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Extension of algebraic theories

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I. INTRODUCTION

In his dissertation [3], Lawvere characterizes "equationally definable classes of algebras" by "algebraic categories". For example, a group is a product preserving functor from a specific category $\mathcal{A}$ into the category of sets $\mathcal{S}$. $\mathcal{A}$ is called the algebraic theory of groups and exhibits the underlying algebraic structure of the theory, i.e., it contains the n-ary operations and identities necessary to define a group. The category $\mathcal{A}^b$ of groups has as objects all product preserving functors from $\mathcal{A}$ into $\mathcal{S}$ and the morphisms (group homomorphisms) are the natural transformations between these functors. From the category $\mathcal{A}^b$ of all groups, the identities and operations are recovered by the use of a particular functor $U_{\mathcal{A}}$, called the underlying-set functor.

In this paper we will expand an algebraic theory $\mathcal{A}$ so that it exhibits not only the operations and identities, but some of the homomorphisms, functions, objects and constructions which we encounter when working with the theory. For example, given the theory of groups, we have associated with each group: the commutator subgroup, the quotient of the group modulo its commutator subgroups, the natural homomorphisms associated with the above groups, the free group on the underlying set of the group and the set inclusion of the group into the free group. Since we can consider these entities for every group, we want our expanded theory of groups to reflect them. We feel that in some natural way the above entities are induced by the theory and so we should be able to enlarge the theory to encompass them.

We will construct a complete category such that every object and morphism will be obtained from the original algebraic theory. The category of expanded theories will have the same properties as the original category of theories, and in fact, the two categories will be equivalent.

To illustrate the structure of the new theory, we will construct the commutator subgroup for any given group. After this, we will take a slight turn away from algebraic entities induced by our theory to consider some notions relating topologies to our algebraic theory.

Topologies can be put on various algebras to make their operations continuous. We will consider an example of this process for general algebras. This example has been studied for specific algebras such as rings and modules. In generalizing the example, we will illustrate some of the underlying properties. Then we will exhibit some general methods of inducing topologies on algebras such that the existing homomorphisms and/or operations are continuous.

Chapter II contains most of the basic definitions and theorems needed in the remainder of the paper. We assume knowledge of the definitions of a category and a functor. Only covariant functors will be used. The category $\mathcal{S}$ of sets satisfies the standard Zermelo-Fraenkel axioms for set theory, along with the axiom of choice and the axiom of regularity.

The construction of our extended theories is presented in Chapter III, and is the most important part of the paper.

Chapter IV contains the commutator subgroup example and the construction of a universal object called the special coequalizer.

Chapter V deals with topologies induced on algebras by their algebraic structure.

II. PRELIMINARIES

Given a category X, $|X|$ will denote the family of objects of X and $x \in |X|$ will mean that x is an object of the category X. $f \in X$ will indicate that f is a morphism of X. For categories X and Y, $X \xrightarrow{F} Y$ will refer to a covariant functor F with domain X and codomain Y. Given functors $X \xrightarrow{F} Y$ and $X \xrightarrow{G} Y$, a <u>natural transformation</u> α from F to G, to be indicated by $F \xRightarrow{\alpha} G$, is a family of morphisms $\{\alpha_x\}_{x \in |X|}$ such that for all $f \in X$, $\alpha_{x'} \circ F(f) = G(f) \circ \alpha_x$ where $f : x \longrightarrow x'$. Sometimes $F(x)$ and $F(f)$ will be denoted by Fx and Ff respectively. $\hom_X(x,x')$ will stand for the family of morphisms in X with domain x and codomain x'. The <u>dual category</u> X^{op} of the category X is defined such that $|X^{op}| = |X|$ and $\hom_{X^{op}}(x,x') = \hom_X(x',x)$. Defining the dual category allows the use of covariant functors exclusively.

Let A and X be categories. An <u>adjunction</u> from X to A is a triple $(F,G,\varphi) : X \longrightarrow A$, where F and G are functors $X \underset{G}{\overset{F}{\rightleftarrows}} A$ while φ is a natural transformation such that for $x \in |X|$ and $a \in |A|$, $\varphi_{x,a}$ is a bijection

$$\hom_A(Fx,a) \xrightarrow{\varphi_{x,a}} \hom_X(x,Ga).$$

In this situation, F is called a <u>left adjoint</u> for G. The naturality of φ in the definition means that for any $h \in X$ and $k \in A$ where $h : x_1 \longrightarrow x_2$ and $k : a_1 \longrightarrow a_2$, the following diagram commutes:

$$\begin{tikzcd}
\hom_A(Fx_2, a_1) \arrow[r, "\varphi_{x_2,a_1}"] \arrow[d, "k \circ __ \circ F(h)"] & \hom_X(x_2, Ga_1) \arrow[d, "Gk \circ __ \circ h"] \\
\hom_A(Fx_1, a_2) \arrow[r, "\varphi_{x_1,a_2}"] & \hom_X(x_1, Ga_2)
\end{tikzcd}$$

, i.e., for $Fx_2 \xrightarrow{f} a_1$, $\varphi_{x_1,a_2}(k \circ f \circ F(h)) = Gk \circ \varphi_{x_2,a_1}(f) \circ h$.

If $S : D \longrightarrow C$ is a functor and $c \in |C|$, a <u>universal arrow</u> from c to S is a pair $\langle r, u\rangle$ with $r \in |D|$, $u : c \longrightarrow Sr$, $u \in C$ such that for every pair $\langle d, f\rangle$ with $d \in |D|$, $f : c \longrightarrow Sd$, $f \in C$ there exists a unique $f' : r \longrightarrow d$, $f' \in D$ such that

$$\begin{tikzcd}
c \arrow[r, "u"] \arrow[dr, "f"'] & Sr \arrow[d, "Sf'"] \\
 & Sd
\end{tikzcd} \quad \text{commutes.}$$

An adjunction $(F, G, \varphi) : X \longrightarrow A$ is completely determined by the functors F, G and a natural transformation

$\eta : I_X \Longrightarrow GF$, where I_X is the identity functor on X, such that for each $x \in |X|$, $\langle Fx, \eta_x \rangle$ is a universal arrow from x to G. Given $f : Fx \longrightarrow a$, φ and η are related by the following identities: $\varphi_{x,a}(f) = G(f)\eta_x$ and $\eta_x = \varphi_{x,Fx}(1_{Fx})$.

The following example illustrates the definitions of adjunction and universal arrow. Let $\mathcal{S}$ be the category of sets and $\mathcal{A}^b$ the category of groups. There exists the functor $\mathcal{S} \xrightarrow{F} \mathcal{A}^b$ which assigns to each set the free group on the set and the functor $\mathcal{A}^b \xrightarrow{U_{\mathcal{A}}} \mathcal{S}$ which assigns to each group the underlying set of the group. F is a left adjoint for $U_{\mathcal{A}}$. $\eta_x : x \longrightarrow U_{\mathcal{A}} Fx$ is the inclusion function of the set x into the underlying set of the free group Fx. For $Fx \xrightarrow{\alpha} a$, $\varphi_{x,a}(\alpha) = U_{\mathcal{A}} \alpha \circ \eta_x$. The fact that $\langle Fx, \eta_x \rangle$ is a universal arrow in this example means that given a set x and a function $x \xrightarrow{f} U_{\mathcal{A}} d$ where $d \in |\mathcal{A}^b|$, there exists a unique $\alpha : Fx \longrightarrow d$ such that

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, "f"'] & U_{\mathcal{A}} Fx \arrow[d, "U_{\mathcal{A}} \alpha"] \\ & U_{\mathcal{A}} d \end{tikzcd}$$

commutes.

A <u>functor category</u> C^I is defined for categories I and C to be the category whose objects are the functors with domain I and codomain C and whose morphisms are the natural transformations between them. Sometimes $\hom_{C^I}(f,g)$ will be denoted $\mathrm{Nat}_{C^I}(f,g)$.

A functor $\Delta : C \longrightarrow C^I$ is defined such that for $c \in |C|$, $\Delta c(i) = 1_c$ for all $i \in I$ and for $c \xrightarrow{\alpha} c'$, $\alpha \in C$, $\Delta\alpha_i = \alpha$ for all $i \in |I|$.

A universal arrow $\langle r,u\rangle$ from $F \in |C^I|$ to $\Delta : C \longrightarrow C^I$ is called a colimit (direct limit) diagram for the functor F. It consists of an object $r \in |C|$ written $\varinjlim F$, together with a natural transformation $u : F \Longrightarrow \Delta r$ which is universal among all natural transformations $\tau : F \Longrightarrow \Delta c$ for all $c \in |C|$.

If $|I| = \{1,2\}$ and $I = \{1_1, 1_2, 1 \xrightarrow{\alpha} 2, 1 \xrightarrow{\beta} 2\}$, then $F(2) \xrightarrow{u_2} \varinjlim F$ is called the coequalizer of $F(\alpha)$ and $F(\beta)$.

A limit diagram for a functor $F \in |(C^I)^{op}| = |C^I|$ is a universal arrow $\langle r,\nu\rangle$ from F to Δ^{op} where $\Delta^{op} : C^{op} \longrightarrow (C^I)^{op}$, $r = \varprojlim F \in |C|$ and $\nu : \Delta r \Longrightarrow F$.

If $|I| = \{1,2\}$, $I = \{1_1, 1_2, 1 \xrightarrow{\alpha} 2, 1 \xrightarrow{\beta} 2\}$, then $\varprojlim F \xrightarrow{\nu_1} F(1)$ is called the equalizer of $F(\alpha)$ and $F(\beta)$.

Suppose X^Y is a functor category and $F : I \longrightarrow X^Y$. Let $G_y \equiv F(\ )y : I \longrightarrow X$ for all $y \in |Y|$. If $\varinjlim G_y$ exists for all $y \in |Y|$, then $\varinjlim F$ exists and can be constructed as follows. Let $\langle \varinjlim G_y, u_y \rangle$ be a universal arrow from G_y to $X \xrightarrow{\Delta} X^I$ for all $y \in Y$. Define $(\varinjlim F)(y) \equiv \varinjlim G_y$ for each $y \in Y$ and if $\alpha : y \longrightarrow y'$ in Y, define $(\varinjlim F)(\alpha)$ to be the unique morphism in X such that $(\varinjlim F)(\alpha) \circ u_{yi} = u_{y'i} \circ F(i)(\alpha)$ for all $i \in I$. Pictorially,

$$
\begin{tikzcd}
 & G_y(i) = F(i)(y) \arrow[dl, "G_y(s) = F(s)_y"'] \arrow[dr, dashed, "F(i)(\alpha)"] \arrow[rr, "u_{yi}"] & & (\varinjlim F)(y) = \varinjlim G_y \arrow[dr, "(\varinjlim F)(\alpha)"] & \\
G_y(j) = F(j)(y) \arrow[urrr, "u_{yj}"] \arrow[dr, "F(j)(\alpha)"'] & & F(i)(y') \arrow[rr, dashed, "u_{y'i}"] \arrow[dl, dashed, "F(s)_{y'}"] & & (\varinjlim F)(y') = \varinjlim G_{y'} \\
 & G_{y'}(j) = F(j)(y') \arrow[urrr, "u_{y'j}"'] & & &
\end{tikzcd}
$$

where $i \xrightarrow{s} j$, $s \in I$.

