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# Positivity in function algebras

Jason Ekstrand
*Iowa State University*

**Positivity in function algebras**

by

**Jason Ekstrand**

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Program of Study Committee:

Justin Peters, Major Professor

Tathagata Basak

Yiu Tung Poon

Paul Sacks

Eric Weber

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## ABSTRACT

In this dissertation, we consider algebras of holomorphic functions on the unit disc and other domains. We begin with the disc algebra $\mathcal{A}(\mathbb{D})$ which is a well-studied example in the field of Banach and operator algebras. To this algebra can be applied an involution $f \mapsto f^*$ given by $f^*(z) = \overline{f(\bar{z})}$. With this involution, $\mathcal{A}(\mathbb{D})$ becomes a Banach $*$-algebra that is not a $C^*$-algebra. We study the positive elements of this Banach $*$-algebra and compare them to the classical $C^*$-algebra case. In particular, we use the classical BSF factorization on $H^p(\mathbb{D})$, to show that $f = g^*g$ for some $g \in \mathcal{A}(\mathbb{D})$ if and only if $f([-1,1]) \subseteq \mathbb{R}_+$. A similar result is proved for $H^p(\mathbb{D})$; $1 \leq p \leq \infty$. These results are then extended, first to holomorphic functions on an annulus, and then to holomorphic functions on any domain $G$ that is symmetric with respect to the real line and where $\partial G$ is the union of finitely many disjoint Jordan curves. Connections are also made between these results and the representation theory of holomorphic function algebras.

# CHAPTER 1. INTRODUCTION

The proper setting for the problem this dissertation aims to study is in the field of Banach and operator algebras. In order to state our problem and put it in its proper context, we must first discuss some of the fundamental results in the field. We begin with an overview of normed spaces and continue on to discuss Banach algebras, Hilbert spaces, and $C^*$-algebras.

## 1.1 Background

### 1.1.1 Normed linear spaces

We begin our discussion of operator theory with a discussion of *normed linear spaces.* Let $X$ be a linear (vector) space. (Unless otherwise noted, we will assume that all vector spaces are over the complex numbers.) A *norm* on $X$ is a map $n : X \to \mathbb{R}_+$ that satisfies the following properties:

1. $n(\alpha x) = |\alpha| n(x)$ for all $x \in X$ and all $\alpha \in \mathbb{C}$,

2. $n(x) = 0$ if and only if $x = 0$,

3. $n(x + y) \leq n(x) + n(y)$.

This third property is commonly referred to as *subadditivity* or the *triangle inequality.* A linear space together with a norm is called a *normed linear space.* We usually denote the norm of an element $x \in X$ by $\|x\| = n(x)$. If multiple spaces are involved we may write $\| \cdot \|_X$ for clarity.

Let $x_1, x_2, x_3, \ldots$ be a sequence in a normed space $X$. We say that the sequence $\{x_n\}$ *converges to* $x$ and write $x_n \to x$ if, for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ so that, for every $n \geq N$,

$$\|x_n - x\| < \varepsilon.$$

We say the sequence $\{x_n\}$ *converges* if there is some $x \in X$ so that $x_n \to x$ (the element $x$ is necessarily unique). We say that the sequence $\{x_n\}$ is *Cauchy* if, for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that, for every $m, n \geq N$,

$$\|x_n - x_m\| < \varepsilon.$$

Finally, the space $X$ is *complete* if every Cauchy sequence converges. A complete normed linear space is also called a *Banach space*.

Let $X$ be a linear space and $Y$ subspace of $X$. Then we may form the *quotient space* $X/Y$ where the vectors are sets of the form $x + Y = \{x + y : y \in Y\}$ and linear combinations are given by $\alpha(x_1 + Y) + (x_2 + Y) = (\alpha x_1 + x_2) + Y$. The map $x \mapsto x + Y$ is called the *natural projection* and is frequently denoted by $\pi$. If $X$ is a normed space and $Y$ is norm-closed, we can give $X/Y$ the norm

$$\|x + Y\| = \inf_{y \in Y} \|x + y\|.$$

(The norm-closure of $Y$ is needed to ensure $\|x + Y\| = 0$ implies $x \in Y$.) Since $0 \in Y$, the projection map $\pi$ is norm-decreasing. If $X$ is a Banach space, so is $X/Y$ [1, p. 45].

Let $X$ and $Y$ be normed linear spaces and let $T : X \to Y$ be a linear map (also called an *operator*). Then we define the quantity

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|T(x)\|_Y}{\|x\|_X} = \sup_{\substack{x \in X \\ \|x\|=1}} \|T(x)\|_Y \tag{1.1}$$

where $\|T\|$ may take on the value $+\infty$ if the supremum is infinite. If $\|T\| < \infty$, we say that the operator $T$ is *bounded* and we have that $\|T(x)\|_Y \leq \|T\|\|x\|_X$ for all $x \in X$. We denote by $B(X, Y)$, the set of all bounded linear operators from $X$ to $Y$. It is straightforward to show that $B(X, Y)$ is a linear space and that the map $T \mapsto \|T\|$ as given in (1.1) is a norm on $B(X, Y)$; this norm is called the *operator norm*. If $Y$ is a Banach space, then $B(X, Y)$ is also a Banach space [1, p. 45].

There are two special cases of $B(X, Y)$ that are of particular interest. One is the case where $Y$ is one-dimensional, i.e., $Y = \mathbb{C}$. In this case, the linear maps from $X$ to $Y$ are called *linear functionals* and the space $B(X, Y)$ is called the *dual space* of $X$ and is denoted $X^*$.

Second, is the case where $X$ and $Y$ are the same space and we shorten the notation from $B(X,X)$ to $B(X)$. This case is interesting because any two elements $S,T \in B(X)$ have the same domain and codomain and we can form the composition $S \circ T$ (or simply $ST$). This gives the space $B(X)$ additional algebraic structure beyond simply being a vector space. Given two operators $S,T \in B(X)$ and $x \in X$, we have

$$\|STx\|_X \leq \|S\|_{B(X)}\|Tx\|_X \leq \|S\|_{B(X)}\|T\|_{B(X)}\|x\|_X$$

so $\|ST\| \leq \|S\|\|T\|$. This inequality is of particular importance because it gives a relationship between the norm of $B(X)$ and composition of operators. This is of paramount importance when studying $B(X)$ from an algebraic perspective.

### 1.1.2 Banach algebras

As just mentioned, the space $B(X)$ for some Banach space $X$ has an interesting normed algebraic structure. The basics of this algebraic structure are encapsulated in the concept of a *Banach algebra*.

The most basic algebraic object we will require is simply called an *algebra*. An *algebra* is a vector space $\mathcal{A}$ together with a map

$$(a,b) \mapsto ab.$$

that is associative and bilinear (i.e., satisfies the usual distributive laws $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for every $a,b,c \in \mathcal{A}$). The algebra $\mathcal{A}$ is called *unital* if there is some element $1 \in \mathcal{A}$ so that $1a = a1 = a$ for every $a \in \mathcal{A}$. Such an element is unique and is called the *unit* of $\mathcal{A}$. The algebra $\mathcal{A}$ is called *commutative* or *abelian* if $ab = ba$ for every $a,b \in \mathcal{A}$.

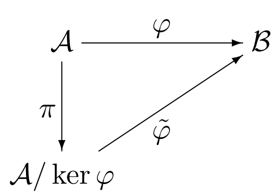
Examples of algebras abound. In Galios theory, an algebra of particular importance is the algebra of polynomials over a (possibly finite) field; this algebra is both abelian and unital. One non-abelian example is the algebra of all $n \times n$ matrices over a field. Of interest to functional analysts, is the algebra $\mathcal{C}_0(\mathbb{R})$ of continuous functions which vanish at infinity, i.e. $\lim_{x\to\pm\infty} f(x) = 0$. This last algebra does not have a unit because the constant function 1, while continuous, is not in the algebra since $\lim_{x\to\infty} 1 = 1$.

Let $\mathcal{A}$ be an algebra. Then a set $\mathcal{B}$ is a *subalgebra* of $\mathcal{A}$ (written $\mathcal{B} \leq \mathcal{A}$) if $\mathcal{B}$ is a vector subspace of $\mathcal{A}$ and, for every $a, b \in \mathcal{B}$, $ab \in \mathcal{B}$. Given two algebras $\mathcal{A}$ and $\mathcal{B}$, a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called an *(algebra) homomorphism* if it respects multiplication, i.e., $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in \mathcal{A}$. If $\varphi$ is bijective, it is called an *isomorphism.* Two algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* (written $\mathcal{A} \cong \mathcal{B}$) if there exists an isomorphism between them. If $\varphi$ is injective, then $\mathcal{A} \cong \varphi(\mathcal{A}) \leq \mathcal{B}$ and $\varphi$ is called an *embedding* of $\mathcal{A}$ into $\mathcal{B}$.

A *left* (resp. *right*) *ideal* of an algebra $\mathcal{A}$ is a vector subspace $I$ of $\mathcal{A}$ such that $ab \in I$ (resp. $ba \in I$) for all $a \in \mathcal{A}$ and $b \in I$. An *ideal* of $\mathcal{A}$ is a subspace $I \subseteq \mathcal{A}$ such that $I$ is both a left and right ideal. For any homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, $\ker \varphi$ is an ideal. This is true because, given $a \in \mathcal{A}$ and $b \in \ker \varphi$, $\varphi(ab) = \varphi(a)0 = 0 = 0\varphi(a) = \varphi(ba)$.

Given an ideal $I$ of $\mathcal{A}$, we may form the quotient algebra $\mathcal{A}/I$ of equivalence classes of the form $a + I$ where the multiplication is given by $(a + I)(b + I) = ab + I$. The natural projection $\pi : \mathcal{A} \to \mathcal{A}/I$ given by $\pi(a) = a + I$ is a homomorphism with $\ker \pi = I$. Therefore, every ideal is the kernel of some homomorphism. As with any other algebraic discussion, we now state the first isomorphism theorem.

**Theorem 1.1.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then $\varphi(\mathcal{A})$ is a subalgebra of $\mathcal{B}$, $\ker \varphi$ is an ideal of $\mathcal{A}$, and there is a unique homomorphism $\tilde{\varphi} : \mathcal{A}/\ker \varphi \to \mathcal{B}$ making the following diagram commute:*




*Furthermore, if $\varphi$ is surjective, then $\tilde{\varphi}$ is an isomorphism. In particular, $\mathcal{A}/\ker \varphi \cong \varphi(\mathcal{B})$.*

While algebras, as defined above, are interesting in their own right, what is of more interest to is are algebras with a reasonable norm. A *normed algebra* is an algebra $\mathcal{A}$ together with a norm $\|\cdot\|$ that is submultiplicative, i.e.,

$$\|ab\| \leq \|a\|\|b\|.$$

If the algebra is complete with respect to its norm (i.e., is a Banach space), then we call $\mathcal{A}$ a *Banach algebra.*

One important effect of the above norm condition is that multiplication is continuous. To see this, let $a_n \to a$ and $b_m \to b$. Then $\|a_n\| \to \|a\|$ so $\|a_n\|$ is bounded and

$$\|a_nb_m - ab\| = \|a_nb_m - a_nb + a_nb - ab\| = \|a_n(b_m - b) + (a_n - a)b\| \leq \|a_n\|\|b_m - b\| + \|a_n - a\|\|b\| \to 0$$

as $n, m \to \infty$ so $a_nb_m \to ab$.

If $X$ is a Banach space, then we showed above that $\|ST\| \leq \|S\|\|T\|$ for every $S, T \in B(X)$ so $B(X)$ is a normed algebra. We also mentioned above that $B(X, Y)$ is complete if $Y$ is complete so $B(X) = B(X, X)$ is a Banach algebra. The algebra $B(X)$ is one of the primary motivating examples for the study of Banach algebras and provides much of the intuition. Another example of Banach spaces arises from continuous functions.

**Example 1.1.2.** Let $\Omega$ be a locally compact Hausdorff space. Then a continuous function $f : \Omega \to \mathbb{C}$ is said to *vanish at infinity* if, for every $\varepsilon > 0$ there is a compact set $K \subseteq \Omega$ so that $|f(x)| < \varepsilon$ for every $x \notin K$. Let $\mathcal{C}_0(\Omega)$ be the space of continuous complex-valued functions on $\Omega$ that vanish at infinity. For $f \in \mathcal{C}_0(\Omega)$, $f$ is clearly bounded because there is some compact space $K$ so that $|f| < 1$ off $K$ and, since $K$ is compact, $f$ is bounded on $K$. Since every function in $C_0(\Omega)$ is bounded, we can define the supremum norm

$$\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)|$$

which is finite for each $f \in \mathcal{C}_0(\Omega)$. Given $f, g \in \mathcal{C}_0(\Omega)$, $fg$ vanishes at infinity and

$$\|fg\|_\infty = \sup_{\omega \in \Omega} |(fg)(\omega)| = \sup_{\omega \in \Omega} |f(\omega)||g(\omega)| \leq \left(\sup_{\omega \in \Omega} |f(\omega)|\right)\left(\sup_{\omega \in \Omega} |g(\omega)|\right) = \|f\|_\infty\|g\|_\infty$$

so $\mathcal{C}_0(\Omega)$ is a normed algebra. Since the uniform limit of continuous functions is continuous, $\mathcal{C}_0(\Omega)$ is complete with respect to $\|\cdot\|_\infty$ and is therefore a Banach algebra.

The algebra $\mathcal{C}_0(\Omega)$ is unital if and only if $\Omega$ is compact. This is because, if $\Omega$ is compact, then the constant function 1 vanishes at infinity. If $\Omega$ is not compact, then there is no compact $K \subseteq \Omega$ so that $|1| < 1/2$ off $K$; therefore $1 \notin \mathcal{C}_0(\Omega)$.

As we will soon see, all abelian Banach algebras have a connection to $\mathcal{C}_0(\Omega)$ for some locally compact Hausdorff space $\Omega$.

**Example 1.1.3.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane and let $\mathbb{D}^-$ be its closure. (For a set $X \subseteq \mathbb{C}$, we will use the notation $X^-$ to denote its norm-closure throughout this dissertation.) Then space $\mathcal{H}(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ is an algebra. Let $\mathcal{A}(\mathbb{D})$ be the subspace of $\mathcal{H}(\mathbb{D})$ consisting of those functions $f$ that have a continuous extension to $\mathbb{D}^-$. Since $\mathbb{D}^-$ is compact and every $f \in \mathcal{A}(\mathbb{D})$ is continuous on $\mathbb{D}^-$, we can give $\mathcal{A}(\mathbb{D})$ the supremum norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

Since uniform limits of holomorphic functions are holomorphic, $\mathcal{A}(\mathbb{D})$ is complete with respect to $\|\cdot\|_\infty$ so $\mathcal{A}(\mathbb{D})$ is a Banach algebra. This algebra is called the *disc algebra* and has been the topic of a considerable amount of study.

An element $a$ of a unital algebra $\mathcal{A}$ is said to be *invertible* if there is some element $b \in \mathcal{A}$ so that $ab = ba = 1$. This element $b$ is called the *inverse* of $a$ and is written $a^{-1}$. To show that an element is invertible, it is sufficient to show that it has both left and right inverses, i.e. there are some $b, c \in \mathcal{A}$ so that $ab = ca = 1$. That the left and right inverses are the same is automatic because $c = c1 = c(ab) = (ca)b = 1b = b$. Inverses may not always exist. For example, on the space $\mathcal{C}_0([-1,1])$ the function $f(z) = z$ is not invertible because $f(0) = 0$ and, for any $g \in \mathcal{C}_0([-1,1])$, $(fg)(0) = f(0)g(0) = 0$ so $fg \neq 1$. For a unital Banach algebra $\mathcal{A}$, there are conditions under which when we can know an element $a \in \mathcal{A}$ is invertible.

**Theorem 1.1.4.** *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$. If $\|a\| < 1$ then $1 - a$ is invertible and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

*Proof.* First, we need to establish that the sum given above converges. Since any finite sum $\sum_{n=1}^{N} a^n$ is well-defined, we need only show that the tail can be made arbitrarily small. However,

$$\left\| \sum_{n=N}^{\infty} a^n \right\| \leq \sum_{n=N}^{\infty} \|a^n\| \leq \sum_{n=N}^{\infty} \|a\|^n$$

which can be made arbitrarily small because it is a geometric series and $\|a\| < 1$. Since multiplication is continuous,

$$(1-a)\left(\lim_{N\to\infty} \sum_{n=0}^{N} a^n\right) = \lim_{N\to\infty} (1-a) \sum_{n=0}^{N} a^n = \lim_{N\to\infty} 1 - a^{n+1} = 1$$

since $\|a^{n+1}\| \leq \|a\|^{n+1} \to 0$ as $n \to \infty$ since $\|a\| < 1$. □

Given a unital algebra $\mathcal{A}$ and $a \in \mathcal{A}$, we define the *spectrum* of $a$ to be the set

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible}\}.$$

Let us consider this definition in light of two of the examples of Banach algebras that we gave above. For a square matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$, $\sigma(A)$ is simply the set of eigenvalues of $A$. If $\Omega$ is a compact Hausdorff space and $f \in \mathcal{C}_0(\Omega)$, then $\sigma(f) = f(\Omega)$. It is often useful to think of the spectrum as a generalization of the eigenvalues of a matrix or the range of a function.

If an element $a \in \mathcal{A}$ is invertible, then $\sigma(a^{-1}) = \sigma(a)^{-1} = \{\lambda^{-1} : \lambda \in \sigma(a)\}$. This is because, if $\lambda \notin \sigma(a)$ then $\lambda - a$ is invertible with some inverse $b$ and

$$-ba\lambda(\lambda^{-1} - a^{-1}) = -ba(1 - \lambda a^{-1}) = -b(a - \lambda) = b(\lambda - a) = 1$$

Similarly, $(\lambda^{-1} - a^{-1})(-\lambda ab) = 1$ so $\lambda^{-1} - a^{-1}$ is invertible and $\lambda^{-1} \notin \sigma(a^{-1})$. Finally, since $(a^{-1})^{-1} = a$, $\lambda \notin \sigma(a^{-1})$ implies $\lambda^{-1} \notin \sigma(a)$.

For an element $a$ of a unital algebra $\mathcal{A}$, we define the *spectral radius* of $a$, denoted $\rho(a)$, by

$$\rho(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

For a general algebra $\mathcal{A}$ the quantity above may be infinite. However, if $\mathcal{A}$ is a Banach algebra, $\rho(a) \leq \|a\|$. This is because, given $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$, $\|\lambda^{-1}a\| = |\lambda|^{-1}\|a\| < 1$ and, by theorem 1.1.4, $\lambda - a = \lambda(1 - \lambda^{-1}a)$ is invertible so $\lambda \notin \sigma(a)$. Not only do we have a bound, we can also use norms to compute $\rho(a)$ exactly.

**Theorem 1.1.5.** [2, p. 10] *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then*

$$\rho(a) = \inf_n \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}$$

This gives us that the spectrum of an element is bounded. It does not, however, tell us that the spectrum is non-empty or anything else about its topology.

**Theorem 1.1.6.** [2, p. 9] *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then $\sigma(a)$ is non-empty and compact.*

As an easy corollary of this, we get the Gelfand-Mazur theorem which tells when a Banach algebra is actually equal to its field of scalars.

**Corollary 1.1.7** (Gelfand-Mazur)**.** [2, p. 9] *If $\mathcal{A}$ is a unital Banach algebra in which every non-zero element is invertible then $\mathcal{A} = 1\mathbb{C}$.*

*Proof.* Since $\mathcal{A}$ is unital, $1\mathbb{C} \subseteq \mathcal{A}$. Let $a \in \mathcal{A}$. Since $\sigma(a)$ is non-empty, there is some $\lambda \in \sigma(a)$ and $\lambda 1 - a$ is not invertible. By hypothesis, $\lambda 1 - a = 0$ and therefore $a = \lambda 1$. □

Without any more machinery, we can state the well-known spectral mapping theorem. The version below is stated for polynomials and elements of an arbitrary unital algebra. Similar results exist (also bearing the name "spectral mapping theorem") for holomorphic functions on a neighborhood of $\sigma(a)$ and elements of a Banach algebra, continuous functions and normal elements of a $C^*$-algebra, and several other cases. The version below can be stated (and proved) with only the machinery we have developed so far.

**Theorem 1.1.8** (Spectral Mapping Theorem)**.** [2, p. 7] *Let $\mathcal{A}$ be a unital algebra and $a \in \mathcal{A}$. Then, if $\sigma(a) \neq \emptyset$ and $p(z)$ is a polynomial,*

$$p(\sigma(a)) = \sigma(p(a)).$$

Thus far, all of our definitions and theorems regarding spectra have required that the algebra $\mathcal{A}$ be unital. For any algebra $\mathcal{A}$ we may form the unitization $\tilde{\mathcal{A}}$ of $\mathcal{A}$ by starting with the vector space $\tilde{A} = A \oplus \mathbb{C}$ and defining the multiplication

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \mu\lambda).$$

We have a natural embedding of $\mathcal{A}$ into $\tilde{\mathcal{A}}$ given by $a \mapsto (a, 0)$ If $\mathcal{A}$ is a normed algebra, we may extend the norm on $\mathcal{A}$ to $\tilde{A}$ by

$$\|(a, \lambda)\| = \|a\| + |\lambda|$$

making $\tilde{A}$ into a normed algebra. In this case, $\mathcal{A}$ is a norm-closed subalgebra of $\tilde{\mathcal{A}}$. If $\mathcal{A}$ is a Banach algebra, then $\tilde{\mathcal{A}}$ is also a Banach algebra. We can use the unitization to define a spectrum even in the case where $\mathcal{A}$ is non-unital. For a non-unital algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$, we define the spectrum of $a$ as $\sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}((a, 0))$.

We will continue our discussion of Banach algebras in the following subsections with a discussion of Gelfand theory.

### 1.1.3 Gelfand theory

Let $\mathcal{A}$ be an abelian Banach algebra. A *multiplicative functional* is a linear map $\tau : \mathcal{A} \to \mathbb{C}$ that preserves multiplication i.e., $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in \mathcal{A}$. We denote by $\Omega(\mathcal{A})$ the set of all non-zero multiplicative functionals on $\mathcal{A}$. An equivalent definition of a multiplicative functional is an algebra homomorphism from $\mathcal{A}$ to $\mathbb{C}$. Before we concern ourselves with the existence of such maps, there are a few things we should say about them.

**Theorem 1.1.9.** *Let $\mathcal{A}$ be a unital Banach algebra and $\tau \in \Omega(\mathcal{A})$. Then $\tau(1) = \|\tau\| = 1$.*

*Proof.* First, by the definition of $\Omega(\mathcal{A})$, $\tau \neq 0$. Also, $\tau(a) = \tau(1a) = \tau(1)\tau(a)$ for every $a \in \mathcal{A}$ since $\tau$ is a homomorphism; since $\tau(a) \neq 0$ for some $a \in \mathcal{A}$, $\tau(1) = 1$. Let $a \in \mathcal{A}$. Then

$$\tau(a - \tau(a)1) = \tau(a) - \tau(\tau(a)1) = \tau(a) - \tau(a)\tau(1) = 0$$

so $a - \tau(a)1$ is not invertible and $\tau(a) \in \sigma(a)$. Therefore, $|\tau(a)| \leq \|a\|$ and this holds for all $a \in \mathcal{A}$ so $\|\tau\| \leq 1$. Since $\tau(1) = 1$, $\|\tau\| = 1$. □

Also, it is fairly easy to see that multiplicative linear functionals are entirely determined by their kernels. Suppose $\tau_1, \tau_2 \in \Omega(\mathcal{A})$ with $\tau_1 \neq \tau_2$. Then there is some $a \in \mathcal{A}$ such that $\tau_1(a) \neq \tau_2(a)$ and

$$\tau_1(a - \tau_1(a)) = \tau_1(a) - \tau_1(a) = 0 \neq \tau_2(a) - \tau_1(a) = \tau_2(a - \tau_1(a)).$$

Then $a - \tau_1(a) \in \ker \tau_1$ but $a - \tau_1(a) \notin \ker \tau_2$ so $\ker \tau_1 \neq \ker \tau_2$. Therefore, $\tau_1 = \tau_2$ if and only if $\ker \tau_1 = \ker \tau_2$.

In order to say more about multiplicative functionals and to establish that $\Omega(\mathcal{A})$ is non-empty, we need to look at the ideals of $\mathcal{A}$. In particular, we need to consider those ideals which are *maximal*. For an algebra $\mathcal{A}$, an ideal $I$ of $\mathcal{A}$ is said to be *proper* if $I \neq \mathcal{A}$ and $I$ is said to be *maximal* if there does not exist an ideal $J$ with $I \subsetneq J \subsetneq \mathcal{A}$. If $\mathcal{A}$ is unital then the ideal $I$ is proper if and only if $1 \notin I$. If $\mathcal{A}$ unital and abelian then, for any $a \in \mathcal{A}$, the set $a\mathcal{A}$ is an ideal of $\mathcal{A}$ and $a\mathcal{A}$ is proper if and only if $a$ is not invertible.

**Theorem 1.1.10.** *Let $\mathcal{A}$ be a unital algebra. Then every proper ideal of $\mathcal{A}$ is contained in a maximal ideal.*

*Proof.* Suppose that $I$ is a proper ideal of $\mathcal{A}$. Let $\mathcal{I}$ be the partially ordered set

$$\mathcal{I} = \{J : J \text{ is a proper ideal of } \mathcal{A}, I \subseteq J\}$$

ordered by inclusion. Let $\mathcal{C}$ be any chain in $\mathcal{I}$ and let $U = \bigcup_{J \in \mathcal{C}} J$. Let $a, b \in U$. Then $a \in J_a$ and $b \in J_b$ for some $J_a, J_b \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, $J_a \subseteq J_b$ or $J_b \subseteq J_a$ and $a + b \in J_a \cup J_b \subseteq U$ since $J_a$ and $J_b$ are subspaces. Also, for any $c \in \mathcal{A}$, $ca \in J_a \subseteq U$ since $J_a$ is an ideal of $\mathcal{A}$. Therefore, since this holds for all $a, b \in U$, $U$ is an ideal of $\mathcal{A}$. Furthermore, since $1 \notin J$ for all $J \in \mathcal{C}$, $1 \notin U$ and $U$ is proper. We have shown that every chain in $\mathcal{I}$ has an upper bound $U$ in $\mathcal{I}$. Therefore, by Zhorn's lemma [3, p. 53], $\mathcal{I}$ has a maximal element $M$. By our definition of the set $\mathcal{I}$, $M$ must be an ideal of $\mathcal{A}$ with $I \subseteq M$ and, since $M$ is maximal in $\mathcal{I}$, $M$ must be maximal as an ideal of $\mathcal{A}$. □

In Banach algebras, we also care about the topological properties of the ideal. In order for the quotient algebra $\mathcal{A}/I$ to be a Banach algebra, we need $I$ to be closed. As it turns out, this is automatically true for maximal ideals of Banach algebras.

**Theorem 1.1.11.** *Let $\mathcal{A}$ be a unital Banach algebra and let $I$ be an ideal of $\mathcal{A}$. If $I$ is proper, then so is its closure $I^-$.*

*Proof.* Suppose $I$ is proper and let $b \in I$. If $\|1 - b\| < 1$ then $b$ is invertible by theorem 1.1.4 and, for every $a \in \mathcal{A}$, $a = a(b^{-1}b) = (ab^{-1})b \in I$ which is a contradiction since $I$ is proper. Therefore, $\|1 - b\| \geq 1$ for all $b \in I$ so $1 \notin I^-$ and $I^-$ is proper. □

**Corollary 1.1.12.** *Let $\mathcal{A}$ be a unital Banach algebra. Then every maximal ideal is closed.*

Now that we have established the existence and norm-closure of maximal ideals, we need a way to construct multiplicative functionals from them. The following is a well-know theorem in ring theory. (If you forget the vector-space interaction between an algebra and its field of scalars, every algebra is also a ring.)

**Theorem 1.1.13.** *Let $\mathcal{A}$ be an abelian unital algebra and $I$ be an ideal of $\mathcal{A}$. Then $I$ is maximal if and only if $\mathcal{A}/I$ is a field.*

*Proof.* Since $\mathcal{A}$ is abelian, $\mathcal{A}/I$ is also abelian. Suppose $\mathcal{A}/I$ is a field. Let $J$ be an ideal of $\mathcal{A}$ with $I \subsetneq J$ and let $a \in J \setminus I$. Since $a \notin I$ and $\mathcal{A}/I$ is a field, $a + I$ is invertible in $\mathcal{A}/I$. Let $b \in \mathcal{A}$ so that $(a+I)(b+I) = ab + I = 1 + I$. Since $a \in J$, $ab \in J$ and, since $I \subseteq J$ and $I$ is a subspace of $\mathcal{A}$, $ab + I \subseteq J$. In particular, $1 \in J$ so $c = 1c \in J$ for all $c \in \mathcal{A}$ and $J = \mathcal{A}$.

Now suppose simply that $I$ is maximal. Let $a + I \in \mathcal{A}/I$ and let $J = (a+I)(\mathcal{A}/I)$. Then $J$ is an ideal of $\mathcal{A}/I$ and $\pi^{-1}(J)$ is an ideal of $\mathcal{A}$ with $I \subseteq \pi^{-1}(J)$ where $\pi : \mathcal{A} \to \mathcal{A}/I$ is the natural projection map. Since $I$ is maximal, either $\pi^{-1}(J) = I$ or $\pi^{-1}(J) = \mathcal{A}$. If $\pi^{-1}(J) = I$ then $J = \{0 + I\}$ and $a + I = 0 + I$. Now suppose that $\pi^{-1}(J) = \mathcal{A}$. Then $\pi(1) \in J$ and there is some $b + I \in \mathcal{A}/I$ so that $(a+I)(b+I) = 1 + I$. In other words, $(a + I)$ is invertible. Therefore, for each $a + I \in \mathcal{A}/I$, either $a + I = 0 + I$ or $a + I$ is invertible, so $\mathcal{A}/I$ is a field. □

The above theorem tells us that, if $M$ is a maximal ideal then $\mathcal{A}/M$ is a field. It does not, however, tell us that $\mathcal{A}/M \cong \mathbb{C}$, the field of scalars. We do know that, if $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, then so $\mathcal{A}/I$. If $\mathcal{A}$ is a Banach algebra, $\mathcal{A}/M$ is a Banach algebra since $M$ is closed and we may apply the Gelfand-Mazur theorem (Corollary 1.1.7) to $\mathcal{A}/M$ to get that $\mathcal{A}/M \cong \mathbb{C}$.

To every maximal ideal $M$ we may then associate a multiplicative functional $\tau_M$ constructed by composing the natural projection $\pi : \mathcal{A} \to \mathcal{A}/M$ and the isomorphism $\mathcal{A}/M \cong \mathbb{C}$. The multiplicative functional $\tau_M$ is unique because linear functionals are determined entirely by their kernels and $\ker \tau_M = M$. For this reason, if $\mathcal{A}$ is a unital abelian Banach algebra, $\Omega(\mathcal{A})$ is sometimes called the *maximal ideal space of* $\mathcal{A}$.

If the algebra $\mathcal{A}$ is abelian and unital, there is a very useful relationship between $\Omega(\mathcal{A})$ and the spectrum $\sigma(a)$ of some element $a \in \mathcal{A}$.

**Theorem 1.1.14.** *Let $\mathcal{A}$ be a unital abelian Banach algebra. Then, for every $a \in \mathcal{A}$,*

$$\sigma(a) = \{\tau(a) : \tau \in \Omega(\mathcal{A})\}.$$

*Proof.* Fix $a \in \mathcal{A}$. In the proof of theorem 1.1.9, we showed that $\tau(a) \in \sigma(a)$ for every $\tau \in \Omega(\mathcal{A})$. Let $\lambda \in \sigma(a)$ and let $J_\lambda = (\lambda - a)A$. Since $\mathcal{A}$ is abelian, $J_\lambda$ is an ideal of $\mathcal{A}$. Since $\lambda \in \sigma(a)$, $\lambda - a$ is not invertible and $1 \notin J_\lambda$ so $J_\lambda \neq \mathcal{A}$ and there is some maximal ideal $M_\lambda$ with $J_\lambda \subseteq M_\lambda \subsetneq \mathcal{A}$. Let $\tau_\lambda$ be the multiplicative functional associated with $M_\lambda$. Then $(\lambda - a) \in J_\lambda \subseteq M_\lambda = \ker(\tau_\lambda)$ so $0 = \tau_\lambda(\lambda - a) = \lambda - \tau_\lambda(a)$ and $\tau_\lambda(a)$. □

We now have enough spectral theory to develop the theory of the Gelfand transform. For any $a \in \mathcal{A}$, let $\hat{a} : \Omega(\mathcal{A}) \to \mathbb{C}$ be the function given by $\hat{a}(\tau) = \tau(a)$. This function $\hat{a}$ is called the *Gelfand transform* of $a$. In order for the Gelfand transform to be useful, we need to know something about the topology of $\Omega(\mathcal{A})$. We give $\Omega(\mathcal{A})$ the weak-$*$ topology inerited from $\mathcal{A}^*$ which is the weakest topology on $\Omega(\mathcal{A})$ so that $\hat{a}$ is continuous for all $a \in \mathcal{A}$. With this topology, the set $\Omega(\mathcal{A})$ is actually compact.

