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SOME STABILITY PROPERTIES OF LINEAR OPERATORS IN BANACH SPACES

by

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NOTATION

A, B, T, U, V: Linear operators.

X, Y: Complex Banach spaces.

D(A): The domain of A.

R(A): The range of A.

N(A): The set of all x in D(A) such that $Ax = 0$.

$\alpha(A)$: The dimension of N(A).

$\beta(A)$: The dimension of Y/R(A).

dim M: The dimension of the linear manifold M.

γ: $1/\|B^{-1}\|$.

E(U,V): The set of all x_0 in X for which there is a sequence having the property that $Ux_{n+1} = Vx_n$.

S: The collection of all T with norm less than $\gamma/2$.

E: $\bigcap_{T\in S} E(B,T)$.

M_0: X.

M_n: $V^{-1}UM_{n-1}$.

J(U,V): $\bigcap_{n=0}^{\infty} M_n$.

K(U,V): $\bigcap_{n=0}^{\infty} UM_n$.

I. INTRODUCTION

In this paper, we shall consider a closed linear operator A with domain $D(A)$ contained in a complex Banach space X and range $R(A)$ contained in a complex Banach space Y. We shall denote the null space of A by $N(A)$, the dimension of $N(A)$ by $\alpha(A)$, and the dimension of $Y/R(A)$ by $\beta(A)$. For the special case when the domain and range of A are in the same space (i.e. $Y = X$), Gohberg (2) has shown that if Γ is any connected open set contained in the open set $\{\lambda : \alpha(A-\lambda I)<\infty$ and $R(A-\lambda I)$ is closed$\}$ and if $\Gamma_o = \{\lambda \epsilon \Gamma : \alpha(A-\lambda I) = \alpha_o\}$ where $\alpha_o = \min\{\alpha(A-\lambda I) : \lambda \epsilon \Gamma\}$ then $\{\lambda \epsilon \Gamma : \lambda \notin \Gamma_o\}$ has no accumulation point in Γ. Markus (10) has shown that $\bigcap_{n=1}^{\infty} R((A-\lambda I)^n)$ is independent of λ in Γ_o. For λ_o in Γ_o, he has also shown that if $y \in \bigcap_{n=1}^{\infty} R((A-\lambda_o I)^n)$ then $y \in R(A-\lambda I)$ for all λ in Γ_o and if $y \notin \bigcap_{n=1}^{\infty} R((A-\lambda_o I)^n)$ then $\{\lambda \epsilon \Gamma_o : y \epsilon R(A-\lambda I)\}$ has no accumulation point in Γ_o.

We would like to study in more detail and in a more general setting the phenomenon which occurs at the points of Γ_o as distinguished from the other points of Γ. A characterization of the points of Γ_o which does not explicitly involve $\alpha(A-\lambda I)$ is an easy consequence of the work of Homer (6) and Graber (5). For each λ_o in Γ, there exists a closed linear operator B, which is a restriction of A, such that $R(B-\lambda_0 I) = R(A-\lambda_0 I)$ and $(B-\lambda_0 I)^{-1}$ exists and is continuous. The characterization is that $\lambda_0 \varepsilon \Gamma_0$ if and only if $R(B-\lambda I) = R(A-\lambda I)$ for all λ in some neighborhood of λ_0.

We proceed to investigate the problem of when $R(B-\lambda I) = R(A-\lambda I)$ for all λ in some neighborhood of λ_0 where λ_0 can be taken to be zero without any loss of generality. This study is independent of any consideration of the dimension of $N(A-\lambda I)$. Another appropriate generalization would be to consider the possibility of Y not being equal to X. Thus we can no longer consider the identity operator so we shall replace it by a closed linear operator T such that $D(T) \supset D(A)$. If $D_0(A)$ is taken to be D(A) with the new norm $\|x\|_A = \|x\| + \|Ax\|$, then $D_0(A)$ is a Banach space because A is closed. It is obvious that the operator A_0 induced on $D_0(A)$ by A is continuous. Now the operator T_0 defined on $D_0(A)$ by $T_0 x = Tx$ is clearly a linear operator with range in Y. If we let $\{x_n\} \subset D_0(A)$ be a sequence such that $\|x_n - x\|_A \to 0$ and $\|T_0 x_n - y\| \to 0$, then $\|x_n - x\| \to 0$ and $\|Tx_n - y\| \to 0$. Since T is closed, $Tx = y$ and hence $T_0 x = y$. Therefore, T_0 is a closed linear operator defined on all of the Banach space $D_0(A)$ and hence, by the Closed Graph

Theorem, T_0 is continuous. Thus there is no loss of generality if we consider A and T to be continuous linear operators defined on all of X with ranges in Y.

In consideration of this background, the object of study in Chapter II is the determination of when $R(B-T) = R(A-T)$ for all T with sufficiently small norm where B is a closed linear operator such that B is one-to-one, $R(B) = R(A)$, and $Bx = Ax$ for all x in D(B). We consider the linear operator $(B-T)B^{-1}$ which is a mapping of R(A) into R(A-T). Now $R[(B-T)B^{-1}] = R(B-T)$ and hence if $(B-T)B^{-1}$ maps R(A) onto R(A-T) then $R(B-T) = R(A-T)$. We show that if A has closed range then R(B-T) is closed for any T with sufficiently small norm. Goldman (4) has shown that unless $\alpha(A) < \infty$ or $\beta(A) < \infty$ then there exists an operator T with arbitrarily small norm such that R(A-T) is not closed. Thus, for such a T, R(B-T) could not equal R(A-T). Because of this, we temporarily shift our attention to the consideration of a fixed continuous linear operator T with $D(T) = X$ and consider the operator $(B-\lambda T)B^{-1}$ which maps R(A) into $R(A-\lambda T)$. We show that if R(A) is closed then $(B-\lambda T)B^{-1}$ is a closed, continuous, linear operator. Also, we show that there exists a neighborhood of zero such that $(B-\lambda T)B^{-1}$ is one-to-one and relatively open for all λ in this neighborhood. For two linear operators U and V with domains in X and ranges in Y, we define a linear manifold E(U,V) which consists of all x_0 in X for which there exists a sequence $\{x_n\}$ in X having the property that $Ux_{n+1} = Vx_n$. We then show that $R(B-\lambda T) = R(A-\lambda T)$ for all λ in some neighborhood of zero if and only if $N(A) \subset E(A,T)$. We conclude the chapter with the result that $R(B-T) = R(A-T)$ for all T of sufficiently small norm if and only if $\alpha(A) = 0$ or $\beta(A) = 0$. In proving this result we employ a set E which consists of all those elements common to E(B,T) for every T of sufficiently small norm.

