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A new approach to representations of the Lorentz group

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A NEW APPROACH TO REPRESENTATIONS OF THE LORENTZ GROUP

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TABLE OF CONTENTS

I. INTRODUCTION

A. Previous Work

Symmetries have long been used to make predictions about the behavior of a system of particles, even when the form of the interaction between particles is not known. If the symmetry can be expressed as invariance of physical quantities under a group of transformations, the behavior of the system will be greatly restricted.

The most widely applied group of transformations has been the group of rotations and reflections in three dimensions. The usual procedure is to express physical amplitudes as a sum of irreducible representations of this group. Products of amplitudes may be expressed in terms of irreducible representations with the aid of the Clebsch-Gordan series and recoupling coefficients. This procedure is known as harmonic analysis and provides a systematic approach to symmetry groups.

While the importance of the Lorentz group in physics is well established, the unitary representations of the Lorentz group have been used infrequently in the solution of physical problems. One reason is that the various functions necessary for harmonic analysis on the Lorentz group have not been studied as thoroughly as in the case of the rotation group.

The finite dimensional representations of the Lorentz group have long been known. These representations are obtained from the representations of the rotation group in four dimensions by the "unitary trick." Since the Lorentz group is noncompact any unitary representation must be infinite dimensional. In 1945 Dirac (1) pointed out several unitary representations and posed the problem of classifying all irreducible unitary representations. In 1947 Harish Chandra (2) proved that in unitary representations the two invariants of the Lorentz group have the values

$$J \cdot K = \ell\nu \quad , \qquad J^2 - K^2 = \ell^2 - \nu^2 - 1$$

where ℓ is half integral and ν is real. In the case $\ell = 0$ ν may also be pure imaginary with $|\nu|<1$.

Gel'fand and Naimark (3) (4) constructed representations of the Lorentz group on the space of complex valued functions $f(z)$, $z = x + iy$ for which

$$\iint |f(z)|^2 \, dx \, dy < \infty .$$

While the space of complex valued functions is convenient from a mathematical standpoint, it has not been used in the solution of physical problems. The main disadvantage is that the complex variable z has no immediate physical interpretation. In physical problems one encounters functions of one or more four vectors which transform according to the Lorentz group. For example scalar fields satisfy the transformation law

$$\phi(x) \rightarrow \phi(\Lambda x).$$

Dolginov et al. (5)(6)(7), derived unitary representations of the Lorentz group in a form which is more useful to physicists. The results, which were obtained by analytic continuation of unitary representations of the four dimensional rotation group, are not in the most convenient form. In 1965 Ström (8) obtained certain matrix elements by integrating differential relations. The general matrix element was then to be found by ladder operators. In 1967 Ström (9) found an expression for the general matrix element by a different method.

In the above work states in a representation are labeled by the rotation group. However, the Lorentz group also contains the conformal group in a plane and the three dimensional Lorentz group as subgroups. In certain problems it may be convenient to take a basis in which states are labeled by one of the latter subgroups. For example, it would be more convenient to expand a function of a light-like four vector in terms of the conformal group basis since the conformal group in a plane is the little group of a light-like vector. A number of advantages of the conformal group basis are discussed by Chang and O'Raifeartaigh (10). In addition they derive the transformation coefficient relating angular momentum states and conformal group states for the case $j = m$.

In this thesis the angular momentum and conformal group bases are studied. The relationship between these bases is expressed by the Iwasawa decomposition of the Lorentz group. This decomposition implies that every Lorentz transformation may be written as the product of a rotation and a conformal group element. This relationship provides a simple derivation for many of the functions needed for harmonic analysis on the Lorentz group.

B. Work Done in This Thesis

Chapter 2 contains a number of well known results concerning Lie groups which will be required in later chapters. Chapter 3 is a summary of well known results concerning the Lorentz group and Poincaré group.

In chapter 4 representations of the Lorentz group are constructed in a conformal group basis. The matrix elements of a general Lorentz transformation are derived in this basis. Finally the expansion coefficient connecting a conformal group state with an angular momentum state is derived. These results are original with the exception of the expansion coefficient f^j_m for the case $j = m$ which has also been calculated by Chang and O'Raifeartaigh (10).

