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On 8p-dimensional Hopf algebras with the Chevalley property

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On $8p$-dimensional Hopf algebras with the Chevalley property

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DEDICATION

To my parents, for their never-ending love, support and encouragement.

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ABSTRACT

In this dissertation, we study the classification of Hopf algebras of dimension 24, and more generally, $8p$ where p is an odd prime. In particular, we show if a 24-dimensional Hopf algebra has a nontrivial grouplike element, then it must also have a nontrivial skew primitive element. Further, a nonsemisimple $8p$-dimensional Hopf algebra cannot contain a semisimple Hopf subalgebra of dimension 8. Finally, we classify the nonsemisimple Hopf algebras of dimension $8p$ with the Chevalley property. In particular, we find such a Hopf algebra must either be pointed, or have a coradical of dimension $4p$. Further the $4p$-dimensional coradical can only be isomorphic to the dual of the dihedral group algebra, the dual of the dicyclic group algebra, or the self-dual, noncommutative semisimple Hopf algebra A_+ of dimension $4p$.

CHAPTER 1. INTRODUCTION

Hopf algebras are bialgebra structures, both an algebra and coalgebra, that possess an endomorphism (referred to as the antipode) satisfying certain conditions. They were first introduced in 1941 by Heinz Hopf in the context of algebraic topology [27]. It wasn't until the 1960s and the introduction of Sweedler's book [55] that a formal theory of Hopf algebras began. In 1975, Irving Kaplansky presented ten conjectures as an appendix to lecture notes [28]. These conjectures have served as a focus for much of the development of research on Hopf algebras. Several conjectures have been shown to be true, and several have been proven false with the discovery of counterexamples. Despite progress made since first being introduced, there are still conjectures that have not yet been solved.

1.1 Kaplansky's conjectures

In this section, we will introduce four of Kaplansky's conjectures, as found in [28, A.2], as well as the progress that has been made in proving them true or false.

> **Conjecture 1:** *If C is a Hopf subalgebra of the Hopf algebra B, then B is a free left C-module.*

This conjecture has been shown to not be true by several infinite-dimensional counter-examples. In particular, Radford constructed a counter-example in [47]. However, for the finite-dimensional case, the conjecture was proven true in 1989 by Nichols and Zoeller with what is now known as the Nichols-Zoeller Theorem [46]. In particular, for the finite-dimensional case, this result is an extension of Lagrange's Theorem for finite groups, indicating that if K is a Hopf subalgebra of H, then $\dim K$ divides $\dim H$.

For the remaining conjectures, let H be a finite-dimensional Hopf algebra over an algebraically closed field $\Bbbk$.

Conjecture 5: *If H or H^* is semisimple, the square of the antipode is the identity.*

This conjecture was proven true when $\Bbbk$ has characteristic zero in 1988 by Larson and Radford in [30] and [31].

Conjecture 8: *If the dimension of H is prime, then H is commutative and cocommutative.*

In 1994, Zhu proved this conjecture to be true when the base field has characteristic zero in [59]. He showed that all Hopf algebras of prime dimension must be isomorphic to group algebras when $\Bbbk$ is of characteristic zero.

Conjecture 10: *There are only a finite number (up to isomorphisms) of Hopf algebras of a given dimension.*

Since the group algebras are one of the most common examples of Hopf algebras, Kaplansky's tenth conjecture is a natural extension of the fact that there are a finite number of isomorphism classes of groups of a given order. In fact, in 1997, Ştefan proved this conjecture true for semisimple Hopf algebras [53]. However, this conjecture was disproven for the general case shortly thereafter in [4], [7], and [24]. Later, Etingof and Gelaki additionally found that there exists an infinite family of non-quasi-isomorphic Hopf algebras of dimension 32 in [16]. To date, this is the smallest such dimension with a counterexample.

While not true in general, the tenth conjecture sparked a movement to classify all finite-dimensional Hopf algebras of small dimension. A natural question to ask is if an infinite family of Hopf algebras with dimension less than 32 can be found. In the next section, we will present an overview of the classifications of Hopf algebras with dimension less than 32, including the single remaining case, dimension 24. A comprehensive table outlining the current status of classifications of Hopf algebras with dimension less than 100 can be found in [9].

1.2 Classifications of Hopf algebras with small dimension

In the last few decades, completed classifications of some small dimensional Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic zero have been established. As previously mentioned, Zhu showed in [59] that all Hopf algebras of dimension p, where p is prime, must be *trivial*. That is, they are isomorphic to a group algebra or the dual of a group algebra. For any p, we know there is only one group algebra, which is additionally isomorphic to its dual, and thus, up to isomorphism, there is only one Hopf algebra of prime dimension for each prime. Ng showed in [43] that Hopf algebras with dimension p^2 must be either trivial or the Taft algebras. In particular, if semisimple, the Hopf algebra must be a group algebra [36] and if nonsemisimple, it must be pointed (c.f. [43]) and hence a Taft algebra by [3]. As there are only $p-1$ Taft algebras and 2 group algebras of dimension p^2 for each prime p, we know there are $p+1$ Hopf algebras of dimension p^2. Next, we consider the classifications of Hopf algebras of dimension p^3. When $p=2$, Masuoka found there is only one nontrivial, semisimple Hopf algebra [35]. Further, Ştefan found the four isomorphism classes of nontrivial, pointed Hopf algebras of dimension 8 in [54] and Williams and Ştefan established there are no other Hopf algebras of this dimension in [58] and [54]. If p is an odd prime, Masuoka found there are $p-1$ nontrivial, semisimple Hopf algebras of dimension p^3, while Andruskiewitsch and Schneider listed in [4] the nonsemisimple pointed p^3-dimensional Hopf algebras. Beattie and García completed the classification of dimension 27 Hopf algebras by showing they must be either semisimple or pointed in [21] and [10].

Further, Masuoka showed in [34] that semisimple Hopf algebras of dimension $2p$, with p an odd prime, must also be trivial. Beattie and Dăscălescu first established in [6] that 14-dimensional Hopf algebras must be semisimple. Ng later proved that nonsemisimple Hopf algebras of dimension $2p$ cannot exist for any prime p in [44]. Therefore, we know all Hopf algebras of dimension $2p$ are trivial and thus there are only 3 of them up to isomorphism: $\Bbbk\mathbb{Z}_{2p}$, the dihedral group algebra, and its dual.

Now, we consider dimension $2p^2$, where p is an odd prime. Natale showed in [38] there are only two nontrivial semisimple Hopf algebras of dimension $2p^2$, which are duals of each other, constructed by Masuoka in [36]. Andruskiewitsch and Natale found there are $4(p-1)$ pointed Hopf algebras of dimension $2p^2$ in [2, A.1]. Finally, while D. Fukuda was able to complete the classification of 18-dimensional Hopf algebras in [18], Hilgemann and Ng completed the generalized classification of Hopf algebras of dimension $2p^2$ in [26].

Next, we let p, q, r be distinct primes. Etingof and Gelaki were able to show in [15] that semisimple Hopf algebras of dimension pq must be trivial and Ng further proved that for $p < q \leq 4p + 11$, a pq-dimensional Hopf algebra must be semisimple in [45]. The classification has not yet been completed for a general p and q. Considering the pq^2 case, N. Fukuda found in [20] that a semisimple Hopf algebra of dimension 12 must either be trivial or one of two self-dual, noncommutative Hopf algebras, which will be discussed in more detail later in this work. In the pq^2 case, partial classifications were completed for the semisimple Hopf algebras by Gelaki, Masuoka, and Natale ([23], [36], [38], [39], [41]), but a complete classification was obtained in [17] by Etingof, Nikshych and Ostrik. Further, Andruskiewitsch and Natale found there were $4(q-1)$ nontrivial pointed Hopf algebras of dimension pq^2, found in [2, A.1], and Natale was able to complete the classification for dimension 12 in [40] while Cheng and Ng completed the classification of dimension 20, 28 and 44 in [13]. However, the pq^2-dimensional case is not completed in general. Finally, we examine Hopf algebras with dimension pqr, and in particular, dimension 30. Natale showed in [42] that 30-dimensional semisimple Hopf algebras must be trivial and D. Fukuda showed any such Hopf algebra must be semisimple [19]. Beattie and García were able to demonstrate that in the general pqr case, a Hopf algebra cannot have the Chevalley property, but the classification is far from complete.

At this stage, the only dimensions less than 32 whose classifications have not been presented are 16 and 24. Kashina was able to show in [29], there are exactly 16 nontrivial semisimple Hopf algebras of dimension 16. Caenepeel, Dăscălescu and Raianu found there are 29 pointed Hopf algebras of dimension 16 in [11], while in [12], it was shown by Călinescu, Dăscălescu, Masuoka, and Menini that there are only two 16-dimensional nonpointed Hopf algebras with the Chevalley property. García and Vay were able to complete the 16 dimension classification by organizing such Hopf algebras into five categories [22].

We are now left with dimension 24 being the smallest dimension in which there is not a complete, finite classification. Graña began this classification by classifying Nichols algebras of dimension 24 in [25]. These Nichols algebras can then be lifted to find the pointed Hopf algebras of dimension 24. Further, using Ştefan's result, we know there are only a finite number of semisimple Hopf algebras of dimension 24 [53]. The classification of nonpointed 24-dimensional Hopf algebras with the Chevalley property will be outlined later in this dissertation.

1.3 Overview of dissertation

In this dissertation, unless otherwise noted, $\Bbbk$ will be used to denote a field and H will be a finite-dimensional Hopf algebra. We will begin in Chapter 2 by reviewing and introducing several preliminary definitions, notations and general results that are useful in our proofs found in later chapters. We continue with several general results about Hopf algebras of dimension 24, and then more generally, $8p$ where p is an odd prime. In Chapter 4, after introducing the Yetter-Drinfeld category and the Radford biproduct, also known as bosonization, we highlight our main result, in the classification of nonpointed, nonsemisimple Hopf algebras of dimension $8p$ with the Chevalley property. We find there are only finitely many such Hopf algebras and are able to explicitly construct them using the structure of their coradical. We close with a look at what is remaining to be done in the classification of 24-dimensional (and more generally $8p$-dimensional) Hopf algebras as well future work to complete this classification.

CHAPTER 2. PRELIMINARIES

In this chapter, we will introduce several basic definitions as well as results that are well known but will be useful in later chapters. The readers are referred to [55], [37] and [49] for further reference for these preliminaries. Throughout this dissertation, $\Bbbk$ will be used to denote a field and all vector spaces, algebras, coalgebras and the tensor products $\otimes$ are defined over the same field $\Bbbk$ unless otherwise noted.

2.1 Basic definitions

Definition 2.1.1. An *algebra structure* over a field $\Bbbk$ is a triple (A, μ, η) where A is a $\Bbbk$-space and $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ are $\Bbbk$-linear maps satisfying the associativity and unitary properties illustrated in the commutative diagrams found in Figure 2.1.

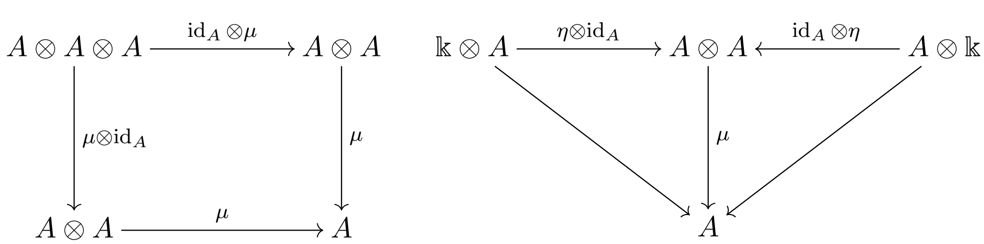


Figure 2.1: Commutative Diagrams for an Algebra

We say an algebra is *commutative* if for all $a, a' \in A$, $\mu(a \otimes a') = \mu(a' \otimes a)$. Further, if $B \subset A$ such that $\mu(B \otimes B) \subset B$, then we say B is a *subalgebra* of A.

The function μ is often called the *multiplication* of the algebra A, and $\mu(a \otimes b)$ is simply denoted by ab whenever there is no ambiguity. The element $\eta(1) = 1_A$ is called the *identity* of A. When the context is clear, we will simply say A is a $\Bbbk$-algebra for the algebra structure (A, μ, η).

Example 2.1.2. A basic example of an algebra structure is the group algebra. If G is a group, the *group algebra*, denoted $\Bbbk G$, contains elements of the form $\sum_{g \in G} a_g g$ with $a_g \in \Bbbk$ such that all but finitely many are 0. Multiplication and the identity are inherited from the underlying group.

Example 2.1.3. Further, if A and B are $\Bbbk$-algebras, then $A \otimes B$ is the *tensor algebra* where multiplication is given by

$$(a \otimes b)(a' \otimes b') = (aa' \otimes bb')$$

for $a, a' \in A$ and $b, b' \in B$ and the identity is given by $1_A \otimes 1_B$. In particular, we note that $A \otimes A$ is an algebra.

Next, we can consider maps between algebras.

Definition 2.1.4. Let (A, μ_A, η_A) and $(A', \mu_{A'}, \eta_{A'})$ be algebra structures over $\Bbbk$. An *algbera map* $f : A \to A'$ is a $\Bbbk$-linear map of the underlying vector spaces such that the diagrams in Figure 2.2 commute.

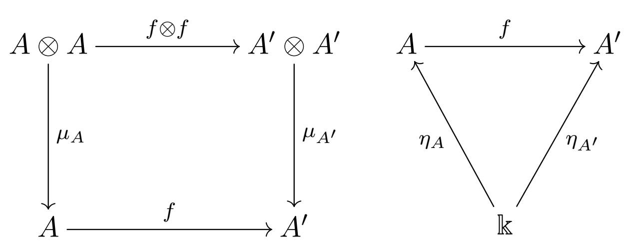


Figure 2.2: Commutative Diagrams for an Algebra Map

The commutative diagrams can be summarized as the following conditions:

$$f(ab) = f(a)f(b) \text{ and } f(1_A) = 1_{A'} \text{ for all } a, b \in A.$$

Now, we define the dual notion of an algebra, called a coalgebra. A coalgebra can be thought of as simply reversing the arrows in Figure 2.1. We present a formal definition below.

Definition 2.1.5. A *coalgebra structure* (over $\Bbbk$) is a triple (C, Δ, ϵ) where C is a $\Bbbk$-space and $\Delta : C \to C \otimes C$ and $\epsilon : C \to \Bbbk$ are $\Bbbk$-linear maps satisfying the coassociativity and counitary properties illustrated in Figure 2.3.

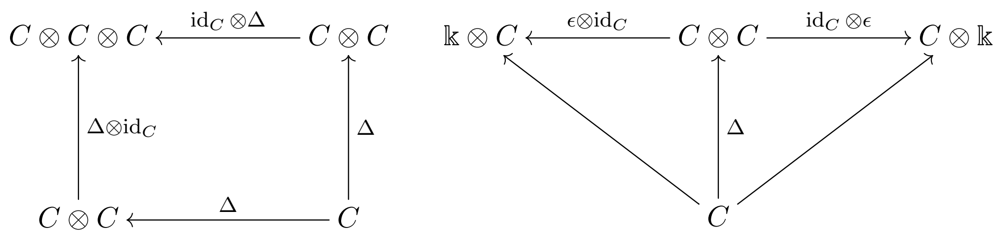


Figure 2.3: Commutative Diagrams for a Coalgebra

The coalgebra structure (C, Δ, ϵ) over $\Bbbk$ will simply be called a coalgebra C with *comultiplication* Δ and *counit* ϵ. As with algebras, when the context is clear, we will write C is a $\Bbbk$-coalgebra for the coalgebra structure (C, Δ, ϵ). Adopting the Sweedler-Heinemann notation (cf. [55]) with the summation suppressed, we will write $\Delta(c) = c_1 \otimes c_2$ for $c \in C$.

Define the twist map $\tau : C \otimes C \to C \otimes C$ by $\tau(c \otimes c') = c' \otimes c$. Then, we say a coalgebra is *cocommutative* if $\Delta = \tau \circ \Delta$ or $c_1 \otimes c_2 = c_2 \otimes c_1$ for all $c \in C$. Further, if $D \subset C$ such that $\Delta(D) \subset D \otimes D$, then we say D is a *sub-coalgebra* of C.

Example 2.1.6. Revisiting our group algebra example, we can define a coalgebra structure on $\Bbbk G$ by $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$. Using these definitions to trace elements through the diagrams found in Figure 2.3, one can easily verify that $\Bbbk G$ is a coalgebra.

In fact, elements that behave as though they are elements in some group algebra are very useful. We define them below.

Definition 2.1.7. Let (C, Δ, ϵ) be a coalgebra over $\Bbbk$. Then $c \in C$ is *grouplike* if

$$\Delta(c) = c \otimes c \text{ and } \epsilon(c) = 1.$$

The set of grouplike elements of a coalgebra C is denoted by $G(C)$. We can further say an element $c \in C$ is (g, h)-*skew primitive* if $\Delta(c) = g \otimes c + c \otimes h$ where $g, h \in G(C)$.

Example 2.1.8. As with algebras, if C and D are coalgebras, then $C \otimes D$ is a coalgebra with comultiplication defined by

$$\Delta(c \otimes d) = (c_1 \otimes d_1) \otimes (c_2 \otimes d_2)$$

and the counit defined by $\epsilon(c \otimes d) = \epsilon_C(c)\epsilon_D(d)$ for $c \in C$ and $d \in D$, where ϵ_C and ϵ_D are the counits of C and D respectively. In particular, $C \otimes C$ is a coalgebra.

As with algebras and coalgebras, we dualize our definition of algebras maps to define a map between two coalgebras.

Definition 2.1.9. Let $(C, \Delta_C, \epsilon_C)$ and $(C', \Delta_{C'}, \epsilon_{C'})$ be coalgebra structures over $\Bbbk$. A *coalgebra map*, $g : C \to C'$, is a $\Bbbk$-linear map of the underlying vector spaces such that the diagrams in Figure 2.4 commute.