In Chapter III, we consider the linear manifold J(U,V) employed by Graber (5) and Kato (7). We show that E(U,V) is not always equal to J(U,V) and then give conditions under which they are the same.

In Chapter IV, we generalize some results of Graber (5) and Kato (7). Graber (5) has shown that if T(N(A)) is closed and strongly disjoint from R(A), then there exists a neighborhood of zero such that $R(A-\lambda T)$ is closed for all λ in this neighborhood. Kato (7) has shown that if N(A) is contained in J(A,T) then there exists a neighborhood of zero such that $R(A-\lambda T)$ is closed for all λ in this neighborhood. We show that, if $N(A) = L \oplus M$ where L and M are closed linear manifolds, $TM \subset R(A)$, TL and $R(A)$ are strongly disjoint, and $T(N(A))$ is closed, then there exists a neighborhood of zero such that $R(A-\lambda T)$ is closed for all λ in this neighborhood provided that $M \subset J(A,T)$ or the dimension of M is finite.

II. STABILITY PROPERTIES

In this chapter, we shall let A be a continuous linear operator defined on the complex Banach space X with range in the complex Banach space Y. We shall let B be a closed linear operator such that B is one-to-one, $R(B) = R(A)$, and $Bx = Ax$ for all x in $D(B)$.

For a continuous linear operator T defined on X with range in Y, we wish to determine sufficient conditions for the linear operator $(B-T)B^{-1}$ to be a one-to-one, closed, continuous, relatively open mapping of $R(A)$ onto $R(A-T)$. The definition of a relatively open operator is given below in Definition 2.1.

We know that, if $(B-T)B^{-1}$ is to be closed and continuous, then its domain which is $R(A)$ must be closed. Hence we shall assume that A has closed range.

Definition 2.1: A linear operator U defined on a Banach space X is said to be relatively open if and only if for every open set $G \subset X$ there exists an open set $H \subset Y$ such that $UG = H \cap R(U)$.

If the range of U is closed, then it is a Banach space. Hence the following lemma is a consequence of the Interior Mapping Principle which can be found in Dunford and Schwartz

(1, p. 55) as well as in a number of other books.

Lemma 2.2: Let U be a continuous linear operator defined on X with range in Y. If R(U) is closed, then U is relatively open.

The following lemma is the converse of Lemma 2.2 and was proved by the author before discovering that its proof is a consequence of a number of results of Kelley and Namioka (8). The proof is included here because of its concise nature.

Lemma 2.3: Let U be a continuous linear operator defined on X with range in Y. If U is relatively open, then R(U) is closed.

Proof: Let $X' = X/N(U)$ and let σ be the canonical mapping of X onto X'. Since U is continuous with $D(U) = X$, U is closed and hence N(U) is closed. Thus X' is a Banach space with norm $\|x'\| = \inf \{ \|x+t\| : t \in N(U) \}$. Clearly σ is continuous since $\|\sigma x\| = \|x'\| = \inf \{ \|z\| : z \in x' \} \leq \|x\|$.

We shall now let the linear operator U' be defined by $U'x' = Ux$ where x belongs to x' and show that U' is closed. Suppose that $\{x'_n\}$ is a sequence in X' which converges to x' and suppose that $\{U'x'_n\}$ converges to y. Since $x'_n \to x'$, we have that $\| x_n' - x' \| = \| (x_n - x)' \| \to 0$. For each n, $\| (x_n - x)' \| = \inf \{ \| x_n - x + t \| : t \in N(U) \}$ and thus, for each n, there exists a t_n in N(U) such that $\| x_n - x + t_n \| < 2 \| (x_n - x)' \|$. Therefore, $\| x_n - x + t_n \|$ converges to zero as $n \to \infty$. Hence $x_n + t_n \to x$ and $U(x_n + t_n) = Ux_n = U'x_n' \to y$. Since U is closed, we have that x belongs to D(U) and so x' belongs to D(U'). Also Ux = U'x' = y. Hence U' is closed. Since U' is closed and $(U')^{-1}$ exists, we have that $(U')^{-1}$ is also closed.

Now let G' be any open set in X'. Since σ is continuous, we have that $G = \{ x \in X : \sigma x \in G' \}$ is open in X. Now U is relatively open and hence UG = U'G' is a relatively open set in Y. Hence U' is relatively open. Thus, since $(U')^{-1}$ exists, we have that $(U')^{-1}$ is continuous. Now $(U')^{-1}$ is closed and continuous and, therefore, its domain which is R(U') = R(U) is closed.

We now combine Lemma 2.2 and Lemma 2.3 in the following theorem.

Theorem 2.4: Let U be a continuous linear operator defined on X with range in Y. Then U is relatively open if and only if R(U) is closed.

Remark 2.5: B is a restriction of a continuous linear operator and hence B is continuous. Since B is closed and one-to-one, B^{-1} exists and is closed. Thus, since B has closed range, B^{-1} is continuous.

Definition 2.6: We define $\gamma = 1/\|B^{-1}\|$.

Theorem 2.7: $(B-T)B^{-1}$ is closed and continuous. Furthermore, if $\|T\| < \gamma/3$, then $(B-T)B^{-1}$ is one-to-one and relatively open.

Proof: Since B and T are continuous, B - T is continuous. As pointed out in Remark 2.5, B^{-1} is continuous and hence $(B-T)B^{-1}$ is continuous. Since the domain of $(B-T)B^{-1} = R(A)$ is closed, we have that $(B-T)B^{-1}$ is closed.

From Definition 2.6, it follows that $\|Bx\| \geq \gamma \|x\|$ for all x in D(B). Hence, if $\|T\| < \gamma/3$, then $\|(B-T)x\| = \|Bx-Tx\| \geq \|Bx\| - \|Tx\| > (\gamma-\gamma/3)\|x\| = (2\gamma/3)\|x\|$ for all x in $D(B-T) = D(B)$. Hence B - T is one-to-one and has closed range whenever $\|T\| < \gamma/3$. Thus B - T and B^{-1} are both one-to-one and hence $(B-T)B^{-1}$ is one-to-one whenever $\|T\| < \gamma/3$. Now $R[(B-T)B^{-1}] = R(B-T)$ and, therefore, $(B-T)B^{-1}$ has closed range for all T satisfying $\|T\| < \gamma/3$. Thus by Theorem 2.4, $(B-T)B^{-1}$ is relatively open whenever $\|T\| < \gamma/3$.

From Goldman (4), we know that unless $\alpha(A) < \infty$ or

$\beta(A) < \infty$ there exists a T satisfying $\| T \| < \gamma/3$ such that $R(A-T)$ is not closed. Hence, for such a T, $(B-T)B^{-1}$ cannot map $R(A)$ onto $R(A-T)$.