Matrix elements of the Lorentz group in an angular momentum basis are considered in chapter 5. Expressions for these matrix elements have been derived by several authors (7)(10). However, the method of section A is original and results in several new expressions for the matrix elements. In addition the symmetries of matrix elements are very easy to derive by this method. In section B a new integral representation for the matrix elements is derived and is used in section C to derive the asymptotic form of the matrix elements. The orthogonality relations for the matrix elements have been listed by Dolginov and Moskalev (7). An alternative proof of the orthogonality relations is presented in section D.

In chapter 6 harmonic functions for the Poincaré group are considered. In section A harmonic functions in an angular momentum basis are derived. Any wave function may be expanded in terms of these functions. In section B a corresponding set of Lorentz harmonics is derived in a conformal group basis. The results of chapter 6 are original with the exception of the angular momentum basis for the case of spin zero which has been derived by (11)(12)(13).

This work suggests a number of possible directions for future work. In order to expand functions of space-like four vectors it would be useful to have matrix elements in a basis labeled by the 3 dimensional Lorentz group, O(2,1). The expansion coefficients relating the O(2,1) basis to the conformal group and rotation group bases would be useful. Finally the Clebsch-Gordan coefficients for the Lorentz group have been studied very little. Most of the available results are in terms of the space of functions of a complex variable.

II. LIE GROUPS

A. Introduction

An abstract group is a set of elements G which possess a law of composition that associates with every ordered pair of elements x,y a unique element z in G. The following conditions must be satisfied.

1. Associative law:

$$x(yz) = (xy)z \tag{2.1}$$

2. Unit element: There exists in G a left unit e such that for any x in G

$$ex = x \quad . \tag{2.2}$$

3. Inverse: Every element x in G possess a left inverse x^{-1} such that

$$x^{-1}x = e \quad . \tag{2.3}$$

The group postulates imply that e is also a right unit and x^{-1} is a right inverse.

Groups do not arise in physics as abstract groups but as transformations on vector spaces. They may be discrete groups such as the rotations and translations which are symmetries of crystal fields or they may be continuous groups.

The continuous groups which have occurred in physics have been Lie groups for example, the rotation group, the Lorentz group, and SU(3). Lie groups have an analytic structure which simplifies the study of their properties.

In this chapter we consider properties of Lie groups which will be required in later chapters. We begin with the definition of a Lie group.

A group G is a Lie group if:

1. Elements of G in a neighborhood of the unit element can be put in a one to one correspondence with points $x^i (i = 1, \ldots, r)$ in a region of a r dimensional Euclidian space with the unit element corresponding to the origin.

2. The coordinates z^i of the product of elements x,y are analytic functions of x^i, y^i.

$$z^i = \varphi^i(x^1, \ldots, x^r; y^1, \ldots, y^r) \equiv \varphi^i(x,y) \quad . \quad (2.4)$$

The group postulates impose several conditions on the functions φ^i.

Since

$$\varphi^i(0,y) = y^i \qquad \text{and} \qquad \varphi^i(x,0) = x^i \qquad (2.5)$$

the functions φ^i have the Taylor series

$$\begin{aligned}\varphi^i(x,y) &= x^i + y^i + a^i_{jk}\, x^j y^k \\ &+ g^i_{jk\ell}\, x^j x^k y^\ell + h^i_{jk\ell} x^j y^k y^\ell \\ &+ r\end{aligned} \qquad (2.6)$$

where we have assumed summation on repeated indices and the remainder r is of the order four in x and y.

The commutator

$$q = x\, y\, x^{-1} y^{-1} \qquad (2.7)$$

of two elements x,y will play a special role in the development of Lie groups. In order to derive the Taylor series for the commutator, we first find a second degree approximation for x^{-1} in terms of x.

