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Characterizations of quasi-Frobenius rings

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CHARACTERIZATIONS OF QUASI-FROBENIUS
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CHARACTERIZATIONS OF QUASI-FROBENIUS RINGS

by

Edgar Andrews Rutter, Jr.

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I. INTRODUCTION

The theory of Frobenius and quasi-Frobenius rings (see Definition 3.1) was begun by Nakayama (16 and 17) in connection with his work on representations of algebras and, at his hands and those of many other mathematicians, has developed into a very extensive body of knowledge. Quasi-Frobenius rings have, from the start, been connected with the idea of "duality" but in most early papers on that theory duality meant either lattice theoretic duality or, when the ring Λ under consideration was an algebra over a commutative ring K , the duality of K modules. The study of quasi-Frobenius rings from the point of view of the duality of Λ modules (see Definition 2.1) seems to have been undertaken for the first time by Morita and Tachikawa (15) and independently but slightly later by Dieudonné' (4). Bass (2) made use of the canonical homomorphism of a module into its double dual (see p. 6) in his investigation of the relationship between the various homological dimensions of a ring and introduced the terms torsionless and reflexive to describe those modules for which this canonical homomorphism is a monomorphism and an isomorphism, respectively (see Definitions 2.2 and 2.3). This work by Bass stimulated Jans (11) to make a study of

duality theory in the case of Noetherian rings. Among his results was a characterization of the Noetherian rings over which every torsionless module is reflexive. Jans also showed that a quasi-Frobenius ring could be characterized as a Noetherian ring with the property that all finitely generated modules (both left and right) are reflexive. These results of Jans provided the initial motivation for the research contained in this thesis.

Chapter II contains a survey of the basic definitions and results from duality theory which will be used through out the rest of the paper.

In Chapter III, we show that a quasi-Frobenius ring can be characterized as a Noetherian ring with the property that every finitely generated left module is torsionless and present an example due to Nakayama(18) which shows that this result is the best possible along these lines. We also consider the relationship of this result to a characterization of quasi-Frobenius rings due to Ikeda and Nakayama (9) which applies only to algebras of finite rank. In addition, rings with the property that all finitely generated left modules are submodules of free left modules are shown to be quasi-Frobenius rings. We conclude this chapter by proving that

in the case of a right principal ideal ring, it suffices to assume that every cyclic left module is torsionless in order to show that it is a quasi-Frobenius ring.

The fourth chapter is devoted to proving that an Artinian ring all of whose cyclic left modules are reflexive must be a quasi-Frobenius ring. In the process, we obtain several other characterizations of these rings.

The fifth chapter contains an alternate approach to proving the main result of Chapter IV which extends it to Noetherian rings. The technique used for this also yields another characterization of quasi-Frobenius rings which is novel in that all of the hypotheses are placed upon the radical of the ring. The rest of this chapter is devoted to applications of this latter result.

All of the characterizations presented here have in common the fact that they are phrased in terms of the dual or are obtained through applications of the duality theory. They also share the property that all of the hypotheses, other than the chain conditions, concerning the ring or its modules are on the same side and are not left-right symmetric as has been the case in much of the previous work in this area.

The limitations imposed by space have compelled us to assume that the reader is familiar with the basic notions and notation of homological algebra as contained, for example, in Northcott (19) or Cartan and Eilenberg (3). As is customary in this area, we assume that all rings have a multiplicative identity and that all modules are unitary.

II. PRELIMINARY MATERIAL

In this chapter, we present a summary of the basic duality theory to be used through out this paper. Most of this material is taken from Chapter 5 of Jans (13) or Section 5 of Jans (12), but it also includes results from Bass (2) and Dieudonne' (4). For the most part, proofs have been presented in this chapter only if they have been omitted in the source from which the material is taken and might cause the reader difficulty.

We shall always use Λ to denote a ring, ${}_s\Lambda$ to denote the ring Λ regarded as a left Λ module, and Λ_d to denote the ring Λ regarded as a right module over itself.

Definition 2.1. Let M be a left Λ module. The dual of M , which is denoted M^* , is the right Λ module $\text{Hom}_\Lambda(M, {}_s\Lambda)$, where module multiplication is given by the rule

$$(f\lambda)(m) = (f(m))\lambda, \quad f \in M^*, \lambda \in \Lambda, m \in M.$$

If M is a right Λ module, then the dual of M is a left Λ module, where module multiplication is defined by

$$(\lambda f)(m) = \lambda(f(m)), \quad f \in M^*, \lambda \in \Lambda, m \in M.$$

Once we have the dual M^* of a module M , we may form the

double dual $M^{**} = (M^*)^*$ of M . If M is a left Λ module (right Λ module), so is its double dual. There is a natural Λ homomorphism of M into M^{**} ,

$$\sigma_M: M \rightarrow M^{**},$$

defined by the rule

$$[\sigma_M(m)](f) = f(m),$$

where $m \in M$, $f \in M^*$.

Definition 2.2. A Λ module M is torsionless if and only if σ_M is a Λ monomorphism.

Definition 2.3. A Λ module M is reflexive if and only if σ_M is a Λ isomorphism.

If we have the diagram $A \xrightarrow{f} B$, then this induces

$$\text{Hom}_{\Lambda}(B, \Lambda) \xrightarrow{f'} \text{Hom}_{\Lambda}(A, \Lambda),$$

where $f' = \text{Hom}(f, i)$ and i denotes the identity map on Λ , or, conforming with the notation introduced above, $B^* \xrightarrow{f^*} A^*$.

It is a straightforward matter to verify that f^* is a module homomorphism and that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \sigma_A \downarrow & & \downarrow \sigma_B \\ A^{**} & \xrightarrow{f^{**}} & B^{**} \end{array},$$

where $f^{**} = (f^*)^*$.

We now give some properties of $(*)$.

(2.4) If $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is an exact sequence of Λ modules and Λ homomorphisms, so is $0 \rightarrow C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*$, i.e., $(*)$ is a left exact functor.

(2.5) ${}_s\Lambda^* \cong \Lambda_d$ and

$$\Lambda_d^* \cong {}_s\Lambda.$$

These mappings are given by $f \rightarrow f(1)$ and will be denoted by ρ_s and ρ_d , respectively. The fact that they are Λ isomorphisms is easily verified.

Theorem 2.6. If P is a finitely generated, projective, left Λ module (right Λ module), then

(1) P^* is a finitely generated, projective, right Λ module (left Λ module), and

(2) P is reflexive.

Proposition 2.7. ${}_s\Lambda$ and Λ_d are reflexive. This proposition is a special case of Theorem 2.6 which has been stated separately for emphasis.

The next proposition gives an alternate characterization of torsionless modules which is easier to visualize than the definition.

Proposition 2.8. A Λ module M is torsionless if and

only if for each $0 \neq m \in M$ there is an $f \in M^*$ such that $f(m) \neq 0$.

Proposition 2.9. If M is a torsionless Λ module and K is a submodule of M , then K is torsionless.

Proposition 2.10. If M is a left Λ module (right Λ module), then M^* is a torsionless right Λ module (left Λ module).

We associate annihilators with subsets of the ring Λ as follows:

Definition 2.11. Let U be a subset of Λ . Then the right annihilator of U (in Λ), which is denoted by $d(U)$, is the set

$$\left\{ \lambda : \lambda \in \Lambda \text{ and } u\lambda = 0 \text{ for all } u \in U \right\}$$

Similarly, the left annihilator of U (in Λ) is defined by

$$s(U) = \left\{ \lambda : \lambda \in \Lambda \text{ and } \lambda u = 0 \text{ for all } u \in U \right\}$$

The following basic properties of the annihilators are immediate from the definitions.

Proposition 2.12. Let U and V be subsets of the ring Λ . Then (1) $s(U)$ and $d(U)$ are left and right ideals of Λ , respectively,

$$(2) \ s(d(U)) \supseteq U \text{ and } d(s(U)) \supseteq U, \text{ and}$$

(3) if $U \supset V$, then $d(V) \supset d(U)$ and $s(V) \supset s(U)$.

The notation of the next proposition is the same as that of (2.5).

Proposition 2.13. For any exact sequence $\Lambda_d \xrightarrow{\eta} \Lambda_d/R \rightarrow 0$, where R is a right ideal of Λ and η is the natural map of Λ_d onto Λ_d/R , the dual sequence $0 \rightarrow (\Lambda_d/R)^* \xrightarrow{\eta^*} \Lambda_d^*$ is exact. Furthermore, ρ_d carries $\text{Im}(\eta^*)$ onto $s(R)$ and hence induces an isomorphism between $(\Lambda_d/R)^*$ and $s(R)$.

Proof: The exactness of the dual sequence is immediate from (2.4).