We do not wish to restrict ourselves to the condition that either $\alpha(A)$ or $\beta(A)$ is finite. Hence we will select some fixed continuous linear operator T defined on X and consider the operator $(B-\lambda T)B^{-1}$. From the proof of Theorem 2.7, we clearly see that if $R(A)$ is closed then $(B-\lambda T)B^{-1}$ is closed and continuous. Furthermore, for all λ satisfying $|\lambda| \; \| T \| < \gamma/3$, we have that $(B-\lambda T)B^{-1}$ is one-to-one and relatively open.

We shall now determine a necessary and sufficient condition for $(B-\lambda T)B^{-1}$ to be a mapping from $R(A)$ onto $R(A-\lambda T)$ for all λ satisfying $|\lambda| < \delta$ where δ is a positive constant.

Definition 2.8: For two linear operators U and V with domains in X and ranges in Y, we define the linear manifold $E(U,V) = \{x \in X:$ there exists a sequence $\{x_n\}$ in X such that $Ux_n = Vx_{n-1}$ where $x_0 = x\}$.

We note that such a sequence $\{x_n\}$ is contained in $E(U,V)$.

Definition 2.9: We define $F(U,V) = UE(U,V)$.

Theorem 2.10: If $N(A) \subset E(A,T)$, then $E(B,T) = E(A,T)$.

Proof: Clearly $E(B,T) \subset E(A,T)$. Let $x_0 \in E(A,T)$. Then there exists a sequence $\{u_{0,n}\}$ such that $Au_{0,n+1} = Tu_{0,n}$ where $u_{0,0} = x_0$. Clearly $TE(A,T) \subset R(A) = D(B^{-1})$. Let $x_1 = B^{-1}Tx_0$ and let $u_{1,1} = x_1 - u_{0,1}$. Thus $Au_{1,1} = Ax_1 - Au_{0,1} = Bx_1 - Tu_{0,0} = Tx_0 - Tx_0 = 0$. Now $u_{1,1} = x_1 - u_{0,1} \in E(A,T)$ so there exists a sequence $\{u_{1,n}\}$, $n = 1, 2, 3, \ldots$, such that $Au_{1,n+1} = Tu_{1,n}$. Now $x_1 = u_{0,1} + u_{1,1} \in E(A,T)$ and, therefore, $Tx_1 \in D(B^{-1})$. We shall now complete the proof by induction. Assume that for $k \geq 1$ there exists a set $\{x_0, x_1, x_2, \ldots, x_k\}$ and for each x_j, $j = 1, 2, 3, \ldots, k$, there exists a sequence $\{u_{j,n}\}$, $n = j, j+1, j+2, \ldots$, such that $Au_{j,n+1} = Tu_{j,n}$. Also $u_{j,j} = x_j - \sum_{i=0}^{j-1} u_{i,j}$, $Bx_j = Tx_{j-1}$, and $Tx_k \in D(B^{-1})$. We shall now let $x_{k+1} = B^{-1}Tx_k$ and show that $Tx_{k+1} \in D(B^{-1})$. Define $u_{k+1,k+1} = x_{k+1} - \sum_{i=0}^{k} u_{i,k+1}$. Thus $Au_{k+1,k+1} = Ax_{k+1} - \sum_{i=0}^{k} Au_{i,k+1} = Tx_k - \sum_{i=0}^{k} Tu_{i,k} = T(x_k - \sum_{i=0}^{k-1} u_{i,k}) - Tu_{k,k} = Tu_{k,k} - Tu_{k,k} = 0$. Therefore, $u_{k+1,k+1} \in E(A,T)$ and so does $x_{k+1} = u_{k+1,k+1} + \sum_{i=0}^{k} u_{i,k+1}$. Hence $Tx_{k+1} \in D(B^{-1})$ and we let $x_{k+2} = B^{-1}Tx_{k+1}$. Thus we have by induction that there exists a sequence $\{x_n\}$ such that $Bx_{n+1} = Tx_n$ and, therefore, $x_0 \in E(B,T)$. Hence $E(A,T) = E(B,T)$.

<u>Theorem 2.11</u>: If $N(A) \subset E(A,T)$, then $T(N(A)) \subset R(B-\lambda T)$ for every λ satisfying $|\lambda| \| B^{-1}T \| < 1$.

Proof: By Theorem 2.10, $N(A) \subset E(B,T)$. Let $x \in N(A)$. Then there exists a sequence $\{x_n\}$ such that $Bx_{n+1} = Tx_n$ where $x = x_0$. Let $u_\lambda = \sum_{j=1}^{\infty} \lambda^{j-1} x_j$ which converges since $|\lambda| \| B^{-1}T \| < 1$. Thus $u_\lambda \in D(B)$ since $D(B)$ is closed. Now $(B-\lambda T)u_\lambda = (B-\lambda T)\sum_{j=1}^{\infty} \lambda^{j-1}x_j = Bx_1 + \sum_{j=2}^{\infty} \lambda^{j-1}Bx_j - \sum_{j=1}^{\infty} \lambda^{j}Tx_j =$ $Bx_1 + \sum_{j=1}^{\infty} \lambda^{j}Bx_{j+1} - \sum_{j=1}^{\infty} \lambda^{j}Tx_j = Bx_1 = Tx_0 = Tx$. Hence $Tx \in$ $R(B-\lambda T)$ and, therefore, $T(N(A)) \subset R(B-\lambda T)$.

<u>Theorem 2.12</u>: If $N(A) \subset E(A,T)$, then $R(A-\lambda T) = R(B-\lambda T)$ for every λ satisfying $|\lambda| \| B^{-1}T \| < 1$.

Proof: Clearly $R(B-\lambda T) \subset R(A-\lambda T)$ so let $y \in R(A-\lambda T)$. Then there exists an x in X such that $(A-\lambda T)x = y$. Now $x = z + u$ where $z \in N(A)$ and $u \in D(B)$. By Theorem 2.11, $T(N(A)) \subset R(B-\lambda T)$ and hence there exists a u_1 in $D(B)$ such that $(B-\lambda T)u_1 = Tz$. Thus $(B-\lambda T)(u-\lambda u_1) = (B-\lambda T)u -$ $\lambda(B-\lambda T)u_1 = (B-\lambda T)u + (A-\lambda T)z - (A-\lambda T)z - \lambda(B-\lambda T)u_1 =$ $(A-\lambda T)x - (A-\lambda T)z - \lambda Tz = y + \lambda Tz - \lambda Tz = y$. Hence

$y \in R(B-\lambda T)$ and, therefore, $R(A-\lambda T) = R(B-\lambda T)$.

Theorem 2.13: If there exists a sequence $\{\lambda_j\}$ such that $\lambda_j \to 0$, $\lambda_j \neq 0$ for every j, and if $y \in R(B-\lambda_j T)$ for all j, then $y \in F(B,T)$.