Dually, $\varprojlim F$ can be constructed.

A category X is <u>small</u> if $|X|$ is a set and for any $x, x' \in |X|$, $\hom_X(x,x')$ is a set. A colimit (limit) of $F : I \longrightarrow C$ is called a <u>small</u> <u>colimit</u> (small limit) if I is small.

If X is a complete category, i.e., all small limits and small colimits of functors into X exist, them X^Y is complete.

If W is a category such that $|W| \subset |C|$ and $W \subset C$, then W is called a <u>subcategory</u> of C. Given a complete category X and a subcategory W of X^Y then the <u>completion</u> $\hat{W}$ of W is the smallest complete subcategory of X^Y which contains W.

Let $K : M \longrightarrow C$ and $c \in |C|$, the <u>comma category</u> K/c has as objects all morphisms $Km \xrightarrow{f} c$ and as morphisms from $Km \xrightarrow{f} c$ to $Km' \xrightarrow{f'} c$ all commutative triangles
$$\begin{tikzcd} Km \arrow[rr, "Kh"] \arrow[dr, "f"'] & & Km' \arrow[dl, "f'"] \\ & c & \end{tikzcd}$$
where $f, f' \in C$ and $h \in M$. The composition of morphisms is indicated below.

$$\begin{tikzcd} Km \arrow[r, "Kh"] \arrow[rr, bend left, "K(h'h)"] \arrow[dr, "f"'] & Km' \arrow[r, "Kh'"] \arrow[d, "f'"] & Km'' \arrow[dl, "f''"] \\ & c & \end{tikzcd}$$

There exists a functor $K/c \xrightarrow{P} M$ called the <u>projection</u> such that

$$P\left(\begin{tikzcd} Km \arrow[rr, "Kh"] \arrow[dr, "f"'] & & Km' \arrow[dl, "f'"] \\ & c & \end{tikzcd}\right) = h.$$

Some of the material from Lawvere's dissertation [3] will be needed. The following is a brief summary of that material, where the notation is that used in [5]. A summary of Lawvere's dissertation appears in [4].

An <u>algebraic theory</u> is a category $\mathscr{A}$ having as objects $A^o \equiv 1, A, A^2, A^3, \ldots, A^n, \ldots;$ where $n \in \omega$, the first infinite ordinal. There exists n morphisms $A^n \xrightarrow{\pi_i^{(n)}} A$, $i \in n$, which form a categorical product in the theory $\mathscr{A}$, i.e., given any n morphisms $A^m \xrightarrow{\theta_i} A$, $i \in n$ in $\mathscr{A}$ there exists a unique morphism $\langle \theta_o, \theta_1, \ldots, \theta_{n-1} \rangle$ making

$$\begin{tikzcd} A^m \arrow[rr, "\langle \theta_o, \theta_1, \ldots, \theta_{n-1} \rangle"] \arrow[dr, "\theta_i"'] & & A^n \arrow[dl, "\pi_i^{(n)}"] \\ & A & \end{tikzcd}$$

commute for any $i \in n$. Morphisms $A^n \longrightarrow A$ in $\mathscr{A}$ are called <u>n-ary operations</u> and morphisms $A^m \longrightarrow A^n$ are called <u>generalized operations</u>. Recalling that $\mathscr{S}$ is the category of sets, the <u>algebraic category</u> $\mathscr{A}^b$ associated with $\mathscr{A}$ is defined to be the full subcategory of $\mathscr{S}^{\mathscr{A}}$ whose objects are the product preserving functors in $|\mathscr{S}^{\mathscr{A}}|$. <u>Full</u> means that $\hom_{\mathscr{A}^b}(f,g) = \hom_{\mathscr{S}^{\mathscr{A}}}(f,g)$ for any

$f, g \in |\mathcal{A}^b|$. The category T is defined such that the objects of T are the algebraic theories and the morphisms are those functors $\mathcal{A} \xrightarrow{f} \mathbb{B}$ such that $f(\pi_i^{(n)}) = \pi_i^{(n)}$ for each $i \in n \in \omega$. A functor $f : \mathcal{A} \longrightarrow \mathbb{B}$ as above determines an <u>algebraic</u> <u>functor</u> $\mathbb{B}^b \xrightarrow{f^b = \_\_ \circ f} \mathcal{A}^b$ where

$$f^b\left(\begin{tikzcd} B \arrow[r, bend left, "h"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{S} \end{tikzcd}\right) = \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "hf"] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\Downarrow\alpha f"] & \mathcal{S} \end{tikzcd} \qquad \text{and}$$

$(\alpha f)_{A^n} \equiv \alpha_{f(A^n)}$. There exists the functor $\mathcal{A}^b \xrightarrow{U_{\mathcal{A}}} \mathcal{S}$ such that $U_{\mathcal{A}}(f \overset{\alpha}{\Longrightarrow} g) = f(A) \xrightarrow{\alpha_A} g(A)$. $U_{\mathcal{A}}$ has the left adjoint $F_{\mathcal{A}} : \mathcal{S} \longrightarrow \mathcal{A}^b$ where $F_{\mathcal{A}}(s)$ is a free $\mathcal{A}$-algebra on the set s. Note the objects of $\mathcal{A}^b$ are called <u>$\mathcal{A}$-algebras</u> and the morphisms (natural transformations) are called <u>$\mathcal{A}$-algebra</u> <u>homomorphisms</u>. $U_{\mathcal{A}}$ is called the <u>underlying-set</u> <u>functor</u> and $F_{\mathcal{A}}$ is called the <u>free</u> <u>functor</u> associated with $U_{\mathcal{A}}$. The following diagram commutes for any theory morphism $\mathcal{A} \xrightarrow{f} \mathbb{B}$.

$$\begin{tikzcd} \mathbb{B}^b \arrow[rr, "f^b"] \arrow[dr, "U_{\mathbb{B}}"'] & & \mathcal{A}^b \arrow[dl, "U_{\mathcal{A}}"] \\ & \mathcal{S} & \end{tikzcd}$$

The <u>semantics</u> <u>functor</u> $\mathrm{Sm} : T^{op} \longrightarrow (\mathrm{Cat}, \mathcal{S})$ is defined by

$$\mathrm{Sm}(\mathcal{B} \xrightarrow{f} \mathcal{A}) = \begin{tikzcd} \mathcal{B}^\flat \arrow[rr, "f^\flat"] \arrow[dr, "U_\mathcal{B}"'] & & \mathcal{A}^\flat \arrow[dl, "U_\mathcal{A}"] \\ & \mathcal{S} & \end{tikzcd}$$

where the objects of $(\mathrm{Cat}, \mathcal{S})$ are functors from any category into $\mathcal{S}$ and the morphisms are commutative triangles

$$\begin{tikzcd} X \arrow[rr, "\theta"] \arrow[dr, "U"'] & & X' \arrow[dl, "U'"] \\ & \mathcal{S} & \end{tikzcd}$$

of functors. Sm has a left adjoint St called the <u>structure</u> <u>functor</u> such that given $X \xrightarrow{U} \mathcal{S}$, St U is the full subcategory of $(\mathrm{Cat}, \mathcal{S})$ with objects $\{U^n\}_{n \in \omega}$, where $U^n : X \longrightarrow \mathcal{S}$ is the n-th cartesian power of U. Note that $U^n \xrightarrow{\varphi} U$ assigns an operation to every value of U such that all morphisms of X are homomorphisms with respect to it. In particular, if $X = \mathcal{A}^\flat$ and $U = U_\mathcal{A}$, then for $f \overset{\alpha}{\Longrightarrow} g$ in $\mathcal{A}^\flat$, the following diagram commutes:

$$\begin{tikzcd}[column sep=huge, row sep=large]
U^n_{\mathscr{A}}(f) = f(A)^n \arrow[r, "\alpha_{A^n} = U^n_{\mathscr{A}}(\alpha)"] \arrow[d, "\varphi_f"'] & g(A)^n = U^n_{\mathscr{A}}(g) \arrow[d, "\varphi_g"] \\
U_{\mathscr{A}}(f) = f(A) \arrow[r, "\alpha_A = U_{\mathscr{A}}(\alpha)"'] & g(A) = U_{\mathscr{A}}(g)
\end{tikzcd} .$$

This indicates that α is a homomorphism with respect to the operation φ. In fact, St $U_{\mathscr{A}}$ is isomorphic to $\mathscr{A}$, i.e.,

$$\hom_{\mathscr{A}}(A^n, A^m) \cong \mathrm{Nat}_{\mathscr{S}^{\mathscr{A}^b}}(U^n_{\mathscr{A}}, U^m_{\mathscr{A}})$$

for any $n, m \in \omega$. The isomorphism $\mathscr{A} \xrightarrow{\eta_{\mathscr{A}}} (\mathrm{St}\ \mathrm{Sm}\,\mathscr{A}) = \mathrm{St}\ U_{\mathscr{A}}$ is defined such that for any $h \in \mathscr{A}$ and $f \in |\mathscr{A}^b|$, $\eta_{\mathscr{A}}(h)_f \equiv f(h)$.

It will be noted at this time that $\mathscr{S}$ is complete and that $\mathscr{A}^b$ is complete for any $\mathscr{A} \in |T|$.

