The optimal capacity expansion policy for a firm

Amitabha Sen
Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Economics Commons

Recommended Citation
https://lib.dr.iastate.edu/rtd/4267

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
SEN, Amitabha, 1933-
THE OPTIMAL CAPACITY EXPANSION POLICY FOR A
FIRM.

Iowa State University, Ph.D., 1970
Economics, general

University Microfilms, A XEROX Company, Ann Arbor, Michigan
THE OPTIMAL CAPACITY EXPANSION POLICY FOR A FIRM

by

Amitabha Sen

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Economics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1970
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER I. INTRODUCTION</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Case for Incorporating Sales Constraints in the Theory of the Firm</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER II. SURVEY OF LITERATURE</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Notions of Fixed, Semi-fixed and Variable Cost: The Concept of Capacity</td>
<td>14</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER III. MODELS OF OPTIMAL CAPACITY EXPANSION</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-product Firm</td>
<td>32</td>
</tr>
</tbody>
</table>

| CHAPTER IV. HICKSIAN MODEL OF A MULTI-PRODUCT FIRM | 46 |

| CHAPTER V. PROGRAMMING MODEL OF A MULTI-PRODUCT FIRM | 53 |

| CHAPTER VI. THE MODEL | 73 |

| CHAPTER VII. THE OPTIMAL CAPACITY EXPANSION POLICY FOR A MULTI-PRODUCT FIRM | 87 |

| CHAPTER VIII. BALANCED GROWTH PATH FOR AN EXPANDING FIRM | 112 |

| LITERATURE CITED | 154 |

| ACKNOWLEDGEMENTS | 159 |
CHAPTER I. INTRODUCTION

A Case for Incorporating Sales Constraints in the Theory of the Firm

Traditional marginalist theory of the firm, as it evolved from the time of Cournot, is mathematically speaking, most elegant in the case of pure unilateral monopoly. Implicit in the deterministic, partial equilibrium framework, in which the optimization problem for a firm is posed, is a distinction between the interdependence of the first order of smallness and that of a higher order. The whole economy is divided into industries producing similar products. There are two types of interdependence among these industries that are brought into prominence in a general equilibrium analysis. In the first place, the industry may produce intermediate goods that are used as inputs by the other industries. But even when it produces only final goods, it competes with other industries for the consumers' dollar. Secondly, each industry is directly or indirectly related with other industries, as it is shown in an input-output table, because of its purchase of inputs from other industries and productive services from the household sectors.

However, any marginal change in the price or offer variation in any particular industry (which, itself, is of first-order of smallness, by definition) is diffused, in a random manner, over the rest of the economy so that the impact felt by any firm, or a group of firms, not in the same industry, could, at most, be of only the second-order of smallness. Under such conditions the ceteris paribus assumption about the rest of the economy holds (Olson and McFarland, 1962).
A pure monopolist is the only firm in the industry. Hence, the rest of the economy is a passive shock absorber of any chain reaction pattern transmitted throughout the economy by a marginal change in its price-output policy. As there is no feedback, the monopolist is faced with a known determinate demand function relating the price that would clear the market with the amount offered by it. Assuming that the firm purchases all its inputs in a pure competitive market, it would have a determinate cost function as well. Such a situation is tailor-made for the application of infinitesimal calculus. If the demand function depends also on the rate of change of price, we obtain another tailor-made situation for the application of variational calculus (Evans, 1930; Tintner, 1942).

In all other market situations, a determinate result is obtained only by introducing some behavioristic ritual as a supplement to (or as one may argue, as a substitute for) the usual profit maximizing rule. In the case of a duopoly or oligopoly situation such behavioristic ritual (or decision rule, as it is called in a game-theoretic context) is made explicit. However, even if it is not made explicit, it permeates through the entire analytical scaffolding.

Take the case of a large group of symmetrical firms producing branded but close substitute products. One decision rule is that each firm would produce that output at which its marginal cost is equated with marginal revenue based on a ceteris paribus 'dd' curve facing it. However, if all the firms in the industry (including new entrants, if any) behave according to the same decision rule, the expectation of each firm is always falsified unless the group equilibrium is reached, as the market evolves.
according to the _mutatis mutandis_ "DD" curve with a much flatter slope. It may be noted that the expectations would be fulfilled at group equilibrium only if they are generated by the same decision rule as before. The position of such a firm is therefore not much different from a Cournot-type duopolist.

The ingenuity of Chamberlin (1965) lies in seeing that such an "as if monopolist" elasticity-induced price-quantity adjustment on the basis of the "dd" curve is more reasonable in the above large-group case than in an oligopoly, or duopoly situation, though ultimately it has the effect of wiping out any profit in equilibrium, an "as if competitive" outcome. As the effect of any marginal change of price-output policy of a firm would be spread thin over the entire group, it would tend to become of second-order of smallness as the number of firms in the group becomes larger and larger. So there would not be any oligopolistic reaction on the part of other firms in the group as a result of the action of any particular firm. Since the state of the market, as reflected by the "DD" curve, is a result of their joint action, it would not do to argue that any firm in a depressed market would not cut price "for the fear of spoiling the market" since the firm does not have to pay a penalty commensurate with its own contribution to the tilting of the "DD" curve. In the absence of any collusive arrangement each firm would follow its own self-interest. Similarly, in a sellers' market, each firm may "make hay while the sun shines" by going for a "fleeting surplus profit," charging an unduly high price and thus attracting more firms into the industry than would be optimal for its long-run interests (Paul, 1954). However, the reasonableness of such
a behavioristic ritual has not gone unchallenged in the polemical literature on the "doctrine of excess capacity" (Harrod, 1952, 1967; Hicks, 1954; Hahn, 1955; Edwards, 1955; Lydell, 1955).

The decision rule implied in the "pure competitive behavior" of a firm is even simpler, namely each firm will take the ruling price, \( p \), as parametrically given. Under such a decision rule, the equilibrium of the industry can be studied in isolation from the equilibrium of the firm.

Assume that each firm has a determinate cost function that is convex and piecewise-smooth, i.e., has continuous derivative with respect to output except at some corner points, where both the left-hand and right-hand derivatives exist, within its admissible region of output variation. The lower end-point of the admissible region is given by zero output at which the right-hand derivative of the cost function exists. The upper end-point of the admissible region is given by the capacity output of the firm at which the left-hand derivative of the cost function exists. The right-hand derivative of the cost function may be called \( f_{mc} \), the forward marginal cost, and the left-hand derivative of the cost function may be called backward marginal cost or \( b_{mc} \). Since each firm is a price-taker (i.e., a price-fixed quantity adjuster) at any price, \( p \), it will supply a definite quantity. Let \( S_i(p) \) be the amount supplied by the \( i \)th producer at price \( p \), and \( \bar{S}_i \) be its capacity output. \( S_i(p) = 0 \) if \( f_{mc} \) is higher than \( p \) at \( S_i = 0 \) and throughout the admissible region of output variation for the \( i \)th firm. \( S_i(p) = \bar{S}_i \), if \( b_{mc} \) at \( \bar{S}_i \) is less than or equal to \( p \). In all other cases, \( b_{mc} \leq p \leq f_{mc} \) for some \( \bar{S}_i \).
0 < \tilde{S}_i < \bar{S}_i. If \text{bmc} \neq \text{fmc}, then \tilde{S}_i = S_i(p). If \text{bmc} = \text{fmc}, then we need a further check to see whether at \tilde{S}_i(-) just below \tilde{S}_i, \text{fmc} is less than \text{p} or not. If yes, then \tilde{S}_i = S_i(p) once again. If not, then there would be some other \hat{S}_i, \hat{S}_i > \tilde{S}_i, for which \text{bmc} \leq \text{p} \leq \text{fmc} and \hat{S}_i = S_i(p); or \text{bmc} \leq \text{p} up to \bar{S}_i, when S_i(p) = \bar{S}_i. By aggregating over all the firms in the industry we get the supply of the industry at a given price,

\begin{equation}
(1) \quad S(p) = \sum_{i=1}^{n} S_i(p),
\end{equation}

where \( n \) is the number of firms (indexed by \( i \)) in the industry.

By varying the price parametrically, we generate the supply function, \( S(p) \), of the industry which might be assumed to be continuous and single-valued\(^1\) with the following properties:

\begin{enumerate}
  \item \( \overline{S} \geq S(p) \geq 0; \ p \geq 0 \)
  \item \( S(p) = 0 \) for \( 0 \leq p \leq p \)
  \item \( S(p) = \overline{S} \) for \( p \geq \overline{p} > 0 \)
\end{enumerate}

where \( p \) is the highest price at which no supply will be forthcoming into the market and \( \overline{p} \) is the lowest price at which supply reaches its upper limit \( \overline{S} \).

If the marginal cost of the "marginal firm" is positive, \( p \) will exist. \( \overline{S} \) may be interpreted as the sum total of the output capacities

\(^1\)We shall make the distinction between a "function" and a "correspondence." The adjective "single-valued" is therefore redundant.
(technical output limits) of the firms in the industry in a given time period

\[ (3) \quad \bar{S} = \sum_{i=1}^{n} S_i. \]

Let there be one and only one price at which the market will take the quantity \( S > 0 \). Define \( \bar{p} = \bar{v}(S) \) as the market clearing price of \( S \). An equilibrium price is defined as the price \( p^* \) at which \( p^* = \bar{v}(S(p^*)) \) and \( S(p^*) > 0 \).

**Proposition 1**

Let \( \bar{v}(S) \) be continuous in \( S, 0 < S < \bar{S} \). An equilibrium price (at which the expectations are realized) will exist if the following conditions hold:

\[ (4) \quad \lim_{S \to 0^+} \bar{v}(S) > p; \quad \lim_{S \to S^-} \bar{v}(S) < \bar{p} \]

**Proof**

Define \( F(p) = \bar{v}(S(p)) - p, p \in (p, \bar{p}) \)

Since \( p \) is the highest price at which \( S(p) = 0 \), \( S(p) > 0 \) for \( p \in (p, \bar{p}) \). Hence, \( F(p) \) is defined and continuous in \( p \in (p, \bar{p}) \) as \( \bar{v}(S) \) is defined and continuous on \((0, \bar{S})\).

For any \( \epsilon > 0 \), \( F(p + \epsilon) = \bar{v}(S(p + \epsilon)) - p - \epsilon \)

As \( \epsilon \to 0^+ \)

\[ F(p + \epsilon) = \bar{v}(0^+) - p > 0; \quad \text{by} \quad 4. \]

Hence, there exists some \( \delta > 0 \), for which \( F(p + \delta) > 0 \), since \( F(p) \) is continuous on \((p, \bar{p})\).
At \( \bar{p} \); \( F(\bar{p}) = \Psi(S(\bar{p})) - \bar{p} = \Psi(S) - \bar{p} \leq 0 \)

since \( \Psi(S) \) is right continuous at \( \bar{S} \). If the equality in 5 holds then
\( \bar{p} = p^* \) and \( S(\bar{p}) = S > 0. \)

If the inequality holds in 5, then by the intermediate value theorem there exist a \( p^* \), at which \( F(p^*) = 0, \ p^* \in (p + \delta, \bar{p}). \) since \( F(p) \) is continuous on \( (p + \delta, \bar{p}). \)

But then \( \Psi(S(p^*)) = \bar{p} \). So \( \hat{p} = p^* \), and \( S(\hat{p}) > 0 \), as \( \hat{p} > p. \)

Let \( E = \{ p^* | p^* = \Psi(S(p^*)) \} \)

At any \( p \not\in E \), the pure competitive firm is not much different from a Cournot duopolist, whose expectations are always falsified at a price not belonging to an equilibrium set. Moreover, if the adjustment process is lagged, as in cobweb models, \( p \) may not converge to some \( p^* \in E \) as \( t \to \infty. \)

The usual justification for the "competitive behavior" of the firm, however, assumes that adjustment of \( S_i \) to \( p \) is instantaneous. Moreover, the output supplied by the firm is always an infinitely small portion of the total industrial output. Hence, the market-clearing price, \( \tilde{p} \), is not affected by any change in the \( i^{th} \) firm's output.

To put it mathematically:

\[ (6) \quad \frac{\partial \tilde{p}}{\partial S_i} = 0, \ 0 \leq S_i \leq \bar{S}_i; \text{ for any } i = 1, 2, \ldots n. \]

This immediately suggests that \( \Psi(S) \) has horizontal segments and

\[ \frac{\partial \tilde{p}}{\partial S_i} \]

is evaluated within such a segment of \( \Psi(S) \) function. Since \( S = \sum_{i=1}^{n} S_i; \)
\[ \frac{\partial S}{\partial S_i} = 1 \text{ and} \]

\[ (7) \frac{\partial \tilde{p}}{\partial S_i} = \psi'(S) \frac{\partial S}{\partial S_i} = \psi'(S). \]

(We are assuming that \( n \) is finite. Otherwise \( \Sigma S_i \) would not converge.)

If the capacity constraint is removed, there is no guarantee that as \( S_i \) is increased it would not hit any corner point of \( \psi(S) \). Moreover if \( S_i \) is not bounded it cannot always remain an infinitely small portion of \( S \) as \( S_i \) is increased since \( S_i \) and \( S \) are both real, i.e., belong to a field that is Archemedean.

To sum up, since \( \psi(S) \) cannot be constant in \( S \) for \( 0 < S \leq \bar{S} \), it can at most have some horizontal segments. The competitive behavior of a firm is plausible only if the underlying supply (or cost) conditions imposes a capacity limit for the firm.\(^1\) However, even then a sales restraint exists up to which the firm can dispose of any quantity at a constant price—if the adjustment of output to price is instantaneous. If not, the market clearing price \( \tilde{p} \) will depend on \( \Sigma S_i \), which is not known to the firm, nor can be controlled by it. This anomaly in competitive behavior is discussed by Arrow (1959) and Dickson (1961). The reason for this anomaly lies in \( \psi' \). Any marginal change in the output supplied by the \( i^{th} \) firm is of the same order of smallness as a marginal change in the supply of the industry. Therefore \( \psi'(S) \) does not involve \( n \), the number of firms in the industry.

\(^1\)This is clearly stated by Bishop (1952).
Moreover, it may be noted that if \( Y(S) \) has horizontal segments then it is not invertible. So the demand function in its usual sense does not exist, as more or less than a specific quantity can be sold at the same price.

A somewhat different type of demand function is increasingly being used in operational research literature and has also penetrated economic literature, especially in the formulation of capacity expansion policy for a firm. In these models there is a demand function, which is a univariate function of time. In inventory models, this means that the firm produces according to order and the order rate is independent of the parameters in the action space of the firm. In stochastic versions, the analysis is generally restricted to the case of a probability distribution of demand which does not change over time or to the case of a fixed order rate but the time at which it occurs is a random variable.

This type of demand function has been used by Chenery (1952), Smith (1961), Arrow and his associates (1958), Manne and his associates (1967), and Zabel (1963), to determine a program of optimum growth of capacity for the firm. However, the interpretation of the demand function is not the same in each case. Chenery, Smith and Zabel regard \( d(t) \) as fixed output requirements over time. Chenery, however, focuses attention on a special situation in which the time derivative of \( d(t) \) is a constant. In both cases, the demand can neither be backlogged nor can be met from inventory accumulated from previous production. So \( d(t) \) is also the optimum output program over time and the problem for the firm is to minimize the cost of satisfying the output requirements by choosing an optimum capacity policy. In Manne type models, demand
can be backlogged at a finite (penalty) cost which is equivalent to assuming that the deficit can be met by purchasing from the market (or importing from abroad). A somewhat similar idea also appears in Dantzig's (1963) two-stage programming under uncertainty in which "permanent feasibility" is maintained. In Arrow type models d(t) provides a sales constraint up to which the firm can sell at a given market price but beyond which it cannot sell at any positive price. In the context of a multi-product firm, Kornai and Liptak (1962) have introduced a similar idea by providing for some "market acceptance limits" for different brands of a product of the same firm. Hans Brems (1961) has also suggested incorporating sales constraints (and keeping price frozen) in a programming framework. In his model, the quality and selling efforts needed to sell one unit of a product are fully specified by a "process" the elements of which specify the physical units of the \( i^{th} \) input per physical unit of output of the \( j^{th} \) process, i.e., to produce and sell one unit of the \( j^{th} \) process.

Besides the usefulness of the concept of a sales constraint in a multi-product context, one chief merit of such a concept (and the type of demand function it implies) is its easy adaptability to a theory of growth of a firm in a partial equilibrium framework. Since the firm does not have any price or selling policy, it simplifies the theory. At any time the firm operates under two constraints. One is the sales limit, growing exogenously over time and not amenable to control. The other is its capacity limit, a state variable, but amenable to control by some control variables which determine the rate of change of capacity over time.
But what is the market structure under which such a demand function would be plausible? In the first place the firm is like a pure monopolist with a determinate demand function, i.e., it is not bothered by industry effects. If the price is set independently of the firm's control (by a marketing board) then the firm is a price-taker. However, the firm can also be a price-maker, if it sets a price and waits for orders to pile up. The essential feature is that the firm has no "price" or selling policy worth the name and the sales limit is independent of the firm's action parameters.

Mills (1962) tried to bridge the gap between such a demand function and the usual demand function of economic textbooks by bringing in inventory and uncertainty. The objective function is accordingly modified so as to maximize the mathematical expectation of profit.

We shall follow here an alternative route. We shall assume that all goods produced by the firm are nonstorable. If the price function is not time-dependent, i.e., constant over time, this would result in no loss of generality as there would be no speculative motive for holding inventories. Besides, no stochastic elements would be introduced. We should, in their place, introduce a marketing cost function. One part of the marketing expense is proportional to the amount sold. The other part would be a quadratic function of the ratio of amount sold to the sales limit. In the rest of this section, a justification for this type of marketing cost function will be given.

Let $d$ be the sales limit and $x$ be the amount sold. Consider the logarithmic penalty function $m \log \left(1 - \frac{x}{d}\right)$, $m > 0$. When $x = 0$, the penalty
is zero. When \( x = d \), the penalty is \(-\infty\). If this penalty function is added to the net revenue function and the firm maximizes their algebraic sum then \( x \) will always be less than \( d \). The logarithmic penalty function, however, imposes an infinite penalty as \( x \to d \) from left. Its first derivative with respect to \( x \) is \(-m \left( \frac{1}{d} - \frac{1}{x} \right)\) which also tends to minus infinity as \( x \to d \). We can call \( x/d \) as the market penetration index.

Let us expand the above penalty function: We get

\[
m \log \left(1 - \frac{x}{d}\right) = m \left[ -\frac{x}{d} - \frac{1}{2} \frac{x^2}{d^2} - \frac{1}{3} \frac{x^3}{d^3} \ldots \right]
\]

provided \( x < d \). Since all the terms are negative, we get an underestimate of the penalty function if we keep only up to quadratic terms, i.e.,

\[
m \left[ -\frac{x}{d} - \frac{1}{2} \frac{x^2}{d^2} \right], \quad 0 \leq x < d
\]

As \( x \to 0 \), the penalty is zero as before. If \( x \to d \), then the penalty is \( \frac{3m}{2} \), which is finite if \( m \) is finite.

The interpretation of this penalty function is simple. Suppose the firm produces at a given point but sells in different markets located uniformly over space. By drawing Von Thünen type concentric circles with the center at the production location, it can easily be shown that the marketing cost will become larger as the firm tries to sell more. In the first place transport cost may be assumed to be proportional to the Euclidean distance between the market location and the production location. Moreover, the further it goes, the more it will encroach on the markets of other producers located nearby. Note that if we make \( p \) a function of \( x \), then as \( x \) increases not only the price will decrease for the marginal units but for all units. However,
by introducing marketing cost, the net price (i.e., price net of marketing cost) will still be decreasing as the marketing cost is an increasing function of x. But in the markets near the production location, the net price will remain the same.

To recapitulate, a sales constraint is implicit even for a pure competitive firm. But this never becomes binding because cost or other constraints become binding before the sales constraint is reached. In the operations research literature the sales constraint is introduced explicitly and the cost situation is not such that an optimal size of firm exists in a long-run situation. By introducing a marketing cost function a compromise between the two approaches is possible.
CHAPTER II. SURVEY OF LITERATURE

The Notions of Fixed, Semi-fixed and Variable Cost: The Concept of Capacity

In the Walrasian view, the firm is a pure "hirer-seller" with no scarce factors private to it. As a result, all its costs are variable (Kuennε, 1963). Usually, however, the total cost of a firm is written as the sum of fixed costs, a constant and variable cost.

Now why does this constant appear? A simple answer would be that these are fixed charges like taxes which are liabilities to the firm imposed by social institutions. Variable costs are then the minimum cost of a given output subject to the constraints of the production function and the price function of inputs that "went" into the product, or simply all costs that the firm would be interested in minimizing. Such a viewpoint can be found in Samuelson (1965).

There are two problems with the above cost classification. In the first place, even contractual or statutory obligations might vary with the designated inputs and outputs. Secondly, most economists regard costs as events that occur with factors of production in use (Noyes, 1941). We are thus led to the more fundamental question, what is cost of production for a firm? In economic literature a straightforward answer to the above question is avoided. It is argued that the concept of cost has different shades of meaning and it is a relative concept. Yet one can distinguish between two distinct notions of cost (Fellner, 1960).
1. The cost of production of a given output is the value to other producers of the resources which are used to produce it. A clear exposition of this concept of cost can be found in Lewis (1949), Stigler (1952), and Machlup (1952). This is "cost" in a welfare-theoretic sense.

2. The human efforts and sacrifices which are involved in making the factors available for use, i.e., real cost.

Such theoretical notions of cost can be of value only in a general equilibrium analysis. In a partial equilibrium it is almost imperative to adopt a more simplistic notion of cost, namely, the total expense of production that is incurred on account of factors of production. A distinction is often made between fixed and variable productive services (Carlson, 1965; Stigler, 1952). Fixed productive services flow from fixed productive factors that give rise to "inescapable, contractual" cost in a given period though it is still cost, being escapable in a longer period. Here again we can distinguish between two lines of thought. The popular view is that "fixed factors" are those factors that for some reason or other cannot be varied even though they are "scarce." These factors are the "microeconomic counterparts" of land and other primary factors in general equilibrium analysis. Carlson, on the other hand, gives the impression that he is thinking of "discontinuity factors" which are only step-wise variable. He does not make a distinction between a fixed factor (which does not vary with the amount produced) and its service-flow (which depends on the rate of utilization of the factor). But in his mono-periodic theory, the firm is assumed to have all its resources in liquid forms.
at the beginning of the period and use up all its productive services, bought at the beginning of the period, within the production period. Thus it appears that the optimum amount of a fixed productive factor, for at least a given range of output variation, is the same as otherwise in a mono-periodic situation the firm would not purchase more or less of the "fixed" productive factor than what is "optimally" necessary. Similarly, in his poly-periodic theory, Carlson makes a sharp distinction between the fixed and variable nature of productive service and the durable (i.e., which appears in more than one period) and non-durable nature of productive service.

Most economists, I presume, associate with the notion of fixed costs, a certain immobility or irreversibility of factors that is completely alien to Carlson's notion of fixed cost. Unfortunately, both Marshall (1961) and J. M. Clark (1923) who popularized such a concept of "supplementary" or "overhead" cost are also great believers in the relativity of cost concepts. To Marshall, which factors are to be regarded as fixed and which are variable depends on the length of time one considers. Sometimes this may lead to a logical circle. Thus, Viner (1931) defines the short run as the period which is long enough to permit any desired change of output technologically possible without altering the "scale of the plant." He then defines the "scale of plant" as the group of factors which are fixed in amount in the short run.

Like Marshall, J. M. Clark also illustrates how we may choose from a host of "ad hoc" cost concepts in different real-life situations, confronted with various kinds of managerial problems. In recent years
the above line of approach has further been extended by Dean (1951) and a group of French economists (Lessourne, 1963).

Since we have a definite end in view, namely to formulate the theory of growth of a firm, it is not unreasonable to expect that we should, once for all, adopt a cost-classification scheme and stick to it. With this purpose in view, let us start with the simplest situation, i.e., a single-product firm subject to the constraint of a production function when the price functions of factors are known.

First of all, we make the old-fashioned distinction between (i) current inputs whose consumption is contemporaneous with current production and (ii) capital goods, which contribute to the production process by their mere presence. They are imperfectly substitutable for each other.

We assume that all current input items have a perfect market, in the sense that (i) their purchase price is always the same as their selling price, (ii) there is no lag of delivery, the purchase of these items can be postponed to the last moment if required and (iii) an organized future market exists where prices are so adjusted that at any time there is no gain from speculative holding of these items.

Under the above conditions it would never pay the firm to have an excess supply (or inventory) of these items.

Let us define an operational period as a week. No change in prices of current inputs can take place within a week. Besides at the end of the week there are no goods-in-process. We assume that output produced is nonstorabel, hence sold at the end of the week.
Since all current-cost items that are made available to the firm are consumed, they are inputs, i.e., their available supply over the week is equal to their total use. So both current inputs and output have the dimensionality of units per week.

Capital goods are stock inputs of varying degrees of durability (requiring regular maintenance and replacement, if necessary) and are irreversible in the sense that though they are freely variable at the time of their installation, once they are installed they can no longer be varied except by installing a parallel line or by replacing with a larger line, in the upward direction, or allowing the forces of mortality to work on them in the downward direction.

Let us now consider the production function of the firm. The most usual form of production function expresses the flow of output as a function of flow of inputs in a momentary production. Each flow of service has the dimension of the unit of service per moment. The flow of output has also the same dimension.

A departure from the above classical position is often made to deal with the special case of fixed coefficients of production or limitational inputs (Georgescu-Roegen, 1966; Danø, 1966). The production function, implicit in the L-shaped isoquants, for instance, is a relation of output and the available "supply" of factors of production. When we consider "the week" as the operational period, the above two lines of approach do not make any difference with respect to current inputs. Haavelmo (1960) discusses the concept of capacity in relation to both classical and nonclassical types of production function.
If we measure the capital goods or the stock inputs by their physical size and both output and current inputs by their respective units per week, we get a hybrid kind of production function. Such a stock-flow production function has been used by Smith (1961) and Sen and Sengupta (1969). Smith's production function is a simplification of engineering type production functions. A similar type of production function, which also makes a distinction between divisible capital goods and indivisible capital goods, has been used by Chenery (1952).

The virtue of this type of production function is, that under certain simplifying assumptions, it helps to illustrate the various concepts of capacity of the firm. Let us introduce the simplifying assumptions first.

A1. All capital goods (plants) (i) are of constant efficiency with a fixed life $L$, determined by technical considerations, (say, by obsolescence rate), and hence there is no user cost; (ii) they have no scrap value.

A2. Each firm has one plant, so that there is no ambiguity about the (physical) size of a plant, denoted by $k$. We assume that $L$ is measured in weeks and it is a positive integer.

Define

$$E(w) = E(q(w), k) = a(w) + v(x) \cdot x(w) + c(w), \quad 0 \leq q(w) \leq \tilde{q}(k)$$

where $a(w)$ is the cost per week which is independent of the plant size,
k and the output rate, q(w), i.e., fixed cost in Carlson's sense (a(w) ≥ 0); x(w) = x(q(w), k) is the physical dimension of all current inputs used up contemporaneously with output produced in a given week, q(w); and v(x) is its appropriate price index, assumed to be independent of time (x(w) ≥ 0; v(x) > 0). c(w) is the current cost equivalent of c(k), the (initial) cost of purchasing (or constructing) k, i.e., the cost of the capital stock k, allocated to the given week; and \( \bar{q}(k) \), is the technical upper limit, assumed to be a monotonically increasing function of k, the plant size. Equation 8 defines a one-parameter family of short-run cost functions (parameterized by k, the size of the plant) defined for weekly output rate q(w) where 0 ≤ q(w) ≤ \( \bar{q}(k) \). For a given k, E(w) becomes the usual short-run cost function. Let r, 0 < r < 1, be the rate of interest (as well as the rate of discount for the firm) that can be earned on a perfectly riskless security worth one dollar. Then one way of defining c(w) would be

(9)  \[
c(w) = r \cdot c(k) \cdot \frac{1}{1 - (1+r)^{-L}}
\]

If L \( \to ^{\infty} \), we get c(w) = rc(k), which is Smith's definition of "current cost" of maintaining the "presence" of a unit of the capital good in production. If r is replaced by the normal rate of profit (Mrs. Robinson's term (1969), indicating the rate at which no entry or exit would be made into the industry) or "the standard rate of profit" (Harrod's term (1952) meaning thereby the rate at which the firm would be on the point of indifference regarding investing in the same industry) we get alternative methods of deriving "supplementary cost"
which gives U-shapedness to the average cost curve, $E(w)/q(w)$.

We prefer 9 because it can simply be interpreted as "amortization cost." Assuming all amortization funds are invested at the market rate of interest, $c(w)$ accumulated over L weeks would be equal to $c(k)$. Secondly, the above way of allocating $c(k)$ seems reasonable if the "earning power" of $k$ is the same in each week, which would be true by virtue of the following assumption.

A3. Each plant is designed (by an engineer) with a particular output rate ($q_o(w)$) in mind which the firm produces under identical demand conditions over L weeks.

The optimum plant size for the designed rate of output is therefore given by the following cost minimization problem:

$$
(10) \quad \text{Min } \sum_{w=1}^{L} \frac{E(q_o(w), k)}{(1+r)^w} = \left[ \frac{1 - (1+r)^{-L}}{r} \right] \cdot \text{Min } E(q_o(w), k).
$$

We assume:

A4. $k$ can be designed for any specific rate of output $q_o(w)$ to be produced over L weeks and is a strictly increasing function of $q_o(w)$.

Suppose the solution of the above minimization problem is given by $k_o$ for a given $q_o(w)$. By varying $q_o(w)$, parametrically, we can generate the (nonspecific) extremal, $k_o = F(q_o(w))$, which, by A3, is invertible. It may be noted that $F$ is independent of $k$, as it is minimized out in 10. Hence for each value of $q_o(w) \geq 0$, we can derive the following Engineers' cost function:
(11) \( E_o(w) = E(v_o(w), P(q_o(w)). \)

It can easily be seen that since 10 implies \( \partial E(q_o(w), k_o)/\partial k = 0, \) the Engineer's cost function is the envelope cost function of the one parameter family of short-run cost functions, 8 ,

(11a) \( \partial E(q(w), k)/\partial k = 0 \)

\( E(w) = E(q(w), k) \)

Hence, 11 is also what is called the "long-run" cost function. By A3. \( E_o(w) \) is strictly increasing in \( q_o(w), q_o(w) \geq 0. \) Suppose that it is also concave up to \( 0 < q_o(w) \leq q_{oo}(w) \) and convex after \( q_{oo}(w), \) which is the inflection point of \( E_o(w), \) i.e., the point where "scale economies" are exhausted.

