial guess as we ran the Newton’s Method along all possible wavenumber values and we were able to solve for dispersion relation for the low viscous fluid. We then use the information on the low viscous fluid to implement a parameter continuation approach that allowed us to run Newton’s Method by now slowly incrementing the viscosity values and update each initial guess until we were able to arrive at the viscosity value that was needed for the investigation. Once we arrived at the objected viscosity parameter again we ran the Newton’s Method with parameter continuation approach in order properly solve Eq. (4.15) for higher viscosity fluid.

Next, we present some basic components in the strategy for solving the dispersion relation. Using the vector notation for Eqs. (4.17)-(4.18) and representing that system as $F(X) = 0$ with $X := (s, f)$ allows us to implement Newton’s Method by properly taking into account the several parameters that variate along the simulation process. Parameters that variate include the fluid viscosity $\nu^*$, uniform and non-uniform applied field $\delta$, intensity of the electric field $\Omega$, and the wave number $k$. Here we present the primary structure of the Newton’s Method with the parameter continuation approach coded in Mathematica Software

Algorithm/Pseudocode:

For loop structure with parameter continuation approach for $\nu^*$

    Nested For loop structure for each $\delta$, $\Omega$ and $k$

    Initial guess $X$
For $i = 1$ to Max. number of iterations.

$$X_{\text{new}} = X - \text{InverseJacobian}[F(X)]F(X)$$

If $||X_{\text{new}} - X||_2 < Tol$, Break

$$X = X_{\text{new}}$$

End

End

End

For the low viscous fluid we used (water glycerol mixture) $\nu^* = 0.60764$ and higher viscous fluid (glycerol) $\nu^* = 9.05384$ we used the following $\omega(k) := \alpha k$ and $\omega(k) := \eta k$ respectively. Here both $\alpha$ and $\eta$ were given specific constant values to compute different phase fluid velocities depending on the fluid non-dimensional viscosity. We were able to solve the problem of linear combined spatial and temporal instability for different magnitudes of the applied electric field under a uniform and non-uniform applied electric field (See Figures 1-2). We stored the data for the linear problem in matrices that were later used for search and detection of certain modes of instability that satisfy the triad resonance conditions. We then use such modes to investigate the nonlinear problem associated with the nonlinear wave interactions.

4.2.4 Nonlinear Wave Theory and Triad Resonant Modes

In this section, we apply the theory of weakly nonlinear wave theory under dyad or triad resonant modes [1,14,21,22]. We start by considering the following $h = 1 + h_1$, $v = 0 + v_1$, $\sigma = 1 + \sigma_1$ and $E = E_b + E_1$ and substitute these in Eqs.(4.2)-(4.5) and we keep linear terms on left hand side and products of up to two perturbation quantities on the right hand side. For simplicity and proper arrangement of equations we drop the subscript labels and obtain
\[
\frac{2\partial h}{\partial t} + \frac{\partial v}{\partial z} = -2h \frac{\partial h}{\partial t} - 2h \frac{\partial v}{\partial z} - 2v \frac{\partial h}{\partial z} \tag{4.19}
\]

\[
\frac{\partial \sigma}{\partial t} + \sigma_0 \frac{\partial h}{\partial t} + \sigma_0 \frac{\partial v}{\partial z} + \frac{1}{2} K^* \left[2\Omega \frac{\partial h}{\partial z} + \frac{\partial E}{\partial z} \right] = - \frac{\partial (h\sigma)}{\partial t} - \sigma_0 v \frac{\partial h}{\partial z} - \sigma_0 \frac{\partial v}{\partial z} - \sigma_0 \frac{\partial v}{\partial z} - v \frac{\partial \sigma}{\partial z} \tag{4.20}
\]

\[
\frac{1}{2} K^* \left[2 \frac{\partial (Eh)}{\partial z} + \Omega \frac{\partial (h^2)}{\partial z} \right] \tag{4.21}
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial h}{\partial z} - \frac{\partial^2 h}{\partial z^2} - \frac{\Omega}{4\pi} \frac{\partial E}{\partial z} - 8\pi \sigma_0 \frac{\partial \sigma}{\partial z} - \frac{2}{\sqrt{\beta}} \left[\Omega \sigma + \sigma_0 (E + \Omega h) \right] - 3\nu \frac{\partial^2 v}{\partial z^2} = -v \frac{\partial v}{\partial z} + \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial h}{\partial z} \right)^2 + \frac{1}{4\pi} E \frac{\partial E}{\partial z} + 8\pi \frac{\partial \sigma}{\partial z} + \frac{2}{\sqrt{\beta}} \left[\Omega \sigma_0 h^2 - \Omega \sigma h - \sigma_0 Eh + E\sigma \right] + 6\nu \frac{\partial}{\partial z} \left( h \frac{\partial v}{\partial z} \right) - 6\nu h \frac{\partial^2 v}{\partial z^2} \tag{4.22}
\]

Next we consider the solution form for Eqs.(4.19)-(4.22) in the following form \( (h, v, \sigma, E) = \epsilon (h_1, v_1, \sigma_1, E_1) + \epsilon^2 (h_2, v_2, \sigma_2, E_2) \), which provides different order of perturbation magnitudes that are required to treat the nonlinear problem in a triad resonant setting. We also introduce the slowly varying variables for both space and time that are required to capture the nonlinear wave interactions\[1,14\]. Letting the different order perturbations have the following form

\[
(h_1, v_1, \sigma_1, E_1) = \sum_{n=1}^{3} (h_{1n}(z_s, t_s), v_{1n}(z_s, t_s), \sigma_{1n}(z_s, t_s), E_{1n}(z_s, t_s)) A_n(z_s, t_s)e^{[(f_n+i\omega_n)t+(s_n+i\kappa_n)z]} + c.c \tag{4.23}
\]

\[
(h_2, v_2, \sigma_2, E_2) = \sum_{n=1}^{3} (h_{2n}(z_s, t_s), v_{2n}(z_s, t_s), \sigma_{2n}(z_s, t_s), E_{2n}(z_s, t_s)) e^{[(f_n+i\omega_n)t+(s_n+i\kappa_n)z]} + c.c \tag{4.24}
\]