Let $f \in (\Lambda_d/R)^*$. Then $\eta^*(f) = f\eta$ and $\rho_d(f\eta) = f(\eta(1)) = f(1+R)$. Since $r + R \in \text{Ker}(f)$ for all $r \in R$, we have $0 = f(r+R) = f(1+R)r$ and hence $f(1+R) \in s(R)$, i.e., ρ_d carries $\text{Im}(\eta^*)$ into $s(R)$.

Let $t \in s(R)$ and define a Λ homomorphism f_t from Λ_d into itself by the condition $f_t: \lambda \rightarrow t\lambda$. Since $t \in s(R)$, $\text{Ker}(f_t) \supset R$ and hence the mapping $\bar{f}_t: \Lambda_d/R \rightarrow \Lambda_d$ defined by $\bar{f}_t(\lambda+R) = f_t(\lambda)$ is a well defined Λ homomorphism, i.e., $\bar{f}_t \in (\Lambda_d/R)^*$. Now $\eta^*(\bar{f}_t) = f_t$ and $\rho_d(f_t) = t$, so ρ_d carries $\text{Im}(\eta^*)$ onto $s(R)$. This completes the proof of the proposition.

It is also possible to define annihilator relations between the subsets of any Λ module M and its dual M^* by

means of the natural pairing $M \times M^* \rightarrow \Lambda$. This is done as follows:

Definition 2.14. Let U be a subset of a Λ module M . Then the annihilator of U with respect to M^* , $\text{Ann}(U, M^*)$, is the set

$$\left\{ f: f \in M^* \text{ and } f(u) = 0 \text{ for all } u \in U \right\}.$$

Similarly, if U' is a subset of M^* , the set

$$\left\{ m: m \in M \text{ and } f(m) = 0 \text{ for all } f \in U' \right\}$$

is called the annihilator of U' with respect to M and is denoted by $\text{Ann}(U', M)$.

We now give some of the basic properties of these annihilators.

Proposition 2.15. Let M be a Λ module, U and V be subsets of M , and U' and V' be subsets of M^* . Then

(1) $\text{Ann}(U, M^*)$ and $\text{Ann}(U', M)$ are submodules of M^* and M , respectively,

(2) $\text{Ann}(\text{Ann}(U, M^*), M) \supset U$ and $\text{Ann}(\text{Ann}(U', M), M^*) \supset U'$,
and

(3) (a) if $U \supset V$, $\text{Ann}(U, M^*) \subset \text{Ann}(V, M^*)$, and

(b) if $U' \supset V'$, $\text{Ann}(U', M) \subset \text{Ann}(V', M)$.

The two types of annihilators described above agree on subsets of the ring. To be more precise:

Proposition 2.16. Let U be a subset of the ring Λ . Then (1) $s(d(U)) = \text{Ann}(\text{Ann}(U, {}_s\Lambda^*), {}_s\Lambda)$, and (2) $d(s(U)) = \text{Ann}(\text{Ann}(U, \Lambda_d^*), \Lambda_d)$.

Proof: Since the two parts are similar, we shall only verify (1). Let $f \in \text{Ann}(U, {}_s\Lambda^*)$. Then for any $u \in U$, $0 = f(u) = uf(1)$ and hence $f(1) \in d(U)$. Consequently, $f(\lambda) = \lambda f(1) = 0$ for all $\lambda \in s(d(U))$ and hence $\text{Ann}(\text{Ann}(U, {}_s\Lambda^*), {}_s\Lambda) \supset s(d(U))$.

Let $a \in \text{Ann}(\text{Ann}(U, {}_s\Lambda^*), {}_s\Lambda)$ and let $b \in d(U)$. The mapping $g: \Lambda \rightarrow \Lambda$ defined by $g: \lambda \rightarrow \lambda b$ belongs to ${}_s\Lambda^*$ and since $b \in d(U)$, $g \in \text{Ann}(U, {}_s\Lambda^*)$. Therefore, $0 = g(a) = ab$ and hence $s(d(U)) \supset \text{Ann}(\text{Ann}(U, {}_s\Lambda^*), {}_s\Lambda)$. This completes the proof of the proposition.

Definition 2.17. Let M be a Λ module and K a submodule of M . Then K is closed in M if and only if $\text{Ann}(\text{Ann}(K, M^*), M) = K$.

Proposition 2.18. Let M be a Λ module and K a submodule of M . Then the quotient module M/K is torsionless if and only if K is closed in M .

Proof: Assume that M/K is torsionless. We need only

verify that $\text{Ann}(\text{Ann}(K, M^*), M) \subset K$ since the reverse inclusion is always valid. Let $m \in M - K$. Then it follows from Proposition 2.8 that there exists an $\bar{f} \in (M/K)^*$ such that $\bar{f}(m+K) \neq 0$. Let $f = \bar{f}\eta$, where η is the natural map of M onto M/K . Then $f \in M^* \cap \text{Ann}(K, M^*)$ and $f(m) = \bar{f}(m+K) \neq 0$. Therefore, $m \notin \text{Ann}(\text{Ann}(K, M^*), M)$. Since this is true for any $m \in M - K$, we have the desired inclusion.

Assume that K is closed in M and let $m \in M - K$. Since $m \notin \text{Ann}(\text{Ann}(K, M^*), M)$, there is a $f \in \text{Ann}(K, M^*)$ which is such that $f(m) \neq 0$. But $\text{Ker}(f) \subset K$ and hence the identity $\bar{f}(t+K) = f(t)$ defines a Λ homomorphism of M/K into Λ . It can now be seen from Proposition 2.8 that M/K is torsionless. This completes the proof.

For future convenience, we rephrase this proposition for the special case of the ring.

Proposition 2.19. Let L be a left ideal of Λ (R be a right ideal of Λ). Then ${}_s\Lambda/L$ (Λ_d/R) is a torsionless left Λ module (right Λ module) if and only if $s(d(L)) = L$ ($d(s(R)) = R$).

Proposition 2.20. Let Λ be a ring which satisfies the ascending chain condition on right ideals. If $s(d(L)) = L$ for every left ideal L of Λ , then Λ satisfies the descending

chain condition on left ideals.

Proof: Let $\{L_\alpha\}_{\alpha \in A}$ be an arbitrary chain of left ideals in Λ . It is immediate from Proposition 2.12 that $\{d(L_\alpha)\}_{\alpha \in A}$ is an ascending chain of right ideals in Λ and hence terminates after a finite number of steps. Consequently, the original chain also terminates after a finite number of steps since it equals the chain $\{s(d(L_\alpha))\}_{\alpha \in A}$. This completes the proof of the proposition.

III. TORSIONLESS MODULES AND QUASI-FROBENIUS RINGS

In this chapter, the duality concepts introduced in the preceding chapter will be used to obtain some new characterizations of quasi-Frobenius rings (Definition 3.1). We shall, therefore, begin with a definition of these rings. Before turning to this, however, we mention that throughout this paper we will use the term Noetherian ring to refer to a ring which is assumed to satisfy the ascending chain condition on both left and right ideals and left or right Noetherian for a ring which is assumed to satisfy the ascending chain condition only on left or right ideals. The terms Artinian, left Artinian, and right Artinian will be used in an entirely analogous manner for rings assumed to satisfy the descending chain condition on the appropriate one sided ideals.

Definition 3.1. A ring Λ is a quasi-Frobenius ring if and only if it is a Noetherian ring and the relations

$$(1) s(d(L)) = L, \text{ and}$$

$$(2) d(s(R)) = R$$

hold for all left ideals L and right ideals R of Λ .

This class of rings was first introduced by Nakayama (17) as an outgrowth of his work on representations of

algebras. They have many remarkable properties including a number of characterizations which appear very different in nature from the above definition. We shall encounter several of these in the sequel. The definition given above is not Nakayama's original description of these rings but is an alternate characterization which appears in the same paper.

Before we can prove our first theorem, we shall require a number of preliminary definitions and lemmas. We shall have some occasion to make use of cardinal and ordinal numbers. In this connection, we will use the following notation: If S is a set, \bar{S} denotes the cardinal number of S and if α is an ordinal number, $\bar{\alpha}$ denotes the cardinal number of any set of order type α .

Lemma 3.2. For any infinite cardinal number τ , $\tau^2 = \tau$. This is a well known property of cardinal numbers. A proof can be found in Halmos' book (8, p. 97).

The next lemma is also known but we have included a proof as we have been unable to locate a suitable reference.

Lemma 3.3. Let $\{S_a\}_{a \in A}$ be a family of sets indexed by a set A and let τ be an infinite cardinal number. Then if $\bar{A} \leq \tau$ and $\bar{S}_a \leq \tau$ for every $a \in A$, $\overline{\bigcup_{a \in A} S_a} \leq \tau$.