Since in geometrical arguments it is more convenient to deal with "average curves" we may state the following results, without proof, as they are already stated or proved (more or less rigorously) in economic literature.

R1. Divide \( E_o(q_o(w)) \) by \( q_o(w). \) This gives us long-run average cost which is the envelope cost function (or k-discriminant locus) of the one parameter family of short-run average cost curve \( E(q(w), k)/q(w). \) The point at which the above two curves would be tangential to each other is also where 8 is tangential to 11 , or namely at \( q_o(w), \) (the conjugate point).

R2. The point \( q_{oo}(w) \) is the minimum point of the long-run average cost curve as defined in R1.
R3. Define $q_1(k)$ as the minimum point of the average short-run cost curve for a given $k$. It can be shown that the marginal short-run cost would be equal to average short-run cost at $q_1(k)$. In other words

$$\frac{\partial E(q(w), k)}{\partial q(w)} = \frac{E(q(w), k)}{q(w)}$$

both evaluated at $q(w) = q_1(k)$.

R4. Consider $q_{oo}(w)$ and find the $k_{oo}$ that corresponds to it, in $k_{oo} = F(q_{oo}(w))$. If for a firm $k = k_{oo}$, then $q_{oo}(w)$ is also the same as $q_1(k_{oo})$, i.e., the minimum point of the short-run average cost curve $E(q(w), k_{oo})/q(w)$.

R5. Given any arbitrarily given $\bar{k}$ it is possible that the output produced is not the same as the output for which it is designed. We can still find the output rate for which $\bar{k}$ is designed by inverting the $F$ function and evaluating it at $\bar{k}$. It would create no confusion if we also call it $q_o(w)$.

Since $F$ is invertible (by A4), $q_o(w) = f^{-1}(\bar{k})$. But $\bar{k} = k_o = F(q_o(w))$.

It can be seen that $q_o(w)$ is the conjugate point of $E(q(w), \bar{k})$ as defined in R1. By varying $k$ parametrically, we get $q_o(k)$. Economic literature is replete with various concepts of capacity of (i) a plant, (ii) a firm, (iii) an industry and lastly (iv) an economy as a whole under the conditions of sustained simultaneous operations. Let us illustrate some of the concepts of capacity of a firm, for our one-product, one-plant firm, assuming $v(x)$ and $L$ are given, but the demand conditions are subject to change so that $q(w)$ may not be equal to $q_o(w)$ over all $L$ weeks.
a. The simplest (and economically speaking, simplistic) concept of capacity of a firm with a given plant k is the output rate $q(k)$ at which

$$\left( \frac{\partial (q(w), k)}{\partial x(w)} \right) + = 0$$

b. The output rate $q_1(k)$. J. M. Clark (1923) seems to be the earliest economist to refer to this concept. But it got full treatment from J. M. Cassels (1937).

c. The output rate $q_0(k)$. The first clear exposition is given by Hahn (1955).

d. The point $q_2(k)$ at which average variable cost, i.e., $v(x) \cdot x(w)/q(w)$ reaches its minimum. Cassels considered it but only Eiteman (1947) strongly advocates it.

e. From 8, $q(w)$ for a given k is a function of $x(w)$, i.e., $q(w) = q(x(w))$. Let $\bar{x}(w)$ be considered the "normal operating conditions" for a firm of size k. Then $q_3(k) = q(\bar{x}(w))$ is perhaps closest to the concept of "practically attainable" capacity of a firm, which, once again, can be found in J. M. Clark, but seems to have been popularized by the Brookings Institution (Nourse, et al., 1934).

A simple measure of $\bar{x}(w)$ would be the time average over L weeks, i.e., $\bar{x}(w) = \frac{1}{L} \sum_{w=1}^{L} q(w)$. More sophisticated ways of deriving $\bar{x}(w)$ would be to take (i) the (ensemble) average of $x(w)$ for different firms in the industry producing the same product with a given plant size or (ii) the weighted average of $x(w)$ as in (i) but with different plant size
in a normal year, or (iii) some combination of ensemble and time
average or (iv) some weighted average of the subjective estimates
of experts about what is normal $x(w)$.

Since we are going to use a concept of capacity, which is, at
least, as naive as $q(k)$, we owe an explanation as to why the other
concepts of capacity are not very useful, if we remove our simplifying
assumptions.

Under identical demand conditions, $q(w)$ would tend to be equal
to $q_0(w)$ and $k$ to $k_0$. Assuming no external effects, under perfectly
atomistic competition, with free mobility of resources, $k$ would also
tend to $k_\infty$ and $q(w)$ to $q_\infty(w)$ and $E_0(w)$ would be equal to the price
of $q(w)$. At this point $q_1(k) = q_2(k) = q_\infty$.

If $q_0(w) < q_\infty(w)$, then since all scale economies are not
realized (as it would be under perfectly competitive behavior of
the firm), we can, at most, reach a second best. $k$ would tend to
become equal to $k_0 < k_\infty$. But at $k = k_0$, $q_1(k) > q_0(k)$. But at
the point $q(w) = q_1(k)$, $k = k_0$, cost can be reduced by building a
more elaborate plant, namely $F(q_1)$. Hence, in the long-run, $q_1(k)$
cannot become the second best if $q_1(k) > q_0(k)$. Only at $k = k_0$
and $q(w) = q_0(k)$, is the cost minimizing proportions between $k$ and $x(q)$
reached. The above point was driven home by Harrod (1952), with
the help of envelope cost functions. (J. M. Clark also recognized
that it may pay to use a more expensive equipment, even if it cannot
be kept busy all the time, if it results in big savings in direct
costs, in preference to a small plant. But he did not see that
q(w) > q_1(k) is a sufficient but not a necessary condition for a change of k.

q_2(k) does not have any normative significance unless we assume (as Eiteman does) that average variable cost declines continuously up to q(k) at which it is left continuous. But then q_2(k) = q(k) = q_1(k) (with a usual U-shaped average cost curve q_2(k) < q_1(k)). Moreover q_2(k) is also (by Eiteman's definition) the output rate for which k is designed.

Similarly q_3(k) does not have any normative significance (for it is based on actual output rates) unless it serves as a good approximation for some other concept of capacity. Steindl (1952) took q_3(k) as a good approximation for q_1(k) which, we have shown, cannot serve as "second best" unless it is also "the best."

Remove the assumption of "identical demand conditions." Since the optimum plant would now be designed for more than one output rate, it cannot be characterized by its size alone. We have also to characterize the optimum degree of flexibility property built into the plant. Evidently the plant designed under fluctuating demand conditions will be more flexible (or adaptable) than a plant designed under identical demand conditions.

There are two ways to get around the above difficulties. Instead of measuring capacity in terms of output, we may measure it in terms of input. We may assume that at any moment (the smallest unit of time) the plant generates a uniform and continuous flow of services whose dimension (as measured by a flow meter) is a unit of the service per moment. The capacity of the plant is total availability of its
services in a week.

An alternative procedure would be to assume that for any plant, the operating cost \( (v(x) \cdot x(w)) \) is proportional to \( q(w) \), for all \( q(w), 0 \leq q(w) < q \). One possible situation when this can happen is when \( v(x) \) is independent of \( x(w) \) and \( x(w) \) is proportional to \( q(w) \), a product shadow. Sen and Sengupta (1969) have shown that if the firm has more than one plant of "equal efficiency" and if there are no interplant economies, then the ratio of variable costs to output will approach a constant asymptotically as the number of plants increases. This results from applying Joshep's (1938) theorem to a multiplant firm.\(^1\)

The reasoning is simple. For each plant, there is an optimum proportioning of current inputs and the plant. If all the plant's average-cost curves are U-shaped then this point corresponds to the minimum point of this plant-average cost curve. Let the output that results at this point be called plant-capacity output. The capacity of the firm, \( \bar{q}(w) \), is the number of plants times the plant-capacity. So long as \( q(w) \) is less than \( \bar{q}(w) \), the optimum proportioning between current inputs and plants would be maintained by bringing more and more plants into use as output expands.

It may be noted that even when such "physical divisibility" does not exist, the plant is perfectly segmentable on a time scale, if it generates a homogeneous flow of services at each moment. So an optimum proportioning between machine hours and current inputs can be maintained by keeping the plant idle if necessary.

\(^1\)For a more rigorous treatment, see Samuelson (1967).
Two objections might be raised:

1. As the plant (or machine) is used over a continuous stretch of time its efficiency would be reduced.

2. Plants installed at different dates (hence of different vintages) may require different amounts of current inputs per unit of output.

Al. takes care of such possible objections. It may be mentioned that even if \( x(w) \) is not proportional to \( q(w) \), \( v(x) \cdot x(w) \) can be proportional to \( q(w) \), if \( v(x) \) falls whenever \( x(w) \) rises more than proportionately to \( q(w) \), such that \( v(x) \cdot x(w) \) remains a fixed proportion of \( q(w) \). This can happen if the firm can put "monopsonistic" pressure on the factor markets from which \( x(w) \) are bought as \( x(w) \) gets larger or more simply, if the firm can get a "quantity discount" (i.e., a price discount when a larger quantity is bought). The latter situation arises whenever the production of \( x(w) \) is subject to economies of scale.

Assuming that \( v(x) \cdot x(w) \) is proportional to \( q(w) \), let us also assume that \( q \) is a strictly increasing function of \( k \), the size of the plant. Moreover the constant factor of proportionality \( v(x) \cdot x(w)/q(w) \) is assumed to be independent of \( k \). Thus, we assume away the problem of the optimum proportioning of \( k \) and \( x(w) \) for a given rate of output (or a range of output). Since \( c(k) \) is a strictly increasing function of \( k \), the only reason for building a larger plant would be to increase \( q(k) \). Hence, \( q(k) \) can be said to be equal to \( q_o(k) \), in the sense that the size of the plant must be justified by how large a \( q(k) \) is planned for. Another way of
stating the same fact is \( \frac{dq(w)}{dk} \), the marginal productivity of \( k \) is zero for all \( q(w) < \overline{q}(k) \) and is only positive at \( q(w) = \overline{q}(k) \). Lastly if \( a(w) > 0 \), \( q_1(k) = \overline{q}(k) \). Suppose the firm has more than one stock inputs, \( k_1, k_2, \ldots, k_m \). For each stock input we associate \( \overline{q}(k_j) \), \( j=1, 2, \ldots, m \). Let \( u \) denote the proportionality factor, \( v(x) \cdot x(w) / q(w) \). Henceforth, we denote \( v(x) \cdot x(w) = u \cdot q(w) = V(w) \).

We assume that not any two of stock-inputs are in substitution relationship to each other. Hence \( \overline{q}(k_j) \), \( j=1, 2, \ldots, m \) and \( q(x) = \frac{V(x)}{u} \) are all limitational functions. Measuring each of these by their availability to the firm, the production function can be written as,

\[
q(w) = \min (\overline{q}(k_1), \overline{q}(k_2), \ldots, \overline{q}(k_m), \frac{V(w)}{u})
\]

Now \( V(w) = q(w) \cdot u \), hence, there cannot be any excess capacity with respect to current inputs. The same cannot be said of stock inputs. Hence, if for any \( j \), \( \overline{q}(k_j) > q(w) \), there is an excess capacity of \( k_j \).

Needless to say, this results from complete irreversibility of stock inputs, i.e., once they are purchased, they cannot be disposed of (or leased to other firms) at any positive price.

While \( k_j \)'s cannot be sold in the market, they are subject to a constant rate of depreciation. But the rate of depreciation could be very low.

We shall assume that it is perfectly possible to add to any \( k_j \). Since all variable costs are proportional to output, independently of

\[\text{We assume that } \overline{q}(k_j) \text{ is bounded at } k_j = 0, \text{i.e., } k_j \text{ is needed for producing } q.\]
30

\( k_j \)'s, the capacity of any \( j^{th} \) resource is additive, i.e., \( q(k_j + \Delta k_j) = q(k_j) + q(\Delta k_j) \), for all \( j=1, 2, \ldots, m \).

If for any \( k_j, q(k_j) = q(w) \), then that \( j^{th} \) factor can also be called the minimum or limitative factor. If \( q(w) = q(k_j) = q(k^1) \); \( j \neq 1 \), then both the \( j^{th} \) resource and the \( 1^{st} \) resource are limitational factors (Georgescu-Roegen, 1966).

A growing firm, i.e., a firm for which \( q(w) \) is growing over time would ultimately adjust \( k_j \)'s so that it is on its limitational line, i.e., \( q(w) = q(k_j) \) for all \( j \).

The completely irreversible case is unrealistic, for in many realistic situations the capital goods can be sold at a positive price or they can be leased to others, if necessary. However, the selling price may be much lower than the purchasing price. Even disregarding the costs of disentangling the unused portion of an organic resource, there are installation costs or costs of assembling, etc. that cannot be covered. Lastly, for a growing firm, what is surplus capacity now may be required capacity later.

We may now introduce the concepts of "fixed costs," "semi-fixed costs" and variable costs for our one-product firm. Variable costs are associated with current inputs. They are proportional to output rates, whatever may be the level of stock inputs. We now introduce factor functions; \( k_j = k_j(q(w)) \). \( k_j \) is the minimum level of stock input (\( j^{th} \) input) for producing \( q(w) \). Fixed costs are associated with fixed factors, i.e., factors for which \( k_j, j=1, 2, \ldots, m \) are either constants throughout or do not vary within a given range.
The semi-fixed costs are associated with semi-fixed factors for which \( k_j = k_j(q(w)) \) is a monotonically increasing function of \( q(w) \). For these factors, the factor functions can be inverted to obtain limitation functions, \( q(k_j) \).

Both fixed and semi-fixed factors are irreversible so that their rates of decrease are bounded below by their rates of depreciation. But they can be varied continuously both at the time of their installation and at the end of each week by building parallel facilities.
CHAPTER III. MODELS OF OPTIMAL CAPACITY EXPANSION

Single-Product Firm

An analytical appraisal of some of the models of optimal capacity expansion policy for a single-product firm is attempted in this chapter. This analysis is intended to be illustrative rather than comprehensive. A more detailed treatment can be found in Sengupta and Sen (1969).

It is assumed that only a single nonstorable output is produced. A demand function \( d(t) \) exists but the interpretation of this demand function is not the same in different models.

The models are divided into two groups:
1. capacity-expansion models with scalar optimization of the decision variables;
2. capacity-expansion models using the functional equation approach of Bellman (1957) to determine the optimal intertemporal path.

Chenery model (1952)

The assumptions of the Chenery model are:
1. \( d(t) \) is given such that its time derivative \( d(t) = g \), where \( g \) is a constant \( (g > 0) \);
2. all capital units are of constant efficiency type up to the end of the planning period, \( T(T < \infty) \);
3. demand at time 0, i.e., \( d_0 = 0 \).
The production function of the Chenery model can be written as:

\[ q = q(x, X, k_{ij}) \]

where \( q \) = the output produced, \( x \) = the physical dimension of variable costs (flow inputs), \( X \), the number of capital inputs of divisible nature and \( k_{ij} \) denotes an indivisible factor where \( i \) = the physical size of the factor and \( j \) = the number of units of the factor; \( j \) can assume only integral (non-negative) values.

From this production function (which is invariant over time and hence can be written without time subscripts), we can draw a series of cost curves.

- \( C_L \) = the long-run cost function showing the minimum cost of producing any given \( q \) by varying \( x, X, i \) and \( j \). (As there are economies of scale \( j=1 \), so long as we move on \( C_L \).)
- \( C_1 \) = intermediate cost curve, varying \( x \) and \( X \), given \( i \) and \( j \).
- \( C(i) \) = intertemporal cost curve showing the minimum cost of producing \( q \), by varying \( x, X \) and \( j \), given \( i \).
- Plant curve = similar cost function by varying \( x \) alone, given \( X, i \) and \( j \).

Now, owing to the nature of the demand curve, profit maximization will require installation of the same size of plant as previously existing and at equally spaced time intervals. So the only relevant decision variable is the size of plant--which in turn will be uniquely related to the time interval between two subsequent installations. Since one \( C(i) \) corresponds to a given value of \( i \), the objective of the entrepreneur is to minimize total discounted cost with respect to \( i \) under the constraint of given output requirements.
In his more specific model, Chenery made the additional assumptions: 
Variable costs, i.e., cost associated with \( x \) can be ignored; 
\( X \) is assumed to be fixed (i.e., no process flexibility), so that 
\( C(i) \) would consist of a series of stair steps. Besides, 
the long-run cost function is specified by the relation 
\[ C_1 = b \, s^\alpha \]
where \( \alpha \) and \( b \) are constants. (\( b > 0, 0 < \alpha < 1 \)), and \( s \) = 
the scale of the plant of size \( i \) measured in terms of output. 
(This output is the same as capacity.) Maximum output of 
the plant is uniquely determined, for \( X \) is now fixed and \( j \) 
must be one since \( \alpha < 1 \)—the only variable being \( x \).

Next, Chenery approximates the stair-step cost function \( C(i) \) as a 
linear function of \( s \), i.e., by \( C_t \) say:
\[ C_t = a_0 + a_1 \, s/2 + a_1 \, t \, g \]
where \( a_0 \) = the portion of total cost that does not depend on output, and 
\( a_1 = (C_1 - a_0)/s \).

The minimizing condition for total discounted cost is then
\[ \frac{b \alpha}{2} (\sigma)^{\alpha-1} + b(\alpha-1) \, t(\sigma)^{\alpha-2} + a_1 \, t/\sigma^2 = 0 \]
where \( \sigma \) is the scale of plant measured in "years" so that \( \sigma \, g = s, g = 1 \).

Assuming \( a_1 \) and \( b \) are given constants, 13 gives a relation among 
\( r, T \) and \( \hat{\sigma} \), the optimum scale of plant. Of them the first three are 
parameters and the last one is the decision variable. Keeping any two 
of the parameters fixed, it is possible to show the parametric variation 
of \( \hat{\sigma} \) due to the change of any one of the three parameters. Specifically, 
it can be shown that \( \hat{\sigma} \) is a decreasing function of \( \alpha \), given \( r \) and \( T \).
Since a lower value of $\alpha$ indicates greater economies of scale, Chenery's model provides theoretical support to the policy of building capacity ahead of demand when substantial economies of scale exists. Chenery has also shown that with process flexibility (i.e., when $X$ is variable), the profitability of building capacity ahead of demand is increased and under fluctuating demand conditions the optimal capacity expansion policy will be biased in favor of over-expansion.

**Manne-Erlenkotter model (Manne, 1961; 1967)**

The assumptions of the model are:

1. $d(t)$ is growing at a constant annual rate $g$. Furthermore $d_0 =$ initial capacity;
2. all capital units are of constant efficiency infinite-durability type;
3. the planning period is infinite;
4. all operating costs are proportional to output;
5. the $C$ function (i.e., capacity-construction cost function) is stationary;
6. there is a penalty (shortage) cost for the failure to meet $d(t)$ and the rate of penalty cost is strictly proportional to the size of the backlog $z$, measured in years.

The object of the entrepreneur is to minimize

$$C(\sigma, \bar{z}) = \frac{1}{1 - e^{-r\sigma}} \left[ \rho \cdot \int_{0}^{\bar{z}} z e^{-rz} dz + e^{-r\bar{z}} C(\sigma) \right]$$

where $\rho$ is the penalty cost per unit of backlog $z$ and $\bar{z}$ is the optimum trigger level for backlog in demand, so that whenever $z$ grows to $\bar{z}$, a new
facility is built. Both \( z \) and \( \bar{z} \) are like \( \sigma \), expressed in years, so that \( zg \) is the amount of backlog that is permitted, both in terms of output. \( C(\sigma) \) is the construction cost of building a plant of scale \( \sigma \). The decision variables are \( \sigma \) and \( \bar{z} \).

Erlenkotter has shown

R1. An admissible plant size \( \sigma \), must satisfy the following relation

\[
C(\sigma) \left| \int_{t=0}^{\infty} \sigma \rho e^{-rt} dt = \frac{\sigma g}{r} \right.
\]

If no value of \( \sigma \) satisfies 15, no plant will be built. (A similar lower limit can be found for any nondecreasing demand function.)

R2. For all admissible plant sizes

\[
\bar{z}(\sigma) = \frac{r C(\sigma)}{\rho g} \leq \sigma .
\]

Using these results and simplifying, it can be shown that an equivalent problem to 14 is to minimize

\[
[a, \bar{z}(\sigma)] = \frac{\sigma - \bar{z}}{2} \cdot \frac{1 - e^{-r\bar{z}}}{1 - e^{-r\sigma}} .
\]

Taking the log of 17 and differentiating with respect to \( \sigma \), and setting it to zero gives

\[
\frac{e^{r\bar{z}} - 1}{e^{r\sigma} - 1} = \frac{r C'(\sigma)}{\rho} .
\]

An important special case is when \( C(\sigma) = b \sigma^\alpha \), where as before \( b > 0 \) and \( 0 < \alpha < 1 \).
The minimum admissible plant size will then be

\[ \min (k) = \left( \frac{r_b}{\rho} \right)^{1/(1-\alpha)}. \]

The minimizing condition is

\[ \frac{\alpha r_z}{r_z (e^r - 1)} - \frac{r}{e^r - 1} = 0 \]

i.e.,

\[ \frac{r \sigma}{e^r - 1} / \frac{r_z}{e^{r_z} - 1} = \alpha. \]

If the penalty cost is very high \((p \to \infty)\), \(z\) drops out and (21) reduces to

\[ \alpha = \frac{r \sigma}{e^r - 1}. \]

This is an alternative version of Chenery's specific model with an infinite planning horizon and continuous discounting. Since the planning period, \(T\), is no longer a parameter, we are left with only two parameters, \(r\) and \(\alpha\). A higher \(r\) leads to lower \(\sigma\) and a higher \(\alpha\) to a higher \(\sigma\).

An important feature of the Manne-Erlenkotter model is its regeneration point property—which states that between any two building dates there is a point at which demand equals capacity installed up to that point. In other words, no plant will be constructed when excess capacity exists. Besides from (19) it is evident that for all admissible plant sizes, the temporary phase for which demand is backlogged is less in duration than the number of years required for the growth of demand to equal the optimum scale of a plant. Thus, the assumption of finite penalty cost makes build-
ing capacity ahead of demand profitable under more general cost conditions
(i.e., concavity of \( C_1 \) need not be assumed) and at the same time stipulates
the condition that demand will catch up to capacity before any further
addition to capacity.

Srinivasan's model (Manne, 1967)

The assumptions of this model are similar to a continuous discount-
ing, infinite horizon version of Chenery model as described before. The
only difference is in the nature of demand function—namely

\[
\frac{d(t)}{d(t)} = g' \text{ for all } t. \text{ Besides as before } d(0) = \text{ initial capacity.}
\]

Srinivasan then establishes the result that it is optimal to construct
plants at each point of a sequence of equally spaced time points,
but now the size of plants to be constructed will grow exponentially.

The decision variable is taken as \( \bar{t} \), the time interval between
any two successive plant installations and the objective is to minimize

\[
(23) \quad C(\bar{t}) = \sum_{n=0}^{\infty} e^{-n\bar{r}\bar{t}} b \cdot \left[ d(0)(e^{G\bar{t}} - 1) e^{ng'\bar{t}} \right]^\alpha
\]

Assuming \( r > \alpha g' \) and dropping the constant term an equivalent
problem to (23) is to choose \( \bar{t} \) to minimize

\[
(24) \quad [e^{G'(\bar{t})} - 1]^\alpha / [1 - e^{-(r-\alpha g')\bar{t}}]
\]

The optimum \( \bar{t} \) which can be shown to be unique is given by

\[
(25) \quad \alpha G'(e^{h\bar{t}} - 1) = h(1 - e^{-G'\bar{t}})
\]

where \( h = r - \alpha g' > 0 \) by assumption.
It hardly needs mentioning that Srinivasan, like Chenery assumes that demand requirements must be met which is equivalent to assuming infinite penalty cost.

All the above models employ classical optimization procedures to determine the structure of the optimum values. In the Chenery and Manne-Erlenkotter models the optimum size of plant is the decision variable and since it is independent of the initial capacity, by applying renewal theory (or the idea of regeneration points) it can be shown that the same scale of plant will be built at equidistant time intervals. In Srinivasan's model the decision variable is the timing of installation of new plants and since in this model also the optimum size of plant is independent of initial capacity, the time phasing will be the same, though the size of plant will be growing. Besides either due to the concavity of the cost function, (Chenery, Srinivasan cases) or due to the assumption of finite penalty cost for backlogging (Manne-Erlenkotter case) the optimum sequence of time points at which plants are added must be discrete, i.e., separated from one another by finite time interval. But in every case the optimum time interval has an upper limit, i.e., it is not profitable to build capacity infinitely ahead of demand. If the cost of construction of capacity is strictly proportional to incremental capacity and if the penalty cost is infinite, it would be always profitable to wait for demand to increase before any further addition to capacity and there would be continuous addition to capacity at the rate of increase of demand.
Capacity expansion models with optimal path approaches

We shall now consider the capacity-expansion policy models which aim to develop algorithm for finding out the optimal path of capital expansion. We shall, however, discuss only the economic aspects of these models rather than their computational aspects.

Arrow, Beckmann and Karlin model (Arrow, et al., 1958)

The assumptions of this model are:

1. \( d(t) \) is given as a function of time but its time rate of change \( d(t) \) is not constant;
2. all capital units are of constant efficiency type with infinite durability. (Strictly speaking, they assume that maintenance costs are proportional to output.)
3. \( y_t \), the maximum (capacity) output at \( t \) is uniquely determined by \( k_t \), the aggregate size of capital equipments at \( t \).
4. \( V_t = m q_t \) defined only for values of \( q_t \leq y_t \), \( m \) is a constant \( (m > 0) \);
5. \( y_0 \) = capacity at time \( t = 0 \).
6. \( 0 \leq y(t) \leq M \) for all \( t \), where \( M \) is a given constant; \( (M < \infty) \) and \( y(t) \) is the rate of change of capacity at \( t \).

Since capacity and cost are both additive we have

\[
(26) \quad y_t = y_0 + \int_{0}^{t} y(\tau) \, d\tau \quad \text{and}
\]

\[
(27) \quad C = \int_{0}^{T} c \, y(t) \, e^{-rt} \, dt, \quad \text{where } c \text{ is a constant, } (c > 0) \text{ and } C \text{ is total discounted cost over the planning horizon.}
\]
In their system \( q_t = \min (d_t, y_t) \) is measured in units of net profitability. The object of the entrepreneur is to choose an optimum capacity schedule \((y_1, y_2, \ldots, y_T)\) such that

\[
G = \int_0^T [q_t e^{-rt} - c y_t e^{-rt}] dt \text{ is maximized.}
\]

An equivalent way to write the objective function is

(28) \[ \text{max} \min H(v, u) \text{ where} \]

\[
H(v, u) = \int_0^T [d_t (1 - u(t)) + y_t u(t) - c y_t] e^{-rt} dt
\]

subject to the following constraints

1. \( y_t \leq M \)
2. \( 0 \leq u(t) \leq 1 \).

(29) \[ \text{Let } \phi (t) = \int_0^t u(\tau) e^{-r\tau} d\tau - c e^{-rt}. \]

Then, the optimal capacity expansion policy would be

R1. \( y_t = 0 \) for all \( t \), i.e., \( \bar{y}(t) = y_0 \) if \( rc \geq 1 \);

R2. if \( 0 < rc < 1 \), the optimal path \((0, T)\) can be divided into subintervals of the following types.

a. \( \phi(t) > 0; y_t = M \), i.e., maximum expansion.

b. \( \phi(t) < 0; y_t = 0 \), i.e., no expansion.

c. \( \phi(t) = 0; y_t = d_t \); so that \( y_t = d_t \) throughout this subinterval.

The economic meaning of the results is when \( rc \geq 1 \), the interest on the cost of expansion of capacity by one unit is as large as or greater than unit profitability. Hence, the condition R1 holds. \( u(\tau) \)
will be one when demand exceeds capacity and zero when capacity expands

demand. So the integral \[ \int_{t}^{T} u(\tau) \, d\tau \] is a measure of the set of times
at which the firm will produce at capacity from \( t \) to \( T \). Hence, the

present value of returns from adding one more unit of capacity at \( t \)
is given by the integral in 29 and its cost is \( ce^{-rt} \). \( \phi(t) \) is
therefore the marginal profitability or the (discounted) profitability
at \( t \) of adding a marginal unity to capacity. Hence 2a and 2c. Lastly,

when \( 0 < y_t < M \), \( u(t) = r c < 1 \) and \( y_t = d_t \) and \( y_t = d_t \) as in 2c.

In this model \( d(t) \) is given as a function of time but the optimum
output program does not coincide with it. In other words the model
allows for negative excess capacity in the sense that for some
periods \( d(t) \) may exceed the output produced and there is no penalty
cost associated with it. Besides, negative excess capacity may exist
even when marginal profitability is positive, since it is impossible
to add new capacity above a certain level. This restraint is similar
to expansion cost associated with the first of the four factors in
Baumol's model (1967). But here this expansion cost is infinite when \( y_t = M \)
and zero below that level. This restraint is necessary, for otherwise
there would be an infinite expansion of capacity if marginal profita-

bility is positive. Since the cost function, \( C \), is linear, there is
no building ahead of demand.

Since, in this model, output is measured in units of profitability,
(i.e., net revenue), the existence of maintenance cost, proportional
to output, does not make any difference. We have only to redefine unit
of output so that it is also net of maintenance cost of capital per
unit of output. So assumption 2 is not really necessary. We need only
to assume that the capital unit has no fixed lifetime and its service period can be extended up to the end of the planning period by incurring maintenance cost which is proportional to output.

**Manne-Veinott model (Manne, 1967)**

The assumptions of this model are:

1. \( d(t) \) is given subject to the condition that \( d(t) \geq 0 \) for all \( t = 0, \ldots, T-1 \), and \( \sum_{t=0}^{T-1} d(t) > 0 \);

2. all capital equipments are of constant efficiency and infinite durability type;

3. \( C_t = c_t y_t \) where \( C_t \) is the cost of increment in capacity at \( t \) and \( y_t \) is the increment in capacity. This function is assumed to be concave for all \( t \);

4. \( y_0 = d_0 \), i.e., initial capacity is equal to initial demand;

5. \( y_t \geq 0 \).