Given a functor $Y \xrightarrow{G} X$ and a functor category $U_o \subset Y^X$ then there exists a category $G(U_o)$ such that $H \in |G(U_o)|$ if and only if $H = GF$ for some $F \in |U_o|$ and

$$\begin{tikzcd}
X \arrow[r, bend left=50, "H = GF"] \arrow[r, bend right=50, "H' = GF'"'] \arrow[r, phantom, "\Downarrow\alpha"] & X
\end{tikzcd}$$

is in $G(U_o)$ if and only if

$$\alpha_X = (G\tau)_X \equiv G(\tau_X) \quad \text{for some} \quad X \overset{F}{\underset{F'}{\underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow\tau}}}} Y \quad \text{in} \quad U_0.$$

Since $\mathcal{A}^b$ is complete, given any $f \in |\mathcal{A}^b|$, the n-fold product f^n exists in $|\mathcal{A}^b|$. The n-fold product in $\mathcal{A}^b \subset \mathcal{L}^{\mathcal{A}}$ is the same as the product in $\mathcal{L}^{\mathcal{A}}$. Because $\mathcal{L}$ is complete $f^n(\tau) = (f(\tau))^n = \langle f(\tau),\ldots,f(\tau)\rangle$, for any $\tau \in \mathcal{A}$. Given $F \in |(\mathcal{B}^b)^{\mathcal{A}^b}|$, because $\mathcal{B}^b$ is complete, the product F^n exists in $|(\mathcal{B}^b)^{\mathcal{A}^b}|$ and

$$F^n\left(\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{L} \end{tikzcd}\right) = \begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F(f)^n"] \arrow[r, bend right, "F(g)^n"'] \arrow[r, phantom, "\Downarrow F(\alpha)^n"] & \mathcal{L} \end{tikzcd} \quad \text{for any } \alpha \in \mathcal{A}^b.$$

<u>Lemma</u>: Let $\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{A}^b \end{tikzcd}$ be given where τ is a transformation, not necessarily natural. Then τ is natural if for any $f \overset{\alpha}{\Longrightarrow} g$ in $\mathcal{A}^b$, the following diagram commutes:

$$\begin{tikzcd} F(f)(A) \arrow[r, "(\tau_f)_A"] \arrow[d, "F(\alpha)_A"'] & G(f)(A) \arrow[d, "G(\alpha)_A"] \\ F(g)(A) \arrow[r, "(\tau_g)_A"] & G(g)(A) \end{tikzcd} \quad .$$

__Pf__: Given any $f \overset{\alpha}{\Longrightarrow} g$, it must be shown that

$$\begin{tikzcd} F(f) \arrow[r, Rightarrow, "\tau_f"] \arrow[d, Rightarrow, "F(\alpha)"'] & G(f) \arrow[d, Rightarrow, "G(\alpha)"] \\ F(g) \arrow[r, Rightarrow, "\tau_g"'] & G(g) \end{tikzcd}$$

commutes. This diagram commutes

if and only if

$$\begin{tikzcd} F(f)(A^n) \arrow[r, "(\tau_f)_{A^n}"] \arrow[d, "F(\alpha)_{A^n}"'] & G(f)(A^n) \arrow[d, "G(\alpha)_{A^n}"] \\ F(g)(A^n) \arrow[r, "(\tau_g)_{A^n}"'] & G(g)(A^n) \end{tikzcd}$$

commutes for every $n \in \omega$.

Obviously the previous diagram commutes for $n = 0$. For $n \neq 0$, the following diagram exists where the commutative figures are indicated by ///. The figures commute because $F(f)$, $F(g)$, $G(f)$ and $G(g)$ are product preserving functors and τ_f and τ_g are natural transformations. The p_i's are the i-th projections where i is fixed but arbitrary.

$$\begin{tikzcd}
F(f)(A) \arrow[rrr, "(\tau_f)_A"] \arrow[ddd, "F(\alpha)_A"'] & & & G(f)(A) \arrow[ddd, "G(\alpha)_A"] \\
& F(f)(A^n) \arrow[ul, "p_i"] \arrow[r, "(\tau_f)_{A^n}"] \arrow[d, "F(\alpha)_{A^n}"'] & G(f)(A^n) \arrow[ur, "p_i"'] \arrow[d, "G(\alpha)_{A^n}"] & \\
& F(g)(A^n) \arrow[dl, "p_i"] \arrow[r, "(\tau_g)_{A^n}"'] & G(g)(A^n) \arrow[dr, "p_i"'] & \\
F(g)(A) \arrow[rrr, "(\tau_g)_A"'] & & & G(g)(A)
\end{tikzcd}$$

By hypothesis, the outer diagram commutes. Consider the family $\{p_i(\tau_g)_{A^n} F(\alpha)_{A^n} : i \in n\}$ of maps. Since $G(g)(A^n)$ is a product there exists a unique morphism h making

$$\begin{tikzcd}
F(f)(A^n) \arrow[r, "h"] \arrow[dr, "p_i(\tau_g)_{A^n} F(\alpha)_{A^n}"'] & G(g)(A^n) \arrow[d, "p_i"] \\
& G(g)(A)
\end{tikzcd}$$

commute for each $i \in n$. But from the previous large diagram, both $(\tau_g)_{A^n} F(\alpha)_{A^n}$ and $G(\alpha)_{A^n}(\tau_f)_{A^n}$ make the diagram commute, hence $(\tau_g)_{A^n} F(\alpha)_{A^n} = G(\alpha)_{A^n}(\tau_f)_{A^n}$. Thus τ is a natural transformation.

III. EXTENDED ALGEBRAIC THEORIES

In this chapter an arbitrary algebraic theory $\mathcal{A}$ will be enlarged to a complete category $\mathcal{A}_\omega$. $\mathcal{A}_\omega$ will be called an <u>extended algebraic theory</u>. A category T_ω of the extended theories will be defined so that T_ω and T will be isomorphic categories in the sense that there exists functors $T_\omega \xrightarrow{F} T$ and $T \xrightarrow{F_1} T_\omega$ such that FF_1 and F_1F are identity functors. Since $\mathcal{A}$ will be a full subcategory of $\mathcal{A}_\omega$ and T will be isomorphic to T_ω, the properties of $\mathcal{A}$ and the $(St, Sm, \varphi) : (Cat, \mathcal{L}) \longrightarrow T^{op}$ adjunction will be preserved.

Given $\mathcal{A} \in |T|$ and $\mathcal{A}^b \xrightarrow{U_{\mathcal{A}}} \mathcal{L}$, consider the category $St\ U_{\mathcal{A}} \subseteq \mathcal{L}^{\mathcal{A}^b}$. $St\ U_{\mathcal{A}}$ will be identified with $\mathcal{A}$ since they are isomorphic. Let $U_0 \equiv St\ U_{\mathcal{A}}$. Assuming that U_k has been defined for $k \in \omega$, let $\hat{U}_k$ be the completion of U_k in $\mathcal{L}^{\mathcal{A}^b}$, i.e., $\hat{U}_k$ is closed under small limits and small colimits. It is possible to find $\hat{U}_k$ since $\mathcal{L}$ is complete. $\hat{U}_k$ is small because U_k is small.

For $C \overset{\alpha}{\Longrightarrow} U_{\mathcal{A}}^n$ in $\hat{U}_k$, a corresponding morphism

$$\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "F_{\mathcal{A}} C"] \arrow[r, bend right, "Id^n"'] \arrow[r, phantom, "\Downarrow\hat{\alpha}"] & \mathcal{A}^b \end{tikzcd}$$

will be induced where Id is the identity functor. The properties below are true since $F_{\mathcal{A}}$ is the left adjoint of $U_{\mathcal{A}}$, see Chapter II:

1) $\mathrm{Nat}_{\mathcal{A}^b}(F_{\mathcal{A}}x, f) \xrightarrow[\varphi_{x,f}]{} \hom_{\mathcal{L}}(x, U_{\mathcal{A}}f)$ is a bijection for all $x \in |\mathcal{L}|$ and $f \in |\mathcal{A}^b|$,

2) for all $x \in |\mathcal{L}|$, $\langle F_{\mathcal{A}}x, \eta_x \rangle$ is a universal arrow from x to $U_{\mathcal{A}}$, where $\eta_x = \varphi_{x, F_{\mathcal{A}}x}(1_{F_{\mathcal{A}}x})$, i.e., given any $f \in |\mathcal{A}^b|$ and function $x \xrightarrow{\sigma} U_{\mathcal{A}}f$, there exists a unique morphism $F_{\mathcal{A}}x \xRightarrow[\varphi^{-1}_{x,f}(\sigma)]{} f$ such that

$$\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[dr, "\sigma"'] & U_{\mathcal{A}}F_{\mathcal{A}}x \arrow[d, "U_{\mathcal{A}}(\varphi^{-1}_{x,f}(\sigma))"] \\
 & U_{\mathcal{A}}f
\end{tikzcd}$$

commutes.

Using 1), for any $f \in |\mathcal{A}^b|$,
$\hom_{\mathcal{L}}(Cf, U^n_{\mathcal{A}}f) = \hom_{\mathcal{L}}(Cf, U_{\mathcal{A}}(f^n)) \cong \mathrm{Nat}_{\mathcal{A}^b}(F_{\mathcal{A}}Cf, f^n)$. Define $\hat{\alpha}_f \equiv \varphi^{-1}_{Cf, f^n}(\alpha_f) : F_{\mathcal{A}}Cf \Longrightarrow f^n$. To prove that $\hat{\alpha} \equiv \{\hat{\alpha}_f : f \in |\mathcal{A}^b|\}$ is a natural transformation, it is sufficient, by the lemma in Chapter II, to show that

$$
\begin{tikzcd}
U_{\mathcal{A}}F_{\mathcal{A}}C(f) \arrow[r, "U_{\mathcal{A}}(\hat{\alpha}_f)"] \arrow[d, "U_{\mathcal{A}}F_{\mathcal{A}}C(\tau)"'] & U_{\mathcal{A}}(f^n) \arrow[d, "U_{\mathcal{A}}(\tau^n)"] \\
U_{\mathcal{A}}F_{\mathcal{A}}C(g) \arrow[r, "U_{\mathcal{A}}(\hat{\alpha}_g)"'] & U_{\mathcal{A}}(g^n)
\end{tikzcd}
$$

commutes for any $\tau : f \Longrightarrow g$ in $\mathcal{A}^b$. Consider the following diagram.

$$
\begin{tikzcd}
C(f) \arrow[r, "\eta_{Cf}"'] \arrow[d, "C(\tau)"'] \arrow[rr, bend left, "\alpha_f"] & U_{\mathcal{A}}F_{\mathcal{A}}C(f) \arrow[r, "U_{\mathcal{A}}(\hat{\alpha}_f)"'] \arrow[d, "U_{\mathcal{A}}F_{\mathcal{A}}C(\tau)"'] & U_{\mathcal{A}}(f^n) \arrow[d, "U_{\mathcal{A}}(\tau^n)"] \\
C(g) \arrow[r, "\eta_{Cg}"'] \arrow[rr, bend right, "\alpha_g"'] & U_{\mathcal{A}}F_{\mathcal{A}}C(g) \arrow[r, "U_{\mathcal{A}}(\hat{\alpha}_g)"'] & U_{\mathcal{A}}(g^n)
\end{tikzcd}
$$

The top and bottom figures commute because $U_{\mathcal{A}}$ and $F_{\mathcal{A}}$ are adjoints. The outer diagram and left square commute because α and η are natural transformations. Since $\langle F_{\mathcal{A}}Cf, \eta_{Cf}\rangle$ is a universal arrow and the right square commutes when preceded by η_{Cf}, the uniqueness of the morphism from $U_{\mathcal{A}} F_{\mathcal{A}} Cf$ to $U_{\mathcal{A}}(g^n)$ implies that the right square commutes. Hence $\hat{\alpha}$ is a natural transformation.