where \((h_{1n}, v_{1n}, \sigma_{1n}, E_{1n})\) are constants, \((h_{2n}(z_s, t_s), v_{2n}(z_s, t_s), \sigma_{2n}(z_s, t_s), E_{2n}(z_s, t_s))\) are functions of slowly varying time and space, \(c.c\) represents the complex conjugate of the preceding expressions and \(A_n(z_s, t_s)\) are the amplitude functions that carry the information on the nonlinear wave interactions of the triad resonant modes. Making use of Eqs.(4.23)-(4.24) along with \( \frac{\partial}{\partial z} := \frac{\partial}{\partial z} + \epsilon \frac{\partial}{\partial z} + \epsilon^2 \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial t} := \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial t} \), where \( z_s := \epsilon z \) and \( t_s := \epsilon t \), in Eqs.(4.19)-(4.22) gives the following system
\[ 2 \frac{\partial h_2}{\partial t} + \frac{\partial v_2}{\partial z} = -2h_1 \frac{\partial h_1}{\partial t} - 2h_1 \frac{\partial v_1}{\partial z} - 2v_1 \frac{\partial h_1}{\partial z} - \frac{\partial v_1}{\partial z} - 2 \frac{\partial h_1}{\partial t}, \]  \hspace{1cm} (4.25) \\
[\frac{\partial \sigma_2}{\partial t} + \sigma_0 \frac{\partial h_2}{\partial z} + \sigma_0 \frac{\partial v_2}{\partial z} + \frac{1}{2} K^* [2 \partial h_2 + \partial E_2] = - \partial (h_1 \sigma_1) - \sigma_0 v_1 \frac{\partial h_1}{\partial t} - \partial v_1 \frac{\partial h_1}{\partial z} - \sigma_0 h_1 \frac{\partial v_1}{\partial z} - \sigma_0 \frac{\partial v_1}{\partial t} - \frac{1}{2} K^* [2 \partial (h_1) + \partial (E_1) - \sigma_0 \frac{\partial h_1}{\partial z} - \frac{\partial \sigma_1}{\partial t}, \] \hspace{1cm} (4.26) \\
[\frac{\partial v_2}{\partial t} + \frac{\partial h_2}{\partial z} - \frac{\partial^2 h_2}{\partial z^2} - \frac{\Omega}{\partial} \frac{\partial E_2}{\partial z} - 8 \pi \sigma_0 \frac{\partial \sigma_2}{\partial z} - \frac{2 \sqrt{\beta}}{\sigma} \left[ \Omega \sigma_2 + \sigma_0 E_2 + \Omega \sigma_0 h_2 \right] - 3 \nu^* \frac{\partial^2 v_2}{\partial z^2} = -v_1 \frac{\partial v_1}{\partial z} + \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial h_1}{\partial z} \right)^2 + \frac{1}{4 \pi} \frac{E_1}{\partial \frac{\partial E_1}{\partial z}} + 8 \pi \sigma_1 \frac{\partial \sigma_1}{\partial z} + \frac{2 \sqrt{\beta}}{\sigma} \left[ \Omega \sigma_0 h_1^2 - \Omega \sigma_1 - \sigma_0 E_1 h_1 + E_1 \sigma_1 \right] + 6 \nu^* \frac{\partial}{\partial z} \left( \frac{\partial v_1}{\partial z} \right), \] \\
\left( h_1 \frac{\partial v_1}{\partial z} \right) - 6 \nu^* \frac{\partial v_1}{\partial z^2} - \frac{\partial h_1}{\partial z} + 3 \left( \frac{\partial^2 \sigma_1}{\partial z^2} \right) h_1 + \frac{\Omega}{\sigma} \frac{E_1}{\partial \frac{\partial E_1}{\partial z}} + 8 \pi \sigma_0 \frac{\partial \sigma_1}{\partial z} + 6 \nu^* \frac{\partial}{\partial z} \left( \frac{\partial v_1}{\partial z} \right) v_1 - \frac{\partial v_1}{\partial t} \right] \] \hspace{1cm} (4.27) \\
E_2 + \text{Ln} \left[ \frac{100}{89k} \left( \frac{\partial^2 h_2}{\partial z^2} + \frac{\partial^2 E_2}{\partial z^2} \right) + 4 \pi \sqrt{\beta} \left( \frac{\partial \sigma_2}{\partial z} + \sigma_0 \frac{\partial h_2}{\partial z} \right) \right] = \text{Ln} \left[ \frac{100}{89k} \left( \frac{\beta}{2} \left( \frac{\partial^2 (2h_1 E_1)}{\partial z^2} \right) + \Omega \frac{\partial^2 h_1}{\partial z^2} \right) - 4 \pi \sqrt{\beta} \frac{\partial (\sigma_1 + \sigma_0 h_1)}{\partial z} \right] \hspace{1cm} (4.28) \\

In order to treat Eqs.(4.25)-(4.28) we require the solution from section 2.1 Eq.(4.15) and seek for solutions that satisfy the triad resonance conditions under perfect resonance \((k_3, \omega_3) = (k_1, \omega_1) + (k_2, \omega_2) \) [1,14]. Each solution associated with each particular fluid is in the form of \((k_n^*, \omega_n^*, s_n^*, f_n^*)\) for the wavenumber, real frequency, spatial growth rate and temporal growth rate along with the parameters used in Eq.(15) for \(n = 1, 2, 3\). We implemented a code in Mathematica to search for resonance modes in the matrices solutions we stored for Eq.(4.15) in section 2.1 and we were able to detect several modes that satisfy triad resonance conditions. We then used such modes to investigate their nonlinear wave interactions.

We now continue with the nonlinear problem formulation and for this we will require the solutions of the constant amplitude values in Eq.(4.23) \((h_{1n}, v_{1n}, \sigma_{1n}, E_{1n})\) for \(n = 1, 2, 3\). We formulate the problem by making use of Eqs.(4.19)-(4.22) and linearizing with respect to the amplitude of the perturbation along with the amplitude function to produce the following linear systems.
$$(ik_n + s_n)v_{1n} + 2h_{1n}(f_n + i\omega_n) = 0$$  \(4.29\)

$$E_{1n}K^*(ik_n + s_n) + 2v_{1n}(ik_n\sigma_0 + s_n\sigma_0) + 2\sigma_{1n}(f_n + i\omega_n) + 2h_{1n}(f_n\sigma_0 + i\sigma_0\omega_n + ik_nK^*\Omega + K^*s_n\Omega) = 0$$  \(4.30\)

$$v_{1n} \left( f_n + 3k_n^2v^* - 6ik_nsv^* - 3s_n^2v^* + i\omega_n \right) + E_{1n} \left( \frac{-2\sigma_0}{\sqrt{\beta}} \frac{-ik_n\Omega}{4\pi} - \frac{s_n\Omega}{4\pi} \right) + \sigma_{1n}(-8ik_n\pi\sigma_0 - 8\pi s_n\sigma_0 - \frac{2\Omega}{\sqrt{\beta}}) + h_{1n} \left( ik_n^3 + s_n + 3k_n^2s_n - s_n^3 - ik_n(-1 + 3s_n^2) + \frac{2\sigma_0\Omega}{\sqrt{\beta}} \right) = 0$$  \(4.31\)

$$4i\pi(k_n - is_n)\sqrt{\beta}\sigma_{1n} \ln \left[ \frac{100}{89k_n} \right] + h_{1n}(k_n - is_n)\sqrt{\beta} \left( 4i\pi\sigma_0 + (k_n - is_n)\sqrt{\beta}\Omega \right) \ln \left[ \frac{100}{89k_n} \right] + \frac{1}{2}E_{1n}(2 + (k_n - is_n)^2\beta\ln \left[ \frac{100}{89k_n} \right]) = 0$$  \(4.32\)

Using the triad resonant modes we detected and prescribing suitable values for \((h_{1n})\) for \(n = 1, 2, 3\) according to the weakly nonlinear theory\cite{1,14} we can solve the above linear systems.

Now we set up the problem for the adjoint variables \((h_{1n}^{(a)}, v_{1n}^{(a)}, \sigma_{1n}^{(a)}, E_{1n}^{(a)})\) that are required by the related adjoint problem for the nonlinear problem. The strategy for solving for the adjoint variables is to use the inner product, \((x, y) = x^\top y\), where the overline denotes the complex conjugate. Using the inner product of the linear system formed by Eqs.(4.29)-(4.32) against \((h_{1n}^{(a)}, v_{1n}^{(a)}, \sigma_{1n}^{(a)}, E_{1n}^{(a)})\) gives the single equation (by using the left hand sides of Eqs.(4.29)-(4.32))

$$(L.H.S_{Eq.(4.29)})h_{1n}^{(\overline{a})} + (L.H.S_{Eq.(4.30)})v_{1n}^{(\overline{a})} + (L.H.S_{Eq.(4.31)})\sigma_{1n}^{(\overline{a})} + (L.H.S_{Eq.(4.32)})E_{1n}^{(\overline{a})} = 0$$  \(4.33\)

we factor Eq.(4.33) with the help of Mathematica in the following form \((adj.eq1)h_{1n} + (adj.eq2)v_{1n} + (adj.eq3)\sigma_{1n} + (adj.eq4)E_{1n} = 0\) and we simultaneously solve the four equations \((adj.eq1) - (adj.eq4)\), which are lengthy and will not be given, to zero to obtain the solution to the adjoint variables. We now have both the regular amplitude variables and the adjoint amplitude variables for \(n = 1, 2, 3\).

We now proceed with the nonlinear system of PDEs from Eqs.(4.19)-(4.22) and due to the size of the system we will work each case separately for \(n = 1, 2, 3\). Using Eqs.(4.23)-(4.24) in
the nonlinear Eqs. (4.19)-(4.22) and keeping on the right hand sides only terms that resonate\[^{14}\]

for the case \(n=1\), we obtain

\[
L.H.S_{\text{Eq.}(4.29)} = (b_1) \frac{\partial A_1}{\partial z_s} + (c_1) \frac{\partial A_1}{\partial t_s} + (d_1) \overline{A_2} A_3 e^{[(f_2 + f_3 - f_1)t + (s_2 + s_3 - s_1)z]} \tag{4.34}
\]

\[
L.H.S_{\text{Eq.}(4.30)} = (b_2) \frac{\partial A_1}{\partial z_s} + (c_2) \frac{\partial A_1}{\partial t_s} + (d_2) \overline{A_2} A_3 e^{[(f_2 + f_3 - f_1)t + (s_2 + s_3 - s_1)z]} \tag{4.35}
\]

\[
L.H.S_{\text{Eq.}(4.31)} = (b_3) \frac{\partial A_1}{\partial z_s} + (c_3) \frac{\partial A_1}{\partial t_s} + (d_3) \overline{A_2} A_3 e^{[(f_2 + f_3 - f_1)t + (s_2 + s_3 - s_1)z]} \tag{4.36}
\]