**Theorem 1.1.15.** *If $\mathcal{A}$ is a unital abelian Banach algebra then $\Omega(\mathcal{A})$ is a compact Hausdorff space with respect to the weak-$*$ topology.*

*Proof.* The topology on $\mathcal{A}^*$ is Hausdorff because the continuous functions $\hat{a}$ separate the points of $\mathcal{A}^*$. Let $S$ be the unit ball in $\mathcal{A}^*$. Then we know $S$ is compact by the Banach-Alaoglu theorem [4, p. 130]. Also, $\Omega(\mathcal{A}) \subset S$ by theorem 1.1.9. For each $a, b \in \mathcal{A}$, let $F_{a,b} : \mathcal{A}^* \to \mathbb{C}$ be given by $F_{a,b}(\phi) = \hat{a}(\phi)\hat{b}(\phi) - \widehat{(ab)}(\phi) = \phi(a)\phi(b) - \phi(ab)$. Clearly, $F_{a,b}$ is continuous since the map $\phi \mapsto \phi(a)$ is continuous in the weak-$*$ topology on $\mathcal{A}^*$ for all $a \in \mathcal{A}$. Finally, we see that $\Omega(\mathcal{A}) = \bigcap_{a,b \in \mathcal{A}} F_{a,b}^{-1}(\{0\})$ which is closed. Therefore, $\Omega(\mathcal{A})$ is a closed subset of a compact Hausdorff space and must be compact. □

Define the function $\varphi : \mathcal{A} \to \mathcal{C}(\Omega(\mathcal{A}))$ by $\varphi(a) = \hat{a}$. This provides us a natural map from our arbitrary unital abelian Banach algebra $\mathcal{A}$ into the continuous functions on a compact Hausdorff space. This is useful because continuous functions are substantially easier to study than the more generic setting of Banach algebras. This map, called the *Gelfand representation of $\mathcal{A}$* has a number of nice properties which can be summed up as follows:

**Theorem 1.1.16.** *Let $\mathcal{A}$ be a unital abelian Banach algebra. Then the Gelfand transform $\varphi : \mathcal{A} \to \mathcal{C}(\Omega(\mathcal{A}))$ is a norm-decreasing homomorphism so that, for each $a \in \mathcal{A}$, $\sigma(a) = \hat{a}(\Omega(\mathcal{A}))$ and $\rho(a) = \|\hat{a}\|_\infty$.*

*Proof.* First, observe that, for any $a, b \in \mathcal{A}$ and for any $\tau \in \Omega(\mathcal{A})$,

$$\varphi(ab)(\tau) = \tau(ab) = \tau(a)\tau(b) = \varphi(a)(\tau)\varphi(b)(\tau)$$

and

$$\varphi(a+b)(\tau) = \tau(a+b) = \tau(a) + \tau(b) = \varphi(a)(\tau) + \varphi(b)(\tau)$$

so $\varphi$ an algebra homomorphism. By theorem 1.1.14, $\sigma(a) = \{\tau(a) : \tau \in \Omega(\mathcal{A})\} = \hat{a}(\Omega(\mathcal{A}))$. Also, $\rho(a) = \max\{|\lambda| : \lambda \in \sigma(a)\} = \max\{|\lambda| : \lambda \in \hat{a}(\Omega(\mathcal{A}))\} = \|\hat{a}\|_\infty$. Finally, for any $a \in \mathcal{A}$, $\|\varphi(a)\|_\infty = \rho(a) \leq \|a\|$ so $\phi$ is norm-decreasing. □

### 1.1.4 Hilbert spaces

Given a vector space $X$, an *inner product* is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ that satisfies the following for all $x, y, z \in X$ and $\alpha \in \mathbb{C}$:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry),
2. $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$ (linearity in the first component),
3. $\langle x, x \rangle \geq 0$ for all $x \in X$ and $\langle x, x \rangle = 0$ if and only if $x = 0$ (positive definiteness).

(For real vector spaces, a similar definition can be made by simply replacing $\mathbb{C}$ with $\mathbb{R}$ above.) Conjugate symmetry, together with linearity in the first component implies that $\langle x, \alpha y + z \rangle = \bar{\alpha}\langle x, y \rangle + \langle x, z \rangle$ (conjugate linearity in the second component). A vector space, together with it's inner product, is called a *inner product space.*

Every inner product space is also a normed space with the norm given by $\|x\| = \langle x, x \rangle^{1/2}$. For any $x, y \in X$ and $\alpha \in \mathbb{C}$,

$$\langle x + \alpha y, x + \alpha y \rangle = \langle x, x \rangle + 2\Re\bar{\alpha}\langle x, y \rangle + |\alpha|^2 \langle y, y \rangle. \tag{1.2}$$

Substituting $\alpha = -\frac{\langle x,y \rangle}{\langle y,y \rangle}$ and applying positive definiteness,

$$0 \leq \langle x + \alpha y, x + \alpha y \rangle = \langle x, x \rangle - 2\frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2 \langle y, y \rangle}{\langle y, y \rangle^2} = \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$$

Multiplying both sides by $\langle y, y\rangle$, re-arranging terms, and taking a square root yields the well-known *Cauchy-Schwartz inequality*:

$$|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2} = \|x\|\|y\|$$

Setting $\alpha = 1$ and substituting the Cauchy-Schwartz inequality into (1.2) yields

$$\|x+y\|^2 = \langle x+y, x+y\rangle \leq \langle x, x\rangle + 2\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2} + \langle y, y\rangle = \big(\|x\| + \|y\|\big)^2$$

which is the triangle inequality. The other two properties of norms are trivial to verify.

We have shown that every inner product space is also a normed space. An inner product space that is complete with respect to its norm (i.e., a Banach space) is called a *Hilbert space*. Hilbert spaces have been the subject of a substantial amount of study because their properties closely mirror those of the finite dimensional space $\mathbb{C}^n$. While every linear space has a basis (this follows from Zorn's lemma), every Hilbert space admits an orthonormal basis[4, p. 14]. In particular, if the Hilbert space $\mathcal{H}$ is separable, there is a countable set of vectors $\{e_n\} \subset \mathcal{H}$ so that, for every $x \in \mathcal{H}$,

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n \quad \text{and} \quad \|x\| = \left(\sum_{n=1}^{\infty} \langle x, e_n\rangle^2\right)^{1/2}.$$

In this way, Hilbert spaces act somewhat like an infinite dimensional version of $\mathbb{C}^n$ under the 2-norm. In fact, every separable Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{N})$.

Another important property of Hilbert spaces is given by the *Riesz representation theorem*:

**Theorem 1.1.17.** [4, p. 13] *Let $\mathcal{H}$ be a Hilbert space and, for each $u \in \mathcal{H}$, let $\varphi_u$ be the linear functional given by $\varphi_u(x) = \langle x, u\rangle$. Then every bounded linear functional $\varphi \in \mathcal{H}^*$ is of the form $\varphi_u$ for some $u \in \mathcal{H}$. Furthermore, the map $u \mapsto \varphi_u$ is isometric.*

This is an extremely powerful result. The map $u \mapsto \varphi_u$ in the Riesz representation theorem is a conjugate-linear isomorphism of $\mathcal{H}$ with $\mathcal{H}^*$. For any normed linear space $X$, we can embed $X$ into $(X^*)^*$ by the map $x \mapsto x^*$ where $x^*(\phi) = \phi(x)$. For Hilbert spaces, this map is simply the Riesz map applied twice so $\mathcal{H} \cong (\mathcal{H}^*)^*$. If $\mathcal{H}$ is real-linear, then the map $u \mapsto \varphi_u$ is a linear isomorphism and $\mathcal{H} \cong \mathcal{H}^* \cong (\mathcal{H}^*)^*$.

Given a linear operator $T \in \mathcal{B}(\mathcal{H})$, the Reisz representation theorem also allows us to define an operator $T^* \in \mathcal{B}(\mathcal{H})$ with the property that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

This operator $T^*$ is called the *adjoint* of $T$. If our Hilbert space is the space $\mathbb{C}^n$, with the usual inner product $\langle u, v\rangle = v^*u$ then $T^*$ corresponds directly to the conjugate transpose of the matrix of $T$. That $T$ is linear is a direct computation. For any $x \in \mathcal{H}$,

$$\|Tx\| = \left\langle Tx, \frac{Tx}{\|Tx\|}, \right\rangle \leq \sup_{\|y\|=1} \langle Tx, y\rangle = \sup_{\|y\|=1} \langle x, T^*y\rangle \leq \sup_{\|y\|=1} \|x\|\|T^*y\| = \|x\|\|T^*\|$$

so $\|T\| \leq \|T^*\|$. Since $T = T^{**}$, we may reverse the inequality, and we have that $\|T\| = \|T^*\|$.

There is another important relationship between the norm of an operator and its adjoint. For any operator $T \in B(\mathcal{H})$ and any $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|x\|\|T^*Tx\| \leq \|T^*T\|$$

so, taking the supremum on the left-hand side over all $x$ with $\|x\| = 1$, we have $\|T\|^2 \leq \|T^*T\|$. Applying the norm property $\|ST\| \leq \|S\|\|T\|$, we have

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

so $\|T^*T\| = \|T\|^2$. This equality is interesting because, even though proving it heavily relied on the properties of Hilbert spaces and their inner products, it can be stated entirely algebraically without any reference to the underlying space $\mathcal{H}$.

Before we finish our discussion of Hilbert spaces, we should mention an important class of algebras derived from them called *operator algebras*.

**Definition 1.1.18.** A normed algebra $\mathcal{A}$ is an *operator algebra* if there is some Hilbert space $\mathcal{H}$ so that $\mathcal{A}$ is a normed-closed subalgebra of $\mathcal{B}(\mathcal{H})$.

Obviously, any norm-closed subalgebra of an operator algebra is, itself, an operator algebra. Note that the above definition says nothing about the adjoint operation $T \mapsto T^*$. The only requirements of an operator algebra are that it must be norm-closed (i.e., a Banach algebra) and that it is a subalgebra of $\mathcal{B}(\mathcal{H})$. For an operator algebra $\mathcal{A} \leq \mathcal{B}(\mathcal{H})$, every element $a \in \mathcal{A}$

has some adjoint $a^* \in \mathcal{B}(\mathcal{H})$ but $a^*$ may not live in $\mathcal{A}$. An operator algebra $\mathcal{A}$ is said to be self-adjoint if $a^* \in \mathcal{A}$ for every $a \in \mathcal{A}$. We will discuss self-adjoint operator algebras, also called $C^*$-algebras, in more detail in the next section.

### 1.1.5 $C^*$-algebras

Given an algebra $\mathcal{A}$, an *involution* on $\mathcal{A}$ is a conjugate-linear map $a \mapsto a^*$ so that

$$(a^*)^* = a \quad \text{and} \quad (ab)^* = b^*a^*$$

for all $a, b \in \mathcal{A}$. An algebra together with such a map is called a *$*$-algebra*. If $\mathcal{A}$ is also a Banach algebra, then $\mathcal{A}$ is called a *Banach $*$-algebra*. We say that the set $\mathcal{B}$ is a $*$-subalgebra of $\mathcal{A}$ if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ and $b^* \in \mathcal{B}$ for all $b \in \mathcal{B}$. A homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism if it preserves the adjoint, i.e., if $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}$.

Given an element $a$ of a $*$-algebra $\mathcal{A}$, $a$ is called *hermitian* or *self-adjoint* if $a^* = a$ and $a$ is called *normal* if $a^*a = aa^*$. If the $\mathcal{A}$ is unital and $a^*a = 1$ then $a$ is said to be *unitary*. If $\mathcal{A}$ is unital, the element 1 is both hermitian and unitary because

$$1^* = 11^* = (1^*)^*1^* = (11^*)^* = (1^*)^* = 1 \quad \text{and} \quad 1^*1 = 1^* = 1.$$

This implies that, for any complex number $\lambda \in \mathbb{C}$, $(\lambda 1)^* = \bar{\lambda}1^* = \bar{\lambda}1$.

The involution also interacts nicely with the spectrum. If $a \in \mathcal{A}$ is invertible with inverse $b$ then $a^*$ must have inverse $b^*$. In particular, if $\lambda \notin \sigma(a)$ then $\lambda - a$ is invertible so $(\lambda - a)^* = \bar{\lambda} - a^*$ is invertible and $\bar{\lambda} \notin \sigma(a^*)$. Therefore, for any $a \in \mathcal{A}$,

$$\sigma(a^*) = \sigma(a)^* = \{\bar{\lambda} : \lambda \in \sigma(a)\}.$$

**Definition 1.1.19.** A *$C^*$-algebra* is a Banach $*$-algebra $\mathcal{A}$ in which, for each $a \in \mathcal{A}$,

$$\|a^*a\| = \|a\|^2.$$

The equality given above is frequently called the $C^*$-condition. It is actually sufficient to simply assume that $\|a^*a\| \geq \|a\|^2$. From this weaker condition and properties of normed algebras, we have that

$$\|a\|\|a\| = \|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\|$$

so, for $a \neq 0$, $\|a\| \leq \|a^*\|$ and, by symmetry, $\|a^*\| \leq \|a^{**}\| = \|a\|$ so $\|a^*\| = \|a\|$. If $a = 0$ then $a^* = 0$ by because the involution is conjugate-linear. Therefore the involution is isometric. Also, $\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\| \leq \|a\|^2$ so we get the equality given above.

This condition on the norm is very powerful. For example, let $\mathcal{A}$ be a $C^*$-algebra and consider the unitary elements of $\mathcal{A}$. If $u \in \mathcal{A}$ is unitary then $1 = \|1\| = \|u^*u\| = \|u\|^2$ so $\|u\| = 1$. Furthermore, $\rho(u) \leq \|u\| = 1$ so $|\lambda| \leq 1$ for all $\lambda \in \sigma(u)$. However, by what we showed in our section on Banach algebras, $\sigma(u)^{-1} = \sigma(u^{-1}) = \sigma(u^*)$ and $|\lambda^{-1}| \leq \rho(u^*) \leq \|u^*\| = 1$ for all $\lambda \in \sigma(u)$. Therefore, $|\lambda| = 1$ for all $\lambda \in \sigma(u)$ and $\sigma(u) \subseteq \mathbb{T}$.

Now, let $h \in \mathcal{A}$ be hermitian. Then we can form the element $e^{ih}$ by using the convergent power series $\sum_{n=0}^{\infty} \frac{(ih)^n}{n!}$. Since the exponential function converges absolutely for all complex numbers, $e^a$ is well-defined for all $a \in \mathcal{A}$. Also, multiplication of exponential functions of commuting Banach algebra elements behaves in much the same way as multiplication of exponentials of complex numbers (this can be easily verified by direct computation). Using these facts,

$$(e^{ih})^* e^{ih} = e^{(ih)^*} e^{ih} = e^{-ih^*} e^{ih} = e^{-ih^* + ih} = e^0 = 1.$$

so $e^{-ih}$ is a unitary and $\sigma(e^{ih}) \subseteq \mathbb{T}$. Let $\lambda \in \sigma(h)$ and let $b = \sum_{n=1}^{\infty} i^n (h - \lambda)^{n-1}/n!$. Then

$$e^{ih} - e^{i\lambda} = (e^{ih - i\lambda} - 1)e^{i\lambda} = (h - \lambda)be^{i\lambda}.$$

Since $h - \lambda$ is not invertible and $h$ and $b$ commute, $e^{ih} - e^{i\lambda}$ is not invertible. Therefore, $e^{i\lambda} \in \sigma(e^{ih}) \subseteq \mathbb{T}$ and $\lambda \in \mathbb{R}$. Since this holds for all $\lambda \in \sigma(h)$, $\sigma(h) \subseteq \mathbb{R}$. These two facts about the spectra of hermitian and unitary elements both require the $C^*$ norm condition; they do not necessarily hold in general Banach $*$-algebras.

Before we proceed much further, let us consider a few examples of Banach and $C^*$-algebras.

**Example 1.1.20.** Let $\Omega$ be a locally compact Hausdorff space. We already know that $\mathcal{C}_0(\Omega)$ is a Banach algebra. The conjugation map $f \mapsto \bar{f}$ can be easily shown to be an involution. Since $\bar{f}f = |f|^2$, $\|\bar{f}f\| = \|f\|^2$ and $\mathcal{C}_0(\Omega)$ is a $C^*$-algebra. As we will soon see, every abelian $C^*$-algebra is isomorphic to $\mathcal{C}_0(\Omega)$ for some locally compact space $\Omega$.

**Example 1.1.21.** Let $\mathcal{H}$ be a Hilbert space. Then we already know that $B(\mathcal{H})$ is a Banach algebra because $\mathcal{H}$ is complete. We also showed above that, for any $T \in B(\mathcal{H})$, $\|T^*T\| = \|T\|^2$.

Therefore, $B(\mathcal{H})$ is a $C^*$-algebra. In fact, every self-adjoint operator algebra $\mathcal{A}$ is a $C^*$-algebra because it inherits the $C^*$-condition from $B(\mathcal{H})$.

One very powerful tool in any algebraic setting is *representation theory.* The goal of representation theory is to map an algebraic object about which we know little onto one that is well-studied. In this way, we can use what we know about the well-studied algebra to learn more about the algebra we are actually studying. For a $*$-algebra $\mathcal{A}$, a *representation* of $\mathcal{A}$ is a pair $(\mathcal{H}, \varphi)$ where $\mathcal{H}$ is a Hilbert space and $\varphi : \mathcal{A} \to B(\mathcal{H})$ is a $*$-homomorphism. This is useful because it is frequently easier to reason about operators on vectors than about general algebra elements. The representation $(\mathcal{H}, \varphi)$ is called *faithful* if $\varphi$ is injective. A faithful representation is even more powerful because it allows us to embed an isomorphic copy of $\mathcal{A}$ in $B(\mathcal{H})$ and $\mathcal{A}$ can be considered as a $*$-subalgebra of $B(\mathcal{H})$.

A famous result of Gelfand and Naimark [2, p. 94] is that every $C^*$-algebra has a faithful representation. (This representation is also isometric.) While the construction of the universal representation is outside the scope of this dissertation, it is worth mentioning because it tells us that every $C^*$-algebra can be thought of as a norm-closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In other words, every $C^*$-algebra is a self-adjoint operator algebra. It also means that the inequality $\|T\|^2 \leq \|T^*T\|$ we derived above using properties of Hilbert spaces, together with the inequality for normed algebras, encapsulates the entire relationship between the norm and the algebraic structure of a $C^*$-algebra.

We will finish our brief introduction to $C^*$-algebras with the following example of a $*$-algebra that is not quite a $C^*$-algebra.

**Example 1.1.22.** Let $\mathcal{A}(\mathbb{D})$ be the disc algebra as described in example 1.1.3. We know that $\mathcal{A}(\mathbb{D})$ is norm-closed and it is obviously a subalgebra of the $C^*$-algebra $\mathcal{C}(\mathbb{D})$ so $\mathcal{A}(\mathbb{D})$ is an operator algebra. The map

$$f \mapsto f^*; \quad f^*(z) = \overline{f(\bar{z})} \tag{1.3}$$

gives an involution on $\mathcal{A}(\mathbb{D})$. The Banach algebra $\mathcal{A}(\mathbb{D})$ together with this involution is a Banach $*$-algebra. It is also easy to see that this involution is isometric, i.e., $\|f^*\| = \|f\|$ for all $f \in \mathcal{A}(\mathbb{D})$. Consider the elements $f(z) = z + i$. By a simple calculation, $(f^*f)(z) = (z - i)(z + i) = z^2 + 1$

and, since norms in $\mathcal{A}(\mathbb{D})$ are supremums over the disc $\mathbb{D}$, $\|f^*f\| = 2 \neq 2^2 = \|f\|^2$. Therefore, this algebra is *not* a $C^*$-algebra.

### 1.1.6 $C^*$-algebras and Gelfand theory

We now wish to apply the Gelfand theory to $C^*$-algebras. Given an abelian unital $C^*$-algebra $\mathcal{A}$, theorem 1.1.16 gives a norm-decreasing algebra homomorphism $\varphi : \mathcal{A} \to \mathcal{C}(\Omega(\mathcal{A}))$. While both $\mathcal{A}$ and $\mathcal{C}(\Omega(\mathcal{A}))$ are $C^*$-algebras (c.f. example 1.1.20), this theorem is stated in the setting of Banach algebras, and this map need not preserve the involution structure of the algebra. Our first task will be to recover that structure.

**Theorem 1.1.23.** *Let $\mathcal{A}$ be a $C^*$-algebra and let $\tau \in \Omega(\mathcal{A})$. Then $\tau$ is a $*$-homomorphism.*

*Proof.* Let $a \in \mathcal{A}$. Let $h = \frac{1}{2}(a + a^*)$ and $k = \frac{i}{2}(a - a^*)$. Then $a = h + ik$ where both $h$ and $k$ are hermitian. By theorem 1.1.14, $\tau(h) \in \sigma(h) \subseteq \mathbb{R}$ and, similarly, $\tau(k) \in \mathbb{R}$. Then

$$\tau(a) = \tau(h + ik) = \tau(h) + i\tau(k) \quad \text{and} \quad \tau(a^*) = \tau((h + ik)^*) = \tau(h - ik) = \tau(h) - i\tau(k).$$

Since $a \in \mathcal{A}$ was arbitrary, $\tau(a^*) = \overline{\tau(a)}$ for all $a \in \mathcal{A}$. □

Since the multiplicative functionals are also $*$-homomorphisms, we can quite easily recover the $*$-structure of the Gelfand map. Let $\varphi : \mathcal{A} \to \mathcal{C}(\Omega(\mathcal{A}))$ and let $a \in \mathcal{A}$. Then

$$\varphi(a^*)(\tau) = \tau(a^*) = \overline{\tau(a)} = \overline{\varphi(a^*)(\tau)}$$

for all $\tau \in \Omega(a)$. Therefore, if we give $\mathcal{C}(\Omega(\mathcal{A}))$ the usual involution $f \mapsto \overline{f}$, the above calculation shows that $\varphi(a^*) = \varphi(a)^*$ and $\varphi$ is a $*$-homomorphism.

Next, consider theorem 1.1.5 which gave us an explicit formula for the spectral radius of an element in terms of a limit. For hermitian elements of a $C^*$-algebra, this result can be sharpened a good deal.

**Theorem 1.1.24.** *Let $\mathcal{A}$ be a $C^*$-algebra and let $a \in \mathcal{A}$ be hermitian. Then $\rho(a) = \|a\|$.*

*Proof.* Since $a$ is hermitian, $\|a^2\| = \|a^*a\| = \|a\|^2$. Therefore,

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{k\to\infty} \|a^{2^k}\|^{2^{-k}} = \lim_{k\to\infty} \|a\| = \|a\|.$$ □

This is interesting because, while $\|a\|$ is a norm property, $\rho(a)$ is purely algebraic. This means that, for a given algebra (not necessarily $*$-algebra), there is only one norm under which it could possibly be a $C^*$-algebra. The property above is only on the hermitian elements, but for an arbitrary $a \in \mathcal{A}$,

$$\|a\| = \sqrt{\|a\|^2} = \sqrt{\|a^*a\|} = \sqrt{\rho(a^*a)}.$$

We now have all of the pieces required for the Gelfand theory for $C^*$-algebras.

**Theorem 1.1.25.** *Let $\mathcal{A}$ be an unital abelian $C^*$-algebra. Then the Gelfand representation $\varphi : \mathcal{A} \to \mathcal{C}(\Omega(\mathcal{A}))$ is an isometric $*$-isomorphism.*

*Proof.* We already know from theorem 1.1.16 that $\varphi$ is a norm-decreasing $*$-homomorphism with $\rho(a) = \|\hat{a}\|_\infty$. Using the same logic we used above,

$$\|a\|^2 = \|a^*a\| = \rho(a^*a) = \|\varphi(a^*a)\|_\infty = \|\varphi(a)^*\varphi(a)\|_\infty = \|\varphi(a)\|_\infty^2$$

since $\mathcal{C}(\Omega(\mathcal{A}))$ is also a $C^*$-algebra. Since $\varphi$ is isometric, $\ker\varphi = 0$ and $\varphi$ is injective. The image of $\varphi$, $\varphi(\mathcal{A})$ is obviously a self-adjoint $*$-subalgebra of $\mathcal{C}(\Omega(\mathcal{A}))$. Since $\varphi$ is isometric and $\mathcal{A}$ is a Banach algebra, $\varphi(\mathcal{A})$ norm-closed. Since $\mathcal{A}$ is unital, $\varphi(\mathcal{A})$ contains the constants. Also, for $\tau_1, \tau_2 \in \Omega(\mathcal{A})$, if $\hat{a}(\tau_1) = \hat{a}(\tau_2)$ for all $\hat{a} \in \varphi(\mathcal{A})$ then $\tau_1 = \tau_2$; in other words, $\varphi(\mathcal{A})$ separates the points of $\Omega(\mathcal{A})$. Therefore, by the Stone-Weierstrass theorem [4, p. 145], $\varphi(\mathcal{A}) = \mathcal{C}(\Omega(\mathcal{A}))$ and $\varphi$ is an isomorphism. □

One important consequence of this stronger version of the Gelfand theory is that the multiplicative functionals on $\mathcal{A}$ separate the points of $\mathcal{A}$. Let $\mathcal{A}$ be an abelian unital $C^*$-algebra and suppose $a, b \in \mathcal{A}$ with $\tau(a) = \tau(b)$ for every $\tau \in \Omega(\mathcal{A})$ then $\hat{a}(\tau) = \hat{b}(\tau)$ for every $\tau \in \Omega(\mathcal{A})$ and $\hat{a} = \hat{b}$ as functions on $\Omega(\mathcal{A})$. Since $\mathcal{A} \cong \mathcal{C}(\Omega(\mathcal{A}))$, this implies that $a = b$. This is not necessarily true in Banach algebras.

Everything we proved about the Gelfand representation was done in the setting of unital abelian Banach and $C^*$-algebras. While the requirement that $\mathcal{A}$ be unital makes the proofs somewhat simpler, it is not necessary. If the Banach algebra $\mathcal{A}$ is not unital, $\Omega(\mathcal{A})$ is only locally compact instead of compact and $\varphi : \mathcal{A} \to C_0(\Omega(\mathcal{A}))$ rather than $\mathcal{C}(\Omega(\mathcal{A}))$. With those two changes, theorems 1.1.16 and 1.1.25 still hold even for non-unital abelian Banach and $C^*$-algebras.

Our two primary examples of $C^*$-algebras were $B(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space and $\mathcal{C}_0(\Omega)$ for some locally compact space $\Omega$. Earlier, we discussed the Gelfand-Naimark representation which allows us to embed any $C^*$-algebra isometrically into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The theorem above gives us the corresponding result for $\mathcal{C}_0(\Omega)$; namely that, for any abelian $C^*$-algebra $\mathcal{A}$, $\Omega(\mathcal{A})$ is locally compact and $\mathcal{A}$ is isometrically isomorphic to $\mathcal{C}_0(\Omega(\mathcal{A}))$.

This brings us to a very important tool in the study of $C^*$-algebras called the *functional calculus*. For a subset $S$ of a $C^*$-algebra $\mathcal{A}$, we define $C^*(S)$ to be the smallest norm-closed $*$-subalgebra of $\mathcal{A}$ containing $S$. This is also called the $C^*$-algebra *generated by* $S$. If $a$ is a normal element of some $C^*$-algebra $\mathcal{A}$, then $C^*(a,1)$ is abelian since since it is the norm-closure of the polynomials in $a$ and $a^*$ and $a$ and $a^*$ commute. In order for this to be useful, we need the following theorem:

**Theorem 1.1.26.** [2, p. 41] *Let $\mathcal{B}$ be a $C^*$-subalgebra of a unital $C^*$-algebra $\mathcal{A}$ with $1_{\mathcal{A}} \in \mathcal{B}$. Then, for all $b \in \mathcal{B}$, $\sigma_{\mathcal{A}}(b) = \sigma_{\mathcal{B}}(b)$.*

While the proof of this theorem is outside the scope of the current discussion, it's consequences are easily applicable. In short, this means that, for any $a \in \mathcal{A}$, we can restrict to our algebra to $C^*(a,1)$ without changing the spectrum of $a$. Now, let $\mathcal{A}$ be a unital $C^*$-algebra and fix a normal element $a \in \mathcal{A}$. By what we proved above, the gelfand map $\varphi : C^*(a,1) \to C_0(\Omega(C^*(a,1)))$ is an isometric $*$-isomorphism. Consider the map $\hat{a} : \Omega(C^*(a,1)) \to \sigma(a)$. This map is continuous and we know from theorem 1.1.14 that $\hat{a}$ is surjective. Furthermore, this map is actually a homeomorphism.

**Theorem 1.1.27.** *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $a \in \mathcal{A}$ be normal. Then $\hat{a} : \Omega(C^*(a,1)) \to \sigma(a)$ is a homeomorphism.*

*Proof.* We already know that $\hat{a}$ is continuous and surjective. Let $\tau_1, \tau_2 \in \Omega(C^*(a,1))$ with $\hat{a}(\tau_1) = \hat{a}(\tau_2)$. Then $\tau_1(a) = \tau_2(a)$ and $\tau_1(p(a)) = \tau_2(p(a))$ for any complex polynomial $p(z)$. By the continuity of $\tau_1$ and $\tau_2$, $\tau_1 = \tau_2$ on all of $C^*(a,1)$. Since $\tau_1$ and $\tau_2$ were arbitrary, $\hat{a}$

is also injective. Finally, since $\sigma(a)$ and $\Omega(\mathcal{C}^*(a,1))$ are both compact and Hausdorff, $\hat{a}$ is a homeomorphism. $\square$

Since $\Omega(C^*(a,1))$ and $\sigma(a)$ are homeomorphic, the $C^*$-algebras $\mathcal{C}(\Omega(C^*(a,1)))$ and $\mathcal{C}(\sigma(a))$ are easily seen to be isometrically isomorphic. This brings us to the following theorem which is the basis for the functional calculus:

**Theorem 1.1.28.** *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $a \in \mathcal{A}$ be normal. Then there is a unique unital $*$-homomorphism $\varphi : \mathcal{C}(\sigma(a)) \to \mathcal{A}$ so that $\varphi(z) = a$ where $z$ is the embedding of $\sigma(a)$ into $\mathbb{C}$. Furthermore, $\varphi$ is isometric and $\varphi(\mathcal{C}(\sigma(a))) = C^*(a,1)$.*

*Proof.* First, we show $\varphi$ is isometric if it exists. Suppose $\varphi : \mathcal{C}(\sigma(a)) \to \mathcal{A}$ is any $*$-homomorphism with $\varphi(z) = a$. Then for any polynomial $p$ in $z$ and $\bar{z}$, $\varphi(p(z)) = p(\varphi(z)) = p(a)$ and, by the spectral mapping theorem, $\sigma(\varphi(p(z))) = \sigma(p(a)) = p(\sigma(a))$ so $\rho(\varphi(p(z))) = \|p(z)\|_\infty$ when $p(z)$ is considered as an element of $\mathcal{C}(\sigma(a))$. Finally,

$$\|\varphi(p(z))\| = \sqrt{\rho(\varphi(p(z))^*\varphi(p(z)))} = \sqrt{\rho(\varphi(\bar{p}(z)p(z)))} = \sqrt{\|\bar{p}(z)p(z)\|} = \sqrt{\||p(z)|^2\|} = \|p(z)\|$$

so $\varphi$ is isometric on a dense subset of $\mathcal{C}(\sigma(a))$ and must therefore be isometric on all of $\mathcal{C}(\sigma(a))$. This also shows that $\varphi$ is unique if it exists.

Let $\psi : C^*(a,1) \to \mathcal{C}(\Omega(\mathcal{C}(a,1)))$ be the Gelfand representation of $C^*(a,1)$. Define $\varphi : \sigma(a) \to \mathcal{A}$ by $\varphi(f) = \psi^{-1}(f \circ \hat{a})$. Then $\varphi$ is a $*$-homomorphism because $\psi^{-1}$ is an isomorphism and the map $f \mapsto f \circ \hat{a}$ is clearly a $*$-homomorphism. Also, $\varphi(z) = \psi^{-1}(z \circ \hat{a}) = \psi^{-1}(\hat{a}) = a$. Clearly, $\varphi(\mathcal{C}(\sigma(a))) \subseteq C^*(a,1)$ since $\mathcal{C}(\sigma(a))$ is the uniform closure of the polynomials on $\sigma(a)$. Since $\mathcal{C}(\sigma(a))$ contains the polynomials in $z$ and $\bar{z}$, $\varphi(\mathcal{C}(\sigma(a)))$ contains all of the polynomials in $a$ and $a^*$ and, since $\varphi$ is isometric, $\phi(\mathcal{C}(\sigma(a)))$ is closed so $\phi(\mathcal{C}(\sigma(a))) = C^*(a,1)$. $\square$

This theorem gives us the basics of the functional calculus on normal elements of $C^*$-algebras. For a normal element $a$ of a $C^*$-algebra $\mathcal{A}$, if $f \in \mathcal{C}(\sigma(a))$, the theorem above allows us to define an element $f(a)$ by $f(a) = \varphi(f)$ where $\varphi : \mathcal{C}(\sigma(a)) \to \mathcal{A}$ is the homomorphism given above. This notation is standard in $C^*$-algebra theory and is very useful. This notation is also entirely consistent with the previous notation $p(a)$ where $p$ is a polynomial. This mapping has the very important property that, for any $f \in \mathcal{C}(\sigma(a))$ and any $\tau \in \Omega(\mathcal{A})$,

$$\tau(f(a)) = f(\tau(a)).$$

Since $\tau$ is a $*$-homomorphism, this obviously holds if $f$ is a polynomial. For a general continuous function $f$ on $\sigma(a)$, $f = \lim_{n\to\infty} p_n$ for some polynomials $p_n$ and

$$\tau(f(a)) = \tau(\varphi(f)) = \tau(\varphi(\lim_{n\to\infty} p_n)) = \lim_{n\to\infty} \tau(\varphi(p_n)) = \lim_{n\to\infty} \tau(p_n(a)) = \lim_{n\to\infty} p_n(\tau(a)) = f(\tau(a))$$

since both $\varphi$ and $\tau$ are continuous. This shows us better picture of the relationship between $\sigma(a)$ and $\Omega(C^*(a,1))$ which we already knew were homeomorphic.

As an example of this functional calculus, we can now prove a more general version of the spectral mapping theorem.

**Theorem 1.1.29** (Spectral Mapping Theorem)**.** *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $a \in \mathcal{A}$ be normal. Then, for any $f \in \mathcal{C}(\sigma(a))$,*

$$\sigma(f(a)) = f(\sigma(a))$$

*and $g(f(a)) = (g \circ f)(a)$ whenever $g \in \mathcal{C}(f(\sigma(a)))$.*

*Proof.* Fix $f \in \mathcal{C}(\sigma(a))$ and let $\varphi : \mathcal{C}(\sigma(a)) \to \mathcal{C}^*(a,1)$ as given in theorem 1.1.28. Then, for any $\tau \in \Omega(C^*(1,a))$, $\tau(\varphi(f)) \in \sigma(f)$ since $\tau \circ \varphi \in \Omega(\mathcal{C}(\sigma(a)))$ and, since $\tau$ was arbitrary, $\sigma(\varphi(f)) \subseteq \sigma(f)$ by theorem 1.1.14. Since $\varphi$ is an isomorphism, this works in reverse and $\sigma(f) \subseteq \sigma(\varphi(f))$. Therefore, since $\varphi(f) = f(a)$ and $\sigma(f) = f(\sigma(a))$, $\sigma(f(a)) = f(\sigma(a))$. For the last part we need only observe that, for every $\tau \in \Omega(C^*(a,1))$,

$$\tau(g(f(a))) = g(\tau(f(a))) = g(f(\tau(a))) = (g \circ f)(\tau(a)) = \tau((g \circ f)(a))$$

and, since the multiplicative functionals separate the points of $C^*(a,1)$, $g(f(a)) = (g \circ f)(a)$. $\square$

Another important topic in $C^*$-algebras is the study of the *positive* elements of a $C^*$-algebra. If $a \in \mathcal{A}$ is hermitian then we showed earlier that $\sigma(a) \subseteq \mathbb{R}$.

**Definition 1.1.30.** Let $\mathcal{A}$ be a $C^*$-algebra and let $a \in \mathcal{A}$. Then $a$ is *positive* if $a$ is hermitian and $\lambda \geq 0$ for every $\lambda \in \sigma(a)$.

As an easy application of the functional calculus, we can show that every positive element has a positive square root.