The result, which follows from 2.6, is

$$(x^{-1})^i = -x^i + a^i_{jk}\, x^j x^k + 0(3) \quad . \qquad (2.8)$$

Relations 2.6, 2.7, and 2.8 imply that the commutator has the Taylor series

$$q^i = (a^i_{jk} - a^i_{kj}) x^j y^k + 0(3) \quad . \qquad (2.9)$$

The numbers

$$c^i_{jk} = a^i_{jk} - a^i_{kj} \qquad (2.10)$$

are called the structure constants of the group G. The structure constants clearly satisfy the relation

$$c^i_{jk} = - c^i_{kj} \quad . \qquad (2.11)$$

A further condition on the structure constants arises from the associative law

$$\varphi^i[x,\varphi(y,z)] = \varphi^i[\varphi(x,y),z] \quad . \qquad (2.12)$$

If we substitute the Taylor series for φ^i in 2.12 we have the condition

$$a^i_{jk} a^j_{\ell m} - a^i_{\ell j} a^j_{mk} = h^i_{\ell mk} + h^i_{\ell km} - g^i_{\ell mk} - g^i_{m\ell k} \quad . \qquad (2.13)$$

If we antisymmetrize 2.13 with respect to the indicies k,ℓ,m and use definition 2.10, we find

$$c^i_{kj} c^j_{\ell m} + c^i_{\ell j} c^j_{mk} + c^i_{mj} c^j_{k\ell} = 0 \quad . \qquad (2.14)$$

It has been proven (14) that Lie groups with the same structure constants are isomorphic in some neighborhood of the unit element and that there exists a Lie group for every set of numbers c^i_{jk} satisfying 2.11 and 2.14.

B. Lie Algebras

We now consider the relationship between the structure constants and a Lie algebra. In order to define the Lie algebra of a group we consider all one dimensional submanifolds $x^i(s)$ (curves) of the group containing the unit element at $s = 0$.

The tangents to these curves at $s = 0$

$$\xi^i = \frac{dx^i}{ds}\Big|_{s=0} \tag{2.15}$$

form a r dimensional vector space. This vector space becomes the Lie algebra for the group when we define a suitable product of two vectors.

If $x^i(s)$ and $y^i(s)$ are curves in G with tangent vectors η^i and ξ^i we define the Lie product

$$\zeta^i = [\eta,\xi]^i \tag{2.16}$$

to be the tangent of the commutator

$$z^i(s') = [x(s)y(s)x^{-1}(s)y^{-1}(s)]^i \quad \text{where } s' = s^2 \quad . \tag{2.17}$$

The Lie product is therefore given by

$$\zeta^i = [\eta,\xi]^i = c^i_{jk}\eta^j\xi^k \quad . \tag{2.18}$$

In order to simplify the notation we introduce a basis E_i for the Lie algebra and make the following definitions.

$$\eta = E_i\eta^i \qquad\qquad [E_j,E_k] = c^i_{jk}E_i \quad . \tag{2.19}$$

Conditions 2.11 and 2.14 on the structure constants imply two conditions on the Lie product.

$$\begin{aligned} &1. \quad [\eta,\xi] = -[\xi,\eta] \\ &2. \quad [[\eta,\xi],\zeta] + [[\xi,\zeta],\eta] + [[\zeta,\eta],\xi] = 0 \quad . \end{aligned} \tag{2.20}$$

Having defined a Lie algebra, we list several important facts about Lie algebras and Lie groups.

In order to state the following definitions more concisely we define the Lie product of sets $\underline{R},\underline{G}$ to be the set

$$[\underline{R},\underline{G}] = \{\text{the set of all } \zeta = [\eta,\xi] \quad \text{where} \quad \eta\in\underline{R},\xi\in\underline{G}\} \quad .$$

Definitions:

A Lie algebra $\underline{G}$ is said to be Abelian if

$$[\underline{G},\underline{G}] = 0 \quad .$$

A subset $\underline{R}$ of a Lie algebra $\underline{G}$ is said to be a subalgebra if

$$[\underline{R},\underline{R}]\subset\underline{R} \quad .$$

A subalgebra $\underline{R}$ of a Lie algebra $\underline{G}$ is called an ideal if

$$[\underline{G},\underline{R}]\subset\underline{R} \quad .$$

A Lie algebra is called semisimple if it has no nonzero Abelian ideals.

We note several properties of Lie groups which are reflected in their Lie algebras.

1. Abelian groups have Abelian Lie algebras.
2. If $\underline{R}$ is the Lie algebra of a subgroup R and $\underline{G}$ is the Lie algebra of the group G then $\underline{R}$ is a subalgebra of $\underline{G}$.
3. If R is an invariant subgroup of G then the Lie algebra $\underline{R}$ of R is an ideal in $\underline{G}$.