Proof: Let C be a set with $\bar{C} = \tau$. Then for each

$a \in A$, there exists a function f_a which maps C onto S_a . Define a mapping $g: C \times A \rightarrow \bigcup_{a \in A} S_a$ by $g((c, a)) = f_a(c)$ for each pair $(c, a) \in C \times A$. Clearly, g maps $C \times A$ onto $\bigcup_{a \in A} S_a$ and hence $\overline{\bigcup_{a \in A} S_a} \leq \overline{C \times A}$. But $\overline{C \times A} = \overline{C} \cdot \overline{A} = \tau \cdot \overline{A} \leq \tau^2 = \tau$ (Lemma 3.2). This completes the proof of the lemma.

Definition 3.4. Let M be a Λ module. The rank of M , which is denoted by \overline{M} , is the smallest cardinal number for which M has a set of generators of this cardinal.

The next several definitions and lemmas involve the notion of a direct limit of Λ modules. The complexity of this concept precludes us from including a discussion of it; however, several references are available. For instance, one may consult Eilenberg and Steenrod (6) or Rotman (20). Actually we shall only make use of a very special type of direct limit as is explained in Lemma 3.7.

Definition 3.5. A ring Λ is called left perfect if and only if a direct limit of projective left Λ modules is a projective left Λ module.

The class of left perfect rings was first studied by Bass (2). However, instead of giving Bass's original definition of these rings, which involves concepts for which we shall have no use, we selected one of the alternate

characterizations given by him in (2). The next lemma is also due to Bass (2).

Lemma 3.6. If Λ is either left or right Artinian, Λ is left perfect.

Lemma 3.7. Let M be a Λ module and $\{M_\alpha\}_{\alpha \in A}$ be a family of submodules of M indexed by a set A of ordinal numbers. Then if for all $\alpha, \alpha' \in A$, $\alpha < \alpha'$ implies that $M_\alpha \subset M_{\alpha'}$, and if $\bigcup_{\alpha \in A} M_\alpha = M$, then M is the direct limit of the M_α 's. This lemma is immediate from the definition of a direct limit.

Lemma 3.8. Let M be a left Λ module. If M is a submodule of a free left Λ module F , there exists a free submodule F' of F which contains M and satisfies the following conditions:

- (1) if \overline{M} is finite, $\overline{F'}$ is finite, and
- (2) if \overline{M} is infinite, $\overline{F'} \leq \overline{M}$.

Proof: Let G be a set of generators for M with $\overline{G} = \overline{M}$ and let B be a basis for F . For each $g \in G$, let B_g be the subset of B consisting of precisely those elements which have non-zero coefficients in the Λ linear expansion of g in terms of B . Then for each $g \in G$, $\overline{B_g}$ is finite. Hence it is immediate from Lemma 3.3 and the fact that B is a basis for

F that the submodule F' of F generated by the set $\bigcup_{g \in G} B_g$ satisfies the requirements of the lemma.

Lemma 3.9. If Λ is a left perfect ring with the property that every finitely generated left Λ module is a submodule of a free left Λ module, then every left Λ module is a submodule of a free left Λ module.

Proof: Suppose that the lemma is false. Then there must exist a smallest cardinal number τ for which there is a module M with $\overline{M} = \tau$ which cannot be imbedded in a free left Λ module. It is immediate from the hypotheses that τ is an infinite cardinal.

Let G be a set of generators for M with $\overline{G} = \overline{M} = \tau$. Let γ be the smallest ordinal associated with the cardinal τ . Since τ is an infinite cardinal, γ is a limit ordinal, i.e., γ does not have an immediate predecessor. The set of all ordinals strictly less than γ is a set of order type γ and cardinal $\overline{\gamma} = \tau$. Let $\alpha \rightarrow g_\alpha$ be a one-to-one correspondence between the set of all ordinals strictly less than γ and the set G . For each ordinal $\alpha < \gamma$, define M_α to be the submodule of M generated by the subset $G_\alpha = \{g_\beta : \beta < \alpha\}$ of G . Then the family $\{M_\alpha\}_{\alpha < \gamma}$ is an inclusion chain of submodules of M with the property that $M = \bigcup_{\alpha < \gamma} M_\alpha$. Furthermore, for each

$\alpha < \gamma$, $\overline{M}_\alpha < \tau$ since the set G_α is ordered by order type α which is strictly less than γ and hence has cardinal strictly less than τ .

We now show that the proof of this lemma will be complete if we can construct a family $\{F_\alpha\}_{\alpha < \gamma}$ of free left Λ modules having the following properties:

(1) for all $\alpha, \beta < \gamma$, $\alpha < \beta$ implies that F_α is a submodule of F_β ,

(2) for each $\alpha < \gamma$, $F_\alpha \cap M = M_\alpha$, where this is to be understood to mean that M_α is a submodule of F_α , and

(3) for each $\alpha < \gamma$, if $\overline{\alpha}$ is finite then \overline{F}_α is finite and if $\overline{\alpha}$ is infinite then $\overline{F}_\alpha \leq \overline{\alpha}$.

The third condition is a technical restriction which will be used later in proving that such a family exists and is not of importance at this point.

Let us, therefore, assume temporarily that we have shown the existence of such a family. Then by Lemma 3.7, the module $F = \bigcup_{\alpha < \gamma} F_\alpha$ is the direct limit of the F_α 's and hence is projective since Λ is assumed to be a left perfect ring. This means, of course, that F is a direct summand of a free left Λ module. However, it is immediate from Condition 2 above that $M = \bigcup_{\alpha < \gamma} M_\alpha$ is a submodule of F and

hence of a free Λ module which is a contradiction.

We, therefore, turn our attention to constructing such a family of free left Λ modules. We proceed by transfinite induction. It is clear from the way in which the M_α 's were defined that we can take $F_0 = 0$. It, therefore, remains to show for each $0 < \alpha < \gamma$, that if F_β is defined for all $0 \leq \beta < \alpha$, then F_α can also be defined. We must consider two cases:

Case I (α not a limit ordinal). Let $\alpha - 1$ be the immediate predecessor of α and define Rel to be the submodule

$$\left\{ (m, -m) : m \in M_{\alpha-1} \right\}$$

of the module $F_{\alpha-1} \oplus M_\alpha$. There exist Λ homomorphisms f_1 and f_2 from $F_{\alpha-1}$ and M_α , respectively, into $(F_{\alpha-1} \oplus M_\alpha)/\text{Rel}$ which are defined by composing the usual injections into the direct sum with the natural mapping onto $(F_{\alpha-1} \oplus M_\alpha)/\text{Rel}$. It is a straightforward matter to verify that Rel has been defined in such a way that:

- (A) f_1 and f_2 are both monomorphisms,
- (B) f_1 and f_2 are identical on $F_{\alpha-1} \cap M_\alpha = M_{\alpha-1}$, and
- (C) $\text{Im}(f_1) \cap \text{Im}(f_2) = f_1(M_{\alpha-1}) = f_2(M_{\alpha-1})$.

It is also immediate from the way in which $M_{\alpha-1}$ was defined

that the rank of $(F_{\alpha-1} \oplus M_{\alpha})/\text{Rel}$ is less than or equal to $\overline{F}_{\alpha-1} + \overline{\alpha}$ which, in view of (3) above, is finite if $\overline{\alpha}$ is finite and less than or equal to $\overline{\alpha-1} + \overline{\alpha} = \overline{\alpha}$ if $\overline{\alpha}$ is infinite. It, therefore, follows from Lemma 3.8 that there exists a free Λ module H_{α} which contains $(F_{\alpha-1} \oplus M_{\alpha})/\text{Rel}$ and is such that \overline{H}_{α} satisfies the requirements of Condition 3 above. Hence if we identify both $F_{\alpha-1}$ and M_{α} with their images in H_{α} under f_1 and f_2 , respectively, which can be done by virtue of Properties A, B and C above, we obtain the desired F_{α} .

Case II (α a limit ordinal). Let $S_{\alpha} = \bigcup_{\beta < \alpha} F_{\beta}$. Since α is a limit ordinal, $M_{\alpha} = \bigcup_{\beta < \alpha} M_{\beta}$ and hence $S_{\alpha} \cap M = (\bigcup_{\beta < \alpha} F_{\beta}) \cap M = \bigcup_{\beta < \alpha} (F_{\beta} \cap M) = M_{\alpha}$. It will, therefore, suffice to show that S_{α} can be imbedded in a free left Λ module with rank less than or equal to $\overline{\alpha}$. But by Lemma 3.8 and the way in which τ was chosen, all that is necessary for this is to show that $\overline{S}_{\alpha} \leq \overline{\alpha}$. Now for each $\beta < \alpha$, F_{β} has a set of generators B_{β} which satisfies the requirements of Condition 3 and, since the set of ordinals strictly less than α is of order type α and cardinal $\overline{\alpha}$, it is immediate from Lemma 3.3 that the set $B = \bigcup_{\beta < \alpha} B_{\beta}$ is a set of generators for S_{α} with $\overline{B} \leq \overline{\alpha}$. This completes the proof of the lemma.

We have now developed enough preliminary material to be

able to state and prove the main theorem of this chapter.