**Theorem 1.1.31.** *Let $\mathcal{A}$ be a unital $C^*$-algebra and let $a \in \mathcal{A}$ be positive. Then there is a positive element $b \in \mathcal{A}$ so that $a = b^2$.*

*Proof.* Since $\sigma(a) \geq 0$, $f(x) = \sqrt{x} \in \mathcal{C}(\sigma(a))$. Let $\varphi : \mathcal{C}(\sigma(a)) \to \mathcal{A}$ be the map given in theorem 1.1.28 and let $b = f(a) = \varphi(f)$. Then, $b^2 = \varphi(f^2) = \varphi(z) = a$ since $f^2 = z$ on $\sigma(a)$. Also, since $a$ is hermitian and $f$ can be approximated using only real polynomials, $f(a)$ is also hermitian. Finally, $\sigma(f(a)) = f(\sigma(a)) \geq 0$ so $b = f(a)$ is positive. □

It is natural to ask where these positive elements come from. Let $\mathcal{A}$ be a unital $C^*$-algebra and let $a \in \mathcal{A}$. Obviously, $a^*a$ is hermitian. If $a \in \mathcal{A}$ is normal then, for every $\tau \in \Omega(C^*(a, 1))$, $\tau(a^*a) = \overline{\tau(a)}\tau(a) = |\tau(a)|^2 \geq 0$ so $\sigma(a^*a) \geq 0$ and $a^*a$ is positive. A deeper result is that $a^*a$ is positive even if $a$ is not normal.

**Theorem 1.1.32.** [2, p. 46] *Let $\mathcal{A}$ be a $C^*$-algebra and let $a \in \mathcal{A}$. Then $a^*a$ is positive.*

The previous two theorems put together show us that an element $a$ in a $C^*$-algebra $\mathcal{A}$ is positive if and only if it is of the form $b^*b$ for some $b \in \mathcal{A}$. This equivalence is fundamental to the study of $C^*$-algebras. It is interesting in its own right, but it also plays a fundamental role in the construction of the Gelfand-Naimark representation that we mentioned earlier. The primary objective of this dissertation is to prove this same result on a class of function algebras that are not $C^*$-algebras.

## 1.2 Proposed research

We now have enough background material that we can set our problem in its proper context. Consider example 1.1.22 of the disc algebra with involution $f \mapsto f^*$; $f^*(z) = \overline{f(\bar{z})}$. While $\mathcal{A}(\mathbb{D})$ has been the subject of a considerable amount of study as a Banach algebra, less is known about the involution. In this dissertation, we study some the properties of this involution and, in particular, characterize the *positive elements* of $\mathcal{A}(\mathbb{D})$. After characterizing the positive elements of $\mathcal{A}(\mathbb{D})$, we extend these results to the similarly defined Banach $*$-algebras on the annulus and on even more general domains.

First, let us consider where the Banach $*$-algebra $\mathcal{A}(\mathbb{D})$ fits in the landscape of other well-known algebras. We already showed above that $\mathcal{A}(\mathbb{D})$ is a Banach $*$-algebra that is *not* a $C^*$-algebra. Considered as a Banach algebra (ignoring the involution), $\mathcal{A}(\mathbb{D})$ is a subalgebra of $\mathcal{C}(\mathbb{D}^-)$ making $\mathcal{A}(\mathbb{D})$ an operator algebra. The $C^*$-envelope of $\mathcal{A}(\mathbb{D})$ (smallest $C^*$-algebra containing $\mathcal{A}(\mathbb{D})$) is actually $\mathcal{C}(\mathbb{T})$ which is a strict subalgebra of $\mathcal{C}(\mathbb{D}^-)$. The embedding of $\mathcal{A}(\mathbb{D})$ into $\mathcal{C}(\mathbb{T})$ is given by the restriction map $f \mapsto f|_{\mathbb{T}}$. This relationship between a holomorphic function and its boundary values will be of great importance to us as it forms the basis of Fourier analysis on the circle. The algebra $\mathcal{A}(\mathbb{D})$ can also be considered as a $*$-subalgebra of $\mathcal{C}([-1,1])$. The restriction map $f \mapsto f|_{[-1,1]}$ is a $*$-homomorphism from $\mathcal{A}(\mathbb{D})$ into $\mathcal{C}([-1,1])$ and, by the identity theorem from complex analysis, this map is injective. While this map is continuous, it is *not* bounded away from zero and the embedding of $\mathcal{A}(\mathbb{D})$ into $\mathcal{C}([-1,1])$ loses the norm structure of $\mathcal{A}(\mathbb{D})$. However, the $*$-algebra structure is preserved and this relationship will be important for us because the concept of positivity, which we wish to study, is well-defined on $\mathcal{C}([-1,1])$ since it is a $C^*$-algebra.

For a general $*$-algebra $\mathcal{A}$, a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is said to be *positive* if $\varphi(a^*a) \geq 0$ for every $a \in \mathcal{A}$ [6, p. 799]. It is well-known that the positive functionals on $\mathcal{A}(\mathbb{D})$ with this involution are precisely the functionals of the form

$$\varphi(f) = \int_{-1}^{1} f(x)\, d\mu(x)$$

where $\mu$ is a positive real Borel measure on $[-1,1]$. The proof for this is not difficult and is used as a homework problem in [5, p. 289]. What has not been characterized, are the positive elements of $\mathcal{A}(\mathbb{D})$. Indeed, it is not even clear what definition should be used.

For a $*$-algebra $\mathcal{A}$ (with no assumption of a norm), an element $a \in \mathcal{A}$ is usually said to be positive if it can be written as a sum $a = \sum_{i=1}^{n} b_i^* b_i$ for some $b_i \in \mathcal{A}$ [6, p. 798]. However, this definition is awkward in $\mathcal{A}(\mathbb{D})$ since it has no obvious relationship to the norm. For instance, it is not obvious whether or not the set of positive elements is closed. On the other hand, $\mathcal{A}(\mathbb{D})$ is a Banach algebra and so $\sigma(f)$ is well-defined and compact for every $f \in \mathcal{A}(\mathbb{D})$ so one might be inclined to try and apply the $C^*$ definition and say $a$ is positive if $\sigma(f) \geq 0$. However, this condition is too restrictive to be useful. For instance, if $f(z) = z^2$ then $f = g^*g$ where $g(z) = z$ so we would like $f$ to be positive but $\sigma(f) = f(\mathbb{D}^-) = \mathbb{D}^-$. More generally, $\sigma(f) = f(\mathbb{D}^-)$ so $\sigma(f) \geq 0$ implies that $f$ is real-valued which, since $f$ is holomorphic, implies that $f$ is constant.

In order to solve this problem, we turn to one of the two embeddings we discussed above, namely the map $\mathcal{A}(\mathbb{D}) \to \mathcal{C}([-1,1])$ given by $f \mapsto f|_{[-1,1]}$. Since this embedding preserves the $*$-algebraic structure of $\mathcal{A}(\mathbb{D})$, it is natural to to look to $\mathcal{C}([-1,1])$ for information on the positive elements of $\mathcal{A}(\mathbb{D})$. In particular, we will say that $f \in \mathcal{A}(\mathbb{D})$ is *positive* if $\sigma(f|_{[-1,1]}) = f([-1,1]) \geq 0$. Obviously, given $g \in \mathcal{A}(\mathbb{D})$, $(g^*g)([-1,1]) \geq 0$ because $(g^*g)(x) = \overline{g(\bar{x})}g(x) = |g(x)|^2$ for any $x \in [-1,1]$. Also clear is that any sum of positive elements is positive, so this definition is no stronger than the usual $*$-algebra definition of positivity. What is more surprising is that these two definitions are equivalent.

**Theorem 1.2.1.** *For each $f \in \mathcal{A}(\mathbb{D})$, the following are equivalent:*

1. *$f$ is positive, i.e., $f([-1,1]) \geq 0$,*
2. *$f = g^*g$ for some $g \in \mathcal{A}$,*
3. *$f = \sum_{i=1}^n g_i^* g_i$ for some $g_1, \ldots, g_n \in \mathcal{A}$,*
4. *$f = \lim_{n\to\infty} f_n$ where each $f_n$ is of the form given in 3.*

That (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) is obvious. Showing that (1) $\Rightarrow$ (2) is much more difficult and is the primary objective of this dissertation.

### 1.2.1 Basic definitions and theorems

Before we can prove theorem 1.2.1, we need lay the ground work with some basic theorems and definitions. Because we will need it later, many of of these will be stated in substantially more generality.

Let $G$ be a domain in $\mathbb{C}$. We will say that $G$ is *symmetric* if $G = G^* = \{\bar{z} : z \in G\}$. For any holomorphic function $f$ on a symmetric domain $G$, the function $f^*$ given by $f^*(z) = \overline{f(\bar{z})}$ is well-defined and holomorphic. We denote by $\mathcal{H}(G)$ the set of all holomorphic functions on

$G$. This, together with the involution $f \mapsto f^*$ gives us a $*$-algebra of functions. An element $h \in \mathcal{H}(G)$ is *hermitian* if $h = h^*$ or, equivalently, if $h(\bar{z}) = \overline{h(z)}$ for all $z \in G$. The following theorem gives equivalent conditions for $h \in \mathcal{H}(G)$ to be hermitian in terms of only it's values on the real line.

**Theorem 1.2.2.** *Let $G$ be a non-empty symmetric domain and let $f \in \mathcal{H}(G)$. Then the following are equivalent:*

1. *$f$ is hermitian, i.e. $f^* = f$,*

2. *$f(G \cap \mathbb{R}) \subseteq \mathbb{R}$,*

3. *$f(I) \subseteq \mathbb{R}$ for some open real interval $I \subseteq G$.*

*Proof.* That (1) $\Rightarrow$ (2) is immediate. By the definition of a complex domain, $G$ is connected so $G \cap \mathbb{R} \neq \emptyset$. Also, $G \cap \mathbb{R}$ and open in $\mathbb{R}$ so it contains an open real interval and (3) follows from (2).

Now suppose (3) and let $a \in I$. Since $f(I) \subseteq \mathbb{R}$, the coefficients of the Taylor series for $f$ at $a$ are real-valued. Since the Taylor series converges to $f$ on some disc $D$ containing $a$, $f = f^*$ on $D$. Since $f^*$ is well-defined and holomorphic on $G$, $f = f^*$ on $G$ by the identity theorem. $\square$

Tying this into our discussion above, if we look at $h \in \mathcal{A}(\mathbb{D})$, this says that $h = h^*$ if and only if $h|_{[-1,1]}$ is real-valued. Therefore, the hermitian elements of $\mathcal{A}(\mathbb{D})$ are precisely those elements which map to hermitian elements of $\mathcal{C}([-1,1])$. (This fact can also be derived directly from the injectivity of the map $f \mapsto f|_{[-1,1}$.) While nowhere close to a proof, this at least suggests that looking at the map from $\mathcal{A}(\mathbb{D})$ to $\mathcal{C}([-1,1])$ is in the right direction.

Given a symmetric domain $G$, we define the algebra $H^\infty(G)$ of all bounded holomorphic functions on $G$ and the algebra $\mathcal{A}(G)$ of bounded holomorphic functions on $G$ that have continuous extension to $G^-$. Clearly from the definition, $\mathcal{A}(G)$ is a subalgebra of $H^\infty(G)$. Under the uniform norm and the involution $(\cdot)^*$, both $\mathcal{A}(G)$ and $H^\infty(G)$ are Banach $*$-algebras. In the case where the domain $G$ is bounded, the boundedness requirement on $\mathcal{A}(G)$ is redundant since $G^-$ is compact and every continuous function on a compact set is bounded. However, in the case that $G$ is unbounded, this is important for two reasons. First, it ensures that $\mathcal{A}(G)$ is a subalgebra of $H^\infty(G)$. Second, and more importantly, if $\partial G$ is compact, the boundedness of $f \in \mathcal{A}(G)$ implies that $f(\infty)$ is a removable singularity and the function $f(1/z)$ is also well-defined. This will be extremely important when we try to generalize our results to other, possibly unbounded, domains.

**Definition 1.2.3.** Let $G$ be a symmetric domain and let $f \in \mathcal{H}(G)$. We say $f$ is *positive* if

$$f(G \cap \mathbb{R}) \geq 0.$$

Since $\mathcal{A}(G)$ and $H^\infty(G)$ are both subalgebras of $\mathcal{H}(G)$, we can apply the same definition to them. Also, it is clear from theorem 1.2.2 that every positive element of $\mathcal{H}(G)$ is hermitian. For any $g_1, \ldots, g_n \in \mathcal{H}(G)$,

$$\left(\sum_{i=1}^{n} g_i^* g_i\right)(x) = \sum_{i=1}^{n} g_i^*(x) g_i(x) = \sum_{i=1}^{n} \overline{g_i(x)} g_i(x) = \sum_{i=1}^{n} |g_i(x)| \geq 0$$

so every element of $\mathcal{H}(G)$, $\mathcal{A}(G)$, or $H^\infty(G)$ that is positive according to the usual $*$-algebra definition is also positive with respect to definition 1.2.3. We will use this definition of positivity throughout this dissertation.

# CHAPTER 2. POSITIVITY ON THE DISC

We begin our study of positivity in function algebras with the disc algebra $\mathcal{A}(\mathbb{D})$ as the theory is somewhat simpler there. We will then extend the techniques used in the disc to study algebras of holomorphic functions on more complicated domains.

Let $f$ be a positive holomorphic function on $\mathbb{D}$ that is non-vanishing. Then we know from complex analysis [7, p. 95] that $f$ has a holomorphic logarithm, i.e., there is some holomorphic function $h$ on $\mathbb{D}$ such that $f(z) = e^{h(z)}$ for all $z \in \mathbb{D}$. Since $f$ is positive, the complex part of $h$ is a constant multiple of $2\pi i$ on $(-1, 1)$. Therefore, we may choose $h$ so that $h$ is real-valued on $(-1, 1)$ without changing $f$. By theorem 1.2.2, this implies that $h$ is hermitian. Clearly, if $h$ has continuous extension to $\mathbb{D}^-$ then $g = e^{h/2}$ also has continuous extension to $\mathbb{D}^-$ and $f = g^*g$. However, the requirement that $f$ be continuous on $\mathbb{D}^-$ is not sufficient to guarantee that $h$ is continuous on $\mathbb{D}^-$. For example, the function $f(z) = 1 - z$ is non-vanishing on $\mathbb{D}$ but $\log(1 - z)$ has a branch point at $z = 1$ so every logarithm $h$ of $f$ is discontinuous at 1.

The situation gets even more complex if we allow the function $f$ to have zeros in $\mathbb{D}$. In the general case, a function $f \in \mathcal{A}(\mathbb{D})$ may have infinitely many zeros in $\mathbb{D}$ and every one of those zeros is a branch point of the multi-valued function $\sqrt{f}$. In this case it is a challenge even to find a holomorphic function $g$ with $f = g^*g$, much less to ensure that $g$ has continuous extension to $\mathbb{D}^-$.

In order to handle these difficulties, we require the classical BSF factorization theory on the disc. Therefore, before we go on with proofs about positivity in $\mathcal{A}(\mathbb{D})$ we need to cover a good deal more background material.

## 2.1 Factorization theory

### 2.1.1 Holomorphic functions and their boundary values

In order to study functions on the disc, we must first see how a holomorphic (or harmonic) function on $\mathbb{D}$ relates to its boundary values on the circle $\mathbb{T}$. Given a function $f$ on the disc, we define the functions $f_r : [-\pi, \pi] \to \mathbb{C}$ by $f_r(t) = f(re^{it})$ for each $r < 1$. Whenever it is well-defined to do so, we extend this notation to $r = 1$ to represent the values of $f$ on the circle. Since not all holomorphic functions on $\mathbb{D}$ have meaningful boundary values, we must always be careful to ensure that $f_1$ is well-defined.

Our study of boundary values begins with the well-known *Cauchy integral formula* from complex analysis [7, p. 73]. It can be stated as follows:

**Theorem 2.1.1** (Cauchy integral formula)**.** *Let $G \subseteq \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be holomorphic. For any open ball $B(a, r)$ with $\overline{B(a, r)} \subseteq G$, $f$ is infinitely differentiable on $B(a, r)$ and*

$$f^{(n)}(a) = \frac{n!}{2\pi} \int_{-\pi}^{\pi} \frac{f(re^{it})}{(re^{it} - a)^{n+1}} re^{it}\, dt \tag{2.1}$$

*for $n \geq 0$. Furthermore, for all $z \in B(a, r)$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n.$$

If we restrict the above theorem to the discs centered at the origin and use the notation we defined above, equation 2.1 becomes

$$f^{(n)}(0) = \frac{n!}{2\pi} \int_{-\pi}^{\pi} \frac{f_r(t)}{(re^{it})^{n+1}} re^{it}\, dt = r^{-n} \frac{n!}{2\pi} \int_{-\pi}^{\pi} f_r(t) e^{-int}\, dt$$

and the $n$'th coefficient of the power series expansion for $f$ about 0 is given by

$$\frac{f^{(n)}(0)}{n!} = r^{-n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(t) e^{-int}\, dt$$

for any $r \in (0, 1)$.

Now suppose that the $f_r$ are uniformly bounded in $L^1[-\pi, \pi]$, i.e. that there is some $M \in \mathbb{R}$ so that $\|f_r\|_1 \leq M$ for all $r < 1$. Let $T$ be the compact Hausdorff space given by taking $[-\pi, \pi]$ and identifying the endpoints and let $B$ be the set of all measures $\mu$ on $T$ with $\|\mu\| \leq M$. For each $r \in (0,1)$, let $\mu_r$ be the measure on $T$ given by $\mu_r(E) = \int_E f_r \, dm$ where $m$ represents Lebesgue measure. Then $\mu_r$ is a finite complex Borel measure on $T$ with $\|\mu_r\| = \|f_r\|_1 \leq M$ so $\mu_r \in B$. Note that, since the span of the functions $e^{-int}$ is uniformly dense in $\mathcal{C}(T)$, the space $\mathcal{C}(T)$ is separable. By the Banach-Alaoglu theorem [4, p. 130], $B$ is compact with respect to the weak-$*$ topology and, since $\mathcal{C}(T)$ is separable, $B$ is metrizable [4, p. 134] so $B$ is sequentially compact. In particular, for any sequence $r_n \to 1$, $\{\mu_{r_n}\}$ is a sequence of measures in $B$ so there is some subsequence $\{n_k\}$ and some measure $\mu \in B$ so that $\mu_{r_{n_k}} \to \mu$ in the weak-$*$ topology.

Now suppose that $r_k$ be any sequence in $(0,1)$ with $r_k \to 1$ so that $\mu_{r_k}$ converges to some measure $\mu$ in the weak-$*$ topology on measures. Then, for any $n \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} d\mu(t) = \lim_{k\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} d\mu_{r_k}(t) = \lim_{k\to\infty} \frac{r_k^n f^{(n)}(0)}{n!} = \frac{f^{(n)}(0)}{n!}.$$

Notice that the quantity on the right is independent of the choice of our sequence $r_k$. Therefore, if $r_k'$ is another sequence in $(0,1)$ with $r_k' \to 1$ and $\mu_{r_k'} \to \mu'$, we must have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} d\mu(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int} d\mu'(t)$$

for all $n \in \mathbb{Z}$. Since the linear span of the functions $e^{-int}$ is uniformly dense in the continuous functions on $T$, we have that $\int f \, d\mu = \int f \, d\mu'$ for any continuous $f$ on $T$. Therefore, by the Riesz representation theorem for continuous functions [8, p. 40], $\mu = \mu'$. We have just shown that, given any sequence $r_n \to 1$ there is a subsequence $n_k$ so that $\mu_{r_{n_k}} \to \mu$ where $\mu$ is independent of the sequence $r_n$. Therefore, as $r \to 1$, $\mu_r \to \mu$ in the weak-$*$ topology on measures on $T$.

What the above discussion shows is is that, while the boundary value function $f_1$ is not clearly well-defined, we do have a well-defined boundary measure as long as the $f_r$ are bounded in $L^1[-\pi,\pi]$ as $r \to 1$. The $L^1$-boundedness of the functions $f_r$ is is important because, without it, the boundary measure might not be finite if we could even define it at all. Another key component of our discussion was the fact that we have an explicit formula for the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_r(t)e^{int}dt$. This allowed us to show that each of the subsequences actually converged to the same measure.

### 2.1.2 Fourier series

Let us forget, for a moment, about holomorphic functions and focus merely on functions on the interval $[-\pi, \pi]$. The integral quantity that was so important in our discussion of boundary measures gives us what are called the *Fourier coefficients* of the function or measure.

**Definition 2.1.2.** Let $f \in L^1[-\pi, \pi]$. For $n \in \mathbb{Z}$, the $n^{\text{th}}$ *Fourier coefficient of* $f$, denoted $\hat{f}(n)$, is given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt.$$

The *Fourier series* for $f$ is the formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

The Fourier series is said to be *formal* because, while we can write down the infinite series, it need not converge. If the above definition is applied to $f_r$ for some holomorphic function $f$ on $\mathbb{D}$, then the Fourier series for $f_r$ is the power series for $f$ on the circle of radius $r$ and therefore converges. However, this need not happen in general.

It is clear that, given a function $f \in L^1[-\pi, \pi]$, the Fourier coefficients for $f$ are well-defined. What is also true, is that the Fourier coefficients uniquely determine the function. To see this, we first consider the partial sums $s_k(t)$ given by

$$s_k(t) = \sum_{n=-k}^{k} \hat{f}(n) e^{int}.$$

As discussed above, the $s_k$ need not converge. However, using these $s_k$, we can form the *Cesaro means* $\sigma_n$ of the Fourier series for $f$ by

$$\sigma_n(t) = \frac{1}{n}\big(s_0(t) + \cdots + s_{n-1}(t)\big). \tag{2.2}$$

While the $s_k$ may not converge in any meaningful way, the Cesaro means do:

**Theorem 2.1.3.** [9, p. 18] *Let $f \in L^p[-\pi, \pi]$ for $1 \le p < \infty$. Then the Cesaro means $\sigma_n$ for the Fourier series for $f$ converge to $f$ in $L^p[-\pi, \pi]$. If $f$ is continuous and $f(-\pi) = f(\pi)$, then $\sigma_n \to f$ uniformly.*

This tells us that, for $f \in L^p(\mathbb{T})$, the Fourier coefficients for $f$ uniquely determine $f$ up to a set of Lebesgue measure zero. This also implies that the polynomials in $z$ and $\bar{z}$ are dense in $L^p(\mathbb{T})$. There is one important case missing from the above theorem, namely the case where $p = \infty$. For this, we don't quite get $L^p$ convergence but rather get weak-$*$ convergence.

**Theorem 2.1.4.** [9, p. 19] *Let $f \in L^\infty[-\pi, \pi]$. Then the Cesaro means $\sigma_n$ for the Fourier series for $f$ converge to $f$ in the weak-$*$ topology on $L^\infty[-\pi, \pi]$.*

For a finite complex Borel measure $\mu$ on $[-\pi, \pi]$, we define the Fourier coefficients of $\mu$ by

$$\hat{\mu}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} d\mu(t).$$

In this case, we can get a similar uniqueness and convergence result.

**Theorem 2.1.5.** *Let $\mu$ be a finite complex Borel measure on $[-\pi, \pi]$ and let $\sigma_n$ be the Cesaro means for the Fourier series for $\mu$. Then $\sigma_n dt \to d\mu$ in the weak-$*$ topology on measures on $[-\pi, \pi]$. In other words, for each $h \in \mathcal{C}[-\pi, \pi]$,*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} h(t)\sigma_n(t)\ dt = \int_{-\pi}^{\pi} h(t)\ d\mu(t).$$

It is worth noting that none of the above convergence results rely in any way on the function or measure coming from a holomorphic function. The only requirement was that the measure (or induced measure in the case of a function) be of bounded variation. Before we finish our discussion of Fourier series, there are a couple of useful properties of Fourier series that are worth discussing. First, the Fourier transform is linear, i.e., $(\widehat{f + \alpha g})(n) = \hat{f}(n) + \alpha\hat{g}(n)$. This comes directly from the linearity of the integral. Second, given the Fourier series of a function, we can directly compute the Fourier coefficients of the conjugate function:

$$\widehat{\bar{f}}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \bar{f}(t)e^{-int}\ dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(t)e^{int}}\ dt = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{int}\ dt} = \overline{\hat{f}(-n)}.$$

In particular, $\Re(f) = \frac{1}{2}(f + \bar{f})$ so $\widehat{\Re(f)}(n) = \frac{1}{2}\Big(\hat{f}(n) + \overline{\hat{f}(-n)}\Big)$ for any $f \in L^1[-\pi, \pi]$.

### 2.1.3 Convolutions

Given any function $f \in L^1[-\pi, \pi]$ and any finite complex Borel measure $\mu$ on $[-\pi, \pi]$, we can form the *convolution* $f * \mu$ by

$$(f * \mu)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) d\mu(s)$$

where $f$ is considered to be $2\pi$-periodic. This newly formed function is in $L^1[-\pi, \pi]$ because

$$\begin{aligned}\int_{-\pi}^{\pi} |(f * \mu)(t)| \, dt &= \int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) d\mu(s) \right| dt \leq \int_{-\pi}^{\pi} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t-s)| \, d|\mu|(s) \right] dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \int_{-\pi}^{\pi} |f(t-s)| dt \right] d|\mu|(s) = \frac{1}{2\pi} \|f\|_1 |\mu|([-\pi, \pi]) < \infty.\end{aligned}$$

Another easily shown fact about Fourier coefficients is that convolution of functions corresponds to multiplication of Fourier coefficients. If $f \in L^1[-\pi, \pi]$ and $\mu$ is a finite Borel measure, we can use an exchange of integrals to obtain

$$\begin{aligned}\widehat{(f * \mu)}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f * \mu)(t) e^{-int} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) d\mu(s) \right] e^{-int} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) e^{-int} dt \right] d\mu(s) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) e^{-in(t-s)} dt \right] e^{-ins} d\mu(s) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(n) e^{-ins} d\mu(s) = \hat{f}(n) \hat{\mu}(n).\end{aligned}$$

The exchange of integrals is justified because both $\mu$ and $e^{-int} dt$ are finite measures.

Given a function $f$ on $\mathbb{D}$, we define the convolution $f * \mu$ by $(f * \mu)(re^{it}) = (f_r * \mu)(t)$ or

$$(f * \mu)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(ze^{-is}) d\mu(s).$$

If the function $f$ is holomorphic then $f * \mu$ is also holomorphic regardless of the measure $\mu$. To see this, fix $w \in \mathbb{D}$ and let $|w| < r_1 < 1$. Since $f$ is continuously differentiable on $\mathbb{D}$, there is some $M \in \mathbb{R}$ so that $|f'(z)| \leq M$ for $|z| \leq r_1$. Let $z_n \to w$ with $|z_n| \leq r_1$ for all $n$. Then, for all $t \in [-\pi, \pi]$,

$$\frac{f(z_n e^{-it}) - f(we^{-it})}{z_n - w} \to f'(we^{-it})$$

and the left-hand side is bounded by $M$. Therefore, by the bounded convergence theorem,

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(z_n e^{-it}) - f(we^{-it})}{z_n - w} \, d\mu(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(we^{-it}) \, d\mu(t).$$

Since this holds for all such sequences $z_n \to w$, the function $f * \mu$ is differentiable at $w$ and, since $w$ was arbitrary, $f$ is holomorphic by Goursat's theorem [7, p. 100]. Similar properties can be derived for convolutions with harmonic functions.

**Definition 2.1.6.** A function $u : G \to \mathbb{C}$ is *harmonic* if it is twice differentiable and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where $z = x + iy$.

Since the derivatives in the above definition are real partial derivatives, it is clear that $u : G \to \mathbb{C}$ is harmonic if and only if its real and imaginary parts are each harmonic. It is easily shown, by a simple calculation, that every holomorphic function is also a harmonic function. Therefore, the real and imaginary parts of a holomorphic function are harmonic. In the case where the domain of the function is actually a disc, we can say substantially more.

**Theorem 2.1.7.** [7, p. 43] *Let $u : D \to \mathbb{R}$ where $D$ is some disc in the complex plain. Then $u$ is harmonic if and only if it is the real part of some holomorphic function.*

If $u : \mathbb{D} \to \mathbb{C}$ is harmonic and $\mu$ is a finite Borel measure on $[-\pi, \pi]$ then $u * \mu$ is also harmonic. In the case where $u$ and $\mu$ are both real-valued, $u = \Re[f]$ for some holomorphic function $f$ on $\mathbb{D}$ and $u * \mu = \Re[f * \mu]$ is harmonic. If $u$ and $\mu$ are complex-valued then we can write $u = w + iv$ and $\mu = \nu + i\eta$ where $w$, $v$, $\nu$, and $\eta$ are real-valued. Then

$$u * \mu = w * \nu + i(v * \nu) + i(w * \eta) - v * \eta$$

which is harmonic because any linear combination of harmonic functions is also harmonic.

Next, we define the function $P : \mathbb{D} \to \mathbb{R}$ by

$$P(re^{i\theta}) = P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}.$$

This function is called *Poisson's kernel.* When considered as a function on the disc, Poisson's kernel is harmonic because

$$\frac{1 - r^2}{1 - 2r\cos(\theta) + r^2} = \Re\left[\frac{1 - 2ir\sin(\theta) + r^2}{1 - 2r\cos(\theta) + r^2}\right] = \Re\left[\frac{(1 + re^{it})(1 - re^{-it})}{(1 - re^{it})(1 - re^{-it})}\right] = \Re\left[\frac{1 + re^{it}}{1 - re^{it}}\right]$$

and the function on the right is the real part of the holomorphic function $\frac{1+z}{1-z}$. Next, we will compute the Fourier coefficients of $P_r$. First, observe that

$$P_r(t) = \Re\left[\frac{1+re^{it}}{1-re^{it}}\right] = \Re\left[\frac{2-(1-re^{it})}{1-re^{it}}\right] = \Re\left[\frac{2}{1-re^{it}} - 1\right] = 2\Re\left[\frac{1}{1-re^{it}}\right] - 1$$

We can rewrite this last term using conjugates as

$$2\Re\left[\frac{1}{1-re^{it}}\right] - 1 = \frac{1}{1-re^{it}} + \overline{\left(\frac{1}{1-re^{it}}\right)} - 1 = \frac{1}{1-re^{it}} + \frac{1}{1-re^{-it}} - 1.$$

Since $r < 1$, the two fractions on the right can be written as geometric series and we have

$$P_r(t) = \frac{1}{1-re^{it}} + \frac{1}{1-re^{-it}} - 1 = \sum_{n=0}^{\infty} r^n e^{int} + \sum_{n=0}^{\infty} r^n e^{-int} - 1 = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int}$$

and this series converges uniformly in $t$ for all $r < 1$. Since this power series converges uniformly to $P_r$ it is also the Fourier series of $P_r$ and $\hat{P}_r(n) = r^{|n|}$ for all $n \in \mathbb{Z}$. Also, since $P_r$ is non-negative, this immediately implies that $\|P_r\|_1 = 2\pi\hat{P}_r(0) = 2\pi$.

### 2.1.4 Boundary values of harmonic functions

Suppose that $f : \mathbb{D} \to \mathbb{C}$ is harmonic such that $\|f_r\|_1$ are bounded as $r \to 1$. We showed above that if $f$ is holomorphic then the $f_r$ converge in the weak-$*$ topology to a boundary measure $\mu$. The same proof also works if $f$ is merely harmonic because the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_r(t)e^{int}dt$ can still be expressed as a continuous function of $r$. More specifically, if $f$ is real-valued and harmonic and $n > 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_r(t)e^{int}dt = \hat{f}_r(n) = \widehat{\Re[g_r]}(n) = \frac{1}{2}\left(\hat{g}_r(n) - \overline{\hat{g}_r(-n)}\right) = \frac{1}{2}\frac{r^n g^{(n)}(0)}{n!}$$

where $g$ is holomorphic such that $f = \Re[g]$. Similar calculations can be performed for $n < 0$ and $n = 0$. If $f$ is not real-valued, then we can compute the Fourier coefficients for the real and imaginary parts of $f$ and combine them using the linearity property of the Fourier transform. In any case, we get the explicit formula required to extend our proof of the existence of a boundary measure to the harmonic case.

For a harmonic function $f$, this also shows us an important relationship between the Fouier coefficients of $f_r$ and the radius. In particular, $\hat{f}_r(n) = r^{|n|}\hat{f}_1(n)$ for all $r < 1$. (Even if $f_1$

does not exist, the polynomial relationship between $r$ and $\hat{f}_r(n)$ still holds.) Therefore, if $f, g : \mathbb{D} \to \mathbb{C}$ are harmonic and there is some $r_0 < 1$ so that $\hat{f}_{r_0}(n) = \hat{g}_{r_0}(n)$ for all $n \in \mathbb{Z}$, then $\hat{f}_r(n) = \hat{g}_r(n)$ for all $r < 1$ and $n \in \mathbb{Z}$. Applying theorem 2.1.3, this implies that $f_r = g_r$ for all $r < 1$ so $f = g$.

We now have enough pieces to finish our discussion of boundary values. Let $\mu$ be a finite complex Borel measure on $[-\pi, \pi]$ and define the function $f : \mathbb{D} \to \mathbb{C}$ by

$$f(re^{i\theta}) = (P_r * \mu)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(t) P_r(\theta - t)\, dt.$$

This function is called the *Poisson integral of* $\mu$. We know that $f$ is harmonic because it is the convolution of a harmonic function with a measure. We also know that $\|f_r\|_1$ is bounded as $r \to 1$ because $\|f_r\|_1 = \frac{1}{2\pi}\|P_r\|_1\|\mu\| = \|\mu\|$. Using the weak-$*$ convergence of the $f_r$ to $\mu$, we can see that $\hat{f}_r(n) = r^{|n|}\hat{\mu}(n)$ for all $r < 1$ and all $n \in \mathbb{Z}$.

Going about this from the other direction, suppose we start with a harmonic function $f : \mathbb{D} \to \mathbb{C}$ with $\|f_r\|$ bounded as $r \to 1$. Then we know that the $f_r$ converge in the weak-$*$ topology to some boundary measure $\mu$. If $g$ is the Poisson integral of $\mu$ then we know, by the multiplicative property of Fourier coefficients under convolution, that $\hat{g}_r(n) = \hat{P}_r(n)\hat{\mu}(n) = r^{|n|}\hat{\mu}(n)$. However, these are precisely the Fourier coefficients of $f_r$, so $g = f$.

This gives us a way to get from a harmonic function that is bounded in $L^1$ to a measure on the boundary and back again. The ability to completely characterize a harmonic function by its boundary values is crucial to the study of harmonic and holomorphic functions on the disc. Thus far, we have only discussed the general case where the boundary is given by a measure. Depending on the behavior of the $f_r$ as $r \to 1$, we can say a lot more.

**Theorem 2.1.8.** [9, p. 33] *Let $f : \mathbb{D} \to \mathbb{C}$ be harmonic. Then*

1. *For $1 < p \leq \infty$, $f$ is the Poisson integral of a function in $L^p[-\pi, \pi]$ if and only if $\|f_r\|_p$ is bounded as $r \to 1$.*
2. *$f$ is the Poisson integral of a function in $L^1[-\pi, \pi]$ if and only if the $f_r$ converge in $L^1$.*
3. *$f$ is the Poisson integral of a continuous function on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ if and only if the $f_r$ converge uniformly.*

4. *$f$ is the Poisson integral of a finite complex Borel measure on $[-\pi, \pi]$ if and only if the $f_r$ are bounded in $L^1$-norm.*

With the exception of the very strong case where $f_r \to f_1$ uniformly, the above theorem only talks about $L^p$ or weak-$*$ convergence. While enough for many discussions, it is frequently useful to have a more concrete picture of what happens as you approach the boundary. The following theorem by Fatou tells us that, at the points where the boundary measure is differentiable, we actually get pointwise convergence.