Cartan's criterion for a Lie algebra to be semisimple will be important in the following work. To state it we introduce the symmetric bilinear form (Killing form) $B(\xi,\eta)$ on the Lie algebra $\underline{G}$ as follows: If E_i is a basis for $\underline{G}$ with

$$[E_i, E_j] = C^k_{ij} E_k \qquad \xi = \xi^i E_i \quad , \qquad \eta = \eta^i E_i \quad , \tag{2.21}$$

we define the Killing form to be

$$B(\xi, \eta) = \xi^i \eta^\ell C^k_{ij} C^j_{\ell k} \quad . \tag{2.22}$$

Cartan's criterion: $\underline{G}$ is semisimple if and only if the form B(,) is nondegenerate, (i.e. the determinant of $C^k_{ij} C^j_{\ell k}$ is nonzero).

C. Invariant Integration

The topological properties of a Lie group are related to the Killing form on the Lie algebra of the group. To see this relationship we consider integration on a group.

We say a Lie group G is compact if for every real continuous function f defined on G we can define an integral

$$I = \int f(x) \ \rho(x) dx \tag{2.23}$$

where ρ is independent of f and I has the following properties:

1. $$\int \rho(x) dx = 1 \tag{2.24}$$
2. If $f(x) \geq 0$ but is not identically zero
$$\int f(x) \ \rho(x) dx > 0 \quad . \tag{2.25}$$
3. If a is an element of G
$$\int f(ax) \ \rho(x) dx = \int f(x) \ \rho(x) dx \quad . \tag{2.26}$$

The integral I is called the invariant integral on G.

If $z^i = \varphi^i(x,y)$ is the law of composition for the Lie group, the function

$$\rho(x) = \frac{1}{\left| \det \frac{\partial \varphi^i(x,y)}{\partial y^j} \right|_{y=0}} \tag{2.27}$$

satisfies 2.25 and 2.26. To see this we define w = ax and change the variable of integration.

$$\int f(w)\rho(a^{-1}w)\,dx = \int f(w)\rho[\varphi(a^{-1},w)]\left|\det \frac{\partial\varphi^i(a^{-1},w)}{\partial w^j}\right| dw \quad . \qquad (2.28)$$

Since

$$\frac{\partial\varphi^i}{\partial y^j}[\varphi(a^{-1},w),y]\Big|_{y=0} = \frac{\partial\varphi^i}{\partial y^j}[a^{-1},\varphi(w,y)]\Big|_{y=0}$$

$$= \frac{\partial\varphi^i}{\partial w^k}(a^{-1},w)\,\frac{\partial\varphi^k}{\partial y^j}(w,y)\Big|_{y=0} \quad , \qquad (2.29)$$

we have

$$\rho[\varphi(a^{-1},w)] = \frac{\rho(w)}{\left|\det \frac{\partial\varphi^i}{\partial w^j}(a^{-1},w)\right|} \qquad (2.30)$$

which completes the proof.

If the integral of 2.27 exists the group is compact since 2.24 may be satisfied by normalizing 2.27. If G is compact I has the following additional properties (14).

$$\int f(xa)\rho(x)\,dx = \int f(x)\rho(x)\,dx$$

$$\int f(x^{-1})\rho(x)\,dx = \int f(x)\rho(x)\,dx \qquad (2.31)$$

The relationship of the Killing form on the Lie algebra to the topology of the group is seen in following theorems (15).

If G is the Lie algebra of a compact group, the Killing form on G is negative semidefinite, i.e. $B(\eta,\eta) \leq 0$ for all $\eta \in$ G.

(Weyl) If G is semisimple then G is compact if and only if the Killing form on G is negative definite, i.e. $B(\eta,\eta) < 0$ for all $\eta \in$ G.

If 2.24 does not converge the group is said to be noncompact.

D. Representations of Lie Algebras and Lie Groups

Suppose that to each element ξ of a Lie algebra there corresponds a linear transformation $A(\xi)$ of a vector space U, which in general is of different dimension than the Lie algebra, with the properties:

$$\begin{aligned} &A(\xi + \eta) = A(\xi) + A(\eta) \\ &A(c\xi) = cA(\xi) \quad , \qquad c \text{ a scalar} \\ &A([\xi,\eta]) = A(\xi)A(\eta) - A(\eta)A(\xi) \quad . \end{aligned} \tag{2.32}$$

Such a correspondence of operators is called a representation of the Lie algebra $\underline{G}$.