Define $\hat{\hat{U}}_k$ as the completion in $(\mathcal{A}^b)^{\mathcal{A}^b}$ of the category generated by $F_{\mathcal{A}}(\hat{U}_k)$ and all $\hat{\alpha}$ as constructed above. Recall the definition of the category $F_{\mathcal{A}}(\hat{U}_k)$ from Chapter II and note that the codomain of α is an object of St $U_{\mathcal{A}}$. Since $\mathcal{A}^b$ is a complete category the above completion exists in $(\mathcal{A}^b)^{\mathcal{A}^b}$. Define U_{k+1} to be the category generated in $\mathcal{L}^{\mathcal{A}^b}$ by $U_{\mathcal{A}}(\hat{\hat{U}}_k)$ and $\hat{U}_k$. Let $\mathcal{A}_\omega$ be the completion of $\bigcup_{i\in\omega} U_i$ in $\mathcal{L}^{\mathcal{A}^b}$. $\mathcal{A}_\omega$ is called the extended theory of $\mathcal{A}$ = St $U_{\mathcal{A}}$.

The following lemma will be used to define the category of extended theories.

<u>Lemma</u> ([6], page 143): If M is a full subcategory of a category C, $M \xrightarrow{K} C$ is the insertion functor and $M \xrightarrow{T} A$ is a functor such that each composite $K/c \xrightarrow{P} M \xrightarrow{T} A$ has a colimit in A, for each $c \in |C|$, then there exists a functor $C \xrightarrow{S} A$, unique up to natural equivalence, with $SK = T$ such that the identity natural transformation $1 : T \Longrightarrow SK$ is part of a universal arrow $\langle S,1\rangle$ from T into A^K.

It will be recalled that $K/c \xrightarrow{P} M$, the projection from the comma category K/c into M, was defined in Chapter II. $A^K : A^C \longrightarrow A^M$ is defined by
$A^K(g \stackrel{\beta}{\Longrightarrow} g') = gK \stackrel{\beta K}{\Longrightarrow} g'K$.

Given $\mathcal{A} \xrightarrow{f} \mathcal{B}$ where $f \in T$ and $\mathcal{B} \xrightarrow{I_{\mathcal{B}}} \mathcal{B}_\omega$ and $\mathcal{A} \xrightarrow{I_{\mathcal{A}}} \mathcal{A}_\omega$ are the inclusion embeddings, let $M = \mathcal{A}$, $C = \mathcal{A}_\omega$, $T = I_{\mathcal{B}}f$, $A = \mathcal{B}_\omega$ and $K = I_{\mathcal{A}}$ in the lemma. Since $I_{\mathcal{A}}/c$, for any $c \in |\mathcal{A}_\omega|$, is small and $\mathcal{B}_\omega$ is complete, $I_{\mathcal{A}}/c \xrightarrow{P} \mathcal{A} \xrightarrow{I_{\mathcal{B}}f} \mathcal{B}_\omega$ has a colimit in $\mathcal{B}_\omega$. Hence the previous lemma implies that there exists $f_\omega \equiv S : \mathcal{A}_\omega \longrightarrow \mathcal{B}_\omega$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[r, "I_{\mathcal{A}} = K"] \arrow[d, "f"'] & \mathcal{A}_\omega \arrow[d, "f_\omega"] \\ \mathcal{B} \arrow[r, "I_{\mathcal{B}}"'] & \mathcal{B}_\omega \end{tikzcd}$$ commutes.

The category T_ω is defined to be the category of all objects $\mathcal{A}_\omega$ and functors f_ω obtained from T by the above process. It is clear that the association $f \longmapsto f_\omega$ defines a functor $T \xrightarrow{F_1} T_\omega$ which is an isomorphism in the sense that there exists another functor $T_\omega \xrightarrow{F} T$ such that F_1F and FF_1 are identity functors.

For $\mathcal{A} \xrightarrow{f} \mathcal{L}$ in $\mathcal{A}^b$, the previous lemma can be applied with $M = \mathcal{A}$, $C = \mathcal{A}_\omega$, $T = f$, $A = \mathcal{L}$ and $K = I_{\mathcal{A}}$ to obtain a functor $\mathcal{A}_\omega \xrightarrow{f_\omega \equiv S} \mathcal{L}$. Because, for each $f = T$, the natural transformation $1 : T \Longrightarrow SK$ is part of a universal arrow $\langle S, 1\rangle$ from $T = f$ into $A^K = \mathcal{L}^{I_{\mathcal{A}}}$, $f \xRightarrow{\theta} f'$ in $\mathcal{A}^b$ induces a unique $f_\omega \xRightarrow{\theta_\omega} f'_\omega$ in $\mathcal{L}^{\mathcal{A}_\omega}$ making

$$\begin{array}{ccc} f = T & \xRightarrow{\ 1\ } & SK = \mathcal{L}^{I_{\mathcal{A}}}(f_\omega) \\ \theta \Big\Downarrow & & \Big\Downarrow \mathcal{L}^{I_{\mathcal{A}}}(\theta_\omega) \\ f' = T' & \xRightarrow{\ 1\ } & S'K = \mathcal{L}^{I_{\mathcal{A}}}(f'_\omega) \end{array}$$

commute. Hence the category $\mathcal{A}^b_\omega$ will be defined to be the subcategory of $\mathcal{L}^{\mathcal{A}_\omega}$ obtained from $\mathcal{A}^b$ by the above process. $\mathcal{A}^b$ and $\mathcal{A}^b_\omega$ are isomorphic. If $\mathcal{A}^b$ and $\mathcal{A}^b_\omega$ are identified in the construction of Lawvere's comma category $(\mathrm{Cat}, \mathcal{L})$, then $(F_1^{op}\mathrm{St}, \mathrm{Sm}\ F^{op}, \bar{\varphi}) : (\mathrm{Cat}, \mathcal{L}) \longrightarrow T_\omega^{op}$ is an adjunction where $T_\omega^{op} \underset{F_1^{op}}{\overset{F^{op}}{\rightleftarrows}} T^{op}$ come from the isomorphisms $T_\omega \underset{F_1}{\overset{F}{\rightleftarrows}} T$, and $\bar{\varphi}$ is the obvious extension of φ in the adjunction $(\mathrm{St}, \mathrm{Sm}, \varphi) : (\mathrm{Cat}, \mathscr{A}) \longrightarrow T^{op}$.

The original algebraic theories of Lawvere have now been extended to exhibit, not only the operations and identities, but natural consequences of them. Moreover, the method used did not destroy the structure of the original theories but embeds the theories in extended theories. The original morphisms between theories have not been lost but emerge when the extended theory morphisms are restricted to the original theories. The St-Sm adjunction has also been preserved. Hence the constructions have enriched the theories without destroying their original properties.

IV. ALGEBRAIC APPLICATIONS

Some of the considerations that shaped the construction of the extended theory will be shown here as well as some of the uses of the theory. The following example demonstrates how the commutator subgroup of a group and the corresponding quotient homomorphism, from the group onto the group modulo the commutator subgroup, appear in the extended theory.

To provide direction and clarification, a few remarks in the notation of classical algebra might be appropriate. For each group G, there is the operation $h : G \times G \longrightarrow G : (x,y) \longmapsto xyx^{-1}y^{-1}$ and the operation $G \times G \xrightarrow{0} G$ which takes everything to the identity. Now the projection $p : G \longrightarrow \frac{G}{[G]} : x \longmapsto x[G]$, where $[G]$ is the commutator subgroup is in some sense the "group coequalizer" of h and 0, i.e., it is a factor of every homomorphism q out of G such that $qh = q0$. It is not necessarily the set coequalizer of h and 0. For if $G \xrightarrow{c} C$ is the set coequalizer then $c^{-1}(c(0)) = h(G \times G)$ but $h(G \times G) \subset p^{-1}(0)$ is not necessarily a subgroup since the set of commutators need not form a subgroup. There are no morphisms in the algebraic theories of groups or sets, or even in the extended theory of sets which correspond to the homomorphism p. However, the extended theory of groups contains such a morphism.

In general, given an algebraic theory $\mathcal{A}$, there exists the functor $U \equiv U_{\mathcal{A}} : \mathcal{A}^b \longrightarrow \mathcal{S}$. For each pair $(\alpha,\beta) \in \mathrm{Nat}_{\mathcal{S}^{\mathcal{A}^b}}(U^n,U) \times \mathrm{Nat}_{\mathcal{S}^{\mathcal{A}^b}}(U^n,U)$, consider all natural transformations ν, $\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\nu"] & \mathcal{A}^b \end{tikzcd}$, where Id is the identity functor and G is an arbitrary functor, with the property that

$$\begin{tikzcd} & U \arrow[dr, Rightarrow, "U\nu"] & \\ U^n \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] & & UG \\ & U \arrow[ur, Rightarrow, "U\nu"'] & \end{tikzcd}$$

commutes, where $(U\nu)_x \equiv U(\nu_x)$ for all $x \in |\mathcal{A}^b|$. Call the class of these natural transformations $U(\alpha,\beta)$. A natural transformation $\lambda \in U(\alpha,\beta)$ will be called a <u>special</u> <u>coequalizer</u> of α and β if given any natural transformation $\nu \in U(\alpha,\beta)$ there exists a unique natural transformation $\tau \in (\mathcal{A}^b)^{\mathcal{A}^b}$ making

$$\begin{tikzcd} \mathrm{Id} \arrow[rr, Rightarrow, "\nu"] \arrow[dr, Rightarrow, "\lambda"'] & & G \\ & F \arrow[ur, Rightarrow, "\tau"'] & \end{tikzcd}$$

commute.

<u>Theorem</u>: For each pair $(\alpha,\beta) \in \mathrm{Nat}_{\mathcal{S}^{\mathcal{A}^b}}(U^n,U) \times \mathrm{Nat}_{\mathcal{S}^{\mathcal{A}^b}}(U^n,U)$, there exists a special coequalizer λ of α and β.