\[
L.H.S_{\text{Eq.}(4.32)} = (a_4) \frac{\partial^2 A_1}{\partial z_s^2} + (b_4) \frac{\partial A_1}{\partial z_s} + (c_4) \frac{\partial A_1}{\partial t_s} + (d_4) \overline{A_2} A_3 e^{[(f_2 + f_3 - f_1)t + (s_2 + s_3 - s_1)z]} \tag{4.37}
\]

where the overline in the amplitude function \(A_1(z_s,t_s)\) denotes the complex conjugate, coefficients \((b_i, c_i, d_i)\) for \(i = 1 \text{ to } 4\) and \(a_4\) will not be given for the above equations at this stage, but will be given once we have arrived at final set of equations for \(n = 1, 2, 3\). We now consider the Eqs. (4.34)-(4.37) for the case \(n=1\) in vector-matrix form which can be written as \(Lx = N\) with its related homogeneous adjoint problem \(L^{(a)} x^{(a)} = 0\). Here \(x\) represents the vector containing the constant amplitude variables \((h_{1n}, v_{1n}, \sigma_{1n}, E_{1n})\), \(L\) represents the coefficient matrix and \(N\) represents the right hand side vector. We then then can apply the solvability conditions\[^{13}\] for the above system. Pairing \(N\) against \(x^{(a)}\) gives the following \((N, x^{(a)}) = (Lx, x^{(a)}) = (x, L^{(a)} x^{(a)}) = (x, 0) = 0\), hence we obtain the solvability condition for Eqs. (4.34)-(4.37) for \(n=1\) as \((N, x^{(a)}) = 0\) which gives the following nonlinear PDE

\[
(a_4 \ast E^{(\pi)}_{11}) \frac{\partial^2 A_1}{\partial z_s^2} + (b_1 \ast h^{(\pi)}_{1n} + b_2 \ast v^{(\pi)}_{1n} + b_3 \ast \sigma^{(\pi)}_{1n} + b_4 \ast E^{(\pi)}_{11}) \frac{\partial A_1}{\partial z_s} + (c_1 \ast h^{(\pi)}_{1n} + c_2 \ast v^{(\pi)}_{1n} + c_3 \ast \sigma^{(\pi)}_{1n} + c_4 \ast E^{(\pi)}_{11}) \frac{\partial A_1}{\partial t_s} + (d_1 \ast h^{(\pi)}_{1n} + d_2 \ast v^{(\pi)}_{1n} + d_3 \ast \sigma^{(\pi)}_{1n} + d_4 \ast E^{(\pi)}_{11}) \overline{A_2} A_3 e^{[(f_2 + f_3 - f_1)t + (s_2 + s_3 - s_1)z]} = 0 \tag{4.38}
\]

We follow an analogous approach when \(n=2\) and \(n=3\) to find the solvability conditions for Eqs. (4.34)-(4.37). We also write Eq. (4.38) in compact form to express the system of nonlinear system of PDEs that govern the evolution of the amplitude functions \(A_1(z_s,t_s), A_2(z_s,t_s)\) and \(A_3(z_s,t_s)\).
where the coefficients \((C_i, D_i, G_i)\) for \(i = 1 \text{ to } 4\) are given on Appendix C. Eqs. \((4.39)-(4.41)\) govern the dynamics of the nonlinear wave interactions for instabilities that evolve in time and space. This concludes the weakly nonlinear wave theory for the triad resonant modes. In the next section we consider numerical approach to solve the above system and present the typical results obtained from the dyad resonant problem.

### 4.2.5 Governing PDE System for Amplitude Functions and its Solution

This section we discuss the numerical algorithm that we used to solve Eqs.(4.39)-(4.41). We implemented the Method of Lines (MOL) to solve the above system. We used a backward in time finite difference discretization for the time derivative appearing in the system and then we prescribed the initial condition along the axial direction which in turn transformed the PDE system into a system of ordinary differential equations (ODEs). We then iterate along time and allow at each time step for the solutions to evolve along the z axial direction by clamping the amplitude functions and their derivatives at the initial z axial direction. Using the appropriate slowly varying dependent variables throughout the PDE system and approximating the time derivative in Eqs.(4.39)-(4.41) we obtain

\[
(C_1) \frac{\partial^2 A_1}{\partial z_s^2} + (C_2) \frac{\partial A_1}{\partial z_s} + (C_3) \frac{\partial A_1}{\partial t_s} + (C_4) \overline{A_2} A_3 e^{[(f_2+f_3-f_1)z + (s_2+s_3-s_1)]} = 0 \tag{4.39}
\]

\[
(D_1) \frac{\partial^2 A_2}{\partial z_s^2} + (D_2) \frac{\partial A_2}{\partial z_s} + (D_3) \frac{\partial A_2}{\partial t_s} + (D_4) \overline{A_1} A_3 e^{[(f_1+f_3-f_2)z + (s_1+s_3-s_2)]} = 0 \tag{4.40}
\]

\[
(G_1) \frac{\partial^2 A_2}{\partial z_s^2} + (G_2) \frac{\partial A_2}{\partial z_s} + (G_3) \frac{\partial A_1}{\partial t_s} + (G_4) A_1 A_2 e^{[(f_1+f_2-f_3)z + (s_1+s_2-s_3)]} = 0 \tag{4.41}
\]

The Eqs.(4.42)-(4.44) are can now be treated ODEs in terms of the dependent variables \((z_s, t_s)\). Here \(t_s\) will be fixed depending on the iteration step along the time variable. In the
notation above for the time discretization we label \( A_{t0} \) as the previous known stage with respect to time and \( A_{t1} \) will represent the amplitude function \( A_1(z_s,t_s) \) for a given \( t_s \) according to the time step selected, which complies with the C.F.L condition, and the iteration process. The same is applied for the second amplitude function \( A_2(z_s,t_s) \) and \( A_3(z_s,t_s) \). The boundary conditions and initial conditions used for closing the system of PDEs and numerically solving for the amplitude functions are chosen according to the weakly nonlinear wave theory [1,14,22,24]. Using the notation described about the above amplitude functions, we now proceed to present the algorithm used to solve the Eqs.(4.39)-(4.41).

**Algorithm/Pseudocode:**

Discretize time and space according to C.F.L condition \( \Delta t/(\Delta x)^2 \leq 1/2 \)

Initialize: \( A_{t0}^1(z_s,t_s), A_{t0}^2(z_s,t_s), A_{t0}^3(z_s,t_s), A_1(0,t_s), A_2(0,t_s), A_3(0,t_s), A_1'(0,t_s), A_2'(0,t_s) \) and \( A_3'(0,t_s) \)

For \( i = 1 \) to \( n \)

Solve ODE system from Eqs.(4.42)-(4.44) using Runge-Kutta Method

Assign \( A_{t0}^i(z_s,t_s) := A_1(z_s,t_s), A_{t0}^i(z_s,t_s) := A_2(z_s,t_s) \) and \( A_{t0}^i(z_s,t_s) := A_3(z_s,t_s) \)

End

The algorithm was implemented in Mathematica Software and the solutions obtained were transformed back to the regular space and time variables. These solutions were then used to construct perturbation quantities in the form of Eq.(4.23) to provide a study and graphical representation of the nonlinear wave interactions of the triad resonant modes studied for this investigation.

### 4.3 Simulation Results and Discussion

In the present study we considered the nonlinear investigation of the combined spatial and temporal instability of electrically driven jets with finite electrical conductivity and under the presence of both uniform and non-uniform applied electric field. We consider two types of
fluids for the jet, which can be representative as those that can be used in the experimental investigation for the problem such as water-glycerol mixture and glycerol [9,10]. For such fluids, we set representative parameter values to be \( K^* = 19.60 \), \( \nu^* \) (glycerol) = 9.05384, \( \nu^* \) (water-glycerol) = 0.60764, and \( \beta = 77 \). We also include experimental and simulations values \( \sigma_0 = 0.0 \) for the constant applied field and \( \sigma_0 = 0.10 \) for the variable applied field. We also set \( \epsilon = 0.01 \) which is required to change the regular dependent variables for space and time to slowly varying and to capture the dynamics of the nonlinear wave interactions[1,14]. For this investigation we consider several different magnitudes of the applied electric field \( \Omega = 0.5, 1.0, 1.5 \) and 2.3. For the triads, we consider two cases in which one case is composed of a single real frequency function and the other case is composed from three different real frequency functions. The multiple triad mode with three functions is composed of a stationary mode and two non-stationary modes and the other triad mode contains a single non-stationary mode. The modes are directly connected to the type of real frequency \( \omega \) used according to the viscosity value of the fluid. We found that higher viscosity fluid required higher phase velocities in order to find a convergent numerical solution to the governing dispersion relation of the jet flow system.