Proof: By deleting a finite number of terms, we may assume that $|\lambda_j| \, \| B^{-1}T \| < 1/2$. Let $\{x_j\}$ be a sequence in $D(B)$ such that $(B-\lambda_j T)x_j = y$. Therefore, $Bx_j = \lambda_j Tx_j + y$ and hence $x_j = B^{-1}(\lambda_j Tx_j + y)$. Thus $\| x_j \| \leq \| B^{-1} \| (|\lambda_j| \, \| T \| \, \| x_j \| + \| y \|) \leq \| x_j \| /2 + \| B^{-1} \| \, \| y \|$ or $\| x_j \| \leq 2 \| B^{-1} \| \, \| y \|$. Since $\| x_j \|$ is bounded, $Bx_j \to y$ in $R(B)$. Let $u_0 = B^{-1}y$. Then $B[(x_j - u_0)/\lambda_j] = Tx_j + y/\lambda_j - y/\lambda_j = Tx_j$ or $[(x_j - u_0)/\lambda_j] = B^{-1}Tx_j$. Since $\| B^{-1}Tx_j \|$ is bounded, we have that $x_j \to u_0$. We shall now assume the induction hypothesis that $u_0, u_1, u_2, \ldots, u_n$ satisfy $Bu_0 = y$, $Bu_k = Tu_{k-1}$, $k = 1, 2, 3, \ldots, n$, and $\lim_{j\to\infty} w_{j,n} = u_n$ where $w_{j,n} = (1/\lambda_j)^n \, [x_j - \sum_{k=0}^{n-1} (\lambda_j)^k \, u_k]$. Then $Bw_{j,n+1} = Tw_{j,n}$ so $w_{j,n+1} = B^{-1}Tw_{j,n} \to u_{n+1}$ as $j \to \infty$ and $Bu_{n+1} = Tu_n$ since B and T are continuous. Therefore, by induction $u_0 \in E(B,T)$ and $y = Bu_0 \in BE(B,T) = F(B,T)$.

Theorem 2.14: If there exists a sequence $\{\lambda_j\}$ where $\lambda_j \to 0$, $\lambda_j \neq 0$ for every j, such that $R(A-\lambda_j T) = R(B-\lambda_j T)$, then $N(A) \subset E(A,T)$.

Proof: Let $x \in N(A)$. Then $(A-\lambda_j T)(-x/\lambda_j) = Tx \in R(A-\lambda_j T) = R(B-\lambda_j T)$. Thus by Theorem 2.13, we have that $Tx \in F(B,T) = BE(B,T)$. Hence there exists an x_0 in $E(B,T)$ such that $Tx = Bx_0$ and there exists a sequence $\{x_n\}$ such that $Bx_{n+1} = Tx_n$. Let $x = z_0$ and $x_i = z_{i+1}$, $i = 0, 1, 2, \ldots$. Thus we have a sequence $\{z_n\}$ such that $Bz_{n+1} = Tz_n$. Therefore, $z_0 = x$ is in $E(B,T)$. Hence $N(A) \subset E(B,T) \subset E(A,T)$.

Theorem 2.15: There exists a constant $\delta > 0$ such that $(B-\lambda T)B^{-1}$ is a mapping of $R(A)$ onto $R(A-\lambda T)$ for all λ satisfying $|\lambda| < \delta$ if and only if $N(A) \subset E(A,T)$.

Proof: Suppose that there exists a constant $\delta > 0$ such that $R(A-\lambda T) = R(B-\lambda T)$ for all λ satisfying $|\lambda| < \delta$. Then clearly there exists a sequence $\{\lambda_j\}$ such that $\lambda_j \to 0$, $\lambda_j \neq 0$ for every j, and λ_j satisfies $|\lambda_j| < \delta$. Hence $R(A-\lambda_j T) = R(B-\lambda_j T)$ and thus, by Theorem 2.14, we have that $N(A) \subset E(A,T)$. The rest of the proof follows from Theorem 2.12.

We shall now return to the case where T is an arbitrary continuous linear operator defined on X with range in Y.

Definition 2.16: We define $S = \{ T : \|T\| < \gamma/2 \}$.

Definition 2.17: We define $E = \bigcap_{T \in S} E(B,T)$.

<u>Theorem 2.18</u>: $N(A) \subset E$ if and only if $\alpha(A) = 0$ or $\beta(A) = 0$.

Proof: Suppose that $\alpha(A) = 0$. Then clearly $N(A) \subset E(B,T)$ for all T in S. Hence if $\alpha(A) = 0$ then $N(A) \subset E$.

Now suppose that $\beta(A) = 0$. Let T be any member of S and let $x_0 \in N(A)$. Then $Tx_0 \in Y = R(B)$ and thus there exists an x_1 in X such that $Bx_1 = Tx_0$. Now assume the induction hypothesis that there exists an x_k in $X = D(T)$ such that $Bx_k = Tx_{k-1}$. Now $Tx_k \in R(B)$ and hence there exists an x_{k+1} in X such that $Bx_{k+1} = Tx_k$. Hence by induction $x_0 \in E(B,T)$ and, therefore, $N(A) \subset E(B,T)$ for all T in S. Thus, if $\beta(A) = 0$, we have that $N(A) \subset E$.

We now suppose that $\alpha(A) > 0$ and $\beta(A) > 0$. Let z be a non-zero element in N(A) and let y be any element in Y which does not belong to R(A). There exists a linear manifold M such that $X = M \oplus [z]$ where [z] is the linear span of z. Each element x in X may be written as $x = u + \lambda z$ where u is in M and λz is in [z]. Define T by $Tx = T(u+\lambda z) = Tu + \lambda Tz = Au + \lambda y$. Hence $R(T) = R(A) \oplus [y]$. Now $Tz = y \notin R(A) = R(B)$ and thus N(A) is not contained in E(B,T) for any such B.

The proof is now complete.

<u>Theorem 2.19</u>: $R(A-T) = R(B-T)$ for every T in S if and only if $\alpha(A) = 0$ or $\beta(A) = 0$.

Proof: We first suppose that $R(A-T) = R(B-T)$ for every T in S. Now let T be any member of S. Then $(1/j)T \in S$ for every j and hence $R(A-T/j) = R(B-T/j)$. Thus, by Theorem 2.14, $N(A) \subset E(A,T) = E(B,T)$. Since T was arbitrary, $N(A) \subset E$. Hence, by Theorem 2.18, $\alpha(A) = 0$ or $\beta(A) = 0$.