Once again additivity of capacity, i.e., 26 holds. The objective of the firm is to find an optimal feasible capacity-expansion schedule defined by the vector \( \mathbf{y} = (y_0, y_1, \ldots, y_{T-1}) \), to minimize

\[
(30) \quad C([y_t]) = \sum_{t=0}^{T-1} c_t \left( y_t + \sum_{t=0}^{T-1} \rho_t \max(0, - z_t) \right)
\]

where \( \rho_t = \) the (temporary) penalty cost proportional to negative excess capacity.

Let \( N_t \) be the end of period excess capacity at \( t \), i.e., the difference between the cumulative values of \( y_t \) and \( d_t \). The feasibility
requirements on \( \dot{y}_t \) vector is given by \( \dot{y}_t \geq 0 \) and \( \dot{\Omega}_T = 0 \).

A point of regeneration of capacity expansion schedule \( y^* \) is said to occur at those points at which \( \dot{\Omega}_t = 0 \). It is then proved that there is an optimal capacity schedule which has the regeneration point property, as defined previously, and the schedule with that property can be searched efficiently with a dynamic programming recursion to find one that is optimal.

In many respects the above model is a generalization of the Manne-Erlenkotter model. While it generalizes the specific forms of demand and cost functions, it also arrives at less economically meaningful results.

In recent years a number of capacity expansion policy models have appeared and it is not possible to give a comprehensive account of all these models. The purpose of several models not considered here is to extend the basic models presented in this paper to more general conditions and to undertake either a kind of sensitivity analysis to show the variations of the optimal plant size and its time phasing as one or another parameters is changed or to evolve a computing procedure to find the optimal capacity expansion path. Some of these extensions are quite straightforward. Thus, instead of a deterministic demand function, a probabilistic growth of demand function has been considered by Manne even in his 1961 article. The extensions do not alter the nature of optimal policies though they may require some additional condition.

Another line that has been explored by Manne and his associates is the case of more than one producing area, instead of a single-producing
area. In this case, however, the extensions generally give less definite results and the computing procedure is only combinatorial, enumerative and less efficient.

One usual feature of the manufacturing industries that is systematically ignored by the policy models considered here is the fact that very few firms produce a single output. Especially when extra capacity exists, it can be utilized by adding one more product to the list of products of the firm.
CHAPTER IV. HICKSIAN MODEL OF A MULTI-PRODUCT FIRM

In the Hicksian model of a multi-product and multi-factor firm, the factors are treated as negative outputs. Consequently there are no sign restrictions on the decision variables, \( q_1, q_2, \ldots, q_n \) which are related by a transformation function

\[
(31) \quad f(q_1, q_2, \ldots, q_n) = 0.
\]

The transformation function (a scalar function of a vector, \( q = \{ q_1, q_2, \ldots, q_n \} \)) is assumed to be twice differentiable, i.e., possesses continuous first and second order partial derivatives. Moreover, both the prices of products and the factors (negative products) are parametrically given. No sales constraints or similar constraints on the availability of any factor is imposed. We shall also assume that there is a perfect market for all \( q_i \)'s, in the sense of Chapter II. Hence, we do not have to make a distinction between the availability (or supply) and consumption of a factor. Moreover, all products are sold. The net revenue that the firm maximizes is \( \sum_{i=1}^{n} p_i q_i \), where \( p_i \) is the price of \( i^{th} \) output.

The problem before the firm (in vector notation) is

\[
\text{max } p'q = z \quad \text{subject to } f(q) = 0
\]

where \( p = \{ p_1, p_2, \ldots, p_n \} \); \( p_i > 0, i=1, 2, \ldots, n \).

Once an optimal \( q^0 \) is chosen we can distinguish between outputs and inputs, accordingly as \( q_i \) is positive or negative. It may be noted that for many firms output produced may also be used as inputs. So what is output and what is input is not known beforehand.
A few comments may be made about the transformation function. It has \( n-1 \) degrees of freedom in the sense that once \( n-1 \) of the \( q_i \)'s are chosen the residual one can be read off from \( f(q) = 0 \). \( f(q) \) incorporates all technological knowledge so that given any combination of \( n-1 \) \( q_i \)'s the rest is maximized algebraically.

The maximizing output-input rates \( q^o \) are given by

\[
(32a) \quad p_i = \mu f_i \quad (i=1, 2, \ldots, n)
\]
\[
(32b) \quad f(q) = 0
\]

where \( \mu \) is the Lagrangian multiplier associated with \( f \); provided that \( f \) is quasi-concave (to be defined) not only at \( q^o \) but at any \( q \). Only in the latter case we can guarantee that \( q^o \) is a global (rather than local) maximum. It may be noted that Hicks(1946) requires \( f \) to be unique only up to such monotonically increasing transformations for which the origin is a fixed point. Since any monotonic increasing function of a quasi-concave function is also quasi-concave, such a change of scale will preserve quasi-concavity.

Quasi-concavity is familiar to the economists in the form of sign restrictions on the principal minors of the bordered Hessians of the function. One definition of quasi-concavity, without assuming differentiability, is that the function \( f(q) \) is quasi-concave if the set \( \{ q \mid f(q) > \gamma \} \) is convex for any scalar \( \gamma \). Geometrically this means that in the \( n+1 \) dimensional vector space \( \{ q, z \} \), if a hyperplane (\( n \)-dimensional) parallel to the \( q \)-plane is passed through the vertical \( z \) at \( \gamma \), then the hypersphere \( f(q) \) lying above the hyperplane, projected on the \( q \)-plane would be a convex set. In the two-dimensional case any
elongated S-shaped curve which is convex in some part, concave in some part, but monotonically increasing, is quasi-concave; on the other hand any U-shaped curve will not be quasi-concave since we can draw a line parallel to the x-axis and the portion of U lying above the line, when projected on the x-axis will be disconnected and hence not a convex set.

Arrow and Enthoven (1961) have shown that if \( f \) has an increasing marginal rate of transformation when \( f'_q > 0 \) for \( i=1, 2, \ldots, n \), between any pair of variables \( q_i, q_j \), \( i \neq j \) or between any two distinct composite variables, then \( f \) is quasi-concave. It may also be noted that if \( f \) is quasi-concave then for any set of prices, there would exist a global maximum that satisfy (32).

Increasing marginal rate of transformation (remembering our sign conventions) imply:

1. decreasing marginal rate of technical substitution between factors (i.e., for any \( i, j, i \neq j \), such that \( q_i, q_j < 0 \)).
2. decreasing marginal product for all factor-product combinations, (i.e., for any \( i, j, i \neq j \), such that \( q_i \cdot q_j > 0 \)).
3. increasing marginal rate of product transformation, i.e., increasing marginal cost of one product in terms of another as the degree of assortment becomes biased in favor of the product, (i.e., for any \( i, j, i \neq j \), such that \( q_i, q_j > 0 \)).

Similarly 32a implies:

1. the price ratio of any two products must equal the marginal rate of product transformation between any products;
2. the price ratio of any two factors must equal the marginal rate of technical substitution between two products;

3. the price ratio between any factor-product combination must equal marginal product of the factor in terms of particular factor-product combination.

To get economically meaningful results, Hicks also requires that \( f \) is not homogenous (of degree zero) in \( q_i \)'s. He therefore suggested that there is some fixed production opportunity to which \( z \) is to be imputed. Since this production opportunity is fixed and not variable, it can be excluded from the transformation function, \( f \), though its presence will determine the form of the function.

Let us assume that the firm has some semi-fixed factors \( k_1, k_2, \ldots, k_m \) which, in the given period are at the levels \( \bar{k}_1, \bar{k}_2, \ldots, \bar{k}_m \) and let us measure them with a positive sign. Incorporating them in the transformation function, the first problem in this chapter, becomes:

\[
\begin{align*}
\text{(33a)} & \quad \max_{\mathbf{q}} \sum_{i=1}^{n} p_i q_i = z \\
\text{(33b)} & \quad F(q_1, q_2, \ldots, q_n, k_1, k_2, \ldots, k_m) = 0 \\
\text{(33c)} & \quad k_j = \bar{k}_j \quad j=1, 2, 3, \ldots, m.
\end{align*}
\]

Setting up the Lagrangian:

\[
\begin{align*}
\text{(34)} & \quad L = \sum_{i=1}^{n} p_i q_i - \mu F + \sum_{j=1}^{m} \lambda_j (\bar{k}_j - \bar{k}_0)
\end{align*}
\]

and differentiating \( L \) with respect to \( q_1, \ldots, q_n \) and \( k_1, \ldots, k_m \) and setting them to zero, we get
\[ (35a) \quad p_i = \mu F_i; \quad i = 1, 2, \ldots, n \]
\[ (35b) \quad \lambda_j = -\mu F_j; \quad j = 1, 2, \ldots, m \]

where \( F_i \) indicates the partial derivative of \( F \) with respect to \( q_i \) and \( F_j \), its partial derivative with respect to \( k_j \). (Almost similar results are obtained by Nerlove (1958).)

By solving \( 35a, \ 35b \) and \( 33b, \ 33c \), we obtain \( q_i^0, (i = 1, 2, \ldots, m), \omega^0, \) and \( \lambda_j^0 (j = 1, 2, \ldots, m) \) and \( k_j^0 (j = 1, 2, \ldots, m) \). We assume that each of them is a continuous and differentiable function of \( (k_1, k_2, \ldots, k_m) \) in some neighborhood of an initial vector point \( \bar{k}^0 = (\bar{k}_1^0, \bar{k}_2^0, \ldots, \bar{k}_m^0) \). (Each of these functions has continuous partial derivative in each \( k_j, j = 1, 2, \ldots, m \)).

Now \( z = \sum_{i=1}^{n} p_i q_i^0 \).

Differentiating \( z \) with respect to \( k_j \), we get

\[ (36) \quad \frac{\partial z}{\partial k_j} = \sum_{i=1}^{n} p_i \frac{\partial q_i^0}{\partial k_j}, \quad j = 1, 2, \ldots, m. \]

Similarly, differentiating \( 33b \) and \( 33c \) and remembering \( \frac{\partial k_i}{\partial k_j} = 0; \ j \neq 1 \) (as they are not in substituional relationship, by definition), we get

\[ (37) \quad \sum_{i=1}^{n} f_i \frac{\partial q_i^0}{\partial k_j} + b_j \frac{\partial k_j^0}{\partial k_j} = 0 \quad j = 1, 2, \ldots, m \]
Multiply 37 by \( -\mu \) and 38 by \( \lambda_j \) and add them to the left hand side of 36. We get

\[
\frac{\partial z}{\partial k_j} = \sum_{i=1}^{n} (p_i - \mu f_i) \cdot \frac{\partial q_i^0}{\partial k_j} + \left[ -\mu F_j - \lambda_j \right] \cdot \frac{\partial k_j^0}{\partial k_j} + \lambda_j = 0
\]

\( j=1, 2, \ldots, m. \)

Hence, using 35a, 35b, we get

\[
\frac{\partial z}{\partial k_j} = \lambda_j, \ j=1, 2, \ldots, m.
\]

So \( \lambda_j \)'s can be given an economic interpretation. \( \lambda_j \) is the marginal revenue productivity of one more unit of \( k_j \), evaluated at \( \overline{k} = \overline{k}^0 \). \( \lambda_j \)'s are also called the opportunity costs of the semi-fixed factors, \( \overline{k}_j \)'s.

Multiply \( \lambda_j \) by \( \overline{k}_j \) and summing over \( j \)'s, we get,

\[
\sum_{j=1}^{m} \lambda_j \overline{k}_j = \sum_{j=1}^{m} \lambda_j k_j^0 \quad \text{(by (35c))}
\]

\[
= -\mu \sum_{j=1}^{m} F_j k_j^0 \quad \text{(by (35b)).}
\]

From (33b),

\[
\sum_{i=1}^{n} F_i q_i^0 + \sum_{j=1}^{m} F_j k_j^0 = 0,
\]

up to first order approximation, since \( F \) is differentiable.

Combining, we get,
\[
\sum_{j=1}^{m} \lambda_j k_j = -\mu \sum_{i=1}^{n} F_i q_i^o \\
= \sum_{i=1}^{n} p_i q_i^o = z \text{ (by 35a).}
\]

Hence, the total inputed value of the stock of semi-fixed resources is the same as total net revenue of the firm.

Note that to derive 42 we did not assume homogeneity of \( F \). If we want an approximation of a higher order, the homogeneity of \( F \) of any degree will suffice.

Unfortunately, this is the furthest we can go with a Hicksian type transformation function. Equation 35b indicates the optimal value of \( \lambda_j \) but it involves \( \mu \) which cannot be given any economic interpretation.

In the usual single-product, multi-factor theory of a firm the associated Lagrange multiplier can be given an economic interpretation, namely that of marginal cost. Nerlove suggested one way of approach. He solves the transformation function explicitly in terms of a (numeraire) commodity. But this loses the symmetry of the problem.

In the next section, we shall show how the separable convex programming approach effectively tackles the problem and expresses the opportunity cost of a semi-fixed resource in terms of the structural characteristics of the model, without being involved in an "undetermined multiplier" which cannot be given any economic meaning.
CHAPTER V. PROGRAMMING MODEL OF A MULTI-PRODUCT FIRM

In a programming model, a criterion function, \( f \), (usually a real-valued function) defined on a convex set (usually a vector space) is optimized on a decision space (usually a manifold characterized by the nonnegativity of a vector function, \( g \), on the same vector space). Further assumptions on the function \( f \) and the mapping \( g \) yield special types of programming, such as linear, quadratic, concave, etc. In the above sense, few optimizing models of a firm can escape from being programming models. By a programming model of a multi-product firm, we, however, mean that type of model in which the concept of a "process" plays a central role. We, first of all, give a generalized version of a programming model. By making simplifying assumptions we then deduce as special cases some illustrative models of a multi-product firm and compare them with the Hicksian type models.

We consider a model of production where \( N \) goods are "processed" with \( m \) factors of production. All factors are "stocks," i.e., they contribute to the production process by their mere presence. At any "moment" (the smallest unit of time) they generate a uniform and continuous flow of services whose dimension (as measured by a flow meter) is unit of the service per moment. They are not in substitution relationship, i.e., the service of one stock unit cannot be substituted for the service of another. (There may be more than one stock input of "equal efficiency.") \( b_j \) \((j=1, 2, \ldots, m)\) is the "capacity" of the \( j^{th} \) factor, i.e., the total availability of the services of the \( j^{th} \) factor in a week, the operational period. It is equal
to the rate of flow (number of units of service per moment) multiplied by the number of moments in a week, and hence has the dimensionality of unit per week (unit/moment \times moment/week).

The flow of services from any \( j \)th factor is considered "free" up to its limits of availability, \( b_j \). It may be possible to impute "price" with the service of the \( j \)th factor (and on that basis, "rental" per unit of the \( j \)th factor). But it is not market price (or rental since the possibility of leasing the \( j \)th factor to outside or hiring it from outside does not exist within a week) and it does not appear in the objective function. In the latter part of this chapter, the above "irreversibility" assumption with respect to factors would be relaxed.

To concentrate on the production aspect of the firm, we assume that all the goods have a "perfect" market as defined before.

There are \( r \) processes, each indexed by \( k \). The intensity of the \( k \)th process is indicated by \( x_k \). It is measured by an ordinal scale, unique only up to strictly increasing monotonic transformation with origin set at the zero level of intensity. \( x_k \) can only take nonnegative values, i.e., each process is irreversible.

An activity \( x \) is an \( r \)-dimensional vector, belonging to \( \mathbb{R}^r \), the (cartesian) product set of \( r \) copies of \( \mathbb{R} \), the set of real numbers. The \( k \)th element of \( x \) is \( x_k \), the intensity of the \( k \)th process, i.e., \( x = \{ x_1, x_2, \ldots, x_r \} \). \( X \) denotes the set of all activities, i.e., that subset of the nonnegative orthant of \( \mathbb{R}^r \), to which \( x \) must belong.

Define \( m + N \) scalar (single-valued) functions:

\[
\begin{align*}
f_i &: X \to \mathbb{R}; \ i \in I_N \\
g_j &: X \to \mathbb{R}; \ j \in I_m
\end{align*}
\]
where $I_n$ is the index set of positive integers of dimensionality $n$.

For any $x \in X$, and for a given $i \in I^+_n$, $f_i^+(x)$, if positive, uniquely determines the amount of the $i^{th}$ good (output) produced by the activity $x$, if negative, the amount of the $i^{th}$ good (input) consumed by the activity $x$, all measured per week. There is a market price, $p_i > 0$, for each $i^{th}$ good, $i \in I^+_N$, at which the firm can either purchase or sell any amount it likes of the $i^{th}$ good, without any sales constraint or any limit on its availability. We assume that all outputs are sold and all inputs are purchased in the right amounts. In a "perfect market" this "no inventory" assumption does not result in any loss of generality as the "net worth" of the firm would remain the same anyway. The dimension of $p_i$ is dollar per unit of the $i^{th}$ good, $i \in I^+_N$.

The net revenue from $x$ (or more strictly speaking, the aggregated quasi-rents of the capital outfit, $b_1, b_2, \ldots, b_m$) is:

$$\sum_{i=1}^{n} p_i f_i(x) = p'f(x)$$

where $p = \{p_1, p_2, \ldots, p_N\}$; $f(x) = \{f_1(x), f_2(x), \ldots, f_N(x)\}$.

Similarly for all $x \in X$, and $j \in I^+_m$, $g_j(x)$ uniquely determines the rate of flow of service from the $j^{th}$ factor multiplied by the moments per week, the $j^{th}$ factor is utilized by the activity vector $x$. Its dimension is unit per week (unit/moment x moment/week).

The problem before the firm is:

\begin{align*}
(I.1) & \quad \max_{x \in X} p'f(x) = z \\
(I.2) & \quad \text{subject to } g(x) \leq b
\end{align*}
where \( g(x) = \{g_1(x), g_2(x), ..., g_m(x)\} \) and \( b = \{b_1, b_2, ..., b_m\} \).

Equation (1.1) indicates that the \( i^{th} \) good, \( i \in I_N \), contributes to "net revenue" at the rate \( p_i \). Since the price vector, \( p \), is positive, \( p_i f_i(x) \) is positive, if \( f_i(x) \) is positive, i.e., if the \( i^{th} \) good is an output and \( p_i f_i(x) \) is negative, if \( f_i(x) \) is negative, i.e., if the \( i^{th} \) good is an input. The relation (1.2) indicates that for a particular \( x \) chosen that kind of perfect balancing may not be achieved so that all the factors are utilized up to their capacities. Hence, excess capacity (rate of flow x idle time) might exist for a particular factor. Its dimension is also unit of the service per week. Furthermore, such excess capacity is "costless," a consequence of irreversibility of the factor supplies. \( z \) is a scalar and is measured in dollars per week. We introduce the following assumptions:

AA.1 \( f_i(x) \) is process-wise separable, \( i \in I_N \), i.e.,
\[
f_i(x) = \sum_{k=1}^{r} f_{ik}(x_k), \quad i \in I_N
\]

AA.2 \( g_j(x) \) is process-wise separable, \( j \in I_m \), i.e.,
\[
g_j(x) = \sum_{k=1}^{r} g_{jk}(x_k), \quad j \in I_m
\]

AA.3 \( X \) is the nonnegative orthant of \( \mathbb{R}^r \).

AA.4 For each process \( k, k \in I_r \), there is at most one \( i, i \in I_N \), for which \( f_{ik} > 0 \).

AA.5 AA.1 holds and \( f(x) = Bx \) where
\[
B = \begin{pmatrix}
\begin{array}{cccc}
b_{11} & b_{12} & \cdots & b_{1r} \\
b_{21} & b_{22} & \cdots & b_{2r} \\
\vdots & \vdots & & \vdots \\
b_{N1} & b_{N2} & \cdots & b_{Nr}
\end{array}
\end{pmatrix}
\]

AA.6  AA.2 holds and \( g(x) = Ax \) where

\[
A = \begin{pmatrix}
\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1r} \\
a_{21} & a_{22} & \cdots & a_{2r} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mr}
\end{array}
\end{pmatrix}
\]

AA.1 and AA.2 imply that the effects of different processes \( x_k, k \in I_r \) are distinguishable. The processes do not support or impede one another, however, they are not completely independent because they share the services of an inelastically given supply of factors with fixed capacity limits, \( b \).

It may be noted that AA.2 does not necessarily mean that the factors are physically divisible among the process, without any loss of efficiency. Even when the factor is physically indivisible and immobile, its flow of service may be divisible (allocable) among the processes by time segmentation. Separability is the obverse side of additivity. Hence, AA.1 implies that the total amount of the \( i^{th} \) good, \( i \in I_N \), produced by the activity vector \( x \), is the algebraic sum of the amounts produced by the \( r \) processes at the indicated level of intensities, \( x_k \), \( k \in I_r \). Similarly AA.2 implies that the total amount of the service capacity of \( j^{th} \) factor, \( j \in I_m \), utilized by the activity vector, \( x \), is the sum of the consumption rates of its service by each of the \( r \)
processes at the indicated level of intensities. AA.3 implies that each process can either be used to any positive extent or not used at all, i.e., an entire half-line \((0,\infty)\) belongs to each process. By AA.3, each process inherits the "perfect divisibility" property of real half-line. AA.4 implies there is no joint production of more than one output by the same process, \(k, k \in I_r\). AA.5 implies constant coefficients of production or consumption of each \(i\)th good, \(i \in I_N\), by each process. AA.6 implies that the technical rates of utilization of the service of each \(j\)th factor, \(j \in I_m\), are fixed for each \(k\)th process, \(k \in I_m\).

We now consider (1) a concave programming problem, (2) a separable concave programming problem, and (3) a linear programming problem--in the context of a multi-product firm. Since we are interested in economic implication, we may make a "stronger" (but economically meaningful) assumption than what is necessary to guarantee the results.

**Concave-programming problem**

We assume AA.3 holds. The problem I becomes,

\[
\max I \sum_{i=1}^{N} p_i f_i(x)
\]

(II)

subject to \(g_j(x) \leq b_j, j \in I_m; x \geq 0, k \in I_r\)

We assume

1. \(f_i(x)\) is a concave and differentiable function of \(x, x \in X, i \in I_N\).
2. \( g_j(x) \) is a convex and differentiable function of \( x, x \in X, i \in I_N \). Define \( F(x) = \sum_{i=1}^{N} p_i f_i(x) \). Since \( p_i \geq 0 \), \( F(x) \) will also be a concave and differentiable function in \( x \). Since stocks are measured with positive sign, \( b \geq 0 \). Let us assume that \( b > 0 \), and \( g(\phi) = 0 \), where \( \phi \) is the null vector, i.e., \( \phi \in \mathbb{R}^r \). (At the zero level of activity no factor of production is utilized.)

Hence, the problem II satisfies Slater's condition (1951), since \( \phi \in X \) and \( b - g(\phi) > 0 \). Suppose an optimal \( x^0 \) exists for a given \( b^0 \). Then \( z(b^0) \) is finite. It can be shown that at \( b = b^0 \), the following right-hand and left-hand derivatives will exist (Uzawa, 1958; Balinsky and Baumol, 1968).

\[
\left( \frac{\partial z}{\partial b_j} \right)_+ = \lim_{\Delta b_j \to 0^+} \frac{z(b^0 + \Delta b_j e_j) - z(b^0)}{\Delta b_j}
\]

\[
\left( \frac{\partial z}{\partial b_j} \right)_- = \lim_{\Delta b_j \to 0^-} \frac{z(b^0 + \Delta b_j e_j) - z(b^0)}{\Delta b_j}
\]

where \( e_j \) is the \( j \)th elementary vector.

\( \left( \frac{\partial z}{\partial b_j} \right)_+ \) is the forward marginal revenue productivity and \( \left( \frac{\partial z}{\partial b_j} \right)_- \) is the backward marginal revenue productivity of the \( j \)th factor, \( j \in I_m \), evaluated at \( b = b^0 \).

Moreover, \( x^0 \) is optimal if and only if there exists a vector \( y^0 = \{ y_1^0, y_2^0, \ldots, y_m^0 \} \) such that the following Kuhn-Tucker conditions hold,

\[
\left( \frac{\partial F}{\partial x_k} \right)_+^0 \leq \sum_{j=1}^{m} y_j^0 \left( \frac{\partial g_j}{\partial x_k} \right)_+^0, k \in I^1_r.
\]

\( ^1(\ )^0 \) implies that the partial derivatives are evaluated at \( x=x^0 \).
with equality for such \( k \) for which \( x_k^0 \) is positive, i.e. a scope for reducing \( x_k \) further exists.

\[
g_j(x_k^0) \leq b_j, \ j \in I_m
\]

with equality for such \( j \) for which \( y_j^0 > 0 \).

Lastly, \( y_j^0, j \in I_m \) is an upper bound of forward marginal productivity and a lower bound of backward marginal productivity of \( j \)th factor, \( j \in I_m \):

\[
\left( \frac{\partial z}{\partial b_j} \right)_+ \leq y_j^0 (b^0) \leq \left( \frac{\partial z}{\partial b_j} \right)_-, \ j \in I_m
\]

with equality replacing inequalities where the left-hand and right-hand derivatives are equal at \( b = b^0 \).

**Separable concave programming model**

We assume AA.1, AA.2 and AA.3 hold.

Define

\[
c_k(x_k) = \sum_{i=1}^{N} p_{ik} f_i(x_k), \ k \in I_r
\]

\[
F_k(x_k) = c_k(x_k) x_k
\]

The problem I becomes

\[
(III) \max \sum_{k=1}^{r} F_k(x_k)
\]

\[
\text{subject to} \sum_{k=1}^{r} g_{jk}(x_k) \leq b_j, \ j \in I_m
\]

\[
x_k \geq 0, \ k \in I_r
\]

Assumptions 1 and 2 of problem II are replaced by:

3. \( F_k(x_k) \) is a concave and differentiable function of \( x_k, k \in I_r \);
4. \( g_{jk}(x_k) \) is a convex and differentiable function of \( x_k, j \in I_m, k \in I_r \).

We further assume (as before) \( b > 0 \) and \( g_{jk}(0) = 0, j \in I_m, k \in I_r \).

For a given \( b = b^0 > 0 \), an optimum \( x^0 \geq 0 \) exists if and only if, there exists an \( y^0 \geq 0 \), such that

\[
\left( \frac{\partial F(x_k)}{\partial x_k} \right)^0 - \sum_j y^0 \left( \frac{\partial g_{jk}}{\partial x_k} \right)^0 \leq 0, k \in I_r
\]

and the equality holding for such \( k \) for which \( x^0_k > 0 \)

\[
\sum_{k=1}^r g_{jk}(x^0_k) \leq b_j, j \in I_m
\]

with the equality for such \( j \), for which \( y^0_j > 0 \).

\( y^0_j \), once again will be an upper bound of marginal revenue productivity of the \( j^{th} \) resource in the forward direction and a lower bound of the same in the backward direction. It may now be interpreted also as "efficiency" price associated with the \( j^{th} \) constraint at which a transfer of the \( j^{th} \) factor as between different processes should take place so that the best "use" is attained for this factor relative to all opportunities and all constraints in the system.

To illustrate this property of \( y^0 \), we create the "fiction" of a divisionalized firm. Each division is assigned one process. Besides these \( r \) divisions, there is a central division, the custodian of all \( m \) factors from which the different divisions requisition their requirements. The manager of the \( k^{th} \) division is asked to requisition \( g_{jk}[x_k(t)], j \in I_m, k \in I_r \), where \( x_k(t) \) is chosen to maximize

\[
\pi(x_k) = F_k(x_k) - \sum_{j=1}^m y_j(t) g_{jk}(x_k), \text{ and}
\]
\( y_j(t) \) is the "provisional" price of the service of the \( j^{th} \) factor, \( j \in I_m \), charged by the central division at \( t^{th} \) iteration. For each \( y(t) = \{y_1(t), y_2(t), \ldots, y_m(t)\} \), the corresponding \( x_k(t) = x_k[y(t)] \) is determined. To justify his existence the manager maintains zero-profit intensity of the process at a constant level, but increases the intensity of the process if it is profitable, and decreases the intensity of the process if it is unprofitable unless \( x_k = 0 \) already. Once the requisitions are made, prices are revised by the following rule (Whinston, 1964).

\[
\frac{dy_j(t)}{dr} = \begin{cases} 
0 & \text{if } y_j(t) = 0 \text{ and } b_j - \sum_{k=1}^{r} g_{jk}[x_k(t)] > 0 \\
L \left\{ \sum_{k=1}^{r} g_{jk}[x_k(t)] - b_j \right\} & \text{otherwise}, 
\end{cases}
\]

where \( L \) is a positive constant. In other words, the price is raised if the resource is in excess demand, lowered (unless it is zero already) if in excess supply. When "efficiency" prices \( y^o \) is established, \( \frac{dy_j(t)}{dr} = 0 \), for each \( j \in I_m \), and \( x_k^o \pi(x_k^o) = 0 \), i.e., if \( x_k^o \) is positive, \( \pi(x_k^o) = 0 \). So the requisition orders are not changed. The central division, then and only then, allocates to each division, \( g_{jk}(x_k^o), j \in I_m, k \in I_r \); and absorbs any surplus that is not allocated. Such a process of allocation is called "fiction" for most of the fixed and semi-fixed resources are not like "spare parts" which can be requisitioned from the center but "immobile." The possibility of transferring units of such a factor from one process to another still exists since men and materials appropriate to one process can be moved more easily. If the mountain does not come to Mohammed, Mohammed would go to the mountain.
However, such a switching of one resource from one process to another might give rise to "relocation cost" as in Pfouts' (1961) and Dhrymes' (1964) models.

It may be noted that \( F_k(0) = 0 \), \( k \in I_r \). So no process will be operated at a positive level unless \( F_k(x^0_k) \geq 0 \). But since \( \pi(x^0_k) = 0 \), for \( x^0_k > 0 \), \( F_k(x^0_k) = 0 \) with \( x^0_k > 0 \), only if \( g_{jk}(x^0_k) = 0 \) for each \( j \in I_m \), i.e., the process does not require any of the factors, even when operated at a positive level. If we exclude that case \( F_k(x^0_k) > 0 \), if \( x^0_k > 0 \). \( F(x^0_k) \) is the contribution of each process towards net revenue, i.e., net revenue is process wise separable.

**Linear programming problem**

We now assume AA.3, AA.5, and XX.6 hold,

\[
c_k(x_k) = \sum_{i=1}^{N} p_{ik} b_i = c_k, \ k \in I_r.
\]

If the process intensity is measured by a scale unique up to linear transformation for each \( k \in I_r \), then a further simplication is possible by setting the origin at the zero level of intensity, as before, and choosing the unit of the \( k^{th} \) process in such a way that \( c_k = l_k(k \in I_r) \).