**Theorem 2.1.9** (Fatou)**.** [9, p. 34] *Let $\mu$ be a finite complex Borel measure on $[-\pi, \pi]$ and let $f$ be the harmonic function given by the Poisson integral of $\mu$, i.e.,*

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\ d\mu(\theta).$$

*Let $\theta_0$ be any point where $\mu$ is differentiable with respect to Lebesgue measure. Then*

$$\lim_{r \to 1} f_r(\theta_0) = \left(\frac{d\mu}{d\theta}\right)(\theta_0) = 2\pi\mu'(\theta_0).$$

At this point we can now substantiate a couple of claims we made in the introduction. First was that we claimed that $\mathcal{A}(\mathbb{D})$ is a subalgebra of $\mathcal{C}(\mathbb{T})$. This is now obvious since we have now seen that every function in $\mathcal{A}(\mathbb{D})$ is completely defined by its boundary values. Also, by the maximum modulus principle of complex analysis (c.f. [7, p. 128]), a function in $\mathcal{A}(\mathbb{D})$ takes on its maximum absolute value on the boundary so the norm on $\mathcal{A}(\mathbb{D})$ is the same as the norm on $\mathcal{C}(\mathbb{T})$. We also claimed that $\mathcal{C}(\mathbb{T})$ is actually the $C^*$-envelope of $\mathcal{A}(\mathbb{D})$ (the smallest $C^*$-algebra containing $\mathcal{A}(\mathbb{D})$.) Let $t : \mathcal{A}(\mathbb{D}) \to \mathcal{C}(\mathbb{T})$ be the embedding given by $f \mapsto f|_{\mathbb{T}}$. We know that $t(\mathcal{A}(\mathbb{D}))$ contains all of the complex polynomials. Applying the involution $f \mapsto \bar{f}$, we can see that that any $C^*$-algebra containing $t(\mathcal{A}(\mathbb{D}))$ must contain all the polynomials in $z$ and $\bar{z}$. Since, for any $f \in \mathcal{C}(\mathbb{T})$, the Cesaro means of $f$ are polynomials in $z$ and $\bar{z}$, we can see that any $C^*$-algebra containing $t(\mathcal{A}(\mathbb{D}))$ must be all of $\mathcal{C}(\mathbb{T})$.

We should also, at this point, revisit the topic of Cesaro means. Let $f : \mathbb{D} \to \mathbb{C}$ be harmonic. Thanks to the polynomial relationship between $r$ and $\hat{f}_r(n)$, $\hat{f}_1(n)$ is always well-defined even if $f_1$, as a function, is not. We can reformulate the Cesaro means as

$$\sigma_n(z) = \frac{1}{n}(s_0(z) + \cdots + s_{n-1}(z)) \quad \text{where} \quad s_k(z) = \sum_{n=1}^{k} \hat{f}_1(-n)\bar{z}^n + \sum_{n=0}^{k} \hat{f}_1(n)z^n. \tag{2.3}$$

For each $0 < r < 1$, the $(\sigma_n)_r$ are the Cesaro means for $f_r$ as originally formulated in (2.2). Suppose, for the moment, that $f_1$ is well-defined and continuous. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned}|\sigma_n(re^{i\theta}) - f(re^{i\theta})| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (\sigma_n(e^{it}) - f(e^{it})) P_r(\theta - t)\ dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n(e^{it}) - f(e^{it})| P_r(\theta - t)\ dt \leq \|(\sigma_n)_1 - f_1\|_\infty.\end{aligned}$$

Since $(\sigma_n)_1 \to f_1$ uniformly, this implies that $\sigma_n \to f$ uniformly on $\mathbb{D}^-$. Without the assumption that $f_1$ exists and is continuous, we can replace $f(z)$ with $f(rz)$ for some $r < 1$ to see that $\sigma_n \to f$ uniformly on any compact subset of $\mathbb{D}$.

There is one more convolution we need to mention before we finish this section. Consider the function $H_r(t) = \frac{1+re^{it}}{1-re^{it}}$. We used this function to show that $P$ was harmonic because $H$ is a holomorphic function with $P_r = \Re[H_r]$. Let $\mu$ be any finite real-valued Borel measure on $[-\pi, \pi]$. Then

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_r(\theta - t)\ d\mu(t)$$

is holomorphic because $H$ is holomorphic and

$$\Re\big[f(re^{i\theta})\big] = \Re\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} H_r(\theta - t)\ d\mu(t)\right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\ d\mu(t).$$

Therefore, $f$ is a holomorphic function whose real part has boundary measure $\mu$. This construction will be crucial to our discussion of factorization theory in the following sections.

### 2.1.5 Hardy spaces

We now have the background to introduce an important class of function spaces that we will use for the remainder of this dissertation.

**Definition 2.1.10.** For $1 \leq p \leq \infty$, *Hardy space* $H^p$ is defined by

$$H^p = \{f_1 \in L^p[-\pi, \pi] : \hat{f}_1(n) = 0 \text{ for all } n < 0\}$$

or, equivalently,

$$H^p = \{f \in \mathcal{H}(\mathbb{D}) : \|f_r\|_p \text{ is bounded as } r \to 1\}.$$

The two definitions given above are equivalent. If $f_1 \in L^p[-\pi,\pi]$, $1 \le p \le \infty$, then the Poisson integral of $f_1$ yields a harmonic function $f$ with $f_r \to f_1$ in $L^p[-\pi,\pi]$. Since $\hat{f}_1(n) = 0$ for all $n < 1$, the Cesaro means of the Fourier series for $f_1$ as given in (2.3) contain no $\bar{z}$ terms and are therefore holomorphic. Since the Cesaro means converge uniformly on compact subsets of $\mathbb{D}$ to $f$ this implies that $f$ is holomorphic.

For the other direction, let $1 \le p \le \infty$, and let $f \in \mathcal{H}(\mathbb{D})$ with $\|f_r\|_p$ bounded as $r \to 1$. If $p > 1$ then theorem 2.1.8 implies that $f_r \to f_1$ in $L^p[-\pi,\pi]$ as $r \to 1$. For any $n \in \mathbb{Z}$,

$$\hat{f}_r(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})e^{-int}\,dt = \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(re^{it})e^{-i(n+1)t} ie^{it} dt = \frac{1}{2\pi i}\oint_{\mathbb{T}} f(rz)z^{-1-n}dz.$$

If $n < 0$ then, since $f$ is holomorphic, the right-most integral above is a closed-curve line integral of a holomorphic function, so $\hat{f}_r(n) = 0$. Some care must be taken with the case $p = 1$. For that, we will need the following theorem:

**Theorem 2.1.11** (F. and M. Riesz)**.** [9, p. 47] *Let $\mu$ be a finite Borel measure on $[-\pi,\pi]$ so that*

$$\int e^{-int}\,d\mu(t) = 0 \text{ for } n = -1, -2, -3, \ldots.$$

*Then $\mu$ is absolutely continuous with respect to Lebesgue measure.*

This gives us the final piece in our equivalence. If $f$ is holomorphic with $\|f_1\|$ bounded as $r \to 1$, then the $f_r$ converge in the weak-$*$ topology to a boundary measure $\mu$. By the same calculation we did for $p > 1$, $\hat{f}_r(n) = 0$ for $n < 0$ and, by weak-$*$ convergence, the same holds for the Fourier coefficients of $\mu$. Therefore, by the above theorem, $\mu$ is absolutely continuous with respect to Lebesgue measure and the Radon-Nikodym derivative $f_1 = \frac{d\mu}{dm}$ is the desired boundary value function.

At this point, we need to reintroduce the disc algebra $\mathcal{A}(\mathbb{D})$ (originally introduced as example 1.1.3) and discuss a few of its properties. In the previous section, we saw that, if $f_1$ is a continuous function on $[-\pi,\pi]$ with $f_1(-\pi) = f_1(\pi)$, then the Poisson integral $f$ of $f_1$ is harmonic and $f_r \to f_1$ uniformly. Therefore, we can formulate the disc algebra equivalently as an algebra on the circle as follows:

$$\mathcal{A}(\mathbb{D}) = \{f_1 \in \mathcal{C}[-\pi,\pi] : f_1(-\pi) = f_1(\pi) \text{ and } \hat{f}_1(n) \text{ for all } n < 0\} \tag{2.4}$$

One useful property of the disc algebra is that the real parts of the functions $f$ in $\mathcal{A}(\mathbb{D})$ are dense in the real-valued $L^p$ functions on $[-\pi, \pi]$. To see this, let $f_1 \in L^p[-\pi, \pi]$ be real-valued and let $f$ be the Poisson integral of $f_1$. Then $f$ real-valued since $P_r$ is real-valued for each $r$ and, since $f$ is harmonic, $f$ is the real part of some holomorphic function $g$ on $\mathbb{D}$. Then, for each $r < 1$, $\hat{g}_r(n) = 0$ for all $n < 0$ so $g_r \in \mathcal{A}(\mathbb{D})$ in the sense of (2.4). Also, $\Re(g_r) = f_r \to f_1$ as $r \to \infty$ in $L^p$ so $f$ is in the $L^p$-closure of $\mathcal{A}(\mathbb{D})$. Algebras with this density property are called Dirchlet algebras.

Given an element $z = re^{i\theta} \in \mathbb{D}$, we will define the subalgebra $\mathcal{A}_z(\mathbb{D})$ of $\mathcal{A}(\mathbb{D})$ by

$$\mathcal{A}_z(\mathbb{D}) = \{f \in \mathcal{A}(\mathbb{D}) : f(z) = 0\}$$

This algebra has a similar property, namely that the real parts of the functions in $\mathcal{A}_z(\mathbb{D})$ are dense in the functions $f \in L^p[-\pi, \pi]$ with $f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t)\ dt = 0$. To see this, let $f \in L^p[-\pi, \pi]$ with $f(z) = 0$ and let $\varepsilon > 0$. Choose $g \in \mathcal{A}(\mathbb{D})$ with $\|f - g\|_1 < \varepsilon$. Then the function $h = g - g(z)$ is in $\mathcal{A}_z(\mathbb{D})$ and

$$\|f - h\| = \|f - g\| + |g(z)| \leq \varepsilon + \varepsilon \|P_r\|_\infty$$

and, since $P_r$ is bounded for any $r < 1$, we can make the right hand side arbitrarily small. These density facts are crucial in the proofs of Szegö's theorem and Jensen's inequality on $H^p(\mathbb{D})$ which follow. We begin with Szegö's theorem which can be stated as follows:

**Theorem 2.1.12** (Szegö)**.** *Let $\mu$ be a finite positive Borel measure that is absolutely continuous and let $h$ be the derivative of $\mu$ with respect to Lebesgue measure. Then*

$$\inf_{f \in \mathcal{A}_0(\mathbb{D})} \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - f(t)|^2 d\mu = \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \log h(t)\ dt\right].$$

As stated above, Szegö's theorem primarily concerns the $L^2(\mu)$ distance between the element 1 of $\mathcal{A}(\mathbb{D})$ and the subalgebra $\mathcal{A}_0(\mathbb{D})$. While this is an interesting fact in its own right, we are more concerned with what it says about the integral of $\log h$. A full proof of the above theorem may be found in [9, p. 48]. For our purposes, we only require half of the above equality, but in a more general form:

**Theorem 2.1.13.** *Let $h \in L^1[-1,1]$ be non-negative. Then, for any $z = re^{i\theta} \in \mathbb{D}$,*

$$\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log h(t) P_r(\theta - t) dt\right] \geq \inf_{f \in \mathcal{A}_z(\mathbb{D})} \frac{1}{2\pi}\int_{-\pi}^{\pi} h(t)|1 - f_1(t)|^2 P_r(\theta - t) dt.$$

*Proof.* First fix $z = re^{i\theta} \in \mathbb{D}$. For ease of notation, we will let $d\mu = \frac{1}{2\pi} P_r(\theta - t)dt$ on $[-\pi, \pi]$. Then, for any $f \in H^1(\mathbb{D})$, $f(z) = \int f \; d\mu$. Since $h \in L^1[-\pi, \pi]$ and $\mu$ is bounded by a constant multiple of Lebesgue measure, $h$ is $\mu$-integrable. Suppose, for the moment, that $\log h$ is $\mu$-integrable. Let $\lambda = \int \log h \; d\mu$ and let $g = \lambda - \log h$. Then $g(z) = \int g \; d\mu = 0$ and

$$\int he^g \; d\mu = \int e^\lambda \; d\mu = e^\lambda = \exp\left[\int \log h \; d\mu\right].$$

Let $g_n$ be a sequence of $L^\infty$ functions with $\int g_n \; d\mu = 0$ and $g_n \to g$ monotone (i.e., $g_{n+} \to g_+$ and $g_{n-} \to g_-$ monotone increasing). Then, by the monotone convergence theorem, $\int he^{g_n} \; d\mu \to \int he^g \; d\mu$. Since the real parts of the functions in $\mathcal{A}_z(\mathbb{D})$ (in the sense of (2.4)) are dense in the $L^\infty$ functions which vanish at $z$ we may, for any $g_n$, choose a sequence $f_{n,k} \in \mathcal{A}_z(\mathbb{D})$ so that $\Re(f_{nk}) \to g_n$ pointwise a.e. on the boundary as $k \to \infty$. Therefore,

$$\exp\left[\int \log h \; d\mu\right] \geq \inf_{\substack{g \in L^1[-\pi,\pi] \\ g(z)=0}} \int he^g \; d\mu = \inf_{\substack{g \in L^\infty[-\pi,\pi] \\ g(z)=0}} \int he^g \; d\mu = \inf_{f \in \mathcal{A}_z(\mathbb{D})} \int he^{\Re(f_1)} \; d\mu.$$

Now suppose that $\log h$ is not $\mu$-integrable. Then, since $\log h < h$, $\int \log h \; d\mu = -\infty$. Let $\varepsilon > 0$. Then $\log(\varepsilon + h)$ is $\mu$-integrable and

$$\exp\left[\int \log(h + \varepsilon) \; d\mu\right] \geq \inf_{f \in \mathcal{A}_z(\mathbb{D})} \int (h + \varepsilon)e^{\Re(f_1)} \; d\mu \geq \inf_{f \in \mathcal{A}_z(\mathbb{D})} \int he^{\Re(f_1)} \; d\mu.$$

Since the right-hand side is independent of $\varepsilon$, we may let $\varepsilon$ tend to zero, and we have

$$\exp\left[\int \log h \; d\mu\right] \geq \inf_{f \in \mathcal{A}_z(\mathbb{D})} \int he^{\Re(f_1)} \; d\mu.$$

Finally, observe that, for any $g \in \mathcal{A}_z(\mathbb{D})$, $e^{g(z)} = e^0 = 1$ so $e^g = 1 - f$ for some $f \in \mathcal{A}_z(\mathbb{D})$. Also, $e^{2\Re(g)} = |e^g|^2$, so

$$\exp\left[\int \log h \; d\mu\right] \geq \inf_{f \in \mathcal{A}_z(\mathbb{D})} \int he^{\Re(f_1)} \; d\mu = \inf_{f \in \mathcal{A}_z(\mathbb{D})} \int h|1 - f_1|^2 \; d\mu$$

Substituting our definition of $d\mu$ back into the equation yields the desired inequality. $\square$

With this in place, we can now go on to prove a very important result in the study of $H^p$ spaces. The following theorem gives us a relationship between the logarithm of a function and the logarithm of its boundary values. This is sometimes referred to as Jensen's inequality because of its similarity to Jensen's inequality from real analysis.

**Theorem 2.1.14.** *Let $f$ be any function in $H^1$. Then $\log|f(e^{i\theta})|$ is Lebesgue integrable and*

$$\log|f(z)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{i\theta})|P_r(t-\theta)d\theta$$

*for any $z = re^{it} \in \mathbb{D}$.*

*Proof.* Again, fix $z = re^{it} \in \mathbb{D}$ and let $d\mu = \frac{1}{2\pi}P_r(\theta - t)dt$. First assume that $f \in H^2$. Then, applying theorem 2.1.13 to $|f|^2$, we have

$$\exp\left[\int \log|f|^2 \, d\mu\right] \geq \inf_{g\in\mathcal{A}_z(\mathbb{D})} \int |1-g|^2|f|^2 \, d\mu.$$

However, for each $g \in \mathcal{A}_z(\mathbb{D})$, $(1-g)f = f - fg = f(z) - p$ for some $p \in \mathcal{A}_z(\mathbb{D})$. Therefore,

$$\int |1-g|^2|f|^2 \, d\mu = \int |f - fg|^2 \, d\mu = \int |f(z)-p|^2 \, d\mu \geq \left|\int (f(z)-p)^2 \, d\mu\right|$$
$$= \left|\int f(z)^2 - 2p + p^2 \, d\mu\right| = \left|\int f(z)^2 \, d\mu\right| = |f(z)^2|$$

since $\int p \, d\mu = \int p^2 \, d\mu = 0$. Therefore, since the quantity on the right does not depend at all on our choice of $g$,

$$\int \log|f|^2 \, d\mu \geq \log|f(z)|^2.$$

Now suppose, merely, that $f \in H^1$. Let $f_n$ be a sequence in $H^2$ with $f_n(z) = f(z)$ and $f_n \to f$. Let $\varepsilon > 0$. Then $\log(\varepsilon + |f_n|) \to \log(\varepsilon + |f|)$ in $L^1$ so

$$\int \log(\varepsilon + |f|)d\mu = \lim_{n\to\infty} \int \log(\varepsilon + |f_n|)d\mu \geq \log(\varepsilon + |f(z)|) \geq \log(|f(z)|).$$

Finally, we let $\varepsilon \to 0$ and apply the monotone convergence theorem. □

**Corollary 2.1.15.** *Let $f$ be any function in $H^1$ that is not the zero function. Then $\log|f(e^{i\theta})|$ is Lebesgue integrable and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{i\theta})| \, d\theta \geq \log|f(0)|$$

*Proof.* The inequality is already given to us by applying theorem 2.1.14 at the point $z = 0$. We need only show that $\log|f_1|$ is Lebesgue integrable. Let $m$ be the multiplicity of the zero of $f$ at $z = 0$ and let $g(z) = z^{-m} f(z)$. Then $|f(e^{i\theta})| = |g(e^{i\theta})|$ for all $\theta$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(e^{i\theta})|\, d\theta \geq \log|g(0)| > -\infty. \qquad \square$$

### 2.1.6 Factorization for $H^P$ functions

We now have the machinery required to discuss the factorization theory in $H^p(\mathbb{D})$. Let $f \in H^1(\mathbb{D})$. From corollary 2.1.15 in the previous section, we know that $\log|f_1|$ is integrable. Define the function $F : \mathbb{D} \to \mathbb{C}$ by

$$F(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})|\, d\theta\right]. \tag{2.5}$$

This definition is convenient when working with $F$ as a function of the complex variable $z$. However, certain properties of $F$ are more easily seen if we look at $F$ in polar coordinates:

$$F(re^{it}) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+re^{it}}{e^{i\theta}-re^{it}} \log|f(e^{i\theta})|\, d\theta\right] = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H_r(t-\theta)\log|f(e^{i\theta})|\, d\theta\right]$$

where $H_r$ is the kernel discussed at the end of section 2.1.4. Since $\Re(H_r) = P_r$ and $\frac{1}{2\pi}P_r(t)dt$ is a positive measure of mass 1, we may apply the classical form of Jensen's inequality from real analysis to obtain

$$|F(re^{it})| = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\log|f_1(\theta)|\, d\theta\right] \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)|f_1(\theta)|\, d\theta$$

and, by an exchange of integrals,

$$\begin{aligned}\int_{-\pi}^{\pi} |F(re^{it})|dt &\leq \int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)|f_1(\theta)|\, d\theta\right] dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi} P_r(t-\theta)dt\right]|f_1(\theta)|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f_1(\theta)|\, d\theta\end{aligned}$$

so $F \in H^1(\mathbb{D})$. Taking a logarithm, we have

$$\log|F(re^{it})| = \Re\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H_r(t-\theta)\log|f(e^{i\theta})|\, d\theta\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\log|f_1(\theta)|\, d\theta.$$

Therefore, $\log|F|$ is a harmonic function with boundary values given by $\log|f_1|$. Applying theorem 2.1.9 to both $\log|F|$ and $F$, we have

$$|F_1(t)| = \left|\lim_{r\to 1} F_r(t)\right| = \exp\left(\lim_{r\to 1}\log|F_r(t)|\right) = \exp\left(\log|f_1(t)|\right) = |f_1(t)|$$

almost everywhere. Finally, we may apply theorem 2.1.14, to see that

$$\log|f(re^{it})| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(e^{it})|P_r(\theta - t)\ dt = \log|F(re^{it})|$$

so $|f| \leq |F|$. Since $F$ is defined as the exponential function composed with a holomorphic function, $F$ is non-vanishing on $\mathbb{D}$. A function of the form given in (2.5) is called an *outer function*. More specifically,

**Definition 2.1.16.** An *outer function* is a function $F \in \mathcal{H}(\mathbb{D})$ of the form

$$F(z) = \lambda \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}k(\theta)\ d\theta\right]$$

where $k \in L^1[-\pi,\pi]$ is real-valued and $|\lambda| = 1$.

Since $F$ is non-vanishing, the function $g = f/F$ is well-defined and holomorphic. Since $|f| \leq |F|$, $|g| \leq 1$ on $\mathbb{D}$ and, since $|f| = |F|$ on $\mathbb{T}$, $|g| = 1$ on $\mathbb{T}$. A function with these properties is called and *outer function*:

**Definition 2.1.17.** An *inner function* is a function $g \in \mathcal{H}(\mathbb{D})$ such that $|g| \leq 1$ on $\mathbb{D}$ and $|g| = 1$ almost everywhere on $\mathbb{T}$.

We have now factored $f$ into the product of an outer function and an inner function. For any factorization $f = gF$ where $g$ is an inner function and $F$ is an outer function, we know that $|F| = |f_1|$ on $\mathbb{T}$ so the function $k$ in the above definition must be given by $\log|f_1|$. Therefore, the factorization is unique up to the constant $\lambda$ of modulus 1. Our next objective is factorize $f$ into more pieces by splitting the inner function into a *Blaschke product* and a *singular function*

**Definition 2.1.18.** A *Blaschke product* is a holomorphic function $B$ of the form

$$B(z) = z^{p_0}\prod_{n=1}^{\infty}\left[\frac{\bar{\alpha}_n}{|\alpha_n|}\frac{\alpha_n - z}{1-\bar{\alpha}_n z}\right]^{p_n} \tag{2.6}$$

where $p_0, p_1, p_2, \ldots$ are non-negative integers, the $\alpha_n$ are distinct, and $\sum_n p_n(1-|\alpha_n|) < \infty$.

The condition that $\sum_n p_n(1-|\alpha_n|) < \infty$ is necessary for the convergence of the infinite product. The following theorem, which we will not prove here, shows that this condition is both necessary and sufficient for the product $B(z)$ to converge. A proof of this theorem can be found in [9, p. 64].

**Theorem 2.1.19.** *The product in (2.6) converges uniformly on compact subsets of* $\mathbb{D}$ *if and only if* $\sum_n p_n(1-|\alpha_n|) < \infty$. *The resulting function* $B(z)$ *is an inner function whose zeros are precisely* $\alpha_n$ *with multiplicity* $p_n$.

Of more immediate interest to us is the following theorem which shows when the zeros of a bounded holomorphic function have this convergence property. Again, the proof can be found in [9]. However, we include it here because the proof provides intuition that will be useful in later sections when we start considering domains other than $\mathbb{D}$.

**Theorem 2.1.20.** *Let* $f \in H^\infty(\mathbb{D})$ *be non-zero and let* $\{\alpha_n\}$ *be the sequence of unique zeros of* $f$ *in* $\mathbb{D}$ *and let* $\{p_n\}$ *be their multiplicities. Then*

$$\sum_n p_n(1-|\alpha_n|) < \infty.$$

*Proof.* Since $f$ is bounded, we may, without loss of generality, assume that $|f| \leq 1$ on $\mathbb{D}$. Assume for the moment that $f(0) \neq 0$. For each $k \in \mathbb{N}$, let $B_k(z)$ be the finite product

$$B_k(z) = \prod_{n=1}^{k} \left[\frac{\alpha_n - z}{1-\bar{\alpha}_n z}\right]^{p_n}.$$

Since $|\alpha_n| < 1$ for each $n$, the function $B_k$ is well-defined and holomorphic on the set $D_k = \{z \in \mathbb{C} : |z| < |\alpha_n|^{-1} \text{ for each } n \leq k\}$ which contains $\mathbb{D}^-$. For $|z| = 1$,

$$\left|\frac{\alpha_n - z}{1-\bar{\alpha}_n z}\right| = \frac{|\alpha_n - z|}{|1-\bar{\alpha}_n z|} = \frac{|\alpha_n - z|}{|\bar{z}-\bar{\alpha}_n \bar{z} z|} = \frac{|\alpha_n - z|}{|\bar{z}-\bar{\alpha}_n|} = 1$$

so $|B_k| = 1$ on $\mathbb{T}$. By the maximum modulus principle, $|B_k| \leq 1$ on all of $\mathbb{D}^-$ so $B_k$ is an inner function. Since $f$ has a zero of multiplicity $p_n$ at each $a_n$, the function $g_k = f/B_k$ has only removable singularities. Since $B_k$ is holomorphic on $D_k$, it is continuous on $\mathbb{D}^-$ and $(B_k)_r \to (B_k)_1$ uniformly as $r \to 1$. Therefore, since $|B_k| = 1$ on $\mathbb{T}$ and $f_r \to f_1$ in $L^1$, the functions $(g_k)_r = f_r/(B_k)_r$ converge in $L^1$ to $(g_k)_1$ and $\|(g_k)_1\|_\infty \leq 1$ (since $f \leq 1$ on $\mathbb{D}^-$). Applying the Poisson integral formula,

$$|g_k(re^{i\theta})| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} g_k(e^{it})P_r(\theta-t)\,dt\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g_k(e^{it})\right| P_r(\theta-t)\,dt \leq \|(g_k)_1\|_\infty \leq 1$$

so $|g_k| \leq 1$ on $\mathbb{D}$. Since $g_k = f/B_k$, this implies that $|f| \leq |B_k|$ on $\mathbb{D}$. Therefore,

$$0 < |f(0)| \leq |B_k(0)| = \prod_{n=1}^{k} |\alpha_n|^{p_n}$$

and, since this $|f(0)|$ is independent of $k$ and $|\alpha_k| \leq 1$ for all $k$, this implies that the infinite product $\prod_{n=1}^{\infty} |\alpha_n|_n^p$ converges. However, the convergence of this product is equivalent to the convergence of the infinite sum $\sum_{n=1}^{\infty} p_n(1 - |\alpha_n|)$.

Finally, consider the case where $f(0) = 0$. In this case, since $f$ is not the zero function, the zero at $z = 0$ has some finite multiplicity $p_0$ and $f(z) = z^{p_0} g(z)$ for some holomorphic function $g$ with $g(0) \neq 0$. Then $g$ is also a bounded function and we may then apply the theorem to $g$. Since $g$ has the same zeros as $f$ except for the one at $z = 0$, this is sufficient to prove the general case. □

When a Blaschke product converges, we can actually say a bit more. In the above two theorems we only consider the Blaschke product on the disc. In fact, the Blaschke product is defined on almost the entire complex plane. The following theorem will be very useful when we talk about continuity properties of factorizations. The proof is omitted because it follows almost directly from the proof of 2.1.19.

**Theorem 2.1.21.** [9, p. 68] *The Blaschke produce with zeros $\{\alpha_n\}$ converges the entire complex plain except on the closure of the set $\{1/\bar{\alpha}_n\}$.*

There is another interesting fact about Blaschke products that wasn't stated in the above theorem but none the less comes out of its proof. Namely, if $f \in \mathcal{H}(\mathbb{D})$ is bounded by 1 and $B$ is the Blaschke product formed from the zeros of $f$, then $|f| \leq |B|$ on $\mathbb{D}$. This is because $|f| \leq |B_k|$ for the partial product $B_k$ for each $k$ and, since $f$ is independent of $k$, the inequality holds in the limit. Using this fact, we can see that $|f/B| \leq 1$ so $f/B$ is a well-defined holomorphic function on $\mathbb{D}$. More precisely, we have the following theorem.

**Theorem 2.1.22.** *Let $f \in H^\infty(\mathbb{D})$ be non-zero. Then $f$ can be written uniquely in the form $f = Bg$ where $B$ is a Blaschke product and $g \in \mathcal{H}(\mathbb{D})$ is bounded and non-vanishing.*

Now suppose that $f$ is an inner function. In this case, $|f| \leq |B| \leq 1$ so $|f| \leq |f/B| \leq 1$ and the function $g = f/B$ is also an inner function. This new inner function is special in that is non-vanishing.

**Definition 2.1.23.** An inner function that is both non-vanishing and is positive at the origin is called a *singular function.*

While it is not quite as obvious as in the case of outer functions, inner functions can also be written in terms of an exponential function and an integral. While the outer function was specified in terms of an absolutely continuous measure, an inner function is given by a similar integral involving a real-valued singular measure.

**Theorem 2.1.24.** [9, p. 66] *Let $g$ be an inner function without zeros which is positive at the origin. Then there is a unique singular positive measure $\mu$ on $[-\pi, \pi]$ so that*

$$g(z) = \exp\left[-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\ d\mu(\theta)\right]$$

We now have all of the pieces required for the classical BSF factorization of functions in $H^p(\mathbb{D})$. Given $f \in H^1(\mathbb{D})$, we can factor $f$ as $f = gF_1$ where $F_1$ is an outer function and $g$ is an inner function. We can then factor $g$ as $Bh$ where $B$ is a Blaschke product and $h$ is a non-vanishing inner function. We can then find a constant $\lambda$ of modulus 1 so that $S = \lambda h$ is a singular function. Finally, letting $F = \lambda F_1$, $F$ is another outer function and $f = BSF$. This gives us the following theorem:

**Theorem 2.1.25.** *Let $f \in H^p$ be non-zero. Then $f$ can be written uniquely as $f = BSF$ where $B$ is a Blaschke product, $S$ is a singular function, and $F$ is an outer function.*

The BSF factorization allows us to split a function in $H^1(\mathbb{D})$ into three functions each containing different information about the function $f$. The Blaschke product gives us the zeros of $f$ while the outer function gives us information about how $f$ behaves on the boundary.

**Theorem 2.1.26.** [9, p. 69] *Let $f \in H^1$. Then $f \in H^p$ if and only if the outer part of $f$ is in $H^p$ for $1 \le p \le \infty$. If $f$ is continuous on $\mathbb{T}$ then so is the outer part.*

The exact nature of the singular part is a bit more mysterious. However, if we combine it with the outer part, we get

$$FS(z) = \lambda\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}k(\theta)\ d\theta - \int\frac{e^{i\theta}+z}{e^{i\theta}-z}\ d\mu(\theta)\right] = \lambda\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ d\nu\right]$$

where the absolutely continuous part of $\nu$ is given by $d\nu_{cont} = k(\theta)d\theta$ and the singular part of $\nu$ is given by $\nu_\perp = -2\pi\mu$. In this sense $S$ is, as indicated by the name, the singular part of the function $FS$.

## 2.2 Positivity in $\mathcal{A}(\mathbb{D})$ and $H^p(\mathbb{D})$

We now have the tools required to solve the problems presented at the beginning of this chapter, namely the continuity of boundary values and the factorization of functions with zeros. We begin with the simpler case where the function is non-vanishing. As discussed before, given a positive function $f \in \mathcal{A}(\mathbb{D})$ which is non-vanishing, we can write $f$ as $f = e^h$ for some hermitian holomorphic function $h$ which may not have continuous boundary values. While $h$ may not be continuous on the entire circle, there is a particular compact measure-zero set $K$ so that $h$ is continuous on $\mathbb{T} \setminus K$ and this, as we will see, is sufficient.

**Lemma 2.2.1.** *Suppose $h : \mathbb{D} \to \mathbb{C}$ is continuous and that there is a continuous function $F : \mathbb{D}^- \to \mathbb{C}$ with $F = e^h$ on $\mathbb{D}$. If $K$ is the set of zeros of $F$ on $\mathbb{T}$ then $h$ can be continuously extended to $\mathbb{D}^- \setminus K$.*

*Proof.* Write $h = u + iv$ where $u$ and $v$ are real-valued and let $K$ be the set of zeros of $F$ on the circle. Since $u(z) = \log |F(z)|$ on $\mathbb{D}$, it is easy to see that $u$ can be extended to $\mathbb{D}^- \setminus K$. It suffices to show that, for any $z_0 \in \mathbb{T} \setminus K$, $v$ can be continuously extended to a neighborhood of $z_0$ that is open in $\mathbb{D}^-$. By replacing $F$ with $G(z) = \alpha F(\beta z)$ for some $\alpha, \beta \in \mathbb{C}$ where $\alpha \neq 0$ and $|\beta| = 1$, we need only show that, if $F(1) = 1$, then $v(z)$ can be continuously extended to a neighborhood of 1 that is open in $\mathbb{D}^-$.

Suppose $F(1) = 1$. Let $a : \mathbb{C} \to \mathbb{R}$ be the usual branch of the argument and observe that $a(1) = 0$ and $a$ is continuous on a neighborhood of 1. Since $F$ is continuous on $\mathbb{D}^-$ and $a$ is continuous on a neighborhood of 1, there is some $\delta > 0$ so that $a \circ F$ is continuous on $\mathbb{D}^- \cap B_\delta(1)$. Since $v$ is continuous on $\mathbb{D}$, $v - a \circ F$ is continuous on $\mathbb{D} \cap B_\delta(1)$. Also, for every $z \in \mathbb{D}$, $v(z) - a(F(z)) = 2k\pi$ for some $k \in \mathbb{Z}$. Therefore, since $\mathbb{D} \cap B_\delta(1)$ is connected and $v - a \circ F$ is continuous, $v - a \circ F$ is constant and there is some fixed $k \in \mathbb{Z}$ so that $v - a \circ F = 2k\pi$ on $\mathbb{D} \cap B_\delta(1)$. Let $v = a \circ F + 2k\pi$ on $\mathbb{T} \cap B_\delta(1)$. Then $v$ is clearly continuous $\mathbb{D}^- \cap B_\delta(1)$. □

Even though the function $h$ above is not continuous on all of $\mathbb{D}^-$, continuity on $\mathbb{D}^- \setminus K$ is sufficient to show that $e^{h/2}$ has continuous extension to $\mathbb{D}^-$. With this, we can now prove our first positivity result on $\mathcal{A}(\mathbb{D})$.