Now suppose that $\alpha(A) = 0$ or $\beta(A) = 0$. Then it follows from Theorem 2.18 that $N(A) \subset E$. Thus for every T in S we have that $N(A) \subset E(B,T)$. Hence $N(A) \subset E(A,T)$. Now by Theorem 2.12 we have that $R(A-\lambda T) = R(B-\lambda T)$ for every T in S whenever λ satisfies $|\lambda| \|T\| < \gamma$ or $|\lambda| < 2$ and in particular for $\lambda = 1$.

III. A COMPARISON OF E(U,V) WITH J(U,V)

In this chapter, we shall let U and V be two linear operators with domains in X where $D(U) \subset D(V)$ and with ranges in Y. Instead of E(U,V), Kato (7) and Graber (5) have employed a similar linear manifold J(U,V) which is defined below. We shall show by means of an example that these two linear manifolds are not always equal. We shall also give conditions under which they are equal.

Kato (7) and Graber (5) have used the following two definitions.

<u>Definition 3.1</u>: $M_0 = X$ and $M_n = M_n(U,V) = V^{-1}(UM_{n-1})$, for $n = 1, 2, 3, \ldots$, where $V^{-1}(G) = \{x \in X : Vx \in G\}$.

<u>Definition 3.2</u>: $J(U,V) = \bigcap_{n=0}^{\infty} M_n$ and $K(U,V) = \bigcap_{n=0}^{\infty} UM_n$.

We shall now give an example where E(U,V) is not equal to J(U,V).

<u>Example 3.3</u>: Let $X = Y = \ell_1$ and let $V = I$. Denote the elements in the space by $x = (x(1), x(2), x(3), \ldots)$. Define U by $U(x) = (y(1), y(2), y(3), \ldots)$ where $y(1) = \sum_{n=1}^{\infty} x(2^n)$, $y(2^n-1) = 0$, $n = 2, 3, 4, \ldots$, and $y(2^n+j-1) = x(2^n+j)$, $n = 1, 2, 3, \ldots$, $j = 1, 2, 3, \ldots, 2^n - 1$. Then

$$\|y\| = \sum_{k=1}^{\infty} |y(k)| = |y(1)| + \sum_{k=2}^{\infty} |y(k)| \leq \sum_{k=1}^{\infty} |x(k)| = \|x\| .$$

Hence U is a continuous linear operator with $D(U) = X$ and $\|u\| = 1$. It follows that U is also closed. Now $E(U,I) = \{0\}$ but clearly $(1, 0, 0, 0, \ldots)$ is in $J(U,V)$.

<u>Theorem 3.4</u>: $E(U,V) = J(U,V)$ if and only if $U(D(U) \cap J(U,V)) = K(U,V)$.

Proof: Assume that $E(U,V) = J(U,V)$. Clearly $U(D(U) \cap J(U,V)) \subset K(U,V)$ so let $y \in K(U,V)$. Now $J(U,V) = \bigcap_{n=0}^{\infty} M_n = \bigcap_{n=1}^{\infty} M_n = \bigcap_{n=1}^{\infty} V^{-1}(UM_{n-1}) = V^{-1}(\bigcap_{n=0}^{\infty} UM_n) = V^{-1}K(U,V)$ and hence $VJ(U,V) = K(U,V)$. Thus there exists an x in $J(U,V)$ such that $Vx = y$. x is in $E(U,V)$ and hence there exists an x_1 such that $Ux_1 = Vx = y$. Clearly $x_1 \in E(U,V) = J(U,V)$ and hence $y \in U(D(U) \cap J(U,V))$. Thus $U(D(U) \cap J(U,V)) = K(U,V)$.

Now assume that $U(D(U) \cap J(U,V)) = K(U,V)$. Clearly $E(U,V) \subset J(U,V)$ so let $x_0 \in J(U,V)$. Then $Vx_0 \in K(U,V) = U(D(U) \cap J(U,V))$. Hence there exists $x_1 \in J(U,V)$ such that $Ux_1 = Vx_0$. Now assume the induction hypothesis that $Ux_n = Vx_{n-1}$ for x_n in $J(U,V)$ and Vx_{n-1} in $K(U,V)$. Then $Vx_n \in U(D(U) \cap J(U,V))$ and hence there exists an x_{n+1} in $J(U,V)$ such that $Ux_{n+1} = Vx_n$. Thus by induction, there exists a sequence $\{x_n\}$ such that $Ux_{n+1} = Vx_n$ and hence $x_0 \in E(U,V)$. Therefore, $E(U,V) = J(U,V)$.

Theorem 3.5: If there exists an integer m such that $N(U) \cap M_m = N(U) \cap M_k$ for all $k > m$, then $E(U,V) = J(U,V)$.

Proof: Clearly $U(D(U) \cap J(U,V)) \subset K(U,V) = \bigcap_{n=0}^{\infty} UM_n$. Let $y \in K(U,V)$. Then, for every n, $y \in UM_{n+1}$. Thus for each n there exists an x_n in $D(U) \cap M_n$ such that $Ux_n = y$. Let $z_k = x_m - x_{m+k}$. Then $z_k \in M_m$ and $U(x_m - x_{m+k}) = y - y = 0$. Hence $z_k \in N(U) \cap M_m = N(U) \cap M_{m+i}$, $i = 1, 2, 3, \ldots$. Therefore, $z_k \in N(U) \cap M_{m+k}$ and so $x_m = z_k + x_{m+k} \in M_{m+k}$. This holds for $k = 1, 2, 3, \ldots$ and hence $x_m \in D(U) \cap J(U,V)$. Since $Ux_m = y$, $y \in U(D(U) \cap J(U,V))$. Therefore, $K(U,V) = U(D(U) \cap J(U,V))$ and hence, by Theorem 3.4, $E(U,V) = J(U,V)$.

Corollary 3.6: If $N(U) \subset J(U,V)$, then $E(U,V) = J(U,V)$.

Proof: Since $N(U) \subset J(U,V)$, we have that $N(U) \cap M_0 = N(U) \cap M_k$ for all $k > 0$ and hence, by Theorem 3.5, $E(U,V) = J(U,V)$.

Corollary 3.7: If $\alpha(U)$ is finite, then $E(U,V) = J(U,V)$.

Proof: If $\alpha(U)$ is finite, it is obvious that there exists an integer m such that $N(U) \cap M_m = N(U) \cap M_k$ for all $k > m$. Thus $E(U,V) = J(U,V)$.

Corollary 3.8: If $\beta(U)$ is finite and V is one-to-one, then $E(U,V) = J(U,V)$.