Theorem 3.10. Let Λ be a Noetherian ring. Then the following conditions are all equivalent:

- (1) Λ is a quasi-Frobenius ring,
- (2) every left Λ module is a submodule of a free left Λ module,
- (3) every left Λ module is torsionless,
- (4) every finitely generated left Λ module is torsionless, and
- (5) every finitely generated left Λ module is a submodule of a free left Λ module.

Proof: The equivalence of Conditions 1 and 2 has been established by Morita, Kawada, and Tachikawa (14).

That (2) implies (3) may be shown as follows: Let M be any left Λ module. Then there exists a free left Λ module F containing M . By Proposition 2.9, it suffices to show that F is torsionless. Let $\{b_\alpha\}_{\alpha \in A}$ be a basis for F and let $0 \neq x \in F$. Then there exist $b_{\alpha(i)}$ and $0 \neq \lambda_i \in \Lambda$ ($1 \leq i \leq k$) such that $x = \sum_i \lambda_i b_{\alpha(i)}$ ($1 \leq i \leq k$). Let $f: F \rightarrow {}_s\Lambda$ be the Λ module homomorphism induced by the conditions $f(b_{\alpha(1)}) = 1$ and $f(b_\alpha) = 0$ for all $b_\alpha \neq b_{\alpha(1)}$. Then $f(x) = \lambda_1 \neq 0$ and hence it follows from Proposition 2.8 that F is torsionless.

It is obvious that Condition 3 implies Condition 4, so we now show that (4) implies (5). Let M be any finitely generated left Λ module. Then there exists a free left Λ module F of finite rank and an epimorphism $\alpha: F \rightarrow M$. Hence the sequence $F \xrightarrow{\alpha} M \rightarrow 0$ is exact. Dualizing this sequence yields the exact sequence $0 \rightarrow M^* \rightarrow F^*$ of right Λ modules. Since F is finitely generated, it follows from Theorem 2.6 that F^* is finitely generated and hence so is M^* . Let g_1, g_2, \dots, g_k be a finite set of generators for M^* . Then, since M is torsionless, it follows from Proposition 2.8 that for each $m \neq 0$ in M , there is a g_j ($1 \leq j \leq k$) such that $g_j(m) \neq 0$. Consequently, the mapping $m \rightarrow (g_1(m), g_2(m), \dots, g_k(m))$ of M into the free left Λ module $\sum_{i=1}^k \Lambda_i$ ($1 \leq i \leq k$) is a monomorphism.

In view of what has been shown above, the proof of the theorem will be complete if we show that Condition 5 implies Condition 2. Hence, by virtue of Lemma 3.9, we need only prove that Λ is left perfect. Lemma 3.6 will be used for this, i.e., we shall show that Λ satisfies the descending chain condition on left ideals. Since we are assuming that every finitely generated left Λ module is a submodule of a free left Λ module, it is immediate, by the same reasoning

used above to show that (2) implies (3), that every finitely generated and hence, a fortiori, every cyclic left Λ module is torsionless. Consequently, Proposition 2.19 implies that $s(d(L)) = L$ for every left ideal L of Λ . Hence Proposition 2.20 implies that Λ satisfies the descending chain condition on left ideals. This completes the proof of the theorem.

In an abstract which appeared in a recent issue of the "Notices of the American Mathematical Society", Faith and Walker (7) have announced a number of characterizations of quasi-Frobenius rings which include the equivalence of Conditions 1 and 2 of Theorem 3.10 without any finiteness assumptions on the ring. Another result in this abstract states in part; "if every cyclic left Λ module is a submodule of a free left Λ module, Λ is left Artinian". Combining these two results with Lemmas 3.6 and 3.9, we obtain the following theorem:

Theorem 3.11. The following conditions are equivalent:

- (1) Λ is a quasi-Frobenius ring,
- (2) every left Λ module is a submodule of a free left Λ module, and
- (3) every finitely generated left Λ module is a submodule of a free left Λ module.

In light of the above results, it seems natural to wonder if Theorem 3.10 can be improved further. That is to say, can Condition 4 of Theorem 3.10 be weakened to the assumption that each cyclic left Λ module is torsionless? By virtue of Proposition 2.19, this is equivalent to asking if a Noetherian ring with the property that $s(d(L)) = L$ for every left ideal L of Λ must be a quasi-Frobenius ring. The answer to this question is "No" as is shown by the following example due to Nakayama (18):

Example 3.12. Let $\Omega = K(X)$ be the field of rational functions in X over a field K . Sending X into X^2 generates an isomorphism ρ of Ω with the subfield $\Omega^\rho = K(X^2)$. We will need the fact that $\Omega = \Omega^\rho \oplus \Omega^\rho X$, where it is to be understood that this is a direct sum decomposition of the additive group of Ω or of Ω regarded as a vector space over the subfield Ω^ρ .

Let A be a vector space over Ω with basis b_1, b_2 .

Define a partial multiplication in A as follows:

For all $\omega, \bar{\omega}$ in Ω ,

- (i) $(\omega b_1)(\bar{\omega} b_1) = (\omega \bar{\omega}) b_1$, (ii) $(\omega b_1)(\bar{\omega} b_2) = (\omega \bar{\omega}) b_2$,
 (iii) $(\omega b_2)(\bar{\omega} b_1) = (\omega \bar{\omega}^\rho) b_2$, and (iv) $(\omega b_2)(\bar{\omega} b_2) = 0$.

The above relations together with requirement that the distributive laws hold extend this multiplication to all of A

and make A into a ring with unit. However, A is not an algebra over Ω because of (iii).

It is easy to verify that A , N , and 0 exhaust the left ideals of A , where $N = \Omega b_2$ is the radical of A and is irreducible as a left ideal of A . But $N = \Omega b_2 = \Omega^p b_2 \oplus (\Omega^p X) b_2$ can be decomposed into a direct sum of proper right ideals and hence A has five right ideals: A , N , $\Omega^p b_2$, $(\Omega^p X) b_2$ and 0 .

It requires only straightforward calculations to show that $s(d(A)) = s(0) = A$, $s(d(N)) = s(N) = N$, and $s(d(0)) = d(0) = A$. While on the other hand, $d(s(N)) = d(s(\Omega^p b_2)) = d(s((\Omega^p X) b_2)) = d(N) = N$. Consequently, A is both a Noetherian and an Artinian ring, every cyclic left Λ module is torsionless, but A is not a quasi-Frobenius ring.

It is interesting to note, however, that the above question is answered in the affirmative by the following theorem due to Ikeda and Nakayama (9), if the finiteness assumptions on the ring are strengthened.

Theorem 3.13. Let Λ be an algebra of finite rank over a field K . Then Λ is a quasi-Frobenius ring if and only if $s(d(L)) = L$ for every left ideal L of Λ .

We will now discuss the connection between Theorems 3.10

and 3.13, and Example 3.12 from a slightly different point of view. The latter theorem and the example show that the annihilator relations assumed in the definition of a quasi-Frobenius ring (Definition 3.1) need only be required on one side in the case of an algebra of finite rank but that this is not true for Noetherian or Artinian rings. Now it seems reasonable to wonder in what if any ways Theorem 3.13 can be extended to Noetherian rings and, in view of the fact that the annihilators for modules introduced in terms of the dual reduce to the usual annihilators in the case of subsets of the ring (see Definitions 2.11 and 2.14 and Proposition 2.16), they would seem to provide a vehicle for attempting such extensions. That is to say, we might look for a characterization of quasi-Frobenius rings, valid for Noetherian rings as well as algebras, under the assumption that the submodules of some class of modules are closed (see Definition 2.17), where the class of modules under consideration consists entirely of modules of the same hand.

Probably the most natural possibilities along this line are the following:

(3.14) $\text{Ann}(\text{Ann}(K, F^*), F) = K$ for every submodule K of every finitely generated, free left Λ module F ,

(3.15) $\text{Ann}(\text{Ann}(K, M^*), M) = K$ for every submodule K of every finitely generated left Λ module M ,

(3.16) $\text{Ann}(\text{Ann}(K, F^*), F) = K$ for every submodule K of every free left Λ module F , and

(3.17) $\text{Ann}(\text{Ann}(K, M^*), M) = K$ for every submodule K of every left Λ module M .

Notice that (3.14) is implied by all the others while (3.17) implies all the others.

It is immediate from Proposition 2.18 that (3.17) is equivalent to Condition 3 of Theorem 3.10. Furthermore, since every finitely generated left Λ module is a quotient module of a finitely generated free left Λ module, Proposition 2.18 also implies that (3.14) is equivalent to Condition 4 of Theorem 3.10. Thus we can rephrase part of Theorem 3.10 as follows:

Corollary 3.18. Let Λ be a Noetherian ring. Then the following conditions are equivalent:

- (1) Λ is a quasi-Frobenius ring,
- (2) $\text{Ann}(\text{Ann}(K, F^*), F) = K$ for every submodule K of every finitely generated, free left Λ module F , and
- (3) $\text{Ann}(\text{Ann}(K, M^*), M) = K$ for every submodule K of every left Λ module M .