**Theorem 2.2.2.** *Let $f \in \mathcal{H}(\mathbb{D})$ be positive with no roots in $\mathbb{D}$. Then, for every integer $n > 0$ there is a unique positive function $g \in \mathcal{H}(\mathbb{D})$ such that $f = g^n$. If $f \in H^p(\mathbb{D})$ for some $1 \le p \le \infty$, then $g \in H^{np}(\mathbb{D})$. If $f \in \mathcal{A}(\mathbb{D})$, then $g \in \mathcal{A}(\mathbb{D})$.*

*Proof.* Fix $n \in \mathbb{N}$. Because $\mathbb{D}$ is simply connected, there is a function $h \in \mathcal{H}(\mathbb{D})$ so that $f = e^h$ on $\mathbb{D}$. Since $f|_{(-1,1)}$ is nonnegative, $h|_{(-1,1)}$ takes only values of the form $x + 2k\pi i$ where $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. Since $h$ is continuous on $(-1,1)$, the imaginary part of $h|_{(-1,1)}$ is constant. By adding an integer multiple of $2\pi i$, we may choose $h$ to be real-valued on $(-1,1)$ without altering $f$. Let $g = e^{h/n}$ on $\mathbb{D}$. Then $g^n = f$ on $\mathbb{D}$ and, since $h$ is real-valued on $(-1,1)$, $g|_{(-1,1)} \ge 0$. Also $|g|^n = |f|$ so $g \in H^{np}(\mathbb{D})$. Since $h$ is real-valued on $(-1,1)$, $g$ is positive.

Now suppose that $f$ has continuous boundary values. By lemma 2.2.1, we may extend $h$ to $\mathbb{D}^- \setminus K$ where $K$ is the set of roots of $f$ in $\mathbb{T}$. Define $g$ on $\mathbb{T}$ by $g = e^{h/n}$ for $z \in \mathbb{T} \setminus K$ and $g = 0$ on $K$. The extended function $g$ is obviously continuous on $\mathbb{D}^- \setminus K$. For any $z_0 \in K$, $\lim_{z\to z_0} |g(z)| = \lim_{z\to z_0} \sqrt[n]{|f(z)|} = 0$, so $\lim_{z\to z_0} g(z) = 0 = g(z_0)$; this shows that $g$ is continuous on $\mathbb{D}^-$.

To see that $g$ is unique we need simply observe that, for any positive $g$ with $g^n = f$, $g$ must be the non-negative real $n^{\text{th}}$ root of $f$ on $(-1,1)$ and apply the identity theorem. □

The above proof relies heavily on the fact that the function $f$ is non-vanishing. In general, a function $f \in \mathcal{A}(\mathbb{D})$ may have infinitely many zeros in $\mathbb{D}$. In this case, it is not obvious that $f = g^*g$ for any $g \in \mathcal{H}(\mathbb{D})$, much less that $g$ should have continuous boundary values. To solve this, we need to first prove a few results about continuous boundary values and Blaschke products. We must be careful here because Blaschke products, in general, do not have continuous boundary values. One of the results of theorem 2.1.26 is that, if $f \in \mathcal{A}(\mathbb{D})$, then the outer part of $F$ also has continuous boundary values. Our first result concerning Blaschke products is slight variation on this result.

**Theorem 2.2.3.** *Let $f \in \mathcal{A}(\mathbb{D})$ and decompose $f$ as $f = gB$ where $g \in H^\infty(\mathbb{D})$ and $B$ is a Blaschke product. Then $g \in \mathcal{A}(\mathbb{D})$ and $g$ has the same zeros on $\mathbb{T}$ as $f$.*

*Proof.* Let $K$ be the set of roots of $f$ on $\mathbb{T}$. Then $K$ contains all the accumulation points of the roots of $f$. We know, from theorem 2.1.21 that $B$ is holomorphic on $\mathbb{C} \setminus \{1/\bar{\alpha} : f(\alpha) = 0\}^-$. In particular, this means that $B$ is continuous on $\mathbb{D}^- \setminus K$ and, since $B$ is an inner function, $|B| = 1$ on $\mathbb{T} \setminus K$. Therefore, we may define $x : \mathbb{T} \to \mathbb{C}$ by $x(z) = f(z)/B(z)$ for all $z \in \mathbb{T} \setminus K$ and $x = 0$ on $K$. Obviously, $x$ is well-defined and continuous on $\mathbb{T} \setminus K$ and, since $|B| = 1$ on $\mathbb{T} \setminus K$, $|x| = |f|$ on $\mathbb{T}$. Therefore, for $z_0 \in K$, $\lim_{z \to z_0} |x(z)| = \lim_{z \to z_0} |f(z)| = 0$ so $\lim_{z \to z_0} x(z) = 0 = x(z_0)$ and $x$ is continuous on $\mathbb{T}$. Also, by corollary 2.1.15, $\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(e^{it})| dt$ is finite so $\lambda(K) = 0$ where $\lambda$ is arc-length on the circle. Therefore, $x$ is almost-everywhere the boundary values of $g$ and $g$ is the Poisson integral of $x$. Since $x$ is continuous on $\mathbb{T}$, $g$ is continuous on $\mathbb{D}^-$ by theorem 2.1.8. Since $|g| = |x| = |f|$ on $\mathbb{T}$, $g$ has the same zeros on $\mathbb{T}$ as $f$. □

The above theorem, together with theorem 2.1.22 from our earlier discussion allows us to factor a function $f \in \mathcal{A}(\mathbb{D})$ as $f = gB$ where $g \in \mathcal{A}(\mathbb{D})$ is non-vanishing on $\mathbb{D}$ and $B$ is a Blaschke product. This allows us to split our problem into two much easier problems. The first, which we have already solved, is taking an $n^{\text{th}}$ root of a non-vanishing positive function in $\mathcal{A}(\mathbb{D})$. The second (which we will attend to shortly) is in factoring a Blaschke product. However, in order for working with the Blaschke product and the non-vanishing part separately to be useful, we need to be able to recombine the two pieces and get a continuous function in the end. This requires a bit of care and the next theorem shows how we can do this given a few restrictions.

**Theorem 2.2.4.** *Let $f \in \mathcal{A}(\mathbb{D})$ and let $B$ be a Blaschke product such that $f(z) = 0$ whenever $z$ is a limit point of the roots of $B$. Then $fB \in \mathcal{A}(\mathbb{D})$.*

*Proof.* Let $K$ be the set of limit points of the roots of $B$. Then, by theorem 2.1.21, $B$ is continuous on $\mathbb{D} \setminus K$. Define $x : \mathbb{T} \to \mathbb{C}$ by $x(z) = f(z)B(z)$ for $z \in \mathbb{T} \setminus K$ and $x = 0$ on $K$. By the same argument we made in the proof of theorem 2.2.3, $x$ is continuous on $\mathbb{T}$, $K$ has measure zero, $x$ is almost everywhere the boundary values of $fB$. Therefore, by theorem 2.1.8, $fB \in \mathcal{A}(\mathbb{D})$. □

The requirement that $f$ have a zero at each of the limit points of the roots of $B$ is crucial. Without it, the resulting function will have a discontinuity on the boundary. If we factor $f$ as $f = gB$ where $B$ is a Blaschke product, one of the things that falls out of the proof of theorem 2.2.3 is that, while the function $g$ may not vanish on $\mathbb{D}$, it must take on a value of zero at those points on the boundary that are limit points of the roots of $B$. This is also true of the $n^{\text{th}}$ root of $g$ obtained using theorem 2.2.2. This allows us to take a square root of the non-vanishing part and the Blaschke product part seperately and then put them together in the end.

**Theorem 2.2.5.** *Let $B$ be the Blaschke product. If $B$ has the same roots as some positive $f \in \mathcal{H}(\mathbb{D})$, then there is another Blaschke product $B_+$ with $B = B_+^* B_+$.*

*Proof.* Suppose that $f \in \mathcal{H}(\mathbb{D})$ is positive and that $B$ has the same roots as $f$. Let $\alpha$ be a root of $f$. Then, since $f$ is hermitian, $f(\bar{\alpha}) = f^*(\bar{\alpha}) = \overline{f(\alpha)} = 0$ and

$$\begin{aligned}\lim_{z\to\bar{\alpha}_n} (z-\bar{\alpha}_n)^{-k} f(z) &= \lim_{z\to\bar{\alpha}_n} (z-\bar{\alpha}_n)^{-k} f^*(z) = \lim_{z\to\bar{\alpha}_n} (z-\bar{\alpha}_n)^{-k} \overline{f(\bar{z})} \\ &= \lim_{z\to\bar{\alpha}_n} \overline{(\bar{z}-\alpha_n)^{-k} f(\bar{z})} = \lim_{z\to\alpha_n} \overline{(z-\alpha_n)^{-k} f(z)}\end{aligned}$$

so $\alpha$ and $\bar{\alpha}$ are both roots of $f$ of the same multiplicity. If $\alpha$ is real

$$\lim_{z\to\alpha} (z-\alpha)^{-p_\alpha} f(z) = c \neq 0$$

where $p_\alpha$ is the multiplicity of $\alpha$. Since $f$ is non-negative on the real line, taking a limit from the right along the real axis reveals that $c > 0$. If $p_\alpha$ were odd, then the limit from the left along the real axis would be negative but the right and left-hand limits must agree, so $p_\alpha$ must be even. Let $\{\beta_n\}_{n\geq 1}$ be the sequence of roots of $f$ with non-negative imaginary part and let $p_n$ be their multiplicities. For convenience of notation, we will assume that $\beta_0 = 0$ (with a multiplicity of zero if needed). For each $n$, let $q_n = p_n$ if $\Im[\beta_n] > 0$ and $q_n = p_n/2$ if $\beta_n \in \mathbb{R}$. Define the Blaschke product $B_+$ by

$$B_+(z) = z^{q_0} \prod_{n\geq 1}^{\infty} \left[ \frac{\bar{\beta}_n}{|\beta_n|} \frac{\beta_n - z}{1 - \bar{\beta}_n z} \right]^{q_n}.$$

This new Blaschke product contains exactly half of the factors of $B$. This implies that $B_+$ converges because each factor is bounded above by 1 so $|B| \leq |B_+| \leq 1$. Furthermore, the product $B_+^*$ is given by

$$B_+^*(z) = z^{q_0} \prod_{n\geq 1}^{\infty} \left[ \frac{\beta_n}{|\beta_n|} \frac{\bar{\beta}_n - z}{1 - \beta_n z} \right]^{q_n}.$$

and contains the other half of the factors of $B$. Therefore, $B = B_+^* B_+$. $\square$

It is worth noting that the above theorem makes no norm restrictions on the function $f$. In the factorization theory discussed in the previous chapter, we required that $f$ be bounded in order to get a Blaschke product with the same roots as $f$. In the above theorem, we assume that such a Blaschke product exists and then only care about the placement and the multiplicities of the zeros of $f$. Making this distinction allows us to substantially simplify the proof of our next theorem.

We can now have the machinery to prove theorem 1.2.1. What we will actually prove is a generalization that also includes the $H^p$ spaces. While $H^p(\mathbb{D})$ is not an algebra and the product of two functions in $H^p(\mathbb{D})$ may not be in $H^p(\mathbb{D})$, the theorem can be stated as a factorization of a function in $H^p(\mathbb{D})$ as a product of functions in $H^{2p}(\mathbb{D})$. The proof is mostly just putting the pieces together from the other theorems we have just proved.

**Theorem 2.2.6.** *Let $f \in H^p(\mathbb{D})$ for some $1 \leq p \leq \infty$. Then $f$ is positive if and only if there exists $g \in H^{2p}(\mathbb{D})$ so that $f = g^* g$. If $f \in \mathcal{A}(\mathbb{D})$ then $g$ may also be chosen to be in $\mathcal{A}(\mathbb{D})$.*

*Proof.* The reverse direction and the case where $f = 0$ are both trivial. Suppose that $f$ is positive and $f \neq 0$. Let $f = BSF$ where $F$ is an outer function, $S$ is a singular function and $B$ is a Blaschke product. Observe that $F$ and $S$ are both non-vanishing on $\mathbb{D}$ and so $B$ contains exactly the same roots as $f$ including multiplicities. Therefore, by theorem 2.2.5 there is another Blaschke product $B_+$ with $B = B_+^* B_+$. In particular, this implies that $B$ is positive and, since $f$ is positive, $SF$ must also be positive. (This can be seen by looking at the factorization $f = BSF$ evaluated on $(-1, 1)$.) Since $SF$ is positive and non-vanishing, theorem 2.2.2 implies that $SF$ has a unique positive square root which we will denote $\sqrt{SF}$. Letting $g = B_+\sqrt{SF}$, it is clear that $f = g^* g$. Also,

$$|g|^2 = |B_+\sqrt{SF}|^2 \leq |\sqrt{SF}|^2 = |SF| \leq |F|$$

and, since $F \in H^p(\mathbb{D})$ by theorem 2.1.26, $g \in H^{2p}(\mathbb{D})$.

Finally, suppose that $f$ has continuous extension to $\mathbb{D}$. Then, by theorem 2.2.3, so does $SF$ and, by the second half of theorem 2.2.2, $\sqrt{SF}$ is also in $\mathcal{A}(\mathbb{D})$. Since $|\sqrt{SF}| \leq \sqrt{|F|}$, $SF$ has the same zeros on $\mathbb{T}$ as $F$ and, in particular, $\sqrt{SF}(z) = 0$ whenever $z$ is a limit point of the roots of $f$. Therefore, since all of the roots of $B_+$ are also roots of $f$, we may apply theorem 2.2.4 to see that $g \in \mathcal{A}(\mathbb{D})$ □

It is worth noting that the factorization given in the above theorem is in no way unique. Consider the sequence of functions $\{g_n\}_{n\geq 1} \subseteq \mathcal{A}(\mathbb{D})$ given by

$$g_n(z) = \frac{z-(1+1/n)i}{z+(1+1/n)i}.$$

Then $\|g_n\|_\infty = g_n(-i) = n(2+1/n)$ is unbounded as $n \to \infty$ but $g_n^* g_n = 1$ for all $n \geq 1$. Also, for any $f \in H^\infty(G)$ factored as $f = g^*g$, replacing $g$ with $gg_n$, we get an unbounded sequence of factorizations for $f$.

Before we conclude this section, we should say a bit about the BSF factorization as it relates to positive and hermitian functions. We have already discussed the relationship between positivity and Blaschke products, but what about singular or outer functions? For the sake of simplicity, the above proof neatly sidesteps those questions. We will say a bit about them now.

Let $f \in H^1(\mathbb{D})$ be hermitian and factor $f$ as $f = BSF$. Define $h : \mathbb{D} \to \mathbb{C}$ by

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})|\, d\theta.$$

Then the outer part of $f$ is given by $F = \lambda e^h$. Also, $h$ is hermitian because

$$\begin{aligned} h^*(z) = \overline{h(\bar{z})} &= \overline{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+\bar{z}}{e^{i\theta}-\bar{z}} \log|f(e^{i\theta})|\, d\theta\right)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-i\theta}+z}{e^{-i\theta}-z} \log|f(e^{i\theta})|\, d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-i\theta}+z}{e^{-i\theta}-z} \log|f(e^{-i\theta})|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|f(e^{i\theta})|\, d\theta = h(z). \end{aligned}$$

Therefore, the outer function $F_1 = e^h$ is *positive*. We do not yet know if $F = \lambda e^h$ is positive because $\lambda$ may not be 1. In the proof of 2.2.5 we saw that $f$ being hermitian implied that the roots of $f$ come in conjugate pairs with equal multiplicity. We only needed the positivity of $f$ to show that the real roots had even multiplicity. Therefore, $f$ being hermitian is sufficient to know that the Blaschke product

$$B(z) = z^{p_0} \prod_{n=1}^{\infty} \left[\frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1-\bar{\alpha}_n z}\right]^{p_n}$$

is hermitian. Since both $F$ and $B$ are hermitian, we know that $S$ must also be hermitian and, since $S$ is non-vanishing and $S(0) > 0$ by definition, $S$ must be positive. Since $F$ and $B$ are both hermitian, this implies that $\lambda = \pm 1$ so either $F$ or $-F$ is positive. If we further assume that $f$ is positive then $B$ is also positive so $F$ is positive and $\lambda = 1$.

## 2.3 Algebraic properties of $\mathcal{A}(\mathbb{D}, *)$

Before we finish our discussion of $\mathcal{A}(\mathbb{D})$ and continue on to more complex domains, we wish to discuss a few other algebraic properties of $\mathcal{A}(\mathbb{D})$ with our involution. Because we will be going back and forth between algebras we will, for ease of notation, use $\mathcal{A}(\mathbb{D})$ to denote the usual disc algebra without the involution and $\mathcal{A}(\mathbb{D}, *)$ to denote the disc algebra with involution. We have already mentioned that $\mathcal{A}(\mathbb{D}, *)$ is Banach $*$-algebra that is not a $C^*$-algebra. While $\mathcal{A}(\mathbb{D}, *)$ does not have the $C^*$ condition, it does inherit a different and very useful norm condition from $\mathcal{A}(\mathbb{D})$, namely that $\|f\| = \rho(f)$ for all $f \in \mathcal{A}(\mathbb{D})$ where $\rho(f)$ is the spectral radius of $f$. This is because $\sigma(f) = f(\mathbb{D}^-)$ and so both the norm and the spectral radius of $f$ are simply the maximum modulus of $f$.

### 2.3.1 Automorphisms of $\mathcal{A}(\mathbb{D}, *)$

First, we wish to classify all of the automorphisms of $\mathcal{A}(\mathbb{D}, *)$. Let $\varphi : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\mathbb{D})$ be an isomorphism that does not necessarily respect the involution. First note that $\varphi$ is automatically isometric because $\|\varphi(f)\| = \rho(\varphi(f)) = \rho(f) = \|f\|$ since isomorphisms preserve the spectral radius. Also, for any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$,

$$\varphi(p(z)) = \varphi(a_0 + a_1 z + \cdots + a_n z^n) = a_0 + a_1 \varphi(z) + \cdots + a_n \varphi(z)^n = p(\varphi(z)).$$

Let $f \in \mathcal{A}(\mathbb{D})$. Since the polynomials are dense in $\mathcal{A}(\mathbb{D})$, there is some sequence $p_n$ of polynomials with $p_n \to f$. Then

$$\varphi(f(z)) = \varphi\left(\lim_{n\to\infty} p_n(z)\right) = \lim_{n\to\infty} \varphi(p_n(z)) = \lim_{n\to\infty} p_n(\varphi(z)) = f(\varphi(z))$$

so any automorphism $\varphi$ of $\mathcal{A}(\mathbb{D})$ is of the form $f \mapsto f \circ \varphi(z)$. Also, since $\varphi$ is an automorphism it has an inverse and $\varphi(z) \circ \varphi^{-1}(z) = \varphi(\varphi^{-1}(z)) = z = \varphi^{-1}(\varphi(z)) = \varphi^{-1}(z) \circ \varphi(z)$ so the function $\varphi(z)$ is a holomorphic automorphism of the disc which is continuous on the boundary. Furthermore, it is well known in complex analysis (c.f., [7, p. 132]) that the holomorphic automorphisms of disc are precisely the functions of the form

$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$$

for some $\alpha \in \mathbb{D}$. This is the unique automorphism of $\mathbb{D}$ that maps $\alpha$ to the origin. For the converse, it is easy to see that, if $\psi : \mathbb{D} \to \mathbb{D}$ is of the above form, then it is a holomorphic automorphism $\mathbb{D}$ and $f \mapsto f \circ \psi$ is an automorphism of $\mathcal{A}(\mathbb{D})$.

Now suppose that $\varphi_\alpha$ is an automorphism of $\mathcal{A}(\mathbb{D}, *)$, i.e., an automorphism of $\mathcal{A}(\mathbb{D})$ that preserves the involution. Let $h \in \mathcal{A}(\mathbb{D})$ be given by $h(z) = z$. Then $\varphi_\alpha(h) = h \circ \varphi_\alpha = \varphi_\alpha$. Also, since $h$ is hermitian, $\varphi_\alpha(h)$ is hermitian and

$$\varphi_\alpha(z) = \varphi_\alpha^*(z) = \overline{\varphi_\alpha(\bar{z})} = \overline{\left(\frac{\bar{z}-\alpha}{1-\bar{\alpha}\bar{z}}\right)} = \frac{z-\bar{\alpha}}{1-\alpha z}$$

so $\alpha$ must be real-valued. Conversely, if $\alpha$ is real-valued, $\varphi_\alpha$ is clearly hermitian and, for any $f \in \mathcal{A}(\mathbb{D})$,

$$\varphi_\alpha(f^*)(z) = f^*(\varphi_\alpha(z)) = \overline{f(\overline{\varphi_\alpha(z)})} = \overline{f(\varphi_\alpha(\bar{z}))} = (f \circ \varphi_\alpha)^*(z) = \varphi_\alpha(f)^*(z)$$

so $\varphi_\alpha$ is a automorphism of $\mathcal{A}(\mathbb{D}, *)$. Therefore, the automorphisms of $\mathcal{A}(\mathbb{D}, *)$ are precisely the compositions of the form $f \mapsto f \circ \varphi_\alpha$ where $\alpha \in (-1, 1)$.

### 2.3.2 Representation theory of $\mathcal{A}(\mathbb{D}, *)$

It is also worth taking a look at the representation theory of $\mathcal{A}(\mathbb{D}, *)$ and how that interacts with this idea of positivity. First, we will consider the one-dimensional $*$-representations of $\mathcal{A}(\mathbb{D}, *)$. A one-dimensional $*$-representation of a $*$-algebra is a homomorphism from the algebra to the $1 \times 1$ complex matrices so it is just a multiplicative linear functional which also preserves the involution. The multiplicative linear functionals on $\mathcal{A}(\mathbb{D})$ are precisely the functionals of the form $f \mapsto f(\alpha)$ for some $\alpha \in \mathbb{D}$, i.e., the pointwise evaluations. For $\alpha \in \mathbb{D}$, let $\hat{\alpha} : \mathcal{A}(\mathbb{D}, *) \to \mathbb{C}$ be given by $\hat{\alpha}(f) = f(\alpha)$. If $\hat{\alpha}$ is a $*$-homomorphism, then

$$\overline{f(\bar{\alpha})} = f^*(\alpha) = \hat{\alpha}(f^*) = \overline{\hat{\alpha}(f)} = \overline{f(\alpha)}$$

and, since this holds for all $f \in \mathcal{A}(\mathbb{D}, *)$, $\alpha$ must be real-valued. The converse is easy to see, namely that if $\alpha \in [-\pi, \pi]$ then $\hat{\alpha}$ is a $*$-homomorphism. This gives us another equivalent condition for positivity:

**Theorem 2.3.1.** *Let $f \in \mathcal{A}(\mathbb{D}, *)$. Then $f$ is positive if and only if $\varphi(f) \geq 0$ for every one-dimensional $*$-representation $(\mathbb{C}, \varphi)$ of $\mathcal{A}(\mathbb{D}, *)$.*

This equivalence is of particular interesting because it also holds in $C^*$-algebras. If $\mathcal{A}$ is a $C^*$-algebra then an element $a \in \mathcal{A}$ is positive if and only if $\sigma(a) \geq 0$. However, in a $C^*$-algebra, every multiplicative functional automatically preserves the involution (theorem 1.1.23), so

$$\begin{aligned}\sigma(a) &= \{\tau(a) : \tau \text{ is a multiplicative linear functional on } \mathcal{A}\} \\ &= \{\varphi(a) : \varphi \text{ is a one-dimensional } *\text{-representation of} \mathcal{A}\}\end{aligned}$$

for every $a \in \mathcal{A}$. Therefore, $a$ is positive in $\mathcal{A}$ if and only if $\varphi(a) \geq 0$ for every one-dimensional $*$-representation $\varphi$ of $\mathcal{A}$.

We can extend this a bit further to include any $*$-representation of $\mathcal{A}(\mathbb{D}, *)$. Let $f \in \mathcal{A}(\mathbb{D}, *)$ be positive and let $(\mathcal{H}, \varphi)$ be a $*$-representation of $\mathcal{A}(\mathbb{D}, *)$. Then, by theorem 2.2.6, $f = g^*g$ for some $g \in \mathcal{A}(\mathbb{D}, *)$ and $\varphi(f) = \varphi(g^*g) = \varphi(g)^*\varphi(g)$ is positive in $B(\mathcal{H})$. Conversely, suppose that $f \in \mathcal{A}(\mathbb{D}, *)$ is such that $\varphi(f)$ is positive in $B(\mathcal{H})$ for every $*$-representation $(\mathcal{H}, \varphi$ of $\mathcal{A}(\mathbb{D}, *)$. Then, in particular, $\varphi(f) \geq 0$ for every one-dimensional $*$-representation $(\mathbb{C}, \varphi)$ of $\mathcal{A}(\mathbb{D}, *)$ which, by what we said above, implies that $f$ is positive. Therefore, we may add the following additional equivalent conditions to theorem 1.2.1:

5. *$\varphi(f) \geq 0$ for every one-dimensional $*$-representation $(\mathbb{C}, \varphi)$ of $\mathcal{A}(\mathbb{D}, *)$*

6. *$\varphi(f)$ is positive in $B(\mathcal{H})$ for every $*$-representation $(\mathcal{H}, \varphi)$ of $\mathcal{A}(\mathbb{D}, *)$*

Another interesting outcome of our two equivalent definitions of positivity in $\mathcal{A}(\mathbb{D}, *)$ is that the set $\{g^*g : g \in \mathcal{A}(\mathbb{D}, *)\}$ is both convex and norm-closed. Neither of these facts is non-obvious in the setting of $C^*$-algebras and they are even less obvious here. However, with the help of theorem 2.2.6, they become trivial.

# CHAPTER 3. POSITIVITY ON THE ANNULUS

Next, we turn to considering function algebras on an annulus. Fix $0 < r_0 < 1$ and define the annulus $A = \{z \in \mathbb{C} : r_0 < |z| < 1\}$. We consider the algebra $\mathcal{A}(A)$ of all holomorphic functions on $A$ with continuous extension to $A^-$. As we will see, most of the important results we obtained on the disc still hold on the annulus, though the proofs are somewhat more complicated. Before we begin, we need to develop some factorization theory on the annulus. This brief study is guided primarily by the work of Sarason in [10] and Nevanlinna in [12].

## 3.1 Factorization theory

### 3.1.1 Boundary values

Let $D_\infty$ denote the unbounded disc $D_\infty = \{z \in \mathbb{C} : |z| > r_0\}$. Then, by taking a Laurant expansion, we can write any function $f \in \mathcal{H}(A)$ as $f = g + h$ where $g \in \mathcal{H}(\mathbb{D})$ and $h \in \mathcal{H}(D_\infty)$ with $\lim_{z\to\infty} h(z)$ well-defined and finite. (Note that, because $\lim_{z\to\infty} h(z)$ exists, $h(r_0/z)$ is well-defined holomorphic function on $\mathbb{D}$.) Since $g$ and $h$ are holomorphic on $\mathbb{D}$ and $D_\infty$ respectively, they are continuous and their restrictions to $A$ have continuous extension to $r_0\mathbb{T}$ and $\mathbb{T}$ respectively. Therefore $f$ has a continuous extension to $A^-$ if and only if $g$ and $h$ both have continuous extension to the closure of their domains. Applying the function theory of the disc, if $\|f_r\|_p$ are bounded for $r_0 < r < 1$ then the $f_r$ converge in $L^p(\mathbb{T})$ as $r \to 1$ and as $r \to r_0$. Also worth mentioning is that, for $f \in H^1(\mathbb{D})$,

$$\|f_r\|_1 = \|g_r + h_r\|_1 \leq \|g_r\|_1 + \|h_r\|_1 \leq \|g_1\|_1 + \|h_{r_0}\|_1$$

so we can bound all the $L^1$ norms of $f$ in terms of $L^1$ norms of the boundaries of $g$ and $h$.

We recover a holomorphic function on the annulus from its boundary values as follows: Given $r_0 < r_1 < r_2 < 1$, we may apply Cauchy's integral formula to obtain

$$f(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(r_2e^{it})}{r_2e^{it} - w} r_2e^{it}dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(r_1e^{it})}{r_1e^{it} - w} r_1e^{it}dt \tag{3.1}$$

for $r_1 < |w| < r_2$. Now suppose that $\|f_r\|_1$ is bounded for $r_0 < r < 1$. As mentioned above, $f_r \to f_{r_0}$ as $r \to r_0$ and $f_r \to f_1$ as $r \to 1$ in $L^1(\mathbb{T})$. For a fixed $w$ with $r_1 < |w| < r_2$, $\frac{re^{it}}{re^{it}-w}$ is bounded for $r_0 < r < r_1$ and $r_2 < r < 1$. Therefore, we may let $r_1$ tend towards $r_0$ and $r_2 \to 1$ and we have

$$f(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(e^{it})}{e^{it} - w} e^{it}\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(r_0e^{it})}{r_0e^{it} - w} r_0e^{it}\, dt. \tag{3.2}$$

Let us again consider the decomposition $f = g + h$ where $g \in \mathcal{H}(\mathbb{D})$ and $h \in \mathcal{H}(D_\infty)$ with continuous extension to $\infty$. If we apply (3.2) to $g$, we see that the right-hand side of the difference is zero because it is the closed-curve integral of a holomorphic function on a simply connected domain. On the other hand, consider what happens if we apply (3.2) to $h$. For $R > 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(Re^{it})}{Re^{it} - w} Re^{it}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(Re^{it}) \frac{R}{R - we^{-it}}\, dt$$

and, as $R \to \infty$, $h(Re^{it}) \to h(\infty)$ and $\frac{R}{R-we^{it}} \to 1$ uniformly in $t$. Therefore, since (3.1) holds on $D_\infty$ and is independent of $r_2$, we may let $r_1$ tend towards $r_0$ and $r_2 \to \infty$ and we have

$$h(w) = h(\infty) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(r_0e^{it})}{r_0e^{it} - w} r_0e^{it}dt.$$

Therefore, when we decompose $f$ as $f = g + h$ in this way, (3.2) becomes

$$f(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g(e^{it})}{e^{it} - w} e^{it}\, dt + h(\infty) - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(r_0e^{it})}{r_0e^{it} - w} r_0e^{it}dt.$$

### 3.1.2 Poisson-Jensen formula

Given a measurable complex-valued function $f$ and some $r > 0$ so that $r\mathbb{T}$ is contained in the domain of $f$, we define the quantity $L(f; r)$ by

$$L(f; r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |f(re^{it})|\, dt.$$

If $\|f_r\|_1 < \infty$ then it is easily seen that $L(f; r) < \infty$ but, if $|f_r|$ is small, we may have $L(f; r) = -\infty$. One important result from the study of meromorphic functions on a disc is the

Poisson-Jensen formula which gives an explicit expression of $L(f;r)$ in terms of the roots and poles of $f$. We follow a construction similar to that found in [12, ch. 5].

Let $w$ be a function that is meromorphic on some disc $D$ of radius $R$. For the moment assume that $w(0)$ is non-zero and finite. Then $\log|w|$ is harmonic on $D$ except at the roots and poles of $w$. Let $\{a_n\}$ denote the roots of $w$ and let $\{b_n\}$ denote it's poles each repeated according to multiplicity. Fix $0<\rho<R$ such that $w$ is finite and non-vanishing on the circle of radius $\rho$. For $\alpha \in D$, let

$$g(z,\alpha) = \frac{\rho^2 - \bar{\alpha} z}{\rho(z-\alpha)}$$

and observe that, for $|z| = \rho$,

$$|g(z,\alpha)| = \frac{|\rho^2 - \bar{\alpha} z|}{|\rho(z-\alpha)|} = \frac{|\rho^2 \bar{z} - \bar{\alpha} z\bar{z}|}{|\rho\bar{z}(z-\alpha)|} = \frac{\rho^2|\bar{z}-\bar{\alpha}|}{\rho^2|z-\alpha|} = 1.$$

Using these functions $g$, we can remove the roots and poles of $w$ so

$$w(z)\left(\prod_{|a_n|<\rho} g(z,a_n)\right)\left(\prod_{|b_n|<\rho} \frac{1}{g(z,b_n)}\right)$$

is holomorphic and non-vanishing in $\mathbb{D}$. Taking a logarithm of the above,

$$h(z) = \log|w(z)| + \sum_{|a_n|<\rho} \log|g(z,a_n)| - \sum_{|b_n|<\rho} \log|g(z,a_n)|$$

is harmonic in $D$. Therefore, using Poisson integration, we have

$$h(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|w(\rho e^{it})| \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(t-\theta) + r^2}\, dt$$

since $\log|g(\rho e^{it},\alpha)| = 0$ for all $t$. Evaluating at $r=0$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|w(\rho e^{it})|\, dt &= \log|w(0)| + \sum_{|a_n|<\rho} \log|g(0,a_n)| - \sum_{|b_n|<\rho} \log|g(0,a_n)| \\ &= \log|w(0)| + \sum_{|a_n|<\rho} \log\frac{r}{|a_n|} - \sum_{|b_n|<\rho} \log\frac{r}{|b_n|}.\end{aligned}$$

Now we remove the restriction that $w(0)$ is non-zero and finite. In some neighborhood of 0, we can write

$$w(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots$$

Then $w(z)z^{-\lambda}$ is non-zero and finite at $z = 0$ and, applying the above formula to $w(z)z^{-\lambda}$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|w(\rho e^{it})|\, dt + \lambda \log\rho = \log|c_\lambda| + \sum_{0<|a_n|<\rho} \log\frac{\rho}{|a_n|} - \sum_{0<|b_n|<\rho} \log\frac{\rho}{|b_n|}.$$

Re-arranging terms yields

$$L(w;\rho) = \log|c_\lambda| + \sum_{0<|a_n|<\rho} \log\frac{\rho}{|a_n|} - \sum_{0<|b_n|<\rho} \log\frac{\rho}{|b_n|} - \lambda\log\rho \tag{3.3}$$

For a holomorphic function $f$ and $r_1 \le r_2$, define $N_0(f; r_1, r_2)$ and $N_\infty(f; r_1, r_2)$ to be the number of zeros and poles respectively of $f$ in the annulus $\{z \in \mathbb{C} : r_1 < |z| \le r_2\}$. Then we can rewrite the sums in the above equation using integrals as follows:

$$\sum_{0<|a_n|<\rho} \log\frac{\rho}{|a_n|} = \sum_{0<|a_n|<\rho} \int_{r=|a_n|}^{\rho} \frac{1}{r}\, dr = \int_{r=0}^{\rho} \frac{N_0(w,0,r)}{r}\, dr$$

and, similarly for the poles of $w$,

$$\sum_{0<|b_n|<\rho} \log\frac{\rho}{|a_n|} = \sum_{0<|b_n|<\rho} \int_{r=|b_n|}^{\rho} \frac{1}{r}\, dr = \int_{r=0}^{\rho} \frac{N_\infty(w,0,r)}{r}\, dr.$$

Therefore, we may re-write (3.3) as

$$L(w;\rho) = \log|c_\lambda| + \int_{r=0}^{\rho} \frac{N_0(w,0,r)}{r}\, dr - \int_{r=0}^{\rho} \frac{N_\infty(w,0,r)}{r}\, dr - \lambda\log\rho. \tag{3.4}$$

Notice that all of the terms in the right-hand side are clearly finite and continuous in $\rho$ even for those $\rho$ where $w$ has a pole or a root. Therefore, $L(w;\rho)$ is finite and continuous in $\rho$.