Proof: Let $L_i = N(U) \cap M_i$, $i = 0, 1, 2, \ldots$, and suppose that there exists no positive integer m such that $L_m = L_n$ for all $n > m$. Then there exists an increasing sequence of positive integers $\{k(i)\}$, $i = 1, 2, 3, \ldots$ such that $L_{k(i)} \neq L_{k(i)+1}$. Thus there exists a sequence $\{x_i\}$, $i = 1, 2, 3, \ldots$, such that $x_i \in L_{k(i)}$ but $x_i \notin L_{k(i)+1}$. Hence $x_i \in M_{k(i)}$ but $x_i \notin M_{k(i)+1}$. Now, for each j, $x_j \in M_{k(j)} = V^{-1}UM_{k(j)-1}$ and hence there exists $z(j, k(j) - 1) \in M_{k(j)-1}$ such that $Vx_j = Uz(j, k(j) - 1)$. Since $x_j \notin M_{k(j)+1}$, $z(j, k(j) - 1) \notin M_{k(j)}$. In a similar manner we get an element $z(j, k(j) - 2)$ in $M_{k(j)-2}$ which does not belong to $M_{k(j)-1}$ such that $Vz(j, k(j) - 1) = Uz(j, k(j) - 2)$. Continuing this process a finite number of times we finally get an element $z(j,0)$ in M_0 which does not belong to M_1 such that $Vz(j,1) = Uz(j,0)$. Since $D(U) \subset D(V)$, we have that $z(j,0) \in D(V)$. Now $R(U) = UM_0 = VM_1$ and so $Vz(j,0) \notin R(U)$ since $z(j,0) \notin M_1$. Thus for each j we get a set $\{z(j,n)\}$, $n = 0, 1, 2, \ldots, k(j) - 1$, such that $Uz(j,n) = Vz(j,n+1)$ where $z(j, k(j)) = x_j$, $z(j,n) \in M_n$ but $z(j,n) \notin M_{n+1}$, and $Vz(j,0) \notin R(U)$. Let r be any arbitrary positive integer. We will now show that the elements $Vz(j,0)$, $j = 1, 2, 3, \ldots, r$, are linearly independent. Let $z = \sum_{j=1}^{r} \alpha_j z(j,0)$ and suppose that $Vz = 0$. Since V is one-to-one, $z = 0$. Now $V^{-1}Uz(j,n) = z(j,n+1)$, $n = 0, 1, 2, \ldots, k(j)-1$. Thus, after r applications of $V^{-1}U$ to the equation $z = 0$, we get $\alpha_r x_r = 0$ and since $x_r \neq 0$ we get $\alpha_r = 0$. Similarly we can show that $\alpha_{r-1}, \alpha_{r-2}, \ldots, \alpha_1$ must all be zero. Thus $z(j,0)$ and hence $Vz(j,0)$, $j = 1, 2, 3, \ldots, r$, are all linearly independent. Now suppose that w_1 is any element of the linear span of $\{z(j,0)\}$, $j = 1, 2, 3, \ldots, r$, which belongs to M_1. Hence $w_1 = \sum_{j=1}^{r} \alpha_j z(j,0) \in M_1$. Thus $V^{-1}Uw_1 = w_2 \in M_2$, $V^{-1}Uw_2 = w_3 \in M_3$, $\ldots$, $V^{-1}Uw_{k(r)} = \alpha_r x_r \in M_{k(r)+1}$. Since $x_r \notin M_{k(r)+1}$, we have that $\alpha_r = 0$. Similarly we can show that $\alpha_{r-1}, \alpha_{r-2}, \ldots, \alpha_1$ must all be zero. Hence $w_1 = 0$. Thus we have shown that the linear span of $\{z(j,0)\}$, $j = 1, 2, 3, \ldots, r$, intersected with M_1 is $\{0\}$. Since V is one-to-one, the linear span of $\{Vz(j,0)\}$, $j = 1, 2, 3, \ldots, r$, intersected with $VM_1 = UM_0 = R(U)$ is $\{0\}$. Since r is arbitrary, we have contradicted the fact that $\beta(U)$ is finite. Hence there exists a positive integer m such that $N(U) \cap M_m = N(U) \cap M_n$ for all $n > m$. Therefore, by Theorem 3.5, we have that $E(U,V) = J(U,V)$.

The following example provides a case where $E(U,V) = J(U,V)$ but there is no positive integer m such that $N(U) \cap M_m = N(U) \cap M_k$ for all $k > m$. This is, therefore, an example which shows that the converse of Theorem 3.5 does not hold.

<u>Example 3.9</u>: Let $X = Y = \ell_2$ and let $V = I$. Let $x = (x(1), x(2), x(3), \ldots)$ and define U by $Ux = (y(1), y(2), y(3), \ldots)$ where $y(k) = 0$ if $k = (2n-1)(n+1)$ or $k = n(2n+3)$, $n = 1, 2, 3, \ldots$, and $y(k) = x(k+1)$ otherwise. Then $E(U,I) = J(U,I) = \{0\}$. However, there exists no positive integer m such that $N(U) \cap M_m = N(U) \cap M_k$ for all $k > m$.

IV. OPERATORS WITH CLOSED RANGE

Throughout this chapter, we shall consider two continuous linear operators A and T defined on a Banach space X with ranges in a Banach space Y. Also, we shall assume that A has closed range. Clearly $A-\lambda T$ is continuous with $D(A-\lambda T) = X$ and, therefore, $A-\lambda T$ is a closed linear operator for every λ.

Definition 4.1: Two linear manifolds L and M are strongly disjoint if and only if there exists a positive constant k such that $\|x+y\| \geq k\|x\|$ for all $x \in L$ and $y \in M$.

Kato (7, p. 297) has shown that if $N(A) \subset J(A,T)$[1], then there exists a positive constant ρ such that $A-\lambda T$ has closed range for all λ satisfying $|\lambda| < \rho$. Also, Graber (5, p. 9) has shown that, if $T(N(A))$ is closed and $T(N(A))$ and $R(A)$ are strongly disjoint, then there exists a positive constant ρ such that $A-\lambda T$ has closed range for all λ satisfying $|\lambda| < \rho$.

In this chapter, we shall prove a theorem which is a generalization of these two results.

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[1]The condition that $V(A:T) = \infty$ used by Kato is clearly equivalent to $N(A) \subset J(A,T)$.

Lemma 4.2: If two linear manifolds L and M in X are strongly disjoint and $L \oplus M$ is closed, then L and M are closed.

Proof: Let $x \in \bar{L}$. There exists a sequence $\{x_n\} \subset L$ such that $x_n \to x$. Since $\{x_n\} \subset L$, $\{x_n\} \subset L \oplus M$. But $L \oplus M$ is closed and hence $x \in L \oplus M$. Thus $x = u + v$ for some $u \in L$ and $v \in M$. Hence $\| x_n - x \| = \| x_n - u - v \| \to 0$. Now $x_n - u \in L$ and $v \in M$ and, since L and M are strongly disjoint, we have that $\| x_n - u - v \| \geq k \| v \|$ for some positive constant k. Hence $v = 0$ and $x = u \in L$. Therefore, L is closed. Similarly we can show that M is closed.