(The dimension of \( c_k \) is dollar per unit of intensity of \( k^{th} \) process).

We shall, however, not make such a simplification.

The problem before the firm is

\[
\max c'x = p'Bx
\]

\[(IV) \quad \text{subject to } Ax \leq b\]

\[x \geq 0\]
where \( c = \{c_1, c_2, \ldots, c_r\} \).

We now assume that \( b \geq 0, A \geq 0 \), and some \( a_{jk} \) is positive for each \( k \in I_r \). Then an optimal \( x^o \) must exist.

**Proof:** The set \( \{ x | Ax \leq b, x \geq 0 \} \) is not empty as \( \emptyset \) is a point in the set. Write the dual to (IV)

\[
\min b'y \\
\text{subject to } yA' \geq c' \\
y \geq 0.
\]

By choosing \( y \) as large as necessary (e.g., \( y = \max(0, \frac{c_k}{\max_j a_{jk}}) \)) it can be shown that \( \{ y | yA' \geq c', y \geq 0 \} \) is not empty. Hence, an optimal \( x^o \geq 0 \) and corresponding \( y^o \geq 0 \) exist such that

\[
c' - y^o'A \leq 0
\]

with equality for such \( x^o_k > 0, k \in I_r \)

\[
Ax^o \leq b
\]

with equality for such \( y^o_j > 0, j \in I_m \).

Moreover, for any \( b \geq 0 \), \( z(b) \) will be a concave, piece-wise linear function in \( b \). Its partial derivatives where they exist, e.g., \( \frac{\partial z}{\partial b_j} \) are equal to \( y^o_j, j \in I_m \). Where they do not exist,

\[
\frac{\partial z}{\partial b_j^+} \leq y^o_j \leq \frac{\partial z}{\partial b_j^-}
\]

since the left-hand and right-hand derivatives will always exist. At the optimal \( x^o \), the \( y^o \) vector can be used to calculate the imputed cost
of $k^{th}$ process,

$$z_k = a_{k1}y_1 + a_{k2}y_2 + \ldots + a_{km}y_m.$$ 

If $z_k < c_k$, the $k^{th}$ process will not be utilized and $z_k - c_k$ is the cost of introducing the $k^{th}$ process at the unit level evaluated at $z(b) = b'y^0 = c'x^0$. Besides only those processes would be utilized for which $z_k = c_k$.

So far we have made no distinction between input and output. This is not mere fancy. When the processes are essentially interdependent (i.e., not separable) some good may be an output for a firm at a particular value of $x$ and input for some other value of $x$, the activity vector. (For a farmer, seeds may be sold at a low intensity of farming and purchased from the market at a high intensity of farming.)

When the processes are separable, we only require to know what is input and output for a particular process. Even then for a nonlinear process there is one difficulty. Let $f_i(x_k) = a_i(x_k^2 - x_k^0)$, where $a_i > 0$. Then by our sign convention, the $i^{th}$ good is an output for $0 < x_k < 1$ and input for $x_k > 1$ for the $k^{th}$ process! Perhaps this can be avoided by defining an "economic region" for a particular process. We shall however, make the following assumption:

AA.7 AA.1 holds, $f_i(x_k) = 0$ at $x_k = 0$ and $f_i$ is monotonic in $x_k$ for all $i \in I_n$ and $k \in I_r$.

It is easy to see that for any particular process, $k$, inputs and outputs are unambiguously determined, if AA.7 holds, by their signs at any $x_k > 0$, unless $f_i(x_k) = 0$. Moreover, in the linear case, AA.7
Let AA.2, AA.3, AA.4 and AA.7 hold. We also omit from the list of processes those for which \( f_i(x_k) \geq 0 \) for any \( i \in I_n \), and for any value of \( x_k > 0 \). Such a process can never contribute towards net revenue. By AA.4, \( f_i(x_k) > 0 \) for at most one \( i \). Let \( n \) be the number of processes for which \( f_i(x_k) > 0 \) for some value of \( x_k > 0 \). We renumber the processes so that \( i^{th} \) good is produced by the \( i^{th} \) process. If the same good is produced by two different processes they are considered as separate goods.

Define \( q_i = f_i(x_i) \), \( i \in I_n \). Let us choose \( q_i \) as the index of intensity of the \( i^{th} \) process. We also assume that none of the \( i^{th} \) good, \( i \in I_n \), is being consumed by any process, \( i \in I_n \). So there are outputs, and there are current inputs which are purchased from the market. Let the index of all current inputs be \( j \), \( j = m+1, \ldots, M \), and \( g_{ji}(q_i) \) is the amount of the \( j^{th} \) current input consumed for producing \( q_i \), \( j = m+1, \ldots, M \).

Define \( v_{mi} = g_{ji}(q_i) \), \( j \in I_M \), \( i \in I_n \). The production function of \( q_i \), \( i \in I_n \) can be written as

\[
q_i = q_i(v_{i1}, v_{2i}, \ldots, v_{Mi}).
\]

Thus, the "process" in programming literature looks similar to a "production function" in neoclassical literature. But the similarity is only apparent. All the arguments in the function \( q_i \) are parametrized by \( q_i \), i.e., all the factors are product shadows. So we have only one degree of freedom. As soon as \( q_i \) is chosen all \( v_{ji}, j \in I_M \) are determined. However, when the same good is produced by more than one process,
we have more than one degree of freedom to obtain a given level of output. By taking convex combinations of the processes we can produce the same level of output by alternative factor combinations. But continuous factor substitutions would not be possible even in this case. Only in the case of infinite processes we have \( v_{ji}, j \in I_M, i \in I_n \) as all independent decision variables. The optimizing conditions would then be different.

To bring out this difference let us relax the assumptions of (1) irreversibility of durable inputs, and (2) no limit on the availability of the current inputs. We assume that the firm starts with a given supply of current inputs. \( b_j \) is the availability of the \( j^{th} \) current input to the firm in the week, \( j = m+1, m+2, \ldots, M \). Let \( w_j \) be the price at which the firm can sell the \( j^{th} \) factor. For \( j = 1, 2, \ldots, m \), \( w_j \) is the rental obtained by the firm by leasing one unit of service of the \( j^{th} \) factor to the outsiders. For \( j = m+1, \ldots, M \), \( w_j \) is the price at which the market would take the \( j^{th} \) current input. We shall assume that \( b_j \), \( j = 1, 2, \ldots, M \) is given, i.e., it is not possible to augment the supply of the \( j^{th} \) factor by either outright purchase, \( j = m+1, \ldots, M \), or by hiring the service of the \( j^{th} \) factor, \( j = 1, 2, \ldots, m \), from the market. \( w_j \) can be given an alternative interpretation. Let us assume that at the beginning of the week all resources are with the central division. The cost of transferring any \( j^{th} \) factor to all product divisions is the same. Then \( w_j \) is the cost of allocating the \( j^{th} \) factor to a product division, that could be saved by the firm by no such allocation. We assume \( w_j \geq 0 \).
We now define a selling activity. Let $q_{n+j}$ be the amount of the factor, $j$, that is sold in the market at $w_j$, $j \in I_M$. The programming problem before the firm is

$$\max_{q_1, q_2, \ldots, q_{n+M}} \sum_{i=1}^{n} c_i q_i + \sum_{j=1}^{M} w_j q_{n+j} = z$$

subject to

$$\sum_{i=1}^{n} g_{ji}(q_i) + q_{n+j} \leq b_j, \quad j = 1, 2, \ldots, M$$

$$q_i \geq 0, \quad i = 1, 2, \ldots, n+M.$$

We assume

1. $g_{ji}(q_i)$ is a convex and differentiable function of $q_i$, $i \in I_n$, $j \in I_M$.
2. $b_j > 0$, and $g_{ji}(0) = 0$, $j \in I_M$, $i \in I_n$.

The above is a separable concave programming problem, which satisfies Slater's condition at $q_i = 0$, $i \in I_{n+M}$. Hence, for a given $b = \{b_1, b_2, \ldots, b_M\}$, $b > 0$, an optimum $q^0 \geq 0$ exists, if and only if, there exists an $y^0 \geq 0$, such that,

$$c_i = \sum_{j=1}^{M} y^0_j \left( \frac{\partial g_{ji}}{\partial q_i} \right)^0 \leq 0, \quad i = 1, 2, \ldots, n$$

and the equality holds for such $i$, for which $q_i^0 > 0$;

$$w_j - y_j^0 \leq 0, \quad j = 1, 2, \ldots, M$$

and the equality holds for such $j$ for which $q_{n+j}^0 > 0$;

$$\sum_{i=1}^{n} g_{ji}(q_i^0) + q_{n+j}^0 \leq b_j, \quad j = 1, 2, \ldots, M$$

and the equality holds for such $j$, for which $y_j^0 > 0$. 

From 44, 45 and 46 we obtain

\begin{equation}
\sum_{i=1}^{n} c_i q_i^o - \sum_{i=1}^{n} \sum_{j=1}^{M} \left( \frac{\partial g_j}{\partial q_i} \right) q_j^o = 0
\end{equation}

\begin{equation}
\sum_{j=1}^{M} w_j q_{n+j} = \sum_{j=1}^{M} y_j q_{n+j}
\end{equation}

\begin{equation}
\sum_{j=1}^{M} \sum_{i=1}^{n} g_{ji}(q_i^o) y_j^o + \sum_{j=1}^{M} q_{n+j} y_j^o = \sum_{j=1}^{M} b_j y_j^o
\end{equation}

The new element is relation 45. It shows that the imputed price \( y_j^o \) for any resource cannot fall below \( w_j \), for any \( j \)th factor. Moreover, if \( w_j > 0 \), \( y_j^o > 0 \) and we have from 46,

\[ \sum_{i=1}^{n} g_{ji}(q_i^o) + q_{n+j} = b_j \]

i.e., the available supply of resources will either be used in production, or be sold (or leased) at a positive price if such a possibility exists. Similarly from 45, \( w_j = y_j^o \), for any resource sold (or leased) outside.

Using 48 and 49, \( z \) can be expressed as

\[ z = \sum_{i=1}^{n} c_i q_i^o + \sum_{j=1}^{M} w_j q_{n+j} \]

\[ = \sum_{i=1}^{n} c_i q_i^o + \sum_{j=1}^{M} b_j y_j^o - \sum_{j=1}^{M} g_{ji}(q_i^o) y_j^o \]

\[ = \sum_{j=1}^{M} b_j y_j^o + \sum_{i=1}^{n} \left[ c_i q_i^o - \sum_{j=1}^{M} y_j^o g_{ji}(q_i^o) \right] \]

Let us make the usual assumptions, namely,

1. \( q_{n+j} \) is unrestricted in sign for \( j=m+1, m+2, ..., M \), i.e.,
current inputs can either be purchased or sold at a given
price \( w_j > 0 \), \( j = m+1, \ldots, M \).

2. \( w_j = 0 \) for \( j = 1, 2, \ldots, m \), i.e., durable capital goods cannot
be leased to outside. Hence, \( q_{n+j}^o = 0 \) for \( j = 1, 2, \ldots, m \).

Now 45 should be replaced by

\[
(45') \quad w_j = y_j^o, \quad j = m+1, m+2, \ldots, M
\]
\[
y_j^o \geq 0, \quad j = 1, 2, \ldots, m.
\]

And from 46

\[
(50) \quad \sum_{j=1}^{m} \sum_{i=1}^{n} g_{ji}(q_i^o) y_j^o = \sum_{j=m+1}^{M} b_j y_j^o.
\]

Using 50, \( z \) can be written as

\[
z = \sum_{j=m+1}^{M} b_j w_j + \sum_{i=1}^{n} \left\{ c_i q_i^o - \sum_{j=m+1}^{M} w_j g_{ji}(q_i^o) \right\}.
\]

The first term is the net worth of the firm at the beginning of the week
and the second term is the net value added by production within the week.
It may be noted 45' implies that for any current input the marginal
revenue productivity is equal to its price for all \( j \).

Let us now compare the above model with a model of multi-product
firm in which the "production function" replaces the process.

The problem before the firm is

\[
\max \sum_{i=1}^{n} c_i q_i - \sum_{i=1}^{n} \sum_{j=1}^{M} w_j v_{ji}
\]

subject to \( q_i - q_i(v_{1i}, v_{2i}, \ldots, v_{Mi}) = 0 \), \( i = 1, 2, \ldots, n \)
\[
\sum_{i=1}^{n} v_{ji} \leq b_j, \quad j = 1, 2, \ldots, m.
\]
where $q_i \geq 0$, $i \in I_n$, $v_{ji} \geq 0$, $j \in I_M$, $i \in I_n$. The Lagrangian would be

\[
L = \sum_{i=1}^{n} c_i q_i - \sum_{i=1}^{M} \sum_{j=1}^{n} w_{ji} v_{ji} + \sum_{i=1}^{n} \lambda_i [q_i - q_i(v_i)] + \sum_{j=1}^{M} \sum_{i=1}^{n} \mu_j (b_j - \sum_{i=1}^{n} v_{ji})
\]

where $v_i = v_1^i, v_2^i, \ldots, v_M^i$. It may be noted that $w_{ji}, j=1, 2, \ldots, m,$ now should be interpreted as allocation cost of the $j^{th}$ resource to the $i^{th}$ production division. If an optimum $q^o, v_1^o, v_2^o, \ldots, v_n^o$ exist then there would exist $\lambda_i^o, i=1, 2, \ldots, n$ and $\mu_j^o, j=1, 2, \ldots, m$, such that,

\begin{align*}
(51) & \quad c_i \leq \lambda_i, \quad i=1, 2, \ldots, n \\
(52) & \quad w_{ji} - \lambda_i \frac{\partial q_i}{\partial v_{ji}} - \mu_j \leq 0 \\
(53) & \quad q_i - q_i(v_i) = 0 \\
(54) & \quad b_j - \sum_{i=1}^{n} v_{ji} \leq 0
\end{align*}

with equality for those $i$, for which $q_i^o > 0$

with equality for those $(ji)$ for which $v_{ji}^o > 0$

\begin{align*}
(53) & \quad q_i - q_i(v_i) = 0 \\
(54) & \quad b_j - \sum_{i=1}^{n} v_{ji} \leq 0
\end{align*}

with equality for those $j$, for which $\mu_j^o > 0$.

$\mu_j^o \geq 0; y_j^o$ is unrestricted in sign, $j=1, 2, \ldots, m$. 
Such differences appear because now the decision variables are \( n + nM \), though they are related by \( n \) production relations and \( m \) inequalities. We have thus more degrees of freedom.

The above discussion of programming type models is not intended to be a survey of existing models of multi-product firms. By concentrating on "production aspects" of a multi-product firm, we have ignored quadratic-programming type models (e.g., Dorfman's models, 1951) or other types of models in which price is not assumed to be given and selling is considered a separate activity (Farrel, 1954; Brems, 1961). Moreover, it ignores stochastic considerations (Dhrymes, 1964). Our purpose has been to present a variety of programming models to indicate what we gain or lose by alternative sets of assumptions.
CHAPTER VI. THE MODEL

We now formulate a simple model of a multi-product firm, which we hope, would be regarded as more suitable for explicit dynamization than the existing models of multi-product firm. Much of the development would be straightforward in view of the related discussion of one-product firms in the first two chapters.

We adopt the following conventions:

1. products will be indexed by $i$;
2. resources will be indexed by $j$;
3. inputs are measured with positive sign. So costs also appear with positive signs.

Consider a firm producing $n$ number of products, $q_1, q_2, \ldots, q_n$. For each product there is a sales limit $d_i$, $i \in I_n$, up to which the firm can dispose of any amount that it is willing to produce at a given (gross) market price, $p_i$. Anything produced over the sales limit cannot be sold in the market. The price functions, and the corresponding revenue functions, are given by:

\begin{align*}
(55) \quad p_i(q_i) &= p_i; \quad 0 < q_i \leq d_i \\
(56) \quad R_i(q_i) &= p_i q_i; \quad 0 < q_i \leq d_i \\
&= p_i \cdot \frac{q_i}{d_i}; \quad q_i \geq d_i \\
&\quad i \in I_n.
\end{align*}

We shall make the "no inventory" assumption. All outputs are sold. Moreover, the firm will not produce anything that cannot be marketed.
So \( q_i \) is also the amount sold of the \( i^{th} \) good, \( i \in I_n \). As a consequence the amount produced of the \( i^{th} \) good would be subject to the following constraints

\[
0 \leq q_i \leq d_i, \quad i \in I_n.
\]

There are two types of cost associated with producing and marketing any commodity. In the first place, there is variable cost, \( V_i \), associated with producing the \( i^{th} \) commodity which is assumed to be proportional to the amount produced, \( q_i \):

\[
V_i(q_i) = u_i q_i; \quad q_i \geq 0, \quad u_i > 0, \quad i \in I_n.
\]

Economically speaking, \( u_i \), the constant of proportionality, is the marginal as well as the average variable cost for the \( i^{th} \) product, \( i \in I_n \).

As regards the marketing cost, we shall make a slight departure from Chapter I. We assume that the marketing cost associated with the selling of the \( i^{th} \) commodity is given by the following quadratic function,

\[
M_i(q_i, d_i) = m_{i1} \frac{q_i}{d_i} + \frac{1}{2} m_{i2} \frac{q_i^2}{d_i^2}
\]

where

\[
0 \leq q_i \leq d_i, \quad m_{i1}, m_{i2} > 0, \quad i \in I_n.
\]

If we make \( m_{i1} = m_{i2} \), we get the marketing cost function that we obtained in Chapter I. However, Equation 59 does not result in any additional complications and there is no reason why the cost of marketing (for a given \( d_i \)) proportional to the amount sold \( q_i < d_i \),
and the cost of marketing proportional to \( q_i^2 \) should have the same constant of proportionality.

Differentiating \( M_i(q_i, d_i) \) partially with respect to \( q_i \) and \( d_i \) and differentiating them again with respect to \( q_i \), we get

\[
(60a) \quad \frac{\partial M_i(q_i, d_i)}{\partial q_i} = \frac{m_{i1}}{d_i} + \frac{m_{i2}q_i}{d_i^2}, \quad 0 \leq q_i < d_i, \ i \in \mathbb{I}_n
\]

\[
(60b) \quad \frac{\partial M_i(q_i, d_i)}{\partial d_i} - \frac{m_{i1}q_i}{d_i^2} - \frac{m_{i2}q_i^2}{d_i^3}, \quad 0 \leq q_i < d_i, \ i \in \mathbb{I}_n
\]

\[
(60c) \quad \frac{\partial^2 M_i(q_i, d_i)}{\partial^2 q_i} = \frac{m_{i2}}{d_i^2}, \quad 0 \leq q_i < d_i, \ i \in \mathbb{I}_n
\]

\[
(60d) \quad \frac{\partial^2 M_i(q_i, d_i)}{\partial q_i \partial d_i} - \frac{m_{i1}}{d_i^2} - \frac{2m_{i2}q_i}{d_i^3}, \quad 0 \leq q_i < d_i, \ i \in \mathbb{I}_n
\]

The above relations indicate that the marketing cost is a convex and increasing function of \( q_i \), and hence, the marginal marketing cost is also an increasing function of \( q_i \), \( q_i < d_i \).

Both marketing cost and marginal marketing cost are reduced with an increase in \( d_i \), \( q_i \) being evaluated at a point where \( q_i < d_i \).

Subtracting 58 and 59 from 56 for each \( i \in \mathbb{I}_n \) and summing over \( i \), we get the objective function of the firm

\[
(61) \quad \max_{q_1, q_2, \ldots, q_n} \sum_{i=1}^{n} \left[ p_i q_i - u_i q_i - \frac{m_{i1}q_i}{d_i} - \frac{1}{2} \frac{m_{i2}q_i^2}{d_i^2} \right]
\]

\[ 0 \leq q_i \leq d_i, \ i \in \mathbb{I}_n. \]

In 61 we are, in effect, assuming (in addition to the assumptions that are stated explicitly before) that marketing costs are separable
for each product and they are additive. Of course, this does some violence to reality.

a. In the first place, all products (or at least some) might be sold through the same sales organization set up by the firm.

b. The familiarity of the consumers with one product of the firm might create good will and hence push the sales limit for the other products of the firm, hence, reduce marketing costs by 60b.

c. Most buyers like to purchase from the same spot, due to inertia.

As against a, b and c which might create complimentarity of marketing costs of different products, we have to consider that a multi-product firm generally chooses those products which are technically complementary (i.e., lies within its technical horizon) and hence competitive in demand. These two opposing forces may neutralize each other.

There are two types of constraints, subject to which the objective function in 61 would be maximized by the firm for a given period.
In the first place, there are the sales constraints, already incorporated in our model by 57. In addition to them there are the "capacity" constraints. We start with defining capacity as in the last chapter, namely, \( b_j, j \in I_m \) is the total availability of service flow from the \( j^{th} \) resource in a given production period. Later we show how these input capacities give rise to some "conditional" supply restraints for the production of any commodity, given the production levels of other commodities.
We assume that each product is produced by only one process and each process produces only one product, i.e., no joint production is allowed. The processes are indexed by the product index and the intensity of the process is measured by the output produced. Let $g_{ji}(q_i)$ be the scalar function that indicates the consumption rate of the $j^{th}$ productive service by the $i^{th}$ process, when the process intensity is $q_i$, $i \in I_n$, $j \in I_m$. $g_{ji}(q_i)$ can also be thought as the "factor function" i.e., the minimum amount of the productive services of the $j^{th}$ factor that is needed for producing the $i^{th}$ good. We assume that $g_{ji}(q_i)$ is a convex, differentiable function of $q_i$ and $g_{ji}(0) = 0$, $g_{ji}(q_i) \geq 0$, $q_i \geq 0$, $i \in I_n$, $j \in I_m$. As a result, $g_{ji}(q_i)$ would be monotonically increasing in $q_i$, $i \in I_n$, $j \in I_m$.

Given the supply of $m$ productive resources, $b_1, b_2, ..., b_m$, the production possibility set for the firm in the given production period is given by:

$$\sum_{i=1}^{n} g_{ji}(q_i) \leq b_j, \quad j \in I_m \quad q_i \geq 0, \quad i \in I_n.$$  \hspace{1cm} (62)

Combining 61 and 62 we get the following programming problem for the firm in a given production period,

$$\text{max} \quad \sum_{i=1}^{n} \left[ p_i q_i - u_i q_i - \frac{m_i 1 q_i}{d_i} - \frac{1}{2} m_i 2 \frac{q_i^2}{d_i^2} \right]$$

subject to

$$\sum_{i=1}^{n} g_{ji}(q_i) \leq b_j, \quad j = 1, 2, ..., m$$

$$0 \leq q_i \leq d_i, \quad i = 1, 2, ..., m.$$  \hspace{1cm} (P)

Define $c_i = p_i - u_i - \frac{m_i 1}{d_i}$, $i \in I_n$.
\[ q = \{ q_1, q_2, \ldots, q_n \} \]
\[ c = \{ c_1, c_2, \ldots, c_n \} \]
\[ d = \{ d_1, d_2, \ldots, d_n \}, \quad M = \text{diag}\left\{ \frac{m_{12}}{d_1^2}, \frac{m_{22}}{d_2^2}, \ldots, \frac{m_{n2}}{d_n^2} \right\}. \]

The programming problem \( P \) can be written as

\[
\begin{align*}
\max_{q} \quad & c'q - \frac{1}{2} q'Mq \\
\text{subject to} \quad & \sum_{i=1}^{n} g_{ji}(q_i) \leq b_j, \quad j \in I_m \\
& 0 \leq q \leq d.
\end{align*}
\]

We assume \( d > 0 \). Then \( M \) is defined and it is positive definite. Hence, the objective function is strictly concave. It is maximized subject to constraints which are convex and separable. So \( P \) is a separable concave programming problem, since the objective function is also separable. Let us assume that \( b_j > 0, j \in I_m \). Then the feasible set is non-empty and satisfies the Slater's condition at \( q=0 \), since \( g_{ji}(0) = 0, \quad i \in I_n, \quad j \in I_m \). Moreover, it is bounded, because of the sales and nonnegativity restraints. Hence, an optimum \( q^0 \) exists and it is unique. Any strictly concave function has a maximum in a closed convex set and it is unique. Hence an optimal \( \{ y^0, w^0 \} \) exists so that the necessary and sufficient Kuhn-Tucker conditions will hold.

\[
(63a) \quad p_i - u_i - \frac{m_{1i}}{d_i} q_i - \frac{m_{2i} q_i}{d_i^2} - \sum_{j=1}^{m} y_j^0 \left( \frac{\partial g_{ji}}{\partial q_i} \right)^0 - w_i^0 \leq 0, \quad i \in I_n
\]

with equality for those \( i \), for which \( q_i^0 > 0 \).
(63b) \[ b_j - \sum_{i=1}^{n} g_{ji}(q_i) \geq 0, \quad j \in I_m \]

with equality for those \( j \), for which \( y_j^0 > 0 \)

(63c) \[ d_i - q_i \geq 0, \quad i \in I_n \]

with equality for those \( i \), for which \( w_i^0 > 0 \)

(63d) \[ y^0 \geq 0, \quad w^0 \geq 0 \]

where \( y^0 = \{ y_1^0, y_2^0, \ldots, y_m^0 \} \), \( w^0 = \{ w_1^0, w_2^0, \ldots, w_n^0 \} \).

It may be noted that \( b_j \) is the total availability of the flow of services from the \( j^{th} \) resource in the given production period, while \( g_{ji} \), for a given \( j \) and \( i \), indicates the total use made of the \( j^{th} \) resource, or the amount of productive services of the \( j^{th} \) resource consumed, in the production of the \( i^{th} \) good. Let \( v_{ji} = g_{ji}(q_i), \quad i \in I_n, \quad j \in I_m \). Then \( v_{ji} \) is the amount of the \( j^{th} \) productive service consumed in producing \( q_i \).

Ignoring the current inputs (i.e., assuming they are purchased as they are needed in sufficient quantity), the production function for \( i^{th} \) good, in the "classical" sense is

(64) \[ q_i = q_i(v_{1i}, v_{2i}, \ldots, v_{mi}), \quad i \in I_n. \]

This is a parametric production function. Once \( q_i \) is determined \( v_{ij}, \quad j \in I_m \) are all determined. So it appears that we have one degree of freedom. We now follow an alternative route. Let there be a decomposition of the resource vector \( b = b_1, b_2, \ldots, b_m \) into \( m \) n
sets of real numbers, $b_{ji}$, which satisfy the following conditions:

\[(65a) \quad b_{j1} + b_{j2} + \ldots + b_{jn} \leq b_j,\]

\[(65b) \quad b_{ji} > 0, \quad i \in I_n, \quad j \in I_m.\]

Once again, such a decomposition can be illustrated with the "fiction" of a divisionalized firm, where each division produces one product. Given the capital outfit, $b_i = \{b_{i1}, b_{i2}, \ldots, b_{im}\}$ of the $i^{th}$ division, there is a reasonable way of defining output capacity of this capital outfit.

For any given $j$ and $i$, $g_{ji}$ function can have one of the following properties:

1. strictly increasing in $q_i$, i.e., the $j^{th}$ resource is a semi-fixed resource for the $i^{th}$ product,
2. constant throughout for any value of $q_i \geq 0$, i.e., the $j^{th}$ resource is a fixed resource for the $i^{th}$ product.
3. monotonically but not strictly increasing in $q_i$, $q_i \geq 0$.

Now since $g_{ji}(0) = 0$, the function in 2 above would have the property that $g_{ji}(q_i) = 0$ for any value of $q_i \geq 0$, i.e., the $j^{th}$ resource is not needed in the production of the $i^{th}$ commodity. It may be noted that we have already assumed that $g_{ji}$ is a convex and differentiable function of $q_i$. Hence, $g_{ji}$ is continuous and cannot have any corner point and since it is convex, it cannot have any inflection point. Moreover, $g_{ji}$ is monotonic. Hence, any function in 3 above will start increasing at $q_i = 0$ and then slowly approach a horizontal asymptote and from $q_i = q_i^*$, it would coincide with the horizontal asymptote.
Let \( b_{ji} = g_{ji}(q_i) \). Then \( b_{ji} \) is the minimum amount of the \( j^{th} \) resource that is needed for producing any \( q_i \geq \hat{q}_i \).

The function in 1 above is invertible. Let us invert it to derive the limitation function,

\[
\bar{q}_i(b_{ji}) = g_{ji}^{-1}(b_{ji}) \text{ for a given } j \text{ and } i,
\]

\( g_{ji}^{-1} \) is the inverse function \( g_{ji} \) and \( b_{ji} \) is the point at which it is evaluated. Since the inverse function of a monotonically strictly increasing function will inherit the same property, \( \bar{q}_i = 0 \) at \( b_{ji} = 0 \). The function is therefore defined for \( b_{ji} \geq 0 \). For a function having the property 2 above we define

\[
(67) \quad \bar{q}_i(b_{ji}) = +\infty, \quad b_{ji} \geq 0.
\]

For a function having the property 3 above we, first of all, subdivide its domain into two intervals, \( 0 \leq q_i \leq \hat{q}_i, \quad q_i > \hat{q}_i \). For the first closed and bounded interval the function has the property 1 and hence can be inverted to give

\[
(68a) \quad \bar{q}_i(b_{ji}) = q_{ji}^{-1}(b_{ji}), \quad 0 \leq b_{ji} \leq \hat{b}_{ji}.
\]

For the open unbounded interval, we define

\[
(68b) \quad \bar{q}_i(b_{ji}) = +\infty, \quad b_{ji} > \hat{b}_{ji}.
\]

At \( b_{ji} = \hat{b}_{ji} \), \( \bar{q}_i(b_{ji}) = \hat{q}_i \) and it jumps to infinity at \( b_{ji} = +\). Moreover, even for this function \( \bar{q}_i = 0 \) at \( b_{ji} = 0 \), unless \( b_{ji} = 0 \), i.e., it has the property 2 above.
The limitation functions in 66 - 68 have a simple economic interpretation. It denotes the "maximum" amount of \( q_i \) that can be produced with \( b_j \geq 0 \), when all other resources except the \( j^{th} \) resource, including all current inputs that are needed, are "available" in sufficient or more than sufficient quantity. In some cases, as we have seen, the "maximum" does not exist. But this is understood. Ignoring current inputs once again, which have no limit on their availability, the output capacity of a given supply of capital outfit, \( b_i = \{ b_{1i}, b_{2i}, \ldots, b_{mi} \} \), is given by the following nonclassical production function,

\[
(69) \quad q_i(b_i) = \min \{ q_i(b_{1i}), q_i(b_{2i}), \ldots, q_i(b_{mi}) \}.
\]

If at least one of the \( j^{th} \) resource is needed for the production of the \( i^{th} \) commodity, the above minimum will always exist. The above production function is nonclassical because it relates output with available supply of factors. The distinction between the two types of production functions will, however, disappear if, each product division is allocated a "correct" amount of resources, i.e., if for each \( i \in I_m \),

\[
b_{ji} = g_{ji}(q_i), \quad j \in I_m.
\]

Now for our divisionalized firm, the rules of the game are set in such a way that no product division will requisition more productive services than it intends to use in its production at a given set of internal prices. Suppose the \( i_0^{th} \) product division deviates from the above rule of the game. For any \( j^{th} \) resource, for which at the optimal output vector \( q_i^0 \), \( \Sigma_{i=1}^n g_{ji}(q_i^0) = b_j^0 \), this would mean that at the efficient set of price vector \( y^0 \), the \( j^{th} \) resource would be in short supply to the extent of over-requisitioning on the part of the \( i_0^{th} \) sector, say, \( b_{j_0}^i \).
So the price of the $j$th resource would be increased, i.e., the $j$th resource would be charged a positive price, even if at the optimal $y^0, y^0_j$ was zero. (The latter possibility exists if $y^0$ is not uniquely determined, i.e., if there is degeneracy.) Now since at the "efficiency prices" all sectors break even; the $i_{th}$ sector cannot have more than adequate supply of the $j$th resource without showing deficit. Only when the $j$th resource is in excess supply the $i_{th}$ sector can have more of it than needed for production with impunity. However, if we introduce the cost of transferring any resource from the central division to any product division, each resource would have a minimum positive supply price for any product division and hence all excess supply of resources would be absorbed by the central division.