In the cases we consider, a representation of a Lie algebra will determine uniquely a representation of the Lie group with the following properties:

1. For every element s of the Lie group there corresponds a linear transformation D(s) of the vector space U.
2. $D(st) = D(s)D(t)$
3. $D(s^{-1}) = D^{-1}(s)$. (2.33)

If ξ is an element of the Lie algebra, a corresponding element of the Lie group will be represented by

$$D = \exp[A(\xi)] = \sum_n \frac{[A(\xi)]^n}{n!} \quad . \tag{2.34}$$

We will not prove this result but note that if D(s) is a representation of a Lie group and if $s^i(\tau)$ is a one parameter curve in G with

$$\xi^i = \frac{ds^i}{d\tau} \Big|_{\tau=0} \quad , \tag{2.35}$$

then the operators

$$A(\xi) = \frac{d}{d\tau} D[s(\tau)]\Big|_{\tau=0} \tag{2.36}$$

represent the corresponding Lie algebra.

We call a representation reducible if there exists a subspace R of U which is invariant under the transformations D(s). In other words if

$$|\alpha\rangle \subset R \quad \text{then} \quad |\alpha'\rangle = D(s)|\alpha\rangle \in R \quad \text{for all} \quad s \in G. \tag{2.37}$$

If there are no invariant subspaces in U we say the representation is irreducible.

If the vector space U has a Hermitian inner product $\langle\alpha|\beta\rangle$ and if

$$\langle\alpha'|\beta'\rangle = \langle\alpha|\beta\rangle$$

where

$$\begin{aligned} |\alpha'\rangle &= D(s)|\alpha\rangle \\ |\beta'\rangle &= D(s)|\beta\rangle \end{aligned} \tag{2.38}$$

for all $|\alpha\rangle$, $|\beta\rangle \in U$, $s \in G$ then the representation is said to be unitary.

E. Decompositions of Lie Groups

We list two decompositions of semisimple Lie groups which will be used in later chapters. We omit the proofs (16) which are long, since the detailed structure of all semisimple Lie algebras must be considered.

Cartan's theorem:

If $\underline{G}$ is a noncompact semisimple Lie algebra $\underline{G}$ has a direct sum decomposition $\underline{G} = \underline{J} \oplus \underline{K}$ satisfying:

1. $\underline{J}$ is a subalgebra $[\underline{J},\underline{J}] \subset \underline{J}$

   $[\underline{K},\underline{J}] \subset \underline{K}$ $\qquad$ $[\underline{K},\underline{K}] \subset \underline{J}$

2. The Killing form restricted to $\underline{J}$ is negative definite.

   The Killing form restricted to $\underline{K}$ is positive definite.

3. If G is a connected Lie group with Lie algebra $\underline{G}$ then G has the unique decomposition

$$G = RA \tag{2.39}$$

where R is the image of $\underline{J}$ and A is the image of $\underline{K}$ under the exponential map.

If $\underline{K}'$ is a maximum Abelian subalgebra of $\underline{K}$, A can be decomposed as

$$A = R_1 A' R_1^{-1} \tag{2.40}$$

where A' is the image of $\underline{K}'$.

Therefore G has the decomposition

$$G = R_2 A' R_3 \quad . \tag{2.41}$$

The Iwasawa decomposition:

If $\underline{J} \oplus \underline{K}$ is a Cartan decomposition of a noncompact semisimple Lie algebra $\underline{G}$ and $\underline{K}'$ is a maximum Abelian subalgebra of $\underline{K}$ then there is a basis E_i for $\underline{G}$ with the properties:

1. $\underline{K}'$ is diagonal and has real eigenvalues.

$$\begin{aligned} &[\alpha, E_i] = \lambda_i(\alpha) E_i \\ &\alpha \in \underline{K}' \qquad \lambda_i \quad \text{real} \end{aligned} \tag{2.42}$$

2. The set of vectors E_i which do not have identically zero eigenvalues, $\lambda_i(\alpha)$, are called root vectors. This set of root vectors may be decomposed into two nilpotent subalgebras as follows: Let α_o be a regular element of $\underline{K}'$, i.e. $\lambda_i(\alpha_o) \neq 0$. Then

$$\begin{aligned} \underline{N}_+ &= \{E_i; \quad \lambda_i(\alpha_o) > 0\} \\ \underline{N}_- &= \{E_i; \quad \lambda_i(\alpha_o) < 0\} \end{aligned} \tag{2.43}$$

are nilpotent subalgebras of $\underline{G}$.