### 3.1.3 Jensen's inequality

For a function $f \in H^1(\mathbb{D})$, we know from theorem 2.1.14 that $L(f;1) > \infty$ and, if $f(0) \neq 0$ then $L(f;1) \ge \log|f(0)|$. This is sometimes called Jensen's inequality. We will require an analogue of this in the Annulus.

If $f \in \mathcal{H}(A)$, then, for $r_0 < r < 1$, $L(F;r)$ is finite and continuous with respect to $r$. To see this, consider any compact sub-annulus $A' \subseteq A$. Since $f$ has finitely many roots and poles in $A'$, there is some rational function $R(z)$ so that $f = Rg$ on $A$ where $g$ has no roots or poles in $A'$. By the Poisson-Jensen formula, $L(R;r)$ is well-defined, finite, and continuous with respect to $r$. Since $g$ is both bounded and bounded away from zero on $A'$, $L(g;r)$ is finite and continuous with respect to $r$. Finally, $L(f;r) = L(R;r) + L(g;r)$ is finite and continuous with respect to $r$.

Care must be taken, however, with the boundary values. We will show that, for $f \in \mathcal{H}(A)$, $L(f;r)$ is convex with respect to $\log r$, i.e.

$$L(f;r_0^\delta) \leq \delta L(f;r_0) + (1-\delta)L(f;1).$$

This is what Sarason calls Jensen's inequality in the annulus [10, p. 10]. First, we show this inequality for a class of rational functions.

**Lemma 3.1.1.** *Let $R(z)$ be a rational function with no poles in $A^-$. Then*

$$L(R;r_0^\delta) \leq \delta L(R;r_0) + (1-\delta)L(R;1)$$

*for all* $0 \leq \delta \leq 1$.

*Proof.* Applying the Poisson-Jensen formula (3.4) for $\rho = r_0^\delta$, we have

$$L(R;r_0^\delta) = \log|c_\lambda| + \int_0^{r_0^\delta} \frac{N_0(R;0,r)}{r}\,dr - \int_0^{r_0^\delta} \frac{N_\infty(R;0,r)}{r}\,dr - \lambda\delta\log(r_0).$$

The first and last terms in right-hand side of the above equation are obviously linear in $\delta$. Also, since $R$ has no poles for $r_0 \leq r \leq 1$, $N_\infty(R,0,r)$ is constant in this interval and

$$\begin{aligned}\int_0^{r_0^\delta} \frac{N_\infty(R;0,r)}{r}\,dr &= \int_0^{r_0} \frac{N_\infty(R;0,r)}{r}\,dr + \int_{r_0}^{r_0^\delta} \frac{N_\infty(R,0,r_0)}{r}\,dr \\ &= \int_0^{r_0} \frac{N_\infty(R;0,r)}{r}\,dr + N_\infty(R;0,r_0)\int_{r_0}^{r_0^\delta} \frac{1}{r}\,dr \\ &= \int_0^{r_0} \frac{N_\infty(R;0,r)}{r}\,dr + (\delta-1)\log(r_0)N_\infty(R;0,r_0)\end{aligned}$$

which is linear in $\delta$ since the left-hand side of the sum is independent of $\delta$. For the final part of our equation,

$$\int_0^{r_0^\delta} \frac{N_0(R;0,r)}{r}\,dr = \int_0^{r_0} \frac{N_0(R;0,r)}{r}\,dr + \int_{r_0}^{r_0^\delta} \frac{N_0(R,0,r_0)}{r}\,dr$$

where, again, the first part of the sum is constant. Using the change of variables $r = r_0^\delta u^{1-\delta}$, we have $dr = r_0^\delta(1-\delta)u^{-\delta}du$ and

$$\int_{r_0}^{r_0^\delta} \frac{N_0(R,0,r_0)}{r}\,dr = \int_{r_0}^{1} \frac{N_0(R,0,r_0^\delta u^{1-\delta})}{r_0^\delta u^{1-\delta}} r_0^\delta(1-\delta)u^{-\delta}\,du$$
$$= (1-\delta)\int_{r_0}^{1} \frac{N_0(R,0,r_0^\delta u^{1-\delta})}{u}\,du$$

Now, $N_0(R;0,r)$ is increasing in $r$ and $r = r_0^\delta u^{1-\delta} \le u$ for $r_0 \le u \le 1$. Therefore, $N(r;0,r_0^\delta u^{1-\delta}) \le N(R;0,u)$ and

$$\int_{r_0}^{r_0^\delta} \frac{N(R;0,r)}{r}\,dr = (1-\delta)\int_{r_0}^{1} \frac{N_0(R,0,r_0^\delta u^{1-\delta})}{u}\,du$$
$$\le (1-\delta)\int_{r_0}^{1} \frac{N_0(R,0,u)}{u}\,du$$

Therefore, $\int_{r_0}^{r_0^\delta} \frac{N_0(R;0,r)}{r}\,dr$ is convex in $\delta$ since $\int_{r_0}^{r_0^1} \frac{N_0(R;0,r)}{r}\,dr = 0$. Since all of the other parts of $L(R;r_0^\delta)$ are linear, $L(R;r_0^\delta)$ is convex in $\delta$. □

**Theorem 3.1.2.** *Let $f \in H^1(A)$. Then, for all $0 \le \delta \le 1$,*

$$L(f;r_0^\delta) \le \delta L(R;r_0) + (1-\delta)L(R;1).$$

*Proof.* Let $f(z) = g(z) + h(r_0/z)$ where $g,h \in H^1(\mathbb{D})$. Let $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ be the sequences of Cesaro means of $g$ and $h$ respectively. Then $g_n \to g$ and $h_n \to h$ in $L^1(\mathbb{T})$. Also, if $K \subseteq \mathbb{D}$ is a compact, then the Poisson kernel $P_r(\theta)$ is bounded on some compact disc containing $K$ so $g_n \to g$ and $h_n \to h$ uniformly on $K$. Let $f_n(z) = g_n(z) + h_n(r_0/z)$. Then $f_n \to f$ in $L^1(\mathbb{T})$ and $L^1(r_0\mathbb{T})$ and uniformly on compact subsets of $\mathbb{D}$.

Observe that $g_n$ and $h_n$ are polynomials so $g_n$ has no poles and $h_n(r_0/z)$ has a single pole of order $n$ at $z=0$. Therefore, $f_n$ has only the one pole at $z=0$ and, by lemma 3.1.1

$$L(f_n;r_0^\delta) \le \delta L(f_n;r_0) + (1-\delta)L(f_n;1)$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then obviously,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f_n(r_0^\delta e^{it})|\,dt \le \delta\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f_n(r_0 e^{it})|+\varepsilon)\,dt$$
$$+ (1-\delta)\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f_n(e^{it})|+\varepsilon)\,dt$$

Now consider the two integrals on the right of the inequality. We have

$$\int_{-\pi}^{\pi} \log(|f_n(e^{it})| + \varepsilon)\ dt - \int_{-\pi}^{\pi} \log(|f(e^{it})| + \varepsilon)\ dt = \int_{-\pi}^{\pi} \log\left(\frac{|f_n(e^{it})| + \varepsilon}{|f(e^{it})| + \varepsilon}\right) dt$$

and, by Jensen's inequality from real analysis,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\frac{|f_n(e^{it})| + \varepsilon}{|f(e^{it})| + \varepsilon}\right) dt \le \log\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f_n(e^{it})| + \varepsilon}{|f(e^{it})| + \varepsilon}\ dt\right)$$

since $-\log(x)$ is convex. By the generalized Lebesgue dominated convergence theorem with $\frac{|f_n|+\varepsilon}{\varepsilon} \to \frac{|f|+\varepsilon}{\varepsilon}$ as the dominating sequence,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f_n(e^{it})| + \varepsilon}{|f(e^{it})| + \varepsilon} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f(e^{it})| + \varepsilon}{|f(e^{it})| + \varepsilon} = 1.$$

Therefore, putting these together we have that

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f_n(e^{it})| + \varepsilon)\ dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f(e^{it})| + \varepsilon)\ dt$$

and, similarly,

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f_n(r_0 e^{it})| + \varepsilon)\ dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f(r_0 e^{it})| + \varepsilon)\ dt.$$

Also, if we pick $\delta$ so that $f$ has no poles on the circle of radius $r_0^\delta$, then $f_n(r_0^\delta e^{it}) \to f(r_0^\delta e^{it})$ uniformly in $t$ so $\log|f_n(r_0^\delta e^{it}| \to \log|f(r_0^\delta e^{it})|$ uniformly in $t$ and $L(f_n; r_0^\delta) \to L(f; r_0^\delta)$. Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r_0^\delta e^{it})|\ dt \le \delta\frac{1}{2\pi}\int_{-\pi}^{\pi} &\log(|f(r_0 e^{it})| + \varepsilon)\ dt \\ &+ (1-\delta)\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f(e^{it})| + \varepsilon)\ dt.\end{aligned}$$

Since this holds for all $\varepsilon > 0$, letting $\varepsilon \to 0$ the two integrals on the right decrease monotonically and

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r_0^\delta e^{it})|\ dt \le \delta\frac{1}{2\pi}\int_{-\pi}^{\pi} &\log(|f(r_0 e^{it})|)\ dt \\ &+ (1-\delta)\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(|f(e^{it})|)\ dt.\end{aligned}$$

□

### 3.1.4 Blaschke products on the annulus

In order to study Blaschke products on an annulus, we begin by following Sarason's construction [10, p. 15] of Blaschke factors on a universal covering space of $A$. Let $\hat{A} = \{(r,t) \in \mathbb{R}^2 : r_0 < r < 1\}$ and $\varphi(r,t) = re^{it}$; then the pair $(\hat{A}, \varphi)$ makes a universal covering surface for $A$. (The space $\hat{A}$ is considered as a Riemann surface whose conformal structure is given by $\varphi$.) We say that a point $w \in \hat{A}$ *lies above* the point $z \in A$ if $\varphi(w) = z$. Given a meromorphic function $f : \hat{A} \to \mathbb{C}$, we say $f$ is *modulus automorphic* if, for any two points $w_1, w_2 \in \hat{A}$ that lie above the same point $z \in A$, $|f(w_1)| = |f(w_2)|$. For any meromorphic function $f$ on $A$, the function $f \circ \varphi$ is obviously modulus automorphic.

For a modulus automorphic function $F$ on $\hat{A}$ and $r_0 < r < 1$ it is natural to define $L(f;r)$ by

$$L(F;r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(r,t)|\,dt.$$

If $f$ is meromorphic on $A$ and $F = f \circ \varphi$ is the corresponding modulus automorphic function on $\hat{A}$, then $L(F;r) = L(f;r)$, so this notation is consistent. In order to aid in computing the above quantity, we have the following proposition that Sarason calls Jensen's formula.

**Proposition 3.1.3** (Sarason)**.** *Let $F : \hat{A} \to \mathbb{C}$ be modulus automorphic and let $r_0 < r_1 < r_2 < 1$ so that $F$ has no roots or poles with $|z| = r_1, r_2$. Then*

$$\left.\frac{dL(f;r)}{d\log r}\right|_{r=r_2} - \left.\frac{dL(f;r)}{d\log r}\right|_{r=r_1} = N_0(F;r_1,r_2) - N_\infty(F;r_1,r_2) \tag{3.5}$$

*where $N_0(F;r_1,r_2)$ denotes the number of roots and $N_\infty(F;r_1,r_2)$ denotes the number of poles of $F$ in $[r_1,r_2] \times [-\pi,\pi)$.*

*Proof.* By replacing $F(r,t)$ with $F(r,t+\varepsilon)$ for some $\varepsilon$ we may, without loss of generality, assume that $F$ has no roots or poles with $t = -\pi, \pi$.

Under the change of variables $w = \log r + it$, $F(w)$ is holomorphic in the strip $S = \{w \in \mathbb{C} : \log r_0 < \Re(w) < 0\}$ in the usual sense. Let $w_0 \in S$ so that $F$ is holomorphic and non-vanishing in some ball $B$ about $w_0$. Then we can write $F(w) = e^{g(w)}$ for $w \in B$ where $g$ is holomorphic in $B$. In particular,

$$F'(w) = e^{g(w)}g'(w) = F(w)g'(w)$$

so $g'(w) = \frac{F'(w)}{F(w)}$. (Note that derivatives are taken with respect to the complex variable $w$.) Write $g(w) = u(w) + iv(w)$ where $u$ and $v$ are real-valued. Then, by the Cauchy-Riemann equations,

$$\frac{F'(w)}{F(w)} = \frac{\partial}{\partial w} g = \frac{\partial u}{\partial \log r} + i \frac{\partial v}{\partial \log r} = \frac{\partial u}{\partial \log r} - i \frac{\partial u}{\partial t}.$$

Because we are assuming that $F$ has no roots or poles for $r = r_1, r_2$, we may differentiate under the integral and we get

$$\frac{dL(F;r)}{d\log r} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \log r} \log |F(r,t)| \; dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \log r} u(r,t) \; dt$$

where $u$ is as defined above.

Let $\hat{u} = \frac{\partial u}{\partial \log r}$ and $u_t = \frac{\partial u}{\partial t}$. Because $F$ is modulus automorphic, $u(w) = \log |F(w)|$ is $2\pi i$-periodic and so are $\hat{u}$ and $u_t$. Let $C$ be the counter-clockwise curve about the rectangle $[\log r_1, \log r_2] \times [-\pi, \pi]$. Then the integral across the top and bottom cancel each other out and

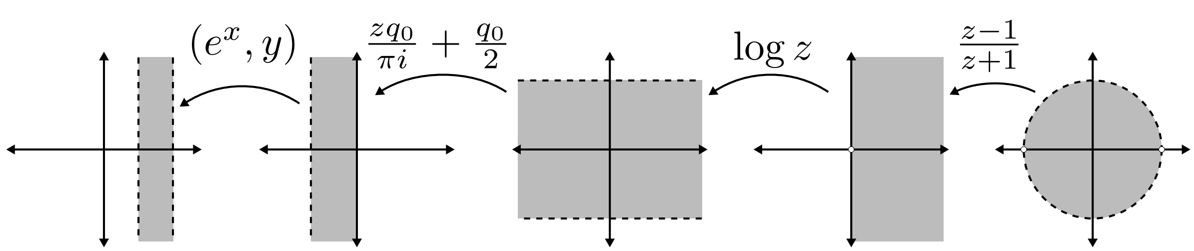
$$\begin{aligned} \left.\frac{dL(f;r)}{d\log r}\right|_{r=r_2} - \left.\frac{dL(f;r)}{d\log r}\right|_{r=r_1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(r_2, t) \; dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{u}(r_1, t) \; dt \\ &= \frac{1}{2\pi i} \oint_C \hat{u}(w) \; dw \\ &= \Re \left[ \frac{1}{2\pi i} \oint_C \hat{u}(w) + i u_t(w) \; dw \right] \end{aligned}$$

However, by the argument priciple and the derivatives we computed above,

$$\frac{1}{2\pi i} \oint_C \hat{u}(w) + i u_t(w) \; dw = \frac{1}{2\pi i} \oint_C \frac{F'(w)}{F(w)} \; dw = N_0(F; r_1, r_2) - N_\infty(F; r_1, r_2).$$

Finally, we observe that $N_0$ and $N_\infty$ are real-valued so (3.5) holds. □

Next, we turn to the construction of Blaschke factors on the covering space $\hat{A}$. Consider a holomorphic bijection $\hat{\psi} : \mathbb{D} \to \hat{A}$. If we call the composite map $\psi = \varphi \circ \hat{\psi}$, then $(\mathbb{D}, \psi)$ is another universal covering surface for $A$. One construction of such a map $\hat{\psi}$ is given by composing simpler conformal maps as follows:



Note that the left-most is holomorphic with respect to the coordinates $(r,t) \mapsto re^{it}$ while the others are holomorphic in the usual sense of complex variables. This yields the following explicit expression for $\hat{\psi}(z) = (r,t)$:

$$r = \sqrt{r_0} \exp\left[\frac{\log r_0}{\pi} \arg\left(\frac{z-1}{z+1}\right)\right], \quad t = -\frac{\log r_0}{\pi} \log\left|\frac{z-1}{z+1}\right|.$$

It is worth noting that a function in $H^1(A)$, when composed with $\psi$ yields a function in $H^1(\mathbb{D})$. To see this, first consider a non-negative harmonic function $u$ on $A$. Then $u \circ \psi$ is a non-negative harmonic function on $\mathbb{D}$ and, by well-known properties of harmonic functions on the disc (c.f. [9, p. 33]), $u \circ \psi$ is bounded in $L^1$ norm. Now let $f \in H^1(A)$ and write $f(z) = g(z) + h(r_0/z)$ where $g, h \in H^1(\mathbb{D})$. Each of $g$ and $h$ can be written as a linear combination of four non-negative harmonic functions. Lifting these to the disc by composing with $\psi$ yeilds that $f \circ \psi$ is the linear combination of eight non-negative harmonic functions on $\mathbb{D}$ and is therefore bounded in $L^1$. Since we already know that $f \circ \psi$ is holomorphic, $f \circ \psi \in L^1(\mathbb{D})$.

We can now construct blaschke factors on the covering space. Fix $a \in A$. Then $\psi - a$ is a bounded holomorphic function with roots exactly at the points of $\mathbb{D}$ which lie above $a$. From function theory on the disc, we know that $\psi - a = gB$ where $g$ is non-vanishing and $B$ is a blaschke product. We define the *Blaschke factor in $\hat{A}$ corresponding to $a$* by $H_a = B \circ \hat{\psi}^{-1}$. This has the explicit expression

$$H_a(r,t) = \prod_{\alpha \in \psi^{-1}(\{a\})} \frac{\bar{\alpha}}{|\alpha|} \frac{\alpha - \hat{\psi}^{-1}(r,t)}{1 - \bar{\alpha}\hat{\psi}^{-1}(r,t)}.$$

**Proposition 3.1.4.** *For any $a \in A$, $H_a$ is modulus automorphic.*

*Proof.* Fix $a \in A$ and write $\psi - a = gB$. Let $\tau : \mathbb{D} \to \mathbb{D}$ be given by $\tau(z) = \hat{\psi}^{-1}(\hat{\psi}_r(z), \hat{\psi}_t(z) + 2\pi)$. Then $\tau$ permutes the roots of $\psi - a$, so $B \circ \tau$ is a holomorphic function that vanishes precisely on $\psi^{-1}(\{a\})$ and is bounded in modulus by 1. Therefore, by well-known properties of Blaschke products, $|B \circ \tau| \leq |B|$. Translating this to $\hat{A}$, we get $|H_a(r, t+2\pi)| \leq |H_a(r,t)|$. Repeating the argument with $\tau^{-1}$ instead of $\tau$ yields $|H_a(r, t-2\pi)| \leq |H_a(r,t)|$ or $|H_a(r, t+2\pi)| \geq |H_a(r,t)|$. Therefore, $H_a$ is modulus automorphic. □

It is worth mentioning that the functions $H_a$ inherit some of the other nice properties of Blaschke products. In particular,

- $|H_a| < 1$
- $H_a$ is holomorphic on $\hat{A}^-$ and $|H_a| = 1$ on $\partial A$
- $H_a$ has a simple root at every point lying above $a$ and none elsewhere.

Each of these follow easily from basic properties of Blaschke products and the fact that $\hat{\psi}$ as constructed is holomorphic on $\hat{A}^-$.

Since $|H_a| = 1$ on $\partial A$, $L(H_a; r_0) = L(H_a; 1) = 0$. Also, $H_a$ has only one root or pole with $-\pi \leq t < \pi$ and it has $r = |a|$. Therefore, by proposition 3.1.3, $\frac{dL(H_a;r)}{d\log r}$ is constant except at $r = |a|$ where it has a jump discontinuity with a difference of 1. Therefore, $L(H_a; r)$ is a piecewise linear equation in $\log r$ given by

$$L(H_a; r) = \begin{cases} m\log r + b_1 & \text{if } r_0 \leq r \leq |a| \\ (m+1)\log r + b_2 & \text{if } |a| \leq r \leq 1 \end{cases}$$

where $m, b_1, b_2$ are constants. Since $L(H_a; 1) = 0$, we immediately have that $b_2 = 0$. Setting the two linear pieces equal at $r - |a|$, we have

$$m\log|a| + b_1 = (m+1)\log|a| = m\log|a| + \log|a|$$

so $b_1 = \log|a|$. Finally, since $L(H_a; r_0) = 0$, $m\log r_0 + b_1 = 0$ so $m = \frac{-\log|a|}{\log r_0}$. This yields the following closed-form expressly for $L(H_a; r)$:

$$L(H_a; r) = \begin{cases} -\frac{\log|a|}{\log r_0}\log r + \log|a| & \text{if } r_0 \leq r \leq |a| \\ \left(1 - \frac{\log|a|}{\log r_0}\right)\log r & \text{if } |a| \leq r \leq 1 \end{cases} \tag{3.6}$$

At this point we diverge from Sarason's work. He goes on to develop a theory of Blaschke products on the covering space $\hat{A}$ and proves several results about the convergence of products made up of the $H_a$. He further goes on to develop a BSF-type factorization for modulus automorphic functions on the covering space $\hat{A}$. While these developments are interesting in their own right, they focus on the covering space $\hat{A}$ and not on the annulus $A$ itself. The following result is a fusion of one of Sarason's results [10, p. 18] and my own research.

**Theorem 3.1.5.** *Let $f$ be a bounded holomorphic function on $A$ that is not identically zero and let $\{a_n\}_{n=1}^{\infty}$ be the set of roots of $f$ repeated according to multiplicity. Then*

$$\sum_{n=1}^{\infty} \min\left(1-|a_n|, 1-\frac{r_0}{|a_n|}\right) < \infty. \tag{3.7}$$

*Proof.* Without loss of generality, assume $|f| < 1$ on $\mathbb{D}$. For each $n \in \mathbb{N}$, let $\{\alpha_{n,k}\}_{k=1}^{\infty}$ be the set of points in $\mathbb{D}$ lying above $a$, i.e. $\psi(\alpha_{n,k}) = a_n$. Then $f \circ \psi$ is a bounded holomorphic funciton on $\mathbb{D}$ with roots $\{\alpha_{n,k}\}_{n,k=1}^{\infty}$ repeated according to multiplicity. Then, from function theory on the disc [9, ch. 5], the Blaschke product

$$B(z) = \prod_{n,k=1}^{\infty} \frac{\bar{\alpha}_{n,k}}{|\alpha_{n,k}|} \frac{\alpha_{n,k}-z}{1-\bar{\alpha}_{n,k}z}$$

converges and $|f \circ \psi| \le |B|$ on $\mathbb{D}$. Since each of the factors in the above product has modulus less than 1 on $\mathbb{D}$, any (possibly infinite) subproduct $P$ must also converge and have $|f \circ \psi| \le |B| \le |P|$ on $\mathbb{D}$. In particular

$$P_j(z) = \prod_{n=1}^{j} \prod_{k=1}^{\infty} \frac{\bar{\alpha}_{n,k}}{|\alpha_{n,k}|} \frac{\alpha_{n,k}-z}{1-\bar{\alpha}_{n,k}z}$$

converges and $P_j \circ \hat{\psi}^{-1} = \prod_{n=1}^{j} H_{a_n}$ for $j \in \mathbb{N}$. Also, $|f \circ \psi| \le |P_j|$ so $|f \circ \varphi| \le |P_j \circ \hat{\psi}^{-1}|$ and

$$L(f;\sqrt{r_0}) \le L(P_j \circ \hat{\psi}^{-1};\sqrt{r_0}) = L\left(\prod_{n=1}^{j} H_{a_n};\sqrt{r_0}\right) = \sum_{n=1}^{j} L(H_{a_n};\sqrt{r_0})$$

Since $L(f;\sqrt{r_0})$ is known to be finite and this holds for all $j$,

$$L(f;\sqrt{r_0}) \le \sum_{n=1}^{\infty} L(H_{a_n};\sqrt{r_0}).$$

If $|a_n| < \frac{r_0}{|a_n|}$ then $|a_n| < \sqrt{r_0}$ and

$$L(H_{a_n};\sqrt{r_0}) = \left(1-\frac{\log|a_n|}{\log r_0}\right)\log\sqrt{r_0} = \frac{1}{2}(\log r_0 - \log|a_n|) = \frac{1}{2}\log\frac{r_0}{|a_n|}.$$

Conversely, if $|a_n| \ge \frac{r_0}{|a_n|}$, then $|a_n| \ge \sqrt{r_0}$ and

$$L(H_{a_n};\sqrt{r_0}) = -\frac{\log|a_n|}{\log r_0}\log\sqrt{r_0} + \log|a_n| = \frac{1}{2}\log|a_n|.$$

Putting these two together we have that

$$L(H_{a_n};\sqrt{r_0}) = \frac{1}{2}\max\left(\log\frac{r_0}{|a_n|}, \log|a_n|\right)$$

and

$$L(f;\sqrt{r_0}) \le \sum_{n=1}^{\infty} L(H_{a_n};\sqrt{r_0}) \le \frac{1}{2}\sum_{n=1}^{\infty} \max\left(\log\frac{r_0}{|a_n|}, \log|a_n|\right).$$

It is well-known from complex analysis that the convergence of the above sum is equivalent to the convergence of (3.7). □

**Theorem 3.1.6.** *Let $f$ be a bounded holomorphic function on $A$ that is not identically zero and let $\{a_n\}_{n=1}^{\infty}$ be the roots of $f$ repeated according to multiplicity. If the limit points of $\{a_n\}_{n=1}^{\infty}$ lie entirely in $\mathbb{T}$, then the Blaschke product*

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|}\frac{a_n - z}{1 - \bar{a}_n z}$$

*converges on $\mathbb{D}$ and $f = gB$ where $g$ is bounded, holomorphic and non-vanishing on $A$. Furthermore, if $f \in \mathcal{A}(A)$ then $g \in \mathcal{A}(A)$ and $f$ and $g$ have the same zeros on $\mathbb{T}$.*

*Proof.* Suppose all of the limit points of $\{a_n\}_{n=1}^{\infty}$ lie in $\mathbb{T}$. By theorem 3.1.5,

$$\sum_{\substack{n\ge 1\\ |a_n|\ge\sqrt{r_0}}} 1 - |a_n| \le \sum_{n=1}^{\infty} \min\left(1 - |a_n|, 1 - \frac{r_0}{|a_n|}\right) < \infty$$

Since the limit points of $\{a_n\}_{n=1}^{\infty}$ lie in $\mathbb{T}$, there must be only finitely many of the $a_n$ with $|a_n| < \sqrt{r_0}$. Therefore, the above inequality implies that $\sum_{n=1}^{\infty} 1 - |a_n| < \infty$. By function theory on the disc, the Blaschke product $B(z)$ (as defined above) converges on $\mathbb{D}$ and has exactly the roots $\{a_n\}_{n=1}^{\infty}$ repeated according to multiplicity. Therefore, $g = f/B$ is well-defined and holomorphic on $A$.

We now show that $g$ is bounded. Since $f$ is bounded, let $M \in \mathbb{R}$ so that $|f| < M$ on $A$. Since $\{a_n\}_{n=1}^{\infty}$ has no limit points on $r_0\mathbb{T}$, $B$ has no roots on $r_0\mathbb{T}$ and there is some $0 < \delta < 1$ so that $|B| > \delta$ on $r_0\mathbb{T}$. For each $k$, let

$$B_k(z) = \prod_{n=1}^{k} \frac{\bar{a}_n}{|a_n|}\frac{a_n - z}{1 - \bar{a}_n z}$$

Then $|B_k|$ is decreasing and $|B| \le |B_k|$. In particular, $|B_k| > \delta$ on $r_0\mathbb{T}$ and $|B_k| = 1$ on $\mathbb{T}$. By uniform continuity of $B_k$ on $\mathbb{D}^-$, let $r_0 < r_1 < r_2 < 1$ so that $|B_k| > \delta$ on $A^- \setminus A_1$ where $A_1 = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Then $f/B_k$ is holomorphic on $A$ with $|f/B_k| = |f|/|B_k| < \frac{M}{\delta}$ on $A \setminus A_1$. By the maximum modulus principle, $|f/B_k| < \frac{M}{\delta}$ on $A_1$ since $f/B_k$ is continuous on

$A_1^-$ and $\partial A_1 \subseteq A \setminus A_1$. Therefore, $|f| < \frac{M}{\delta}|B_k|$ on $A$ and, since $M$ and $\delta$ do not depend on $k$ and $|B_k|$ decreases to $|B|$, $|f| \leq \frac{M}{\delta}|B|$ on $A$.

Now suppose $f$ has a continuous extension to $A^-$ and consider $f$ to be so extended. Since $\{a_n\}_{n=1}^{\infty}$ has no limit points on $r_0\mathbb{T}$, $B(z)$ is bounded away from zero on $r_0\mathbb{T}$ and it is clear that $g$ extends continuously to $A \cup r_0\mathbb{T}$. Let $K$ denote the set of zeros of $f$ in $\mathbb{T}$. Since $L(f;1) > -\infty$, $\lambda(K) = 0$ where $\lambda$ is arc-length measure on $\mathbb{T}$. We know that the Blaschke product converges, not only on $\mathbb{D}$ but on $\mathbb{C} \setminus \{\bar{a}_n^{-1} : n \in \mathbb{N}\}^-$. In particular, $B(z)$ is well-defined and holomorphic on $\mathbb{D} \setminus K$ with with $|B| = 1$ on $\mathbb{T} \setminus K$. Define $x$ on $\mathbb{T}$ by $x = f/B$ on $\mathbb{T} \setminus K$ and $x = 0$ on $K$. Since $g$ is bounded and $x = g$ a.e. $(\lambda)$,

$$g(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{x(e^{it})}{e^{it} - w} e^{it}\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{g(r_0 e^{it})}{r_0 e^{it} - w} r_0 e^{it}\, dt$$

Observe that $x$ is continuous off $K$ since $f$ and $B$ are continuous and non-vanishing off $K$. Also, $|x| = |f|$ on $\mathbb{T}$ since $|B| = 1$ on $\mathbb{T} \setminus K$ and $x = f = 0$ on $K$. For any $\{z_n\} \subseteq \mathbb{T}$ with $z_n \to z \in K$, $|x(z_n)| = |f(z_n)| \to 0 = x(z)$ so $x$ is continuous at every point of $K$. Therefore, since $g$ is the integral of a continuous function on $\partial A$, $g$ has continuous extension to $A^-$. Since $|g| = |x| = |f|$ on $\mathbb{T}$, $g$ and $f$ have the same zeros on $\mathbb{T}$. □

**Corollary 3.1.7.** *Let $f$ be a bounded holomorphic function on $A$ that is not identically zero and let $\{a_n\}$ be the roots of $f$ repeated according to multiplicity. Then the Blaschke products*

$$B_1(z) = \prod_{|a_n| \geq \sqrt{r_0}} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z} \quad \textit{and} \quad B_2(z) = \prod_{|a_n| < \sqrt{r_0}} \frac{a_n}{|a_n|} \frac{r_0/a_n - z}{1 - (r_0/\bar{a}_n) z}$$

*converge and we may decompose $f$ as $f(z) = g(z)B_1(z)B_2(r_0/z)$ where $g$ is bounded, holomorphic, and non-vanishing on $A$. If $f$ has a continuous extension to $A^-$ then so does $g$.*

*Proof.* Let $r_1 < \sqrt{r_0}$ be such that $f$ has no roots $a$ with $r_1 \leq |a| < \sqrt{r_0}$ and let $A_1 = \{z \in \mathbb{C} : r_1 < |z| < 1\}$. Then $f$ is holomorphic on $A_1$ and, if $\{b_n\}$ are the roots of $f$ with modulus greater than or equal to $\sqrt{r_0}$, then $\{b_n\}$ are precicely the roots of $f$ in $A_1$ repeated according to multiplicities. Also, the limit points of $\{b_n\}$ must lie in $\mathbb{T}$ since $f$ has no roots with $|b_n| \in [r_1, \sqrt{r_0})$. Let $B_1(z)$ be the Blaschke product for $\{b_n\}$. By theorem 3.1.6, $B_1$ converges and $g_1 = f/B$ is bounded, holomorphic, and non-vanishing on $A_1$. Since $B_1$ converges in $\mathbb{D}$

and has exactly the roots $\{b_n\}$, $B_1$ is bounded away from zero on $\{z \in \mathbb{C} : |z| \leq r_1\}$. Therefore, $g_1 = f/B_1$ is a bounded holomorphic function on $A$ and non-vanishing in $A_1$.

Let $f_2(z) = g_1(r_0/z)$. Then $f_2(z)$ is a bounded holomorphic function on $A$ and the limit points of the roots of $f_2$ lie entirely in $\mathbb{T}$. By theorem 3.1.6, $f_2 = g_2 B_2$ where $g_2$ is bounded, analytic, and non-vanishing on $A$. Finally, let $g(z) = g_2(r_0/z)$ and we have that $f(z) = g(z)B_1(z)B_2(r_0/z)$ as desired.

If $f$ has continuous extension to $A^-$ then theorem 3.1.6 gives us that $g_1$, $g_2$, and finally $g$ all have continuous extension to $A^-$. □

## 3.2 Positivity in $\mathcal{A}(A)$ and $H^p(A)$

### 3.2.1 Non-vanishing functions on the annulus

When discussing non-vanishing functions on a disc, we had the distinct advantage of a simply connected domain. On a simply connected domain, any non-vanishing holomorphic function $f$ can be written as $e^g$ where $g$ holomorphic. In the annulus, this is not the case. Consider, for instance, $f(z) = z^n$ for any $n \in \mathbb{N}$. This function is clearly non-vanishing on $A$ but any logarithm of $f$ must be discontinuous. We can, however, use the following slightly more general version of this theorem:

**Theorem 3.2.1.** *Let $G$ be a domain and let $f$ be holomorphic on $G$. Suppose that $f$ is non-vanishing and, for every simple closed curve $\gamma$,*

$$\oint_\gamma \frac{f'(z)}{f(z)}\, dz = 0.$$

*Then there exists a holomorphic function $g$ on $G$ so that $f = e^g$.*

*Proof.* Fix $a \in G$. Since $G$ is connected, it is path-connected and, for every $z \in G$ there is some path $\gamma_z$ from $a$ to $z$. Define $g : G \to \mathbb{C}$ by

$$g(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)}\, dw.$$

By our hypothesis, if $\gamma$ and $\gamma'$ are two curves from $a$ to $z$,

$$\int_\gamma \frac{f'(w)}{f(w)}\, dw - \int_{\gamma'} \frac{f'(w)}{f(w)}\, dw = \oint_{\gamma\cup-\gamma'} \frac{f'(w)}{f(w)}\, dw = 0$$

where $-\gamma'$ denotes the curve $\gamma'$ in the opposite direction. Therefore, the definition of $g$ does not depend on the curve chosen. Also,

$$\begin{aligned}\frac{g(z)-g(z_0)}{z-z_0} &= \frac{1}{z-z_0}\left[\int_{\gamma_{z_0}\cup[z_0,z]} \frac{f'(w)}{f(w)}\,dw - \int_{\gamma_{z_0}} \frac{f'(w)}{f(w)}\,dw\right] \\ &= \frac{1}{z-z_0}\int_{[z_0,z]} \frac{f'(w)}{f(w)}\,dw\end{aligned}$$

so $g'(z_0) = \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \frac{f'(z_0)}{f(z_0)}$. In particular, this shows that $g$ is holomorphic. Let $h = e^g$. Then $h$ is holomorphic and non-vanishing on $G$ so $f/h$ is well-defined and holomorphic on $G$ and

$$\begin{aligned}\frac{d}{dz}\frac{f(z)}{h(z)} &= \frac{f'(z)h(z)-f(z)h'(z)}{h(z)^2} = \frac{f'(z)h(z)-f(z)g'(z)h(z)}{h(z)^2} \\ &= \frac{f'(z)-f(z)g'(z)}{h(z)} = 0.\end{aligned}$$

Therefore, $f/h = c$ for some constant $c$ and $f = e^{\log c + g}$. □

In order to better characterize the functions for which we can apply the above theorem, we introduce the concept of the *winding number* of a holomorphic function on the annulus.