It may be mentioned that if all $g_{ji}$ functions have a property 1, i.e., if all the $m \times n$ limitational functions are well defined and bounded then each and every product division will produce on the limitational line, or the confluence region of the productive resources (Frisch, 1965) as given by

\[
q_i(b_i) = \overline{q_i}(b_i) = \overline{q_i}(b_{2i}) = \ldots = \overline{q_i}(b_{mi}).
\]

Generally, however, 70 would not hold for a multi-product firm and some of the limitation functions can become unbounded either at $b_{ji} = 0$ or at $b_{ji} = b_{ji}^*$. The nonclassical production function 69, however, can be written in another form. Let us assume that the output to be produced by all product divisions except the $i_{th}$ division has been decided already. Define $b_{ji}(i_o q) = b_j - \sum_{i \neq i_o} g_{ji}(q_i)$. Then the
production function of \( q_{i0} \) is given by

\[
q_{i0} = \min \left\{ q_{i0}^i, q_{i0}^j, \ldots, q_{i0}^m \right\}.
\]

Equation 71 brings out the conditional nature of production function of any product division. It also shows that we have \( n-1 \) degrees of freedom in producing \( q_{i0} \), for a given "b" vector.

Equation 71 shows why the concept of marginal (revenue) productivity of limitational factors (i.e., factors not in substitution relationship with each other) undergoes a dramatic transformation as we switch over from a single-product firm to a multi-product firm.

With a given b vector the single-product firm has a zero degree of freedom. The forward marginal productivity of a limitational factor is zero while the backward marginal productivity is positive. As soon as we have more than one product, we gain one degree of freedom, the degree of assortment of the two products. While in a single-product case any marginal increase in a limitational factor unless accompanied by an appropriate increase in other limitational factors does not result in any increase in output, hence, revenue, in the case of multi-product firm any marginal increase in one resource that is fully utilized can be accommodated by changing the product mix, i.e., reshuffling resources in different product divisions. As a result forward marginal revenue productivity, \textit{mutatis mutandis}, will turn out, generally, positive for a fully utilized resource and except for some corner points the backward and marginal revenue productivity will coincide. Two points can be made. If we consider relocation costs, then such a concept of marginal revenue productivity
is gross. The relocation cost, however, would be a discontinuous function of changes in the resource vector. Thus, if the firm acquires a capital outfit that exactly fits the requirements of a product-division for a marginal change in its output program then other product divisions might not be disturbed. The consideration of relocation cost lends some importance to a balanced growth path of a firm that would be considered later.

Let us now assume that \( g_{ji}(q_i) = a_{ji}q_i, \ a_{ji} \geq 0, \ i \in \mathcal{I}_n, j \in \mathcal{I}_m \). The problem (P) now becomes a quadratic programming problem:

\[
(Q) \quad \max_{q} \ c'q - \frac{1}{2} q'Mq
\]
subject to \( Aq \leq b \)
\[
0 \leq q \leq d
\]
where
\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn}
\end{pmatrix}
\]

(63a) and (63b) become

\[
(63e) \quad c'q^0 - q^0'M - y^0'A - w^0'q \leq 0
\]
\[
c'q^0 - q^0'Mq^0 - y^0'Aq^0 - w^0'q = 0
\]

(63f) \( b - Aq^0 \geq 0 \)
\[
b'q^0 - q^0'Aq^0 = 0
\]

while (c) and (d) remain unchanged.
\( v_{ji} \) becomes \( a_{ji} q_i, i \in I_n, j \in I_m \). If \( q_{ji} > 0 \), for some \( j, i \), then \( g_{ji} \) satisfies property 1. If \( a_{ji} = 0 \) for some \( j \) and \( i \), \( g_{ji} \) satisfies property 2. These two cases are now mutually exclusive.

\[
- q_i(b_{ji}) = \frac{b_{ji}}{a_{ji}}, \text{ if } a_{ji} \neq 0, \text{ for some } j \text{ and } i.
\]

\[
- q_i(b_{ji}) = +\infty \text{ if } a_{ji} = 0, \text{ for some } j \text{ and } i.
\]

Equation 69 now becomes

\[
q_i(b_{ji}) = \min \left\{ \frac{b_{1i}}{a_{1i}}, \frac{b_{2i}}{a_{2i}}, \ldots, \frac{b_{mi}}{a_{mi}} \right\};
\]

equation 71 becomes

\[
q_{i0} = \min_j \left( \frac{b_j - \sum_{i \neq i_0} a_{ji} q_i}{a_{ji}} \right).
\]

Lastly, writing the Lagrangian for the quadratic programming problem and evaluating at \( q^0, y^0, \) and \( w^0 \), we get

\[
\phi(q^0, y^0, w^0) = c^0 q^0 - \frac{1}{2} q^0 M q^0 + y^0 (b - A q^0) +
\]

\[
- w^0 (d - q^0) = c^0 q^0 - q^0 M q^0 + \frac{1}{2} q^0 M q^0 + y^0 b - y^0 A q^0 +
\]

\[
w^0 d - w^0 q^0 = y^0 b + \frac{1}{2} q^0 M q^0 + w^0 d \quad \text{(by (63d))}.
\]

Hence, the dual to the quadratic programming problem is given by

\[
\min_{q, y, w} y^T b + \frac{1}{2} q^T M q + w^T d
\]

subject to \( A y + M q + w \geq c, y \geq 0, q \geq 0, w \geq 0 \).
CHAPTER VII: THE OPTIMAL CAPACITY EXPANSION POLICY FOR A MULTI-PRODUCT FIRM

In the last chapter, a static one-period optimization model for a multi-product firm was presented. The problem was:

\[
\max c'q - \frac{1}{2} q'Mq \tag{P}
\]

subject to \( \sum_{i=1}^{n} g_{ji}(q_i) \leq b_j, j \in I_m \).

\( 0 \leq q_i \leq d_i, i \in I_n. \)

We now introduce some dynamic equations which will link together the above programming problems into the framework of a dynamic over-period optimization problem. Of course, we have a lot of choice in the specification of the exact nature of these dynamic equations. However, we should try to retain one essential feature of our static one-period model that should not be allowed to evaporate in a steady-state solution of the dynamic system, namely, we should stipulate that the multi-product firm should not degenerate into a specialized single-product firm.

Furthermore, we should not unnecessarily complicate the picture so that the ultimate model becomes sterile, devoid of any interesting results and incapable of giving any insight into the dynamic process of the growth of a multi-product firm. With the latter purpose in view, we introduce a few simplifying assumptions.

Firstly, we assume a given technology, i.e., the scalar functions \( g_{ji}, j \in I_m, i \in I_n \) are not time dependent. Secondly we assume that
d(t) is growing over time exogeneously. However, at each t, d(t) is known, t=0, 1, ..., T-1. The vector c and the matrix M involve d(t) and some other parameters which we assume to be constant over time. So once d(t) is known, c(t) and M(t) are also known.

Suppose the firm starts with an outfit of capital goods b(0). By solving a problem like P, z(0) is determined. For other periods, b(t) is still unknown. Let us assume that the firm solves a problem like P for each possible value of b(t) > 0 in each time period t=1, 2, ..., T-1, where T is the end of planning horizon for the group of managers, who are, at present in charge of the firm. T may be the period up to which the present group of managers expects to retain control.

z(t), t=1, 2, ..., T-1 is therefore a function of b(t) and d(t): q(t) being maximized out. The transformation function we can write as z(b(t), t), where t takes into account the effect of d(t) on z(t). The function z(t) is a piece-wise smooth concave function of b(t), for which at least one sided partial derivatives always exist and they are monotonically decreasing in each b_j(t).

A few words might be in order about the specific role of d(t) in our dynamic model. The optimization problem before the firm is to adjust b(t) to d(t) growing over time so that some preference functional over time is optimized. But once d(t) is introduced it is excluded from our dynamic model and all its effects are incorporated into z(t). The same is true of q(t), the optimum output vector in t. It might thus appear that we have, in effect, a one-good model, the good being dollars, with more than one capital good. We shall, however, claim
that though \( d(t) \) and \( q(t) \) do not appear in the over-period optimization problem explicitly, they are always there, in the one-period optimization problem that determines \( z(t) \).

As regards the vector \( b(t) \), we make two assumptions. In the first place, for each \( j \in I_m \), \( b_j(t) \) is subject to a constant rate of depreciation (evaporative decay) \( \beta_{oj} > 0 \) over time. Besides, at the end of each period, \( b_j(t) \) is increased by \( \beta_j(t) > 0 \) (gross of depreciation) by the firm so that at any time, \( t \), \( b_j(t) \) is given by the following dynamic equation:

\[
\begin{align*}
(72) \quad b_j(t + 1) &= b_j(t) - \beta_{oj} b_j(t) + \beta_j(t); \quad t = 0, 1, 2, \ldots, T-1 \\
&= b_j(o); \quad t = 0; \quad j \in I_m.
\end{align*}
\]

\( b_j(o) \) is the \( j^{th} \) element of the initial vector \( b(o) \) with which the firm starts. \( \beta_j(t), j \in I_m, t = 0, 1, \ldots, T-1 \) are \( mT \) control variables. We assume that for any \( t, t = 0, 1, \ldots, T-1 \), \( \beta_j(t) \) belongs to the closed bounded set:

\[
(73) \quad K[\beta(t)] = \sigma(t) \cdot I(t)
\]

where \( \beta(t) = \left[ \beta_1(t), \beta_2(t), \ldots, \beta_m(t) \right] \in \mathbb{R}^m; \beta(t) \succeq 0. \)

\( I(t) \) is the investible fund at time \( t \). The scalar function \( K \) does not involve \( t \) explicitly and has the following neoclassical properties:

\[
(74) \quad K_j[\beta(t)] > 0; \quad K_{jj}[\beta(t)] > 0 \\
K_j(o) = +\infty; \quad K_j(+) = 0
\]

where \( K_j[\beta(t)] \) is the partial derivative of \( K[\beta(t)] \) with respect to
$g_j(t)$, i.e., the marginal cost of expansion of $b_j(t)$, gross of depreciation. $\sigma(t)$ is the proportion of investible fund that is invested at $t$ for augmenting the resource vector $b(t)$ in the next period. It is also a control variable, subject to the following restriction:

\begin{equation}
0 \leq \sigma(t) \leq 1, \quad t=0, 1, \ldots, T-2, \quad \sigma(T-1) = 1.
\end{equation}

At $t=T-1$, the entire investible fund is invested. A proportion of $z(t)$ is being distributed as dividends, at time $t$, to the owners of the firm:

\begin{equation}
c(t) = \gamma(t)z(t); \quad 0 \leq \gamma(t) \leq 1.
\end{equation}

From 75 and 76, we deduce the investible fund at time $t$, given by the following dynamic equation:

\begin{equation}
I(t) = [1 - \gamma(t)]z(t) + R(t)[1 - \sigma(t-1)]I(t-1),
\end{equation}

\begin{equation}
t=1, 2, \ldots, T-1
\end{equation}

\begin{equation}
T(0) = [1 - \gamma(0)]z(0)
\end{equation}

where $R(t)$ is the interest factor at which one dollar invested at time $t-1$ will be multiplied at $t$, $t=1, 2, 3, \ldots, T-1$.

We now define a utility function on $c(t)$,

\begin{equation}
W(t) = W[\gamma(t)z(t)], \quad t=0, 1, \ldots, T-1.
\end{equation}

$W(t)$ is the current utility at time $t$.

It may be noted that $W[c(t)]$ function does not involve $t$ explicitly, so the form of the utility (or one-period welfare) function
remains stationary over time. For our purpose, we do not need a
cardinal measure of the utility function; so we shall assume that the
utility function is unique only up to increasing linear transformation
for which the origin is fixed at the zero-level of utility (to be de­
 fined). In other words, the utility function can be subjected to
uniform contraction or stretching by multiplying \( W \) by a positive
constant. We also assume that \( W[c(t)] \) is twice-differentiable with
the following properties;

\[
\begin{align*}
(78b) & \quad W(0) = 0 \\
(78c) & \quad W'(c(t)) > 0; \ c(t) \geq 0 \\
(78d) & \quad W''(c(t)) < 0; \ c(t) \geq 0.
\end{align*}
\]

Since the origin is fixed 78b makes sense both mathematically and
economically; no consumption, no utility. Equation 78c implies there is no
saturation and 78d, the concavity of \( W(c(t)) \). Since the first and
second derivatives of \( W(t) \) will preserve signs under the admissible
transformation 78b and 78c also make sense.

At the end of the T-th period, the firm ends up with a capital
outfit \( b(T) \), which the present group of managers hands over to the
succeeding group of managers. We define a scrap value function \( S[b(T)] \)
on the vector-variable \( b(t) \) with properties similar to utility function,

\[
(78e) \quad S(\phi) = 0; \ \nabla S[b(T)] > 0
\]

where

\[
\nabla S[b(T)] = \left( \frac{\partial S}{\partial b_1}, \frac{\partial S}{\partial b_2}, \ldots, \frac{\partial S}{\partial b_m} \right)
\]
(78f) \( S[b(T)] \) is concave in \( b(T) \).

One immediate result of the assumptions 78a-78e is the marginal evaluation of one unit of consumption good now would not be the same as the marginal evaluation of one unit of consumption good one period later if \( c(t) \) is growing over time. This gives one reason for a preference for ante-dating or advancing the timing of consumption which is not related anyway to "pure time preference" or "impatience" for advancing the timing of enjoyment, or the utility of consumption, due to myopia or defective telescopic faculty.

The dynamic model works out as follows:

\[
\begin{align*}
\gamma(o)z(o) & \rightarrow W(o) & \gamma(1)z(1) & \rightarrow W(1) \\
b_o & \rightarrow z(o) \rightarrow I(o) \rightarrow \delta(o) \rightarrow b(1) \rightarrow z(1) \rightarrow I(1) \rightarrow \delta(1) \rightarrow b(2) \\
[1-\sigma(o)]I(o) & \rightarrow R(1)[1-\sigma(o)]I(o) \\
\gamma(T-1)z(T-1) & \rightarrow W(T-1) & S[b(T)] \\
b(T-1) & \rightarrow z(T-1) \rightarrow I(T-1) \rightarrow \delta(T-1) \rightarrow b(T)
\end{align*}
\]

Given the initial vectors \( d(o) \) and \( b(o) \), \( z(o) \) is determined by maximizing \( q(o) \), subject to the restrictions of \( P \). Of this \( z(o) \), \( \gamma(o)z(o) \) is distributed as dividends which results in \( W(o) \), the total utility at \( t=0 \). At \( t=0 \), \( I(o) = [1-\gamma(o)]z(o) \). Of this \( I(o) \), \( \sigma(o)I(o) \) is spent on \( \delta(o) \) subject to the restriction 73. The residual portion is added to \( I(1) \) multiplied by the interest rate factor \( R(o) \).

Once \( \delta(o) \) is determined, \( b(1) \) is determined according to Equation 72. This, together with \( d(1) \), determine \( z(1) \). Once again \( \gamma(1)z(1) \) is distributed generating the flow of utility \( W(1) \) in \( t=1 \). The residual portion is added to \( I(1) \).
I(1) is determined according to Equation 77 and of this $c(1)I(1)$
is invested and so on. The process continues until the $T-1$th period
ends. At $t=T-1$, the entire $I(t-1)$ accumulated from the past plus the contribution of the $T-1$th period to it is invested and $b(T-1)$ and hence $b(T)$
is determined. In the $T$th period, the process comes to an end and
no production takes place, i.e., $q(T)=0$. The utility enjoyed at
$t=T$ is therefore $S[b(T)]$.

We have not yet specified the preference functional over time
that is to be maximized. The simplest assumption is that the firm
would maximize some functional of $W(t)$, $t=0, 1, \ldots, T-1$ and $S(T)$.
It may be noted that $W(t)$ is the current utility enjoyed at $t$, and
$S(T)$ has a similar interpretation. So unless intertemporal comparison
of utility enjoyed at different time periods is possible, we cannot
compare them, let alone add them up.

One possible approach may be to search for points along the
utility-possibility frontier. We have $(m+2)T$ control variables, $b(t),
\gamma(t)$ for $t=0, 1, \ldots, T-1$, and $\sigma(t)$ for $t=0, 1, \ldots, T-1$, of them
one $\sigma(T-1)$ is already determined. Let us choose the other control
parameters subject to all the restraints in the system. Once they
are so chosen they will generate an attainable utility path, $W(0), W(1),
\ldots, W(T-1), S(T)$. Let $\Gamma$ be the set of all such attainable paths.
Define

$$W = \{W(0), W(1), \ldots, W(T-1), S(T)\}$$

Then $W^0$ is efficient if and only if there is no other path $W'$ in $\Gamma$ such that
(79a) \( W' \geq W^o \)

(79b) \( W' \in \Gamma \)

The inequality in 79a implies that \( W'(t) \geq W^o(t) \), \( t=1, \ldots, T-1 \) and \( S'(T) \geq S^o(T) \), and \( W' \neq W^o \). So at least for one period the utility path \( W' \) generates a greater flow of utility than \( W^o \).

Now, since this is a "vector-maximum" problem, under certain conditions of convexity of the set \( \Gamma \) and nonsaturation, this would generate an appropriate price vector \( \{\rho(0), \rho(1), \ldots, \rho(T)\} \) such that \( W^o \) is the solution of the following scalar maximization problem\(^1\)

\[
\max_{t=0}^{T-1} \sum_{t=0}^{T-1} \rho(t) W(t) + \rho(T) S(T)
\]

subject to \( \{W(0), W(1), \ldots, W(T-1), S(T)\} \in \Gamma \).

Since \( \rho(t) \) will be determined only up to a multiplicative constant we may set \( \rho(o) = 1 \), assuming \( \rho(o) > 0 \). We may call \( \rho(t) \) the weightage function. If \( \rho(t) \) satisfies \( 0 < \rho(t) < 1 \) for all \( t, t=1, \ldots, T \), then it can also be viewed as a discount function.

For each point on the utility-possibility frontier we can derive an appropriate weightage function, provided certain constraint qualifications are satisfied so that we can utilize the separating hyperplane theorem of convex bodies. However, it is easy to see that by this procedure we can only obtain a quasi-ordering of all utility paths contained in \( \Gamma \), i.e., we cannot obtain a top level optimum.\(^1\)

\(^1\) Any homothetic transformation will preserve the ordering of the discounted sum.
A simpler approach and the one that we are going to follow is to assume that a weightage function over time is already defined for the firm. Let \( \rho(t) \) be such a weightage function. Once again we set \( \rho(0) = 1 \).

Thus, one unit of utility enjoyed at time \( t = t' \) is equivalent to \( \rho(t') \) units of utility at \( t = 0 \), and this "terms of trade" will remain unaffected by a change of scale of \( W \). If \( 0 < \rho(t) < 1 \), for \( t = 1, \ldots, T \), then \( \rho(t) \) can be called a discount function and the rate of discount at time \( t \) is given by \( \frac{\rho(t-1)}{\rho(t)} - 1 \).

We may also note that if we set \( \rho(t) = 1 \) for all \( t \), we get a "Ramsey" type preference functional. \( \rho(t) \equiv 1 \) implies that there is no preference for advancing the timing of future utilities, i.e., there is no impatience (Koopmans, 1967). If we set \( \rho(t) = 0 \), for \( t = 0, 1, \ldots, T-1 \), and \( \rho(T) = 1 \), then by choosing the \( S \) function appropriately we can reduce the problem to that of maximizing the terminal capital stock \( b(T) \), starting from an initial \( b(0) \).

A third line of approach is also possible. Instead of maximizing a preference functional of \( W(t) \), \( t = 0, \ldots, T-1 \), and \( S(T) \), one may be interested in the rate of growth of one-period utility over time. If \( T \) is infinity, this approach leads to the maximization of some long-run rate of change (Radner, 1966). We shall consider a somewhat related approach in the next chapter.

We start with a simple situation when no transfer of investible funds over time is possible. This means

\[
(81) \quad \gamma(t) + \sigma(t) = 1; \quad 0 \leq \sigma(t) \leq 1, \quad 0 \leq \gamma(t) \leq 1.
\]
The problem before the firm is

$$\max_{c(t), z(t), b(t)} \sum_{t=0}^{T-1} \rho(t)W(c(t)) + \rho(T)S[b(T)]$$

where

$$c(t) = [1-\sigma(t)]z(t) = \gamma(t)z(t)$$

subject to

$$b_j(t+1) = [1-\beta_{o,j}] b_j(t) + \beta_j(t), j \in \{1, \ldots, m\}, \ t=0, 1, 2, \ldots, T-1$$

$$z(t) = z(b(t), t), \ t=0, 1, \ldots, T-1$$

$$K[\sigma(t)] = \sigma(t)z(t).$$

To start with we assume that $$z(b(t), t)$$ is continuously differentiable in each $$b_j(t)$$. We shall relax the assumption later.

The Lagrangian is

$$L = \sum_{t=0}^{T-1} \rho(t)W \left\{ [1-\sigma(t)]z(t) \right\} + \rho(T)S[b(T)]$$

$$+ \sum_{t=0}^{T-1} \rho(t)W(t) \left\{ z(b(t), t) - z(t) \right\}$$

$$+ \sum_{j=1}^{m} \sum_{t=0}^{T-1} \rho(t) \mu_j(t) \left\{ b_j(t+1) - b_j(t)[1-\beta_{o,j}] + \beta_j(t) \right\}$$

$$+ \sum_{t=0}^{T-1} \lambda_t \rho(t) \left\{ \sigma(t)z(t) - K[\beta(t)] \right\}.$$

Assuming necessary regularity conditions so that second order conditions are satisfied and internal maximization, we differentiate $$L$$ with respect to control variables and set them equal to zero.

Differentiation with respect to $$\sigma(t), t=0, 1, \ldots, T-1$$, gives,
\[(82) \quad \rho(t) W'[c(t)]z(t) + \lambda(t)\rho(t)z(t) = 0\]

or

\[(82a) \quad W'[c(t)] = \lambda(t), \quad t=0, 1, \ldots, T-1.\]

Differentiation with respect to \(\beta_j(t), \quad j \in I_m, \quad t=0, 1, \ldots, T-1\) gives,

\[(83) \quad \rho(t)\mu_j(t) - \lambda(t)\rho(t)K_j(t) = 0\]

or

\[(83a) \quad \mu_j(t) = \lambda(t)K_j(t) = W'[c(t)]K_j(t)\]

or

\[(83b) \quad \frac{\mu_j(t)}{K_j(t)} = \lambda(t) = W'[c(t)] \quad (from \ (82a))\]

\[j=1, 2, \ldots, m\]
\[t=0, 1, \ldots, T-1\]

Suppose an optimal \(b^*(t), \quad t=1, 2, \ldots, T\) and \(z^*(t), \quad t=0, 1, 2, \ldots, T-1\) exist. Differentiating \(L\) with respect to these state variables, evaluating at \(b^*(t), \quad t=1, 2, \ldots, T, \quad z^*(t), \quad t=0, 1, 2, \ldots, T-1,\) we get the time path of co-state variables by the adjoint or the perfect foresight equations,

\[(84a) \quad \rho(t) \lambda(t) \frac{\partial z[b(t), t]}{\partial b_j(t)} + \rho(t-1)\mu_j(t-1) - \rho(t)\mu_j(t)(1-\beta_{oj}) = 0\]
\[j \in I_m, \quad t=1, 2, \ldots, T-1\]

\[(84b) \quad \rho(t) \frac{\partial S[b(T)]}{\partial b_j(T)} + \rho(T-1)\mu_j(T-1) = 0; \quad j \in I_m\]

\[(85) \quad \rho(t) W'[c(t)][1-\sigma(t)] - \rho(t)\lambda(t) + \rho(t)\lambda(t)\sigma(t) = 0\]
\[t=0, 1, 2, \ldots, T-1\]
From 82 and 85

\[ (86) \quad \mu(t) = \lambda(t) = W'(c(t)), \quad t = 0, 1, 2, \ldots, T-1. \]

Equation (84a) is a difference equation which can be solved making use of the terminal condition 84b. Using 86 and 84a

\[ \rho(T-2) \mu_j(T-2) = \rho(T-1) \mu_j(t-1) (1-\beta_{o,j}) \]

\[ - \rho(T-1) \lambda(t-1) \frac{\partial z(T-1)}{\partial b_j(T-1)}, \]

or

\[ \mu_j(T-2) = \frac{1}{\rho(T-2)} \left[ - \rho(T) \frac{\partial S[b(T)]}{\partial b_j(T)} (1-\beta_{o,j}) \right. \]

\[ \left. - \rho(T-1) \lambda(t-1) \frac{\partial z(T-1)}{\partial b_j(T-1)} \right] \]

Continuing in this way, we derive

\[ (87a) \quad \mu_j(t) = - \frac{1}{\rho(t)} \left\{ \frac{T-1}{T} \sum_{\tau=t+1}^{T-1} \rho(\tau) W'(c(\tau)) \frac{\partial z(\tau)}{\partial b_j(\tau)} \times \right. \]

\[ \left. (1-\beta_{o,j})^{T-\tau-1} + \rho(T)(1-\beta_{o,j})^{T-\tau-1} \right\}, \quad j \in I_m, \quad t = 1, 2, \ldots, T-1. \]

\[ (87b) \quad \mu_j(T-1) = - \frac{1}{\rho(T-1)} \left\{ \rho(T) \frac{\partial S[b(T)]}{\partial b_j(T)} \right\}, \quad j \in I_m \]

\[ (87c) \quad \rho(t-1) \mu_j(t-1) = \rho(t) \mu_j(t) (1-\beta_{o,j}) - \rho(t) \lambda(t) \times \]

\[ \frac{\partial z[b(t), t]}{\partial b_j(t)}, \quad t = 1, 2, \ldots, T-1. \]
Economic interpretations of the co-state variables are as follows. First consider $\mu_j(T-1)$. The expression within braces in 87b is the discounted marginal contribution of one more unit of $b_j(t)$ at the beginning of the $T$th period, $j \in I_m$. So $\mu_j(T-1)$ is the current cost of foregoing this contribution for not having one more unit of the $j$th type of capital goods, $j \in I_m$, at the beginning of $T$th period.

Consider $|\mu_j(t), t=0, 1, ..., T-2$. A gift of one unit of the $j$th capital to the firm at the beginning of the $t+1$th period is equivalent to the service of one more unit of capital good of the $j$th type in the $t+1$th period, $(1-\beta_{o_j})$ unit of similar good at the $t+2$th period, $(1-\beta_{o_j})^2$ unit of similar good at the $t+3$th period and so on until the $T$th period, when the chapter is closed. Thus, such an once-for-all gift will generate a stream of income flow and the expression within braces in 87a is the discounted value of such an income stream. Now since one unit of utility at $t=0$ is equivalent to $1/\rho(t)$ unit of utility at $t$, $\mu_j(t)$ is the current value of the cost of foregoing such an income stream that would result from a gift of one unit of the $j$th capital good at the end of the $t$th period, $j \in I_m$, $t=0, 1, ..., T-2$. Hence, $\mu_j(t)$ is the current opportunity lost for not increasing $\beta_j(t)$ by one unit, $j \in I_m$, $t=0, 1, ..., T-2$. $K_j(t)$ is the current marginal cost of increasing $\beta_j(t)$ by one unit. Equation 83b indicates that for all types of capital goods, $j \in I_m$, in each time period $t=0, 1, ..., T-1$, the ratio of the opportunity lost for not having one more unit of the $j$th capital good and the marginal cost of one more unit of the $j$th capital good should be equal to $\rho(t)$. Equation 83b therefore indicates the optimum allocation rule of the investible fund $\sigma(t)$ $z(t)$ at
t=0, 1, ..., T-1. \( \lambda(t) \) is the common marginal rate of return that will obtain for each capital good that the firm would lose for not diverting one more unit to capacity expansion. Equation 82a tells us that this common rate of return foregone should be equal to marginal utility of consumption at time \( t \), i.e., \( \lambda(t) \) is the current opportunity cost of one unit of investment good. Equation 87c tells us that the discounted value of the opportunity lost for not having one more unit of capital good of the \( j^{th} \) type (\( j \in I_m \)) at the end of the \( t-1^{th} \) period is equivalent to the discounted value of the opportunity lost for not having \( (1-\beta_{oj}) \) unit of similar good at the end of the \( t^{th} \) period minus the discounted value of the marginal revenue productivity of one unit of capital good of the \( j^{th} \) type multiplied by the marginal utility of consumption in the \( t^{th} \) period, \( t=1, 2, ..., T-1 \). In other words (by changing signs of all expressions of 87c), the discounted value of a gift of one unit of capital of the \( j^{th} \) type (\( j \in I_m \)) at the end of the \( t-1^{th} \) period is equivalent to its marginal contribution to utility stream (discounted) in the \( t^{th} \) period plus a gift of \( (1-\beta_{oj}) \) unit of the capital good at the end of the \( t^{th} \) period, \( t=1, 2, ..., T-1 \).