Which is certainly a reasonable extension of Theorem 3.13 to Noetherian rings.

The next objective is to consider another instance in which the annihilator relations that appear in the definition of a quasi-Frobenius ring can be halved in the manner of Theorem 3.13. First, however, we require a couple of known results. We begin with a theorem of Eilenberg and Nakayama (5).

Theorem 3.19. Let Λ be a left or right Artinian ring. Then Λ is a quasi-Frobenius ring if and only if Λ is right self injective, i.e., Λ_d is an injective right Λ module.

The next result is sometimes referred to as "the injective test theorem". It is proved in Jans's book (13, p. 47).

Lemma 3.20. A right Λ module Q is injective if and only if every diagram of the form

$$\begin{array}{c} Q \\ \nearrow \\ R \subset \Lambda_d, \end{array}$$

can be enlarged to a commutative diagram

$$\begin{array}{ccc} & Q & \\ \nearrow & & \nwarrow \\ R & \subset & \Lambda_d, \end{array}$$

where R is a right ideal of Λ .

Theorem 3.21. Let Λ be a right principal ideal ring. Then Λ is a quasi-Frobenius ring if and only if $s(d(L)) = L$ for every left ideal L of Λ .

Proof: Since Λ is a right principal ideal ring, it is, a fortiori, right Noetherian. It, therefore, follows from Proposition 2.20 that Λ is left Artinian. Therefore, by virtue of Theorem 3.19 and Lemma 3.20, it suffices to show that any Λ homomorphism f from any right ideal R of Λ into Λ can be extended to a Λ module endomorphism of Λ_d .

Since Λ is a right principal ideal ring, $R = a\Lambda$ for some $a \in \Lambda$. Let $a' = f(a)$. It is clear that $d(\{a'\}) \supset d(\Lambda a)$ and hence $\{a'\} \subset s(d(\{a'\})) \subset s(d(\{a\})) \subset s(d(\Lambda a))$ (Proposition 2.12). But, since Λa is a left ideal of Λ , it is immediate from the hypotheses that $s(d(\Lambda a)) = \Lambda a$ and hence a' is in Λa , i.e., $a' = \lambda a$. Thus f is given by a left multiplication and hence can be extended to Λ_d . This completes the proof of the theorem.

Remark 3.22. Example 3.12 shows that a left principal ideal ring which satisfies the annihilator relations of Theorem 3.21 need not be a quasi-Frobenius ring even if it is an Artinian ring. That is to say, the hypotheses of that

theorem cannot be rephrased so that both of the restrictions are placed upon the left ideals of the ring and still yield a characterization of quasi-Frobenius rings.

IV. REFLEXIVE MODULES AND QUASI-FROBENIUS RINGS

The relationship between torsionless modules and quasi-Frobenius rings found in the preceding chapter can be summarized by saying that a quasi-Frobenius ring was characterized as a Noetherian ring all of whose finitely generated left modules are torsionless (Theorem 3.10), and that an example was given (Example 3.12) which showed that a Noetherian ring all of whose cyclic left modules are torsionless need not be a quasi-Frobenius ring. The main objective of this chapter is to show that a quasi-Frobenius ring can be characterized as an Artinian ring all of whose cyclic left modules are reflexive (see Definition 2.3). In the process, several intermediate results are obtained which may be of interest in their own right. The principal point of departure in this chapter is Theorem 4.1 which is due to Dieudonne' (4).

It is known that any finitely generated left (right) module over a left (right) Artinian ring has a composition series and that any time a module has a composition series any two such series must have the same length. For any module M having a composition series, we shall use the notation $\lambda(M)$ to denote the length of any such series in M and refer to $\lambda(M)$ as the length of M .

Theorem 4.1. Let Λ be an Artinian ring. Then Λ is a quasi-Frobenius ring if and only if $\lambda(S^*) \leq 1$ for every simple left and simple right module S over Λ .

Lemma 4.2. Let M be a Λ module having a composition series. Then M is isomorphic to M^{**} if and only if M is reflexive.

Proof: Assume that M is isomorphic to M^{**} . Then $\lambda(M) = \lambda(M^{**})$. By Proposition 2.10, M^{**} is torsionless and hence it is immediate from Proposition 2.8 that M is also. Thus $\lambda(\text{Im}(\sigma_M)) = \lambda(M) = \lambda(M^{**})$. This completes the proof of the lemma.

Lemma 4.3. Let Λ be a left Artinian ring with the property that $s(d(L)) = L$ for every minimal left ideal L of Λ . Then $\lambda(S^*) \leq 1$ for every simple right module S over Λ .

Proof: Since Λ has an identity, any simple right Λ module is isomorphic to a quotient module of the form Λ_d/M , where M is a maximal right ideal of Λ . Its dual is, therefore, isomorphic to $s(M)$ (Proposition 2.13). If $s(M)$ had length ≥ 2 , it would properly contain a minimal left ideal L and hence $d(L) \supset d(s(M)) \supset M$. This would imply in turn that $d(L)$ equals M or Λ , but both of these possibilities lead to contradictions of the assumption that $s(d(L)) = L$. Hence

$\lambda(s(M)) \leq 1$. This completes the proof of the lemma.

The next lemma is a special case of Proposition 4 on page 89 of Northcott (19).

Lemma 4.4. If

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\
 & & \downarrow \alpha & & \downarrow & & \downarrow \gamma \\
 & & D & \rightarrow & E & \rightarrow & F \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

is a commutative diagram of Λ modules and Λ homomorphisms with exact rows and columns, there exists a Λ isomorphism $\delta: D/\text{Im}(\alpha) \rightarrow \text{Ker}(\gamma)$.

Lemma 4.5. Let S be a simple Λ module. Then the following conditions are equivalent:

- (1) S is torsionless,
- (2) $S^* \neq 0$, and
- (3) S is isomorphic to a minimal (one sided) ideal of Λ .

This lemma is immediate from the appropriate definitions and Propositions 2.8 and 2.9.

Theorem 4.6. Let Λ be an Artinian ring. Then Λ is a quasi-Frobenius ring if and only if for every minimal left ideal L of Λ ,

- (1) L is isomorphic to L^{**} , and
- (2) $\lambda(L^*) = 1$.

Proof: The necessity of Conditions 1 and 2 is immediate from Lemmas 4.3 and 4.5. We, therefore, turn to sufficiency.

Let L be any minimal left ideal of Λ , i the identity map of L into Λ , and η the natural map of ${}_s\Lambda$ onto ${}_s\Lambda/L$. Dualizing the exact sequence (A) $0 \rightarrow L \xrightarrow{i} {}_s\Lambda \xrightarrow{\eta} {}_s\Lambda/L \rightarrow 0$, we obtain the exact sequence (B) $0 \rightarrow ({}_s\Lambda/L)^* \xrightarrow{\eta^*} \Lambda^* \xrightarrow{i^*} L^*$. Since L^* is a simple right Λ module, $\text{Im}(i^*) = L^*$ or 0 . But, since $i^*(i_\Lambda) = i_\Lambda i = i$ (i_Λ is the identity map on Λ), which is not the zero element of L^* , $\text{Im}(i^*) = L^*$ and hence the following sequence is exact (C) $0 \rightarrow ({}_s\Lambda/L)^* \xrightarrow{\eta^*} {}_s\Lambda^* \xrightarrow{i^*} L^* \rightarrow 0$. It is immediate from Condition 1 and Lemma 4.2 that L is reflexive. Hence if we dualize Sequence C and use σ to connect it to Sequence A, recalling that ${}_s\Lambda$ is also reflexive, we obtain the following commutative diagram with exact rows and columns.

$$(D) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & L & \rightarrow & {}_s\Lambda & \rightarrow & 0 \\ & & \downarrow \sigma_L & & \downarrow \sigma_{{}_s\Lambda} & & \\ 0 & \rightarrow & L^{**} & \rightarrow & {}_s\Lambda^{**} & \rightarrow & ({}_s\Lambda/L)^{**} \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Diagram D is of the type considered in Lemma 4.4 and hence $L^{**}/\text{Im}(\sigma_L) = 0$ is isomorphic to $\text{Ker}(\sigma_{s\Lambda/L})$. Consequently, $s\Lambda/L$ torsionless. But, by Proposition 2.19, this means that $s(d(L)) = L$.

Since the above reasoning is valid for any minimal left ideal of Λ , it follows from Lemma 4.3 that $\lambda(S^*) \leq 1$ for every simple right Λ module S . Condition 2 and Lemma 4.5 can easily be combined to show that $\lambda(S^*) \leq 1$ for any simple left module S over Λ . It, therefore, follows from Theorem 4.1 that Λ is a quasi-Frobenius ring. This completes the proof of the theorem.