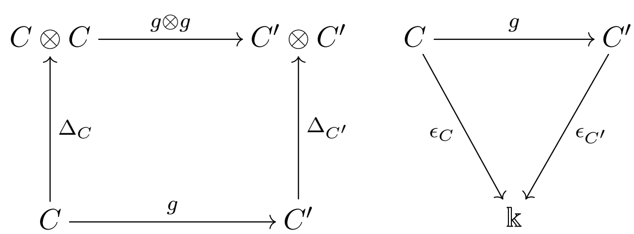


Figure 2.4: Commutative Diagrams for a Coalgebra Map

These commutative diagrams are equivalent to the conditions:

$$g(c)_1 \otimes g(c)_2 = g(c_1) \otimes g(c_2) \text{ and } \epsilon_{C'} \circ g = \epsilon_C \text{ for all } c \in C.$$

Now, since we have defined algebra and coalgebra structures, the next logical step is to consider when a set has both algebra and coalgebra structures. This leads us to the definition of a bialgebra, which requires both structures as well as a compatibility condition.

Definition 2.1.10. A *bialgebra* over $\Bbbk$ is a tuple $(H, \mu, \eta, \Delta, \epsilon)$ satisfying the following:

1. (H, μ, η) is a $\Bbbk$-algebra.

2. (H, Δ, ϵ) is a $\Bbbk$-coalgebra.

3. μ and η are coalgebra maps.

4. Δ and ϵ are algebra maps.

In fact, the last two conditions are equivalent.

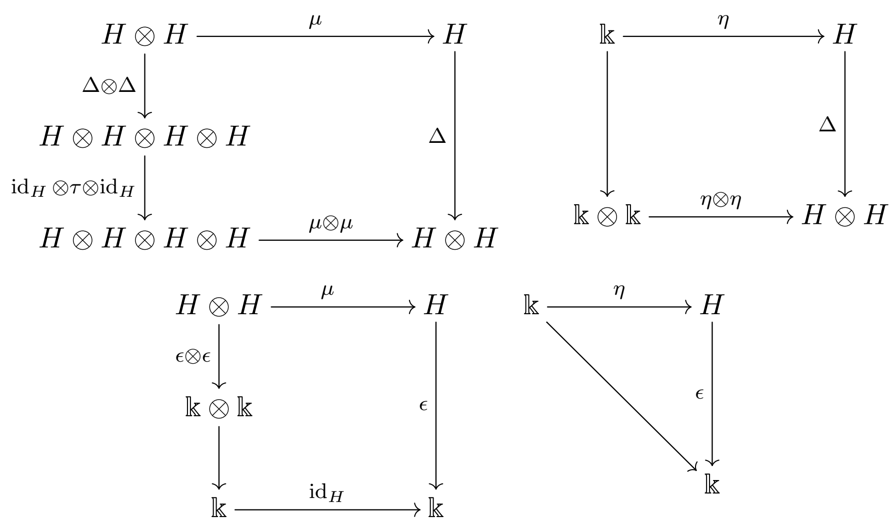


Figure 2.6: Equivalence of Bialgebra Conditions

Using Figure 2.6, we see that the two top-most and two bottom-most commutative diagrams are equivalent to the fact that the comultiplication and counit maps are algebra maps, respectively. Similarly, the two left-most and two right-most commutative diagrams are equivalent to the fact that the multiplication and the unit maps are coalgebra maps, respectively.

From this point forward, we may simply say H is a bialgebra with the understanding that $(H, \mu, \eta, \Delta, \epsilon)$ is a bialgebra for some multiplication μ, unit map η, comultiplication Δ and counit map ϵ.

Remark 2.1.11. In terms of the elements of H, η being a coalgebra is equivalent to 1_H being grouplike, while the fact that the multiplication of H is a coalgebra map (or the comultiplication of H is an algebra map) is equivalent to

$$(ab)_1 \otimes (ab)_2 = a_1 b_1 \otimes a_2 b_2 \text{ for all } a, b \in H\,. \tag{2.1}$$

We say a bialgebra is *commutative* (resp. *cocommutative*) if the underlying algebra (resp. coalgebra) structure is commutative (resp. cocommutative). Further, a subspace A of H is a *sub-bialgebra* if A is both a subalgebra and a sub-coalgebra of H.

Example 2.1.12. To continue with our example of the group algebra, $\Bbbk G$, it is easy to see that the identity element of $\Bbbk G$ is grouplike and $\Delta(gh) = gh \otimes gh = \Delta(g)\Delta(h)$ for $g, h \in G$. Thus, we see that $\Bbbk G$, with the coalgebra structure defined above, is in fact a bialgebra.

Example 2.1.13. If H and H' are bialgebras, then it is also immediate to see that $H \otimes H'$ is a bialgebra with the tensor product algebra and coalgebra structures.

Hopf algebras are simply bialgebras that contain an additional structure, called an *antipode*, whose existence is dependent on the compatibility of the multiplication and comultiplication of the bialgebra.

Definition 2.1.14. Let (A, μ, η) be a $\Bbbk$-algebra and (C, Δ, ϵ) be a $\Bbbk$-coalgebra. Let $f, g \in \mathrm{Hom}_{\Bbbk}(C, A)$. The *convolution* of f and g is $f * g : C \to A$ defined by

$$f * g = \mu \circ (f \otimes g) \circ \Delta \tag{2.2}$$

That is, for $c \in C$, $(f * g)(c) = f(c_1)g(c_2)$.

We note that the convolution product gives us an associative multiplication that creates an algebra structure on $\mathrm{Hom}_{\Bbbk}(C, A)$ with unit $\eta\epsilon$.

Let H be a bialgebra. The inverse S of id_H under the convolution product in $\mathrm{Hom}_{\Bbbk}(H, H)$, if it exists, is called an *antipode* of H, i.e.

$$S * \mathrm{id}_H = \mathrm{id}_H * S = \eta \circ \epsilon\,. \tag{2.3}$$

In other words, S satisfies the *antipode conditions*:

$$S(h_1)h_2 = h_1 S(h_2) = \epsilon(h)1_H \quad \text{for } h \in H. \tag{2.4}$$

Definition 2.1.15. A bialgebra H is called a *Hopf algebra* if H admits an antipode.

Particularly, if an antipode S of H exists, it is unique. This can be seen in Proposition 2.1.16

Proposition 2.1.16. *The antipode of a Hopf algebra is unique.*

Proof. Let H be a Hopf algebra and let S and S' be antipodes of H. Then, we have

$$S = S * \eta\epsilon = S * (\mathrm{id}_H * S') = (S * \mathrm{id}_H) * S' = \eta\epsilon * S' = S'.$$

Thus, the antipode must be unique. ∎

The following proposition lists several useful properties of the antipode, which can be found in [55] and [49].

Proposition 2.1.17. *Let H be a Hopf algebra with antipode S.*

1. *S is an antialgebra automorphism. That is,*

$$S(ab) = S(b)S(a) \text{ and } S(1_H) = 1_H.$$

2. *S is an anti-coalgebra automorphism. That is,*

$$\Delta(S(h)) = S(h_2) \otimes S(h_1) \text{ and } \epsilon(S(h)) = \epsilon(h).$$

3. *If H is commutative or cocommutative, then $S^2 = \mathrm{id}_H$.*

4. *If H is finite-dimensional, then S is bijective.*

Further, we define a *Hopf subalgebra* to be a sub-bialgebra B of H such that $S(B) \subset B$.

Additionally, we can define a *Hopf algebra map* to be a bialgebra map $f : H \to H'$ between Hopf algebras H and H' such that $f \circ S = S' \circ f$, where S and S' are the antipodes of H and H' respectively. However, the later condition is automatically satisfied by any bialgebra map, as presented below and in [49].

Proposition 2.1.18. *Let H and H' be Hopf algebras with antipodes S and S' respectively over $\Bbbk$. Let $f : H \to H'$ be a bialgebra map. Then, $f \circ S = S' \circ f$.*

Proof. We begin by showing $f \circ S$ is an inverse of f in $\mathrm{Hom}_{\Bbbk}(H, H')$. For $h \in H$,

$$f(h_1)f(S(h_2)) = f(h_1S(h_2)) = f(\epsilon(h)1_H) = \epsilon(h)1_{H'}$$
$$f(S(h_1))f(h_2) = f(S(h_1)h_2) = f(\epsilon(h)1_H) = \epsilon(h)1_{H'}$$

It is similarly shown that $S' \circ f$ is an inverse of f in $\mathrm{Hom}_{\Bbbk}(H, H')$.

$$f(h_1)S'(f(h_2)) = f(h)_1S'(f(h)_2) = \epsilon(f(h))1_{H'} = \epsilon(h)1_{H'}$$
$$S'(f(h_1))f(h_2) = S'(f(h)_1)f(h)_2 = \epsilon(f(h))1_{H'} = \epsilon(h)1_{H'}$$

Since inverses are unique in $\mathrm{Hom}_{\Bbbk}(H, H')$, we know $f \circ S = S' \circ f$. □

We say a Hopf algebra map f is a *Hopf algebra isomorphism* if it is bijective.

Since Hopf algebras are vector spaces over $\Bbbk$ with additional structures imposed, we can consider the dual space. That is, the space of $\Bbbk$-linear functionals, i.e. $\Bbbk$-linear mappings, from H to $\Bbbk$. In the next example, we see that if H is a finite-dimensional Hopf algebra, its dual, denoted by H^*, is also a Hopf algebra.

Example 2.1.19. Let H be a finite-dimensional Hopf algebra with antipode S over $\Bbbk$. Then the dual, H^*, is a bialgebra with the operations defined below.

$$\mu_{H^*}(c^* \otimes d^*)(c) = c^*(c_1)d^*(c_2)$$

$$\eta_{H^*}(1) = \epsilon_H$$

$$\Delta_{H^*}(h^*) = h_1^* \otimes h_2^*$$

$$\epsilon_{H^*}(h^*) = h^*(1)$$

where $h^*(hg) = h_1^*(h)h_2^*(g)$ for any $g, h \in H$. Notice that the multiplication defined in the dual is the convolution product introduced in Definition 2.1.14. The antipode of H^* is S^* which is defined by $S^*(h^*) = h^* \circ S$. We see that for $k \in H$ and $h^* \in H^*$.

$$(S^* * \mathrm{id}_{H^*})(h^*)(k) = h_1^*(S(k_1))h_2^*(k_2) = h^*(S(k_1)k_2) = \epsilon(k)h^*(1_{H^*})$$

$$(\mathrm{id}_{H^*} * S^*)(h^*)(k) = h_1^*(k_1)h_2^*(S(k_2)) = h^*(k_1S(k_2)) = \epsilon(k)h^*(1_{H^*})$$

$$\eta_{H^*}\epsilon_{H^*}(h^*)(k) = \epsilon(k)h^*(1_H).$$

Since the antipode conditions are satisfied, H^* is a Hopf algebra.

Example 2.1.20. We now return to our example $\Bbbk G$. We see that defining $S : \Bbbk G \to \Bbbk G$ by $S(g) = g^{-1}$ gives us an endomorphism that satisfies the antipode conditions. Thus, $\Bbbk G$ is a Hopf algebra.

We say a Hopf algebra H is *trivial* if $H \cong \Bbbk G$ or $H \cong \Bbbk^G$, the $\Bbbk$-linear dual of $\Bbbk G$. Further, if H is a bialgebra, it is straightforward to verify that the set of grouplike elements of H forms a monoid with the underlying multiplication of H and $1_H \in G(H)$. If H is a Hopf algebra with an antipode S, then necessarily, $S(g) = g^{-1}$ for all $g \in G(H)$. Hence, $G(H)$ is a group and $\Bbbk G(H)$ is a Hopf subalgebra of H. The Nichols-Zoeller theorem, introduced in [46] and presented below, tells us if K is a Hopf subalgebra of finite-dimensional Hopf algebra H, then $\dim K$ divides $\dim H$. In particular, we note $|G(H)|$ must divide $\dim H$.

Theorem 2.1.21. *Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$, and let B be a Hopf subalgebra. Then H is a free left (or right) B-module.*

The study of $\Bbbk G$ has been useful in the classifications of Hopf algebras. In particular, in 1994, Zhu was able to prove one of Kaplansky's conjectures with the following theorem, found in [59].

Theorem 2.1.22. *Let H be a Hopf algebra over the algebraically closed field $\Bbbk$ of characteristic zero. If $\dim H = p$ for some prime p, then, $H \cong \Bbbk G$ as Hopf algebras, where G is a group of order p.*

Following the terminology of [9], we say a Hopf algebra is of *type* (m, n) if $|G(H)| = m$ and $|G(H^*)| = n$.

Example 2.1.23. If H and H' are Hopf algebras with antipodes S and S' respectively, then $H \otimes H'$ is a Hopf algebra with antipode $S \otimes S'$. In particular, $H \otimes H$ is a Hopf algebra.

We close this section with an example of Hopf algebras that are neither commutative nor cocommutative.

Example 2.1.24. The *Taft algebras*, denoted by $H_{n,q}$ with $n \geq 1$ and $q \in \Bbbk$ an nth root of unity, were introduced by Taft in [56]. As a $\Bbbk$-algebra, $H_{n,q}$ is generated by a and x subject to the conditions $a^n = 1$, $x^n = 0$, and $xa = qax$. The coalgebra structure on $H_{n,q}$ is given by

$$\Delta(a) = a \otimes a, \quad \epsilon(a) = 1, \quad \Delta(x) = 1 \otimes x + x \otimes a, \text{ and } \epsilon(x) = 0\,.$$

It can be verified that this creates a bialgebra structure on $H_{n,q}$. Finally, defining $S : H_{n,q} \to H_{n,q}$ by $S(a) = a^{-1}$ and $S(x) = -xa^{-1}$, we have found an endomorphism that satisfies the antipode conditions, as demonstrated below:

$$(S * \mathrm{id}_H)(a) = S(a)a = a^{-1}a = 1_H = \eta\epsilon(a),$$

$$(\mathrm{id}_H * S)(a) = aS(a) = aa^{-1} = 1_H = \eta\epsilon(a),$$

and

$$(S * \mathrm{id}_H)(x) = S(1)x + S(x)a = x + (-xa^{-1})a = 0 = \eta\epsilon(x),$$
$$(\mathrm{id}_H * S)(x) = 1S(x) + xS(a) = -xa^{-1} + x(a^{-1}) = 0 = \eta\epsilon(x)\,.$$

When $n = 2$ and $q = -1$, this construction is also known as *Sweedler's 4-dimensional Hopf algebra*, first introduced in [55].

2.2 Modules over Hopf algebras

In this section, we introduce the notion of Hopf modules of Hopf algebras H, and the Fundamental Theorem of Hopf Modules. In particular, we present the basic example of the Hopf module H^* over a Hopf algebra H and its dual counterpart.

Definition 2.2.1. Let A be a $\Bbbk$-algebra. A *left A-module* is a vector space M over $\Bbbk$ and with a $\Bbbk$-linear action $\gamma : A \otimes M \to M$ such that $1_A m = m$ and $a(bm) = (ab)m$ for all $m \in M$ and $a, b \in A$, where $\gamma(a \otimes m)$ is simply denoted by am. When the context is clear, we will write M is a left A-module for the module structure (M, γ). Moreover, we will denote by ${}_H\mathcal{M}$ the category of all finite-dimensional left H-modules.

A *right A-module* is defined similarly.

It can easily be seen that if H is a Hopf algebra, H is a left (and right) H-module in which the multiplication of H is the module action. When H is finite-dimensional, we also have Example 2.2.2.

Example 2.2.2. If H is a finite-dimensional Hopf algebra, then one can show directly that H is a left H^*-module under the module action $\rightharpoonup$ defined by

$$h^* \rightharpoonup h = h^*(h_2)h_1. \tag{2.5}$$

Similarly, H is a right H^*-module under the action $\leftharpoonup$ defined by

$$h \leftharpoonup h^* = h^*(h_1)h_2. \tag{2.6}$$

Example 2.2.3. If H is a finite-dimensional Hopf algebra, then H^* is a left H-module under the module action $\succ$ defined by

$$(a \succ h^*)(h) = h^*(ha) = h_1^*(h)h_2^*(a) \tag{2.7}$$

for $a, h \in H$ and $h \in H^*$. This can be shown directly by definition. In view of the previous example, one can simply write $a \succ h^* = h_2^*(a)h_1^*$. Similarly, H^* is a right H-module under the action $\prec$ defined by

$$h^* \prec a = h_1^*(a)h_2^*. \tag{2.8}$$

Example 2.2.4. If M and N are H-modules, then $M \otimes N$ is also an H-module under the action

$$h(m \otimes n) = h_1 m \otimes h_2 n \text{ for all } h \in H, m \in M, n \in N.$$

An H-module M is called *trivial* if $hm = \epsilon(h)m$ for all $h \in H$ and $m \in M$. It is immediate to see that every vector space over $\Bbbk$ can be imposed with a trivial H-module structure. In particular, the trivial H-module $\Bbbk$ is called the unit object of ${}_H\mathcal{M}$.

The next example provides a nontrivial module action on a vector space V.

Example 2.2.5. We let ${}_\sigma V$ denote an H-module on V with the module action $\gamma_\sigma(h \otimes v) = \sigma(h)v$, where σ is an algebra automorphism on H, $h \in H$ and $v \in V$.

Additionally, we present several examples of the duals of vectors spaces as H-modules.

Example 2.2.6. Define $V^\vee$ to be the left H-module with underlying space V^* and module action given by

$$(hf)(x) = f(S(h)x)$$

for $x \in V$ and $f \in V^*$. This is the left dual of the left H-module V.

Example 2.2.7. Similarly, ${}^\vee V$ is the left H-module with underlying space V^* and module action

$$(hf)(x) = f(S^{-1}(h)x).$$

This is the right dual of the left H-module V.

Further, if M is a left H-module, the *H-invariant subspace* of M is defined as

$$M^H = \{m \in M | hm = \epsilon(h)m \ \forall h \in H\}.$$

If M is a right H-module, the H-invariant subspace of M is denoted ${}^H M$. Note that the H-invariant subspace of M is the trivial module, with the trivial module action.

Definition 2.2.8. If A is a $\Bbbk$-algebra and M and N are left A-modules, we say the $\Bbbk$-linear map $f : M \to N$ is an *A-module map* if $f(am) = af(m)$ for all $a \in A$ and $m \in M$.

Next, we dualize the notion of A-modules to define left (or right) C-comodules.