<u>Proof</u>: Given $\alpha,\beta : U^n \Longrightarrow U$, consider $F_{\mathcal{A}}U^n \overset{\hat{\alpha}}{\underset{\hat{\beta}}{\rightrightarrows}} \mathrm{Id}$ as constructed in Chapter III. Let $\mathrm{Id} \xRightarrow{c(\hat{\alpha},\hat{\beta})} C\ (\hat{\alpha},\hat{\beta})$ be the coequalizer of $\hat{\alpha}$, $\hat{\beta}$ in the category $(\mathcal{A}^b)^{\mathcal{A}^b}$ as defined in Chapter II. Note that $U_{\mathcal{A}}(c(\hat{\alpha},\hat{\beta}))$ appears in $U_1 \subset \mathcal{A}_\omega$. It is claimed that $\lambda \equiv c(\hat{\alpha},\hat{\beta})$ is a special coequalizer of α and β.

First the fact that $(U\lambda)\alpha = (U\lambda)\beta$ will be verified. Recall from Chapter III that $\alpha_f = U(\hat{\alpha}_f) \circ \eta_{U^n(f)}$ and $\beta_f = U(\hat{\beta}_f) \circ \eta_{U^n(f)}$. Then $[(U\lambda)\alpha]_f = U(\lambda_f)\alpha_f$ $= U(\lambda_f)U(\hat{\alpha}_f)\eta_{U^n(f)} = U(\lambda_f \circ \hat{\alpha}_f)\eta_{U^n(f)} = U(\lambda_f \circ \hat{\beta}_f)\eta_{U^n(f)}$ $= U(\lambda_f)U(\hat{\beta}_f)\eta_{U^n(f)} = U(\lambda_f)\beta_f = [(U\lambda)\beta]_f$ for every $f \in |\mathcal{A}^b|$. Hence $(U\lambda)\alpha = (U\lambda)\beta$.

Now let $\nu \in U(\alpha,\beta)$ with $\nu : \mathrm{Id} \Longrightarrow F$. $(U\nu)\alpha = (U\nu)\beta$ implies, for each $f \in |\mathcal{A}^b|$, that the upper triangles and outer figure of the following diagram commute:

$$\begin{tikzcd}
& U^n(f) \arrow[dl, "\alpha_f"'] \arrow[d, "\eta_{U^n(f)}"] \arrow[dr, "\beta_f"] & \\
U(f) \arrow[dr, "U(\nu_f)"'] & UF_*U^n(f) \arrow[l, "U(\hat{\alpha}_f)"] \arrow[r, "U(\hat{\beta}_f)"'] & U(f) \arrow[dl, "U(\nu_f)"] \\
& UF(f) &
\end{tikzcd} .$$

Since $\eta_{U^n(f)}$ is part of a universal arrow, the morphism with domain $UF_*U^n(f)$ and codomain $UF(f)$, that makes the outer figure commute, is unique. Hence $U(\nu_f\hat{\alpha}_f) = U(\nu_f\hat{\beta}_f)$ and $\nu_f\hat{\alpha}_f = \nu_f\hat{\beta}_f$. Because f was arbitrary $\nu\hat{\alpha} = \nu\hat{\beta}$. But $Id \overset{\lambda}{\Longrightarrow} C(\hat{\alpha},\hat{\beta})$ is the coequalizer of $\hat{\alpha}$ and $\hat{\beta}$. Therefore there exists a unique τ making the following diagram commute:

$$\begin{tikzcd}
Id \arrow[r, Rightarrow, "\lambda"] \arrow[dr, Rightarrow, "\nu"'] & C(\hat{\alpha},\hat{\beta}) \arrow[d, Rightarrow, "\tau"] \\
& F
\end{tikzcd} .$$

Thus λ is the special coequalizer of α and β.

Lemma: λ in the existence theorem is unique up to isomorphism.

Proof: Suppose λ' exists satisfying the same universal conditions as λ, then there exists τ and τ' such that $\tau\lambda = \lambda'$ and $\tau'\lambda' = \lambda$. This implies that $\tau\tau'\lambda' = \lambda'$ and $\tau'\tau\lambda = \lambda$. By the uniqueness of the morphisms out of C and C' respectively, $\tau'\tau = 1_C$ and $\tau\tau' = 1_{C'}$. Hence τ is an isomorphism.

Note λ is an epimorphism since coequalizers are epimorphisms.

Consider now the commutator subgroup example. Given a homomorphism $g \overset{\delta}{\Longrightarrow} k$ in $\mathcal{A}^b$ where $\mathcal{A}$ is the algebraic theory of groups, there exists a homomorphism $\frac{g}{[g]} \overset{\overline{\delta}}{\Longrightarrow} \frac{k}{[k]}$ where $\frac{g}{[g]}$ is the group g modulo its commutator subgroup $[g]$, $\overline{\delta}_A(x[g(A)]) = \delta_A(x)[k(A)]$ and $[g(A)] \equiv [g](A)$.

If $g \overset{\delta}{\Longrightarrow} k$ and $k \overset{\sigma}{\Longrightarrow} \ell$ then $\overline{\sigma\delta} = \overline{\sigma}\,\overline{\delta}$. A functor $F : \mathcal{A}^b \longrightarrow \mathcal{A}^b$ is thus defined by $F(\delta) = \overline{\delta}$.

If $p_g : g \Longrightarrow \frac{g}{[g]}$ and $p_k : k \Longrightarrow \frac{k}{[k]}$ are the projections such that $(p_g)_A(a) = a[g(A)]$ and $(p_k)_A(b) = b[k(A)]$ for $a \in g(A)$ and $b \in k(A)$, then the following figure commutes:

$$\begin{tikzcd}
g \arrow[r, Rightarrow, "p_g"] \arrow[d, Rightarrow, "\delta"'] & \frac{g}{[g]} \arrow[d, Rightarrow, "\bar{\delta}"] \\
k \arrow[r, Rightarrow, "p_k"'] & \frac{k}{[k]}
\end{tikzcd} .$$

This defines a natural transformation $\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "Id"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "p\Downarrow"] & \mathcal{A}^b \end{tikzcd}$, where

$g \xRightarrow{p_g} \frac{g}{[g]}$ and $F(g) = \frac{g}{[g]}$.

It was stated previously that the projection p_g was in some sense the "group coequalizer" of two operations which were called h and 0. It will be proved that p is the special coequalizer of h and 0.

First it will be shown that h and 0 are operations in $\mathcal{A}$. Since $\mathcal{A}$ has a nullary operation $1 \xrightarrow{e} A$ that picks out the unit and 1 is a terminal object $A^2 \xrightarrow{0} A = A^2 \longrightarrow 1 \xrightarrow{e} A$. $A^2 \xrightarrow{h} A$ is given by the composition of the following morphisms in the theory $\mathcal{A}$:

$$A^2 \xrightarrow{\langle 1_{A^2}, 1_{A^2}\rangle} A^2 \times A^2 \xrightarrow{\langle 1_{A^2}, \langle -1,-1\rangle\rangle} A^2 \times A^2 \xrightarrow{\langle \cdot,\cdot\rangle} A^2 \xrightarrow{\cdot} A$$

where $1_{A^2} : A^2 \longrightarrow A^2$ is the identity, $-1 : A \longrightarrow A$ is the inverse operation, $\cdot$ is the group multiplication and $\langle \alpha, \ldots, \beta \rangle$ denotes the product of the morphisms $\alpha, \ldots, \beta$.

Recall that $\mathcal{A}$ is identified with $\mathrm{St}\ \mathrm{Sm}\,\mathcal{A}$ by the isomorphism $\mathcal{A} \xrightarrow{\eta_{\mathcal{A}}} \mathrm{St}\ \mathrm{Sm}\,\mathcal{A}$ where $\eta_{\mathcal{A}}(h)_f = f(h)$ for any $h \in \mathcal{A}$ and $f \in |\mathcal{A}^b|$. In particular, this identifies $A^2 \xrightarrow{h} A$ with $U_{\mathcal{A}}^2 \xRightarrow{\eta_{\mathcal{A}}(h)} U_{\mathcal{A}}$. To show that p is the special coequalizer of h and 0, it must first be shown that $(U_{\mathcal{A}} p)\eta_{\mathcal{A}}(h) = (U_{\mathcal{A}} p)\eta_{\mathcal{A}}(0)$. Let $U \equiv U_{\mathcal{A}}$. Given any $f \in |\mathcal{A}^b|$ and $(x,y) \in U^2(f)$, $(Up\ \eta_{\mathcal{A}}(h))_f(x,y)$
$= U(p_f)\eta_{\mathcal{A}}(h)_f(x,y) = (p_f)_A f(h)(x,y) = (p_f)_A(xyx^{-1}y^{-1})$
$= e[f(A)] = (p_f)_A f(0)(x,y) = U(p_f)\eta_{\mathcal{A}}(0)_f(x,y)$
$= (Up\eta_{\mathcal{A}}(0))_f(x,y)$. Hence, since f and (x,y) were arbitrary, $(U_{\mathcal{A}} p)\eta_{\mathcal{A}}(h) = (U_{\mathcal{A}} p)\eta_{\mathcal{A}}(0)$.

Now suppose there exists a $\tau : \mathrm{Id} \Longrightarrow G$ such that $(U\tau)\eta_{\mathcal{A}}(h) = (U\tau)(\eta_{\mathcal{A}}(0))$. For any $f \in |\mathcal{A}^b|$, consider the following diagram:

$$\begin{tikzcd}
f(A^2) \arrow[r, shift left, "f(h)"] \arrow[r, shift right, "f(0)"'] & f(A) \arrow[r, "(\tau_f)_A"] \arrow[dr, "(p_f)_A"'] & G(f)(A) \\
 & & F(f)(A) = \dfrac{f(A)}{[f(A)]}
\end{tikzcd} .$$

For $b[f(A)] \in \frac{f(A)}{[f(A)]}$ pick any element $c \in f(A)$ such that $(p_f)_A(c) = b[f(A)]$. Define

$(\sigma_f)_A : \frac{f(A)}{[f(A)]} \longrightarrow G(f)(A)$ by $(\sigma_f)_A(b[f(A)]) = (\tau_f)_A(c)$. It must be shown that $(\sigma_f)_A$ is well-defined. Suppose $(p_f)_A(c) = (p_f)_A(c')$. Then $c[f(A)] = c'[f(A)]$, which implies that $c' = c\, x_o y_o x_o^{-1} y_o^{-1} \cdot \ldots \cdot x_{n-1} y_{n-1} x_{n-1}^{-1} y_{n-1}^{-1}$ for $(x_i, y_i) \in f(A) \times f(A)$, $i \in n$. Since $(\tau_f)_A(d) = e$ when $d \in [f(A)]$ and

$c^{-1}c' = x_o y_o x_o^{-1} y_o^{-1} \cdot \ldots \cdot x_{n-1} y_{n-1} x_{n-1}^{-1} y_{n-1}^{-1} \in [f(A)]$,
$(\tau_f)_A(c^{-1}c') = e$. Hence $(\tau_f)_A(c) = (\tau_f)_A(c')$, because $(\tau_f)_A$ is a homomorphism. Therefore $(\sigma_f)_A$ is well-defined.