In regards to the results for this investigation, we first consider the dispersion relation in Eq.(4.14.1) and solve this using the Newton’s Method by carrying the appropriate parameter continuation approach. The solutions of Eq.(4.14.1) are investigated for both uniform and non-uniform applied electric field. Only those solutions that satisfy the triad resonance condition will be used to perform the nonlinear study [1,14]. Solutions of the dispersion relation exhibit the Rayleigh and Conducting instabilities. In order to provide a qualitative study of the quality of the fiber production we solve for Eqs.(4.42)-(4.44) and use these amplitude functions \( A_1(z,t) , A_2(z,t) \) and \( A_3(z,t) \) to reconstruct the solution in the form of Eq.(4.23) which solves Eqs.(4.19)-(4.22) in a triad sense. This reconstruction for the solution provides a simulation of the time and space evolution process for the fiber fabrication with respect to axisymmetric type of disturbances. We will refer to these simulation plots as the perturbation plots. The results for the solution of the dispersion relation and the main results for the perturbation plots and their time and space evolution for water-glycerol mixture and glycerol jets are briefly presented in the following two sub-sections.
4.3.1 Single Triad Mode for Water-glycerol Mixture Jet

Figs. 4.1 and 4.2 represent the temporal and spatial growth rate $f$ and $s$ versus the axial wave number $k$ for the non-uniform applied electric field ($\sigma_0 = 0.1$), respectively, and for different values of $\Omega$. It can be seen from the Fig. 4.1 that the temporal growth rate $f$ undergoes to a stabilization process as the values of $\Omega$ are increased which gives rise to a classical Rayleigh instability [9,14]. For the largest value of the applied electric field the temporal instability is neutralized and for the wave number values of $k > 0.7$ the temporal growth rate sets at a stable mode. Temporal growth rate $f$ mostly decreases for increased wavenumber values. For the case of constant applied electric field, we detected a small decrease of instability for the temporal growth rate as compared to case where ($\sigma_0 = 0.1$). For the spatial growth rate $s$, we can observe from Fig. 4.2 that by increasing the magnitude of the electric field $\Omega$ the instabilities are enhanced and thus producing a Conducting instability [9,10]. The growth rates $s$ increase for larger wavenumber $k$, specially near $k = 1.0$. For the case of constant applied field, the spatial growth rates are slightly increased as compared to the non-uniform applied field.

For non-uniform applied electric field, we detected several modes of combined spatial and temporal instability that satisfy the triad resonant conditions. Fig. 4.3(a)-(b) contains the three-dimensional solutions for jet radius $h_1(z,t)$ and jet electric field $E_1(z,t)$ along with their corresponding contour plots underneath in Fig. 4.3(c)-(d). Darker regions in contour plots indicate fast transitions with decreasing magnitude and lighter regions indicate increasing magnitude. For the contour plots, the vertical axis is set to be $t$ and the horizontal axis is set to be $z$. It is important for the electric field $E_1(z,t)$ (the driving mechanism) to be increasing and for the jet radius $h_1(z,t)$ to decrease (application importance). Fig. 4.3(a)-(b) demonstrate important characteristics on an unrestricted domain for both electric field and jet radius. For this investigation, we will compare perturbation plots accounting for nonlinear wave interaction(left hand side figures) and in the absence of such interactions (right hand side figures). First we discuss the space evolution (fixing $t$), Fig. 4.3(f) shows that the jet radius (thin-line) grows as $z$ increases while the velocity drops (dotted-line), surface charge (dashed line) and electric field (thick line) decrease. The nonlinear effects considered in Fig. 4.3(e) provide changes in the
perturbation plots that allow for the a decrease in jet radius along with an electric field that increases in magnitude. Second we discuss time evolution (fixing z), Fig. 4.3(f) shows the linear instability in modified by the non-stationary triad modes to produce qualitative changes in the perturbation quantities at a slightly shorter time as shown in Fig. 4.3.(g). The non-stationary modes presented in Fig. 4.3 under the triad interaction changed significantly the trajectories for perturbation quantities into ones of interest for applications.

For uniform applied electric field, we consider another mode that satisfied the triad resonant conditions shown in Fig. 4.4. From the solutions of \( h_1(z,t) \) and \( E_1(z,t) \) it can be seen jet decreases and electric field increases except for some cross-sectional solutions paths \( (t < 2) \). These paths are to be avoided so that the investigation has relevant impact for applications. Selecting different cross-sections from Fig. 4.4(a)-(b) leads to Fig. 4.4(e)-(h). Perturbation quantities for space evolution undergo to significant changes in amplitude and direction due to nonlinear interaction and this shown with a much more reduced \( z \) axis in Fig. 4.4(e) compared to Fig. 4.4(f). For the case of time evolving perturbations, the nonlinear interaction modifies an already thinning jet (see Fig. 4.4(h)) to a thinning jet with an increased amplitude and reduced \( t \) axis by a factor of 2 (see Fig. 4.4(g)).

### 4.3.2 Multiple Triad Mode for Water-glycerol Mixture Jet

For water-glycerol fluid and \( \sigma_0 = 0.1 \), we considered multiple triad modes that included \((\omega_1 = 0, \omega_2 = k, \omega_3 = 0.5k)\) for which fluid phase velocities go from stationary, unitary and non-stationary. Results for \( h_1(z,t) \) and \( E_1(z,t) \) using the multiple triad modes are shown in Fig. 4.5.(a)-(b). The jet radius undergoes reduction expect for \( t > 3 \) and the electric field in the jet is favored for smaller values on time \( (t < 1) \). Compared to single triad mode as in Fig. 4.3 the solutions with multiple triads exhibit greater restriction on their space and time domain. For the space evolution, triad interaction allows for a more favorable mechanics specially in jet radius from thickening to thinning jet (see Fig. 4.5(e)-(f)) and for \( z > 6.8 \) the jet is reduced at an increased rate. Similarly for the time evolution on perturbation plots(see Fig. 4.5.(g)-(h)), the nonlinear interactions modifies the linear instability results and allows for a more realistic mechanism in the jet flow system.
For the case of $\sigma_0 = 0.0$, we considered another set of multiple triad modes for the lower viscosity fluid. The solution paths for $h_1(z,t)$ and $E_1(z,t)$ turn out to be more restrictive (i.e. relevant solution paths need to be chosen in more detail) when compared to the case of $\sigma_0 = 0.1$. Using Fig. 4.6.(a)-(d), we can determine cross-sections for time and space in which jet radius decreases and electric field increases. In Fig. 4.6.(e)-(g), the perturbation quantities are significantly enhanced when the nonlinear interactions are active. The reduction in the $z$ axis and modification of the jet radius and electric field provides a more interesting result with regards the space evolution. For the case of time evolution, it can be seen from Fig. 4.6(g)-(h) that even when there were no significant changes (due to the temporal growth rates that were negative), the nonlinear interactions did not affect the dynamics of interest for the jet flow system shown in Fig. 4.6.(h).

4.3.3 Single Triad Mode for Glycerol Jet

We now provide results for the higher viscosity glycerol fluid and we begin by solving the dispersion relation. Figures 4.7 and 4.8 present the temporal growth rate $f$ and the spatial growth rate $s$ versus the axial wave number for the uniform applied field ($\sigma_0 = 0.0$) for different values of $\Omega$. The temporal growth rate that is shown in Fig. 4.7 shows no variations with respect to changes in the applied electric field and this is due to the small order changes mainly in the $10^{-3}$ for $f$. The temporal growth rate decreases as wavenumber $k$ increases and there is a significant region in which $f$ is found to be stable for $k > 0.45$. For the spatial growth rate $s$, we detected two branches of instabilities and each branch evolved different with respect to changes in $\Omega$. In Fig. 4.8, we detected the classical Rayleigh type of instability which suppresses the instability for increased values in the applied electric field for $k < 0.58$. For $k > 0.58$, we detected another spatial instability branch and for $k < 0.7$ the instability is of Rayleigh type but for $k > 0.7$ $s$ is of the Conducting type, which is enhanced by increasing the strength of $\Omega$ and agrees with the linear results reported in [9]. The second branch of instability for $s$ becomes more dominant as the wavenumber increases and for larger magnitudes of the applied electric field.