A Lie algebra $\underline{L}$ is nilpotent if

$$\begin{aligned} &[\underline{L},\underline{L}] \neq \underline{L} \\ &[\underline{L},[\underline{L},\underline{L}]] \neq [\underline{L},\underline{L}] \\ &\vdots \\ &[\underline{L},[,\ldots,[\underline{L},\underline{L}]\ldots]] = 0 \quad . \end{aligned} \tag{2.44}$$

3. $\underline{K}'\oplus\underline{N}_+$ is a solvable subalgebra. A Lie algebra $\underline{L}$ is solvable if

$$\begin{aligned} &\underline{L}_1 = [\underline{L},\underline{L}] \neq \underline{L} \\ &\underline{L}_2 = [\underline{L}_1,\underline{L}_1] \neq \underline{L}_1 \\ &\vdots \\ &\underline{L}_n = 0 \quad . \end{aligned} \tag{2.45}$$

4. $\underline{G}$ is the direct sum of the subalgebras

$$\underline{J}\oplus\underline{K}'\oplus\underline{N}_+ \quad . \tag{2.46}$$

5. If R,A,N are the connected subgroups corresponding respectively to the subalgebras $\underline{J}$,$\underline{K}'$,$\underline{N}_+$, then

$$G = RAN \tag{2.47}$$

and this decomposition is unique.

III. REPRESENTATIONS OF THE LORENTZ GROUP AND POINCARÉ GROUP

A. The Lie Algebra

The homogeneous Lorentz group is the group of real linear transformations ℓ on the four dimensional space (x^0, x^1, x^2, x^3) which leave the form

$$x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{3.1}$$

invariant. The metric tensor for this space is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 0 \\ -1, & \mu = \nu = 1,2,3 \\ 0, & \mu \neq \nu \end{cases} . \tag{3.2}$$

Invariance of 3.1 implies that a Lorentz transformation ℓ, where

$$x'^\mu = \ell^\mu{}_\nu \, x^\nu ,$$

must satisfy the equation

$$g^{\mu\nu} = \ell^\mu{}_\sigma \, g^{\sigma\tau} \, \ell^\nu{}_\tau \, . \tag{3.3}$$

The Poincaré group contains, in addition to Lorentz transformations, translations a.

$$x'^\mu = x^\mu + a^\mu \tag{3.4}$$

We denote elements of the Poincaré group by (a, ℓ) where a is a 4 vector and ℓ is a 4 x 4 matrix satisfying 3.3 The transformation law

$$x'^\mu = a^\mu + \ell^\mu{}_\nu \, x^\nu \tag{3.5}$$

implies a multiplication law

$$(a, \ell)(a', \ell') = (a + \ell a', \ell\ell') \tag{3.6}$$

for the Poincaré group.

We next derive the Lie algebra for the proper Lorentz group to illustrate the results of chapter 2.

A general Lorentz transformation may be written in the form

$$\ell^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + u^{\mu}{}_{\nu} \quad . \tag{3.7}$$

Only six of the numbers $u^{\mu\nu}$ are independent, since 3.3 implies the following ten relations among the $u^{\mu\nu}$.

$$u^{\mu\nu} + u^{\nu\mu} + u^{\mu\sigma} u^{\nu\gamma} g_{\sigma\gamma} = 0 \tag{3.8}$$

We choose

$$\frac{1}{4} (u^{\mu\nu} - u^{\nu\mu}) \cdot\cdot \tilde{u}^{\mu\nu}$$

to be the six independent parameters of the Lorentz group.

The parameters of the product element

$$\ell(w) = \ell(u)\ \ell(v)$$

follow from the multiplication law 3.6.

$$w^{\alpha\beta} = u^{\alpha\beta} + v^{\alpha\beta} + g_{\gamma\lambda} u^{\alpha\gamma} v^{\lambda\beta} \tag{3.9}$$

In order to find the structure constants of the Lorentz group, we construct the Taylor series for the parameters $\tilde{w}^{\alpha\beta}$.