It is easy to show that $(\sigma_f)_A$ is a homomorphism. $(\sigma_f)_A$ is unique because $(p_f)_A$ is surjective. Since $(\tau_f)_{A^n} = [(\tau_f)_A]^n$ and $(p_f)_{A^n} = [(p_f)_A]^n$, the definition will be extended to define $(\sigma_f)_{A^n} \equiv [(\sigma_f)_A]^n$.

$\sigma_f : F(f) \Longrightarrow G(f)$ can be shown to be a natural transformation using the fact that $[(p_f)_A]^n$ is surjective for each $n \in \omega$. Since $f \in |\mathcal{A}^b|$ was arbitrary, σ is a transformation such that $\sigma p = \tau$. Again σ can be shown to be a natural transformation. Thus p is a special coequalizer of the pair $(\eta_{\mathcal{A}}(h), \eta_{\mathcal{A}}(0))$ which is identified with the pair $(h, 0)$. Recall that $U_{\mathcal{A}} p \in \mathcal{A}_\omega$, the extended theory of groups.

In the construction of $\mathcal{A}_\omega$ where $\mathcal{A}$ is the algebraic theory of groups, p appears and so does
$$\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\overline{0}"] & \mathcal{A}^b \end{tikzcd}$$

where for $f \in |\mathcal{A}^b|$ and $A^n \in |\mathcal{A}|$,

$(\overline{0}_f)_{A^n} = f(A^n) \longrightarrow 1 \xrightarrow{e_{A^n}} F(f)(A^n)$. The equalizer of p and $\overline{0}$ is a functor $F' : \mathcal{A}^b \longrightarrow \mathcal{A}^b$ which picks out the commutator subgroup of every group and $U_{\mathcal{A}} F'$ appears in $|\mathcal{A}_\omega|$.

The construction of the commutator subgroup from the operations suggests that there are many other natural algebraic entities which may be consequences of the theory, some well known and others not extensively studied. The extended theories might well be useful in answering questions about specific algebras. These possibilities will not be pursued at present, in spite of their appeal.

V. TOPOLOGICAL CONSIDERATIONS

Topologies can be put on various algebras so that their operations become continuous. A specific example of this will be considered. Then some general ways of putting topologies on algebras making their operations and/or homomorphisms continuous will be studied.

In various sources, for example [1], [8] and [9], certain groups, rings and modules are endowed with topologies obtained from their algebraic structure. In particular, given a ring R with ideal m, Zariski and Samuel [9] use the families of cosets of m^n for each $n \in \omega$ to form a basis of a topology on the ring for which the operations are continuous with respect to the product topology on R^2. The topologies to be constructed include the above as a special case.

The notational language used will be similar to that used by Pierce [7].

Let Γ be an equivalence relation on a set A. Denote by $\frac{A}{\Gamma}$ the set of equivalence classes. Let $\overline{\Gamma}$ be the natural mapping $A \xrightarrow{\overline{\Gamma}} \frac{A}{\Gamma}$. If A is an $\mathcal{A}$-algebra and Γ is a congruence relation then $\frac{A}{\Gamma}$ is an $\mathcal{A}$-algebra and $\overline{\Gamma}$ an epimorphism. The following lemma will be of use:

<u>Lemma</u> ([7], page 24): Let Γ be an equivalence relation on the set A, then:

(a) $a \in \overline{\Gamma}(a)$,

(b) if $\overline{\Gamma}(a) \cap \overline{\Gamma}(b) \neq \phi$, then $\overline{\Gamma}(a) = \overline{\Gamma}(b)$; and

(c) $\overline{\Gamma}(a) = \overline{\Gamma}(b)$ if and only if $(a,b) \in \Gamma$.

Suppose $\{\Gamma_i : i \in I\}$ is a nonempty family of congruence relations defined on an $\mathcal{A}$-algebra f, $\Gamma_i \subset f(A) \times f(A)$, with the property that for any i, $j \in I$, there exists a $k \in I$ such that $\Gamma_k \subset \Gamma_i \cap \Gamma_j$. Such a family will be called a directed family of congruence relations. The discrete topologies on $\frac{f(A)}{\Gamma_i}$ for each $i \in I$ induce a topology τ on $f(A)$ with the family of all inverse images of open sets as a subbasis of τ, i.e., $\left\{\overline{\Gamma}_i^{-1}(\theta) : i \in I, \theta \subset \frac{f(A)}{\Gamma_i}\right\}$ is a subbasis of τ. By a result which will appear later in this chapter, f with the topology τ is a topological algebra. The topological space is written $(f(A),\tau)$ and τ is said to be induced by the directed family $\{\Gamma_i : i \in I\}$.

<u>Lemma</u>: Let $(f(A),\tau)$ be a topological space where τ is induced by a directed family $\{\Gamma_i : i \in I\}$ of congruence relations. For $x \in f(A)$, $\{\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))\}_{i \in I}$ forms a neighborhood basis of x, i.e., for any neighborhood θ of x there exists an $i \in I$ such that $\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x)) \subset \theta$.

<u>Proof</u>: Let θ_x be an open neighborhood of x. This implies that $\theta_x = \bigcup_{j \in J} \left(\bigcap_{i \in \alpha_j} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(A_i)) \right)$ for some $J \in |\mathcal{J}|$ where α_j is a finite subset of I for all $j \in J$. Then $x \in \bigcap_{i \in \alpha_j} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(A_i))$ for some $j \in J$. Now there exists Γ_k such that $\Gamma_k \subset \bigcap_{i \in \alpha_j} \Gamma_i$ because α_j is finite. Consider any $r \in \overline{\Gamma}_k^{-1}(\overline{\Gamma}_k(x))$. Then $(r,x) \in \Gamma_k \subset \bigcap_{i \in \alpha_j} \Gamma_i$. This implies that $r \in \bigcap_{i \in \alpha_j} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))$ and thus $r \in \bigcap_{i \in \alpha_j} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(A_i))$. Therefore $\overline{\Gamma}_k^{-1}(\overline{\Gamma}_k(x)) \subset \bigcap_{i \in \alpha_j} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(A_i)) \subset \theta_x$. So $\{\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))\}_{i \in I}$ forms a neighborhood basis of x.

<u>Lemma</u>: Let $(f(A), \tau)$ be a topological space where τ is induced by a directed family $\{\Gamma_i : i \in I\}$ of congruence relations. For any $x \in f(A)$, $f(A) \setminus \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))$ is open for each $i \in I$.

<u>Lemma</u>: Let $(f(A), \tau)$ be a topological space where τ is induced by a directed family $\{\Gamma_i : i \in I\}$ of congruence relations. The closure $\overline{S}$ of a subset $S \subset f(A)$ is equal to $\bigcap_{i \in I} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(S))$.

Proof: $x \in \bigcap_{i\in I} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(S))$ if and only if there exists for each $i \in I$, an element $s_i \in S$ such that $x \in \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(s_i))$. Now $x \in \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(s_i))$ for each $i \in I$ if and only if $s_i \in \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))$ for each $i \in I$ which is true if and only if $x \in \overline{S}$.

A topological space (X,θ) is R_1 if and only if $x,y \in X$ and $x \neq y$ imply that either $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$ (the empty set) or $\overline{\{x\}} = \overline{\{y\}}$. (X,θ) is regular, R_2, if and only if for any $x \in X$ and closed $C \subset X$, $x \notin C$ implies that there exists $G \in \theta$ and $H \in \theta$ such that $x \in G$, $C \subset H$ and $G \cap H = \emptyset$. (X,θ) is normal, R_3, if and only if for any two disjoint closed subsets C and C' of X, C and C' have disjoint neighborhoods.

Proposition 1: If $(f(A),\tau)$ is a topological space where τ is induced by a directed family $\{\Gamma_i : i \in I\}$ of congruence relations, then $(f(A),\tau)$ is regular.

Proof: Let x and C be given such that $x \notin C$ and C is closed in $f(A)$. Now $x \notin C = \bigcap_{i\in I} \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(C))$ implies that there is an $m \in I$ such that $x \notin \overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(C))$. Consider $T \equiv \overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(C)) \cap \overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(x))$. Suppose $T \neq \emptyset$. Let $d \in T$, then there exists an $s \in C$ such that $d \in \overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(s))$ and $d \in \overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(x))$. Hence $\overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(x)) = \overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(s))$, and $x \in \overline{\Gamma}_m^{-1}(\overline{\Gamma}_m(C))$. This is a contradiction. Therefore $T = \emptyset$. Since x and C were arbitrary, $(f(A),\tau)$ is regular.

$(f(A),\tau)$ in Proposition 1 need not be normal as the following example will show. The example comes from [2], page 131. $(f(A),\tau)$ will be the product of the topological spaces $(f_1(A),\tau_1)$ and $(f_2(A),\tau_2)$ where τ_1 is the order topology on the set $f_1(A)$ of all ordinal numbers less than the first uncountable ordinal Ω and τ_2 is the order topology on $f_1(A) \cup \{\Omega\} = f_2(A)$. It will now be shown that τ is induced by a directed family of congruence relations.

Let $\mathcal{A}$ be the theory of sets. A congruence relation on an $\mathcal{A}$-algebra, i.e., on a set, is just an equivalence relation because the only n-ary operations are projections. Given a function $h : X \longrightarrow Y$ in $\mathcal{L}$, the inverse images of elements of Y form a partition on X.

Let J be the set of all nonlimit ordinals in $f_1(A)$ and I' be the union of the set products J^n where $n \in \omega$, the first limit ordinal. Let
$I = \{i \in I' : i = (x_o, x_1, \ldots, x_{n-1}), x_o < x_1 < \ldots < x_{n-1}$ for some $n \in \omega\}$. Define for each element
$i = (x_o, x_1, \ldots, x_{n-1}) \in I$, a function $h_{1i} : f_1(A) \longrightarrow \omega$
such that

$$h_{1i}(k) = \begin{cases} 0, & \text{if } k < x_o \\ j, & \text{if } x_{j-1} \leq k < x_j \\ n, & \text{if } k \geq x_{n-1} \end{cases} .$$

The equivalence relations corresponding to the partitions of $f_1(A)$ formed by the functions h_{1i}, $i \in I$, constitute a directed family of congruence relations and induce the order topology τ_1 on $f_1(A)$. Similarly τ_2 is generated by functions $h_{2i} : f_2(A) \longrightarrow \omega$ indexed by the set I as above and defined in the same way as h_{1i}. Finally let $\overline{I}$ be the set $I \times \{1,2\}$ and define functions $h_{i,j} : f_1(A) \times f_2(A) \longrightarrow \omega$ such that $h_{i,j} = h_{ji} \circ p_j$ where p_j is the j-th projection. Then the product topology τ on $f(A) = f_1(A) \times f_2(A)$ is generated by the functions $h_{i,j}$, $(i,j) \in \overline{I}$, in the same manner as τ_1 and τ_2 are generated. It can be shown that τ is not normal.