Lemma 4.3 (Lorch, 9, p. 220): Let L and M in X be two closed linear manifolds. Then L and M are strongly disjoint if and only if $L \cap M = \{0\}$ and $L \oplus M$ is a closed linear manifold.

Throughout the rest of this chapter we will assume that $N(A) = L \oplus M$ where L and M are closed, TL and R(A) are strongly disjoint, $TM \subset R(A)$, and T(N(A)) is closed.

We then have that TL and TM are strongly disjoint and so, by Lemma 4.2, they are closed. Hence Y/TL which we will denote by Y' is a Banach space with norm $\| y' \| = \inf \{ \| y+t \| : t \in TL \}$. Also, L is closed and thus X/L which we will denote by X' is also a Banach space with norm $\| x' \| = \inf \{ \| x+t \| : t \in L \}$.

Definition 4.4: We define τ to be the canonical mapping of X onto X' and μ to be the canonical mapping of Y onto Y'.

Definition 4.5: We define the operator A' from X' into Y' by the equation: $A'x' = \mu Ax = \mu(Ax)$ where $x \in x'$.

We note that A' is a linear operator.

Definition 4.6: We define the operator T' from X' into Y' by the equation: $T'x' = \mu Tx = \mu(Tx)$ where $x \in x'$.

We see that T' is a function since, if we let $\tau u = \tau v = x'$, then $u - v \in L$ so that $Tu - Tv \in TL$ which implies that $\mu Tu = \mu Tv$.

The following lemma is known. However, the proof is given here for completeness.

Lemma 4.7: Let F be a linear manifold contained in X and let G be a closed linear manifold such that $G \subset F$. Let σ be the canonical mapping of X onto X/G. Then σF is closed if and only if F is closed.

Proof: Let F be closed and let $x' \in \overline{\sigma F}$. Then there exists a sequence $\{x_n'\} \subset \sigma F$ such that $x_n' \to x'$. From the definition of the norm in X/G, there exists a sequence $\{z_n\} \subset G$ such that $x_n - x + z_n \to 0$ where $x \in x'$ and $x_n \in x_n'$ for each n. Since $G \subset F$, $\{z_n\} \subset F$. Thus we have that $\{x_n + z_n\} \subset F$ and $x_n + z_n \to x$. Since F is closed, $x \in F$ and hence $x' \in \sigma F$.

Now suppose that σF is closed and let $x \in \overline{F}$. Then there exists a sequence $\{x_n\} \subset F$ such that $x_n \to x$. Clearly $\sigma x_n \to \sigma x$ and $\{\sigma x_n\} \subset \sigma F$. Since σF is closed we have that $\sigma x \in \sigma F$. Thus there exists a z in F such that $\sigma x = \sigma z$ and hence $x - z \in G$. Since $G \subset F$, $x - z \in F$. But $z \in F$ and hence $x \in F$. Thus F is closed.

Lemma 4.8: $R(A')$ is closed.

Proof: $R(A)$ and TL are strongly disjoint and both of them are closed. Hence, by Lemma 4.3, $R(A) \oplus TL$ is closed. Thus, by Lemma 4.7, $\mu(R(A) \oplus TL)$ is closed. But $\mu(R(A) \oplus TL) = \mu(R(A)) = R(A')$.

Lemma 4.9: A' is a closed operator.

Proof: Let $\{x_n'\} \subset X'$ and $\{A'x_n'\} \subset Y'$ be sequences such that $x_n' \to x'$ and $A'x_n' \to y'$. Now $\| x_n' - x' \| \to 0$ and $\| x_n' - x' \| = \| (x_n - x)' \| = \inf \{ \| x_n - x + z \| : z \in L \}$. Thus for each positive integer n there exists $z_n \in L$ such that $\| x_n - x + z_n \| \leq 2 \| x_n' - x' \| \to 0$. Hence $x_n + z_n \to x$. Now, by Lemma 4.8, $R(A')$ is closed and thus there exists an x_o' in $D(A')$ such that $A'x_o' = y'$. Now $A'x_n' \to A'x_o'$ and, therefore, $A'x_n' - A'x_o' = \mu Ax_n - \mu Ax_o = \mu(Ax_n - Ax_o) \to 0$. Hence there exists a sequence $\{ t_n \}$ in TL such that $Ax_n - Ax_o +$

$t_n \to 0$. Now $Ax_n - Ax_o \in R(A)$ and $t_n \in TL$ and, since they are strongly disjoint, we have that $k \| Ax_n - Ax_o \| \leq \| Ax_n - Ax_o + t_n \| \to 0$ for some positive constant k. Thus $Ax_n \to Ax_o$. Since $z_n \in L$ for every n, we see that $A(x_n + z_n) \to Ax_o$. Hence we have that $x_n + z_n \to x$ and $A(x_n + z_n) \to Ax_o$. A is closed since it is continuous and defined on X and thus $x \in D(A)$ and $Ax = Ax_o$. Now $y' = A'x_o' = \mu Ax_o = \mu Ax = A'x'$. Hence $x' \in D(A')$ and $A'x' = y'$. Thus A' is a closed operator.

<u>Lemma 4.10</u>: $R(A'-\lambda T') = \mu(R(A-\lambda T))$.

Proof: Let $y' \in R(A'-\lambda T')$. Then there exists x' in $D(A'-\lambda T')$ such that $(A'-\lambda T')x' = y' = A'x' - \lambda T'x' = \mu Ax - \lambda\mu Tx = \mu(Ax-\lambda Tx) = \mu((A-\lambda T)x)$. Thus $y' \in \mu(R(A-\lambda T))$ and hence $R(A'-\lambda T') \subset \mu(R(A-\lambda T))$.

Now let $y' \in \mu(R(A-\lambda T))$. Then there exists an x in $D(\mu(A-\lambda T))$ such that $(\mu(A-\lambda T))x = y' = \mu Ax - \lambda\mu Tx = A'x' - \lambda T'x' = (A'-\lambda T')x'$. Hence $y' \in R(A'-\lambda T')$ and, therefore, $\mu(R(A-\lambda T)) \subset R(A'-\lambda T')$.

<u>Lemma 4.11</u>: $TL \subset R(A-\lambda T)$ for every $\lambda \neq 0$.

Proof: Let $y \in T(N(A))$. Then there exists an x in $N(A)$ such that $Tx = y$. Thus for every $\lambda \neq 0$, we have that $-x/\lambda \in N(A)$. Hence $(A-\lambda T)(-x/\lambda) = Tx = y$. Thus $y \in$

$R(A-\lambda T)$ and, therefore, $T(N(A)) \subset R(A-\lambda T)$ for $\lambda \neq 0$. Since $TL \subset T(N(A))$, $TL \subset R(A-\lambda T)$.