We antisymmetrize 3.9 on α, β and use 3.8 to obtain a second order approximation for $u^{\alpha\beta}$ and $v^{\alpha\beta}$. The result is

$$\tilde{w}^{\alpha\beta} = \tilde{u}^{\alpha\beta} + \tilde{v}^{\alpha\beta}$$

$$+ (g^{\alpha}{}_{\mu}\, g^{\beta}{}_{\rho} - g^{\beta}{}_{\mu}\, g^{\alpha}{}_{\rho})\, g_{\nu\sigma}\, \tilde{u}^{\mu\nu}\, \tilde{v}^{\sigma\rho} + 0(3). \tag{3.10}$$

We may antisymmetrize the coefficient of $\tilde{u}^{\mu\nu}\, \tilde{v}^{\sigma\rho}$ on μ, ν and ρ, σ and define

$$a^{\alpha\beta}_{\mu\nu,\sigma\rho} = A(\mu,\ \nu) A(\rho,\ \sigma) (g^{\alpha}{}_{\mu}\, g^{\beta}{}_{\rho} - g^{\beta}{}_{\mu}\, g^{\alpha}{}_{\rho})\, g_{\nu\sigma} \tag{3.11}$$

where $A(\lambda,\ \tau)$ antisymmetrizes the indices λ, τ.

The structure constants are then

$$C^{\alpha\beta}_{\mu\nu,\ \sigma\rho} = a^{\alpha\beta}_{\mu\nu,\ \sigma\rho} - a^{\alpha\beta}_{\sigma\rho,\ \mu\nu} \quad . \tag{3.12}$$

We introduce a basis $-iM_{\alpha\beta} = iM_{\beta\alpha}$ for the Lie algebra. The factor $-i$ insures that $M_{\alpha\beta}$ will be Hermitian in a unitary representation of the group. In this basis we have the commutation relations

$$[M_{\mu\nu}, M_{\sigma\rho}] = i(g_{\nu\sigma} M_{\mu\rho} + g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} - g_{\mu\sigma} M_{\nu\rho}) . \tag{3.13}$$

If we introduce a basis $-iP_\mu$ for translations, we find the following commutation relations for the Poincaré group.

$$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu)$$

$$[P_\mu, P_\nu] = 0 \tag{3.14}$$

It follows from the commutation relations that the Lorentz group is a noncompact semisimple group. However the Poincaré Lie Algebra has an Abelian ideal, P_μ, and the group is therefore not semisimple.

In the usual treatment of the Lorentz group operators

$$J^i = \frac{1}{2} \epsilon^{ijk} M^{jk}$$

$$K^i = M^{oi} \tag{3.15}$$

are defined. Physically J^i is the angular momentum (generator of rotations) and K^i is the generator of pure Lorentz transformations. The commutation relations for J and K are

$$[J^i, J^j] = i \epsilon^{ijk} J^k$$

$$[J^i, K^j] = i \epsilon^{ijk} K^k$$

$$[K^i, K^j] = -i \epsilon^{ijk} J^k \quad . \tag{3.16}$$

We see from 3.16 that 3.15 is a Cartan decomposition of the Lie algebra. Cartan's theorem implies that any Lorentz transformation may be written as the product of a pure Lorentz transformation and a rotation.

$$L = e^{-i\vec{\omega}\cdot\vec{J}} \; e^{-i\vec{\alpha}\cdot\vec{K}} = R(\omega)A(\alpha) \tag{3.17}$$

If we choose K_3 as a maximum Abelian subalgebra of $\underline{K}$ we may further decompose 3.17.

$$\begin{aligned} L &= R\, e^{-i\alpha K_3}\, R' \\ &= e^{-i\phi J_3}\, e^{-i\theta J_2}\, e^{-i\Psi J_3}\, e^{-i\alpha K_3}\, e^{-i\beta J_2}\, e^{-i\gamma J_3} \end{aligned} \tag{3.18}$$

B. Representations of the Lorentz Group

We now consider unitary representations of the Lorentz group. The Lorentz group has two independent invariants. These may be chosen as $J^2 - K^2$ and $\vec{J}\cdot\vec{K}$. In an irreducible representation these operators are scalar multiples of the unit operator. Naimark (17) has shown that in unitary representations they have the values

$$\begin{aligned} J^2 - K^2 &= \ell^2 - \nu^2 - 1 \\ \vec{J}\cdot\vec{K} &= \ell\nu \end{aligned} \tag{3.19}$$

where ν is a real number and ℓ is a half integer. If $\ell = 0$, ν may also be a pure imaginary number. The first case is called the principal series, and the second is called the supplementary series.