For non-uniform applied field, we found single triad modes that satisfy the resonance condi-
tions for the combined spatial and temporal instabilities of glycerol and for a larger fluid phase velocity compared to water-glycerol mixture ($\omega = 25k$). It can be seen from Fig. 4.9(a)-(b) that the jet radius $h_1(z,t)$ undergoes increasing regions but then decreases. The electric field $E_1(z,t)$ is found to be increasing with no restrictions on the time or space domain. For the space evolution, the perturbation plots dominate on their non-linear interaction as shown from the linear counterpart. This includes not affecting the dynamics of an already good solution and a reduction on the $z$ space variable that indicates a high level of nonlinear wave interaction (see Fig. 4.9(e)-(g)). On the time evolution, Fig. 4.9(g)-(h) demonstrates that linear results are improved by the weakly nonlinear theory approach. The dominant effects on the glycerol fluid due to nonlinear interactions are noticeable when compared to water-glycerol mixture for the case of $\sigma_0 = 0.0$ (see Fig. 4.3).

For uniform applied field ($\sigma_0 = 0.0$), we detected oscillatory resonant modes that satisfy the triad conditions and these modes enhanced significantly the perturbation quantities but also restricted the space and time domain for which the solutions provide interest for application. The jet radius $h_1(z,t)$ and electric field $E_1(z,t)$ are shown in Fig. 4.10(a)-Fig. 4.10(b) and they both demonstrate to have favorable solution paths for which jet radius decreases and electric field increases. Combining the their contour plots and 3d dimensional plots information we can consider solution paths for both time and space evolution with regards the time and space instabilities. Space evolving instabilities are significantly enhanced with a considerable $z$ axial domain reduction when the resonant modes are considered (see Fig. 4.10(e)-(f)). For evolution in time, the resonant modes favor the realistic features of the linear results by modifying the thickening jet to a thinning jet, reducing the amplitude and direction in the velocity of the jet and generating an increasing electric field (see Fig. 4.10(g)-(h)). The effects of the triad resonant interactions become more dominant for the case of uniform of applied field than compared to non-uniform applied field as shown in Fig. 4.9.

### 4.3.4 Multiple Triad Mode for Water-glycerol Mixture Jet

For glycerol fluid and $\sigma_0 = 0.1$, we considered multiple triad modes that included ($\omega_1 = 0, \omega_2 = 25k, \omega_3 = 10k$) for which fluid phase velocities go from stationary and non-stationary
(two cases). Results for $h_1(z, t)$ and $E_1(z, t)$ using the multiple triad modes are shown in Fig. 4.11(a)-(b). The jet radius undergoes reduction except for $t > 2$ and the electric field in the jet finds the increasing paths for $(t > 1)$. Compared to single triad mode as in Fig. 4.3 the solutions with multiple triads shows a comparable unrestricted domain but for the case of lower viscosity fluid (see Fig. 4.3) the multiple triad mode is favored due to the less restricted solution path domain. For the space evolution, triad interaction with multiple modes significantly enhances the amplitude of the perturbation quantities and reduces the z axial domain as shown in Fig. 4.11(e)-(f). The jet radius is being reduced at a much faster rate and in a very short distance away from the orifice (very important for applications since this implies that less material is to be used for conducting experiments). For the case of time evolution, the perturbation quantities undergo to small changes that allow some improvement in jet radius reduction. The results from these simulations are still of significant importance since space and time evolution are studied in combined form.

For glycerol fluid and $\sigma_0 = 0.0$, we considered a multiple triad mode that satisfied the resonance condition for uniform applied field. This case when the collector plate aligns perfectly with the orifice where the jet is emitted turns out to provided more restrictions for $h_1(z, t)$ and $E_1(z, t)$ when compared to the case of non-uniform field (see Fig. 4.11(a)-(b)). Using the information about the electric field and jet radius as shown in Fig. 4.12(a)-(d), we can provide time and space evolution for the perturbation plots in which the jet flow favor the results relevant for applications. For the case of space evolution, the linear results provide a mechanism of no interest for application as the jet thickens and an electric field that decreases (see Fig. 4.12(f)). The nonlinear wave interactions of the multiple modes we consider modify those results and produce a jet that thins at a very short distance away from orifice ($z = .36$) and it does at a very high rate. The electric field and the velocity undergo oscillations that end up with very high rates of increasing values in magnitude. For the case of time evolution, we obtain moderate modification of the perturbation quantities by considering the nonlinear wave interactions but those modification are very significant. The jet radius under the linear stage undergoes to an increase in the rate of reduction which provides significant relevant change in the fluid flow properties with regards applications in concern. The cases of nonlinear wave
interaction had a more dominant in effect for the case of uniform electric field as compared to non-uniform electric field.

4.3.5 Comparison Between Triad and Dyad Resonance

Comparing the present investigation results with those obtained in [21], we find significant differences in the instabilities (amplitude, interaction and evolution) when we consider triad nonlinear wave interactions to study viscous jets induced by electric fields. Studying combined spatial and temporal instabilities provided a more natural and realistic mathematical modeling for the disturbances in the jet compared to the more restricted case (only temporal instability) reported in [21]. We found under the triad resonance investigation higher level of instabilities that lead to larger rates of increase in magnitude for the electric field $E_1(z,t)$ and jet thickness $h_1(z,t)$ to dramatically decrease at larger rates than those found under the dyad resonance investigation. This investigation demonstrates that under triad resonance much more reduced $z$ axis (distance away from orifice) and $t$ axis (time evolution on jet) are required in order to achieve the desired jet flow mechanism favored in applications.

4.4 Concluding Remarks and Future Research

The main task of this paper was to contribute in the understanding of electrically driven jets and their applications. Our goal was to continue and expand on the understanding that is available at the linear stage [9,10] and make contributions to the nonlinear stage of problem by considering combined space and time evolving instabilities due to disturbances of axysymmetric type in electrically driven jets, which could have a direct impact on the manufacturing process of electrospinning. We carried out mathematical modeling, theoretical and numerical investigation of linear combined spatial and temporal instability as well as the nonlinear interactions of those detected modes that satisfy the triad resonance conditions in electro-hydrodynamic system for axysymmetric electrically forced slender water-glycerol and glycerol jet flows with externally imposed uniform or non-uniform applied electric field.

We found new parameter regimes in which the electrospinning process can benefit by successfully approaching desired properties in the fiber production at earlier $z$-axial and $t$-axis
locations. This results in shortening the electrospinning process by reducing the amount of polymer solution needed and the time involved in the process, which can improve the productivity of the small fibers. We were able to detect nonlinear properties in our investigation that allowed a favorable change in the dynamics of the jet flow that changed from a thickening to a thinning jet. Considering the studied electrically driven jet model in a nonlinear wave theory approach under the triad resonant settings provided a modified mechanism as the one obtained from a classical linear stability theory point of view. This mechanism can be of high interest in practical applications (i.e. electrospinning) to determine necessary means to control and predict instabilities in jets in order to produce higher quality fibers.

Further investigation of nonlinear wave interaction and resonances for the corresponding modes due to the non-axisymmetric disturbances will be good a problem to consider in near future. As is evident from the well-known experiments by Taylor [7,8] that the electrically forced jets can be non-axisymmetric and whip for sufficiently large values of the strength of the electric field and further down-stream away from the jet orifice, we can expect that spatial and temporal instabilities and nonlinear interactions of the modes due to the non-axisymmetric disturbances can dominate over the axisymmetric ones in some domain in the axial direction if $\Omega$ is sufficiently large.