Definition 2.2.9. Let (C, Δ, ϵ) be a coalgebra. Then, a *left C-comodule* is the pair (M, ρ) where M is a vector space over $\Bbbk$ and the coaction $\rho : M \to C \otimes M$ is a $\Bbbk$-linear map such that

$$(\epsilon \otimes \mathrm{id}_M)\rho = \mathrm{id}_M \ \text{ and } (\Delta \otimes \mathrm{id}_M)\rho = (\mathrm{id}_C \otimes \rho)\rho .$$

A *right C-comodule* is defined similarly with the coaction defined by $\rho : M \to M \otimes C$. Using our adaptation of the Sweedler-Heinemann notation (c.f. [55]), if M is a left C-comodule, we will write $\rho(m) = m_{-1} \otimes m_0$ where $m_{-1} \in C$ and $m_0 \in M$. Similarly, if M is a right C-comodule, we will write $\rho(m) = m_0 \otimes m_1$, where again $m_0 \in M$ and now $m_1 \in C$. When the context is clear, we will say M is a left (or right) C-comodule to denote the structure (M, ρ).

As with A-modules, if H is a Hopf algebra, then H is a left and a right H-comodule, where the comultiplication is the coaction.

Example 2.2.10. If M and N are left H-comodules, then $M \otimes N$ is also a left H-comodule under the coaction

$$\rho(m \otimes n) = m_{-1}n_{-1} \otimes (m_0 \otimes n_0).$$

An H-comodule M is called *trivial* if $\rho(m) = 1_H \otimes m$ for all $m \in M$. We can then see that every vector space over $\Bbbk$ can be imposed with a trivial H-comodule structure. In particular, the trivial H-comodule $\Bbbk$ is called the unit object of ${}^H\mathcal{M}$.

Example 2.2.11. Define the left dual $V^\vee$ of the left H-comodule V to be the left H-comodule with underlying space V^* and comodule coaction given by

$$\rho(f) = f_{-1} \otimes f_0 \in H \otimes V^* \text{ such that } h^*(f_{-1})f_0(v) = h^*(S^{-1}(v_{-1}))f(v_0)$$

for $h^* \in H^*$ and $v \in V$.

Example 2.2.12. Similarly, ${}^\vee V$ denotes the right dual of the left H comodule V with underlying space V^* and comodule coaction

$$\rho(f) = f_{-1} \otimes f_0 \in H \otimes V^* \text{ such that } h^*(f_{-1})f_0(v) = h^*(S(v_{-1}))f(v_0)$$

for $h^* \in H^*$ and $v \in V$.

Further, if M is a left H-comodule with the coaction ρ, the *H-coinvariant subspace* of M is defined as

$$M^{coH} = \{m \in M | \rho(m) = 1_H \otimes m\}.$$

If M is a right H-comodule, the *H-coinvariant subspace* of M is denoted ${}^{coH}M$.

Definition 2.2.13. If C is a coalgebra and (M, ρ_M) and (N, ρ_N) are left C-comodules, we say the $\Bbbk$-linear map $f : M \to N$ is a *C-comodule map* if $\rho_N f = (\mathrm{id}_C \otimes f)\rho_M$.

Now that we have definitions for H-modules and H-comodules, we can define an H-Hopf module.

Definition 2.2.14. Let H be a Hopf algebra. A left H-Hopf module is a left H-module M which is also a left H-comodule such that the H-coaction $\rho : M \to H \otimes M$ is an H-module map, i.e. $\rho(hm) = h_1 m_{-1} \otimes h_2 m_0$ for all $h \in H$ and $m \in M$.

A *right H-Hopf module* is defined in a similar way, with the final condition becoming $\rho(mh) = m_0 h_1 \otimes m_1 h_2$.

It is clear that H is a left (and right) H-Hopf module since $\Delta(hh') = (hh')_1 \otimes (hh')_2 = h_1 h'_1 \otimes h_2 h'_2$ for all $h, h' \in H$. Further, a map of left (or right) H-Hopf modules is a linear map that is both an H-module map and an H-comodule map.

Example 2.2.15. If V is a vector space over $\Bbbk$ equipped with the trivial H-module structure, then $V \otimes H$ is a left H-Hopf module with the coaction ρ given by $\rho(v \otimes h) = h_1 \otimes v \otimes h_2$.

We close this section with the Fundamental Theorem of Hopf Modules [55]. It presents a nice relationship between a Hopf algebra and the coinvariants of a Hopf module over that Hopf algebra.

Theorem 2.2.16. *Let H be a Hopf algebra and M a left H-Hopf module. Then, we have*

$$M \cong M^{coH} \otimes H$$

as Hopf modules, where the Hopf module structure of $M^{coH} \otimes H$ follows that of Example 2.2.15.

2.3 Integrals in Hopf algebras

In this section we will introduce the notion of integrals in Hopf algebras, as well as their usefulness including generating a basis for modules and in the construction of a formula for S^4. In this section H will denote a finite-dimensional Hopf algebra with antipode S, comultiplication Δ, and counit ϵ.

Definition 2.3.1. A *left (resp. right) integral* in H is $\Lambda \in H$ such that for all $h \in H$,

$$h\Lambda = \epsilon(h)\Lambda \quad (\text{resp. } \Lambda h = \epsilon(h)\Lambda).$$

We will denote the set of left (resp. right) integrals in H by $\int_H^l$ (resp. $\int_H^r$).

It can be seen that $\int_H^l$ and $\int_H^r$ are left and right ideals of H respectively. Further, a nonzero left (or right) integral generates a one-dimensional left (or right) ideal of H. We say H is *unimodular* if $\int_H^l = \int_H^r$. Since H is finite-dimensional, we know its dual, H^*, is also a Hopf algebra with the counit ϵ_{H^*} given by $\epsilon_H^*(f) = f(1_H)$ for $f \in H^*$. Thus, a left (resp. right) integral in H^* is $\lambda \in H^*$ such that for all $h^* \in H^*$,

$$h^*\lambda = h^*(1_H)\lambda \text{ (resp. } \lambda h^* = h^*(1_H)\lambda).$$

The next theorem, presented in [49], shows the relationship between integrals and the module actions defined in Examples 2.2.2 and 2.2.3.

Theorem 2.3.2. *Let H be a finite-dimensional Hopf algebra.*

1. *Let Λ be a nonzero left or right integral in H. Then, the left (resp. right) H^*-module H under the H^*-module action $\rightharpoonup$ (resp. $\leftharpoonup$) is free with the basis $\{\Lambda\}$.*

2. *Similarly, if λ is a nonzero left or right integral in H^*, then the left (resp. right) H-module H^* under the H-module action $\succ$ (resp. $\prec$) is free with the basis $\{\lambda\}$.*

Further, integrals are used in defining a special type of grouplike elements, called the distinguished grouplike elements, defined below.

Definition 2.3.3. Following [49], we say $g \in H$ is the *H-distinguished grouplike element of H* if for $\lambda \in \int_{H^*}^{r}$ and all $a^* \in H^*$,

$$a^*\lambda = a^*(g)\lambda.$$

Similarly, we say $\alpha \in H^*$ is the *H-distinguished grouplike element of H^** if for $\Lambda \in \int_{H}^{l}$ and for all $a \in H$,

$$\Lambda a = \alpha(a)\Lambda.$$

It should be noted that the H-distinguished grouplike elements are both uniquely determined and do not depend on the choice of λ or Λ. They were extremely useful in the development of Radford's formula for S^4, found in [49], and presented below.

Theorem 2.3.4. *Let H be a finite-dimensional Hopf algebra with antipode S. If $g \in H$ and $\alpha \in H^*$ are the H-distinguished grouplike elements of H and H^* respectively, then for all $a \in H$,*

$$S^4(a) = g(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1})g^{-1}. \tag{2.9}$$

In particular, $S^{4n} = \mathrm{id}_H$ where n is the least common multiple of the orders of g and α.

Larson and Radford show in [30] and [31] that if H is a finite-dimensional Hopf algebra and H^* is semisimple, then H is semisimple with the following theorem.

Theorem 2.3.5. *Let H be a finite-dimensional Hopf algebra with antipode S and over a field $\Bbbk$ of characteristic zero. Then, the following statements are equivalent:*

1. *H is semisimple;*

2. *H is cosemisimple;*

3. *$S^2 = \mathrm{id}_H$;*

4. *$\mathrm{tr}(S^2) \neq 0$.*

Next we present a result of Radford and Schneider about endomorphisms, found in [50]. Firstly, we introduce some notation.

Let $\Lambda \in \int_H^l$ be nonzero. Then, we define

$$\nu_H(f) = S(\Lambda_2)f(\Lambda_1)$$

where f is a linear endomorphism of H. Further, let $l(a)$ denote left multiplication by a. That is, $l(a)(x) = ax$. We similarly use $r(a)$ to denote right multiplication by a.

Proposition 2.3.6. *Let H be a finite-dimensional Hopf algebra with antipode S and $g \in \mathrm{End}_\Bbbk(H)$. Then,*

1. *$\mathrm{tr}(g) = \lambda(\nu_H(g)) = \lambda(S(\Lambda_2)g(\Lambda_1))$, where $\Lambda \in \int_H^l$ and $\lambda \in \int_{H^*}^r$ satisfying $\lambda(\Lambda) = 1$.*

2. *$\nu_H(g) = 0$ if and only if $\mathrm{tr}\,(r(h) \circ g) = 0$ for all $h \in H$.*

3. *If g is an algebra automorphism of H, then $\nu_H(g^{-1}) = 0$ if and only if $\mathrm{tr}\,(l(h) \circ g) = 0$ for all $h \in H$.*

We close this section with two lemmas presented in [45]. Firstly, however, we introduce some notation and definitions, following [45]. For any H-module V, we will let $\mathrm{Soc}(V)$ denote the sum of simple H-submodules of V, called the *socle* of V. Let $P(V)$ denote the *projective cover* of V, which is the smallest projective H-module admitting an epimorphism onto V.

Lemma 2.3.7. *Let H be a finite-dimensional Hopf algebra over an algebraically closed field $\Bbbk$ with H-distinguished grouplike element $\alpha \in H^*$, and V a simple H-module. Then,*

$$\Bbbk_{\alpha^{-1}} \otimes {}^{\vee\vee}V \cong \mathrm{Soc}\, P(V) \text{ and } V \cong \mathrm{Soc}\, P(\Bbbk_\alpha \otimes V^{\vee\vee}). \tag{2.10}$$

Moreover, V is projective if and only if $\dim P(V) < 2\dim V$. In this case, $V \cong \Bbbk_\alpha \otimes V^{\vee\vee}$ as H-modules.

Lemma 2.3.8. *Let $V \in {}_H\mathcal{M}$ such that $V \cong \Bbbk_\beta \otimes V^{\vee\vee}$ for some nontrivial $\beta \in G(H^*)$. Then $\operatorname{Tr}(\tau) = 0$ for $\tau \in \operatorname{Hom}_H(V, {}_\sigma V)$, where $\sigma = S^2 \circ r(\beta)$.*

2.4 The coradical

In this section, we introduce the coradical of a Hopf algebra, which will be a key component of our classification in Chapter 4. We will define a *simple coalgebra* to be a coalgebra that contains no proper sub-coalgebras.

Definition 2.4.1. Let C be a coalgebra. The *coradical* of C, denoted C_0, is the sum of the simple sub-coalgebras of C.

Note that by definition, the coradical is a semisimple sub-coalgebra of C. Since a Hopf algebra has a coalgebra structure, we know there exists a coradical, H_0, of the Hopf algebra H. The structure of this coradical will be significant in our classifications found in later chapters. In particular we will look at Hopf algebras with the Chevalley property, defined below.

Definition 2.4.2. A Hopf algebra H has the *Chevalley property* if H_0 is a Hopf subalgebra. Further, we say H is *pointed* if H_0 is isomorphic to a group algebra.

Note that if H has the Chevalley property, we know from Theorem 2.3.5 that H_0 is not only a cosemisimple, but also a semisimple Hopf algebra.

The structure of the coradical has been useful in previous classifications as well. In particular, in [5], Andruskiewitsch and Vay were able to find a lower bound on the dimension of Hopf algebras with a coradical isomorphic to $\Bbbk^{S_3}$ as stated in the following theorem.

Theorem 2.4.3. *The smallest Hopf algebra with the Chevalley property and a coradical isomorphic to $\Bbbk^{S_3}$ has dimension 72.*

We will now examine how the coradical can generate a filtration of the Hopf algebra as well as the conditions necessary for the associated graded space to be a Hopf algebra. In particular, let H be a finite-dimensional Hopf algebra with the Chevalley property. Then, H_0 denotes the coradical and $H_n \subset H$ can be defined inductively by

$$H_n = \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H), \tag{2.11}$$

and $H_0 \subseteq H_1 \subseteq H_2 \cdots$ form a *coradical filtration* of subspaces of H. Further, Theorem 2.4.4, presented in [37], provides the necessary properties of a *coalgebra filtration* of H.

Theorem 2.4.4. *For all $n \geq 0$, $\{C_n\}$ is a family of sub-coalgebras of H satisfying*

1. $C_n \subset C_{n+1}$ *and* $H = \bigcup_{n \geq 0} C_n$

2. $\Delta C_n \subset \sum_{i=0}^{n} C_i \otimes C_{n-i}$.

An algebra A with an increasing sequence of subspaces $A_0 \subseteq A_1 \subseteq \cdots \subseteq A$ is called a *filtered algebra* if $A = \bigcup_n A_n$ and $A_n A_m \subseteq A_{m+n}$ for all $m, n \in \mathbb{N}$. Associate with the filtered algebra is a graded algebra $\operatorname{gr} A = \bigoplus_{i \geq 0} A_i / A_{i-1}$ assuming $A_{-1} = 0$. The multiplication of $\operatorname{gr} A$ is given by

$$(a + A_{m-1})(b + A_{n-1}) = ab + A_{m+n-1}$$

for $a \in A_m$ and $b \in A_n$. In particular, $\operatorname{gr} A$ is a graded algebra in the following sense.

Definition 2.4.5.

1. An algebra A is called *graded* if there is sequence of subspaces $\{A_n\}_{n \in \mathbb{N}}$ of A such that $A = \bigoplus_{n \in \mathbb{N}} A_n$ and $A_n A_m \subseteq A_{m+n}$ for all $m, n \in \mathbb{N}$.

2. A graded Hopf algebra H is a graded algebra $H = \bigoplus_{n \in \mathbb{N}} H_n$ such that H_n is invariant under the antipode of H and

$$\Delta(H_n) \subseteq \oplus_{i=0}^{n} H_i \otimes H_{n-i}$$

for all $n \in \mathbb{N}$.

Analogous to the graded algebra associated with a filtered algebra, a graded Hopf algebra can be obtained naturally from a Hopf algebra filtration defined as follows:

Definition 2.4.6. Let H be a Hopf algebra with antipode S and $\{K_n\}$. Then, an increasing sequence $\{K_n\}_{n\in\mathbb{N}}$ of subspaces of H is called a *Hopf algebra filtration* if

1. H is a filtered algebra with the filtration $K_0 \subseteq K_1 \subseteq \cdots \subseteq H$.

2. $\{K_n\}_{n\in\mathbb{N}}$ is a coalgebra filtration of H, i.e. $\{K_n\}_{n\in\mathbb{N}}$ satisfies (1) and (2) of Theorem 2.4.4.

3. $S(K_n) \subseteq K_n$ for all $n \in \mathbb{N}$.

The associated grade algebra $\operatorname{gr} H$ of the Hopf algebra filtration $\{K_n\}_{n\in\mathbb{N}}$ is a Hopf algebra with the comultiplication, counit and antipode given by

$$\Delta(a + K_{n-1}) = \sum_{i=0}^{n}(a_1^{(i)} + K_{i-1}) \otimes (a_2^{(i)} + K_{n-i-1})$$

$$\epsilon(a + K_{n-1}) = \epsilon_H(a), \quad S(a + K_{n-1}) = S_H(a) + K_{n-1}$$

for $a \in K_n$, where $a_1 \otimes a_2 = \sum_{i=0}^{n} a_1^{(i)} \otimes a_2^{(i)}$ with $a_1^{(i)} \otimes a_2^{(i)} \in K_i \otimes K_{n-i}$. Note that summation notations are repeatedly suppressed as usual in all these expressions.

We close this section with a lemma presented by Montgomery dictating when $\{H_n\}$ is a Hopf algebra filtration [37].

Lemma 2.4.7. *Let $\{H_n\}$ be the coradical filtration of H. Then, $\{H_n\}$ is a Hopf algebra filtration if and only if H_0 is a Hopf subalgebra of H, or in other words, if H has the Chevalley property.*

Remark 2.4.8. By the preceding lemma, associated to a Hopf algebra H with the Chevalley property is a graded Hopf algebra $\operatorname{gr} H$. We note, however, that some information about H is lost when constructing $\operatorname{gr} H$ as above, so it is possible for two non-isomorphic Hopf algebras to have the same coradical and associated graded Hopf algebra.

2.5 Hopf algebras and tensor categories

In this section, we recall the notion of tensor category theory and the reconstruction theorem which is also known as Tannaka duality. We will assume the basic definitions and theory of abelian categories covered in [32]. We begin with the definition of a monoidal category.