Remark 4.7. The opposite ring of the ring A constructed in Example 3.12 is an Artinian ring which satisfies the conditions of Lemma 4.3 (rephrased for right Artinian rings) and hence Condition 2 of Theorem 4.6, but it is not a quasi-Frobenius ring. There also exist Artinian rings which satisfy Condition 1 of the theorem but are not quasi-Frobenius rings. In fact, by virtue of Corollary 1.3 of Jans (11) and Theorem 3.19, any Artinian ring of injective dimension one as a right module over itself (see Northcott (19) for the notion of injective dimension) will be such a ring. The ring constructed in Problem 5 on page 63 of Jans's book (13) for the

case $n = 1$ can be shown to be such a ring.

Definition 4.8. Let M be a Λ module. Then the projective dimension $\text{Pd}_\Lambda(M)$ is the smallest non-negative integer n for which there exists an exact sequence of the form

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where each P_i is projective. If no such sequence exists, we put $\text{Pd}_\Lambda(M) = \infty$.

Definition 4.9. The left finitistic global dimension of a ring Λ , $\text{LfPD}(\Lambda)$, is the supremum of the set

$$\left\{ \text{Pd}_\Lambda(M) : M \text{ is a finitely generated left } \Lambda \text{ module and } \text{Pd}_\Lambda(M) < \infty \right\}.$$

The next lemma was proved by Jans (11).

Lemma 4.10. Let Λ be a Noetherian ring. Then $\text{LfPD}(\Lambda) = 0$ if and only if for every finitely generated right Λ module M , $M^* = 0$ implies that $M = 0$.

Lemma 4.11. Let Λ be a left Artinian ring with radical N . Then the following conditions are equivalent:

- (1) every simple left Λ module is torsionless, and
- (2) $s(d(N)) = N$.

The analogous result also holds for simple right Λ modules.

Proof: Since Λ is left Artinian, ${}_s\Lambda/N$ is a finite

direct sum, $\sum_1 S_i$, of simple left Λ modules, where the S_i have the property that if S is a simple left Λ module, then S is isomorphic to S_i for some i . Condition 1 is, therefore, equivalent to ${}_S \Lambda/N$ being torsionless. The proof is completed by applying Proposition 2.19.

Lemma 4.12. Let Λ be an Artinian ring. If every simple right Λ module is torsionless and every simple left Λ module is isomorphic to its second dual, then Λ is a quasi-Frobenius ring.

Proof: The lemma will follow from Theorem 4.6, if we can verify that $\lambda(L^*) = 1$ for every minimal left ideal L of Λ . Suppose that $\lambda(L^*) > 1$. Then L^* contains a maximal proper submodule M and L^*/M being a simple right Λ module is torsionless. Therefore, M is closed in L^* (see Definition 2.17). Since L^{**} is isomorphic to L , it is a simple left Λ module. Thus $\text{Ann}(M, L^{**}) = L^{**}$ or 0 . However, L^* , being a dual, is torsionless (Proposition 2.10) and hence both of these possibilities contradict the fact that M is closed in L^* . This completes the proof of the lemma.

Theorem 4.13. Let Λ be an Artinian ring. Then Λ is a quasi-Frobenius ring if and only if

- (1) $\text{LFPD}(\Lambda) = 0$, and

(2) every simple left Λ module is isomorphic to its second dual.

Proof: Suppose that Λ satisfies Conditions 1 and 2 of the theorem. Then it is immediate from Lemmas 4.5 and 4.10 that Λ satisfies the hypotheses of Lemma 4.12 and hence is a quasi-Frobenius ring.

The necessity of Condition 2 is well known. It can be derived easily from Lemmas 4.3 and 4.11. Lemma 4.10 can be used to show the necessity of Condition 1 as follows: It is immediate from Lemma 4.11 that if Λ is a quasi-Frobenius ring, then every simple right Λ module is torsionless and hence has non-zero dual (Lemma 4.5). Let $A \neq 0$ be any finitely generated right Λ module. Then A contains a maximal submodule M and hence A/M is a simple right module. By dualizing the exact sequence $A \xrightarrow{\eta} A/M \rightarrow 0$, where η is the natural mapping of A onto A/M , we see that $A^* \neq 0$.

Lemma 4.12 can also be used to derive the following theorem which is due originally to Morita and Tachikawa (15).

Theorem 4.14. Let Λ be an Artinian ring. Then Λ is a quasi-Frobenius ring if and only if every simple left and every simple right Λ module is isomorphic to its second dual.

The next lemma is immediate from Lemma 22 of Eilenberg

and Nakayama (5).

Lemma 4.15. Let Λ be a left or right Artinian ring with the property that $s(d(L)) = L$ for every left ideal L of Λ . Then every simple right Λ module is torsionless.

Theorem 4.16. Let Λ be an Artinian ring. Then Λ is a quasi-Frobenius ring if and only if

- (1) every cyclic left Λ module is torsionless, and
- (2) every simple left Λ module is isomorphic to its second dual.

Proof: Assume that Λ satisfies Conditions 1 and 2 above. It is immediate from Proposition 2.19 that Λ satisfies the hypotheses of Lemma 4.15 and hence also of Lemma 4.12. Λ is, therefore, a quasi-Frobenius ring.

The necessity of Conditions 1 and 2 has been mentioned several times previously.

This last theorem is in some ways a refinement of the main result for this chapter as discussed in the introduction to the chapter. We now prove the main result of the chapter.

Theorem 4.17. Let Λ be an Artinian ring. Then the following conditions are equivalent:

- (1) Λ is a quasi-Frobenius ring,
- (2) every finitely generated left Λ module is reflexive,

- (3) every cyclic left Λ module is reflexive, and
- (4) every cyclic left Λ module is isomorphic to its second dual.

Proof: It is known that (1) implies (2). A proof may be found in Jans (11). It is obvious that (2) implies (3) and (3) implies (4). Assume, therefore, that Λ satisfies Condition 4. Then, since duals are torsionless (Proposition 2.10), every cyclic left Λ module is torsionless. We can complete the proof by applying Theorem 4.16.

Remark 4.18. Theorem 4.17, particularly the equivalence of Conditions 1 and 2, can be regarded as an alternative to the approach discussed in connection with Corollary 3.18 for obtaining an extension of Theorem 3.13 from algebras to Artinian rings. In that corollary, an extension was presented which was obtained by what amounted to enlarging the class of left modules assumed to be torsionless from the cyclic left modules to all the finitely generated left modules. In Theorem 4.17, attention remains fixed on the class of cyclic left modules but the assumption concerning them is increased to the requirement that they be reflexive instead of merely torsionless.

We note finally that Theorem 4.17 applies to Artinian

rings rather than Noetherian rings as is the case with Corollary 3.18 but that this defect will be remedied in the next chapter (Theorem 5.3).

It seems natural to wonder if Theorem 4.16 can be refined further. To be more specific, is an Artinian ring which satisfies Condition 2 of this theorem necessarily a quasi-Frobenius ring? Unfortunately, we have been unable to resolve this question in general. We have, however, obtained an affirmative answer in a certain special case. This will be considered after we introduce a few more preliminary concepts.

Definition 4.19. Let $\{L_a\}_{a \in A}$ be the family of all minimal left ideals of a ring Λ . Then the left socle of Λ is $\sum_a L_a$ ($a \in A$). The right socle of Λ is defined analogously.

Lemma 4.20. Let Λ be an Artinian ring with radical N . Then the left and right socles of Λ are subsets of $d(N)$ and $s(N)$, respectively.

Proof: Since the two parts are similar, we shall prove only that $d(N)$ contains the left socle. It clearly suffices to show that every minimal left ideal L of Λ is a subset of $d(N)$. Now NL is a left ideal of Λ contained in L so $NL = L$ or 0 . However, since N is nilpotent, the first of these two

possibilities leads to a contradiction. This completes the proof of the lemma.

Following Jacobson (10), we say that Λ is a primary ring if Λ is an Artinian ring whose radical is a maximal (two sided) ideal.

Theorem 4.21. Let Λ be a direct sum of a finite number of primary rings. Then Λ is a quasi-Frobenius ring if and only if every simple left Λ module is isomorphic to its second dual.

Proof: In view of Lemma 4.12, it suffices to show that every simple right Λ module is torsionless. However, for this, it clearly suffices to consider the case where Λ is a primary ring.

Let N be the radical of Λ . Then $d(s(N))$ is a two sided ideal of Λ containing N . Hence $d(s(N)) = N$ or Λ . But, by Lemma 4.20, $s(N)$ contains the right socle of Λ which is not zero since Λ is an Artinian ring. Consequently, $d(s(N)) \neq \Lambda$. The proof is completed by applying Lemma 4.11.

V. FURTHER RESULTS ON QUASI-FROBENIUS RINGS

In this chapter we will give another and more homological proof of the main result of Chapter IV (Theorem 4.17) which enables us to prove it for Noetherian rings. As an outgrowth of this technique, we will obtain another characterization of quasi-Frobenius rings (Theorem 5.9) which is not only of interest in its own right but also yields several interesting corollaries.