<u>Proposition 2</u>: If τ is induced on $f(A)$ by a directed family $\{\Gamma_i : i \in I\}$ of congruence relations, then $(f(A),\tau)$ is Hausdorff if and only if $\bigcap_{i \in I} \Gamma_i = \mathrm{Id}$, where Id is the smallest equivalence relation on $f(A)$.

<u>Proof</u>: Assume $\bigcap_{i \in I} \Gamma_i = \mathrm{Id}$. Suppose $x \neq y$ and $x,y \in f(A)$. Then there exists an $i \in I$ such that $(x,y) \notin \Gamma_i$, and so $\overline{\Gamma}_i(x) \neq \overline{\Gamma}_i(y)$. This implies that $\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x)) \cap \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(y)) = \emptyset$. Hence $(f(A),\tau)$ is Hausdorff since $x \in \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))$ and $y \in \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(y))$.

Otherwise suppose $\bigcap_{i \in I} \Gamma_i \neq \mathrm{Id}$. Then there exists an $(x,y) \in \bigcap_{i \in I} \Gamma_i$ such that $x \neq y$. Note that $\overline{\Gamma}_i(x) = \overline{\Gamma}_i(y)$ for each $i \in I$. Let θ_x be a neighborhood of x. Since $\{\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))\}_{i \in I}$ is a neighborhood basis of x, there is an $i \in I$ such that $\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x)) \subset \theta_x$. But $y \in \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(y)) = \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x)) \subset \theta_x$. Hence the elements x and y cannot be separated. Therefore $(f(A),\tau)$ is not Hausdorff.

Another necessary and sufficient condition on f so that $(f(A),\tau)$ is Hausdorff follows.

<u>Definition</u> ([7], page 42): Let $\{g_i : i \in I\} \subset |\mathcal{A}^b|$. A <u>subdirect product</u> of $\{g_i : i \in I\}$ is any $g \in |\mathcal{A}^b|$ such that g is isomorphic to a subalgebra of the product of the g_i's and such that the restriction of the projections to g are epimorphisms.

<u>Lemma</u> ([7], page 43): Let $f \in |\mathcal{A}^b|$ and $\{\Gamma_i : i \in I\}$ be a nonempty family of congruence relations on $f(A)$. Then f is a subdirect product of $\{\frac{f}{\Gamma_i} : i \in I\}$ if and only if $\bigcap_{i \in I} \Gamma_i = \mathrm{Id}$.

Proposition 2 can now be restated as follows: $(f(A),\tau)$ is Hausdorff if and only if f is a subdirect product of $\{\frac{f}{\Gamma_i} : i \in I\}$.

The example from Zariski and Samuel [9] at the beginning of this chapter has a pseudo-metric topology, as do many other examples. A sufficient condition will be given for a directed family of congruence relations to induce a pseudo-metric topology. Most examples encountered in the literature satisfy this condition.

A pseudo-metric for a set X is a nonnegative real-valued function d on $X \times X$ such that for any $x,y,z \in X$,

(a) $d(x,y) = d(y,x)$,

(b) $d(x,y) + d(y,z) \geq d(x,z)$,

and (c) $d(x,x) = 0$.

If in addition $d(x,y) = 0$ implies that $x = y$, then d is called a metric.

The pseudo-metric topology τ for (X,d) where d is a pseudo-metric has the family $\{\theta_x : \theta_x = \{y : d(x,y) < r\}, r > 0\}$ as a neighborhood basis of x for any $x \in X$.

Let $(f(A),\tau)$ be a topological space where τ is induced by a directed family $\{\Gamma_i : i \in I\}$ of congruence relations with $I = \omega$, the first limit ordinal and $f(A) \times f(A) = \Gamma_o \supset \Gamma_1 \supset \Gamma_2 \supset \cdot \ldots$. For $(x,y) \in f(A) \times f(A)$, let

$$v(x,y) = \begin{cases} \max\{n : (x,y) \in \Gamma_n\}, & \text{if it exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Define $d : f(A) \times f(A) \longrightarrow R : (x,y) \longmapsto e^{-v(x,y)}$ where e is a fixed real number greater than one. It is clear that d is a pseudo-metric. d will be a metric if and only if $\bigcap_{i \in I} \Gamma_i = \mathrm{Id}$.

It will be shown that the pseudo-metric topology for (X,d) is the same as the original topology τ by showing that $T_x \equiv \{\{y : d(x,y) < r\} : r > 0\}$ forms a neighborhood basis of x in the original topology τ. Let $r > 0$ be given. Then $\{y : d(x,y) < r\}$

$= \{y : e^{-v(x,y)} < r\} = \{y : v(x,y) > \log_e \frac{1}{r}\}$

$= \{y : \max\{n : (x,y) \in \Gamma_n\} > \log_e \frac{1}{r}\}$

$= \{y : (x,y) \in \Gamma_k$ where k is the least integer greater than $\log_e \frac{1}{r}\}$

$= \overline{\Gamma}_k^{-1}(\overline{\Gamma}_k(x))$. Hence $T_x \subset \{\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x)) : i \in \omega\}$. Now let $\overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))$ be given. There exists an $r_1 > 0$ such that $\log_e \frac{1}{r_1} > i$. Thus $\{y : d(x,y) < r_1\} \subset \overline{\Gamma}_i^{-1}(\overline{\Gamma}_i(x))$. Therefore T_x forms a neighborhood basis of x in the original topology τ. Since x was arbitrary it can be concluded that $(f(A), \tau)$ is a pseudo-metric topological space.

Algebras with a topology on their underlying sets for which the operations are continuous with respect to the appropriate product topologies have been called topological algebras. In particular if $\mathcal{A}$ is the theory of groups, then an $\mathcal{A}$-algebra for which the operations are continuous is called a topological group.

Proposition 3: Let $\mathcal{A}$ be an algebraic theory. Suppose $\mathcal{A}$-algebras f and g_i, $i \in I$, and homomorphisms $f \xrightarrow{\alpha_i} g_i$ in $\mathcal{A}^b$ are given. In addition assume $(g_i(A), \tau_i)$ is a topological algebra for each $i \in I$. Let $\{(\alpha_i)^{-1}(\theta) : \theta \in \tau_i, i \in I\}$ be a subbasis for a topology τ on $f(A)$. Then all n-ary and generalized operations on f are continuous.

Proof: Let $A^n \xrightarrow{h} A$ be an n-ary operation in the theory $\mathcal{A}$. It must be shown that $f(h)$ is continuous. Recall that $A^n \xrightarrow{p_j} A$ is an operation where p_j is the j-th projection and that $f(p_j)$ and $g_i(p_j)$ are continuous by the definition of the product topology. Now

$$\begin{array}{ccc}
f(A)^n = f(A^n) & \xrightarrow{(\alpha_i)_{A^n}} & g_i(A^n) = g_i(A)^n \\
\Big\downarrow f(p_j) & & \Big\downarrow g_i(p_j) \\
f(A) & \xrightarrow[(\alpha_i)_A]{} & g_i(A)
\end{array}$$

commutes for all j since α_i is a natural transformation.

Let Q be a subbasic open set of $g_i(A^n)$. Then $Q = (g_i(p_j))^{-1}(\theta)$ for some $j \in n$ and $\theta \in \tau_i$. Now $\alpha_{i_{A^n}}^{-1}(Q) = \alpha_{i_{A^n}}^{-1}((g_i(p_j))^{-1}(\theta)) = ((g_i(p_j))\alpha_{i_{A^n}})^{-1}(\theta)$ $= (\alpha_{i_A} f(p_j))^{-1}(\theta)$ which is open since $\alpha_{i_A} f(p_j)$ is continuous. Hence $(\alpha_i)_{A^n}$ is continuous.

Let Q be a subbasic open set in τ, then there exists an $i \in I$ such that $Q = \alpha_{i_A}^{-1}(\theta)$ for some $\theta \in \tau_i$. Now

$$\begin{array}{ccc} f(A^n) & \xrightarrow[\alpha_{i_{A^n}}]{} & g_i(A^n) \\ {\scriptstyle f(h)}\downarrow & & \downarrow{\scriptstyle g_i(h)} \\ f(A) & \xrightarrow[\alpha_{i_A}]{} & g_i(A) \end{array}$$

commutes. As in the proof of the continuity of $(\alpha_i)_{A^n}$, since $g_i(h)$, $\alpha_{i_{A^n}}$ and α_{i_A} are continuous, it can be shown that $f(h)^{-1}(Q)$ is open and hence $f(h)$ is continuous.

Now let $A^n \xrightarrow{k} A^m$ be any generalized operation, it must be shown that $f(k)$ is continuous. k is an m-tuple of n-ary operations, i.e., $k = \langle h_o,\ldots,h_{m-1}\rangle$ where

$$\begin{tikzcd} A^n \arrow[rr, "k"] \arrow[dr, "h_i"'] & & A^m \arrow[dl, "p_i"] \\ & A & \end{tikzcd}$$ commutes.

Given a subbasic open set Q of $f(A^m)$, there exists an $i \in m$ and $\theta \in \tau$ such that $Q = f(p_i)^{-1}(\theta)$. This implies that $f(k)^{-1}(Q) = f(k)^{-1}f(p_i)^{-1}(\theta) = (f(p_i)f(k))^{-1}(\theta)$ $= f(p_ik)^{-1}(\theta) = f(h_i)^{-1}(\theta)$ which is open since $f(h_i)$ is continuous. Therefore $f(k)$ is continuous.

Note at this point that the topological work with congruence relations in the first part of this chapter comes under the hypotheses of the preceding proposition and thus the algebras involved are topological algebras.

If the natural transformations α_i, $i \in I$, of Proposition 3 had codomain f, domain g_i and $I = \{0\}$, then $\tau = \{\beta : (\alpha_{i_A})^{-1}(\beta) \in \tau_i, i \in \{0\}\}$ would be a topology for $f(A)$. Under these conditions the conclusions of Proposition 3 are again valid.