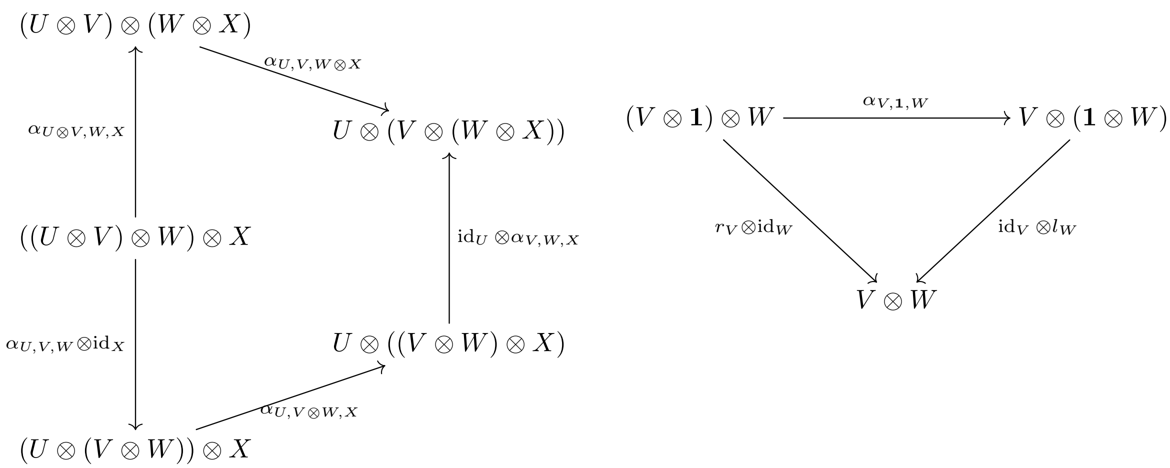


Figure 2.7: Commutative Diagrams for Monoidal Categories

Definition 2.5.1. A *monoidal category* is a category, $\mathcal{C}$, with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *tensor product* of $\mathcal{C}$, a unit object $\mathbf{1}$ in $\mathcal{C}$, and natural isomorphisms $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \to X$, $r_X : X \otimes \mathbf{1} \to X$ that satisfy the pentagon axiom and triangle axiom which are illustrated by the commutative diagrams in Figure 2.7 for all objects U, V, W, X in $\mathcal{C}$. The natural isomorphism α is often referred to as the *associativity constraint* of $\mathcal{C}$, and l, r are called the *unit constraints*.

Example 2.5.2. Our first example of a monoidal category is the category of finite-dimensional vector spaces over $\Bbbk$, denoted $Vect$. The tensor product $\otimes : Vect \times Vect \to Vect$ is simply the usual tensor product of vector spaces over $\Bbbk$ and its unit object is $\Bbbk$. The natural isomorphisms $\Bbbk \otimes V \cong V$ and $V \otimes \Bbbk \cong V$ given by the scalar multiplicaton are the unit constraints, and the associativity constraint $\alpha : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is the usual associativity isomorphism of tensor product defined by

$$\alpha((u \otimes v) \otimes w) = (u \otimes v) \otimes w$$

for $u \in U, v \in V, w \in W$.

Example 2.5.3. Let H be a Hopf algebra over $\Bbbk$. Recall from Example 2.2.4 that the tensor product (as vector spaces over $\Bbbk$) of two H-modules admits a natural H-module structure and $\Bbbk$

is a trivial H-module. The category of left H-modules, denoted H-mod, is a monoidal category with the tensor product, unit object, associativity and unit constraints inherited from $Vect$. Similarly, its full subcategory ${}_H\mathcal{M}$ of left finite-dimensional H-modules is also a monoidal category. The corresponding monoidal categories for right H-modules are analogously defined and denoted by mod-H, $\mathcal{M}_H$.

Example 2.5.4. Let H be a Hopf algebra and M, N left H-comodules. Then $M \otimes N$ admits a natural left H-coaction $\rho : M \otimes N \to H \otimes (M \otimes N)$ given by Example 2.2.10 and unit object $\Bbbk$. Thus, the category of left H-comodules, denoted by H-comod, is a monoidal category with tensor product, unit object, associativity and unit constraints inherited from $Vect$. For the same reason, the category of finite-dimensional left H-comodules, abbreviated as ${}^H\mathcal{M}$, is a monoidal category. The corresponding monoidal categories for right H-comodules are analogously defined and denoted by comod-H, $\mathcal{M}^H$.

Definition 2.5.5. Let V be a object in a monoidal category $\mathcal{C}$.

1. A *left dual* of an object V in a monoidal category $\mathcal{C}$ is a triple $(V^\vee, \mathrm{ev}, \mathrm{db})$ in which $V^\vee$ is an object of $\mathcal{C}$ and $\mathrm{ev} : V^\vee \otimes V \to \mathbf{1}$ and $\mathrm{db} : \mathbf{1} \to V \otimes V^\vee$ satisfying

$$\mathrm{id}_V = (V \xrightarrow{l^{-1}} \mathbf{1} \otimes V \xrightarrow{\mathrm{db} \otimes V} (V \otimes V^\vee) \otimes V \xrightarrow{\alpha} V \otimes (V^\vee \otimes V) \xrightarrow{V \otimes \mathrm{ev}} V \otimes \mathbf{1} \xrightarrow{r} V),$$
$$\mathrm{id}_{V^\vee} = (V^\vee \xrightarrow{r^{-1}} V^\vee \otimes \mathbf{1} \xrightarrow{V^\vee \otimes \mathrm{db}} V^\vee \otimes (V \otimes V^\vee) \xrightarrow{\alpha^{-1}} (V^\vee \otimes V) \otimes V^\vee \xrightarrow{\mathrm{ev} \otimes V^\vee} \mathbf{1} \otimes V^\vee \xrightarrow{l} V^\vee).$$

2. Similarly, a *right dual* of V is a triple $({}^\vee V, \mathrm{ev}', \mathrm{db}')$ in which ${}^\vee V$ is an object of $\mathcal{C}$ and $\mathrm{ev}' : V \otimes {}^\vee V \to \mathbf{1}$ and $\mathrm{db}' : \mathbf{1} \to {}^\vee V \otimes V$ satisfying

$$\mathrm{id}_V = (V \xrightarrow{r^{-1}} V \otimes \mathbf{1} \xrightarrow{V \otimes \mathrm{db}'} V \otimes ({}^\vee V \otimes V) \xrightarrow{\alpha^{-1}} (V \otimes {}^\vee V) \otimes V \xrightarrow{\mathrm{ev}' \otimes V} \mathbf{1} \otimes V \xrightarrow{l} V),$$
$$\mathrm{id}_{{}^\vee V} = ({}^\vee V \xrightarrow{l^{-1}} \mathbf{1} \otimes {}^\vee V \xrightarrow{\mathrm{db}' \otimes {}^\vee V} ({}^\vee V \otimes V) \otimes {}^\vee V \xrightarrow{\alpha} {}^\vee V \otimes (V \otimes {}^\vee V) \xrightarrow{{}^\vee V \otimes \mathrm{ev}'} {}^\vee V \otimes \mathbf{1} \xrightarrow{r} {}^\vee V).$$

Definition 2.5.6. We say the monoidal category $\mathcal{C}$ is *rigid* if every object V of $\mathcal{C}$ has a left and right dual. A *fusion category* is a $\Bbbk$-linear, abelian, semisimple, rigid monoidal category with a finite number of isomorphism classes of simple objects, and the unit object is simple. Here, an object V in a fusion category is called *simple* if $\mathrm{End}_{\mathcal{C}}(V) = \Bbbk$.

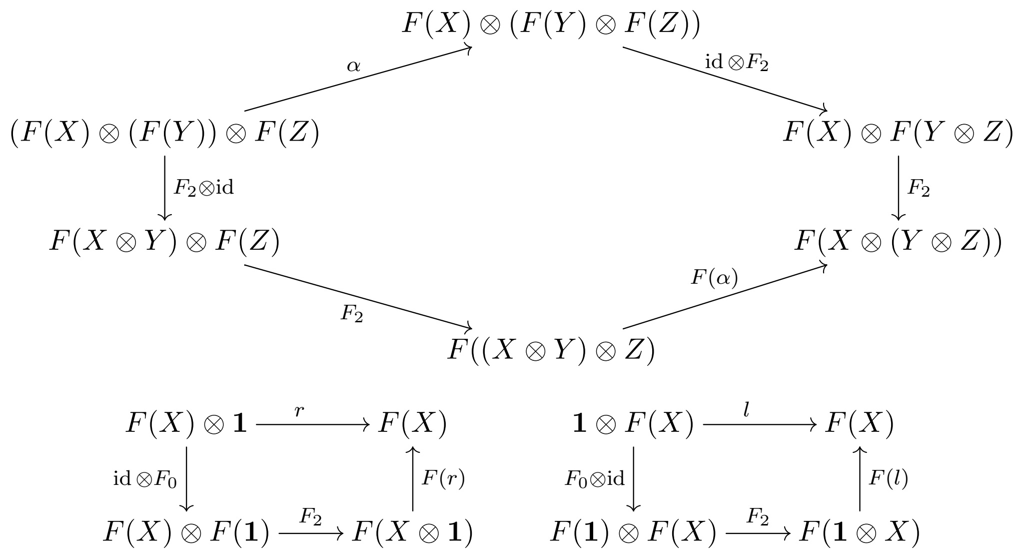


Figure 2.8: Commutative Diagrams for Monoidal Functors

Example 2.5.7. Let H be a Hopf algebra. For any finite-dimensional left H-module V, $(V^\vee, \mathrm{ev}, \mathrm{db})$ and $({}^\vee V, \mathrm{ev}', \mathrm{db}')$ are respectively a left and a right dual of V, where $V^\vee$ and ${}^\vee V$ are defined in Examples 2.2.6 and 2.2.7, ev and ev$'$ are evaluation maps, and db and db$'$ are dual basis maps. Therefore, ${}_H\mathcal{M}$ is a $\Bbbk$-linear, abelian, rigid monoidal category with a simple unit object. If H is finite-dimensional and semisimple over an algebraically closed field $\Bbbk$, then ${}_H\mathcal{M}$ is a fusion category. Similarly, the same properties hold for $\mathcal{M}_H$.

Example 2.5.8. Analogously, ${}^H\mathcal{M}$ and $\mathcal{M}^H$ are $\Bbbk$-linear, abelian, rigid monoidal categories with a simple unit object. In ${}^H\mathcal{M}$, for any finite-dimensional left H-comodule V, $(V^\vee, \mathrm{ev}, \mathrm{db})$ and ${}^\vee V, \mathrm{ev}', \mathrm{db}')$ are respectively a left and a right dual of V where $V^\vee$ and ${}^\vee V$ are defined in Examples 2.2.11 and 2.2.12. This is defined similarly for $\mathcal{M}^H$. Further, if H is finite-dimensional and semisimple over an algebraically closed field $\Bbbk$, then ${}^H\mathcal{M}$ or $\mathcal{M}^H$ is a fusion category.

Definition 2.5.9. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A *monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor equipped with natural isomorphisms $F_2 : F(X)\otimes F(Y) \to F(X\otimes Y)$ and $F_0 : \mathbf{1} \to F(\mathbf{1})$ such that the following diagrams commute for all $X, Y, Z \in \mathcal{C}$: The monoidal functor F is said to be *strict* if F_0 and F_2 are identities. An *monoidal equivalence* is a monoidal functor in which the underlying functor is an equivalence. Two monoidal categories are said to be *monoidally equivalent* if there exists a monoidal equivalence between them.

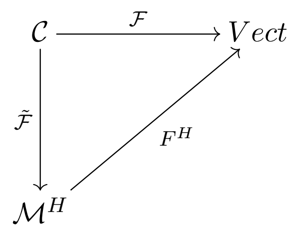
Example 2.5.10. Let H be a finite-dimensional Hopf algebra. The forgetful functor $F^H : \mathcal{M}^H \to Vect$ is a monoidal functor with F_2^H and F_0^H being identities. This monoidal functor is $\Bbbk$-linear, faithful and exact.

Example 2.5.11. Let H be a finite-dimensional Hopf algebra. For each $M \in \mathcal{M}^H$, one can define a left H^*-module $\mathcal{F}M$ where $\mathcal{F}M$ is the $\Bbbk$-linear space M with the H^*-action $f \cdot m = f(m_1)m_0$. For any H-comodules $M, N \in \mathcal{M}^H$, $\phi : M \to N$ is an H-comodule map if, and only if, ϕ is an H^*-module map from $\mathcal{F}M \to \mathcal{F}N$. Thus, one can extend $\mathcal{F}$ to a faithful and full $\Bbbk$-linear functor $\mathcal{F}^H : \mathcal{M}^H \to {}_{H^*}\mathcal{M}$. For $V \in {}_{H^*}\mathcal{M}$ and $v \in V$, one can define $\rho(v) = \sum_i v_i \otimes h_i \in V \otimes H$ such that $f \cdot v = \sum_i f(h_i)v_i$ for all $f \in H^*$. $\overline{V} = (V, \rho)$ is a right H-comodule as $V \in {}_{H^*}\mathcal{M}$. In particular, $\mathcal{F}\overline{V} = V$ and hence $\mathcal{F}$ is an equivalence. In fact, $\mathcal{F}$ is a strict monoidal functor, and we denote this monoidal equivalence by $\mathcal{F}^H : \mathcal{M}^H \to {}_{H^*}\mathcal{M}$.

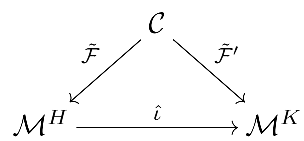
Note that K is a sub-coalegbra of H if, and only if, $i^* : H^* \to K^*$ is an algebra epimorphism where i is the inclusion map. $\mathcal{M}^K$ is an abelian subcategory of $\mathcal{M}^H$, i.e. the inclusion functor is strict monoidal, while ${}_{K^*}\mathcal{M}$ is an abelian subcategory of ${}_{H^*}\mathcal{M}$. Moreover, the image of $\mathcal{M}^K$ under the monoidal equivalence $\mathcal{F}^H : \mathcal{M}^H \to {}_{H^*}\mathcal{M}$ is equal to ${}_{K^*}\mathcal{M}$. In particular, $\mathcal{M}^K$ is closed under the tensor product of $\mathcal{M}^H$ if, and only if, ${}_{K^*}\mathcal{M}$ is closed under the tensor product of ${}_{H^*}\mathcal{M}$

Now, we present the reconstruction theorem, also known as Tannaka Duality (See cf. [57], [33], [51], [52]), and finally, close with a proposition giving a relationship between the Chevalley property and the categories of H-modules and H-comodules.

Theorem 2.5.12. *Let $\mathcal{C}$ be a $\Bbbk$-linear abelian, rigid, monoidal category and let $\mathcal{F} : \mathcal{C} \to Vect$ be a $\Bbbk$-linear, faithful, exact monoidal functor. Then, there exists a universal Hopf algebra H over $\Bbbk$, unique up to isomorphism, and a monoidal equivalence $\tilde{\mathcal{F}} : \mathcal{C} \to \mathcal{M}^H$ such that the following diagram of monoidal functors commutes.*



Here, F^H denotes the forgetful functor and H universal means that if K is a Hopf algebra such that $\mathcal{F}$ factors through $\tilde{\mathcal{F}}' : \mathcal{C} \to \mathcal{M}^K$ and the forgetful functor $F^K : \mathcal{M}^K \to Vect$, then there exists a unique Hopf algebra map $\iota : H \to K$ such that



commutes, where the functor $\hat{\iota} : \mathcal{M}^H \to \mathcal{M}^K$ is the induced functor of ι.

The reconstruction theorem implies the following proposition which gives an equivalence between the Chevalley property and subcategories generated by simple H-modules. However, we will provide a more direct proof of this fact.

Proposition 2.5.13. *Let H be a finite-dimensional Hopf algebra over $\Bbbk$, and H_0 the coradical subcoalgebra of H. The following statements are equivalent.*

1. *H has the Chevalley property.*
2. *${}_{H_0^*}\mathcal{M}$ is a monoidal subcategory of ${}_{H^*}\mathcal{M}$.*
3. *$\mathcal{M}^{H_0}$ is a monoidal subcategory of $\mathcal{M}^H$.*

Proof. To begin, we suppose H has the Chevalley property. Then, we know H_0 is a Hopf subalgebra of H and hence H_0^* is a quotient Hopf algebra of H^*. In particular, the inclusion functor ${}_{H_0^*}\mathcal{M} \to {}_{H^*}\mathcal{M}$ is a monoidal functor and hence ${}_{H_0^*}\mathcal{M}$ is a monoidal subcategory of ${}_{H^*}\mathcal{M}$. In view of Example 2.5.11, $\mathcal{M}^{H_0}$ is a monoidal subcategory of $\mathcal{M}^H$.

Next, we suppose $\mathcal{M}^{H_0}$ is a monoidal subcategory of $\mathcal{M}^H$. In particular, $H_0 \otimes H_0$ is a right H_0-comodule. Thus, for $a, b \in H_0$, $a_1 \otimes b_1 \otimes a_2 b_2 \in H_0 \otimes H_0 \otimes H_0$. Applying $\epsilon \otimes \epsilon \otimes \mathrm{id}_{H_0}$, we see that $ab \in H_0$. Therefore, H_0 is closed under the multiplication of H. Obviously, H_0 is stable under the antipode of H as $S(C)$ is a simple sub-coalgebra of H for any simple sub-coalgebra C of H. This implies H_0 is a Hopf subalgebra and hence H has the Chevalley property. $\square$

When $\Bbbk$ is algebraically closed and H has the Chevalley property, $\mathcal{M}^{H_0}$ or ${}_{H_0^*}\mathcal{M}$ are examples of fusion categories over $\Bbbk$.

CHAPTER 3. NONSEMISIMPLE $8p$-DIMENSIONAL HOPF ALGEBRAS

In this chapter we will cover several known and new results about Hopf algebras of dimension $8p$, where p is an odd prime. These results will hopefully play a role in future classifications of $8p$-dimensional Hopf algebras. For simplicity we will assume the base field $\Bbbk$ is algebraically closed with characteristic 0 for the remainder of this dissertation.

3.1 Useful results

We begin with a section examining several general results that are used for the proofs of later propositions. For this section, we will assume H is a finite-dimensional Hopf algebra.