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Appendix C

The expressions for the coefficients \( C_n(n = 1, 2, 3, 4) \), \( D_m(m = 1, 2, 3, 4) \) and \( G_m(m = 1, 2, 3, 4) \) for Eqs.(4.39)-(4.41) are given below

\[
C_1 = \left( \frac{1}{2} \beta (E_{11} + 2h_{11} \Omega) \right) \log \left( \frac{100}{89k_1} \right) E_{11}^{(\pi)}
\]

\[
C_2 = (-v_{11}) h_{11}^{(\pi)} + \left( -\frac{E_{11} K^*}{2} - v_{11} \sigma_0 - h_{11} K^{* \Omega} \right) v_{11}^{(\pi)} + (h_{11} \left( -1 - 3(k_1 - i s_1)^2 \right) + 6(i k_1 + s_1) v^* v_{11} + 8\pi \sigma_0 \sigma_{11} + \frac{E_{11} \Omega}{4\pi} \sigma_{11}^{(\pi)} + \left( -4\pi \sqrt{\beta} (h_{11} \sigma_0 + \sigma_{11}) \right) \log \left( \frac{100}{89k_1} \right) E_{11}^{(\pi)}
\]

\[
C_3 = (-2h_{11}) h_{11}^{(\pi)} + (-h_{11} \sigma_0 - \sigma_{11}) v_{11}^{(\pi)} + (-v_{11} \sigma_{11}^{(\pi)} + (0) E_{11}^{(\pi)}
\]

\[
C_4 = -2(\{(i k_1 + s_2 + s_3) v_{13} + h_{13}(f_2 + f_3 + i \omega_1)\}) h_{11} - (i(k_1 - i(s_2 + s_3)) (E_{13} K + v_{13} \sigma_0 + h_{13} K^{* \Omega}) - \sigma - 13(f_2 + f_3 + i \omega_1) ) h_{11} - (i(k_1 - i(s_2 + s_3)) (h_{13} \sigma_0 + \sigma_{13}) \sigma_{12} - ((i k_1 + s_2 + s_3) v_{13} + h_{13}(f_2 + f_3 + i \omega_1)) \sigma_{12}^{(\pi)} + \frac{1}{4\pi \sqrt{\beta}} (E_{13}(i k_1 + s_2 + s_3) \sqrt{\beta} - 2E_{13} \sigma_0 + 4h_{13} \sigma_0 \Omega - 2\sigma_{13} \Omega) h_{11} - (6h_{13}(k_2 + i s_2)(k_1 + k_2 - i s_3)v^* h_{12} - (i k_1 + s_2 + s_3) v_{13} \sqrt{\beta} + 2(E_{13} + 4\pi(k_1 + s_2 + s_3) \sqrt{\beta} \sigma_1 - h_{13} \Omega) \sigma_{12}^{(\pi)} + (i k_1 + s_2 + s_3) \sqrt{\beta}(h_{13}(i k_1 + s_2 + s_3) \sqrt{\beta} E_{12} + (4\pi \sigma_{13} + (i k_1 + s_2 + s_3) \sqrt{\beta}(E_{13} + h_{13} \Omega)) h_{11} - 4h_{13} \pi \sigma_{12} \log \left( \frac{100}{89k_1} \right) E_{11}^{(\pi)}
\]

\[
D_1 = \left( \frac{1}{2} \beta (E_{12} + 2h_{12} \Omega) \right) \log \left( \frac{100}{89k_2} \right) E_{12}^{(\pi)}
\]

\[
D_2 = (-v_{12}) h_{12}^{(\pi)} + \left( -\frac{E_{12} K^*}{2} - v_{12} \sigma_0 - h_{12} K^{* \Omega} \right) v_{12}^{(\pi)} + (h_{12}(1 - 3(k_2 - i s_2)^2) + 6(i k_2 + s_2) v^* v_{12} + 8\pi \sigma_0 \sigma_{12} + (E_{12} \Omega)/(4\pi) \sigma_{12}^{(\pi)} + (-4\pi \sqrt{\beta} (h_{12} \sigma_0 + \sigma_{12}) \log \left( \frac{100}{89k_2} \right) E_{12}^{(\pi)}
\]

\[
D_3 = (-2h_{12}) h_{12}^{(\pi)} + (-h_{12} \sigma_0 - \sigma_{12}) v_{12}^{(\pi)} + (-v_{12} \sigma_{12}^{(\pi)} + (0) E_{12}^{(\pi)}
\]

\[
D_4 = -2(\{(i k_2 + s_1 + s_3) v_{13} + h_{13}(f_1 + f_3 + i \omega_2)\}) h_{11} + (h_{13}(i k_2 + s_1 + s_3) \sigma_{11}) h_{12} - (i k_2 + s_1 + s_3) h_{11} - (i k_2 + s_1 + s_3) E_{11} + ((i k_2 + s_1 + s_3) (E_{13} K^* + v_{13} \sigma_0 + h_{13} K^{* \Omega}) + \sigma_{13}(f_1 + f_3 + i \omega_2)) h_{11} + ((i k_2 + s_1 + s_3) (h_{13} \sigma_0 + \sigma_{13}) \sigma_{11} + ((i k_2 + s_1 + s_3) v_{13} + h_{13}(f_1 + f_3 + i \omega_2)) \sigma_{11}^{(\pi)} + \frac{1}{4\pi \sqrt{\beta}} ((E_{13}(i k_2 + s_1 + s_3) \sqrt{\beta} + 8\pi(h_{13} \sigma_0 + \sigma_{13})) E_{11} + 4\pi((k_1 + i s_1)(k_1 + k_2 - i s_3)(h_{13}(i k_2 + s_1 + s_3) + 6v^* v_{13} \sqrt{\beta} - 2E_{13} \sigma_0 + 4h_{13} \sigma_0 \Omega - 2\sigma_{13} \Omega) h_{11} + (6h_{13}(k_1 + i s_1)(k_1 + k_2 - i s_3) v^* - (i k_2 + s_1) v_{13} \sqrt{\beta} + 2(E_{13} + 4\pi(k_1 + s_2 + s_3) \sqrt{\beta} \sigma_1 - h_{13} \Omega) \sigma_{12}^{(\pi)} + (i k_1 + s_2 + s_3) \sqrt{\beta}(h_{13}(i k_1 + s_2 + s_3) \sqrt{\beta} E_{12} + (4\pi \sigma_{13} + (i k_1 + s_2 + s_3) \sqrt{\beta}(E_{13} + h_{13} \Omega)) h_{11} - 4h_{13} \pi \sigma_{12} \log \left( \frac{100}{89k_1} \right) E_{11}^{(\pi)}
\]
\[ + s_3 v_{13}) \sqrt{\beta_{11}} + 2(E_{13} + 4\pi(i k_2 + s_1 + s_3)\sqrt{\beta_{13} - h_{13}\Omega})\sigma_{12}^{(\pi)} - (k_2 - i(s_1 + s_3)) \sqrt{\beta} \\
(h_{13}(k_2 - i(s_1 + s_3)) \sqrt{\beta_{11}} + (4i\pi\sigma_{13} + (k_2 - i(s_1 + s_3)) \sqrt{\beta(E_{13} + h_{13}\Omega)})\bar{h}_{11} + 4ih_{13}\pi\sigma_{11}) \\
\log \left[ \frac{100}{89k_2} \right] E_{12}^{(\pi)} \]  
(C.8)

\[ G_1 = \left( \frac{1}{2} \beta(E_{13} + 2h_{13}\Omega) \log \left[ \frac{100}{89k_3} \right] E_{13}^{(\pi)} \right) \]  
(C.9)

\[ G_2 = -(v_{13}) h_{13}^{(\pi)} + (\frac{E_{13}K^*}{2} - v_{13}\sigma_0 - h_{13}K^*\Omega)v_{13}^{(\pi)} + (h_{13}(-1 - 3(k_1 + k_2 - is_3)^2) + 6i(k_1 + k_2) + s_3) v_{13}^* + 8\pi\sigma_0\sigma_{13} + (E_{13}\Omega)/(4\pi))\sigma_{13}^{(\pi)} - (4\pi\sqrt{\beta}(h_{13}\sigma_0 + \sigma_{13}) \log \left[ \frac{100}{89k_3} \right] E_{13}^{(\pi)} \]  
(C.10)

\[ G_3 = (-2h_{13}) h_{13}^{(\pi)} + (-h_{13}\sigma_0 - \sigma_{13})v_{13}^{(\pi)} + (v_{13})\sigma_{13}^{(\pi)} + (0)E_{13}^{(\pi)} \]  
(C.11)