The preceding topologies were defined on specific $\mathscr{A}$-algebras. One may ask what topologies can be defined on all $\mathcal{A}$-algebras. While this question is very broad, a few partial answers will be given.

<u>Proposition</u> 4: Suppose $\{(G_i,\alpha_i) : i \in I, I \in |\mathcal{L}|\}$

where $\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "Id"] \arrow[r, bend right, "G_i"'] \arrow[r, phantom, "\Downarrow\alpha_i"] & \mathcal{A}^b \end{tikzcd}$ has the property that for each

$f \in |\mathcal{A}^b|$ there exists a topology τ_{if} on $U_{\mathcal{A}}G_i f$ such that $(U_{\mathcal{A}}G_i f, \tau_{if})$ is a topological algebra. If $\{(U_{\mathcal{A}}\alpha_i)_f^{-1}(\theta) : \theta \in \tau_{if}, i \in I\}$ is used as a subbasis for a topology γ_f on $U_{\mathcal{A}} f = f(A)$, then $(f(A),\gamma_f)$ is a topological algebra.

If the natural transformations α_i, $i \in I$, of Proposition 4 had codomain Id, domain G_i and $I = \{0\}$, then $\gamma_f = \{\beta : (U_{\mathcal{A}}\alpha_i)_f^{-1}(\beta) \in \tau_{if}, i \in \{0\}\}$ would be a topology for $f(A)$ and the conclusion of Proposition 4 would again be valid.

In the context of Proposition 4, with $\nu \in \mathcal{A}^b$, $f \overset{\nu}{\Longrightarrow} h$,

$$\begin{tikzcd} f(A) \arrow[r, "\nu_A"] \arrow[d, "U_{\mathcal{A}}\alpha_{i_f}"'] & h(A) \arrow[d, "U_{\mathcal{A}}\alpha_{i_h}"] \\ U_{\mathcal{A}}G_i f \arrow[r, "U_{\mathcal{A}}G_i\nu"'] & U_{\mathcal{A}}G_i h \end{tikzcd}$$ commutes

since $U_{\mathcal{A}}\alpha_i$ is a natural transformation. The question arises as to whether ν_A need be continuous. A quick negative reply is obtained if $I = \{0\}$, $G_o = \mathrm{Id}$ and α_o is the identity natural transformation. Let f be the group of integers with the indiscrete topology τ_1 on $f(A)$ and g be the group of rational numbers with the discrete topology τ_2 on $g(A)$, then the inclusion homomorphism $\nu : f \Longrightarrow g$ is not continuous at A while both $(f(A),\tau_1)$ and $(g(A),\tau_2)$ are topological groups.

The problem of making homomorphisms continuous will now be considered.

<u>Proposition 5</u>: Suppose $\{(G_i,\alpha_i) : i \in I, I \in |\mathcal{L}|\}$

where $\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "U_{\mathcal{A}}"] \arrow[r, bend right, "G_i"'] \arrow[r, phantom, "\Downarrow\alpha_i"] & \mathcal{L} \end{tikzcd}$ has the property that for each

$f \in |\mathcal{A}^b|$ there exists a topology τ_{if} on $G_i f$ such that $G_i f \xrightarrow{G_i\beta} G_i g$ is continuous for each $\beta \in \mathcal{A}^b$. If $\{(\alpha_{i_f})^{-1}(\theta) : \theta \in \tau_{if}, i \in I\}$ is used as a subbasis for a topology γ_f on $f(A) = U_{\mathcal{A}} f$ for each $f \in |\mathcal{A}^b|$, then for $\nu \in \mathcal{A}^b$, $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\nu"] & \mathcal{L} \end{tikzcd}$, ν_A and ν_{A^n} are continuous for all

$n \in \omega$ where the product topology is put on $f(A)^n$.

Proof: First it will be shown that ν_A is continuous. Let Q be a subbasic open set in $h(A)$. Then there exists an $i \in I$ and $\theta \in \tau_{i_h}$ such that $Q = (\alpha_{i_h})^{-1}(\theta)$. Now

$$\begin{tikzcd} f(A) \arrow[r, "\nu_A"] \arrow[d, "(\alpha_i)_f"'] & h(A) \arrow[d, "(\alpha_i)_h"] \\ G_i(f) \arrow[r, "G_i(\nu)"'] & G_i(h) \end{tikzcd}$$

commutes since

α_i is a natural transformation. Note that $(\alpha_i)_f$, $G_i(\nu)$ and $(\alpha_i)_h$ are all continuous. Hence

$\nu_A^{-1}(Q) = \nu_A^{-1}(\alpha_{i_h})^{-1}(\theta) = (\alpha_{i_h}\nu_A)^{-1}(\theta) = (G_i(\nu)\alpha_{i_f})^{-1}(\theta)$

which is open because $G_i(\nu)\alpha_{i_f}$ is continuous. Therefore, since inverse images of subbasic open sets are open, ν_A is continuous.

Using the following commutative rectangle and the continuity of ν_A and the projections, it can be shown that ν_{A^n} is continuous:

$$\begin{tikzcd}
{[f(A)]^n} \arrow[r, "\nu_A^n"] \arrow[d, "p_i"'] & {[h(A)]^n} \arrow[d, "p_i"] \\
f(A) \arrow[r, "\nu_A"'] & h(A)
\end{tikzcd} .$$

If the natural transformations α_i, $i \in I = \{0\}$, of Proposition 5 had codomain $U_{\mathcal{A}}$ and domain G_i, then $\gamma_f = \{\beta : \alpha_{i_f}^{-1}(\beta) \in \tau_{if}, i \in I\}$ would be a topology for $f(A)$ and the conclusion of Proposition 5 would again be valid.

In the context of Proposition 5, the question arises as to whether the operations are continuous with respect to γ_f, if for each $i \in I$, $G_i = U_{\mathcal{A}} F_i$ and $\alpha_i = U_{\mathcal{A}} \delta_i$ for some $\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "F_i"] \arrow[r, bend right, "\mathrm{Id}"'] \arrow[r, phantom, "\Uparrow\delta_i"] & \mathcal{A}^b \end{tikzcd}$. The following is a counterexample.

Let $\mathcal{A}$ be the theory of groups, $I = \{0\}$, $F_o = \mathrm{Id}$ and $\delta_o = 1_{\mathrm{Id}}$. For each $f \in \mathcal{A}^b$ consider the family $\{\Gamma : \Gamma$ is a congruence relation on $f\}$, which corresponds to the family of all normal subgroups of f. For each Γ put the topology on $(\frac{f}{\Gamma})(A)$ consisting of $\{\frac{f}{\Gamma}(A), \{0\}, \emptyset\}$. Then a topology τ_f is induced on $f(A)$ with a basis consisting of all inverse images $\overline{\Gamma}^{-1}(\theta)$ of open sets θ for each $\overline{\Gamma} : f \Longrightarrow \frac{f}{\Gamma}$. The topology τ_f consists of unions of the underlying sets of the normal subgroups of f. Note that every nonempty open set contains the identity $0 \in f(A)$.

Given $(f(A),\tau_f)$, $(g(A),\tau_g)$ and $f \overset{\nu}{\Longrightarrow} g$, ν_A is continuous since the inverse image of a normal subgroup is again a normal subgroup. Thus with $\{(G_i,\alpha_i) : i \in I, I \in |\mathcal{L}|\} = \{(U_{\mathcal{A}}, 1_{U_{\mathcal{A}}})\}$ and the topology τ_f on $U_{\mathcal{A}}(f)$, the hypotheses of Proposition 5 are satisfied. Hence ν_{A^n} is continuous for each $n \in \omega$. Thus the homomorphisms are continuous. But it will now be shown that there are operations which are not continuous.

Let f be a group with at least one proper nontrivial normal subgroup Q. If $(f(A),\tau_f)$ were a topological group, then $f(A)\backslash Q(A)$ would be open. But $(f(A)\backslash Q(A)) \notin \tau_f$ since $f(A)\backslash Q(A) \neq \emptyset$ and $0 \notin f(A)\backslash Q(A)$. Hence $(f(A),\tau_f)$ is not a topological group.

Given an algebraic theory $\mathcal{A}$, one might wonder what natural topologies are induced by $\mathcal{A}$. Considering the extended theory $\mathcal{A}_\omega$, there exist numerous morphisms with $U_{\mathcal{A}}$ as domain or codomain. The above methods can be applied to the morphisms in $\mathcal{A}_\omega$ to obtain naturally induced topologies. These methods began with a family of topologies.

The question of what topologies to begin with is not answered in this paper, but since the discrete topology is always available, and the results of its use can be non-trivial as was evidenced earlier, one can always begin with the discrete topology and obtain some of the natural topologies induced by the extended theory. For example,

given $\begin{tikzcd} \mathcal{A}^b \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "G_i"'] \arrow[r, phantom, "\Downarrow\alpha_i"] & \mathcal{A}^b \end{tikzcd}$, $U_{\mathcal{A}}\alpha_i \in \mathcal{A}_\omega$, $i \in I$ for some I,

putting the discrete topology on $U_{\mathcal{A}}G(f)$ for each $f \in |\mathcal{A}^b|$ and inducing the topology τ_f on $f(A)$ using inverse images of open sets results in both operations and homomorphisms being continuous with respect to the topologies τ_f, i.e., $\mathcal{A}^b$ is isomorphic to a category of topological algebras with continuous homomorphisms.

VI. BIBLIOGRAPHY

1. Bourbaki, N. General topology. Vol. 2. Reading, Mass., Addison-Wesley Publishing Co., Inc. 1966.

2. Kelley, J. L. General topology. Princeton, N.J., D. Van Nostrand Co., Inc. 1955.

3. Lawvere, F. W. Functorial semantics of algebraic theories. Unpublished Ph.D. thesis. New York, N.Y., Library, Columbia University. 1963.

4. Lawvere, F. W. Functorial semantics of algebraic theories. Nat. Acad. Sci. Proc. 50: 869-872. 1963.

5. Lawvere, F. W. Some algebraic problems in the context of functorial semantics of algebraic theories. Lecture Notes in Mathematics 61: 41-61. 1968.

6. MacLane, S. Categorical algebra. Mimeographed notes for the National Science Foundation Advanced Science Seminar. Brunswick, Maine, Mathematics Department, Bowdoin College. 1969.

7. Pierce, R. S. Introduction to the theory of abstract algebras. New York, N.Y., Holt, Rinehart and Winston. 1968.

8. Pontrjagin, L. Topological groups. Princeton, N.J., Princeton University Press. 1939.

9. Zariski, O. and Samuel, P. Commutative algebra. Vol. 2. Princeton, N.J., D. Van Nostrand Co., Inc. 1960.

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