The first several results of this chapter require, either in their statement or proof, some use of the first derived functor Ext_Λ^1 of the functor Hom_Λ and its elementary properties. The limitations of space preclude us from presenting a discussion of these concepts. We refer the reader to Chapter 7 of Northcott's book (19) or Chapter 6 of the book by Cartan and Eilenberg (3) for a general discussion of these ideas. Wherever feasible in the sequel, we shall make specific reference to one of these sources.

The first lemma of this chapter is a special case of Proposition 10, Auslander (1).

Lemma 5.1. Let Λ be a left or right Artinian ring with radical N and let M be a left Λ module. Then the following conditions are equivalent:

(1) $\text{Ext}_{\Lambda}^1[(\text{}_{\text{s}}\Lambda/N), M] = 0$, and

(2) M is an injective left Λ module.

We now prove a rather technical lemma.

Lemma 5.2. Let Λ be a right Artinian ring having radical N . Then Λ is a quasi-Frobenius ring, provided $s(d(N)) = N$ and $\Lambda_d/d(N)$ is a reflexive right Λ module.

Proof: In order to simplify the notation, we denote $\Lambda_d/d(N)$ by Γ_d . Let i be the identity map of $d(N)$ into Λ_d and η the natural mapping of Λ_d onto Γ_d . Dualizing the exact sequence (A) $0 \rightarrow d(N) \xrightarrow{i} \Lambda_d \xrightarrow{\eta} \Gamma_d \rightarrow 0$, we obtain the exact sequence (B) $0 \rightarrow \Gamma_d^* \xrightarrow{\eta^*} \Lambda_d^* \xrightarrow{i^*} (d(N))^*$.

We know from Proposition 2.13 that the natural mapping $\rho_d: \Lambda_d^* \rightarrow \text{}_{\text{s}}\Lambda$ is an isomorphism and carries $\text{Im}(\eta^*)$ onto $s(d(N))$ which is equal to N by hypothesis. Consequently, if we compose ρ_d with the natural map η' of $\text{}_{\text{s}}\Lambda$ onto $\text{}_{\text{s}}\Lambda/N$, we obtain an exact sequence (C) $0 \rightarrow \Gamma_d^* \xrightarrow{\eta^*} \Lambda_d^* \xrightarrow{\alpha} \text{}_{\text{s}}\Lambda/N \rightarrow 0$, where $\alpha = \eta' \rho_d$.

If we dualize (C) and connect it with the first part of the Ext - sequence (Northcott (19), p. 131), we obtain the exact sequence (D) $0 \rightarrow (\text{}_{\text{s}}\Lambda/N)^* \rightarrow \Lambda_d^{**} \rightarrow \Gamma_d^{**} \rightarrow \text{Ext}_{\Lambda}^1[(\text{}_{\text{s}}\Lambda/N), \text{}_{\text{s}}\Lambda] \rightarrow \text{Ext}_{\Lambda}^1(\Lambda_d^*, \text{}_{\text{s}}\Lambda)$. However, since Λ_d^* is a projective left Λ module, $\text{Ext}_{\Lambda}^1(\Lambda_d^*, \text{}_{\text{s}}\Lambda) = 0$ (Northcott (19), p.

p. 131).

If we now use σ to connect Sequences A and D, recalling that Λ_d is always reflexive and that Γ_d is reflexive by hypothesis, we obtain the following commutative diagram with exact rows and columns

$$(E) \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & d(N) & \xrightarrow{i} & \Lambda_d & \xrightarrow{\eta} & \Gamma_d & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ({}_s\Lambda/N)^* & \xrightarrow{\alpha^*} & \Lambda_d^{**} & \xrightarrow{\eta^{**}} & \Gamma_d^{**} & \longrightarrow & \text{Ext}_{\Lambda}^1({}_s\Lambda/N, {}_s\Lambda) & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Since Diagram E is commutative and σ_{Λ_d} and η are epimorphisms, $\text{Im}(\eta^{**}) = \text{Im}(\sigma_{\Gamma_d})$ and hence the sequence

$$(F) \quad 0 \longrightarrow \Gamma_d \xrightarrow{\sigma_{\Gamma_d}} \Gamma_d^{**} \longrightarrow \text{Ext}_{\Lambda}^1({}_s\Lambda/N, {}_s\Lambda) \longrightarrow 0 \text{ is exact.}$$

The exactness of Sequence F, together with the fact that σ_{Γ_d} is an epimorphism, implies that $\text{Ext}_{\Lambda}^1({}_s\Lambda/N, {}_s\Lambda) = 0$. It, therefore, follows from Lemma 5.1 that ${}_s\Lambda$ is an injective left Λ module. The proof is completed by applying the left hand analog of Theorem 3.19.

We are now in a position to give the alternate proof of Theorem 4.17 discussed in the introduction to this chapter. Actually, due to a certain perversity of nature, it was

impossible to phrase Lemma 5.2 in such a way that it could be used to prove both this result and Theorem 5.9 for left hand modules; so we shall actually prove the right hand analog of that theorem.

This presents nothing more than semantic difficulty since all one sided results concerning quasi-Frobenius rings can be phrased on either side due to the left-right symmetry of their definition and could be avoided entirely by appealing to the left hand analog of Lemma 5.2.

Theorem 5.3. Let Λ be a Noetherian ring. Then the following conditions are equivalent:

- (1) Λ is a quasi-Frobenius ring,
- (2) every finitely generated right Λ module is reflexive,
- (3) every cyclic right Λ module is reflexive, and
- (4) every cyclic right Λ module is isomorphic to its second dual.

Proof: Using exactly the same reasoning as in the first part of the proof of Theorem 4.17, we see that the proof reduces to showing that (4) implies (1). Assume, therefore, that Λ satisfies Condition 4. Then, since duals are torsionless, every cyclic right Λ module is torsionless and hence $d(s(R)) = R$ for every right ideal R of Λ (Proposition 2.19).

It, therefore, follows from Proposition 2.20 that Λ is right Artinian. We can now apply Lemma 4.2 to show that every cyclic right Λ module is reflexive and hence that $\Lambda_d/d(N)$ is reflexive. Furthermore, since $d(s(R)) = R$ for every right ideal R of Λ , it follows by the right hand analog of Lemma 4.15 that every simple left Λ module is torsionless and hence that $s(d(N)) = N$ (Lemma 4.11). We can now apply Lemma 5.2 to complete the proof of the theorem.

It is interesting that Lemma 4.15 plays a key role in both proofs of this theorem.

If we examine the proof of Theorem 5.3, we see that it is equally valid for a right Artinian ring, i.e., we have also proved the following theorem:

Theorem 5.4. Let Λ be a right Artinian ring. Then the following conditions are equivalent:

- (1) Λ is a quasi-Frobenius ring,
- (2) every finitely generated right Λ module is reflexive,
- (3) every cyclic right Λ module is reflexive, and
- (4) every cyclic right Λ module is isomorphic to its second dual.

Definition 5.5. A left ideal L of a ring Λ satisfies Shoda's condition if and only if every Λ module homomorphism

of L into ${}_s\Lambda$ can be extended to an endomorphism of ${}_s\Lambda$.

Remark 5.6. In view of the fact that all rings considered in this paper are assumed to have a multiplicative identity, it is clear that a left ideal satisfies Shoda's condition if and only if all of its homomorphisms into the ring are right multiplications by elements of the ring.

Lemma 5.7. Let Λ be a ring and let R be any right ideal in Λ . Then there exists a Λ module homomorphism $\mu_R: \Lambda_d \rightarrow (\Lambda_d/R)^{**}$ with $\text{Ker}(\mu_R) = d(s(R))$. Furthermore, μ_R can be constructed in such a way that it is an epimorphism if and only if $s(R)$ satisfies Shoda's condition.

Proof: We verified in Proposition 2.13 that $(\Lambda_d/R)^*$ is isomorphic to $s(R)$ and, since the duals of isomorphic modules are again isomorphic, it will suffice to construct a Λ homomorphism $\alpha_R: \Lambda_d \rightarrow [s(R)]^*$ which has the desired properties.

Let α_R be defined by the condition that $\alpha_R: \lambda \rightarrow (k \rightarrow k\lambda)$ for all k in $s(R)$ and λ in Λ , i.e., each λ in Λ goes into the right multiplication which it induces on $s(R)$. It is obvious that α_R takes values in $[s(R)]^*$. We will verify that it is in fact a right Λ module homomorphism. For all λ_1 and λ_2 in Λ and k in $s(R)$,

$$[\alpha_R(\lambda_1 + \lambda_2)](k) = k(\lambda_1 + \lambda_2) = k\lambda_1 + k\lambda_2 = [\alpha_R(\lambda_1)](k) +$$

$[\alpha_R(\lambda_2)](k)$ and $[\alpha_R(\lambda_1\lambda_2)](k) = k(\lambda_1\lambda_2) = (k\lambda_1)\lambda_2 =$
 $([\alpha_R(\lambda_1)](k))\lambda_2 = ([\alpha_R(\lambda_1)]\lambda_2)(k)$ (see Definition 2.1).