**Definition 3.2.2.** For $f \in \mathcal{H}(A)$ non-vanishing, we define the *winding number* of $f$ by

$$\operatorname{wn}(f) = \frac{1}{2\pi i}\oint_{\gamma_r} \frac{f'(z)}{f(z)}\,dz$$

where $\gamma_r(t) = re^{it}$ for $t \in [-\pi, \pi]$ and $r_0 < r < 1$.

Note that the winding number is always an integer and is independent of the radius $r$. To see this, observe that $\operatorname{wn}(f) = \operatorname{idx}(f\circ\gamma_r, 0)$; from elementary complex variables, we know that $\operatorname{idx}(f\circ\gamma_r, 0)$ is an integer and, since $f\circ\gamma_r$ is homotopic to $f\circ\gamma_{r'}$ for any $r_0 < r, r' < 1$, it is independent of $r$.

Given $f, g \in \mathcal{H}(A)$ non-vanishing, a simple application of the product rule yields

$$\begin{aligned}\operatorname{wn}(f) &= \frac{1}{2\pi i}\oint_\gamma \frac{(fg)'(z)}{(fg)(z)}\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)g(z)+f(z)g'(z)}{f(z)g(z)}\,dz \\ &= \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)g(z)}{f(z)g(z)}\,dz + \frac{1}{2\pi i}\oint_\gamma \frac{f(z)g'(z)}{f(z)g(z)}\,dz \\ &= \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_\gamma \frac{g'(z)}{g(z)}\,dz \\ &= \operatorname{wn}(f) + \operatorname{wn}(g)\end{aligned}$$

In particular, for any $f \in \mathcal{H}(A)$, the function $z^{-\mathrm{wn}(f)} f(z)$ always has winding number 0.

With this concept of a winding number, we can weaken the hypotheses of theorem 3.2.1 to simply requiring $\mathrm{wn}(f) = 0$ when the domain is the annulus. Let $f \in \mathcal{H}(A)$ be non-vanishing with $\mathrm{wn}(f) = 0$ and let $\gamma$ be any closed curve in $A$. Since $\mathbb{D}$ is simply connected, $\gamma$ is homologous to zero in $\mathbb{D}$. Let $N$ be the index of $\gamma$ about the origin and let $\varphi(t) = \sqrt{r_0} e^{-iNt}$ on $[-\pi, \pi]$. Then the index of $\gamma \cup \varphi$ is zero about the origin. Since the index is continuous on $\mathbb{C} \setminus (\gamma \cup \varphi)$ and $\gamma$ and $\varphi$ both lie in $A$, the index of $\gamma \cup \varphi$ is zero for every point inside the circle of radius $r_0$. Therefore, $\gamma \cup \varphi$ is homologous to zero in $A$ and, since $\frac{f'}{f}$ is holomorphic in $A$, by the argument principle, $\int_{\gamma\cup\varphi} \frac{f'(z)}{f(z)}\, dz = 0$. However, $\int_\varphi \frac{f'(z)}{f(z)}\, dz = 0$ since $\mathrm{wn}(f) = 0$ and $\varphi$ is multiple runs around the circle. Therefore, $\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$ and this holds for any closed curve $\gamma$ in $A$.

We deal with the issue of continuity of the logarithm in exactly the same way as we do in the disc. Since the proof of the following lemma is the same as the version in the disc, we will not repeat it here.

**Lemma 3.2.3.** *Suppose $h : A \to \mathbb{C}$ is continuous and that there is a continuous function $F : A^- \to \mathbb{C}$ with $F = e^h$ on $A$. If $K$ is the set of roots of $F$ on $\partial A$ then $h$ can be continuously extended to $A^- \setminus K$.*

We need to know just a bit more about these winding numbers of holomorphic functions on $A$. In particular, we need to know how $f$ being positive affects its winding number.

**Theorem 3.2.4.** *Let $\gamma : [-\pi, \pi] \to \mathbb{C} \setminus \{0\}$ be a closed curve so that $\gamma(-t) = \overline{\gamma(t)}$ for all $t$. If $\gamma(0) > 0$ and $\gamma(-\pi) > 0$ then $\mathrm{idx}(\gamma, 0)$ is even.*

*Proof.* First, observe that $\gamma$ is homotopic to $\varphi = \frac{\gamma}{|\gamma|}$ since $\gamma$ never takes on the value zero. The new curve $\varphi$ also has the property that $\varphi(-t) = \overline{\varphi(t)}$ for all $t$. Let $\varphi_- = \varphi|_{[-\pi,0]}$ and $\varphi_+ = \varphi|_{[0,\pi]}$ and observe that, since $\gamma(0) > 0$ and $\gamma(\pi) > 0$, $\varphi(0) = \varphi(-\pi) = \varphi(\pi) = 1$ so $\varphi_-$ and $\varphi_+$ are both closed curves. Also,

$$\begin{aligned}
\mathrm{idx}(\varphi_-, 0) &= \frac{1}{2\pi i} \oint_{\varphi_-} \frac{dz}{z} = \frac{1}{2\pi i} \int_{-\pi}^{0} \frac{\varphi'(t)}{\varphi(t)}\, dt = -\frac{1}{2\pi i} \int_0^{\pi} \frac{\varphi'(-t)}{\varphi(-t)}\, dt \\
&= -\frac{1}{2\pi i} \int_0^{\pi} \overline{\left(\frac{\varphi'(t)}{\varphi(t)}\right)}\, dt = \overline{\frac{1}{2\pi i} \int_0^{\pi} \frac{\varphi'(t)}{\varphi(t)}\, dt} = \mathrm{idx}(\varphi_+, 0).
\end{aligned}$$

Therefore, $\mathrm{idx}(\varphi, 0) = \mathrm{idx}(\varphi_-, 0) + \mathrm{idx}(\varphi_+, 0) = 2\mathrm{idx}(\varphi_+, 0)$ which is even. $\square$

**Corollary 3.2.5.** *Let $f \in \mathcal{H}(A)$ be non-vanishing with $f(A \cap \mathbb{R}) \geq 0$. Then* $\mathrm{wn}(f)$ *is even.*

*Proof.* Fix $r_0 < r < 0$ and let $\gamma(t) = f(re^{it})$ for $-\pi \leq t \leq \pi$. Since $f$ is hermitian, $\gamma(-t) = f(re^{-it}) = f(\overline{re^{it}}) = \overline{f(re^{it})} = \overline{\gamma(t)}$ and, since $f$ is non-vanishing and $f(A \cap \mathbb{R}) \geq 0$, $\gamma(0) = f(r) > 0$ and $\gamma(\pi) = f(-r) > 0$. Therefore, by theorem 3.2.4, $\mathrm{wn}(f) = \mathrm{idx}(\gamma, 0)$ is even. □

**Theorem 3.2.6.** *Let $f \in \mathcal{H}(A)$ be non-vanishing with $f(A \cap \mathbb{R}) > 0$. Then there exists a function $g \in \mathcal{H}(A)$ so that $f = g^* g$. Furthermore, if $f \in H^p(A)$, then $g \in H^{2p}(A)$ for $1 \leq p \leq \infty$ and, if $f \in \mathcal{A}(A)$, then $g \in \mathcal{A}(A)$.*

*Proof.* First suppose that $\mathrm{wn}(f) = 0$. By theorem 3.2.1, there is some function $h \in \mathcal{H}(A)$ so that $f = e^h$. Since $f(A \cap \mathbb{R}) \geq 0$, $h$ takes on only integer multiples of $2\pi i$ on $A \cap \mathbb{R}$ and, since $h$ is continuous, $h$ is constant on $(-1, -r_0)$ and $(r_0, 1)$. By adding a multiple of $2\pi$ we may choose $h$ so that $h = 0$ on $(r_0, 1)$. Then, by theorem 1.2.2, $h$ is hermitian. Letting $g = e^{\frac{1}{2}h}$, $g$ is clearly hermitain and $g^* g = g^2 = f$. Also, $|g|^{2p} = |f|^p$ so $f \in H^p(A)$ implies $g \in H^{2p}(A)$. Suppose $f \in \mathcal{A}(A)$. By lemma 3.2.3, $h$ has a continuous extension to $A^- \setminus K$ where $K$ is the set of zeros of $f$ on $\partial A$. Then $g$ is also continuous on $A^- \setminus K$. For any $z_n \to z \in K$, $|g(z_n)| = \sqrt{|f(z_n)|} \to 0$ so $g(z_n) \to 0$. Therefore, defining $g = 0$ on $K$, we can extend $g$ continuously to all of $A^-$.

Now suppose $\mathrm{wn}(f) \neq 0$. By corollary 3.2.5, $\mathrm{wn}(f)$ is even, so there is some $k \in \mathbb{Z}$ so that $\mathrm{wn}(f) = 2k$. Then $f_1(z) = f(z) z^{-2k}$ has winding number zero and, by what we showed above, there is some $g_1 \in \mathcal{H}(A)$ so that $f_1 = g_1^* g_1$. Letting $g(z) = g_1(z) z^k$, we have that $f = g^* g$. Obviously, $f \in H^p(A)$ still implies $g \in H^{2p}(A)$ and, since $z^k$ is holomorphic on $\mathbb{C} \setminus \{0\}$ for any $k \in \mathbb{Z}$, $f \in \mathcal{A}(A)$ still implies $g \in \mathcal{A}(A)$. □

### 3.2.2 Positivity of functions that may have zeros

We now have all of the major results required to discuss positivity of functions on the annulus. Before we can combine theorem 3.2.6 and corollary 3.1.7 to get a factorization result in $H^p(A)$, we need to recover some of the theory of outer functions from the case of the disc.

Given $f \in H^1(\mathbb{D})$, we define the *positive outer function* $F_+(z)$ by

$$F_+(z) = \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \max(0, \log|f(e^{it})|)\, dt\right]. \tag{3.8}$$

A few things are immediately obvious. First, $|f| \leq |F| \leq |F_+|$ where $F$ is the usual outer function associated with $f$. This is because the Poisson kernel is non-negative and $|F| = e^u$ and $|F_+| = e^{u_+}$ where $u$ and $u_+$ are the Poisson integrals of $\log|f|$ and $\max(0, \log|f|)$ respectively. Second, $|1/F_+(z)| = e^{-u_+} \leq 1$ since $u_+$ is non-negative. Third, if $f \in H^p(\mathbb{D})$ for $1 \leq p \leq \infty$ then $F_+ \in H^p(\mathbb{D})$ because $|F_+| = \max(1, |f|)$ a.e. on $\mathbb{T}$. We can use these positive outer functions to prove the following analogue of a classic result in the disc

**Theorem 3.2.7.** *Let $f \in H^p(A)$ for some $1 \leq p \leq \infty$. Then there is a non-vanishing function $F \in H^p(A)$ so that $f/F$ is bounded.*

*Proof.* Write $f(z) = g(z) + h(r_0/z)$ where $g, h \in H^p(\mathbb{D})$. Construct the positive outer functions $G_+$ and $H_+$ from $g$ and $h$ respectively as in (3.8). Then $f(z)/G_+(z)H_+(r_0/z)$ is well-defined and holomorphic on $A$ and

$$\left|\frac{f(z)}{G_+(z)H_+(r_0/z)}\right| \leq \left|\frac{g(z)}{G_+(z)}\right|\left|\frac{1}{H_+(r_0/z)}\right| + \left|\frac{h(r_0/z)}{H_+(r_0/z)}\right|\left|\frac{1}{G_+(z)}\right| \leq 2$$

Now, let $F(z) = G_+(z)H_+(r_0/z)$. Since $G_+$ and $H_+$ are holomorphic on $\mathbb{D}$, there is some $M \in \mathbb{R}$ so that $|G_+(z)|, |H_+(z)| \leq M$ for $|z| \leq \sqrt{r_0}$. Then, if $r \geq \sqrt{r_0}$,

$$\begin{aligned}\int_{-\pi}^{\pi} |F(re^{it})|^p \, dt &= \int_{-\pi}^{\pi} |G_+(re^{it})H_+((r_0/r)e^{-it})|^p \, dt \\ &\leq \int_{-\pi}^{\pi} |G_+(re^{it})|^p M^p \, dt \leq \|G_+\|_p^p M^p\end{aligned}$$

Similarly, if $r \leq \sqrt{r_0}$,

$$\begin{aligned}\int_{-\pi}^{\pi} |F(re^{it})|^p \, dt &= \int_{-\pi}^{\pi} |G_+(re^{it})H_+((r_0/r)e^{-it})|^p \, dt \\ &\leq \int_{-\pi}^{\pi} M^p |H_+((r_0/r)e^{-it})|^p \, dt \leq \|H_+\|_p^p M^p\end{aligned}$$

so $F \in H^p(A)$. □

**Theorem 3.2.8.** *An element $f \in H^p(A)$ is positive if and only if $f = g^*g$ for some $g \in H^{2p}(A)$. Furthermore, if $f$ is continuous on $A^-$, then $g$ may be chosen continuous on $A^-$.*

*Proof.* First, we assume that $f \in H^\infty(A)$. Let $\{a_n\}$ be the set of roots of $f$ repeated according to multiplicity. By corollary 3.1.7, we can decompose $f$ as $f(z) = h(z)B_1(z)B_2(r_0/z)$ where

$B_1$ and $B_2$ are defined by

$$B_1(z) = \prod_{|a_n| \geq \sqrt{r_0}} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z} \quad \text{and} \quad B_2(z) = \prod_{|a_n| < \sqrt{r_0}} \frac{a_n}{|a_n|} \frac{r_0/a_n - z}{1 - (r_0/\bar{a}_n)z}$$

and $h \in H^\infty(A)$. By our discussion in the disc, we can see that $B_1$ and $B_2$ are both positive and that we can write $B_1 = (B_{1+})^* B_{1+}$ and $B_2 = (B_{2+})^* B_{2+}$. Therefore, $B_1(z)B_2(r_0/z)$ is positive, so $h$ is also positive. By theorem 3.2.6, $h = k^* k$ for some $k \in H^\infty(\mathbb{D})$ and, letting $g(z) = k(z)B_{1+}(z)B_{2+}(r_0/z)$, we have that $f = g^* g$. Suppose $f \in \mathcal{A}(A)$. Then, by corollary 3.1.7, $h \in \mathcal{A}(A)$ and, by theorem 3.2.6, $k \in \mathcal{A}(A)$. Also, $B_{1+}$ and $B_{2+}(r_0/z)$ are continuous on $A^-$ except perhaps at the limit points of the $\{a_n\}$ and $k$ vanishes at these points. Therefore, since $B_1$ and $B_2$ are bounded, $g$ is continuous on $A^-$.

Now let $f \in H^p(A)$ for some $1 \leq p < \infty$. By theorem 3.2.7, there is a non-vanishing function $F \in H^p(A)$ so that $f/F$ is bounded. By corollary 3.1.7, we can decompose $f/F$ as $f(z)/F(z) = h(z)B_1(z)B_2(r_0/z)$ as we did above. Since $f$ is positive and $B_1$ and $B_2(r_0/z)$ are positive, $Fh$ is positive. Also, $Fh$ is non-vanishing and, by theorem 3.2.6, there is some $Fh = k^* k$ for some $k \in H^{2p}(A)$. Therefore, writing $g(z) = k(z)B_{1+}(z)B_{2+}(r_0/z)$, we have that $g \in H^{2p}(A)$ and $f = g^* g$. □

# CHAPTER 4. POSITIVITY ON OTHER DOMAINS

Using the same techniques used in the disc and the annulus, we may extend our results to more general domains. In particular, we will consider those domains where the boundary is the union of finitely many disjoint Jordan curves. Careful application of the Caratéodory mapping theorem [11, p. 24] will allow us to extend or earlier results to these domains. If the boundary of the domain is a single Jordan curve, the extension is trivial. Let $G$ be a domain whose boundary is a single Jordan curve. Then $G \cap \mathbb{R} \neq \emptyset$; letting $a \in G \cap \mathbb{R}$, the Caratéodory mapping theorem gives a holomorphic bijection $\phi : G \to \mathbb{D}$ with $\phi(a) = 0$ such that $\varphi$ extends to a homeomorphism from $G^-$ to $\mathbb{D}^-$. Then, for a function $f \in \mathcal{H}(G)$, the function $f \circ \phi^{-1}$ is a holomorphic function on $\mathbb{D}$ that inherits the boundedness and continuity properties of $f$ and we can we can apply the results of chapter 2. For domains with holes, we must be a little more careful. Before we begin our discussion, consider the following example.

**Example 4.0.9.** Let $G$ be the slit disc $\mathbb{D} \setminus (-1, 0]$ and let $f : G \to \mathbb{C}$ be given by $f(z) = z$. Obviously, $f$ has continuous extension to $G^- = \mathbb{D}^-$. Also, $f(G \cap \mathbb{R}) = f((0,1)) \subseteq \mathbb{R}_+$ so $f$ is positive in the sense of definition 1.2.3. Now suppose that $g \in \mathcal{A}(G)$ with $f = g^*g$. By the continuity of $f$ and $g$, $f = g^*g$ on $\mathbb{D}^-$. Therefore,

$$-1/2 = f(-1/2) = g^*(-1/2)g(-1/2) = |g(-1/2)| \geq 0$$

which is a contradiction.

The above example shows us a little of what can go wrong if we are not careful with the domain we choose. For the remainder of this chapter, we will consider domains $G$ where $\partial G$ is the union of finitely many disjoint Jordan curves.

## 4.1 Jordan curves and Riemann maps

Before we can begin to attack our problem on more general domains, we need to first become comfortable Jordan curves and Riemann maps. The solution to our problem in these more general domains comes by transforming the problem into a problem on the disc or the annulus and applying the functional analysis there. There are a few classic theorems will be crucial to our ability to do so. We have already mentioned Jordan curves informally; we will begin with a formal definition.

**Definition 4.1.1.** A Jordan curve $J$ is the image of a continuous injective map $\phi : \mathbb{T} \to \mathbb{C}$.

While Jordan curves are of interest for a variety of reasons in topology, our primary interest in them stems from the well-known Jordan curve theorem.

**Theorem 4.1.2** (Jordan Curve Theorem)**.** [16] *Let $J$ be a Jordan curve. Then $\mathbb{C} \setminus J$ has two components one of which is bounded (called the interior of $J$) and one of which is unbounded (the exterior) and $J$ is the boundary of both.*

This theorem, by itself, is a very deep result. The fact that we can say so much with only the simple topological restrictions of a Jordan curve is fairly amazing. Even more fantastic is Schoenflies' theorem which extends the Jordan curve theorem by allowing us to map the interior and exterior of $\mathbb{C} \setminus J$ to the bounded and unbounded discs.

**Theorem 4.1.3** (Schoenflies)**.** [14] *Let $J$ be a Jordan curve. Then there is a homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ so that $J = \phi(\mathbb{T})$.*

The above two results are entirely topological in nature and deal with only with continuity. For our needs, we will actually need to perform holomorphic, not just continuous, transformations domains. For this, we will need the Riemann mapping theorem.

**Theorem 4.1.4** (Riemann Mapping Theorem)**.** [7, p. 160] *Let $G \subseteq \mathbb{C}$ be a simply connected region that is not the whole plane and let $a \in G$. Then there is a unique holomorphic bijection $\phi : G \to \mathbb{D}$ so that $\phi(a) = 0$ and $\phi'(a) > 0$.*

It is worth noting that, if $J$ is a Jordan curve and $G$ is the interior of $J$ then $G$ is simply connected. To see this, we can apply Schoenflies' theorem to get $\phi : \mathbb{C} \to \mathbb{C}$ and $\phi(\mathbb{D}) = G$. Since $\mathbb{D}$ is obviously simply connected, $G$ must be simply connected. Also, if $G$ is symmetric with respect to $\mathbb{R}$, i.e. if $G^* = \{\bar{z} : z \in G\} = G$ and $a \in G$ is chosen to be real-valued, then the uniqueness property of the Riemann mapping theorem implies that $\phi = \phi^*$. This is because $\phi^* : G \to \mathbb{D}$ is another holomorphic bijection with $\phi^*(a) = 0$ and, by taking the derivative down the real line, $(\phi^*)'(a) = \phi'(a) > 0$.

For our purposes, the Riemann mapping theorem will frequently be insufficient because we need to know something about the boundary of the domain $G$ and not just the open set $G$ itself. For this we require the only slightly less well-known Carathéodory theorem:

**Theorem 4.1.5** (Carathéodory)**.** [11, p. 24] *Let $G \subseteq \mathbb{C}$ be a simply connected region whose boundary is a Jordan curve. Then the Riemann map $\phi : G \to \mathbb{D}$ extends to a homeomorphism $\Phi : G^- \to \mathbb{D}^-$.*

The Riemann and Carathéodory theorems both operate on the interior of the Jordan curve. We will need to be able to work with the exterior as well. For $a \in \mathbb{C}$, let $\Gamma_a : \mathbb{C}\setminus\{a\} \to \mathbb{C}\setminus\{0\}$ be given by $\Gamma_a(z) = \frac{1}{z-a}$. Obviously, $\Gamma_a$ is a holomorphic homeomorphism with $\Gamma_a^{-1}(w) = \frac{1}{w} + a$. Let $J$ be a Jordan curve and let $H$ be the interior of $J$. Let $a \in H$. Then $\Gamma_a(J)$ is also a Jordan curve with exterior $\Gamma_a(H \setminus \{a\})$ and interior $\Gamma_a(\mathbb{C} \setminus H^-) \cup \{0\}$. Then we may apply the Carathéodory and Riemann theorems to get a homeomorphism $\Phi : \Gamma_a(\mathbb{C} \setminus H) \cup \{0\} \to \mathbb{D}^-$ that is holomorphic on $\Gamma_a(\mathbb{C} \setminus H^-)$ and has $\Phi(0) = 0$. Then the function $\Phi' = \Gamma_a^{-1} \circ \Phi \circ \Gamma_a$ is a homeomorphism from $\mathbb{C} \setminus H$ to $\mathbb{C} \setminus \mathbb{D}$ which is holomorphic on $\mathbb{C} \setminus H^-$. In this way we may get a similar result to the Carathéodory theorem for the exterior of a Jordan curve. In either case, whether on the interior of the Jordan curve or the exterior, we will call such a map a *Carathéodory map.*

As an application of Carathéodory maps, we have the following theorem.

**Theorem 4.1.6.** *Let $J$ be a Jordan curve and let $H$ be the interior of $J$. Then there is a continuous function $\gamma : (0, 1] \times [-\pi, \pi] \to \mathbb{C} \setminus H$ so that $\gamma(\{1\} \times [-\pi, \pi]) = J$ and, for each*

*$s \in (0,1)$, $\gamma_s = \gamma(s,\cdot)$ is a closed rectifiable curve in $\mathbb{C} \setminus H^-$ with* $\operatorname{idx}(\gamma_s, a) = 1$*. If $J$ is symmetric w.r.t. $\mathbb{R}$ then $\gamma$ can be chosen so that $\gamma(s,-t) = \overline{\gamma(s,t)}$.*

*Proof.* For the moment, fix $a \in H$. Using what we just did, $\Gamma_a(J)$ is a Jordan curve with interior $\Gamma_a(\mathbb{C} \setminus H^-) \cup \{0\}$ and there is a Carathéodory map $\Phi : \Gamma_a(\mathbb{C} \setminus H) \cup \{0\} \to \mathbb{D}^-$ with $\Phi(0) = 0$. Define $\gamma : (0,1] \times [-\pi,\pi] \to \mathbb{C}$ by $\gamma(s,t) = \Gamma_a^{-1}(\Phi^{-1}(se^{-it}))$. For ease of notation, write $\gamma_s$ for the curve $\gamma(s,\cdot)$. While $\gamma_1$ may not be rectifiable, it is, none the less, well-defined and $\gamma_1 = J$ as sets. For $s < 1$, by a change of variables and our explicit expression for $\Gamma_a$,

$$\begin{aligned}\operatorname{idx}(\gamma_s, a) &= \frac{1}{2\pi i}\int_{\gamma_s} \frac{1}{z-a}\,dz = \frac{1}{2\pi i}\int_{\Phi^{-1}(se^{-it})} \frac{(\Gamma_a^{-1})'(z)}{\Gamma_a^{-1}(z) - a}\,dz = \frac{1}{2\pi i}\int_{\Phi^{-1}(se^{-it})} \frac{-1/z^2}{1/z}\,dz \\ &= \frac{1}{2\pi i}\int_{\Phi^{-1}(se^{-it})} \frac{-1}{z}\,dz = -\frac{1}{2\pi i}\int_{se^{-it}} \frac{(\Phi^{-1})'(z)}{\Phi^{-1}(z)}\,dz = \frac{1}{2\pi i}\int_{se^{it}} \frac{(\Phi^{-1})'(z)}{\Phi^{-1}(z)}\,dz.\end{aligned}$$

By the argument principle, this last quantity is the number (including multiplicities) of zeros of $\Phi^{-1}$ in $\mathbb{D}$. We already know that $\Phi(0) = 0$ and, since $\Phi^{-1}$ is a bijection, $\Phi^{-1}$ has no more zeros. Using the formula for derivatives of inverse functions, $\Phi'(z) = \frac{1}{(\Phi^{-1})'(\Phi(z))}$ so $(\Phi^{-1})'(0) \neq 0$ or else $\Phi'$ would have a pole at zero. Therefore the multiplicity of the zero of $\Phi^{-1}$ at the origin must be 1. Since $\operatorname{idx}(\gamma_s, a)$ is continuous in $a$ on $\mathbb{C} \setminus \gamma_s$, this implies that $\operatorname{idx}(\gamma_s, a') = 1$ for all $a'$ in the interior of $\gamma_s$ and, in particular, for all $a' \in H$ since $\gamma_s \subseteq \mathbb{C} \setminus H^-$.

Now suppose that $J$ is symmetric with respect to $\mathbb{R}$. Then, since $\mathbb{R}$ also divides $\mathbb{C}$ into two components, $H \cap \mathbb{R} \neq \emptyset$. Therefore, since our initial choice of $a \in H$ was arbitrary, we may restrict our choice to $a \in H \cap \mathbb{R}$. Then $\Gamma_a(z) = \frac{1}{z-a}$ is hermitian so $\Gamma_a(J)$ and $\Gamma_a(\mathbb{C} \setminus H) \cup \{0\}$ are also symmetric w.r.t. $\mathbb{R}$ and, since $\Phi(0) = 0$, $\Phi$ is also hermitian. Finally, tracing through the composition $\gamma(s,t) = \Gamma_a^{-1}(\Phi^{-1}(se^{-it}))$, we see that $\gamma(s,-t) = \overline{\gamma(s,t)}$. □

The above construction is just one use we will make of the Riemann, Carathéodory, and Schonflies theorems. In the following sections, we will use these theorems to transform the entire problem of factoring positive functions into a problem on the annulus.

## 4.2 Non-vanishing functions

One of the primary issues we faced in the disc and the annulus was how to get continuous boundary values. In order to solve this problem, we used lemmas 2.2.1 and 3.2.3 which allowed us to get a continuous extension to most of the boundary. One of the key points of the proof of lemma 2.2.1 was a subtle use of the fact that, given a point $z \in \mathbb{T}$, there exists an arbitrarily small open set $U \subseteq \mathbb{C}$ containing $z$ so that $U \cap \mathbb{D}$ is connected. Fortunately, this is a property that is shared by all domains where the boundary is a union of finitely many Jordan curves.

**Lemma 4.2.1.** *Let $G$ be a domain such that $\partial G$ is the union of finitely many disjoint Jordan curves. Then, for every $z \in \partial G$ and every open neighborhood $U$ of $z$ in $\mathbb{C}$, there is an open neighborhood $V \subseteq U$ of $z$ so that $V \cap G$ that is connected.*

*Proof.* Let $z \in \partial G$ and let $U$ be an open neighborhood of $z$ in $\mathbb{C}$. Let $\partial G = \bigcup_{i=1}^{n} J_i$ where the $J_i$ are Jordan curves. Since $z \in \partial G$, there is some $1 \leq k \leq n$ so that $z \in J_k$. Since the curves are disjoint, $z \notin J_i$ for any $i \neq k$. By Schoenflies Theorem (4.1.3, [14]), there is a homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ with $\phi(J_k) = \mathbb{T}$. Since $G$ is connected, $\phi(G)$ lies in only one component of $\mathbb{C} \setminus \mathbb{T}$; call this component $D$. Since $\bigcup_{i \neq k} J_i$ is compact and does not contain $z$, there is some $\delta > 0$ so that $B_\delta(\phi(z)) \subseteq \phi(U)$ and $B_\delta(\phi(z))^- \cap \phi(J_i) = \emptyset$ for all $i \neq k$. Let $N = B_\delta(\phi(z)) \cap \phi(G)$ and $H = B_\delta(\phi(z)) \cap D$. Regardless of whether $D$ is the bounded or infinite unit disc, it is obvious that $H$ is connected and $N \subseteq H$. Also, $N$ is open in $H$ because $N$ is open in $\mathbb{C}$ and $N$ is closed in $H$ because

$$\begin{aligned} N^- \subseteq B_\delta(\phi(z))^- \cap \phi(G)^- &= B_\delta(\phi(z))^- \cap \phi\left(G \cup \bigcup_{n=1}^{n} J_i\right) \\ &= (B_\delta(\phi(z))^- \cap \phi(G)) \cup (B_\delta(\phi(z))^- \cap \phi(J_k)) \end{aligned}$$

and

$$N^- \cap H \subseteq (B_\delta(\phi(z))^- \cap \phi(G) \cap H) \cup (B_\delta(\phi(z))^- \cap \phi(J_k) \cap H) = B_\delta(\phi(z)) \cap \phi(G) = N$$

since $H \cap \phi(J_k) = H \cap \mathbb{T} = \emptyset$. Therefore, since $H$ is connected and $N$ is clearly non-empty, $N = H$ and $N$ is connected. Taking $V = \phi^{-1}(B_\delta(\phi(z)))$, the result follows. □

With the above lemma, the proof of theorem 4.2.2 proceeds in much the same way as the proof of lemma 2.2.1 replacing explicit properties of the disc with the result of the lemma.

**Theorem 4.2.2.** *Let $G$ be such that $\partial G$ is the union of finitely many disjoint Jordan curves. Suppose $h : G \to \mathbb{C}$ is continuous and that there is a continuous function $F : G^- \to \mathbb{C}$ with $F = e^h$ on $G$. If $K$ is the set of zeros of $F$ on $\partial G$ then $h$ can be continuously extended to $G^- \setminus K$.*

*Proof.* Write $h = u + iv$ where $u$ and $v$ are real-valued. Since $u(z) = \log |F(z)|$ on $G$, it is easy to see that $u$ can be extended to $G^- \setminus K$. Fix $z_0 \in \partial G \setminus K$; then $F(z_0) \neq 0$. Let $a$ be a branch of the argument defined in some neighborhood $U$ of $F(z_0)$. Then $F^{-1}(U)$ is open in $G^-$ and $a \circ F$ is continuous on $F^{-1}(U)$. Since $F^{-1}(U)$ is open in $G^-$, there is a set $V$ that is open in $\mathbb{C}$ so that $F^{-1}(U) = V \cap G^-$. By the preceding lemma, there is an open set $W \subseteq V$ containing $z_0$ so that $W \cap G$ is connected. Since $F = e^h$ on $G$, $a(F(z)) - h(z)$ is an integer multiple of $2\pi$ for every $z \in W \cap G$. Since $A \circ F - h$ is continuous on $W \cap G$ and $W \cap G$ is connected, $a \circ F - h$ is constant on $W \cap G$. Therefore, since $a \circ F$ is continuous on $W \cap G^-$, $h$ can be extended continuously to $W \cap G^-$; since $z$ was arbitrary, $h$ can be continuously extended to $G^- \setminus K$. □

The above theorem solves the problem of continuous boundary values, but not the problem of actually taking a logarithm. In the case of the disc, we could take a logarithm of a non-vanishing function because the disc is simply connected. In the annulus we solved this problem by the use of winding numbers. On more general domains, the concept of winding numbers still works but it gets a bit more complicated because there are now multiple holes. In particular, we need to define the concept of the *winding number of $f$ about $H$* where $H$ is a hole in the domain of $f$.

**Definition 4.2.3.** Let $G$ be a domain where $\partial G$ is the union of finitely many disjoint Jordan curves and let $H$ be a hole of $G$ with boundary $J$. Let $\gamma_s(t) = \gamma(s,t)$ be the function given by theorem 4.1.6. Then the *winding number of $f$ about $H$*, denoted $\mathrm{wn}(f;H)$, is given by

$$\mathrm{wn}(f;H) = \lim_{s \to 1} \frac{1}{2\pi i} \oint_{\gamma_s} \frac{f'(z)}{f(z)}\, dz.$$

That the above expression exists isn't entirely trivial. Since the Jordan curves which make up $\partial G$ are compact and disjoint and, since $\gamma_s \to \gamma_1$ uniformly as $s \to 1$, there is some $s_0 < 1$ so that, $\gamma_s \in G$ for all $s_0 < s < 1$. Also, the integral above is continuous in $s$ integer-valued since it is equal to $\operatorname{idx}(f \circ \gamma_s, 0)$. Therefore, for $s > s_0$, the integral is well-defined and constant so the limit exists.