Lemma 3.1.1. *Let C be a 4-dimensional simple coalgebra over $\Bbbk$ and $T : C \to C$ a coalgebra automorphism with finite order. If $\operatorname{Tr} T = 0$, then $\operatorname{ord} T = 2$.*

Proof. Let $n = \operatorname{ord} T$. Ştefan shows in [54, Thm 1.4b] that there is a primitive n-th root $\omega \in \Bbbk$ and a basis $\{e_{ij}\}_{i,j=1,2}$ of C such that $T(e_{ij}) = \omega^{i-j} e_{ij}$. Thus, $\operatorname{Tr} T = (1+\omega)(1+\omega^{-1})$, which is zero if and only if $\omega = -1$. Therefore, $n = 2$. □

Lemma 3.1.2. *If H admits a decomposition of coalgebras, $H = C \oplus C'$, with C simple, then C is invariant under $S^{-2} \circ l(a)$ where a is the H-distinguished grouplike element of H. Moreover, $\operatorname{Tr}_C (S^{-2} \circ l(a)) = 0$.*

Proof. Recall from Theorem 2.3.2 that H is a free left H^*-module of rank 1 under the action $\rightharpoonup$. Let I be a simple right coideal of H such that $\Delta(I) \subset I \otimes C$. Then $C \cong \dim(I) I$ as right H-comodules and hence left H^*-modules. In particular, I is a simple projective H^*-module. By Lemma 2.3.7, $I \cong \Bbbk_{\widehat{a^{-1}}} \otimes I^{**}$ where $\hat{x} \in H^{**}$ denotes the image of $x \in H$ under the natural isomorphism. In particular, $I \cong {}_{\sigma}I$ where $\sigma = (S^*)^2 \circ r(\widehat{a^{-1}})$ (cf. Lemma 2.3.8). Therefore, the annihilator ideal $\operatorname{ann}_{H^*} I$ is invariant under the automorphism σ. Note that $\sigma = \tau^*$, where $\tau = S^2 \circ l(a^{-1})$ is a coalgebra automorphism. Since $\operatorname{ann}_{H^*} I = C^{\perp}$, we find $\sigma(C^{\perp}) = C^{\perp}$ and hence C is invariant under τ.

Now, because $I \cong {}_{\sigma}I$ as H^*-modules, we know $C \cong {}_{\sigma}C$ as H^*-modules. To complete the proof, by Lemma 2.3.7, it suffices to show that $\tau^{-1} : C \to {}_{\sigma}C$ is an H^*-module homomorphism. For $f \in H^*$ and $c \in C$,

$$\sigma(f) \rightharpoonup \tau^{-1}c \; = \sigma(f)(\tau^{-1}c_2)\tau^{-1}c_1 = f\tau\tau^{-1}(c_2))\tau^{-1}c_1 = f(c_2)\tau^{-1}c_1 = \tau^{-1}(f \rightharpoonup c)\,.$$

Therefore, $\tau^{-1} \in \operatorname{Hom}_{H^*}(C, {}_{\sigma}C)$. □

Lemma 3.1.3. *Suppose H admits a pointed Hopf subalgebra A such that $G(H) = G(A)$. Then, H and A have a common distinguished grouplike element.*

Proof. Let λ be a nonzero right integral in H^*. Then, for all $h \in H$, $h \leftharpoonup \lambda = \lambda(h)1$ and $\lambda \rightharpoonup h = \lambda(h)a$ where a is the distinguished grouplike element of H. Since $a \in A$ and $\lambda|_A$ is a right integral of A^*, a is, by definition, the distinguished grouplike element of A. □

The following proposition, found in [8], is an improvement of [9, Prop 2.17] and [26, Prop 1.3].

Proposition 3.1.4. *Let $\pi : H \to A$ be a Hopf algebra epimorphism and assume* $\dim H = 2\dim A$. *Then, $H^{co\pi} = \Bbbk\{1, x\}$ where x is a $(1, g)$-skew primitive element, $g \in G(H)$ and $2|\operatorname{ord} g$. Moreover, if x is trivial then $\operatorname{ord} g = 2$ and H fits into an exact sequence of Hopf algebras $\Bbbk C_2 \hookrightarrow H \twoheadrightarrow A$. If x is nontrivial, then $x^2 = 0$ and H contains a Hopf subalgebra of dimension* $2\operatorname{ord} g$. *In particular,* $4|\dim H$.

3.2 Dimension 24

In this section, we assume H is a Hopf algebra of dimension 24. Beattie and García show in [9] the following theorem, which indicates restrictions on the order of the grouplike elements of H and H^*.

Theorem 3.2.1. *If H is a Hopf algebra of dimension 24 such that H and H^* do not have the Chevalley property, then H is of type $(2,2)$, $(2,4)$, $(4,2)$, $(6,4)$ or $(4,6)$.*

We further examine the grouplike elements and resulting implications in the following proposition.

Lemma 3.2.2. *If $G(H)$ is nontrivial, then H has a nontrivial skew primitive element.*

Proof. It suffices to prove the same statement for H^*.

Suppose $G(H^*)$ nontrivial and H^* has only trivial skew primitive elements. Then, H has a nontrivial 1-dimensional H-module corresponding to the nontrivial grouplike element, and $\text{Ext}(\Bbbk, \Bbbk_\beta) = \text{Ext}(\Bbbk_\beta, \Bbbk) = 0$ for all $\beta \in G(H^*)$ where $\Bbbk_\beta$ denotes the 1-dimensional H-module associated with β and Ext denotes the extension. Thus, $P(\Bbbk)/J^2P(\Bbbk)$ has a simple H-submodule V such that $\dim V \geq 2$.

Claim 1. $\dim V = 2$

Suppose $\dim V \geq 3$. Then, $\dim P(V) \geq 6$ and since $G(H^*)$ is nontrivial, we know $|G(H^*)| \geq 2$. So, we have

$$\dim H \geq |G(H^*)| \dim P(\Bbbk) + \dim V \dim P(V) \tag{3.1}$$

$$\geq 2 \cdot 5 + 3 \cdot 6 > 24, \tag{3.2}$$

a contradiction.

Claim 2. If W is another simple H-module with $\dim W > 1$, then W is projective with $\dim W = 2$.

Since $\dim V = 2$ and $\dim W > 1$, we know $\dim P(V) \geq 5$. Then, as in Claim 1, we note that

$$\dim H \geq |G(H^*)| \dim P(\Bbbk) + \dim V \dim P(V) + \dim W \dim P(W) \tag{3.3}$$

$$\geq 2 \cdot 4 + 2 \cdot 5 + \dim W \dim P(W). \tag{3.4}$$

Thus, we see that $\dim W \dim P(W) \leq 6$. Now, $\dim W > 1$ and $\dim P(W) \geq \dim W$ so we can conclude that $\dim W = \dim P(W) = 2$. In particular, W is projective since $\dim W = \dim P(W)$.

Claim 3. $\dim P(V) \geq 6$.

We know V is the unique nonlinear simple H-module that is not projective by Claim 2, so $V^* \cong V$ and $\Bbbk_\beta \otimes V \cong V \cong V \otimes \Bbbk_\beta$ for all $\beta \in G(H^*)$. Since $P(\Bbbk_\beta) \cong \Bbbk_\beta \otimes P(\Bbbk_\beta)$, V is a simple H-submodule of $P(\Bbbk_\beta)/J^2P(\Bbbk_\beta)$. This implies that $\sum_{\beta \in G(H^*)} \Bbbk_\beta$ is an H-submodule of $P(V)/J^2P(V)$. Knowing this, we can write

$$\dim P(V) \geq |G(H^*)| + 2\dim V \tag{3.5}$$

$$= |G(H^*)| + 4 \geq 6. \tag{3.6}$$

Hence, $\dim P(V) \geq 6$.

Claim 4. $|G(H^*)| = 2$

Suppose $|G(H^*)| \geq 3$. Then, using Equation 3.5 we see that $\dim P(V) \geq 7$. So, we have

$$\dim H \geq |G(H^*)| \dim P(\Bbbk) + 2\dim P(V) \tag{3.7}$$

$$\geq 3 \cdot 4 + 2 \cdot 7 > 24, \tag{3.8}$$

a contradiction.

It should be noted that Claim 3 indicates there is a unique nontrivial grouplike element in H^*.

Claim 5. V is the unique simple H-module with $\dim V > 1$.

Suppose there exists W, a simple H-module with $\dim W > 1$. Then, by Claim 2, W is a 2-dimensional simple projective H-module. In particular, $\dim P(W) = \dim W$. Further, the inequality

$$24 = \dim H \geq 2\dim P(\Bbbk) + 2\dim P(V) + (\dim W)^2 \tag{3.9}$$

$$\geq 2 \cdot 4 + 2 \cdot 6 + 2 \cdot 2 = 24 \tag{3.10}$$

implies that W is the unique simple projective H-module and

$$\dim P(\Bbbk) = 4 \quad \text{and} \quad \dim P(V) = 6. \tag{3.11}$$

Therefore, $W \cong W^*$ and $\Bbbk_\alpha \otimes W \cong W \cong W \otimes \Bbbk_\alpha$, where $\alpha \in G(H^*)$ is the nontrivial grouplike element. Since W is projective, so is $W \otimes W^*$. Therefore, $W \otimes W^*$ can be written as a direct sum of indecomposable projective H-modules by definition of projective module. However, $W \otimes W^*$ maps surjectively onto $\Bbbk$ and $\Bbbk_\alpha$ which implies $P(\Bbbk) \oplus P(\Bbbk_\alpha)$ is a summand of $W \otimes W^*$. This is a contradiction as $\dim(W \otimes W^*) = \dim P(\Bbbk) = \dim P(\Bbbk_\alpha) = 4$. Thus, W cannot exist and V is the unique simple H-module with $\dim V > 1$.

Claim 6. $\dim P(V) = 8$.

Let m be the multiplicity of V as a composition factor of $P(\Bbbk)$. Then, using the argument of Claim 3, m is also the multiplicity of V as a composition factor of $P(\Bbbk_\alpha)$. Since $\Bbbk$ is an algebraically closed field of characteristic zero, $\dim P(V)$ is the multiplicity of V in the composition factors of H (cf. [14, Theorem 54.19]).

By Claims 4 and 5, we have

$$H = P(\Bbbk) \oplus P(\Bbbk_\alpha) \oplus 2P(V).$$

Since $\dim \operatorname{Soc}(P(V)) = \dim V = 2$, we know $\operatorname{Soc}(P(V)) = V$ and hence $[P(V) : V] \geq 2$. Therefore, $\dim P(V) \geq 2m + 4$. Further, we know $\dim P(\Bbbk) \geq 2 + 2m$. Thus, we have

$$\begin{aligned} \dim H &= 2 \dim P(\Bbbk) + 2 \dim P(V) \\ &\geq 2(2 + 2m) + 2(2m + 4) \\ &= 4(2m + 3) \end{aligned}$$

limiting the value of m to 0 or 1. However, if $m = 0$, then V is projective, thus forcing $m = 1$. Now, since we are assuming H^* has only trivial skew primitive elements, we know $\operatorname{Ext}(\Bbbk, \Bbbk_\beta) = \operatorname{Ext}(\Bbbk_\beta, \Bbbk) = \operatorname{Ext}(\Bbbk_\beta, \Bbbk_{\beta'}) = 0$ for all $\beta, \beta' \in G(H^*)$. So, we are guaranteed $\dim P(\Bbbk) = 2 + 2m$. In particular, $\dim P(\Bbbk) = 2 + 2(1) = 4$. Now, we have

$$24 = \dim H = 2 \cdot 4 + 2 \dim P(V),$$

and so $\dim P(V) = 8$.

Claim 7. $P(\Bbbk_\beta) \otimes V \cong P(V) \cong V \otimes P(\Bbbk_\beta)$ for any $\beta \in G(H^*)$.

We know $P(\Bbbk_\beta) \otimes V$ is projective for any $\beta \in G(H^*)$ and maps surjectively onto V. Thus, $P(V)$ is a summand of $P(\Bbbk_\beta) \otimes V$, implying through a dimension comparison that $P(\Bbbk_\beta) \otimes V \cong P(V)$. Similarly, $V \otimes P(\Bbbk_\beta) \cong P(V)$.

Claim 8. H^* has the Chevalley property.

We begin by noting that

$$\dim \operatorname{Hom}_H(P(\Bbbk_\beta), V \otimes V) = \dim \operatorname{Hom}_H(V^* \otimes P(\Bbbk_\beta), V) = \dim \operatorname{Hom}_H(P(V), V) = 1,$$

indicating the multiplicity of $\Bbbk_\beta$ as a composition factor of $V \otimes V$ is 1. Since the dual basis map, $\mathrm{db} : \Bbbk \to V \otimes V^*$, and the evaluation map, $\mathrm{ev} : V^* \otimes V \to \Bbbk$, are nonzero H-module maps, so are $\Bbbk_\beta \otimes \mathrm{db}$ and $\Bbbk_\beta \otimes \mathrm{ev}$ for all $\beta \in G(H^*)$. Therefore, if α is the nontrivial grouplike element of H^*, both $\Bbbk$ and $\Bbbk_\alpha$ are summands of $V \otimes V$, and hence $V \otimes V \cong V \oplus \Bbbk \oplus \Bbbk_\alpha$. By Claim 3, the $\Bbbk$-linear abelian full subcategory $\mathcal{D}$ generated by $\Bbbk, \Bbbk_\alpha$ and V is a fusion subcategory of ${}_H\mathcal{M}$. Then, by Proposition 2.5.13, we know H^* has the Chevalley property.

Claim 9. $(H^*)_0 \cong \Bbbk^{S_3}$.

By the reconstruction theorem (Theorem 2.5.12), $\mathcal{D}$ is monoidally equivalent to ${}_{\overline{H}}\mathcal{M}$ for some semisimple quotient Hopf algebra $\overline{H}$ of H. Since $\dim \overline{H} = 1 + 1 + 2^2 = 6$, it follows from the classification of 6-dimensional semisimple Hopf algebras that $\overline{H} \cong \Bbbk S_3$. Thus, $\Bbbk^{S_3} \cong (H^*)_0$.

However, we know from Proposition 2.4.3 that the smallest Hopf algebra with the Chevalley property and coradical isomorphic to $\Bbbk^{S_3}$ has dimension 72. Thus, we have a contradiction and we can conclude H^* must have a nontrivial skew primitive element. □

Corollary 3.2.3. *If H and H^* do not have the Chevalley property, then both H and H^* have a nontrivial skew primitive element.*

Proof. By Theorem 3.2.1, $|G(H)|$ and $|G(H^*)|$ are even. The statement then follows immediately from Lemma 3.2.2. □

3.3 Dimension $8p$

In this section, we continue our examination of finite-dimensional Hopf algebras. Here, we assume H is a Hopf algebra of dimension $8p$, where p is an odd prime. As with the previous section, the following results provide information about the grouplike elements of H as well as the coradical. To begin, Beattie and García show in [9] the follow proposition, which indicates restrictions on the order of the grouplike elements, similar to the idea of Proposition 3.2.1.

Proposition 3.3.1. *Let H be a nonsemisimple Hopf algebra of dimension $8p$ such that neither H nor H^* is pointed. Then, $|G(H)| \neq 8p, 4p, p$.*

Our next lemma examines conditions on H that indicate when H may have the Chevalley property, which will then fall under our classification to be outlined in the next chapter.

Lemma 3.3.2. *Let H be a nonsemisimple Hopf algebra of dimension $8p$. If K is a $4p$-dimensional semisimple Hopf subalgebra of H, then $K = H_0$. In particular, H has the Chevalley property.*

Proof. To begin, we know we have the projection $\pi : H^* \to K^*$, so we define $R = (H^*)^{co\pi}$. We note that $\dim R = 2$, so R is spanned by $\{1, x\}$ where x is a $(1, g)$-skew primitive element. We can consider two cases: when x is trivial, that is x is a scalar multiple of $1 - g$, or when x is nontrivial.

First, we consider when x is trivial. Then, $g^2 = 1$ or $R = \Bbbk\langle g\rangle$, a Hopf subalgebra which is normal. Then, we have the following exact sequence of Hopf algebras.

$$0 \to \Bbbk\langle g\rangle \to H^* \to K^* \to 0 \tag{3.12}$$

Since $\Bbbk\langle g\rangle$ and K^* are semisimple, H^* must be semisimple, a contradiction. Thus, we know x must be nontrivial.

Since x is nontrivial, $\dim R = 2$, and $R^+ = \ker \epsilon|_R = \Bbbk x$, we know $x^2 = 0$. Further, we note R is left-normal. That is, $h_1 r S(h_2) \in R$ for all $h \in H^*$ and $r \in R$. We further note that

$$\epsilon(h_1 x S(h_2)) = \epsilon(x)\epsilon(h) = 0 \tag{3.13}$$

so we can conclude $h_1xS(h_2) \in \Bbbk x$. Thus, $hx = h_1xS(h_2)h_3 = cxh$ where c is a scalar, implying $H^*x \subset xH^*$. Further, $(R^+H^*)^2 = (xH^*)^2 = xH^*xH^* = x^2H^* = 0$. Since $K^* \cong H^*/R^+H^*$ and K^* is semisimple, $R^+H^* = J(H^*)$. Hence $K \cong (H^*/J(H^*))^* = H_0$. □

Theorem 3.3.3. *Let H be a nonsemisimple Hopf algebra of dimension 2^np where $n \geq 1$ and p is an odd prime. If K is a semisimple Hopf subalgebra of H, then $\dim K \neq 2^n$.*

Proof. Suppose $\dim K = 2^n$. We first claim that p cannot divide the order of S^2. Assume $p|\operatorname{ord} S^2$. Then, due to Radford's formula for S^4 presented in Proposition 2.3.4, we know there exists an H-distinguished grouplike element, a, in either H or H^* such that $p|\operatorname{ord} a$. Let $\alpha = a^{2^i}$ for some i such that $\operatorname{ord} \alpha = p$.

If $\alpha \in H$, then $\Bbbk\langle\alpha\rangle$ is a Hopf subalgebra of H and $p = \dim \Bbbk\langle\alpha\rangle$. Define $\overline{K}$ to be the algebra generated by $C = K \oplus \Bbbk\langle\alpha\rangle$. Then, $\overline{K}$ is a Hopf subalgebra and $\dim \overline{K} > 2^n$, so $\overline{K} = H$. Further, since C is the direct sum of two semisimple Hopf algebras as a vector space, $S^2|_C = \mathrm{id}_C$ and thus, because S^2 is an algebra homomorphism, $S^2 = \mathrm{id}_H$ and we conclude that H must be semisimple, a contradiction. So, we have $\alpha \in H^*$.

Now, consider the exact sequence

$$H^* \xrightarrow{\pi} K^* \to 0.$$

It is easy to see that $\alpha \in (H^*)^{\mathrm{co}\pi}$ and thus $\Bbbk\langle\alpha\rangle \subset (H^*)^{\mathrm{co}\pi}$. However, since the dimension of $(H^*)^{\mathrm{co}\pi}$ is p, we actually have $\Bbbk\langle\alpha\rangle = (H^*)^{\mathrm{co}\pi}$ and thus the exact sequence

$$0 \to \mathbb{Z}_p \to H^* \to K^* \to 0.$$

This implies H^* is semisimple and hence H is semisimple, a contradiction. Thus, we see that p does not divide the order of S^2.