The difference equation, 84a, is nonautonomous, since it involves \( z(t) \), which involves \( t \) explicitly. However, it can be made autonomous on a particular type of growth paths which are called balanced growth paths that we shall discuss in the next chapter. Letting \( T \) go to infinity, the difference equation will have a stationary solution if the following condition holds
\( (88) \quad \mu_j^* = \rho(t) \mu_j^* (1 - \beta_{o_j}) - \rho(t) \lambda(t) \frac{\partial z(t)}{\partial b_j(t)}, \quad j \in I_m. \)

Hence, the stationary value is given by

\[ \mu_j^* = \frac{\rho(t) \lambda(t) \frac{\partial z(t)}{\partial b_j(t)}}{\rho(t)(1 - \beta_{o_j}) - \rho(t-1)} , \quad j \in I_m. \]

It may be noted that since \( \frac{\partial z(t)}{\partial b_j(t)} \) is the marginal revenue productivity of the \( j^{th} \) capital good and \( \lambda(t) \) is equal to the marginal utility of consumption good at \( t \), \( \mu_j^* \) is equal to \( \frac{\rho(t)}{\rho(t)(1 - \beta_{o_j}) - \rho(t-1)} \) times the marginal utility of the \( j^{th} \) capital good multiplied by a term, which if \( \beta_{o_j} = 0 \) (no depreciation) is equal to the reciprocal of the discount rate at time \( t \). The difference equation gives the time path of \( b(t) \), once \( \beta(t) \) is determined. Once \( b(t) \) is determined \( q(t) \) can be determined by solving \( P \), or from the value of \( q(t) \) that corresponds to \( z(b(t), t) \).

Now all capital goods will grow at the same constant rate \( G \), if the following relation holds

\( (89) \quad \frac{\beta_j(t)}{b_j(t)} = 1 - \beta_{o_j} + G, \quad \text{for all } j \in I_m, \quad t = 0, 1, 2, \ldots \).

There are, however, no special reasons to assume that \( 89 \) will be fulfilled.

Let us now consider the following equivalent, one period long-run problem for our over-period optimization problem.
max ρ(t) H(t) = max ρ(t) W\left\{ [1-σ(t)] z(t) \right\}

β(t), σ(t) β(t), σ(t)

+ \sum_{j=1}^{m} [ρ(t-1) \mu_j(t-1) - ρ(t) \mu_j(t)(1-θ_j)] b_j(t+1)

+ \sum_{j=1}^{m} ρ(t) \mu_j(t) \beta_j(t)

subject to

K[\beta(t)] = σ(t) z(t), z(t) = z[b(t),t] for t=0, 1, ..., T-2.

Except for the terminal conditions 84b, 82, 83, 84a, and 85 can be obtained by setting up the Lagrangian corresponding to problem (L) and differentiating it with respect to β(t), σ(t) and b(t), z(t). The Lagrangian can be written as

ρ(t) W\left\{ [1-σ(t)] z(t) \right\} + \sum_{j=1}^{m} [ρ(t-1) \mu_j(t-1) - ρ(t) \mu_j(t)(1-θ_j)] \times

b_j(t+1) + \sum_{j=1}^{m} ρ(t) \mu_j(t) \beta_j(t) + ρ(t) λ(t) \left\{ z[b(t),t] - z(t) \right\} +

λ(t) ρ(t) \{ ρ(t) z(t) - K[\beta(t)] \}.

Let us now relax some of the assumptions we started with. Define r_j(t) = -μ_j(t), j ∈ I_m, r(t) = -λ(t), t=0, 1, 2, ..., T-1. r_j(t) is the marginal evaluation of a gift of capital good of the j\textsuperscript{th} type at the end of the t\textsuperscript{th} period. It is concave and monotonically decreasing function of b_j(t). So as β_j(t) will rise r_j(t) will fall and vice versa.

Suppose at some choice of β_j(t) = \{β_1(t), β_2(t), ..., β_m(t)\}, we have
\[ \frac{\mu_j(t)}{K_j(t)} > \frac{\mu_k(t)}{K_k(t)}, \quad j, k \in I_0, j \neq k, \]

then

\[ \frac{r_j(t)}{K_j(t)} < \frac{r_k(t)}{K_k(t)}, \quad j, k \in I_0, j \neq k. \]

If \( \beta_j(t) \) is decreased then \( K_j(t) \) will fall and \( r_j(t) \) will rise. So assuming that \( \beta_j(t) \), \( j \neq k \), remains fixed one way to restore equality in 83b is to decrease \( \beta_j(t) \), provided \( K_j(t) \), \( j \neq k \) is either nonpositive (i.e., capital goods are not complementary in production) or negligible. However, if \( \beta_j(t) = 0 \), then it cannot be reduced further. However, \( \beta_j(t) \) can be increased. But there is a limit up to which \( \beta_j(t) \), and for that matter any \( \beta_j(t) \), \( j \in I_0 \), can be increased provided by the constraint 73. So ultimately we will reach a situation in each \( t=0, 1, \ldots, T-1, \)

\[ \frac{r_j(t)}{K_j(t)} \leqslant r(t), \quad j \in I_0 \]

and \( \beta_j(t) = 0 \) if the strict inequality holds. Translating in terms of \( \mu_j(t) \) and \( \lambda(t) \), 83b can be written as

\[ \frac{\mu_j(t)}{K_j(t)} \geq \lambda(t), \quad j \in I_0, \quad t=0, 1, \ldots, T-1 \]

and \( \beta_j(t) = 0 \) if the strict inequality hold. It is possible that \( \beta_j(t) \neq 0 \) for only one type of capital good. But the forms of the function \( K(\beta(t)) \) and \( z(b(t), t) \) make it very unlikely.

Consider 82, if \( \lambda(t) > W'(c(t)), \quad t=0, 1, \ldots, T-1 \), then it indicates
that $\sigma(t)$ should be increased. As $\sigma(t)$ is increased $c(t)$ will fall and $W'(c(t))$ will rise. But if $\sigma(t) = 1$ (i.e., $\gamma(t)=0$) then no further rise is possible. Similarly if $\lambda(t) < W'(c(t))$, $\sigma(t)$ should be decreased. But if $\sigma(t) = 0$, $\gamma(t) = 1$) no further decrease is possible.

Lastly, $z(t)$ may not be differentiable in each $b_j(t)$, $j \in I_m$. Equations 87b and 87c then have to be replaced by the following inequalities.

\begin{align*}
(87d) \quad \frac{\partial z[b(t), t]}{\partial b_j(t)} & \leq -\mu_j(t-1) \cdot \frac{\rho(T)}{\rho(t)} \leq \left( \frac{\partial z[b(t), t]}{\partial b_j(t)} \right)_+ \\
(87e) \quad \frac{\partial z[b(t), t]}{\partial b_j(t)} & \leq -\mu_j(t-1) \cdot \frac{\rho(T)}{\rho(t)} - \frac{\mu_j(t)}{\lambda(t)} (1-\delta_{0j}) \\
& \leq \left( \frac{\partial z[b(t), t]}{\partial b_j(t)} \right)_+, \quad t=1, 2, \ldots, T-1, \ j \in I_m.
\end{align*}

Similar changes have to be made in 87a.

Let us now consider the more general model. Once again we start assuming the mathematical regularity conditions and internal maximization. For simplicity, we now drop $z(t)$ as a state variable, replacing it by $z(b(t), t)$, a function of $b(t)$ and $t$, $t=0, 1, 2, \ldots, T-1$. As a result one constraint for each $t$, $t=0, 1, \ldots, T-1$ is dropped.

The Lagrangian is

\begin{align*}
\sum_{t=0}^{T-1} \rho(t) & \begin{cases} \gamma(t) \cdot z[b(t), t] \end{cases} + \rho(T) \cdot S[b(T)] + \\
\sum_{t=0}^{T-1} \sum_{j=1}^{m} \rho(t) \cdot \mu_j(t) \begin{cases} b_j(t+1) - (1-\delta_{0j}) \cdot b_j(t) + \delta_j(t) \end{cases} +
\end{align*}
We first differentiate \( L \) with respect to control variables and set the resulting expressions equal to zero. Differentiation with respect to \( \gamma(t) \), \( t=0, 1, \ldots, T-1 \) gives

\[
\rho(t) W'[c(t)] z[b(t),t] + \rho(t) \eta(t) z[b(t),t] = 0 \quad \text{for } t=0, 1, \ldots, T-1
\]

(90)

\[
\eta(t) = -W'[c(t)] \quad \text{(assuming } z[b(t),t] \neq 0) \quad \text{for } t=0, 1, \ldots, T-1.
\]

(90a)

Differentiation with respect to \( \sigma(t) \), \( t=0, 1, \ldots, T-2 \) gives

\[
\rho(t+1) \eta(t+1) R(t+1) I(t) + \rho(t) \lambda(t) I(t) = 0 \quad \text{for } t=0, 1, \ldots, T-2.
\]

(91)

Assuming \( I(t) \neq 0 \), 90a and 91 give

\[
\dot{\gamma}(t) = \frac{\rho(t+1)}{\rho(t)} W'[c(t+1)] R(t+1), \quad t=0, 1, \ldots, T-2.
\]

(92)

Differentiation with respect \( \delta_j(t) \), \( j \in I_m \), \( t=0, 1, \ldots, T-1 \) gives

\[
\rho(t) \mu_j(t) - \rho(t) \lambda(t) K_j[\delta(t)] = 0
\]

(93a)
\( (93b) \quad \mu_j(t) = \lambda(t) K_j[\beta_j(t)] \) and if \( K_j(t) \neq 0 \)

\( (93c) \quad \frac{\mu_j(t)}{K_j[\beta_j(t)]} = \lambda(t), \ t = 0, 1, \ldots, T-1. \)

Now once the control variables are chosen, \( L \) is differentiated at \( b(t) = b^O(t), t = 1, 2, \ldots, T \) and \( I(t) = I^O(t), t = 0, 1, \ldots, T-1 \) and setting resulting expressions to zero, we derive the adjoint equations for the co-state variables. Differentiation with respect to \( b_j(t), j \in I_m^*, t = 1, 2, \ldots, T-1 \), gives

\[
(94) \quad \rho(t) W'[c(t)] \gamma(t) \frac{\partial \zeta(t)}{\partial b_j(t)} + \rho(t-1) \mu_j(t-1) - \rho(t) \mu_j(t)(1-\beta_{o_j}) + \\
\rho(t) \eta(t)[1-\gamma(t)] \frac{\partial \zeta[b(t), t]}{\partial b_j(t)} = 0, \ t = 1, 2, \ldots, T-1.
\]

Differentiation with respect to \( b_j(T), j \in I_m^* \) gives

\[
(95) \quad \rho(T) \frac{\partial \zeta[b(T), t]}{\partial b_j(T)} + \rho(T-1) \mu_j(T-1) = 0, \ j \in I_m^*.
\]

Differentiation with respect to \( I(t), t = 0, 1, \ldots, T-2 \), gives

\[
(96) \quad \eta(o) + \lambda(o) \sigma(o) = 0.
\]

\[
(97) \quad \rho(t) \eta(t) - \rho(t+1) \eta(t+1) R(t+1)[1-\sigma(t)] + \rho(t) \lambda(t) \sigma(t) = 0, \ t = 1, 2, \ldots, T-2
\]

Using 91 and assuming \( I(t) \neq 0 \), 96 and 97 can be combined by

\[
(98) \quad \rho(t) \eta(t) = \rho(t+1) \eta(t+1) R(t+1), \ t = 0, 1, 2, \ldots, T-2.
\]
Using 91, 98 and 90a, we get

\[(99) \quad \lambda (t) = -\eta (t) = W'[c(t)], \quad t=0, 1, 2, ..., T-2.\]

Differentiation with respect to \(I(T-1)\) gives

\[(100) \quad \rho(T-1) \eta(T-1) + \rho(T-1) \lambda(T-1) \sigma(T-1) = 0\]

or

\[(100a) \quad \eta(T-1) = -\lambda(T-1), \quad \text{since} \quad \sigma(T-1) = 1.\]

So we get by combining 99, 100a and 90a,

\[(101) \quad \lambda(t) = -\eta(t) = W'[c(t)], \quad t=0, 1, ..., T-1.\]

Equation 101 can be used to simplify 95, to give

\[(102) \quad \rho(t-1) \mu_j(t-1) - \rho(t) \mu_j(t)(1-\beta_o) = \rho(t) \lambda(t) \frac{\delta z(b(t),t)}{\delta b_j(t)}, \quad t=0, 1, ..., T-1.\]

Equation 102 corresponds to 84a.

Economic interpretations of \(\mu_j(t)\) and \(\lambda(t)\) remain unchanged. The only new element is \(\eta(t)\) and the following equation.

\[(103) \quad \lambda(t) = \frac{\rho(t+1)}{\rho(t)} W'[c(t+1)] \cdot R(t+1), \quad t=0, 1, ..., T-2.\]

\(\eta(t)\) is the cost of foregoing one more unit of consumption at \(t\) and 101 and 103 tell us that the common rate of return on investment at \(t\), that is foregone by not investing in capital goods should not only be equal to the marginal utility of consumption at \(t\), \(t=0, 1, ..., T-1\), but should also be equal to the opportunity cost of postponing the
investment (or consumption) by one more time unit for \( t = 0, 1, \ldots, T-2 \). This is due to the extra option for the firm due to the possibility of transferring of investible fund over time. It may be noted that though nothing is consumed out of investible funds anything added to \( I(t) \) at \( t-1 \) helps to make an additional sum available for consumption unless \( \gamma(t) = 1 \), since \([1-\gamma(t)]z(t)\) is added to \( I(t) \), at \( t = 0, 1, \ldots, T-1 \).

We shall not discuss what modifications are needed if we relax the assumption of internal maximization as we have already discussed that in detail for our simplified model and the analysis would be largely similar. A few words may be added as regards some of our "economic" assumptions.

In the first place prices are assumed to be given. In a linear model if the "c" vector changes at a scalar rate \( \lambda > 0 \), (i.e., price minus average variable cost for all products changes proportionately) the output vector would remain the same and only \( z \) will change to \( \lambda z \), and \( \gamma^0 \), the dual price vector to \( \lambda \gamma^0 \). Even for our quadratic model this would not be true. However, if we make \( d \rightarrow +\infty \), then our quadratic model Q, in Chapter VI collapses to a linear programming model. However, linearity might be a hindrance rather than a help in a dynamic model of a firm. In the usual linear programming model of a firm, the firm starts with an elastically given supply of scarce resources with potential capacity limits. Since the vector "b" has the dimensionality of \( m \), at most \( m \) linearly independent vectors are needed to express \( b \) as their linear combination. If we make the usual nondegeneracy assumption then \( b \) cannot be expressed as a linear combination of less than \( m \) vectors.
that appear either as a process vector or as a slack vector. In other words the rank of any matrix formed out of b vector and any m-1 vectors of n+m vectors (n process vectors and m slack vectors) should be m. Let us assume that each process produces one product. Then the very fact of the presence of m inelastically given scarce resources, which cannot be substituted for each other in a production process and non-degeneracy forces the firm to be a multi-product one unless in the optimal "basis" m-1 vectors are slack vectors, i.e., m-1 out of a total of m resources that correspond to these slack vectors are not fully utilized. Naturally such a situation is considered unlikely. Similarly a situation where all of the m resources are fully utilized is considered accidental. Thus, the programming theory has the "realistic overtones" of a real world firm which usually produces more than one product and has excess capacity. However, when we take a dynamic view of a multi-product firm, the b vector cannot be considered given. It can be adjusted to producing any optimal output program completely spelled out at the time of planning. The nondegeneracy assumption should then be dropped. There is no reason why b vector cannot suitably be chosen so that it is an exact linear combination of less than m vectors. In fact it may be so adjusted that it is a scalar multiple of one process vector.

Changes in the b vector is considered in the context of planning by Courtillot (1962). However, he assumed that the change in the b vector will be determined by considering only the contribution of this change to revenue in the immediately next period. But if the resource is durable we should also take into account its contribution
to revenue in the subsequent periods. Only if the period in question is long enough to coincide with the payoff period the firm requires, is such a sequential solution of linear programming models possible. Similar approaches can be found in Day's (1963) dynamic land utilization model and in Day and Tinney's (1969) dynamic cobweb models, though the interpretations of the right-hand side of a linear programming model do not correspond to that given above. However, one feature of their recursive programming models is they point out the possibility that in a stationary solution of these dynamic models the multi-product firm may degenerate into a single product firm. There are of course a lot of ways in which such an outcome can be avoided. Technical interdependence of processes may be one possibility. Another possibility is to introduce probabilistic considerations. Diversification then becomes a means of reducing risk and hence increasing the expected utility of a risk-averting producer, and unutilized capacity then becomes reserve capacity to increase system reliability. Another possibility is when \( K[Ø(t)] \) function is concave. To take advantage of economies of scale, building ahead of demand may be profitable and thus creating excess capacity and a desire to explore alternative channels of production. We have introduced the "sales constraints" for that specific purpose. The firm produces more than one product because though it has the ability to expand, it is being approached by an ever-receding saturation point of growth in a particular line of production. The sales limits are growing because of "natural" growth factors, such as the growth of population. Such a view of a firm seems more appropriate for an industrial firm. A firm which, being subject to wide
price swings, diversify as a means of reducing risk and hence to
increase the utility of a risk-averting producer-owner is more likely
to be an agricultural firm. In fact, for an industrial firm, a change
in price has no definite meaning as the quality of the good is changing.

Technical changes are, however, much more important for an industrial
firm. But to a large extent such changes are similar to changes in
b vector, if we consider, as we have done, only limitational factors.
To be definite let us consider the linear programming model once again.
If a_{ji} is reduced by Δa_{ji}, then Δa_{ji}q_{i}^{o} amount of the jth resource will be
released, i.e., we have Δb_{j} = Δa_{ji}q_{i}^{o}. This will lead to an increase
in z by Δa_{ji}q_{i}^{o}y_{j}^{o}, where y_{j}^{o} is the marginal revenue productivity of the jth
resource. It may be noted that if either the jth resource was not fully
utilized before (i.e., y_{j}^{o} = 0) or if q_{i}^{o} = 0, (i.e., the ith commodity was
not produced), z would not be changed unless the reduction in a_{ji} is
large enough to bring ith process in the basis. Since technical change
also entails costs, the effects of technical changes would be similar
to capacity expansion of a particular resource or a group of such
resources.

Lastly, it may be noted that though we have not allowed for external
borrowing, such borrowings can be introduced easily if at any moment
the fund raised from outside, net of costs of borrowing to the firm, is
proportional to the firm's internal investible fund. In that case
C(t) could be greater than one.
CHAPTER VIII: BALANCED GROWTH PATH FOR AN EXPANDING FIRM

In Chapter VI, we presented a quadratic programming model, \( Q \).

If we set \( m_{i1} = 0 \) and \( m_{i2} = m_i \), we obtain the following quadratic programming problem:

\[
\max \quad \sum_{i=1}^{n} \left( p_i - u_i \right) q_i - \frac{1}{2} \frac{m_i}{d_i} q_i^2 = z
\]

\((Q)\)

Subject to \( \sum_{j=1}^{m} a_{ij} q_i \leq b_j, j \in \mathbb{I}_m, \quad 0 \leq q_i \leq d_i, i \in \mathbb{I}_n \)

\( m_i \geq 0; \quad d_i > 0, \quad i \in \mathbb{I}_n; \quad b_j > 0, \quad j \in \mathbb{I}_m \).

Since the constraint set is closed and bounded and the objective function is strictly concave in \( q \), a unique optimal \( q^0 \) will exist.

Suppose \( d \) is increased to \( \tilde{d} = \lambda d \) where \( \lambda > 1 \). If \( b \) is also increased to \( \tilde{b} = \lambda b \), then we have a balanced growth of the firm for it can be shown that \( \tilde{q}^0 = \lambda q^0 \) and \( \tilde{z} = \lambda z \).

**PROOF** Consider the problem \( \tilde{Q} \).

Since \( q^0 \) exists and Slater's condition is satisfied at \( q = 0 \), by the necessary and sufficient Kuhn-Tucker conditions, there exists an optimal vector \( \{y^0, w^0\} \), such that

\begin{align*}
(104a) & \quad p_i - u_i - \frac{m_i q_i^0}{d_i} - \sum_{j=1}^{m} y_j a_{ji} - w_i^0 \leq 0, \quad i \in \mathbb{I}_n \\
(104b) & \quad \sum_{i=1}^{n} \left( p_i q_i^0 - u_i q_i^0 - \frac{m_i}{d_i} q_i^0 + w_i^0 \right) + \sum_{j=1}^{m} \sum_{i=1}^{n} y_j a_{ji} q_j = 0 \\
(104c) & \quad b_j - \sum_{i=1}^{n} a_{ji} q_i^0 \geq 0, \quad j \in \mathbb{I}_m
\end{align*}
Now we make the necessary changes in \( \bar{Q} \), so that \( \tilde{b} \) and \( \tilde{d} \) replace the original \( b \) and \( d \). For this new problem \( \tilde{q}^o \), \( y^o \), \( w^o \) satisfy all the optimality conditions 104a-104f. Hence \( \tilde{q}^o \) must be optimal for the problem \( \bar{Q}(\tilde{b}, \tilde{d}) \). Now substituting in the objective function of \( \bar{Q}(\tilde{b}, \tilde{d}) \), \( \tilde{q}^o \) for \( q \) we get \( z = \lambda z \). Moreover, \( q^o \) is the uniquely determined optimum output vector. A balanced growth path (or the "golden age" path) has attracted considerable attention in the literature on the efficient accumulation problem for the economy. Although, originally, such a growth path was invoked to get around the index number problem (Robinson, 1956), later it has been shown that such a balanced growth path also characterizes the unique "optimal" growth path under certain conditions (Cass, 1965). However, there is no special reason why a firm, facing a problem like \( \bar{Q} \), should follow such a path even when the demand vector, \( d \), grows at a constant rate over time. To bring that out let us consider the following simplified version of the optimal capacity expansion policy model that has been presented in the last chapter.

\[
(i) \quad \max_{t=0} \sum_{\rho} \rho^t z(b(t), d_\rho^t, \bar{t}), \quad 0 < \rho < 1; \quad \frac{1}{\rho} > \lambda > 1; \quad d > 0
\]

\( (M) \)

subject to \( (ii) \) \( b_j(t+1) = b_j(t)(1 - \beta_j) \) + \( \beta_j(t) \), \( j \in I^m \), \( t=0, 1, 2, ... \),
\[ 0 < \delta_{ij} < 1; b_j(0) \text{ is given, } j \in I_m \]

\[ \beta_j(t) \geq 0, j \in I_m; t=0, 1, 2, \ldots, \]

(iii) \[ K[\beta(t)] \leq \sigma \lambda[b(t), \alpha^T], t=0, 1, 2, \ldots, \sigma > 0 \]

where \( \varnothing \) is the null vector, \( K_j \), the partial derivative of \( K[\beta(t)] \) with respect to \( \beta_j(t) \), \( K_{jj} \), the corresponding second partial, \( j \in I_m \), \( t=0, 1, 2, \ldots, m \).

It may be noted that the objective function in \( \bar{M} \) can be derived from the preference functional over time of the last chapter, if we assume

(105a) \[ W[c(t)] \text{ is positive homogeneous in } c(t), \text{ i.e., } \]

\[ W[c(t)] = \alpha c(t), \text{ where } \alpha > 0; \]

(105b) \[ c(t) = \gamma z(t), \gamma > 0; \]

(105c) \[ p(t) = \rho; \]

(105d) \[ T = +\infty. \]

When the above holds, the maximization of the objective function in \( \bar{M} \) will lead to a maximization of the previously considered preference functional. The maximized value would be the same except for a constant factor \( \alpha \gamma > 0 \), which, by an appropriate choice of the unit of the utility scale, can be made unity. Moreover, we may now allow for \( \gamma + \sigma > 1 \) or \( \gamma + \sigma < 1 \). The former possibility will exist if external borrowings (net of costs) proportional to internal financing is
possible, while the latter possibility may arise if the firm has to pay a fixed charge proportionate to $z(t)$, on account of taxes, etc.

The Lagrangian for the problem $M$ is now written as

$$L = \sum_{t=0}^{\infty} \rho^t Z[b(t), d^t]$$

$$+ \sum_{t=0}^{\infty} \sum_{j=1}^{m} \rho^t \mu_j(t) \{ b_j(t+1) - b_j(t) (1-\beta_{o_j}) - \beta_j(t) \}$$

$$+ \sum_{t=0}^{\infty} \rho^t \eta(t) \{ \sigma Z[b(t), d^t] - K[\beta(t)] \}.$$ 

Since $\sigma$, the saving ratio is fixed, the control variables are $\beta_j(t), j \in I_m, t=0, 1, 2, \ldots$.

We assume that all mathematical regularity conditions are satisfied and internal maximization with respect to each $\beta_j(t), j \in I_m, t=0, 1, \ldots$. Differentiating $L$ with respect to $\beta_j(t)$ and setting the resulting expression to zero, we get

$$\mu_j(t) - \eta(t) K_j [\beta(t)] = 0, j \in I_m, t=0, 1, \ldots.$$  

Since $K_j [\beta(t)] > 0$, for any $\beta_j(t) > 0$, and since we are assuming internal maximization with respect to each $\beta_j(t), j \in I_m$, we divide by $K_j [\beta(t)]$ to get

$$\frac{\mu_j(t)}{K_j [\beta(t)]} = \eta(t), j \in I_m, t=0, 1, \ldots.$$  

Now $M$ iii is an inequality. So we have the following conditions,

$$\sigma Z[b(t), d^t] - K[\beta(t)] \geq 0, t=0, 1, \ldots$$  

with strict equality for those $t$, for which $\eta^o(t) > 0$. 
From 106b it appears that if $\eta_0(t) = 0$, then $\mu_j(t) = 0$, $j \in I_m$, for a given $t$, i.e., the firm is not expanding, it is rather decaying unless $\beta_{o_j} = 0$, for each $j \in I_m$. Differentiating with respect to state variables, $b_j(t)$, $j \in I_m$, $t=1, 2, \ldots$, and setting the resulting expressions to zero, we get,

\begin{equation}
\rho^t \frac{\partial z[b(t), d\lambda]}{\partial b_j(t)} + \rho^{t-1} \mu_j(t-1) - \rho^t \mu_j(t)(1-\beta_{o_j}) = 0
\end{equation}

\[ j \in I_m, \ t=1, 2, \ldots \]

The balanced growth path, on the other hand, is characterized by (assuming that $b_j$ does not correspond to a corner point, $j \in I_m$)

\begin{equation}
\frac{\partial z[b(t), d\lambda]}{\partial b_j(t)} = y_j^0, \ j \in I_m.
\end{equation}

Using 108, the difference equation can be written in the following form

\begin{equation}
\rho y_j^0 = \mu_j(t)(1-\beta_{o_j}) - \mu_j(t-1), \ j \in I_m, \ t=1, 2, \ldots.
\end{equation}

The difference equation 109 is autonomous as it does not involve "\(t\)" explicitly. It can be solved in the following way

\begin{equation}
\mu_j(t) = \mu_j(t+1)(1-\beta_{o_j}) - \rho y_j^0
= \rho(1-\beta_{o_j})(\mu_j(t+2)(1-\beta_{o_j}) - \rho y_j^0) - \rho y_j^0.
\end{equation}

Continuing in this way, $\mu_j(t)$ can be expressed as a convergent geometric series except for the last term which can be neglected
since \(0 < \rho < 1\), and \(0 < \beta_{oj} < 1\).

\[
\mu_j(t) = -\rho y_j^o - \rho^2 (1-\beta_{oj}) y_j^o - \rho^3 (1-\beta_{oj})^2 y_j^o - \ldots
\]

or

\[
\mu_j(t) = \frac{\rho y_j^o}{\rho (1-\beta_{oj}) - 1}.
\]

It can easily be seen that \(111a\) does not involve \(t\), and the stationary solution of \(\mu_j(t)\), for each \(t=0, 1, 2, \ldots\), is given by the same expression, i.e.,

\[
\mu_j^* = \frac{\rho y_j^o}{\rho (1-\beta_{oj}) - 1}.
\]

Equation 111 can be given the same economic interpretation that \(\mu_j(t)\) was given in the last chapter. Suppose the firm gets a gift of one more unit of capital good at the end of the \(t\)th period. This would generate a flow of productive services of the \(j\)th type of resource, of one unit in the \(t+1\)th period, of \((1-\beta_{oj})\) unit in the \(t+2\)th period and so on.

Now on a balanced growth path the marginal revenue productivity of the \(j\)th capital good is the same, namely, \(y_j^o\). So the discounted sum of the contribution of one more unit of the \(j\)th capital good at the end of the \(t\)th period is given by

\[
\rho^{t+1} y_j^o + \rho^{t+2} (1-\beta_{oj}) y_j^o + \rho^{t+3} (1-\beta_{oj})^2 y_j^o + \ldots
\]

and since one unit of revenue at \(t=0\) is equivalent to \(1/\rho^t\) unit in the \(t\)th period the current evaluation of the gift is

\[
\rho y_j^o + \rho^2 (1-\beta_{oj}) y_j^o + \ldots.
\]

\(\mu_j(t)\) is therefore the opportunity cost of foregoing such a gift. If we make \(\beta_{oj} = 0\), then we get \(\mu_j(t) = \mu_j^* = \frac{\rho}{\rho - 1} y_j^o\). Hence, the current
evaluation of such a gift in the \( t+1 \)^{th} period from which it can be used

\[
\frac{y_j^{\circ}}{\rho-1} \text{ where } \rho-1 \text{ is the rate of discount.}
\]

Using 111a and 106 b we get,

\[
(113) \quad \frac{\rho y_j^{\circ}}{K_j[\beta(t)].[\beta(1-\beta o_j)-1]} = \eta(t), \quad j \in I_m, \quad t=0, 1, 2, \ldots .
\]

If \( \beta o_j = \beta o \) for all \( j \in I_m \), then 113 shows that on a balanced growth path the marginal cost of expansion of the \( j \)^{th} resource is proportional to its marginal revenue productivity. Moreover, assuming \( K_j[\beta(t)] \neq 0 \) and \( \eta(t) \neq 0 \), (i.e., the firm is expanding and any capital good is not free) we can see that \( y_j^{\circ} \) cannot be zero for any resource. Hence, all resources must be fully utilized on a balanced growth path.