Lemma 4.12: $N(A') = \tau(M)$.

Proof: Let $0' \neq x' \in N(A')$. Then $A'x' = 0' = \mu Ax$. Hence $Ax = 0$ or $Ax \in TL$. But $Ax \in R(A)$ which is strongly disjoint from TL and thus $Ax = 0$. This means that $x \in N(A)$ and thus can be written as $w + z$ where $w \in L$ and $z \in M$. Now $x' = \tau x = \tau(w+z) = \tau z \in \tau M$. Thus $N(A') \subset \tau(M)$.

Now let $x' \in \tau(M)$. Then $x \in M \subset N(A)$ and thus $Ax = 0$. Hence $\mu Ax = 0' = A'x'$ and so $x' \in N(A')$. Hence $\tau(M) \subset N(A')$.

We shall use M_n' to denote the linear manifolds $M_n(A',T')$ which are defined in Chapter III.

Lemma 4.13: $A'M_n' = \mu(AM_n)$ for every non-negative integer n.

Proof: Clearly they are equal for $n = 0$. We shall complete the proof by induction. Assume that $A'M_k' = \mu(AM_k)$ and let $y' \in A'M_{k+1}'$. Then there exists an x_o' in M_{k+1}' such that $y' = A'x_o'$. Now $x_o' \in (T')^{-1}(A'M_k')$ and hence $T'x_o' = A'x_1'$ for some x_1' in M_k'. Since $A'M_k' = \mu(AM_k)$, we have that there exists an x_2 in M_n such that $A'x_1' = \mu Ax_2$. Now $T'x_o' = \mu Tx_o$. Thus $\mu Tx_o = \mu Ax_2$ and so $Tx_o - Ax_2 = Tw$ for some w in L. Hence $T(x_o-w) = Ax_2$ and so $x_o - w = T^{-1}(Ax_2) \in M_{k+1}$.

Therefore, $A(x_o - w) = Ax_o \in AM_{k+1}$ and thus $\mu Ax_o \in \mu(AM_{k+1})$. Since $\mu Ax_o = A'x_o' = y'$, we have that $A'M_{k+1}' \subset \mu(AM_{k+1})$.

Now let $y' \in \mu(AM_{k+1})$. Then there exists an x in M_{k+1} such that $y' = \mu Ax$. Therefore, $x \in T^{-1}(AM_k)$ and so $Tx = Ax_1$ for some x_1 in M_k. Now, by the induction hypothesis, $\mu(AM_k) = A'M_k'$ so there exists an x_2' in M_k' such that $\mu Ax_1 = A'x_2'$. Hence $\mu Ax_1 = \mu Tx = T'x' = A'x_2'$ and so $x' = (T')^{-1}(A'x_2') \in M_{k+1}'$. Thus $A'x' \in A'M_{k+1}'$. Since $y' = \mu Ax = A'x'$, $\mu(AM_{k+1}) \subset A'M_{k+1}'$. Hence $\mu(AM_{k+1}) = A'M_{k+1}'$ and thus by induction $\mu(AM_n) = A'M_n'$ for all non-negative integers n.

<u>Lemma 4.14</u>: If $M \subset J(A,T)$, then $N(A') \subset J(A',T')$.

Proof: By Lemma 4.12, $N(A') = \tau(M)$. Let $x' \in \tau(M)$. Then $x \in M \subset J(A,T) = \bigcap_{n=0}^{\infty} M_n = \bigcap_{n=1}^{\infty} T^{-1}(AM_{n-1})$. Hence, for every $n > 0$, we have that $x \in T^{-1}(AM_{n-1})$ and, therefore, $Tx \in AM_{n-1}$. Thus $\mu Tx \in \mu(AM_{n-1})$ for every $n > 0$ and so $T'x' \in \mu(AM_{n-1})$. Now, by Lemma 4.13, we get that $T'x' \in A'M_{n-1}'$ for every $n > 0$ and consequently $x' \in (T')^{-1}(A'M_{n-1}')$. Hence $x \in \bigcap_{n=1}^{\infty} (T')^{-1}(A'M_{n-1}') = J(A',T')$. Thus $N(A') \subset J(A',T')$.

<u>Theorem 4.15</u>: (Kato, 7, p. 297): If $N(A) \subset J(A,T)$, then there exists a positive constant ρ such that $A - \lambda T$ has closed range for all λ satisfying $|\lambda| < \rho$.

The following theorem is well known and can be found in Gohberg and Krein (3, p. 232) as well as in other sources.

Theorem 4.16: If $\alpha(A) < \infty$, then there exists a positive constant ρ such that $A - \lambda T$ has closed range for all λ satisfying $|\lambda| < \rho$.

Theorem 4.17: If $M \subset J(A,T)$ or if $\dim M < \infty$, then there exists a positive constant ρ such that $A - \lambda T$ has closed range for all λ satisfying $|\lambda| < \rho$.

Proof: First we shall assume that $M \subset J(A,T)$. We know that $R(A-\lambda T)$ is closed for $\lambda = 0$ so assume that $\lambda \neq 0$. From Lemma 4.8 and Lemma 4.9, we have that A' is a closed linear operator with closed range. Since $M \subset J(A,T)$, Lemma 4.14 shows that $N(A') \subset J(A',T')$. Hence A' and T' satisfy the hypothesis of Theorem 4.15 and, therefore, there exists a positive constant ρ such that $A' - \lambda T'$ has closed range for all λ satisfying $|\lambda| < \rho$. By Lemma 4.10, $\mu(R(A-\lambda T))$ is closed for all λ satisfying $|\lambda| < \rho$. Hence, using Lemma 4.11 and Lemma 4.7, we get that $A - \lambda T$ has closed range for all λ satisfying $|\lambda| < \rho$.

Now suppose that $\dim M < \infty$. Again we assume that $\lambda \neq 0$. As in the first part of the proof, we have that A' is a closed operator with closed range. Clearly $\dim \tau(M) \leq$

dim M. By Lemma 4.12, $N(A') = \tau(M)$ and hence $\alpha(A') \leq \dim M < \infty$. Hence A' and T' satisfy the hypothesis of Theorem 4.16 and, therefore, there exists a positive constant ρ such that $A' - \lambda T'$ has closed range for all λ satisfying $|\lambda| < \rho$. Again applying Lemma 4.10, Lemma 4.11, and Lemma 4.7, we have that $A - \lambda T$ has closed range whenever $|\lambda| < \rho$.

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VI. ACKNOWLEDGMENTS

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