Now, again using Proposition 2.3.4, we have that $(S^2)^{2^{n+1}} = \mathrm{id}_H$. In particular, $\operatorname{ord} S^2$ is a 2-power. Let e be the normalized integral of K. That is, $e^2 = e$, or $\epsilon(e) = 1$. We know H is a free right K-module, so

$$H \cong H/HK^+ \otimes K$$

as a right K-module and thus as a vector space,

$$He = H/HK^+ \otimes e$$

implying that $\dim He = p$. Further, we see in [50, Lemma 3] that $\mathrm{Tr}(S^2 \circ r(a)) = 0$ for any $a \in H$. In particular, $\mathrm{Tr}(S^2 \circ r(e)) = 0$. Thus, $\mathrm{Tr}(S^2|_{He}) = 0$ since H is stabilized by S^2. However, we know $\dim He = p$ and the order of S^2 is a 2-power, so by [26, Lemma 1.8], $\mathrm{Tr}(S^2|_{He})$ cannot be 0. Thus, if H is a nonsemisimple Hopf algebra of dimension $2^n p$, then it cannot contain a 2^n-dimensional semisimple Hopf subalgebra.

□

In particular, Theorem 3.3.3 indicates that an $8p$-dimensional Hopf algebra with the Chevalley property cannot have a coradical such that $\dim H_0 = 8$.

CHAPTER 4. NONPOINTED, NONSEMISIMPLE DIMENSION $8p$ HOPF ALGEBRAS WITH THE CHEVALLEY PROPERTY

In this chapter, we will study the nonpointed, nonsemisimple Hopf algebras of dimension $8p$ with the Chevalley property, denoted H. Recall that a Hopf algebra has the Chevalley property when its coradical H_0 is a Hopf subalgebra of H. Associated to such a Hopf algebra is a graded Hopf algebra $\operatorname{gr} H$ given by the coradical filtration of H (cf. Remark 2.4.8). This graded Hopf algebra $\operatorname{gr} H$ can be obtained by a process called bosonization. Utilizing the idea of lifting the bosonization, introduced by Andruskiewitsch and Schneider (cf. [3]), we are able to classify H using the dimension of its coradical. We find that H must either be pointed or have a coradical of dimension $4p$. In particular, we are able to show this coradical of dimension $4p$ must be $\Bbbk^G$, where G is one of either the dihedral or dicyclic groups of order $4p$, or the self-dual noncommutative Hopf algebra A_+ outlined in [1] and described below.

4.1 The Yetter-Drinfeld category and bosonization

In this section, we will introduce the Yetter-Drinfeld category of a bialgebra K over $\Bbbk$, and the Radford biproduct, also known as bosonization. Additionally, we will highlight some useful theorems about bosonizations, particularly when the coradical is used.

We first consider the monoidal category ${}_K\mathcal{M}$ (see Example 2.5.3). We say A is a *left K-module algebra* if it is an algebra in ${}_K\mathcal{M}$. That is, A is an algebra over $\Bbbk$ with a left K-module structure such that the multiplication $A \otimes A \to A$ and the unit map $\Bbbk \to A$ are K-module maps, i.e. for all $k \in K$ and $a, b \in A$, we have

$$k(ab) = (k_1 a)(k_2 b) \text{ and } k1 = \epsilon(k)1. \tag{4.1}$$

Further, if A is a left K-module algebra, then $A \otimes K$ is an algebra over $\Bbbk$ with multiplication defined by

$$(a \otimes k)(a' \otimes k') = a(k_1 a') \otimes k_2 k' \tag{4.2}$$

for all $a, a' \in A$ and $k, k' \in K$ and the unit defined as $1_A \otimes 1_K$. Note that the multiplication is dependent on the module action from A. The associativity and unity axioms of an algebra can easily be verified.

Definition 4.1.1. The *smash product* of A and K, often denoted $A \# K$, is the algebra constructed above.

Similarly, by considering the category ${}^K\mathcal{M}$, we can define a smash coproduct as a left K-comodule C with a coalgebra structure over $\Bbbk$ such that the comultiplication Δ and the counit ϵ are comodule maps, i.e. for all $c \in C$,

$$(c^{(1)})_{-1}(c^{(2)})_{-1} \otimes (c^{(1)})_0 \otimes (c^{(2)})_0 = c_{-1} \otimes (c_0)^{(1)} \otimes (c_0)^{(2)} \tag{4.3}$$

and

$$c_{-1}\epsilon(c_0) = \epsilon(c)1. \tag{4.4}$$

Here, we denote $\Delta(c) = c^{(1)} \otimes c^{(2)}$ to avoid confusion with its K-coactions.

Definition 4.1.2. Let C be a left K-comodule coalgebra. Then, the *smash coproduct*, often denoted $C \natural K$, is $C \otimes K$ with the coalgebra structure defined by

$$\Delta(c \otimes k) = (c^{(1)} \otimes (c^{(2)})_{-1} k_1) \otimes ((c^{(2)})_0 \otimes k_2) \tag{4.5}$$

and

$$\epsilon(c \otimes k) = \epsilon(c)\epsilon(k) \tag{4.6}$$

for all $c \in C$ and $k \in K$.

It can easily be verified that the coalgebra axioms are satisfied. As with the smash product, we note that the comultiplication defined above is dependent on the K-comodule structure of C.

Next, we look to define a notion of what could be considered a "smash biproduct". First, however, we must consider the *Yetter-Drinfeld category*, ${}^K_K\mathcal{YD}$. This is a monoidal category whose objects are (M, μ, ρ) where (M, μ) is a left K-module and (M, ρ) is a left K-comodule with the compatibility condition that for all $k \in K$ and $m \in M$

$$k_1 m_{-1} \otimes k m_0 = (k_1 m)_{-1} k_2 \otimes (k_1 m)_0. \tag{4.7}$$

Let $M, N \in {}^K_K\mathcal{YD}$, $M \otimes N$ admits a left K-module structure as well as a left K-comodule structure given by the Examples 2.2.4 and 2.2.10.

The morphisms of this category are functions of the underlying the vector spaces of objects, which are simply maps of left K-modules and left K-comodules.

Additionally, we note that if M and N are objects in ${}^K_K\mathcal{YD}$, then the map $\sigma_{M,N} : M \otimes N \to N \otimes M$ defined by

$$\sigma_{M,N}(m \otimes n) = m_{-1} n \otimes m_0 \tag{4.8}$$

gives ${}^K_K\mathcal{YD}$ a *prebraiding structure*, i.e. a braiding structure which may not be an isomorphism. Further, when K is a finite-dimensional Hopf algebra, we have a braiding structure with

$$\sigma^{-1}_{M,N}(n \otimes m) = m_0 \otimes S^{-1}(m_{-1}) n \tag{4.9}$$

where S is the antipode of K.

Now, if K is a bialgebra and B is a bialgebra in ${}^K_K\mathcal{YD}$, then $B \otimes K$ is a bialgebra with the smash product and smash coproduct structures. We call this the *Radford biproduct*, or *bosonization*, and denote it $B \star K$. If, further, B and K have antipodes, then $B \star K$ has an antipode defined as

$$S(b \star k) = (1_B \star S_K(b_{-1} k))(S_B(b_0) \star 1_K). \tag{4.10}$$

Thus, we see that $B \star K$ is a Hopf algebra when K and B have antipodes.

The Radford biproduct construction will play a large role in our classification of Hopf algebras of dimension $8p$ with the Chevalley property. Since we are only considering those Hopf algebras H with the Chevalley property, we know H_0 is a Hopf subalgebra, so we can examine $R \star H_0$, where R is an object in ${}^{H_0}_{H_0}\mathcal{YD}$. In particular, Radford showed in [48] the following theorem, which proves to be very useful in constructing the desired Hopf algebras.

Theorem 4.1.3. *If H is a finite-dimensional Hopf algebra with the Chevalley property and R is an object in ${}^{H_0}_{H_0}\mathcal{YD}$, then $\operatorname{gr} H \cong R \star H_0$.*

For the majority of our classification, we will consider a specific bosonization of the graded algebra $\Bbbk[y]/(y^2)$, which we will denote by R_{-1}, and the coradical of H. We note that we can endow R_{-1} with a coalgebra structure defined by

$$\Delta(1) = 1 \otimes 1 \text{ and } \Delta(y) = 1 \otimes y + y \otimes 1. \tag{4.11}$$

Following [12], we define a *Yetter-Drinfeld (YD) datum for R_{-1}* to be a triple (K, g, χ) where K is a Hopf algebra with bijective antipode, $g \in G(K)$, and $\chi : K \to \Bbbk$ an algebra map such that

$$\chi(g) = -1 \text{ and } (\chi \rightharpoonup k)g = g(k \leftharpoonup \chi) \tag{4.12}$$

for all $k \in K$.

We further note that the action and coaction induced by $k \cdot y = \chi(k)y$ and $\delta(y) = g \otimes y$, where $k \in K$, define a left K-module algebra and coalgebra structure (from the action), as well as a left H-comodule algebra and coalgebra structure (from the coaction). Since K is a Hopf algebra, and $R_{-1}(K, g, \chi)$ has an antipode, S defined by $S(1) = 1$ and $S(y) = -y$, we can consider the Hopf algebra created by the bosonization $R_{-1}(K, g, \chi) \star K$.

The following propositions and lemmas are found in [12] and were useful in the completion of our classification.

Proposition 4.1.4. *Let $R_{-1} = R_{-1}(K, g, \chi)$ and $R'_{-1} = R_{-1}(K', g', \chi')$. Then, the following are equivalent.*

1. $(K, g, \chi) \cong (K', g', \chi')$
2. $R_{-1} \star K \cong R_{-1} \star K'$ *as graded Hopf algebras.*

   Additionally, 1 and 2 imply
3. $R_{-1} \star K \cong R_{-1} \star K'$ *as ungraded Hopf algebras.*

Finally, if K and K' are both cosemisimple, then 3 implies 1 and 2.

Lemma 4.1.5. *Let* (K, g, χ) *be a YD datum for* R_{-1}*. Then,* (K^*, χ, g) *is a YD datum for* R_{-1}*. Further, if we write* $R_{-1} = R_{-1}(K, g, \chi)$ *and* $R^*_{-1} = R_{-1}(K^*, \chi, g)$*, then*

$$(R_{-1} \star K)^* \cong R^*_{-1} \star K^* \tag{4.13}$$

as graded Hopf algebras.

Proposition 4.1.6. *Let* H *be a Hopf algebra of dimension* $8p$ *with the Chevalley property. If* $\dim H_0 = 4p$*, then* $\operatorname{gr} H \cong R_{-1}(H_0, g, \chi) \star H_0$ *where*

$$\chi(g) = -1 \tag{4.14}$$

and

$$(\chi \rightharpoonup h)g = g(h \leftharpoonup \chi) \tag{4.15}$$

for all $h \in H_0$*.*

4.2 Results

By the Nichols-Zoeller Theorem (Theorem 2.1.21), we know $\dim H_0$ divides $\dim H$. So, we need only to consider when $\dim H_0 = 2, 4, 8, p, 2p, 4p$. That is, we determine if the coradical of a nonpointed, nonsemisimple Hopf algebra with the Chevalley property is isomorphic to a semisimple Hopf algebra of these dimensions.

4.2.1 Coradical with dimension 2,4,8, or p

Since H_0 must be semisimple, we know if $\dim H_0 = 2, 4, p$, then H is pointed due to H_0 being isomorphic to a group algebra. It was shown in Theorem 3.3.3 that the coradical cannot be 8-dimensional. We present an alternate proof below.

Proposition 4.2.1. *Let* H *be a nonsemisimple Hopf algebra of dimension* $8p$ *with the Chevalley property. Then,* $\dim H_0 \neq 8$*.*

Proof. Suppose $\dim H_0 = 8$. We first claim that $\operatorname{ord} S^2$ is a power of 2. Suppose not. By Radford's formula [49],

$$S^4(x) = \alpha \rightharpoonup axa^{-1} \leftharpoonup \alpha^{-1} \tag{4.16}$$

where a, α are H-distinguished grouplike elements of H and H^* respectively. So, we know $\operatorname{ord}\alpha|8p$ and $\operatorname{ord}a|8p$. Further, we know $\operatorname{ord}S^4 = \operatorname{lcm}(\operatorname{ord}\alpha, \operatorname{ord}a)$. Since we are assuming $\operatorname{ord}S^2$ is not a power of 2, we can say $p|\operatorname{lcm}(\operatorname{ord}\alpha, \operatorname{ord}a)$. Now, we know $a \in H_0$, so if $p|\operatorname{ord}a$, then $p|\dim H_0$, which is not possible under our assumption. Thus, we must have $p|\operatorname{ord}\alpha$.

Now, from the embedding $H_0 \hookrightarrow H$, we have the restriction map $H^* \xrightarrow{\pi} H_0^*$ and since $\dim H_0^* = 8$, we know $\pi(\alpha) = 1$. Then, by definition $\alpha \in H^{*\mathrm{co}H^*}$, so $\langle\alpha\rangle \subset H^{*\mathrm{co}H^*}$. In fact, we know $\dim H^{*\mathrm{co}H^*} = p$ since $\dim H_0^* = 8$, so $H^{*\mathrm{co}H^*} = \mathbb{C}\langle\alpha\rangle$ and thus H^* is semisimple. Hence, by Theorem 2.3.5 H is semisimple. However, if $\dim H = 8p$ and $\dim H_0 = 8$, then by definition, H cannot be semisimple. So, we see that $p \nmid \operatorname{lcm}(\operatorname{ord}\alpha, \operatorname{ord}a)$ and thus, $\operatorname{ord}S^2$ must be a power of 2.

Now, by Theorem 4.1.3 we can write $\operatorname{gr}H \cong R \star H_0$, where $\dim R = p$. Since $\operatorname{ord}S^2$ is a power of 2 and the dimension of R is odd, we know $\operatorname{Tr}(S^2|_R) \neq 0$. Hence, R is semisimple. Thus, by [26, Prop 1.5], with $R \subset H$ and $H_0^* \subset H^*$ such that $\dim H = \dim R \dim H_0^*$ and $\gcd(\dim R, \dim H_0^*) = 1$, we can conclude H is semisimple. From this contradiction, we are able to conclude that $\dim H_0 \neq 8$. □

4.2.2 Coradical with dimension $2p$

Masuoka demonstrated in [34] that any semisimple Hopf algebra of dimension $2p$ is isomorphic to either a group algebra, or the dual of the dihedral group of order $2p$, denoted $\Bbbk^{D_p}$. The following proposition, a result of private communication with Beattie and García, shows that no nonpointed nonsemisimple Hopf algebra of dimension $8p$ with the Chevalley property has a coradical with dimension $2p$.

Proposition 4.2.2. *Let H be a nonsemisimple Hopf algebra of dimension $8p$ with the Chevalley property and $\dim H_0 = 2p$. Then, H is pointed.*

Proof. If H_0 is congruent to a group algebra, then H is pointed. Now, suppose H is nonpointed. That is, $H_0 \cong \Bbbk^{D_p}$. Then, we know $\operatorname{gr}H \cong R \star \Bbbk^{D_p}$ and thus $(\operatorname{gr}H)^* \cong R^* \star \Bbbk D_p$ by Lemma 4.1.5. Further, Proposition 3.3.1 tells us that $|G((\operatorname{gr}H)^*)| \neq 8p, 4p, p$. Since $|G((\operatorname{gr}H)^*)|$ divides $8p$ and $D_p \subset G((\operatorname{gr}H)^*)$, we see that $G((\operatorname{gr}H)^*) = D_p$. We can conclude that $(\operatorname{gr}H)^*$

has a pointed Hopf subalgebra of dimension $4p$ by [9, Prop 4.7]. However, it is shown in [2, A.1] that every pointed Hopf algebra of dimension $4p$ must have its grouplikes isomorphic to C_{2p}, leading to a contradiction. Thus, H must be pointed. □

4.2.3 Coradical with dimension $4p$

We are left only to consider when the coradical is a semisimple Hopf algebra of dimension $4p$. To begin, we make note of a property of the antipode in such a situation.

Proposition 4.2.3. *Let H be a nonsemisimple Hopf algebra of dimension $8p$ with the Chevalley property and $\dim H_0 = 4p$. Let S be the antipode of H. Then, $S^4 = \mathrm{id}_H$.*

Proof. To begin, we note that $S|_{H_0}$ is the antipode of H_0, which is semisimple since it is the coradical. Thus, $S^2|_{H_0} = \mathrm{id}_{H_0}$. Further, since H is nonsemisimple, we know $\mathrm{Tr}(S^2) = 0$. Now, by the coradical filtration, we know $\mathrm{gr}\, H = H_0 \oplus \frac{H}{H_0}$. So,

$$\mathrm{Tr}(S^2) = \mathrm{Tr}(S^2|_{H_0}) + \mathrm{Tr}(S^2|_{\frac{H}{H_0}}) \tag{4.17}$$

and we see that $\mathrm{Tr}(S^2|_{\frac{H}{H_0}}) = -12$. However, $\dim \frac{H}{H_0} = 4p$, so $S^2|_{\frac{H}{H_0}} = -\mathrm{id}_{\frac{H}{H_0}}$. Thus, we conclude $S^4 = \mathrm{id}_H$. □

Now, we will construct Hopf algebras of dimension $8p$ with the Chevalley property using the process of bosonization. This construction will provide us with a graded Hopf algebra as in [12]. To determine all $8p$-dimensional Hopf algebras with the Chevalley property, we will need to consider the liftings of these graded Hopf algebras.