As was the case in the annulus, winding numbers allow us to replace the integral condition in theorem 3.2.1 with a condition that is easier for us to satisfy. In particular, we have the following theorem:

**Theorem 4.2.4.** *Let $G$ be a domain such that $\partial G$ is the union of finitely many disjoint Jordan curves and let $f \in \mathcal{H}(G)$ be non-vanishing. If $\operatorname{wn}(f; H) = 0$ for each hole $H$ of $G$ then there exists a holmomorphic function $g$ on $G$ so that $f = e^g$.*

*Proof.* Enumerate the bounded holes of $G$ as $H_1, \ldots, H_n$ and, for each $i$, let $a_i \in H_i$. For the moment, fix $1 \le i \le n$. Let $\gamma : (0,1] \times [-\pi, \pi]$ be the function given by theorem 4.1.6 and let $s_0 > 0$ so that $\gamma_s \in G$ for all $s_0 < s < 1$. Let $\zeta_i = \gamma_s$ for some $s_0 < s < 1$. Then $\operatorname{idx}(\zeta_i, a) = 1$ for all $a \in H_i$. Also, $\operatorname{idx}(\zeta_i, a) = 0$ for all $a$ in the unbounded component of $\mathbb{C} \setminus \zeta_i$ and, in particular, $\operatorname{idx}(\zeta_i, a) = 0$ for all $a \in H_j$ whenever $i \neq j$.

Now, let $\eta$ be any simple closed curve in $G$. Let $\Gamma = \eta - \sum_{i=1}^{n} \operatorname{idx}(\eta, a_i)\zeta_i$. Then, for any $1 \le i \le n$, and any $a \in H_i$,

$$\operatorname{idx}(\Gamma, a) = \operatorname{idx}(\eta, a) - \sum_{j=1}^{n} \operatorname{idx}(\eta, a_j) \operatorname{idx}(\zeta_j, a) = \operatorname{idx}(\eta, a) - \operatorname{idx}(\eta, a_j) = 0.$$

Obviously, $\operatorname{idx}(\Gamma, a) = 0$ for any $a$ in the unbounded component of $\mathbb{C} \setminus G^-$ (if one exists), so $\Gamma$ is homologous to zero. Since $f$ is non-vanishing, $f'/f$ is holomorphic and $\oint_\Gamma \frac{f'(z)}{f(z)}\, dz = 0$. Also,

$$0 = \frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \oint_\eta \frac{f'(z)}{f(z)}\, dz - \sum_{i=1}^{n} \operatorname{idx}(\eta, a_j) \frac{1}{2\pi i} \oint_{\zeta_i} \frac{f'(z)}{f(z)}\, dz$$
$$= \frac{1}{2\pi i} \oint_\eta \frac{f'(z)}{f(z)}\, dz - \sum_{i=1}^{n} \operatorname{idx}(\eta, a_j) \operatorname{wn}(f; H_i) = \frac{1}{2\pi i} \oint_\eta \frac{f'(z)}{f(z)}\, dz.$$

Since this holds for every simple closed curve $\eta$, the result follows from theorem 3.2.1. $\square$

In order to apply the above theorem we will take a similar approach to what we did in the annulus and divide out by a rational function to get a function that has $\operatorname{wn}(f; H) = 0$ for every hole $H$ of the domain. In order to do this, we will need to know a bit about the interaction between winding numbers and the involution.

**Theorem 4.2.5.** *Let $G$ be a symmetric domain where $\partial G$ is the union of finitely many disjoint Jordan curves, let $f \in \mathcal{H}(G)$, and let $H$ be a hole of $G$. Then* $\operatorname{wn}(f^*; H^*) = \operatorname{wn}(f; H)$*.*

*Proof.* Let $\gamma_s(t)$ be the curve given by theorem 4.1.6 for $f$ and $H$ and let $\zeta_s(t)$ be the curve corresponding to $f^*$ and $H^*$. Let $0 < s_0 < 1$ be such that $\gamma_s, \zeta_s \in G$ for all $s_0 < s < 1$. Let $a \in H$ and observe that $\operatorname{idx}(\gamma_s, a) = 1 = -\operatorname{idx}(\bar{\gamma}_s, \bar{a})$ and $\operatorname{idx}(\zeta_s, a) = 1$; this holds for all $a \in H$. Since $\gamma_s, \zeta_s \in G$ for all $s_0 < s < 1$, all of the other holes of $G$ lie in the exterior of both $\gamma_{s_0}$ and $\zeta_{s_0}$. Fix $s_0 < s < 1$. Then, by what we have just shown, $\operatorname{idx}(\bar{\gamma}_s + \zeta_s, a) = 0$ for all $a \in \mathbb{C} \setminus G^-$; i.e. $\bar{\gamma}_s + \zeta_s$ is homologous to zero. Therefore, since we know the winding number is a real number,

$$\begin{aligned}
\operatorname{wn}(f^*; H^*) &= \frac{1}{2\pi i} \oint_{\zeta_s} \frac{f^{*\prime}(z)}{f^*(z)}\, dz = -\frac{1}{2\pi i} \oint_{\bar{\gamma}_s} \frac{f^{*\prime}(z)}{f^*(z)}\, dz = -\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f^{*\prime}(\bar{\gamma}_s(t))}{f^*(\bar{\gamma}_s(t))} \bar{\gamma}_s'(t)\ d \\
&= -\frac{1}{2\pi i} \int_{-\pi}^{\pi} \overline{\frac{f'(\gamma_s(t))}{f(\gamma_s(t))}} \bar{\gamma}_s'(t)\ dt = \overline{\frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f'(\gamma_s(t))}{f(\gamma_s(t))} \gamma_s'(t)\ dt} \\
&= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f'(\gamma_s(t))}{f(\gamma_s(t))} \gamma_s'(t)\ dt = \frac{1}{2\pi i} \oint_{\gamma_s} \frac{f'(\gamma_s(t))}{f(\gamma_s(t))}\, dz = \operatorname{wn}(f; H).
\end{aligned}$$

□

In order for this to work nicely with positive functions, we will need the following generalization of corollary 3.2.5:

**Theorem 4.2.6.** *Let $G$ be a symmetric domain where $\partial G$ is the union of finitely many disjoint Jordan curves and let $H$ be a hole of $G$ with boundary $J$. Let $f \in \mathcal{H}(A)$ be non-vanishing with $f(G \cap \mathbb{R}) \geq 0$. If $f$ is positive and $H^* = H$ then* $\operatorname{wn}(f; H)$ *is even.*

*Proof.* Let $\gamma$ be the function given by theorem 4.1.6. Let $0 < s_0 < 1$ so that, for all $s_0 < s < 1$, $\gamma_s \in G$. Let $s_0 < s < 1$. Then, since $f$ is hermitian, $f(\gamma_s(-t)) = f(\overline{\gamma_s(t)}) = \overline{f(\gamma_s(t))}$. Since $f$ is non-vanishing with $f(G \cap \mathbb{R}) \geq 0$, $f(\gamma_s(0)) > 0$. For any $s \in (0, 1)$, since $\gamma_s$ is a closed curve with $\gamma(s, -t) = \overline{\gamma(s, t)}$, $\gamma(s, \pi) = \gamma(s, -\pi) \in \mathbb{R}$ and $f(\gamma_s(\pi)) > 0$. Therefore, by theorem 3.2.4, $\operatorname{wn}(f; H) = \operatorname{idx}(f \circ \gamma_s, 0)$ is even.

□

We now have enough pieces that we can begin to prove a version of theorem 1.2.1 in this more general setting. As with the disc and the annulus, our first theorem is restricted to non-vanishing functions.

**Theorem 4.2.7.** *Let $G$ be a symmetric domain where $\partial G$ is the union of finitely many disjoint Jordan curves and let $f \in \mathcal{H}(G)$ be non-vanishing. Then $f$ is positive if and only if there is some $g \in \mathcal{H}(G)$ so that $f = g^* g$. Furthermore, if $f \in H^\infty(G)$ then $g$ may be chosen in $H^\infty(G)$ and, if $f \in \mathcal{A}(G)$, then $g$ may be chosen in $\mathcal{A}(G)$ and $f$ and $g$ have the same zeros on $\partial G$.*

*Proof.* The reverse direction is obvious. Suppose that $f$ is positive, i.e., $f(G \cap \mathbb{R}) \geq 0$. Enumerate the holes of $G$ which non-trivially intersect the upper half-plane as $H_1, \ldots, H_n$. For each $H_i$, observe that, since $H_i$ is connected, either $H_i$ lies entirely in the upper half-plane or $H_i \cap \mathbb{R} \neq \emptyset$. If $H_i \cap \mathbb{R} \neq \emptyset$ then $H_i^*$ is also a hole of $G$ since $G$ is symmetric and $H_i \cap H_i^* \neq \emptyset$ so $H_i = H_i^*$. For each $i$, let $a_i \in H_i$ and $p_i = \operatorname{wn}(f; H_i)$ if $H_i$ lies in the upper half-plane and let $a_i \in H_i \cap \mathbb{R}$ and $p_i = \frac{1}{2}\operatorname{wn}(f; H_i)$ if $H_i^* = H_i$. Define the rational function

$$q(z) = (z - a_1)^{p_1}(z - a_2)^{p_2} \cdots (z - a_n)^{p_n}$$

and note that $q$ is holomorphic and non-vanishing on $G$. The involute $q^*$ is given by

$$q^*(z) = (z - \bar{a}_1)^{p_1}(z - \bar{a}_2)^{p_2} \cdots (z - \bar{a}_n)^{p_n}$$

which is also holomorphic and non-vanishing on $G$. Since the winding number of a product of functions is the sum of the winding numbers (this is trivial to prove),

$$\operatorname{wn}(q; H_i) = \sum_{j=1}^{n} p_j \operatorname{wn}(z - a_j, H_i) = \sum_{j=1}^{n} p_j \lim_{s \to 1} \operatorname{idx}(\gamma_i(s, \cdot), a_j) = \sum_{j=1}^{n} p_j \delta_{ij} = p_i$$

where $\gamma_i$ is the function given by theorem 4.1.6. For any hole $H$ of $G$ in the lower half-plane, $q$ is holomorphic and non-vanishing on some open set containing $H$ so $\operatorname{wn}(q; H) = 0$. From theorem 4.2.5, we know that $\operatorname{wn}(q^*; H^*) = \operatorname{wn}(q; H)$ for any hole $H$. Therefore, for any $i$ where $H_i$ is entirely in the upper half-plane,

$$\operatorname{wn}(q^* q; H_i) = \operatorname{wn}(q^*; H_i) + \operatorname{wn}(q; H_i) = \operatorname{wn}(q; H_i) = p_i = \operatorname{wn}(f; H_i)$$

and

$$\begin{aligned}\mathrm{wn}(q^*q;H_i^*) &= \mathrm{wn}(q^*;H_i^*)+\mathrm{wn}(q;H_i^*) = \mathrm{wn}(q^*;H_i^*) = \mathrm{wn}(q;H_i) = p_i \\ &= \mathrm{wn}(f;H_i) = \mathrm{wn}(f;H_i^*).\end{aligned}$$

For any $i$ where $H_i^* = H_i$,

$$\mathrm{wn}(q^*q;H_i) = \mathrm{wn}(q^*;H_i)+\mathrm{wn}(q;H_i) = \mathrm{wn}(q^*;H_i^*)+\mathrm{wn}(q;H_i) = 2\mathrm{wn}(q;H_i) = 2p_i = \mathrm{wn}(f;H_i).$$

Since every hole $H$ of $G$ is either $H_i$ or $H_i^*$ for some $i$, we have shown that $\mathrm{wn}(q^*q;H) = \mathrm{wn}(f;H)$ for any hole $H$. Since $q^*q$ is non-vanishing on $G$, we may define $F \in \mathcal{H}(G)$ by $F = f/q^*q$ and note that, since $f$ is positive and $q^*q$ is obviously positive, $F$ is positive. Then $\mathrm{wn}(F;H) = \mathrm{wn}(f;H) - \mathrm{wn}(q^*q;H) = 0$ for every hole $H$ of $G$. By theorem 4.2.4, there is some function $h \in \mathcal{H}(G)$ with $F = e^h$. Since $F$ is positive, the complex part of $h$ must only take on values in $2\pi i\mathbb{Z}$ on $G \cap \mathbb{R}$. Therefore, by adding some constant integer multiple of $2\pi i$, $h$ may be chosen so that $h(x) \in \mathbb{R}$ for some $x \in G \cap \mathbb{R}$. Since $G$ is open, $G \cap \mathbb{R}$ is open in $\mathbb{R}$ and there is some open interval $x \in I \subseteq G \cap \mathbb{R}$ and, since the complex part of $h$ takes on only integer multiples of $2\pi i$ on $G \cap \mathbb{R}$, $h|_I$ is real-valued. Therefore, by theorem 1.2.2, $h$ is hermitian. Letting $g = qe^{h/2}$, it is easy to see that $g^*g = q^*qe^{h/2+h^*/2} = f$.

Now suppose that $f$ is bounded. Suppose that $G$ is bounded. Then $q$ is both bounded and bounded away from zero on $G^-$. This implies that $F = f/q^*q$ is bounded and $g = q\sqrt{F}$ is also bounded. Now suppose that $G$ is unbounded. Since $\partial G$ is the union of finitely many Jordan curves, it is compact and there is some $R \in \mathbb{R}$ so that $\{z \in \mathbb{C} : |z| > R\} \subseteq G$. Then $f(1/z)$ is holomorphic on the punctured disc of radius $1/R$ and, since $f(1/z)$ is also bounded ($f$ is bounded by assumption), the singularity at the origin is removable; in particular, $\lim_{z\to\infty} f(z)$ exists. Let $r = \sum_{i=1}^n p_i$. Then $\lim_{z\to\infty} q(z)z^{-r} = 1$ because $q(z)z^{-r}$ is a rational function where the sum of the exponents is zero. Therefore,

$$\lim_{z\to\infty} F(z)z^{2r} = \lim_{z\to\infty} \frac{f(z)}{q^*(z)z^{-r}q(z)z^{-r}} = \lim_{z\to\infty} f(z)$$

since $\lim_{z\to\infty} q(z)z^{-r} = 1$ and

$$\lim_{z\to\infty} |g(z)| = \lim_{z\to\infty} |q(z)e^{h(z)/2}| = \lim_{z\to\infty} |q(z)z^{-r}|\sqrt{|z^{2r}e^{h(z)}|} = \sqrt{\lim_{z\to\infty} |f(z)|}.$$

This implies that $g(z)$ is bounded for $|z| \geq R$ and, by our argument about bounded domains, $g$ is bounded on $\{z \in G^- : |z| \leq R\}$ so $g$ is bounded on $G$.

Finally, suppose that $f \in \mathcal{A}(G)$. Since $q$ is continuous, bounded, and bounded away from zero on $G^-$, $F = f/q^*q \in \mathcal{A}(G)$. By theorem 4.2.2, $h$ extends continuously to $G^- \setminus K$ where $K$ is the set of zeros of $F$ on $\partial G$. We can then define the function $\sqrt{F}$ as $\sqrt{F} = e^{h/2}$ on $G^- \setminus K$ and $\sqrt{F} = 0$ on $K$. Obviously, $\sqrt{F}$ is continuous on $G^- \setminus K$. Since $|\sqrt{F}| = \sqrt{|F|}$ on all of $G^-$, $\lim_{z \to z_0} |\sqrt{F}| = 0$ for $z_0 \in K$. Therefore, $\lim_{z \to z_0} \sqrt{F} = 0$ for all $z_0 \in K$ and $\sqrt{F}$ is continuous on all of $G^-$. Since $q$ is also continuous on $G^-$, $g$ is continuous on $G^-$ and $g \in \mathcal{A}(G)$. Since $q$ is both bounded and bounded away from zero on $\partial G$, the functions $f$, $F$, $\sqrt{F}$, and $g$ all have the same zeros on $\partial G$. □

Unlike theorems 2.2.2 and 3.2.6, the above theorem says nothing about $H^p$ spaces for $p < \infty$. This is because, while $H^\infty(G)$ and $\mathcal{A}(G)$ are fairly obviously defined on $G$, there is no obvious way to choose integration paths through $G$ so as to make $H^p(G)$ well-defined for $p < \infty$.

### 4.3 Positive functions with zeros

We will now attack the issue of positive functions which may have zeros. As with the disc and the annulus, this will be done using Blaschke products. However, unlike the disc and the annulus, we have no concept of a Blaschke product on an arbitrary domain. In order to solve this, we will need to apply the Carathéodory mapping theorem to transform the Blaschke product from the disc into a function on our domain.

It will be easier for this discussion to consider our domain $G$ as a subset of the compactification $C_\infty$ of the complex plane. In order to do so, we need to say a few words about the space $H^\infty(G)$. If our domain $G$ is bounded, then we can simply embed $G$ into $\mathbb{C}_\infty$ without any problems. Suppose $G$ is unbounded. If $\partial G$ is the union of finitely many Jordan curves then $\partial G$ is compact in $\mathbb{C}$, hence bounded by some radius $R$. Then the infinite disc $\{z \in \mathbb{C} : |z| > R\}$ is contained in $G$. Given $f \in H^\infty(G)$, $f(R/z)$ is holomorphic on $\mathbb{D} \setminus \{0\}$ and, since $f(R/z)$ is also bounded, the singularity at zero is removable. Therefore, $f(R/z)$ is holomorphic in $\mathbb{D}$ and the original function $f$ can be extended to $\infty$. (Interestingly, this also implies that the zeros of $f$ are bounded because, if they weren't, then the zeros of $f(R/z)$ would have a limit point at zero and $f(R/z)$ would be identically zero by the identity theorem.) Since this holds for all $f \in H^\infty(G)$, we may embed the unbounded domain $G$ into $\mathbb{C}_\infty$ by adding the point at infinity to $G$ and extend all of the functions in $H^\infty(G)$ accordingly. The algebra $\mathcal{A}(G)$ is then just the subalgebra of $H^\infty(G)$ where each function $f \in \mathcal{A}(G)$ can be continuously extended to $G^-$. Since $\infty \notin \partial G$, this yields the same subalgebra as before.

Since we will be making heavy use of the Carathéodory theorem, we also need to consider Carathéodory maps in $\mathbb{C}_\infty$. Recall our discussion of Carateheódory maps from the exterior of a Jordan curve to the exterior of $\mathbb{T}$. Let $J$ be a Jordan curve in $\mathbb{C}$ (for the moment, we will only consider curves which do not pass through $\infty$). Then $\mathbb{C}_\infty \setminus J$ has two components one of which contains the point at infinity. We will call these components $B$ and $U$ where $\infty \in U$. Let $b \in B$. Since $B$ is also a component of $\mathbb{C} \setminus J$, the regular Riemann mapping theorem gives us a map $\phi : B \to \mathbb{D}$ with $\phi(b) = 0$. By the Carathéodory theorem and the continuity of $1/z$, $\phi$ can be extended to a homeomorphism of $B^-$ and $\mathbb{D}^-$. Now we consider the unbounded component $U$. The map $\Gamma_b(z) = \frac{1}{z-b}$ is a holomorphic bijection from $\mathbb{C} \setminus \{b\}$ to $\mathbb{C} \setminus \{0\}$ and is easily extended to an automorphism of $C_\infty$ by setting $\Gamma_b(\infty) = 0$ and $\Gamma_b(b) = \infty$. Let $u \in U$ (where $u$ may take on the value $\infty$). Then $\Gamma_b(U)$ is the interior of the Jordan curve $\Gamma_b(J)$ in $\mathbb{C}$ and the Riemann mapping theorem yields a holomorphic bijection $\varphi : \Gamma_b(U) \to \mathbb{D}$ with $\varphi(\Gamma_b(u)) = 0$. Then the composition $\varphi \circ \Gamma_b$ is a holomorphic bijection of $U$ to $\mathbb{D}$ taking $u$ to 0 which can be extended to a homeomorphsim of $U$ with $\mathbb{D}^-$ by the Carathéodory theorem and the continuity of $\Gamma_b$. From this point forward, we will freely map either the inside or outside of any Jordan curve to the disc and call such a map a Carathéodory map.

**Lemma 4.3.1.** *Let $G$ be a symmetric domain where $\partial G$ is the union of finitely many disjoint Jordan curves, let $H$ be a component of $\mathbb{C}_\infty \setminus G^-$, and let $\phi : \mathbb{C}_\infty \setminus H \to \mathbb{D}^-$ be a Carathéodory map with $\phi^* = \phi$ if $H^* = H$. Let $f \in H^\infty(G)$ be positive. Then we can factor $f$ as $f(z) = g(z)B^*(\phi^*(z))B(\phi(z))$ where $B$ is a Blaschke product and $g \in H^\infty(G)$ is positive so that the zeros of $g$ have no limit points in $\partial H \cup \partial H^*$. Furthermore, if $f \in \mathcal{A}(G)$ then $g \in \mathcal{A}(G)$ and $g$ has the same zeros on $\partial G$ as $f$.*

*Proof.* First, suppose $A = \{z \in \mathbb{C} : r_0 < |z| < 1\}$ is an annulus such that $A^- \cap \phi(\partial G) = \mathbb{T}$. (That such an annulus exists will be shown later.) Then $A \subseteq \phi(G)$ and the restricted function $\tilde{f} = (f \circ \phi^{-1})|_A$ is an element of $H^\infty(A)$. Since $A^- \cap \phi(\partial G) = \mathbb{T}$ and the limit points of the zeros of $f$ lie in $\partial G$, the limit points of the zeros of $\tilde{f}$ lie entirely in $\mathbb{T}$. Therefore, by theorem 3.1.6, the Blaschke product $B$ corresponding to the zeros of $\tilde{f}$ converges and $\tilde{f}/B$ is bounded on $A$. Since $B$ is holomorphic on $\mathbb{D}$ and the zeros of $\tilde{f}$ have no limit points in $r_0\mathbb{T}$, $B$ is bounded away from zero for $|z| \leq r_0$ and $(f \circ \phi^{-1})/B$ is bounded on all of $\phi(G)$. Composing with $\phi$, we have $f/(B \circ \phi) \in H^\infty(G)$. If $f \in \mathcal{A}(G)$ then $\tilde{f} \in \mathcal{A}(A)$ and, by the second part of theorem 3.1.6, $\tilde{f}/B \in \mathcal{A}(A)$ and $\tilde{f}/B$ has the same zeros as $\tilde{f}$ on $\mathbb{T}$. Since $B$ is continuous and non-vanishing for all $|z| \leq r_0$, we have that $(f \circ \phi^{-1})/B$ is continuous on $\phi(G^-)$ and $(f \circ \phi^{-1})/B$ has the same zeros on $\phi(\partial G) \setminus \mathbb{T}$ as $f \circ \phi^{-1}$. Following through the composition, $f/(B \circ \phi) \in \mathcal{A}(G)$ and $f/(B \circ \phi)$ has the same zeros as $f$ on $\partial G$.

We now consider two cases. First, suppose that $H^* = H$. Since the Jordan curves which make up $\partial G$ are disjoint, $\partial G \setminus \partial H$ is compact and, since $\phi$ is a homeomorphism, $\phi(\partial G \setminus \partial H)$ is a compact subset of $\mathbb{D}$. Therefore, there is some annulus $A = \{z \in \mathbb{C} : r_0 < |z| < 1\}$ so that $A^- \cap \phi(\partial G \setminus \partial H) = \emptyset$. Then, by what we showed above, $g = f/(B \circ \phi) \in H^\infty(G)$ and, if $f \in \mathcal{A}(G)$, $g \in \mathcal{A}(G)$ and $f$ and $g$ have the same zeros on $\partial G$. Since $H^* = H$, $\phi = \phi^*$ by assumption so $\tilde{f}$ (as defined above) is a positive element of $H^\infty(A)$. Therefore, we can split the Blaschke product $B$ as $B = B_+^* B_+$ as we did in the case of the disc and we have $f(z) = g(z)B_+^*(\phi^*(z))B_+(\phi(z))$.

Now consider the case where $H^* \neq H$. Then $(H^*)^- \cap H^- = \emptyset$ and, in particular, $H^- \cap \mathbb{R} = \emptyset$. Since $\mathbb{R} \subseteq \mathbb{C}_\infty \setminus H^-$, $\phi(\mathbb{R} \cup \{\infty\})$ is a compact subset of $\mathbb{D}$. Therefore, there is an annulus $A = \{z \in \mathbb{C} : r_0 < |z| < 1\}$ so that $A^- \cap \phi(\partial G \setminus \partial H) = \emptyset$ (by the same argument we made above) and $A^- \cap \phi(\mathbb{R} \cup \{\infty\}) = \emptyset$. Therefore, by the first part of the proof, $f/(B \circ \phi) \in H^\infty(G)$. Since $A$ is connected and $\phi^{-1}(A^-) \cap \mathbb{R} = \emptyset$, $(\phi^*)^{-1}(A^-) \cap \phi^{-1}(A^-) = \emptyset$ so $B \circ \phi$ and $B^* \circ \phi^*$ have disjoint sets of zeros. Applying the first part of the proof to $f/(B \circ \phi)$ and the Carathéodry map $\phi^*$, we have that $f/(B \circ \phi)(B^* \circ \phi^*) \in H^\infty(G)$. If $f \in \mathcal{A}(G)$ then $f/(B \circ \phi)$ and $g$ are both continuous on $G^-$ and $f$, $f/(B \circ \phi)$, and $g$ all have the same zeros on $\partial G$. □

We now have all of the pieces in place to prove our factorization result in our most general form. At this point, the proof is just a matter of stringing together the pieces we have put together thus far.

**Theorem 4.3.2.** *Let $G$ be a symmetric domain where $\partial G$ is the union of finitely many disjoint Jordan curves and let $f \in H^\infty(G)$. Then $f$ is positive if and only if there is some $g \in H^\infty(G)$ so that $f = g^*g$. Furthermore, if $f \in \mathcal{A}(G)$ then $g$ may be chosen in $\mathcal{A}(G)$.*

*Proof.* The reverse direction and the case where $f = 0$ are both obvious. Suppose that $f$ is positive and not identically zero. Iteratively applying lemma 4.3.1, we can factor $f$ as

$$f(z) = h_1(z)B_1^*(\phi_1^*(z))B_1(\phi_1(z))B_2^*(\phi_2^*(z))B_2(\phi_2(z)) \cdots B_n^*(\phi_n^*(z))B_n(\phi_n(z))$$

where the zeros of $h_1$ have no limit points. We further claim that $h_1$ has only finitely many zeros. If $G$ is bounded, this is trivially true. Now suppose that $G$ is unbounded. Then $\partial G$ is compact so there is some $R \in \mathbb{R}$ so that $\partial G \subseteq \{z \in \mathbb{C} : |z| < R\}$ and $h_1(R/z)$ is a holomorphic function on $\mathbb{D} \setminus \{0\}$. Since $h_1$ is bounded, $h_1(R/z)$ has a removable singularity at the origin. Therefore, if the zeros of $h_1$ were unbounded, the zeros of $h_1(R/z)$ would have zero as a limit point and we would have $h_1 = 0$ by the identity theorem. However, we assumed that $f$ was not identically zero, so $h_1$ must have finitely many zeros.

We know from previous discussions that the zeros of $h_1$ must come in conjugate pairs and have even multiplicity if they lie on the real line. Therefore, we can factor $h_1$ as

$$h_1(z) = h_2(z)(z - a_1)(z - \bar{a}_1)(z - a_2)(z - \bar{a}_2) \cdots (z - a_r)(z - \bar{a}_r)$$

where $h_2$ is non-vanishing. Let $b$ be in the interior of some hole of $G$ and let $q$ be the rational function

$$q(z) = \frac{(z - a_1)(z - a_2) \cdots (z - a_r)}{(z - b)^r}.$$

Then $q$ is bounded on $G$ and $h_1/q^*q$ is also non-vanishing. Applying theorem 4.2.7 to $h_1/q^*q$ we have $h_1/q^*q = h_3^*h_3$ for some $h_3 \in H^\infty(G)$. Letting $g$ be defined as

$$g(z) = h_3(z)q(z)B_1(\phi_1(z))B_2(\phi_2(z)) \cdots B_n(\phi_n(z))$$

we have that $f = g^*g$. Since $h_3$, $q$, and the Blaschke products are all bounded, $g \in H^\infty(G)$. If $f \in \mathcal{A}(G)$, lemma 4.3.1 tells us that $h_1 \in \mathcal{A}(G)$ and, since $q$ is obviously continuous and bounded away from zero on $G^-$, $h_1/q^*q \in \mathcal{A}(G)$ and, by theorem 4.2.7, $h_3 \in \mathcal{A}(G)$. Finally, by the last clause of 4.3.1, the zeros of $h_1$ are the same as the zeros of $f$ on $\partial G$ and, since $q$ is both bounded and bounded away from zero on $\partial G$, $h_1/q^*q$ has the same zeros as $f$ and $h_1$. By theorem 4.2.7, $h_3$ also has the same zeros as $f$ on $\partial G$. Therefore, since the limit points of the zeros of the $B_i(\phi_i(z))$ must be zeros of $f$ on $\partial G$, and so are zeros of $h_3$, the product is continuous and $g \in \mathcal{A}(G)$. □

At this point, we should revisit a seemingly arbitrary restriction we made above. Even though we are primarily working on the compactification $\mathbb{C}_\infty$, we only allowed Jordan curves in $\mathbb{C}$, i.e., those curves which do not pass through the point at infinity. While this makes the proofs easier, it is not necessary. Let $G$ be a connected open subset of $\mathbb{C}_\infty$ where $\partial G$ is the union of finitely many disjoint Jordan curves which may pass through the point at infinity and let $\alpha \in \mathbb{C} \setminus \partial G$. Then $\Gamma_\alpha(z) = \frac{1}{z-\alpha}$ is a holomorphic automorphism of $\mathbb{C}_\infty$ that maps $\alpha$ to the point at infinity. Then $\Gamma_\alpha(G) \setminus \{\infty\}$ is a domain in $\mathbb{C}$ where $\partial\Gamma_\alpha(G) = \Gamma_\alpha(\partial G)$ is the union of finitely many disjoin Jordan curves in $\mathbb{C}$. Then, for a function $f \in H^\infty(G)$, we may then apply theorem 4.3.2 to $f \circ \Gamma_a^{-1} \in H^\infty(\Gamma_a(G) \setminus \{\infty\})$ and compose with $\Gamma_a$ to translate the resulting factorization back to $H^\infty(G)$. With this, we now have our final version of theorem 1.2.1:

**Theorem 4.3.3.** *Let $G \subseteq \mathbb{C}_\infty$ be open and connected so that $\partial G$ is the union of finitely many disjoint Jordan curves in $\mathbb{C}_\infty$. For each $f \in \mathcal{A}(G)$, the following are equivalent:*

1. *$f$ is positive, i.e., $f(G \cap \mathbb{R}) \geq 0$,*
2. *$f = g^*g$ for some $g \in \mathcal{A}$,*
3. *$f = \sum_{i=1}^n g_i^* g_i$ for some $g_1, \ldots, g_n \in \mathcal{A}$,*
4. *$f = \lim_{n\to\infty} f_n$ where each $f_n$ is of the form given in 3.*
5. *$\varphi(f) \geq 0$ for every one-dimensional $*$-representation $(\mathbb{C}, \varphi)$ of $\mathcal{A}(G)$*
6. *$\varphi(f)$ is positive in $B(\mathcal{H})$ for every $*$-representation $(\mathcal{H}, \varphi)$ of $\mathcal{A}(G)$*

The last two conditions in the above theorem come from the representation theory of $\mathcal{A}(G)$. For any $a \in G \cap \mathbb{R}$, the evaluation functional $\varphi_a \in \mathcal{A}(G)^*$ given by $\varphi_a(f) = f(a)$ is a one-dimensional $*$-representation of $\mathcal{A}(G)$ onto $\mathbb{C}$. This means that case (5) implies (1). Case (2) above imples (5) and (6) trivially since, for any $*$-representation $(\mathcal{H}, \varphi)$ of $\mathcal{A}(G)$, $\varphi(g^*g) = \varphi(g)^*\varphi(g)$ is positive in $B(\mathcal{H})$. These last two cases give us some hope that a similar theorem can be proved in a more general algebraic setting.

# CHAPTER 5. FUTURE WORK

There are a number of ways in which the results of this dissertation could potentially be extended. The most obvious would be to try and extend the results to an even broader class of domains. While the class of domains bounded by a finite number of disjoint Jordan curves is fairly broad, it still does not cover the full range of possibilities. For instance, one could consider domains where the boundary curves are allowed to touch or domains with an infinite number of holes. While the proofs given here require the assumption that $\partial G$ is the union of a finite number of disjoint Jordan curves, it not be necessary.

For instance, suppose we removed the assumption that the Jordan curves are disjoint. In this case, we would have two issues that would need to be handled. First is that theorem 4.2.2 relies on the fact that, about every point $z_0 \in \partial G$, there is an arbitrarily small neighborhood $U$ so that $G \cap U$ is connected. Without this, we could have a discontinuity in the logarithm at $z_0$. However, we may be able to fix this by multiplying by $(z - z_0)^p$ for some $p \in \mathbb{Z}$. The second issue is that the zeros of our function may have a limit point that is the intersection of two of the Jordan curves that make up $\partial G$. In this case, it is not clear whether or not the Blaschke product can be made to converge.

While some extension of our results to more general domains looks possible, trying to do so with the techniques presented here leads to an explosion of notation that is probably not worth the trouble. In order to get truly more general results, a new proof technique will most likely be needed. One such approach could be to consider functions on Riemann surfaces. In [13], Voichick and Zalcman briefly develop a factorization theory for a Riemann surface $R$ with boundary $\Gamma$ consisting of finitely many closed analytic curves where $R \cup \Gamma$ lies on one side of $\Gamma$. Given a concept of a symmetric Riemann surface, perhaps the factorization theory they developed could be used to develop a theory of positivity on $\mathcal{A}(R)$ or simply to overcome some of the difficulties of working on $\mathbb{C}_\infty$ directly.

Another potentially interesting extension of these results would be to look at the matrix algebra $\mathcal{M}_{n\times n}(\mathcal{A}(G))$ of matrices whose entries are functions in $\mathcal{A}(G)$. This leads to a fairly natural involution:

$$H \mapsto H^*; \quad H^*(z) = \overline{H^T(\bar{z})}.$$

The functions that we have studied are simply the one-dimensional case of this algebra. One could try and develop a similar theory of positivity that combines the linear algebraic notion of a positive semi-definite matrix with the notion of positivity we have discussed here. For instance, we know that a matrix $A$ is positive semi-definite if and only if there is another matrix $B$ so that $A = B^*B$ (this is because the $n \times n$ matrices form a $C^*$-algebra). Can we find necessary and sufficient conditions for a matrix $F \in \mathcal{M}_{n\times n}(\mathcal{A}(G))$ to be of the form $F = H^*H$ for some $H \in \mathcal{M}_{n\times n}(\mathcal{A}(G))$?

A third potential extension would be to try and find proofs of these results in a more general algebraic setting. While all of our proofs worked directly with functions on $\mathbb{C}_\infty$, the results can be stated entirely in terms of $*$-algebras. We have already commented that the condition $f(G \cap \mathbb{R}) \geq 0$ is equivalent to saying that $\varphi(f) \geq 0$ for every one-dimensional $*$-representation $(\mathbb{C}, \varphi)$ of $\mathcal{A}(G)$. This leads to the question: What are the necessary conditions on a Banach $*$-algebra $\mathcal{A}$ so that, for any $a \in \mathcal{A}$ with $\varphi(a) \geq 0$ for every one-dimensional $*$-representation $(\mathbb{C}, \varphi)$ of $\mathcal{A}$, $a$ is of the form $b^*b$ for some $b \in \mathcal{A}$.

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