Certain features of this balanced growth path appear only when we consider a linear programming model, i.e., we let \( d \) tend to infinity in \( Q \).

The maximization problem before the firm is now,

\[
\max \sum_{i=1}^{n} (\rho_i - u_i)q_i = \sum_{i=1}^{n} c_iq_i = z
\]

(LP)

subject to \( \sum_{j=1}^{m} a_{ji}q_i \leq b_j, \quad j \in I_m, \quad q_i \geq 0, \quad i \in I_n \).

In the matrix-vector notation we used before, LP can be written as

\[
\max c'q = z
\]

subject to \( Aq \leq b \)

\[
q \geq 0
\]
We assume that:

1. \( a_{ji} \) is nonnegative, \( j \in I_m, i \in I_n \) (i.e., \( A \geq 0 \)) and is positive for some \( i \in I_n \) for any given \( j, j \in I_m \),

2. \( b_j > 0, j \in I_m \), i.e., \( b > 0 \).

We have shown that if 1 holds and \( b \geq 0 \) then an optimal \( q^0 \) exists (see Chapter V). This is because both LP and its dual are then feasible. But if \( b_j = 0 \) for some \( j \), then the only feasible point in LP might be \( q = \phi \). In fact if \( a_{ji} \) is positive for some \( j \) for each \( i, i \in I_n \) then the only feasible point in LP is \( q = \phi \) if \( b_j = 0 \).

But if \( a_{ji} = 0 \) for some \( j \) for each \( i \in I_n \) then the \( j^{\text{th}} \) resource is not required by any output program and the constraint associated with it can be dropped, since it is redundant. We now, therefore, assume \( b > 0 \), hoping that this would make some \( q \neq \phi \) optimal for LP.

Let \( b \) be changed to \( b = \lambda b \) in LP, \( \lambda > 1 \). Then it can be shown that \( q^0 \) will increase to \( \tilde{q}^0 = \lambda q^0 \) and \( z \) to \( \tilde{z} = \lambda z \). (In fact this will hold for any \( \lambda > 0 \).)

**Proof** The dual to LP before the change in "b" vector was:

\[
\min \sum_j b_j y_j
\]

\[
\sum_j y_j a_{ji} \geq c_i, \ i \in I_n
\]

As \( b \) is changed to \( \tilde{b} \), \( \tilde{q}^0 = \lambda q^0 \) is feasible for LP (\( \tilde{b} \)) and \( y^0 \) is feasible for D (\( \tilde{b} \)). Furthermore, \( \tilde{b}'y^0 = (\lambda b)'y^0 = \lambda b'y^0 = \lambda z = \lambda c'q^0 = c'\tilde{q}^0 \). So \( \tilde{q}^0 \) must be optimal for LP (\( \tilde{b} \)). Note that if \( q^0 \) were the unique optimum in LP(\( b \)) then \( \tilde{q}^0 \) would be the unique optimum in LP(\( \tilde{b} \)).
We assume that $K[\beta(t)]$ is positive homogeneous in $\beta_1(t), \beta_2(t), \ldots, \beta_m(t)$. Now on a balanced growth path,

\begin{equation}
\begin{aligned}
\beta_j(t+1) &= \lambda \beta_j(t) = \beta_j(t)(1 - \beta_{oj}) + \beta_j(t), \quad j \in I_m \\
t &= 0, 1, \ldots
\end{aligned}
\end{equation}

where $\lambda > 1$; $0 < \beta_{oj} < 1$, $j \in I_m$.

Now (114) holds only if $\beta_{oj} \neq 0$, $j \in I_m$, i.e., the firm is an expanding firm. We have seen that for such a firm $\sigma_j(t) \neq 0$, $j \in I_m$ and $\eta(t) > 0$. $t=0, 1, \ldots$. So (114) will hold with equality.

In the linear programming situation, when $K(\beta(t))$ is positive homogenous and $\beta_{oj} = \beta_0$ for each $j \in I_m$, it can be shown that once the equality in (114) is forced the objective function plays no further role and the rate of growth of the firm is uniquely determined for any given initial vector $b(o)$.

**Proof**: From (114) and setting $\beta_{oj} = \beta_0$, $j \in I_m$

\[ \beta_j(t) = (\lambda - 1 + \beta_0) b_j(t), \quad j \in I_m, \quad t = 0, 1, \ldots \]

\[ K[\beta(t)] = K[\beta_1(t), \beta_2(t), \ldots, \beta_m(t)] \]

\[ = K[(\lambda - 1 + \beta_0) b_1(t), (\lambda - 1 + \beta_0) b_2(t), \ldots, (\lambda - 1 + \beta_0) b_m(t)] \]

\[ = (\lambda - 1 + \beta_0) K[b(t)] = (\lambda - 1 + \beta_0) K[\lambda^t b(o)] \]

\[ = \lambda^t (\lambda - 1 + \beta_0) K[b(o)] \]

Similarly, $z[b(t)] = \lambda^t z[b(o)]$.

Hence from (105a) we obtain that if $\eta(t) \neq 0$,

$\lambda^t z[b(o)] = \lambda^t K[\lambda^t b(o)]$.

Hence,

$\lambda = \frac{z[b(o)]}{K[b(o)]} + \beta_0$.
So once \( b(o) \) is given, \( z[b(o)] \) and hence \( \lambda \) are uniquely determined. We now establish a simple lemma:

**LEMMA 1** Let \( b'(o) = \alpha b^2(o) \), where \( \alpha > 0 \), is a scalar:

and let \( \lambda[b'(o)] \) and \( \lambda[b^2(o)] \) be the associated growth rates. Then \( \lambda[b'(o)] = \lambda[b^2(o)] \).

**PROOF**

\[
\lambda[b'(o)] = \frac{\partial z[b'(o)]}{\partial K[b'^(o)]} + 1 - \beta_o
\]

\[
= \frac{\partial z[\alpha b^2(o)]}{\partial K[\alpha b^2(o)]} + 1 - \beta_o
\]

\[
= \frac{\partial z[b^2(o)]}{\partial K[b^2(o)]} + 1 - \beta_o = \lambda[b^2(o)]
\]

The above lemma shows that \( \lambda \) depends only on the proportionings of \( b_j(o) \), \( j \in I_m \). Now if the firm is given a choice between \( b'(o) \) and \( b^2(o) \) it would always choose \( b'(o) \), if \( \alpha > 1 \) and \( b^2(o) \), if \( \alpha < 1 \) and would be indifferent only if \( \alpha = 1 \). This can easily be seen from the objective function in \( M_i \). However, the choice of \( b(o) \), at the initial period is not costless. In \( M_i \), this cost does not appear since the firm starts with \( b(o) \) (completely amortized). Moreover this cost changes by a scalar rate as \( b(o) \) increases by a scalar rate and \( z[b(o)] \) also increases by the same scalar rate.

So far we have taken for granted that the maximum in \( M_i \) exists. In our quadratic model \( Q \), it appears that since \( d(t) \) is growing at a rate less than \( 1/\rho \), the uniform dampening factor provided by \( \rho \) would be sufficient to make the infinite series in \( M_i \) converge as \( z(t) \) grows.
over time. To be definite let us consider a balanced growth path. Then
\[ z(t) = z(0)\lambda^t \quad \text{and} \quad \sum_{t=0}^{\infty} \rho^t z(t) = \frac{z(0)}{1-\rho}. \]

Such a way of escape does not exist for our linear model since \( d = +\infty \). Hence as \( z(t) \) grows over time (on a balanced or unbalanced growth path) the infinite series in \( M_i \) may not be convergent. We may, however, use the "overtaking" principle (Galé, 1967) for a comparison of infinite programs. Let \( Z^1 \) be a feasible program generating \( z^1(0), z^2(1), \ldots \). Let \( Z^2 \) be another such program which generates \( z^2(0), z^2(1), \ldots \). Now \( Q' \) can be considered "inefficient" in comparison to \( Z^1 \) if there exists a value \( t^* \), such that the following inequality holds:
\[
\sum_{\tau=0}^{t} \rho^\tau z^1(\tau) \geq \sum_{\tau=0}^{t} \rho^\tau z^2(\tau),
\]
for all values of \( t \geq t^* \), i.e., \( t = t^*, t^* + 1, t^* + 2, \ldots \). For balanced growth paths the above principle reduces to a simple rule, namely if \( \lambda_1 > \lambda_2 \), then the growth path at the rate \( \lambda_1 \) is "efficient" in comparison with the growth path at the rate \( \lambda_2 \), whatever may be the initial value of \( z(0) \). Since \( \lambda_1 > \lambda_2 > 1 \), the higher order terms in \( Z^1 \) will dominate the relation 115.

We now reformulate our linear model. We assume that it is required by law that the firm has to grow on a balanced growth (or golden age) path. But the firm has the option, at \( t=0 \), of choosing any initial capital outfit \( b(0) \) subject to a budget constraint. We shall later remove the budget constraint to see how our results are affected thereby.
The problem before the firm is

\[
\max \lambda [b_1(o), b_2(o), \ldots, b_m(o)] = \lambda^* \tag{G}
\]

where \( \lambda [b(o)] = \frac{\sigma z[b(o)]}{K[b(o) + 1]} \)

subject to

\[
K[b(o)] \leq I
\]

\[
b_j \geq 0, \quad j = 1, 2, \ldots, m.
\]

For simplicity we now drop the arguments of the functions \( z, K, \lambda \). We shall write \( y_j \) for \( \frac{\partial z[b(o)]}{\partial b_j(o)} \) and \( K_j \) for \( \frac{\partial K[b(o)]}{\partial b_j(o)} \), \( j \in I_m \).

The Lagrangian is

\[
L = \frac{\sigma z}{K} + 1 - \lambda^* o + \eta(I - K) \tag{116}
\]

Assuming internal maximization with respect to \( b_j \) so that \( b_j^* > 0 \) and differentiability of \( z \) and \( K \) with respect to \( b_j \), \( j \in I_m \) and other regularity conditions as required,

\[
\frac{\partial L}{\partial b_j} = \frac{\sigma y_j K - K_j \sigma z}{K^2} - \eta K_j = 0, \quad j \in I_m. \tag{117}
\]

Moreover we have

\[
I - K \geq 0 \tag{118}
\]

with equality for \( \eta_i > 0 \) and \( \eta_j \geq 0 \).

From 117 we get multiplying throughout by \( \frac{K}{\sigma K_j} \),

\[
\frac{y_j}{K_j} = \frac{z}{K} + \frac{\eta}{\sigma} K, \quad j \in I_m. \tag{119}
\]

Two cases need to be considered:
(119a) \( \eta > 0 \). Then \( I = K \) and \( 119 \) become \( \frac{y_j}{K_j} = \frac{z}{I} + \frac{\eta}{\sigma} I, \ j \in I_m \)

or

(119b) \( \eta = 0 \) and \( \frac{y_j}{K_j} = \frac{z}{K}, \ j \in I_m \).

Now the choice of \( b(o) \) will determine \( b(t) \) for any subsequent period on a balanced growth path. This has the following implication:

(120) \( K_j[\beta(t)] = K_j, \ j \in I_m, \ t=0, 1, 2, \ldots \).

PROOF \( \delta_j(t) = \lambda^t (\lambda-1+\delta_j o) b_j(o) \)

Hence, \( \frac{\partial K[\beta(t)]}{\partial \delta_j(t)} = \lambda^t (\lambda-1+\delta_j o) \cdot \frac{\partial K[b(o)]}{\partial b_j(o)} = K_j, \ j \in I_m, \ t=0, 1, 2, \ldots \).

Equation 120 shows that since \( K[\beta(t)] \) is positive homogeneous in \( \delta_j(t), \ j \in I_m, \) at any time \( t \) the marginal cost of expansion would be equal to the average cost of expansion of capacity of the \( j \)th good so long as \( \delta_j(t) \) is growing by the same rate \( \lambda \) over time for \( t=0, 1, \ldots, j \in I_m \). That \( \delta_j(t) \) would grow by \( \lambda \) can be seen from the following relation:

(121) \( \delta_j(t+1) = \lambda^{t+1} (\lambda-1+\delta_j o) b_j(o) = \lambda \delta_j(t), \ j \in I_m \)

\( t=0, 1, \ldots \).

It may be noted that in deriving 119 from 117, we multiplied throughout by \( K/K_j \). If \( b_j > 0 \) for all \( j \in I_m, K_j > 0 \) (since it is positive homogeneous) and since \( \sigma > 0 \), such a multiplication is permissible. Moreover, \( K > 0 \) for any nontrivial situation.

Let us now consider the situation when there is no budget constraint.
It can easily be seen that in that case 119b holds. This can be checked either directly (i.e., dropping the constraint in G) or by letting I tend to infinite so that I-K>0 will hold and hence \( \eta \) will be zero. Now the relation 119b gives \( m-1 \) independent equations.

**PROOF** Suppose \( \frac{y_j}{K_j} = \frac{z}{K} \), for \( j=1, 2, \ldots, m-1 \) we have to show that \( \frac{y_m}{K_m} = \frac{z}{K} \).

Since \( b_j > 0 \), \( j \in I_m \), \( \frac{b_j y_j}{b_m K_m} = \frac{z}{K} \), \( j=1, 2, \ldots, m-1 \). Hence

\[
\frac{1}{m-1} \sum_{j=1}^{m} \frac{b_j y_j}{b_j K_j} = \frac{z}{K} \quad \text{But } K = \sum_{j=1}^{m} b_j K_j, \quad \text{because of positive homogeneity of } K, \\
\frac{1}{m-1} \sum_{j} b_j K_j
\]

for which Eulers' theorem will hold. Moreover, by the assumption of the differentiability of \( z \) with respect to \( b_j, j \in I_m \), \( y_j \) is the optimum value of the dual variable that corresponds to the \( j^{th} \) resource constraint, \( j \in I_m \) and \( z(o) = \sum_{j=1}^{m} b_j(o)y_j \).

So if there is no budget constraint, there is no unique solution for \( b_j, j=1, 2, \ldots, m \) though the optimum \( \lambda^* \) is uniquely determined, as suggested by Lemma 1. In other words \( b^*(o) \) will be determined unique only up to a multiplicative positive constant and for each such \( b^*(o) \), \( \lambda^* \) would be the same, i.e.,

\[
(122) \quad \lambda^* = \frac{\sigma [b^*(o)]}{K [b^*(o)]} + 1 - \beta_o.
\]

Another thing that is to be noted is 119b does not involve \( \sigma \), though \( \lambda^* \) will be an increasing function of \( \sigma \) (and a decreasing function of \( \beta_o \)), as can be seen from 122.
Let us now consider the case when the budget constraint exists.

Equation 119 can also be written as

\[
\frac{y_1}{K_1} = \frac{y_2}{K_2} = \ldots = \frac{y_m}{K_m} = \frac{z}{K} = \frac{\gamma K}{\sigma}.
\]

Equation 123 follows from the fact that \( \frac{y_j}{K_j} \) is equal to a given constant for each \( j \in I_m \) and hence must be equal to \( z/K \) since both \( z \) and \( K \) are positive homogeneous in \( b_j^* \), \( j \in I_m \), as it has been shown before.

If the budget is not fully utilized, we have \( \gamma = 0 \) and 119b holds once again but with an additional condition \( K < I \). If the budget is fully utilized we have \( K = I \). Even then 119b holds and the indeterminacy disappears. So the optimal conditions can be written in a form that does not involve \( \sigma \), namely

\[
\begin{align*}
\text{(124a)} \quad & \frac{y_1}{K_1} = \frac{y_2}{K_2} = \ldots = \frac{y_m}{K_m} = \frac{z}{K} \\
\text{(124b)} \quad & I = K.
\end{align*}
\]

If \( \gamma > 0 \), \( \gamma \) can be given any of the following interpretations.

1. Since \( \frac{z}{K} = \frac{\gamma K}{\sigma} \), we have \( \sigma z = \gamma K \), but \( \sigma z[b(o)] = K[\delta(o)] \).

Hence \( \gamma = \frac{K[\delta(o)]}{K[b(o)]} = \lambda^{*} - 1 + \delta_o \).

2. \( \gamma = \frac{\partial \lambda^{*}}{\partial I} \), i.e., the marginal contribution of one more unit of investment fund to the optimal rate of growth, other things in the system remaining the same.

Proof: \( \frac{\partial \lambda^{*}}{\partial I} = \sum_{j=1}^{m} \frac{\partial \lambda^{*}[b^{*}(o_j)]}{\partial b_j^*(o)} \cdot \frac{\partial b_j^*(o)}{\partial I} \cdot \frac{\partial \lambda^{*}(o)}{\partial I} \).
Moreover, since $\eta^* > 0$, $I = K$ and

$$
1 = \sum_{j=0}^{m} \frac{\partial K[b^*(o)]}{\partial b_j^*(o)} \cdot \frac{\partial b_j^*(o)}{\partial I}
$$

Hence

$$
\eta_j^* = \eta^* \sum_{j=0}^{m} \frac{\partial K[b^*(o)]}{\partial b_j^*(o)} \cdot \frac{\partial b_j^*(o)}{\partial I}
$$

Hence

$$
\frac{\partial \eta_j^*}{\partial I} = \eta^* + \sum_{j=0}^{m} \left( \frac{\partial b_j^*(o)}{\partial b_j^*(o)} - \eta^* \frac{\partial K[b^*(o)]}{\partial b_j^*(o)} \right) \cdot \frac{\partial b_j^*(o)}{\partial I} = \eta^*.
$$

Since the bracketed expression is $\sum_{j=1}^{m} \frac{\partial L}{\partial b_j^*(o)}$ evaluated at $b^*(o)$ and $\eta^*$.

Now the right-hand side of 119 is positive, $y_j$ must be positive for any $j$ for which $b_j$ (and hence $K_j$) is positive. By the assumption of differentiability of $z(b)$, $y_j = y_j^o = y_j^o(t)$ in $G$ for $t=0, 1, \ldots$, since on a balanced growth path $y_j^o(t)$ will remain the same for any value of $t$. But in the linear program LP, $q_j^o(t)$ may not be uniquely determined, as it would be in $Q$. However, Goldman and Tucker (1956) have shown that if both LP and D are feasible (as they are) then for any $j$, $j \in I_m$, the following alternatives will hold:

1. either $\sum_{i=1}^{n} a_{ij} q_i^o < b_j$ for some optimal $q^o$ and $y_j^o = 0$ for every optimal $y^o$;

---

1With appropriate changes in notation, this is Corollary 2B on page 62, Goldman and Tucker (1956).
2. or \( \sum_{i=1}^{n} a_{ji} q_i^{o} = b_j \) for every optimal \( q^{o} \) and \( y_j^{o} > 0 \) for some optimal \( y^{o}, j \in I_m \).

By the assumption of differentiability of \( z \) with respect to \( b_j, j \in I_m, \) \( y_j^{o} \) will be uniquely determined in \( G \). So the above result shows since \( y_j^{o} > 0, j \in I_m, \) all resources \( j \in I_m \) would be fully utilized for any choice of \( q^{o} \). Furthermore, on a balanced growth path the same goods will be produced for each \( t=0, 1, \ldots \). However, at \( t=0, q^{o}(0) \) may not be uniquely determined. But we can tell what goods will not be produced and what goods may be produced at a positive level by the following rule:

\[(125a) \text{ If } p_i - u_i < \sum_{j=1}^{m} a_{ji} y_j^{o} \text{ for some } i, \text{ then that good will not be produced.}\]

\[(125b) \text{ Only those goods may be produced for some choice of } q^{o}(0) \text{ for which } p_i - u_i = \sum_{j=1}^{m} a_{ji} y_j^{o}.\]

From 120, \( K[b(t)] = K[b^{*}(0)], j \in I_m, t=0, 1, \ldots \). Since \( y_j^{o}(t) = y_j \), the following relation will hold for each \( t, t=0, 1, \ldots \) on a balanced growth path once \( b^{*}(0) \) is optimally determined.

\[(126) \frac{y_j^{o}(t)}{K_j[b(t)]} = z[b^{*}(0)] K[b^{*}(0)], j \in I_m, t=0, 1, 2, \ldots\]

Moreover \( K[b(t)] = (\lambda^* - 1 + \beta^*_o) K[b^{*}(0)] \) and \( sz[b^{*}(0)] = K[b(t)] \).

Hence
Hence, on a balanced growth path the marginal revenue productivity of the $j^{th}$ resource would be proportional to the marginal (and average) cost of expansion of the capacity of the $j^{th}$ resource. Equation 127 therefore gives the firm's "expansion path" over time.

From 125b and 126 we deduce that for any good, $i, i \in I_n$, that is produced at a positive level, we must have

\[
\frac{p_i - u_i}{m} = \frac{1}{\sum \alpha_j K_j [\beta(t)]} \left( \frac{\lambda^{-1+\beta_o}}{\sigma} \right) \frac{z[b(t)]}{K[\beta(t)]},
\]

\[j \in I_m, t=0, 1, \ldots\]

Once again it can be shown that

\[
z[b^*(o)] = \sum_{i=1}^{n} q_i^o (p_i - u_i) = \sum_{i \in I_n} q_i^o (p_i - u_i)
\]

\[q_i^o > 0\]

and

\[
K[b^*(o)] = \sum_{j=1}^{n} b_j^*(o) K_j [b^*(o)] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} q_i^o K_j [b^*(o)]
\]
So far we have assumed that \( \frac{\partial z}{\partial b_j^*(o)} = y_j^0, j \in I_m \). But \( z \) may not be differentiable with respect to \( b_j(o) \) at a corner point (i.e., overdetermined point) and the above partial derivative with respect to \( b_j \) may not exist, \( j \in I_m \).

In that case we have the more general rule, for an optimal choice of \( b_j^*(o), j \in I_m \)

\[
(130) \quad \left( \frac{\partial z[b^*(o)]}{\partial b_j^*(o)} \right)_{+} + K_j[b^*(o)] \leq \frac{z[b^*(o)]}{K[b^*(o)]} \leq \left( \frac{\partial z[b^*(o)]}{\partial b_j^*(o)} \right)_{-} + K_j[b^*(o)], I - K \geq 0, \eta \geq 0, \eta(I-K) = 0.
\]

Now \( y_j^0 \) is not uniquely determined. But by Goldman and Tucker's result we know that if \( y_j^0 > 0, j \in I_m \) for some optimal \( y_j^0 \), then the \( j \)th resource will be fully utilized. Moreover it can be shown that \( y_j^0 > 0 \) for some choice of \( y_j^0 \) if and only if \( \left( \frac{\partial z[b^*(o)]}{\partial b_j^*(o)} \right)_{-} \) is positive for a given \( j \in I_m \).

PROOF As regards the "only if" part we note that

\[
y_j^0 \leq \left( \frac{\partial z[b^*(o)]}{\partial b_j^*(o)} \right)_{-}
\]

Hence if for any \( y_j^0, y_j^0 > 0 \), then

\[
\left( \frac{\partial z[b^*(o)]}{\partial b_j^*(o)} \right)_{-} \text{ must be positive.}
\]
As regards the "if" part we use another result of Goldman and Tucker, which states that if an optimal $q^0$ exists then there exist some $q^0$ and $y^0$ such that

$$(y^0) + (b - Aq) > 0. \text{ } 1$$

Now let us assume that $y^0_j = 0$ for every optimal $y^0$. Then there exists some $q^0$ for which

$$b_j - \sum_{i=1}^{n} a_{ji}q^0_i > 0.$$  

Hence if we reduce $b^j$ and produce $q^0$ as before, $z(q^0)$ will not change. Hence

$$\frac{\partial z(q^0)}{\partial b^j} = 0.$$ 

must be zero. Hence if the left-hand derivative is positive, $y^0_j$ must be positive for some $y^0$.

We now illustrate the maximal balanced growth path of a firm with two processes, each process producing one product and utilizing two resources. The intensity of each process is measured by the amount of output produced. The firm has a given technology. The technological information about the firm is summarized by the following $2 \times 2$ matrix

$$A = (a_1, a_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

---

1This is lemma 4 on page 59, Goldman and Tucker (1956).
where $a_{ji}$ indicates the technically determined utilization rate (the amount of service-flow consumed) of the $j^{th}$ resource by the $i^{th}$ process at the unit level of intensity, i.e., for producing one unit of the $i^{th}$ good, $i=1, 2; j=1, 2$.

$c_i = p_i - u_i$, $i=1, 2$, is the net return per unit of the $i^{th}$ good, or the price of the $i^{th}$ good, $p_i$, net of variable costs per unit of the $i^{th}$ good, $u_i$, $i=1, 2$. We assume that $c_i$ is given to the firm, $i=1, 2$.

We now introduce two other "processes" by the following slack vectors,

$$a_3 = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; a_4 = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

They correspond to keeping unutilized one unit of the $j^{th}$ resource, $j=1, 2$. We set $c_3 = c_4 = 0$, i.e., such processes are costless.

The maximization problem before the firm in $t^{th}$ period is

$$\max c_1 q_1(t) + c_2 q_2(t) = z(t)$$  

subject to

$$a_{11} q_1(t) + a_{12} q_2(t) + q_3(t) = b_1(t)$$

$$a_{21} q_1(t) + a_{22} q_2(t) + q_4(t) = b_2(t)$$

$$q_i(t) \geq 0, i=1, 2, 3, 4,$$

where $b_j(t)$ is the total availability of the services of the $j^{th}$ resource in $t^{th}$ period and $q_3(t)$, $q_4(t)$ are the intensities of the processes $a_3$ and $a_4$ in $t^{th}$ period; $q_i(t)$ is the amount of the $i^{th}$ output produced, $i=1, 2$; $t=0, 1, 2, \ldots$.

We make the following assumptions:
al. \( a_{ij} > 0, \ i = 1, 2; \ j = 1, 2, \)

a2. \( c_i > 0, \ i = 1, 2, \) is positive, i.e., \( p_1 > u_i, \ i = 1, 2, \)

a3. \( |A| \neq 0, \) i.e., \( A \) is of full rank.

Assumptions a1 implies that for any feasible solution for which not both \( q_1(t) \) and \( q_2(t) \) are zero, we require that \( b_1(t) \) and \( b_2(t) \) are positive. Moreover, the resources are nonspecific, i.e., they can be used for either in the production of \( q_1(t) \) or \( q_2(t) \) and both the resources are required for producing any product in positive amounts. The production functions (in the conditional form) of the firm are:

\[
q_1(t) = \min \left( \frac{b_1(t) - a_{12}q_2(t)}{a_{11}}, \frac{b_2(t) - a_{22}q_2(t)}{a_{21}} \right)
\]

\[
q_2(t) = \min \left( \frac{b_1(t) - a_{11}q_1(t)}{a_{12}}, \frac{b_2(t) - a_{21}q_1(t)}{a_{22}} \right)
\]

(131)

Since by a3, \( |A| \neq 0, \) we can make \( |A| \) positive without any loss of generality. For instance if \( |A| \) is negative to start with we switch the numbering of the first two processes and thus making one "inversion" in the associated technological matrix.

Suppose \( b_1(t) > 0 \) and \( b_2(t) > 0 \) are given. Then an optimal \( q^0 \) will exist as we have seen before. So a basis optimal solution will also exist. Since there are four vectors, a basic can be chosen in six possible ways. Let us number them and state the conditions of their feasibility and optimality.

\[ B_1 = A = (a_1, a_2); \ |A| > 0 \]

is feasible if
with at least one inequality strict.

It is also optimal if

\[
\frac{a_{22}}{a_{12}} \geq \frac{b_2(t)}{b_1(t)} \geq \frac{a_{21}}{a_{11}}
\]

with at least one inequality strict.

\[
B_2 = (a_1, a_3) = (a_1, c_1); \ |B_2| = -a_{21} < 0
\]
is feasible if

\[
\frac{a_{22}}{a_{12}} > \frac{a_{21}}{a_{11}} \geq \frac{b_2(t)}{b_1(t)}
\]

and also optimal if

\[
\frac{a_{22}}{a_{12}} \geq \frac{c_2}{c_1}
\]

\[
B_3 = (a_1, a_4) = (a_1, c_2); \ |B_3| = a_{11} > 0
\]
is feasible if

\[
\frac{b_2(t)}{b_1(t)} \geq \frac{a_{21}}{a_{11}}
\]

and also optimal if

\[
\frac{a_{22}}{a_{21}} > \frac{a_{12}}{a_{11}} \geq \frac{c_2}{c_1}
\]
\[ B_4 = (a_2, a_3) = (a_2, e_1); \quad |B_4| = -a_{22} < 0 \]

is feasible if

\[(135a) \quad \frac{a_{22}}{a_{12}} \geq \frac{b_2(t)}{b_1(t)} \]

and also optimal if

\[(135b) \quad \frac{c_2}{c_1} \geq \frac{a_{22}}{a_{21}} > \frac{a_{12}}{a_{11}} \]

\[ B_5 = (a_2, a_4) = (a_2, e_2); \quad |B_5| = a_{12} > 0 \]

is feasible if

\[(136a) \quad \frac{b_2(t)}{b_1(t)} \geq \frac{a_{22}}{a_{12}} > \frac{a_{21}}{a_{11}} \]

and also optimal if

\[(136b) \quad \frac{c_2}{c_1} \geq \frac{a_{12}}{a_{11}} \]

\[ B_6 = (a_3, a_4) = I \]

is feasible but never optimal.

**PROOF** First of all let us indicate why \( B_6 \) cannot be an optimal basis. By simplex criterion, \( \Delta_j = c^T B^{-1} a_j \); \( c_j = 0, j=1, 2, 3, 4 \), for an optimal basis, where \( c_B \) is the truncated \( c = \{ c_1, c_2, c_3, c_4 \} \) vector that corresponds to the basis variables and \( a_j \), any vector, \( j=1, 2, 3, 4 \). Since \( \Delta_j = 0 \) for the basis vector, we
only check whether \( \Delta_j \geq 0 \) for the nonbasis vectors. For
\[ B_6, \quad \Delta_1 = -c_1 \quad \text{and} \quad \Delta_2 = -c_2, \]
where both \( a_1 \) and \( a_2 \) are not in the basis. Hence the optimality criterion is
not satisfied.