Proposition 4.2.4. *Let H be a nonsemisimple nonpointed Hopf algebra with the Chevalley property whose coradical H_0 is a Hopf algebra of dimension $4p$. Then, H is a nontrivial lifting of $\mathrm{gr}\, H$ if and only if $\mathrm{ord}\, g = 4$, where g is the grouplike element as in Proposition 4.1.6. In this case, H is uniquely determined.*

Proof. By Proposition 4.1.6, we have $\mathrm{gr}\, H \cong R_{-1}(H_0, g, \chi)$ with $g \in H_0$ grouplike and $\chi : H \to \Bbbk$, a linear character. From Proposition 3.3.1, we see that $|G(H_0)| = 2$ or 4, so $\mathrm{ord}\, g = 1, 2, 4$. If $\mathrm{ord}\, g = 1$, then g is the trivial grouplike element. We want to consider the nontrivial grouplike elements.

Let $y \in R$ be nontrivial and let $h \in H_0$. Since H is generated by y and H_0, to determine the lifting we need to examine how y and h multiply in H. That is, we consider hy and y^2 in H. We can see that if $\operatorname{ord} g = 2$, then $y^2 = 0$ in H and if $\operatorname{ord} g = 4$, then $y^2 = 0$ or $y^2 = 1 - g^2$ in H by [2]. Further, using the action of R on H, we can lift from the grading to write

$$hy = y(h \leftharpoonup \chi) + \phi(h) \tag{4.18}$$

where $\phi(h) \in H_0$. We know that $S^4 = \mathrm{id}_H$ and $S^2|_{H_0} = \mathrm{id}_{H_0}$. Notice that

$$S^2|_{H_0 y H_0}(hyk) = -hyk.$$

In particular,

$$S^2(hy) = -hy \tag{4.19}$$

and

$$S^2(hy) = S^2(y(h \leftharpoonup \chi)) + S^2(\phi(h)) = -y(h \leftharpoonup \chi) + \phi(h). \tag{4.20}$$

So, from (4.19) and (4.20), we have

$$-hy = -y(h \leftharpoonup \chi) + \phi(h) \tag{4.21}$$

By adding (4.18) and (4.21), we see that

$$0 = 2\phi(h).$$

Thus, $\phi(h) = 0$. Hence, when $\operatorname{ord} g = 2$, the only lifting is trivial and when $\operatorname{ord} g = 4$, there exists exactly one nontrivial lifting. □

It is known that a semisimple Hopf algebra of dimension $4p$ must be isomorphic to one of the group algebras, their duals, or $A_{\pm}$, the self-dual noncommutative, non-cocommutative Hopf algebras described below. We begin by considering the group algebras of dimension $4p$ and their duals. If H_0 is a group algebra, or the dual of an abelian group algebra, then H is pointed. So, we consider when H_0 is the dual of a nonabelian group alegbra. When $p = 3$, we have the alternating group A_4 of order 12. Further, it is known that for a general prime p, there are three additional nonabelian group algebras:

$$D_{2p} = \left\{x, t | x^{2p} = t^2 = 1, xt = tx^{-1}\right\}$$
$$Dic_p = \left\{x, t | x^{2p} = 1, t^2 = x^p, xt = tx^{-1}\right\}$$

and when $p \equiv 1 \mod 4$,

$$E_p = \left\{x, t | x^p = 1, t^4 = 1, txt^{-1} = x^{\xi}\right\}$$

where ξ are elements of order 4 in $(\mathbb{Z}/p\mathbb{Z})^*$.

Proposition 4.2.5. *Let H be a nonsemisimple Hopf algebra of dimension 24 and suppose its coradical H_0 is 12-dimensional. Then, $H_0 \ncong \Bbbk^{A_4}$.*

Proof. Since the grouplike elements of $\Bbbk^{A_4}$ are the linear, or one-dimensional, characters of A_4, we look at the character table of A_4. Let C_1, C_2, C_3, C_4 represent the conjugacy classes of A_4 and let $\chi_1, \chi_2, \chi_3, \chi_4$ be the characters. Then, we have

	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

where $\omega = \frac{-1+\sqrt{3}}{2}$. We note that there are three one-dimensional characters. However, since there must be a grouplike element of even order in H_0 by Proposition 3.3.1 (that is, a linear character of A_4 with even order), it is not possible for $\Bbbk^{A_4}$ to be the coradical of a nonsemisimple 24 dimension Hopf algebra. □

Proposition 4.2.6. *$\Bbbk^{D_{2p}}$ is the coradical of a nonsemisimple Hopf algebra of dimension $8p$.*

Proof. We again use the process of bosonization. To determine the grouplike elements in $\Bbbk^{D_{2p}}$, we first determine the one-dimensional characters of D_{2p}. To begin, it is easily verified that

$$[D_{2p}, D_{2p}] = \left\langle x^2 \right\rangle$$

where $[D_{2p}, D_{2p}]$ denotes the commutator of D_{2p}. Then, since there are p even powers of x, we compute the order of the abelianization of D_{2p} to be 4. Thus, there are 4 one-dimensional characters of D_{2p}. Further, we see the character group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now, to explicitly write these characters, we must look at the conjugacy classes of D_{2p}. It is easily verified that the conjugacy classes are C_1 containing the identity, C_t containing $x^i t$ where i is even, C_{xt} containing $x^i t$ where i is odd, C_{x^p} containing x^p, and for $j = 1, \dots p-1$, C_{x^j} containing x^j and x^{-j}. Thus, there are $p+3$ conjugacy classes. Since the character group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, if $\chi_1, \chi_2, \chi_3, \chi_4$ are the one-dimensional characters, we have the following table.

	C_1	C_t	C_{xt}	C_{x^p}	C_{x^j}
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	1	1	-1	-1	$(-1)^j$
χ_4	1	-1	1	-1	$(-1)^j$

Now, to find a bosonization in which the coradical is $\Bbbk^{D_{2p}}$, we follow the process outlined in [12] and look for $\chi \in \Bbbk^{D_{2p}}$ grouplike (corresponding to the one-dimensional characters of D_{2p}) and a character of $\Bbbk^{D_{2p}}$, g (corresponding to the elements of D_{2p}), such that

$$g(\chi) = -1, \text{ and } (g \rightharpoonup e_h)\chi = \chi(e_h \leftharpoonup g)$$

for all $h \in D_{2p}$. The second condition holds when g is in the center of D_{2p} and thus g must be either 1 or x^p. Now, looking at our table, we see that χ_3 and χ_4 are the only characters of D_{2p} that satisfy both these conditions. So, by Proposition 4.1.6 we have two $8p$-dimensional bosonizations. Namely, $R_{-1}(\Bbbk^{D_{2p}}, \chi_3, x^p) \star \Bbbk^{D_{2p}}$ and $R_{-1}(\Bbbk^{D_{2p}}, \chi_4, x^p) \star \Bbbk^{D_{2p}}$. Next, it is easy to see that $\sigma : \Bbbk^{D_{2p}} \to \Bbbk^{D_{2p}}$ defined by $\sigma(e_x) = e_x$ and $\sigma(e_t) = e_{xt}$ is an automorphism such that $\sigma(\chi_3) = \chi_4$ and $x^p = x^p \circ \sigma$. Thus, by Proposition 4.1.4, we know

$$R_{-1}(\Bbbk^{D_{2p}}, \chi_3, x^p) \star \Bbbk^{D_{2p}} \cong R_{-1}(\Bbbk^{D_{2p}}, \chi_4, x^p) \star \Bbbk^{D_{2p}}$$

both as graded Hopf algebras and as ungraded Hopf algebras. Since χ_3 and χ_4 have order 2 and x^p has order 2, by Proposition 4.2.4 we can conclude the only lifting of this bosonization is trivial. $\square$

Proposition 4.2.7. $\Bbbk^{Dic_p}$ *is the coradical of a nonsemisimple Hopf algebra of dimension* $8p$.

Proof. We again use the process of bosonization. To determine the grouplike elements in $\Bbbk^{Dic_p}$, we first determine the one-dimensional characters of Dic_p. To begin, it is easily verified that

$$[Dic_p, Dic_p] = \langle x^2 \rangle$$

Then, since there are p even powers of x, we compute the order of the abelianization of Dic_p to be 4. Thus, there are 4 one-dimensional characters of Dic_p. Further, we see the character group is isomorphic to $\mathbb{Z}_4$. Now, to explicitly write these characters, we must look at the conjugacy classes of Dic_p. It is easily verified that the conjugacy classes are C_1 containing the identity, C_t containing $x^i t$ where i is even, C_{xt} containing $x^i t$ where i is odd, C_{x^p} containing x^p, and for $j = 1, \ldots p-1$, C_{x^j} containing x^j and x^{-j}. Thus, there are $p+3$ conjugacy classes. Since the character group is isomorphic to $\mathbb{Z}_4$, if $\chi_1, \chi_2, \chi_3, \chi_4$ are the one-dimensional characters, we have the following table.

	C_1	C_t	C_{xt}	C_{x^p}	C_{x^j}
χ_1	1	1	1	1	1
χ_2	1	-1	-1	1	1
χ_3	1	i	$-i$	-1	$(-1)^j$
χ_4	1	$-i$	i	-1	$(-1)^j$

Now, to find a bosonization in which the coradical is $\Bbbk^{Dic_p}$, we again follow the process outlined in [12] and look for $\chi \in \Bbbk^{Dic_p}$ grouplike (corresponding to the one-dimensional characters of Dic_p) and a character of $\Bbbk^{Dic_p}$, g (corresponding to the elements of Dic_p), such that

$$g(\chi) = -1, \text{ and } (g \rightharpoonup e_h)\chi = \chi(e_h \leftharpoonup g)$$

for all $h \in Dic_p$. The second condition holds when g is in the center of Dic_p and thus g must be either 1 or x^p. Now, looking at our table, we see that χ_3 and χ_4 are the only characters of Dic_p that satisfy both these conditions. So, by Proposition 4.1.6 we have two $8p$-dimensional bosonizations. Namely, $R_{-1}(\Bbbk^{Dic_p}, \chi_3, x^p) \star \Bbbk^{Dic_p}$ and $R_{-1}(\Bbbk^{Dic_p}, \chi_4, x^p) \star \Bbbk^{Dic_p}$. Next, it is easy to see that $\sigma : \Bbbk^{Dic_p} \to \Bbbk^{Dic_p}$ defined by $\sigma(e_x) = e_x$ and $\sigma(e_t) = e_{xt}$ is an automorphism such that $\sigma(\chi_3) = \chi_4$ and $x^p = x^p \circ \sigma$. Thus, by Proposition 4.1.4, we know

$$R_{-1}(\Bbbk^{Dic_p}, \chi_3, x^p) \star \Bbbk^{Dic_p} \cong R_{-1}(\Bbbk^{Dic_p}, \chi_4, x^p) \star \Bbbk^{Dic_p}$$

both as graded Hopf algebras and as ungraded Hopf algebras. Since χ_3 and χ_4 have order 4 and x^p has order 2, using Proposition 4.2.4 we can conclude that in addition to the trivial lifting, there is precisely one nontrivial lifting of this bosonization. □

Proposition 4.2.8. *Let H be a Hopf algebra of dimension $8p$. If H_0 is a $4p$-dimensional Hopf algebra, then $H_0 \ncong \Bbbk^{E_p}$.*

Proof. Following [12], we want to find $\chi \in \Bbbk^{E_p}$ grouplike (corresponding to the one-dimensional characters of E_p) and a character of $\Bbbk^{E_p}$, g (corresponding to the grouplike elements of E_p), such that

$$g(\chi) = -1, \text{ and } (g \rightharpoonup e_h)\chi = \chi(e_h \leftharpoonup g)$$

for all $h \in E_p$. The second condition holds only if g is in the center of E_p, thus $g = 1$. However, $1(\chi) = 1$ for all χ. Thus, the necessary conditions for bosonization are not possible. □

We now consider the generalizations of $A_\pm$, as outlined in [1]. Let

$$D_p = \left\{a, b | a^p = b^2 = 1, bab = a^{-1}\right\}.$$

We define $A_+ = \Bbbk^{D_p}[z]$ where $z^2 = 1$ and for $c \in \Bbbk^{D_p}$, $cz = \iota^*(c)z$ where ι is conjugation by b. We can define a coalgebra structure by letting z be grouplike and inheriting the coalgebra structure of $\Bbbk^{D_p}$. A_- is defined similarly, with $z^2 = \Psi$ where $\Psi \in \Bbbk^{D_p}$ is defined by $\Psi(a) = 1$ and $\Psi(b) = -1$. It is shown in [1] that $A_\pm$ are self-dual Hopf algebras.

4.2.3.1 The Hopf algebra A_+

Following our previous method, to determine if A_+ is the coradical of a nonsemisimple, non-pointed Hopf algebra of dimension $8p$, we try to construct a bosonization. We know $G(A_+) = \{1, z, \Psi, z\Psi\}$, so we can examine the idempotent elements that generate one-dimensional ideals to find the linear characters of A_+. Let $z_0 = \frac{1+z}{2}$, $z_1 = \frac{1-z}{2}$ and let $\{e_g\}$ be the standard basis for $\Bbbk^{D_p}$.

To begin, we note that

$$ze_1z_0 = e_1z_0 \text{ and } e_ge_1z_0 = \delta_{g,1}e_1z_0$$

so e_1z_0 generates a one-dimensional ideal. If χ_1 is the corresponding character, we then see that

$$\chi_1(1) = 1,\ \chi_1(z) = 1,\ \chi_1(\Psi) = 1,\ \chi_1(z\Psi) = 1$$

Since (4.14) is not met, no bosonization is possible from this character.

Next, we note that

$$ze_1z_1 = -e_1z_1 \text{ and } e_ge_1z_1 = \delta_{g,1}e_1z_1$$

so e_1z_1 generates a one-dimensional ideal. If χ_2 is the corresponding character, then we see that

$$\chi_2(1) = 1,\ \chi_2(z) = -1,\ \chi_2(\Psi) = 1,\ \chi_2(z\Psi) = -1$$

So, we need to verify our commutativity condition for (z, χ_2) and $(z\chi, \chi_2)$.

$$(\chi_2 \rightharpoonup e_gz^i)z = \left[\sum_{st=g} e_sz^i\chi_2(e_tz^i)\right] z = e_gz^i(-1)^iz = (-1)^ie_gz^{i+1}$$

and

$$z(e_gz^i \leftharpoonup \chi_2) = z\left[\sum_{st=g} \chi_2(e_sz^i)e_tz^i\right] = z(-1)^ie_gz^i = (-1)^ie_{bgb}z^{i+1}$$

Since $e_g \neq e_{bgb}$ for all $g \in D_p$, we cannot have a bosonization using (z, χ_2). Further,

$$(\chi_2 \rightharpoonup e_gz^i)z\Psi = \left[\sum_{st=g} e_sz^i\chi_2(e_tz^i)\right] z\Psi = e_gz^i(-1)^iz\Psi = (-1)^ie_gz^{i+1}\Psi$$

and

$$z\Psi(e_gz^i \leftharpoonup \chi_2) = z\Psi\left[\sum_{st=g} \chi_2(e_sz^i)e_tz^i\right] = z\Psi(-1)^ie_gz^i = (-1)^ie_{bgb}z^{i+1}\Psi$$

Again, since $e_g \neq e_{bgb}$ for all $g \in D_p$, we cannot have a bosonization with $(z\Psi, \chi_2)$.

Further, we note that

$$ze_bz_0 = e_bz_0 \text{ and } e_ge_bz_0 = \delta_{g,b}e_bz_0$$

so e_bz_0 generates a one-dimensional ideal. If χ_3 is the corresponding character, we then see that

$$\chi_3(1) = 1,\ \chi_3(z) = 1,\ \chi_3(\Psi) = -1,\ \chi_3(z\Psi) = -1$$

So, we need to verify our commutativity condition for (Ψ, χ_3) and $(z\Psi, \chi_3)$. That is,

$$(\chi_3 \rightharpoonup e_g z^i)\Psi = \left[\sum_{st=g} e_s z^i \chi_3(e_t z^i)\right] \Psi = e_{gb} z^i \Psi = z^i e_{bg} \Psi$$

and

$$\Psi(e_g z^i \leftharpoonup \chi_3) = \Psi \left[\sum_{st=g} \chi_3(e_s z^i) e_t z^i\right] = \Psi e_{bg} z^i = z^i e_{gb} \Psi$$

but $z^i e_{bg}\Psi \neq z^i e_{gb}\Psi$ for all $g \in D_p$. Thus, there is no bosonization using (Ψ, χ_3). Further,

$$(\chi_3 \rightharpoonup e_g z^i) z\Psi = \left[\sum_{st=g} e_s z^i \chi_3(e_t z^i)\right] z\Psi = e_{gb} z^i z\Psi = e_{gb} z^{i+1} \Psi$$

and

$$z\Psi(e_g z^i \leftharpoonup \chi_3) = z\Psi \left[\sum_{st=g} \chi_3(e_s z^i) e_t z^i\right] = z\Psi e_{bg} z^i = e_{gb} z^{i+1} \Psi$$

so, we can construct the bosonization $R_{-1}(A_+, z\Psi, \chi_3)\#A_+$.

Finally, we see that

$$z e_b z_1 = -e_b z_1 \text{ and } e_g e_b z_1 = \delta_{g,b} e_b z_1$$

so $e_b z_1$ generates a one-dimensional ideal. If χ_4 is the corresponding character, then we see that

$$\chi_4(1) = 1,\ \chi_4(z) = -1,\ \chi_4(\Psi) = -1,\ \chi_4(z\Psi) = 1$$

So, we need to verify our commutativity condition (4.15) for (z, χ_4) and (Ψ, χ_4). That is,

$$(\chi_4 \rightharpoonup e_g z^i) z = \left[\sum_{st=g} e_s z^i \chi_4(e_t z^i)\right] z = (-1)^i e_{gb} z^i z = (-1)^i e_{gb} z^{i+1}$$

and

$$z(e_g z^i \leftharpoonup \chi_4) = z \left[\sum_{st=g} \chi_4(e_s z^i) e_t z^i\right] = (-1)^i z e_{bg} z^i = (-1)^i e_{gb} z^{i+1}$$

so we can construct the bosonization $R_{-1}(A_+, z, \chi_4)\#A_+$. Further,

$$(\chi_4 \rightharpoonup e_g z^i)\Psi = \left[\sum_{st=g} e_s z^i \chi_4(e_t z^i)\right] \Psi = (-1)^i e_{gb} z^i \Psi = (-1)^i z^i e_{bg} \Psi$$

and

$$\Psi(e_g z^i \leftharpoonup \chi_4) = \Psi \left[\sum_{st=g} \chi_4(e_s z^i) e_t z^i\right] = (-1)^i \Psi e_{bg} z^i = (-1)^i z^i e_{gb} \Psi$$

but $(-1)^i z^i e_{bg}\Psi \neq (-1)^i z^i e_{gb}\Psi$ for all $g \in D_p$. Thus, there is no bosonization using (Ψ, χ_4).