\[ G_4 = -2(f_1 h_{11}h_{12} + f_2 h_{11}h_{12} + i h_{12}k_1 v_{11} + i h_{12}k_2 v_{11} + h_{12}s_1 v_{11} + h_{12}s_2 v_{11} + i h_{11}k_1 v_{12} + i h_{11}k_2 v_{12} + h_{11}s_1 v_{12} + h_{11}s_2 v_{12} + i h_{11}h_{12}\omega_1 + i h_{11}h_{12}\omega_2)h_{13}^{(\pi)} - i(E_{12}h_{11}K^*(k_1 + k_2 - i(s_1 + s_2)) + E_{11}h_{12}K^*(k_1 + k_2 - i(s_1 + s_2)) - i((ik_1 + ik_2 + s_1 + s_2)(v_{12}\sigma_{11} + v_{11}\sigma_{12}) + h_{12}((ik_1 + ik_2 + s_1 + s_2)(v_{11}\sigma_0 + h_{11}K\Omega) + \sigma_{11}(f_1 + f_2 + i(\omega_1 + \omega_2)) + h_{11}((ik_1 + ik_2 + s_1 + s_2)v_{12}\sigma_0 + \sigma_{12}(f_1 + f_2 + i(\omega_1 + \omega_2))))v_{13}^{(\pi)} + \frac{1}{4\pi\sqrt{\beta}} ((E_{11}^{(2)}(ik_1 + s_1)\sqrt{\beta} + 8E_{11}\pi(-h_{11}\sigma_0 - \sigma_{11}) + 4\pi(i k_1 + s_1)\sqrt{\beta}((-h_{11}(k_1 - is_1)^2 + 6h_{11}(ik_1 + s_1)^2v_{11}^* - v_{11}^2 + 8\pi\sigma_{11}^2) + 8h_{11}\pi(h_{11}\sigma_0 - \sigma_{11})\Omega)\sigma_{13}^{(\pi)} - (k_1 + k_2 - i(s_1 + s_2))\sqrt{\beta}(E_{12}h_{11}(k_1 + k_2 - i(s_1 + s_2))\sqrt{\beta} + E_{11}h_{12}(k_1 + k_2 - i(s_1 + s_2))\sqrt{\beta} + 4i\pi(h_{12}\sigma_{11} + h_{11}\sigma_{12}) + h_{11}h_{12}(k_1 + k_2 - i(s_1 + s_2))\sqrt{\beta}\Omega) \log \left[ \frac{100}{89k_3} \right] E_{13}^{(\pi)} \]  
(C.12)
Figure Captions

**Fig. 4.1.** The temporal growth rate $f$ versus the axial wave number $k$ and for water-glycerol mixture jet with $\Omega = 0.5$ (thin solid line), 1 (dotted line), 1.5 (dashed line) and 2.3 (thick solid line) and for non-uniform applied electric field ($\sigma_0 = 0.1$).

**Fig. 4.2.** The same as in the Fig. 4.1 but for spatial growth rate $s$ versus wavenumber $k$.

**Fig. 4.3.** Perturbation quantities $h_1$ (thin solid line), $v_1$ (dotted line), $\sigma_1$ (dashed line) and $E_1$ (thick solid line) versus the time variable $t$ and for space variable $z$ for water-glycerol mixture jet. Figs.(4.1a-4.1d) represent the numerical solution for jet radius $h_1(z,t)$ and electric field $E_1(z,t)$. For the contour plots, the vertical axis is set for time $t$ and horizontal axis for space $z$. Darker and lighter regions in the contour plots indicate decreasing and increasing respectively. Figs.(4.1e-4.1h) represent the perturbation plots when nonlinear resonant wave interactions are considered (l.h.s) and in the absence of such nonlinear interactions (r.h.s). Here $t = 5.0$ and $z = 1.8$ for third and fourth pair of sub figures respectively. Fluid parameters and modeling settings for single triad mode (unit phase velocity) $\omega(k) = k, \epsilon = 0.01, \nu^* = 0.60764, \beta = 77.0, K^* = 19.6, \sigma_0 = 0.1, k_1 = \omega_1 = 0.2, k_2 = \omega_2 = 0.3, k_3 = \omega_3 = 0.5, s_1 = 1.5962, s_2 = 1.5953, s_3 = 1.5921, f_1 = 0.3099, f_2 = 0.2911, f_3 = 0.2300$ and $\Omega = 1.0$.

**Fig. 4.4.** Same as in Fig. 4.3 but for $t = 5$ and $z = 0.041$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.3 except for uniform electric field ($\sigma_0 = 0.0$) with triads $k_1 = \omega_1 = 0.2, k_2 = \omega_2 = 0.3, k_3 = \omega_3 = 0.5, s_1 = 1.5815, s_2 = 1.5801, s_3 = 1.5756, f_1 = 0.3890, f_2 = 0.3687, f_3 = 0.3042$ and $\Omega = 1.0$.

**Fig. 4.5.** Same as in Fig. 4.3 but for $t = 1$ and $z = 6.4$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.3 with mixed triad modes ($\omega_1 = 0, \omega_2 = k, \omega_3 = 0.5k$) using $k_1 = 0.2, \omega_1 = 0.0, k_2 = 0.2, \omega_2 = 0.2, k_3 = 0.4, \omega_3 = 0.2, s_1 = 0.5892, s_2 = 1.6179, s_3 = -0.2275, f_1 = -0.2972, f_2 = 0.2624, f_3 = -0.2616$ and $\Omega = 1.5$. 
**Fig. 4.6.** Same as in Fig. 4.4 but for $t = 1$ and $z = 0.05$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.4 with mixed triad modes ($\omega_1 = 0, \omega_2 = k, \omega_3 = 0.5k$) using $k_1 = 0.3, \omega_1 = 0.0, k_2 = 0.3, \omega_2 = 0.3, k_3 = 0.6, \omega_3 = 0.3, s_1 = 0.6331, s_2 = 1.6515, s_3 = 1.0732, f_1 = -0.3346, f_2 = 0.2436, f_3 = -0.3430$ and $\Omega = 2.3$.

**Fig. 4.7.** The same as Fig. 4.1 but for the case of glycerol fluid

**Fig. 4.8.** The same as Fig. 4.2 but for the case of glycerol fluid

**Fig. 4.9.** Same as in Fig. 4.3 but for glycerol fluid ($\nu^* = 9.05384$) with $t = 0.36$ and $z = 0.4$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.3 with non-uniform electric field ($\sigma_0 = 0.1$) and $(\omega(k) = 25k)$ with triads $k_1 = 0.1, \omega_1 = 2.5, k_2 = 0.3, \omega_2 = 7.5, k_3 = 0.4, \omega_3 = 10, s_1 = 0.4607, s_2 = 0.46069, s_3 = 0.46068, f_1 = 5.5167, f_2 = 3.3436, f_3 = 1.4431$ and $\Omega = 1.0$.

**Fig. 4.10.** Same as in Fig. 4.9 but with $t = 0.01$ and $z = 0.0054$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.9 with uniform electric field ($\sigma_0 = 0.0$) and triads $k_1 = 0.1, \omega_1 = 2.5, k_2 = 0.3, \omega_2 = 7.5, k_3 = 0.4, \omega_3 = 10, s_1 = 0.4605, s_2 = 0.4605, s_3 = 0.46051, f_1 = 5.5059, f_2 = 3.3346, f_3 = 1.4348$ and $\Omega = 1.0$.

**Fig. 4.11.** Same as in Fig. 4.9 but with $t = 0.41$ and $z = 0.2$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings with non-uniform electric field ($\sigma_0 = 0.1$) and multiple triad modes ($\omega_1 = 0, \omega_2 = 25k, \omega_3 = 10k$) using $k_1 = 0.3, \omega_1 = 0, k_2 = 0.2, \omega_2 = 5, k_3 = 0.5, \omega_3 = 5, s_1 = -0.3918, s_2 = 0.46078, s_3 = 0.1843, f_1 = -0.0331, f_2 = 4.7179, f_3 = -5.8190$ and $\Omega = 2.3$.

**Fig. 4.12.** Same as in Fig. 4.11 but with $t = 1.25$ and $z = 0.2$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings with uniform electric field ($\sigma_0 = 0.0$) and multiple triad modes ($\omega_1 = 0, \omega_2 = 25k, \omega_3 = 10k$) using $k_1 = 0.3, \omega_1 = 0, k_2 = 0.2, \omega_2 = 5, k_3 = 0.5, \omega_3 = 5, s_1 = -0.3918, s_2 = 0.46078, s_3 = 0.1843, f_1 = -0.0331, f_2 = 4.7179, f_3 = -5.8190$ and $\Omega = 2.3$. 
0.2, \omega_2 = 5, k_3 = 0.5, \omega_3 = 5, s_1 = 0.21969, s_2 = 0.46045, s_3 = 0.1841, f_1 = -0.0354, f_2 = 4.7026, f_3 = -5.8284 and \Omega = 2.3.
Figure 4.1. The temporal growth rate $f$ versus the axial wave number $k$ and for water-glycerol mixture jet with $\Omega = 0.5$ (thin solid line), 1 (square-dotted line), 1.5 (dashed line) and 2.3 (thick solid line) and for non-uniform applied electric field ($\sigma_0 = 0.1$).