It is obvious from the definition of α_R that $\text{Im}(\alpha_R)$ consists of precisely those Λ homomorphisms of $s(R)$ into ${}_s\Lambda$ which are induced by right multiplications and hence α_R is an epimorphism if and only if $s(R)$ satisfies Shoda's condition (see Remark 5.6).

Finally, $\lambda \in \text{Ker}(\alpha_R)$ if and only if $k\lambda = [\alpha_R(\lambda)](k) = 0$ for all k in $s(R)$. Hence $\text{Ker}(\alpha_R) = d(s(R))$.

Lemma 5.8. Let Λ be an Artinian ring and R be any right ideal of Λ . Then Λ_d/R is reflexive if and only if

- (1) $d(s(R)) = R$ and
- (2) $s(R)$ satisfies Shoda's condition.

Proof: Assume that Λ_d/R is reflexive. It follows from Proposition 2.19 that $d(s(R)) = R$. By Lemma 5.7, there exists a Λ homomorphism $\mu_R: \Lambda_d \rightarrow (\Lambda_d/R)^{**}$ with $\text{Ker}(\mu_R) = d(s(R)) = R$. Hence μ_R induces a monomorphism $\bar{\mu}_R: \Lambda_d/R \rightarrow (\Lambda_d/R)^{**}$ with $\text{Im}(\mu_R) = \text{Im}(\bar{\mu}_R)$. Since Λ_d/R is reflexive, $\lambda(\Lambda_d/R) = \lambda((\Lambda_d/R)^{**})$ and, since $\bar{\mu}_R$ is a monomorphism, $\lambda(\Lambda_d/R) = \lambda(\text{Im}(\bar{\mu}_R))$. Consequently, both $\bar{\mu}_R$ and μ_R are epimorphisms and hence it follows from Lemma 5.7 that $s(R)$ satisfies Shoda's condition.

Assume that R satisfies Conditions 1 and 2 above. It follows from Lemma 5.7 that there exists an epimorphism $\mu_R: \Lambda_d \rightarrow (\Lambda_d/R)^{**}$ with $\text{Ker}(\mu_R) = d(s(R)) = R$. Consequently, μ_R induces an isomorphism between Λ_d/R and its second dual. The proof of this lemma is completed by applying Lemma 4.2.

Theorem 5.9. Let Λ be an Artinian ring with radical N . Then Λ is a quasi-Frobenius ring if and only if

- (1) $s(d(N)) = N$, and
- (2) ${}_s N$ satisfies Shoda's condition.

Proof: The necessity of Condition 2 is immediate from Theorem 3.19. Assume, therefore, that Λ satisfies Conditions 1 and 2 above. Consider the right Λ module $\Lambda_d/d(N)$. Then $s(d(N)) = {}_s N$ which satisfies Shoda's condition. Furthermore, $d(s(d(N))) = d(N)$. Hence $\Lambda_d/d(N)$ is reflexive (Lemma 5.8). To complete the proof of the theorem, it remains only to apply Lemma 5.2.

Remark 5.10. Example 3.12 shows that Condition 1 of Theorem 5.9 is not a sufficient condition for an Artinian ring to be a quasi-Frobenius ring. We do not know whether Condition 2 is in general a sufficient condition.

Theorem 5.11. Let Λ be a direct sum of a finite number of primary rings and have radical N . Then Λ is a quasi-

Frobenius ring if and only if ${}_S N$ satisfies Shoda's condition.

Proof: The same reasoning used in the proof of Theorem 4.21 shows that in a ring which satisfies the hypotheses of this theorem $s(d(N)) = N$. Thus this theorem follows from Theorem 5.9.

Corollary 5.12. Let Λ be a ring which is a finite direct sum of primary rings and whose radical N is a principal left and a principal right ideal. Then Λ is a quasi-Frobenius ring.

Proof: By Theorem 38 in Chapter 4 of Jacobson's book (10, p. 76), there is an element w in Λ such that $N = w\Lambda = \Lambda w$. Let $f: {}_S N \rightarrow {}_S \Lambda$ be a Λ module homomorphism and let $w' = f(w)$. Clearly $s(\{w'\}) \subseteq s(\{w\}) \supseteq s(w\Lambda)$ and hence $\{w'\} \subseteq d(s(\{w'\})) \subseteq d(s(w\Lambda))$. But $d(s(w\Lambda)) = d(s(N)) = N = w\Lambda$, and hence w' is in $w\Lambda$, i.e., there is an element λ in Λ such that $w' = w\lambda$. Consequently, f is given by a right multiplication and hence ${}_S N$ satisfies Shoda's condition. Thus this corollary follows from Theorem 5.11.

Corollary 5.13. Let Λ be an Artinian ring with radical N in which every two sided ideal is a principal left and a principal right ideal. Then Λ is a quasi-Frobenius ring.

Proof: It is immediate from Theorem 37 in Chapter 4 of

Jacobson's book (10, p. 75) that such a ring satisfies the hypotheses of Corollary 5.12.

Remark 5.14. It is immediate from this corollary and Proposition 2.20 that a ring in which all two sided ideals are both left and right principal and in particular a commutative principal ideal ring is a quasi-Frobenius ring if and only if it is an Artinian ring.

For the remainder of this chapter, we shall be concerned with commutative rings. Since, in this context, there is no distinction between the left and right annihilators of a subset S of the ring, we shall refer to their common value as the annihilator of S and denote it by S' .

Theorem 5.15. Let Λ be a commutative Artinian ring with radical N . Then Λ is a quasi-Frobenius ring if and only if N satisfies Shoda's condition.

Proof: In view of Theorem 5.9, it suffices to show that $N'' = N$. Let M be any maximal ideal in Λ . Then, since $M'' \supset M$, $M'' = M$ or Λ . But $M = N + \Lambda(1-e)$ for a suitable primitive idempotent e of Λ . Therefore, $M' = (N + \Lambda(1-e))' = N' \cap \Lambda e$. But, by Lemma 4.20, N' contains the socle of Λ , i.e., the sum of all the minimal ideals of Λ and hence $N' \cap \Lambda e \neq 0$. Consequently, $M'' = M$. Furthermore, if I_1 and

I_2 are any two ideals of Λ which are closed in Λ , then $I_1 \cap I_2$ is closed in Λ . For $(I_1 \cap I_2)' \supseteq I_1' + I_2'$ and hence $I_1 \cap I_2 \subset (I_1 \cap I_2)'' \subset (I_1' + I_2')' = I_1'' \cap I_2'' = I_1 \cap I_2$. We can, therefore, verify by means of a finite induction that the intersection of a finite number of ideals all of which are closed in Λ is again closed in Λ . But, since N is the intersection of a finite number of maximal ideals, this completes the proof of the theorem.

Corollary 5.16. Let Λ be a commutative Artinian ring whose radical N is a principal ideal. Then Λ is a quasi-Frobenius ring.

Proof: By virtue of Theorem 5.15, it suffices to show that N satisfies Shoda's condition. However, since $N'' = N$ (see the proof of Theorem 5.15) and is a principal ideal, this can be done by using exactly the same reasoning as in the proof of Corollary 5.12.

Remark 5.17. This corollary is a refinement of Corollaries 5.12 and 5.13 for the case of commutative rings. The ring constructed in Problem 5 on page 63 of Jans's book (13) for the case $n = 1$ shows that this sharper result does not hold in general.

We now give a couple of applications of this corollary.

Example 5.18. Let Z be the ring of integers and n be any positive integer. Then the quotient ring Z_n of integers modulo n is a finite, commutative, principal ideal ring and hence by Corollary 5.16 a quasi-Frobenius ring.

Example 5.19. Let K be a field and $K[X]$ the ring of polynomials over K in one indeterminate. If I is any proper ideal in $K[X]$, it is easy to verify that the quotient ring $K[X]/I$ is a commutative, Artinian, principal ideal ring and hence by Corollary 5.16 is a quasi-Frobenius ring.

Actually these two examples are special cases of the following general result:

Corollary 5.20. Let Λ be a commutative principal ideal domain and let I be any proper ideal in Λ . Then the quotient ring Λ/I is a quasi-Frobenius ring.

Proof: Part of Theorem 32 on page 242 of Zariski and Samuel's book (21) states that any such quotient ring satisfies the descending chain condition on ideals. Thus this corollary follows from Corollary 5.16.

This last corollary can also be derived from Corollary 5.13 or directly from the definition of a quasi-Frobenius ring by making use of the structure theorem for commutative principal ideal rings developed in Section 15 of the fourth chapter of Zariski and Samuel's book (21).

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