We now prove 132a and 132b. If \( B_1 \) is feasible then \( q(t) = B_1^{-1} b(t) \)
\( \succeq 0 \) where \( q(t) = q_1(t), q_2(t) \) and \( b(t) = \{b_1(t), b_2(t)\} \). By
Cramer's rule
\[
q_1(t) = \frac{b_1(t) a_{12}}{|A|} - \frac{a_{21} b_1(t)}{|A|}.
\]
and
\[
q_2(t) = \frac{a_{11} b_1(t)}{|A|} - \frac{b_1(t) a_{22}}{|A|}.
\]
If 132a holds all the determinants are positive, since \(|A|\) is positive
by hypothesis. \( B_1 \) is optimal if \( \Delta_j \geq 0 \) and \( \Delta_4 \geq 0 \). Hence we
require that
\[
[c_1, c_2] B_1^{-1} a_j - c_j \geq 0, \quad j=3, 4.
\]
Since \( c_3 = c_4 = 0 \), we should have the following determinants
nonnegative:
\[
\begin{vmatrix} c_1 & c_2 \\ a_{21} & a_{22} \end{vmatrix}; \quad \begin{vmatrix} a_{11} & a_{12} \\ c_1 & c_2 \end{vmatrix}.
\]
This is guaranteed by 132b. Note that in this case
\[
y_1^0 = \frac{c_1 a_{22} - c_2 a_{21}}{|A|},
\]
and
\[
y_2^0 = \frac{a_{11} c_2 - a_{12} c_1}{|A|}.
\]
To prove 133a and 134b we note that

\[ B_2 = (a_1, e_1); \quad |B_2| = -1/a_{21} \]

\[ q_1(t) = b_2/a_{21} > 0. \]

Hence, \( B_2 \) is feasible if

\[ q_3(t) \geq 0, \text{ i.e., if } \begin{vmatrix} a_{11} & b_1(t) \\ a_{21} & b_2(t) \end{vmatrix} \]

is nonpositive, i.e., if 133a holds.

\[ \Delta_4 = c_2/a_{21} > 0 \]

Hence, \( B_2 \) is also optimal if \( \Delta_2 \) is nonnegative, i.e.,

\[ \Delta_2 = -\frac{1}{a_{21}} \begin{vmatrix} c_1 & 0 \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{12} \\ a_{22} \end{vmatrix} = \frac{c_1 a_{22}}{a_{21}} - c_2 \geq 0 \]

Hence \( \Delta_2 \geq 0 \) if 133b holds.

Similar proofs can be given for the relations 134a - 136b. We now replace \( a_3 \) by a more restrictive assumption.

\( a_4 \). Any 2 x 2 submatrix of

\[ \begin{pmatrix} c_1 & c_2 \\ a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

if of full rank.

Note that if \( a_4 \) holds \( |A| \neq 0 \), so we can make it positive once again. Assumption \( a_4 \) also implies that the following determinants are nonzero,
Let
\[
\begin{vmatrix}
  c_1 & c_2 \\
  a_{11} & a_{12}
\end{vmatrix},
\begin{vmatrix}
  c_1 & c_2 \\
  a_{21} & a_{22}
\end{vmatrix}.
\]

Only \( B_2 \) and \( B_3 \) can be optimal if they are feasible. In both cases \( q_1(t) \) is in the optimal basis, and \( q_2(t) \) would not enter the basis because \( \Delta_2 \) is positive for both \( B_2 \) and \( B_3 \) as basis. With \( B_2 \) as basis \( \Delta_3 = 0, \Delta_4 = \frac{c_1}{a_{11}} \).

With \( B_3 \) as basis \( \Delta_3 = \frac{c_1}{a_{11}}, \Delta_4 = 0. \) Hence at \( t=0 \), let \( \frac{b_2(0)}{b_1(0)} = \frac{a_{21}}{a_{11}}. \)

So that both \( B_2 \) and \( B_3 \) are simultaneously feasible and \( q_3(t) = q_4(t) = 0. \)

At this point
\[
\left( \frac{\partial z}{\partial b_1(o)} \right)_- = \frac{c_1}{a_{11}}; \left( \frac{\partial z}{\partial b_1(o)} \right)_+ = 0
\]
\[
\left( \frac{\partial z}{\partial b_2(o)} \right)_- = \frac{c_1}{a_{21}}; \left( \frac{\partial z}{\partial b_2(o)} \right)_+ = 0.
\]

Equation 138 follows from the fact that any slight increase (decrease) in \( b_1(o) \) would make \( B_2(B_3) \) the only optimal basis and a similar slight increase (decrease) in \( b_2(o) \) would make \( B_3(B_2) \) the only optimal basis.

We now show that when 137 holds, the maximal balanced growth path is given by
\[
\frac{b_2(t)}{b_1(t)} = \frac{a_{21}}{a_{11}}, t=0, 1, 2, \ldots.
\]
PROOF \ From the optimality rule 130 we check to see

\[
\frac{c_1}{K_1 a_{11}} \geq \frac{z[b(o)]}{K[b(o)]} \geq 0; \quad \frac{c_1}{K_2 a_{21}} \geq \frac{z[b(o)]}{K[b(o)]} \geq 0
\]

At \( b_2/b_1 = a_{21}/a_{11} \)

\[
z[b(o)] = \frac{c_1 b_1(o)}{a_{11}} = \frac{c_1 b_2(o)}{a_{21}}
\]

\[
K[b(o)] = K_1 b_1(o) + K_2 a_{21} b_1(o)
\]

\[
= [K_1 + K_2 a_{21} / a_{11}] b_1(o) = [K_1 a_{11} + K_2 a_{21}] \frac{b_1(o)}{a_{11}}
\]

Hence \( \frac{z[b(o)]}{K[b(o)]} = \frac{c_1}{K_1 a_{11} + K_2 a_{21}} > 0 \)

Hence 140 is satisfied since \( K_1 \) and \( K_2 \) are positive.

It may be noted that when 137 holds, the problem \( \ell \) is not much different from the maximization problem of a single-product firm producing with two limitational factors and subject to the following production function, a simple minimum law (Frisch, 1965), obtained from 131 by setting \( q_2(t) = 0 \),

\[
q_1(t) = \min \left( \frac{b_1(t)}{a_{11}}, \frac{b_2(t)}{a_{21}} \right); \ t = 0, 1, 2, \ldots
\]

It can easily be checked that on the limitational line; \( \frac{b_2(t)}{b_1(t)} = \frac{a_{21}}{a_{11}} \left( \frac{\partial q_1(t)}{\partial b_1(t)} \right)_- = \frac{1}{a_{11}}, \left( \frac{\partial q_1(t)}{\partial b_1(t)} \right)_+ = 0, \left( \frac{\partial q_2(t)}{\partial b_2(t)} \right)_- = \frac{1}{a_{21}}, \left( \frac{\partial q_2(t)}{\partial b_2(t)} \right)_+ = 0.

Hence 138 will hold. So both \( b_1(t) \) and \( b_2(t) \) are limitational, any
increase in one of the factors, without a corresponding increase in
the other will contribute nothing to $z(t)$. Hence the solution 139 is
stable.

A similar analysis can be made for the case when

\[
(141) \quad \frac{a_{22}}{a_{21}} < \frac{a_{12}}{a_{12}} < \frac{c_2}{c_1}.
\]

Now only $B_4$ and $B_5$ could be optimal. Only $q_2(t)$ is produced. On
the maximal balanced growth path

\[
(142) \quad \frac{b_2^*(t)}{b_1^*(t)} = \frac{a_{22}}{a_{12}} ; \quad t=0, 1, 2, ...
\]

i.e., $B_4$ and $B_5$ are simultaneously feasible.

\[
(143) \quad \left( \frac{\partial z}{\partial b_1^*(o)} \right)_+ = \frac{c_2}{a_{12}} \left( \frac{\partial z}{\partial b_2^*(o)} \right)_+ = 0
\]

\[
(144) \quad \frac{c_2}{K_1 a_{12}} \geq \frac{c_2}{K_1 a_{12} + K_2 a_{22}} > 0; \quad \frac{c_2}{K_2 a_{22}} \geq \frac{c_2}{K_1 a_{12} + K_2 a_{22}} > 0.
\]

Once again the solution is stable.

The production function is

\[
(131b) \quad q_2(t) = \min \left( \frac{b_1(t)}{a_{12}}, \frac{b_2(t)}{a_{22}} \right).
\]

It may be noted that both 139 and 142 do not involve $K$ function. If
a given budget is given then we should have, $I = K_1 a_{12} + K_2 a_{22}$. So an
unique pair \( b^*_1(o), b^*_2(o) \) and a growth path will be determined. We now consider the situation when,

\[
\frac{a_{22}}{a_{21}} > \frac{c_2}{c_1} > \frac{a_{12}}{a_{11}}.
\]

Now \( B_1, B_2 \) and \( B_5 \) can be optimal. Only \( B_1 \) is optimal if

\[
\frac{a_{22}}{a_{12}} > \frac{b^*_2(o)}{b^*_1(o)} > \frac{a_{21}}{a_{11}},
\]

\( B_1 \) and \( B_5 \) are both optimal if

\[
\frac{b^*_2(o)}{b^*_1(o)} = \frac{a_{22}}{a_{12}} > \frac{a_{21}}{a_{11}} \quad \text{and} \quad \frac{b^*_2(o)}{b^*_1(o)} = \frac{a_{22}}{a_{12}} = \frac{b^*_2(o)}{b^*_1(o)}.
\]

The maximal balanced growth path now depends essentially on the \( K \) function. If 146 holds, then \( z[l(o)] \) is differentiable and

\[
\frac{y_1}{K_1} = \frac{y_2}{K_2} = \frac{z[l(b(o))]}{K[l(b(o))]} = \frac{\lambda^* - \lambda}{\sigma}
\]

i.e.,

\[
\frac{c_1 a_{22} - c_2 a_{21}}{c_2 a_{11} - c_1 a_{12}} = \frac{K_1}{K_2}, \quad \text{given a budget I so that} \quad K[l(b(o))] = I,
\]

an unique maximal balanced growth path will be determined. Moreover, on this maximal balanced growth path

\[
\frac{c_1}{K_1 a_{11} + K_2 a_{21}} = \frac{c_2}{K_1 a_{12} + K_2 a_{22}} = \frac{\lambda^* - \lambda}{\sigma}
\]

If 147 holds then \( B_1 \) and \( B_5 \) are both optimal. If \( b_1 \) increases (decreases) by one unit \( B_1(B_5) \) is the only optimal basis and similarly if \( b_2 \) increases (decreases) by one unit \( B_5(B_1) \) is the only optimal basis.
Now \( z(b(t)) = \frac{c_2 b_1(t)}{a_{12}} \) if we calculate it from \( B_5 \). However, if we calculate it from \( B_1 \) we get the same value.

**Proof**

\[
\begin{align*}
Z(b(t)) &= y_1 b_1(t) + y_2 b_2(t) \\
&= y_1 b_1(t) + y_2 \frac{a_{22}}{a_{12}} b_1(t) \\
&= \frac{b_1(t)}{a_{12}} [y_1 a_{12} + y_2 a_{22}] \\
\end{align*}
\]

But \( y_1 = \frac{c_1 a_{22} - c_2 a_{21}}{|A|} \), \( y_2 = \frac{c_2 a_{11} - c_1 a_{12}}{|A|} \)

Hence, \( z(b(t)) = \frac{b_1(t)}{a_{12}} \cdot \frac{c_2 a_{11} a_{22} - c_1 a_{12} a_{22}}{|A|} \)

Moreover,

\[
\begin{align*}
\frac{\partial Z}{\partial b_1} &= \frac{c_2}{a_{12}} \\
\frac{\partial Z}{\partial b_2} &= \frac{c_1 a_{22} - c_2 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}
\end{align*}
\]

(150)

\[
\begin{align*}
\frac{\partial Z}{\partial b_2} &= \frac{c_2 a_{11} - c_1 a_{12}}{a_{11} a_{22} - a_{12} a_{22}} \\
\frac{\partial Z}{\partial b_2} &= 0.
\end{align*}
\]

Since \( \Delta_3 = y_1; \Delta_4 = y_2 \) when \( B_1 \) is the optimal basis and \( \Delta_3 = \frac{c_2}{a_{12}} ; \Delta_4 = 0 \) when \( B_5 \) is the optimal basis,

\[
K(b(o)) = K_1 b_1(o) + K_2 b_2(o)
\]

\[
= \frac{b_1(o)}{a_{12}} [K_1 a_{12} + K_2 a_{22}]
\]
Hence

\[
\frac{z[b(o)]}{K[b(o)]} = \frac{c_2}{k_1a_{12} + k_2a_{22}}.
\]

So 147 will hold only if

\[
(151) \quad \frac{c_2}{k_1a_{12}} \geq \frac{c_2}{k_1a_{12} + k_2a_{22}} \geq \frac{c_1a_{22} - c_2a_{21}}{k_2(a_{11}a_{22} - a_{12}a_{21})} \geq \frac{c_2}{k_1a_{12} + k_2a_{22}} \geq 0.
\]

If 148 holds then \( B_1 \) and \( B_2 \) can be optimal basis. If \( b_1 \) increases (decreases) by one unit \( B_2(B_1) \) is the only optimal basis and vice versa. By a similar process of reasoning it can be shown that

\[
(152) \quad z(b(t)) = \frac{c_1b_2(t)}{a_{21}}
\]

\[
(153) \quad K[b(o)] = \frac{b_2(o)}{a_{21}} \cdot \left[ k_1a_{11} + k_2a_{21} \right]
\]

\[
(154) \quad \left( \frac{\partial z}{\partial b_1(o)} \right)^- = \frac{c_1a_{22} - c_2a_{21}}{|A|}
\]

\[
\left( \frac{\partial z}{\partial b_1(o)} \right)^+ = 0
\]

\[
\left( \frac{\partial z}{\partial b_2(o)} \right)^- = \frac{c_1}{a_{21}}
\]

\[
\left( \frac{\partial z}{\partial b_2(o)} \right)^+ = \frac{c_{2a_{11}} - c_{1a_{12}}}{|A|}
\]

Hence 147 will hold only if
To complete our analysis we note

1. If \( \frac{b_2(o)}{b_1(o)} > \frac{a_{22}}{a_{12}} > \frac{a_{12}}{a_{11}} \) then when 145 holds, only \( B_5 \) is optimal. Hence \( \left( \frac{\partial z}{\partial b_1(o)} \right) = \frac{c_2}{a_{21}} \left( \frac{\partial z}{\partial b_2(o)} \right) = 0 \).

So \( b_1 \) will be increased and \( b_2 \) decreased till

\[
\frac{b_2(o)}{b_1(o)} = \frac{a_{22}}{a_{12}} > \frac{a_{12}}{a_{11}}, \quad \text{i.e., 147 holds.}
\]

2. If \( \frac{a_{22}}{a_{12}} > \frac{a_{12}}{a_{11}} > \frac{b_2(o)}{b_1(o)} > 0 \) and 145 holds, only \( B_2 \) is optimal and hence \( \left( \frac{\partial z}{\partial b_1(o)} \right) = \frac{c_1}{a_{21}} ; \left( \frac{\partial z}{\partial b_2(o)} \right) = 0 \). Hence \( b_1(o) \) will be decreased and/or \( b_2 \) increased till we hit

\[
\frac{a_{22}}{a_{12}} > \frac{a_{12}}{a_{11}} = \frac{b_2(o)}{b_1(o)} \quad \text{when 148 holds.}
\]

It may be noted that when 145 holds the firm is essentially a two-product firm for which 131 applies. When 146 holds, both \( b_1(t) \) and \( b_2(t) \) have positive marginal revenue productivities both in the forward and backward directions. Any change in \( b_1(t) \) or \( b_2(t) \) alone can be accommodated by changing the relative production levels of
When 147 or 148 holds the firm is on a corner point, i.e., on a knife-edge balance. But even here the option of introducing another product in its production schedule makes the marginal revenue productivity of the more strategic factor positive, even in the forward direction (see 150 and 154). However, it may be noted that the marginal productivity of any particular factor for producing one particular product, keeping the other product at a constant amount is still zero on the maximal balanced growth path, for more than one factor is needed for producing any output and there cannot be any excess capacity of any factor on this particular path. So it is the degree of assortment of two products that gives the firm one extra degree of freedom.

We now relax \( a_4 \) and assume once again \( |A| > 0 \). Let

\[
\begin{vmatrix}
  c_1 & c_2 \\
  a_{11} & a_{12}
\end{vmatrix} = 0; \text{ i.e., } \frac{a_{22}}{a_{21}} > \frac{a_{12}}{a_{11}} = \frac{c_2}{c_1}.
\]

\( B_1, B_2, B_3 \) and \( R_5 \) can now be optimal if any of them is feasible. The feasibility will however depend on \( \frac{b_2(o)}{b_1(o)} \).

We start from a situation when

\[
(156) \quad \omega > \frac{b_2(o)}{b_1(o)} > \frac{a_{22}}{a_{12}} > \frac{a_{21}}{a_{11}}.
\]

\( B_3 \) and \( B_5 \) can be feasible. For \( B_3 \) as the basis, \( \Delta_3 = \frac{c_1}{a_{11}} \), \( \Delta_4 = 0 \).

For \( B_5 \) as the basis, \( \Delta_3 = \frac{c_2}{a_{12}} = \frac{c_1}{a_{11}} \) and \( \Delta_4 = 0 \). Hence
when 156 holds. Hence \( b_1(o) \) (\( b_2(o) \)) would be increased (decreased) till 

\[
\frac{b_2(o)}{b_1(o)} = \frac{a_{22}}{a_{12}}
\]

Now \( B_3 \), \( B_5 \) and \( B_6 \) are feasible. But with \( B_1 \) as the basis 

\[
\Delta_3 = \frac{c_1a_{22} - c_2a_{21}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{c_1}{a_{11}}
\]

\[
\Delta_4 = \frac{c_2a_{11} - c_1a_{12}}{a_{11}a_{22} - a_{12}a_{21}} = 0
\]

So 157 continues to hold and the adjustment process continues till we reach 

\[
\frac{b_2(o)}{b_1(o)} = \frac{a_{21}}{a_{11}}
\]

At this point \( B_1 \) and \( B_2 \) are both optimal.

If we start from the other end, namely 

\[
\frac{a_{22}}{a_{12}} > \frac{a_{21}}{a_{11}} > \frac{b_2(o)}{b_1(o)} > 0
\]

we see that only \( B_2 \) is optimal and since \( \Delta_3 = 0 \), \( \Delta_4 = \frac{c_1}{a_{21}} \)

\[
\frac{\partial z}{\partial b_1(o)} = 0, \quad \frac{\partial z}{\partial b_2(o)} = \frac{c_1}{a_{21}}
\]

if 159 holds. So \( b_2 \) (\( b_1 \)) would be increased (decreased) until we reach 

\[
\frac{b_2(o)}{b_1(o)} = \frac{a_{21}}{a_{11}}
\]
We now claim that the unique minimal balanced growth path is given by

\[(161) \quad \frac{b_2(t)}{b_1(t)} = \frac{a_{21}}{a_{11}}, \quad t=0, 1, \ldots,\]

**PROOF** When 161 holds \((\frac{\partial z}{\partial b_1(o)})_+ = \frac{c_1}{a_{11}}\) and \((\frac{\partial z}{\partial b_1(o)})_- = \frac{c_2}{a_{21}}\); \((\frac{\partial z}{\partial b_2(o)})_+ = 0, (\frac{\partial z}{\partial b_2(o)})_- = \frac{c_1}{a_{21}}\). Hence

\[z[b(o)] = \frac{b_1 c_1}{a_{11}} = \frac{b_2 c_1}{a_{21}} = \frac{c_2 b_1}{a_{12}}\]

and

\[K[b(o)] = K_1 b_1 + K_2 b_2 = \frac{b_1}{a_{11}} [K_1 a_{11} + K_2 a_{21}]\]

Once again the solution is stable and does not depend on K essentially, except for the fact that \(K_1, K_2\) are assumed to be strictly positive.

Similar analysis can be made when

\[(163) \quad \begin{vmatrix} c_1 & c_2 \\ a_{21} & a_{22} \end{vmatrix} = 0 \quad \text{or} \quad \frac{c_2}{c_1} = \frac{a_{22}}{a_{21}} > \frac{a_{12}}{a_{11}}\]

\(B_1, B_2, B_4,\) and \(B_5\) are possible optimal basis.

When 156 holds \(B_5\) is optimal, hence
So $b_1(o) (b_2(o))$ is increased (decreased) till we hit

$$b_2(o) = \frac{a_{22}}{a_{12}} > \frac{a_{21}}{a_{11}}$$

when $B_1$ and $B_5$ are both feasible.

Starting at the other end where 159 holds, we see that $B_2$ and $B_4$ are feasible. Hence

$$\frac{\partial z}{\partial b_1(o)} = \frac{c_1}{a_{21}} = \frac{c_2}{a_{22}}, \quad \frac{\partial z}{\partial b_2(o)} = 0.$$

So $b_1 (b_2)$ would be increased (decreased) until,

$$\frac{a_{22}}{a_{12}} > \frac{a_{21}}{a_{11}} = \frac{b_2(o)}{b_1(o)}.$$

Now $B_1$ becomes feasible but once again, with $B_1$ as the basis, 166 continues to hold and hence the adjustment process stops at the point where 165 holds. Since $B_1$ and $B_5$ are now both optimal, we have at this point

$$\left(\frac{\partial z}{\partial b_1(o)}\right)^- = \frac{c_2}{a_{12}}, \quad \left(\frac{\partial z}{\partial b_1(o)}\right)^+ = 0;$$

$$\left(\frac{\partial z}{\partial b_2(o)}\right)^- = \frac{c_1}{c_{21}} = \frac{c_2}{a_{22}}, \left(\frac{\partial z}{\partial b_2(o)}\right)^+ = 0;$$

$$z(b(o)) = \frac{c_2b_1}{a_{12}} = \frac{c_1b_2}{a_{21}} = \frac{c_2b_2}{a_{22}};$$

$$K(b(o)) = \frac{b_1}{a_{12}} [K_1a_{12} + K_2a_{22}].$$
Hence the optimality conditions are satisfied at $\frac{b^*_2(o)}{b^*_1(o)} = \frac{a_{22}}{a_{12}}$.

\begin{align*}
\frac{c_2}{K_1 a_{12}} & \geq \frac{c_2}{K_1 a_{12} + K_2 a_{22}} > 0; \\
\frac{c_2}{K_2 a_{22}} & \geq \frac{c_2}{K_1 a_{12} + K_2 a_{22}} > 0.
\end{align*}

Once again the solution is stable and does not depend on the $K$ function essentially.

Lastly, we consider the situation when $A$ is singular, i.e. $|A| = 0$.

Let $\frac{a_{12}}{a_{11}} = \frac{a_{22}}{a_{21}} = k > 0$.

\begin{align*}
(169) & \text{ If } \frac{c_2}{c_1} < k, \text{ then } q^o_1 = \min \left[ \frac{b_1}{a_{11}}, \frac{b_2}{a_{21}} \right]; \quad q^o_2 = 0. \\
(170) & \text{ If } \frac{c_2}{c_1} > k, \text{ then } q^o_1 = 0, q^o_2 = \min \left[ \frac{b_1}{a_{12}}, \frac{b_2}{a_{22}} \right].
\end{align*}

If 169 holds $\frac{b^*_1(o)}{a_{11}} = \frac{b^*_2(o)}{a_{21}}$. If 170 holds then $\frac{b^*_1(o)}{a_{12}} = \frac{b^*_2(o)}{a_{22}}$.

PROOF If 169 holds, $z[b(o)] = \frac{c_1 b_1(o)}{a_{11}} = \frac{c_1 b_2(o)}{a_{21}}$;

\begin{align*}
\left( \frac{\partial z}{\partial b_1(o)} \right)_- & = \frac{c_1}{a_{11}}, \quad \left( \frac{\partial z}{\partial b_1(o)} \right)_+ = 0; \\
\left( \frac{\partial z}{\partial b_2(o)} \right)_- & = \frac{c_1}{a_{21}}, \quad \left( \frac{\partial z}{\partial b_2(o)} \right)_+ = 0.
\end{align*}
\[ K[\varphi_1(\theta), \varphi_2(\theta)] = \frac{\varphi_1(\theta)}{a_{21}} [K_{1a_{21}} + K_{2a_{21}}]. \]

Hence the optimality rule

\[
\frac{c_1}{K_{1a_{11}}} < \frac{c_1}{K_{1a_{11}} + K_{2a_{21}}} < 0
\]

(172)

\[
\frac{c_1}{K_{2a_{21}}} < \frac{c_1}{K_{1a_{11}} + K_{2a_{21}}} < 0 \quad \text{holds.}
\]

Similarly, if 170 holds,

\[
z[\varphi(\theta)] = \frac{c_{2\varphi_1}(\theta)}{a_{12}} = \frac{c_{2\varphi_2}(\theta)}{a_{22}};
\]

(173)

\[
\left( \frac{\partial z[\varphi(\theta)]}{\partial \varphi_1(\theta)} \right)_- = \frac{c_2}{a_{12}}; \quad \left( \frac{\partial z[\varphi(\theta)]}{\partial \varphi_2(\theta)} \right)_+ = 0;
\]

\[
\left( \frac{\partial z[\varphi(\theta)]}{\partial \varphi_2(\theta)} \right)_- = \frac{c_2}{a_{22}}; \quad \left( \frac{\partial z[\varphi(\theta)]}{\partial \varphi_2(\theta)} \right)_+ = 0;
\]

(174)

\[ K[\varphi_1^*(\theta), \varphi_2^*(\theta)] = \frac{\varphi_1^*(\theta)}{a_{12}} [K_{1a_{12}} + K_{2a_{22}}]. \]

Hence the optimality rule

\[
\frac{c_2}{K_{1a_{12}}} < \frac{c_2}{K_{1a_{12}} + K_{2a_{22}}} < 0;
\]

\[
\frac{c_2}{K_{2a_{22}}} < \frac{c_2}{K_{1a_{12}} + K_{2a_{22}}} < 0 \quad \text{holds.}
\]

Lastly if \( \frac{c_2}{c_1} = k \) then the firm is in effect producing one product, i.e., the products are economically as well as technically equivalent and no unique solution will exist. We now compare the optimal capacity expansion path of the last chapter with the balanced growth of this chapter.
In the specialized model that we considered in the last chapter no transfer of investible funds over time was allowed. The firm starts with an initial resource vector \( b(0) \) and chooses \( \beta(t) \) and \( \sigma(t) \) optimally over time so that a preference functional over time is maximized. Since the preference functional considered is additive such a growth path satisfies the "optimality" principle. Suppose the firm has chosen an optimal program, completely spelled out at \( t=0 \). The decision vectors \( \beta(t) \) and \( \sigma(t) \) as chosen optimally over time will determine \( b(t) \) over time. Suppose by following this path the firm ends up with \( b(t^\ast) \) at time \( t=t^\ast \). Define

\[
\tau^\ast v(t^\ast) = \sum_{\tau=t^\ast}^{T} \rho(\tau) \ W[c(\tau)] + \rho(T) \ S[b(T)].
\]

Suppose the firm maximizes \( \tau^\ast v(t^\ast) \) from the \( t^\ast \) period onwards with an initial resource vector \( b(t^\ast) \) and subject to the same constraints of the system as before. Then the optimal path of the original problem will coincide with the optimal time path of the truncated problem over \( t^\ast \) to \( T \), i.e., the segment of time the programs share in common. Hence what happened in the past from \( t=0 \) to \( t=t^\ast \) will affect future, \( t^\ast \) to \( T \) only through \( b(t^\ast) \), the state variable at \( t=t^\ast \). Under such conditions as we have seen the efficiency prices associated with any capital good at time \( t \) will depend essentially on behavior of the system (i.e., state and control variables and the exogeneous variable, \( d(t) \)) from the \( t^\ast \) period onwards. If the efficiency prices are known, the over-period optimization problem can be converted into an equivalent one-period (long-run) problem.
All these apply mutatis mutandis to our more general optimal capacity expansion policy model where intertemporal transfer of funds is allowed. Only thing that we have to include is I(t), the investible fund at time t as another state variable. In both cases the efficiency prices are constantly adjusted over time, so that the firm can be viewed as growing on an unbalanced growth path though the future is known beforehand.

A simplified version of such an optimal capacity expansion policy model has been presented in this chapter. It has been shown that if certain conditions are fulfilled then a balanced growth path will exist where the efficiency prices remain stationary over time, since the firm is changing only in scale, so to say.

Lastly, we consider the maximal balanced growth path of a firm under certain simplifying assumptions. The decision variable now is the initial resource vector that the firm should choose at t=0. Once that is chosen, the firm is triggered on a particular balanced growth path over time forever by the rules of the game.

Somewhere in between such a balanced and highly unbalanced growth paths there can be a quasi-balanced growth path, i.e., when the growth path of the firm has a regeneration property. In a linear programming situation suppose the "b" vector after some point of time became proportional to b(t_o) at t=\(t_k\). From \(t_k\), the firm will then grow in the same way until "b" becomes proportional to b(t_o) again. During any such cycle, the firm will switch its basis and/or investment policy. The growth path of the firm can be described by a periodic function of time.
One possible situation in which that can happen is when there are economies of scale so that building ahead of demand becomes profitable. However, such a constant-cycle policy model is not easy to handle in the context of a multi-product firm.

We have assumed a convex k function which makes such economies of scale nonexistent. With the device, the problem remains within the context of a concave programming problem. It may be noted that all the bracket expressions in the Lagrangian are written in such a way that they are concave in the state or control variables involved.

If economies of scale exist, as they do, the K function may become concave. In such cases the optimal value of \( \hat{\beta}_j(t) \), \( j \in I_m \) may be either zero or only bounded by the budget constraint. Such situations are mathematically less tractable than when the neoclassical type of assumptions are made about the K function.

It may be noted that though we have assumed the K function to be smooth, the same assumption was not made about the z function which is only assumed to be piecewise-smooth in each \( b_j, j \in I_m \). One reason for such nonsmoothness is the presence of limitational inputs in the "production process" of any product. It is the latter feature that is brought into prominence in the growth models we have discussed.
LITERATURE CITED


ACKNOWLEDGEMENTS

I should like to express my gratitude to Dr. Jati K. Sengupta, who has guided and inspired me through my studies. He not only suggested the research topic, but the ideas expressed here are also a product of our many discussions. I should also like to thank Dr. Karl A. Fox for his stimulation during my stay at Iowa State University. I am deeply indebted to Dr. George Zyskind, Dr. Erik Thorbecke, Dr. John J. L. Hinrichsen, and Dr. George Seifert for serving on my committee. I would take this opportunity to thank all my fellow inmates of the fourth floor of East Hall and especially to Gene Gruver and Edmond Seay, my office mates, and Mrs. Velma Buttermore, the librarian of the Economics Reading Room. Finally, I would thank my typist, Charlene Carsrud, for patiently wading through my handwriting. Also, I should acknowledge my indebtedness to the National Science Foundation for my support in graduate studies received under the project of Professors Karl A. Fox and Erik Thorbecke.