Figure 4.2. The same as in the Fig. 4.1 but for spatial growth rate $s$ versus wavenumber $k$. 
Figure 4.3. Perturbation quantities $h_1$ (thin solid line), $v_1$ (dotted line), $\sigma_1$ (dashed line) and $E_1$ (thick solid line) versus the time variable $t$ and for space variable $z$ for water-glycerol mixture jet. Figs. (1a-1d) represent the numerical solution for jet radius $h_1(z, t)$ and electric field $E_1(z, t)$. For the contour plots, the vertical axis is set for time $t$ and horizontal axis for space $z$. Darker and lighter regions in the contour plots indicate decreasing and increasing respectively. Figs. (4.1e-4.1h) represent the perturbation plots when nonlinear resonant wave interactions are considered (l.h.s) and in the absence of such nonlinear interactions (r.h.s). Here $t = 5.0$ and $z = 1.8$ for third and fourth pair of sub figures respectively. Fluid parameters and modeling settings for single triad mode (unit phase velocity) $\omega(k) = k$, $\epsilon = 0.01$, $\nu = 0.60764$, $\beta = 77.0$, $k_0 = 19.0$, $\nu_0 = 0.1$, $k_1 = \omega_1 = 0.2$, $k_2 = \omega_2 = 0.3$, $k_3 = \omega_3 = 0.5$, $s_1 = 1.5962$, $s_2 = 1.5957$, $s_3 = 1.5921$, $f_1 = 0.3099$, $f_2 = 0.2911$, $f_3 = 0.2300$ and $\Omega = 1.0$. 
Figure 4.4. Same as in Fig. 4.3 but for \( t = 5 \) and \( z = 0.041 \) for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.3 except for uniform electric field (\( \sigma_0 = 0 \)) with triads \( k_1 = \omega_1 = 0.2, k_2 = \omega_2 = 0.3, k_3 = \omega_3 = 0.5, s_1 = 1.5815, s_2 = 1.5801, s_3 = 1.5756, f_1 = 0.3890, f_2 = 0.3687, f_3 = 0.3042 \) and \( \Omega = 1.0 \).
Figure 4.5. Same as in Fig. 4.3 but for $t = 1$ and $z = 6.4$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.3 with mixed triad modes ($\omega_1 = 0, \omega_2 = k, \omega_3 = 0.5k$) using $k_1 = 0.2, \omega_1 = 0.0, k_2 = 0.2, \omega_2 = 0.2, k_3 = 0.4, \omega_3 = 0.2, s_1 = 0.5892, s_2 = 1.6179, s_3 = -0.2275, f_1 = -0.2972, f_2 = 0.2624, f_3 = -0.2616$ and $\Omega = 1.5$. 
Figure 4.6. Same as in Fig. 4.4 but for $t = 1$ and $z = 0.05$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.4 with mixed triad modes ($\omega_1 = 0, \omega_2 = k, \omega_3 = 0.5k$) using $k_1 = 0.3, \omega_1 = 0.0, k_2 = 0.3, \omega_2 = 0.3, k_3 = 0.6, \omega_3 = 0.3, s_1 = 0.6331, s_2 = 1.6515, s_3 = 1.0732, f_1 = -0.3346, f_2 = 0.2436, f_3 = -0.3430$ and $\Omega = 2.3$. 
Figure 4.7. The same as Fig. 4.1 but for the case of glycerol fluid.

Figure 4.8. The same as Fig. 4.2 but for the case of glycerol fluid.
Figure 4.9. Same as in Fig. 4.3 but for glycerol fluid ($\nu^* = 9.05384$) with $t = 0.36$ and $z = 0.4$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.3 with non-uniform electric field ($\sigma_0 = 0.1$) and ($\omega(k) = 25k$) with triads $k_1 = 0.1, \omega_1 = 2.5, k_2 = 0.3, \omega_2 = 7.5, k_3 = 0.4, \omega_3 = 10, s_1 = 0.4607, s_2 = 0.46069, s_3 = 0.46068, f_1 = 5.5167, f_2 = 3.3436, f_3 = 1.4431$ and $\Omega = 1.0$. 
Figure 4.10. Same as in Fig. 4.9 but with $t = 0.01$ and $z = 0.0054$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings same as in Fig. 4.9 with uniform electric field ($\sigma_0 = 0.0$) and triads $k_1 = 0.1, \omega_1 = 2.5, k_2 = 0.3, \omega_2 = 7.5, k_3 = 0.4, \omega_3 = 10, s_1 = 0.4605, s_2 = 0.4605, s_3 = 0.46051, f_1 = 5.5059, f_2 = 3.3346, f_3 = 1.4348$ and $\Omega = 1.0$. 
Figure 4.11. Same as in Fig. 4.9 but with $t = 0.41$ and $z = 0.2$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings with non-uniform electric field ($\sigma_0 = 0.1$) and multiple triad modes ($\omega_1 = 0, \omega_2 = 25k, \omega_3 = 10k$) using $k_1 = 0.3, k_2 = 0.2, \omega_2 = 5, k_3 = 0.5, \omega_3 = 5, s_1 = -0.3918, s_2 = 0.46078, s_3 = 0.1843, f_1 = -0.0331, f_2 = 4.7179, f_3 = -5.8190$ and $\Omega = 2.3$. 
Figure 4.12. Same as in Fig. 4.11 but with $t = 1.25$ and $z = 0.2$ for 3rd and 4th pair of subfigures respectively. Fluid parameters and modeling settings with uniform electric field ($\sigma_0 = 0.0$) and multiple triad modes ($\omega_1 = 0, \omega_2 = 25k, \omega_3 = 10k$) using $k_1 = 0.3, \omega_1 = 0, k_2 = 0.2, \omega_2 = 5, k_3 = 0.5, \omega_3 = 5, s_1 = 0.21969, s_2 = 0.46045, s_3 = 0.1841, f_1 = -0.0354, f_2 = 4.7026, f_3 = -5.8284$ and $\Omega = 2.3$. 
Bibliography


CHAPTER 5. GENERAL CONCLUSIONS

5.1 General Discussion

We conducted an investigation of electrically driven charged viscous jets with finite electrical conductivity in the presence of either a uniform or non-uniform externally applied electric field. The previous available theoretical and computational results for the electrically driven jets were limited to either linear instability cases or temporal instability alone of some modes with dyad resonance conditions. We were interested in contributing in the more realistic nonlinear regime for this problem with spatial as well as combined spatial and temporal instabilities involved since no results for such cases, which could enhance our understanding of the instability mechanisms that involve in electrically forced jets, were available prior to our investigation. Beside gaining fundamental and basic research understanding about such jet flow instabilities, understanding such instabilities and their roles, in particular, to reduce the jets radius are known to be important in applications to electrospinning. Electrospinning is a technology that uses electric field to produce and control high quality and very small fibers up to nanoscales. We investigated and treated the problem with mathematical modeling, theoretical fluid mechanics and applied mathematics.

We studied the nonlinear spatial instability, nonlinear combined temporal and spatial instabilities in a dyad resonant sense and also nonlinear combined temporal and spatial instabilities in a triad resonance sense of electrically driven jets and their applications. The results that we found revealed new operating modes in which instabilities can be significantly enhanced by the nonlinear wave interactions. The interactions were found to mostly amplify the amplitude of perturbations but allow for the electric field in the jet to grow while allowing for jet thickness to reduce at a higher rate in a shorter time and axial location.
Investigating the cases of combined temporal and spatial instabilities provided a more realistic study since both instabilities naturally exist in experiments. The investigations of dyad and triad resonant nonlinear instability modes in electrically driven jets extend to the available results at the linear stage, contributes at the nonlinear stage for the problem and also improves the understanding to the practical applications of the problem as in the case of electrospinning. These modes provided new and important mechanisms and operating modes very different from those available based on linear theory. The nonlinear resonant instability results are of high interest to the problem of axisymmetric electrically driven jets and their applications to the electrospinning process.

5.2 Recommendation for Future Research

Future work on the area of electrically driven jets should include a natural extension to the area of computational fluid mechanics and it will be of interest to investigate and compare the results performed on this investigation to the results that could be attained numerically by first treating the main modeling equations as a system of conservation laws. The results that are found in this investigation could then be modeled by a numerical method that allows for unrestricted amplitudes in the perturbation quantities. Another problem of interest will be to consider non-axisymmetric disturbances in the jet flow growing both in time and space, which are known to exist far away from jet orifice and for sufficiently large values of the electric field.