Proposition 4.2.9. *A_+ is the coradical of a nonsemisimple, nonpointed Hopf algebra of dimension 8p.*

Proof. By the computations above, we can construct the bosonizations

$$R_{-1}(A_+, z\Psi, \chi_3) \star A_+ \text{ and } R_{-1}(A_+, z, \chi_4) \star A_+.$$

However, we notice that $f : A_+ \to A_+$ defined by $f(e_g z^i) = e_g z^i \Psi^i$ is an automorphism such that $f(z) = z\Psi$ and $\chi_4 = \chi_3 \circ f$. Thus, by Proposition 4.1.4,

$$R_{-1}(A_+, z\Psi, \chi_3) \star A_+ \cong R_{-1}(A_+, z, \chi_4) \star A_+$$

Finally, since χ_3 and χ_4 are order 2 characters, using Proposition 4.2.4, we can conclude that the only lifting of this bosonization is trivial. □

4.2.3.2 The Hopf algebra A_-

To determine if A_- is the coradical of a nonsemisimple, nonpointed Hopf algebra of dimension $8p$, we again try to construct a bosonization. Note that $G(A_-) = \{1, z, \Psi, z\Psi\}$. Using the idempotent elements that generate a one-dimensional ideal, we will construct the linear characters of A_- and verify the conditions (4.14) and (4.15) with the grouplike elements.

To begin, we note that

$$ze_1\left(\frac{1+z}{2}\right) = e_1\left(\frac{1+z}{2}\right) \text{ and } e_g e_1\left(\frac{1+z}{2}\right) = \delta_{g,1} e_1\left(\frac{1+z}{2}\right)$$

so $e_1(\frac{1+z}{2})$ generates a one-dimensional ideal. If χ_1 is the corresponding character, then we see that

$$\chi_1(1) = 1,\ \chi_1(z) = 1,\ \chi_1(\Psi) = 1,\ \chi_1(z\Psi) = 1$$

Since (4.14) is not met, it is not possible to construct a bosonization using this character.

Next, we note that

$$ze_1\left(\frac{1-z}{2}\right) = -e_1\left(\frac{1-z}{2}\right), \text{ and } e_g e_1\left(\frac{1-z}{2}\right) = \delta_{g,1} e_1\left(\frac{1-z}{2}\right)$$

so $e_1(\frac{1-z}{2})$ generates a one-dimensional ideal. If χ_2 is the corresponding character, we then see that

$$\chi_2(1) = 1,\ \chi_2(z) = -1,\ \chi_2(\Psi) = 1,\ \chi_2(z\Psi) = -1$$

Now, we are left to verify (4.15) for (z,χ_2) and $(z\Psi,\chi_2)$. We compute

$$(\chi_2 \rightharpoonup e_g z^j)z = \left[\sum_{st=g} e_s z^j \chi_2(e_t z^j)\right] z = e_g z^j (-1)^j z = (-1)^j e_g z^{j+1}$$

and

$$z(e_g z^j \leftharpoonup \chi_2) = z\left[\sum_{st=g} \chi_2(e_s z^j) e_t z^j\right] = z(-1)^j e_g z^j = (-1)^j e_{bgb} z^{j+1}.$$

Since $(-1)^j e_g z^{j+1} \neq (-1)^j e_{bgb} z^{j+1}$ for all $g \in D_p$, we are unable to construct a bosonization using (z,χ_2). Further, we find we cannot construct a bosonization using $(z\Psi,\chi_2)$ since

$$(\chi_2 \rightharpoonup e_g z^j)z\Psi = \left[\sum_{st=g} e_s z^j \chi_2(e_t z^j)\right] z\Psi = e_g z^j (-1)^j z\Psi = (-1)^j e_g z^{j+1}\Psi$$

and

$$z\Psi(e_q z^j \leftharpoonup \chi_2) = z\Psi\left[\sum_{st=g} \chi_2(e_s z^j) e_t z^i\right] = z\Psi(-1)^j e_g z^j = (-1)^j e_{bgb} z^{j+1}\Psi.$$

Next, we note that

$$z e_b \left(\frac{1+iz}{2}\right) = -i e_b \left(\frac{1+iz}{2}\right), \text{ and } e_g e_b \left(\frac{1+iz}{2}\right) = \delta_{g,b} e_b \left(\frac{1+iz}{2}\right)$$

so $e_b(\frac{1+iz}{2})$ generates a one-dimensional ideal. If χ_3 is the corresponding character, then we see that

$$\chi_3(1) = 1,\ \chi_3(z) = -i,\ \chi_3(\Psi) = -1,\ \chi_3(z\Psi) = i.$$

We verify (4.15) for (Ψ,χ_3) by computing

$$(\chi_3 \rightharpoonup e_g z^j)\Psi = \left[\sum_{st=g} e_s z^j \chi_3(e_t z^j)\right] \Psi = e_{gb} z^j (-i)^j \Psi$$

and

$$\Psi(e_g z^j \leftharpoonup \chi_3) = \Psi\left[\sum_{st=g} \chi_3(e_s z^j) e_t z^j\right] = \Psi(-i)^j e_{bg} z^j.$$

Since $e_{gb} z^j (-i)^j \Psi \neq \Psi(-i)^j e_{bg} z^j$ for all $g \in D_p$, we know we cannot construct a bosonization using (Ψ,χ_3).

Finally, since

$$z e_b \left(\frac{1-iz}{2}\right) = i e_b \left(\frac{1-iz}{2}\right), \text{ and } e_g e_b \left(\frac{1-iz}{2}\right) = \delta_{g,b} e_b \left(\frac{1-iz}{2}\right)$$

we know $e_b(\frac{1-iz}{2})$ generates a one-dimensional ideal. If χ_4 is the corresponding character, then we see that

$$\chi_4(1) = 1,\ \chi_4(z) = i,\ \chi_4(\Psi) = -1,\ \chi_4(z\Psi) = -i$$

We are left to check the commutativity condition for (Ψ, χ_4). We compute

$$(\chi_4 \rightharpoonup e_g z^j)\Psi = \left[\sum_{st=g} e_s z^j \chi_4(e_t z^j)\right] \Psi = e_{gb} z^j (i)^j \Psi$$

and

$$\Psi(e_g z^j \leftharpoonup \chi_4) = \Psi \left[\sum_{st=g} \chi_4(e_s z^j) e_t z^j\right] = \Psi(i)^j e_{bg} z^j,$$

but $e_{gb} z^j (i)^j \Psi \neq \Psi(i)^j e_{bg} z^j$ for all $g \in D_p$, so we are unable to construct a bosonization using (Ψ, χ_4).

Since there are no possible bosonizations when $H_0 \cong A_-$, we have the following conclusion.

Proposition 4.2.10. *Let H be a nonpointed, nonsemisimple Hopf algebra of dimension $8p$ with the Chevalley property. Then, $H_0 \ncong A_-$.*

4.3 Summary of results and future work

Using the results of the previous section as well as other known results, we have completed the classification of nonsemisimple Hopf algebras of dimension $8p$ with the Chevalley property. In particular, we note that there are only finitely many up to isomorphism. A summary of the classification is presented in Table 4.1.

While this completes the classification of the nonpointed, nonsemisimple Hopf algebras of dimension $8p$ with the Chevalley property, the $8p$-dimensional classification overall remains incomplete. In particular, it is still not known whether there is an infinite family of Hopf algebras of dimension 24.

Moving forward, there are several directions to take. Firstly, we look to see if we can explicitly write the 24-dimensional pointed Hopf algebras using Graña's classification of Nichols algebras [25]. Further, we ask if this classification can be extended to the $8p$-dimensional case.

Finally, we look to classify the nonsemisimple Hopf algebras without the Chevalley property of dimension 24, and then more generally dimension $8p$. Here, since the coradical is no longer

Table 4.1: Summary of Classification

$\dim H_0$	*Classifications of H*
1	Impossible
2	H_0 must be a group algebra [59], so H is pointed
4	H_0 must be a group algebra [36], so H is pointed
8	Impossible by Proposition 4.2.1
p	H_0 must be a group algebra [59], so H is pointed
$2p$	· H_0 is either a group algebra or $\Bbbk^{D_p}$ [34] · H is pointed when H_0 is a group algebra · Impossible if $H_0 \cong \Bbbk^{D_p}$ by Prop 4.2.2
$4p$	· H_0 is either a group algebra, dual of a group algebra, or $A_\pm$ [38] · H is pointed when H_0 is a group algebra · H is trivial lifting of bosonization when $H_0 \cong \Bbbk^{D_{2p}}, A_+$ by Proposition 4.2.6 · H is nontrivial lifting of bosonization when $H_0 \cong \Bbbk^{Dic_p}$ by Proposition 4.2.7 · Impossible when $H_0 \cong \Bbbk^{E_p}$ ($p \equiv 1 \mod 4$) by Proposition 4.2.8 · Impossible when $H_0 \cong A_-$ by Proposition 4.2.10

a Hopf subalgebra, we are no longer guaranteed that its dimension must divide 24, or $8p$. Instead, we can approach the problem by breaking the Hopf algebras into cases based on the order and structure of their grouplike elements. In particular, Beattie and García showed in [9] that 24-dimensional Hopf algebras must be one of the following types: (2,2), (2,4), (4,2), (4,6) or (6,4). This result gives us cases to consider, with the hope of either completely classifying or eliminating each of them.

BIBLIOGRAPHY

[1] Nicolás Andruskiewitsch and Sonia Natale. Examples of self-dual Hopf algebras. *J. Math. Sci. Univ. Tokyo*, 6(1):181–215, 1999.

[2] Nicolás Andruskiewitsch and Sonia Natale. Counting arguments for Hopf algebras of low dimension. *Tsukuba J. Math.*, 25(1):187–201, 2001.

[3] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Hopf algebras of order p^2 and braided Hopf algebras of order p. *J. Algebra*, 199(2):430–454, 1998.

[4] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. *J. Algebra*, 209(2):658–691, 1998.

[5] Nicolás Andruskiewitsch and Cristian Vay. Finite dimensional Hopf algebras over the dual group algebra of the symmetric group in three letters. *Comm. Algebra*, 39(12):4507–4517, 2011.

[6] Margaret Beattie and Sorin Dăscălescu. Hopf algebras of dimension 14. *J. London Math. Soc. (2)*, 69(1):65–78, 2004.

[7] Margaret Beattie, Sorin Dăscălescu, and Luzius Grünenfelder. On the number of types of finite-dimensional Hopf algebras. *Invent. Math.*, 136(1):1–7, 1999.

[8] Margaret Beattie and Gastón Andrés García. Classifying Hopf algebras of a given dimension. page math.QA.1206.6529v2.

[9] Margaret Beattie and Gastón Andrés García. Classifying Hopf algebras of a given dimension. In *Hopf algebras and tensor categories*, volume 585 of *Contemp. Math.*, pages 125–152. Amer. Math. Soc., Providence, RI, 2013.

[10] Margaret Beattie and Gastón Andrés García. Techniques for classifying Hopf algebras and applications to dimension p^3. *Comm. Algebra*, 41(8):3108–3129, 2013.

[11] Stefaan Caenepeel, Sorin Dăscălescu, and Şerban Raianu. Classifying pointed Hopf algebras of dimension 16. *Comm. Algebra*, 28(2):541–568, 2000.

[12] Corina Călinescu, Sorin Dăscălescu, Akira Masuoka, and Claudia Menini. Quantum lines over non-cocommutative cosemisimple Hopf algebras. *J. Algebra*, 273(2):753–779, 2004.

[13] Yi-Lin Cheng and Siu-Hung Ng. On Hopf algebras of dimension $4p$. *J. Algebra*, 328:399–419, 2011.

[14] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[15] Pavel Etingof and Shlomo Gelaki. Semisimple Hopf algebras of dimension pq are trivial. *J. Algebra*, 210(2):664–669, 1998.

[16] Pavel Etingof and Shlomo Gelaki. On families of triangular Hopf algebras. *Int. Math. Res. Not.*, (14):757–768, 2002.

[17] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Weakly group-theoretical and solvable fusion categories. *Adv. Math.*, 226(1):176–205, 2011.

[18] Daijiro Fukuda. Classification of Hopf algebras of dimension 18. *Israel J. Math.*, 168:119–123, 2008.

[19] Daijiro Fukuda. Hopf algebras of dimension 30. *J. Algebra Appl.*, 9(1):11–15, 2010.

[20] Nobuyuki Fukuda. Semisimple Hopf algebras of dimension 12. *Tsukuba J. Math.*, 21(1):43–54, 1997.

[21] Gastón Andrés García. On Hopf algebras of dimension p^3. *Tsukuba J. Math.*, 29(1):259–284, 2005.

[22] Gastón Andrés García and Cristian Vay. Hopf algebras of dimension 16. *Algebr. Represent. Theory*, 13(4):383–405, 2010.

[23] Shlomo Gelaki. Quantum groups of dimension pq^2. *Israel J. Math.*, 102:227–267, 1997.

[24] Shlomo Gelaki. Pointed Hopf algebras and Kaplansky's 10th conjecture. *J. Algebra*, 209(2):635–657, 1998.

[25] Matías Graña. On Nichols algebras of low dimension. In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 111–134. Amer. Math. Soc., Providence, RI, 2000.

[26] Michael Hilgemann and Siu-Hung Ng. Hopf algebras of dimension $2p^2$. *J. Lond. Math. Soc. (2)*, 80(2):295–310, 2009.

[27] Heinz Hopf. Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen. *Ann. of Math. (2)*, 42:22–52, 1941.

[28] Irving Kaplansky. *Bialgebras*. Lecture Notes in Mathematics. Department of Mathematics, University of Chicago, Chicago, Ill., 1975.

[29] Yevgenia Kashina. Classification of semisimple Hopf algebras of dimension 16. *J. Algebra*, 232(2):617–663, 2000.

[30] Richard G. Larson and David E. Radford. Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. *J. Algebra*, 117(2):267–289, 1988.

[31] Richard G. Larson and David E. Radford. Semisimple cosemisimple Hopf algebras. *Amer. J. Math.*, 110(1):187–195, 1988.

[32] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[33] Shahn Majid. Tannaka-Kreĭn theorem for quasi-Hopf algebras and other results. In *Deformation theory and quantum groups with applications to mathematical physics (Amherst,*

MA, 1990), volume 134 of *Contemp. Math.*, pages 219–232. Amer. Math. Soc., Providence, RI, 1992.

[34] Akira Masuoka. Semisimple Hopf algebras of dimension $2p$. *Comm. Algebra*, 23(5):1931–1940, 1995.

[35] Akira Masuoka. Semisimple Hopf algebras of dimension $6, 8$. *Israel J. Math.*, 92(1-3):361–373, 1995.

[36] Akira Masuoka. The p^n theorem for semisimple Hopf algebras. *Proc. Amer. Math. Soc.*, 124(3):735–737, 1996.

[37] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[38] Sonia Natale. On semisimple Hopf algebras of dimension pq^2. *J. Algebra*, 221(1):242–278, 1999.

[39] Sonia Natale. On semisimple Hopf algebras of dimension pq^2. II. *Algebr. Represent. Theory*, 4(3):277–291, 2001.

[40] Sonia Natale. Hopf algebras of dimension 12. *Algebr. Represent. Theory*, 5(5):445–455, 2002.

[41] Sonia Natale. On semisimple Hopf algebras of dimension pq^r. *Algebr. Represent. Theory*, 7(2):173–188, 2004.

[42] Sonia Natale. Semisolvability of semisimple Hopf algebras of low dimension. *Mem. Amer. Math. Soc.*, 186(874):viii+123, 2007.

[43] Siu-Hung Ng. Non-semisimple Hopf algebras of dimension p^2. *J. Algebra*, 255(1):182–197, 2002.

[44] Siu-Hung Ng. Hopf algebras of dimension $2p$. *Proc. Amer. Math. Soc.*, 133(8):2237–2242, 2005.

[45] Siu-Hung Ng. Hopf algebras of dimension pq. II. *J. Algebra*, 319(7):2772–2788, 2008.

[46] Warren D. Nichols and M. Bettina Zoeller. A Hopf algebra freeness theorem. *Amer. J. Math.*, 111(2):381–385, 1989.

[47] David E. Radford. On an analog of Lagrange's theorem for commutative Hopf algebras. *Proc. Amer. Math. Soc.*, 79(2):164–166, 1980.

[48] David E. Radford. The structure of Hopf algebras with a projection. *J. Algebra*, 92(2):322–347, 1985.

[49] David E. Radford. *Hopf algebras*, volume 49 of *Series on Knots and Everything*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[50] David E. Radford and Hans-Jürgen Schneider. On the even powers of the antipode of a finite-dimensional Hopf algebra. *J. Algebra*, 251(1):185–212, 2002.

[51] Neantro Saavedra Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972.

[52] Hans-Jürgen Schneider. Normal basis and transitivity of crossed products for Hopf algebras. *J. Algebra*, 152(2):289–312, 1992.

[53] Dragoş Ştefan. The set of types of n-dimensional semisimple and cosemisimple Hopf algebras is finite. *J. Algebra*, 193(2):571–580, 1997.

[54] Dragoş Ştefan. Hopf algebras of low dimension. *J. Algebra*, 211(1):343–361, 1999.

[55] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[56] Earl J. Taft. The order of the antipode of finite-dimensional Hopf algebra. *Proc. Nat. Acad. Sci. U.S.A.*, 68:2631–2633, 1971.

[57] Karl-Heinz Ulbrich. On Hopf algebras and rigid monoidal categories. *Israel J. Math.*, 72(1-2):252–256, 1990. Hopf algebras.

[58] Roselyn E. Williams. *Finite dimensional Hopf algebras*. ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Ph.D.)–The Florida State University.

[59] Yongchang Zhu. Hopf algebras of prime dimension. *Internat. Math. Res. Notices*, (1):53–59, 1994.