The vectors in a representation are labeled by the rotation subgroup. We then have states $|\ell, \nu, j, m\rangle$. In this basis the generators have the following representation (17).

$$J_+|\ell, \nu, j, m\rangle = \alpha^j_{m,\, m+1}\, |\ell, \nu, j, m+1\rangle$$

$$J_-|\ell, \nu, j, m\rangle = \alpha^j_{m-1,\, m}\, |\ell, \nu, j, m-1\rangle$$

$$J_3|\ell, \nu, j, m\rangle = m|\ell, \nu, j, m\rangle$$

$$K_+|\ell, \nu, j, m\rangle = i\alpha^{j+1}_{-m,\, m+1}\, \gamma(j+1;\, \ell, \nu)|\ell, \nu, j+1, m+1\rangle$$

$$\frac{+\nu\ell}{j(j+1)}\ \alpha^{j}_{m,\ m+1}\ |\ell,\ \nu,\ j,\ m+1) + i\alpha^{j}_{m+1,-m}\ \gamma(j;\ell,\nu)\,|\ell,\nu,j-1,\ m+1\rangle$$

$$K_-\,|\ell,\ \nu,\ j,\ m\rangle = -i\alpha^{j+1}_{m-1,-m}\ \gamma(j+1;\ell,\nu)\,|\ell,\ \nu,\ j+1,\ m-1\rangle$$

$$\frac{+\nu\ell}{j(j+1)}\ \alpha^{j}_{m-1,m}|\ell,\nu,\ j,m-1\rangle - i\alpha^{j}_{-m,m-1}\ \gamma(j;\ell,\nu)\,|\ell,\nu,j-1,m-1\rangle$$

$$K_3\,|\ell,\nu,j,m\rangle = -i\alpha^{j+1}_{m,m}\ \gamma(j+1;\ \ell,\ \nu)\,|\ell,\nu,j+1,m\rangle$$

$$\frac{+m\ell\nu}{j(j+1)}\,|\ell,\nu,j,m\rangle + i\alpha^{j}_{m,m}\ \gamma(j;\ell,\nu)\,|\ell,\nu,j-1,m\rangle \tag{3.20}$$

where

$$J_{\pm} = J_1 \pm iJ_2\ ,\quad K_{\pm} = K_1 \pm iK_2$$

$$\gamma(j;\ell,\nu) = \frac{1}{j}\left[\frac{(j^2-\ell^2)(j^2+\nu^2)}{(2j+1)(2j-1)}\right]^{\frac{1}{2}}\ ,\quad \alpha^{j}_{m,n} = [(j-m)(j+n)]^{\frac{1}{2}}\ .$$

We assume that states are normalized

$$\langle \ell,\ \nu,\ j,\ m|\ell,\ \nu,\ j',\ m'\rangle = \delta_{jj'}\,\delta_{mm'} \tag{3.21}$$

and the relative phase of states in a representation is fixed by 3.20. From 3.20 we see that ℓ is the minimum value of angular momentum occurring in a representation.

The problem of finding matrix elements of an arbitrary Lorentz transformation is reduced to finding the matrix elements of pure Lorentz transformations in the z direction by the decomposition 3.18. A general matrix element then has the form

$$\langle \ell,\ \nu,\ j,\ m|L|\ell,\ \nu,\ j',\ m'\rangle = D^{\ell\nu}_{jm,j'm'}(L) =$$

$$\sum_{n=-\min(j,j')}^{\min(j,j')} D^{j}_{mn}(\phi,\theta,\psi)A^{jj'}_{n}(\alpha,\ \ell,\ \nu)\ D^{j'}_{nm'}(0,\ \beta,\ \gamma) \tag{3.22}$$

where

$$A^{jj'}_{n}(\alpha,\ \ell,\ \nu) = \langle \ell,\ \nu,\ j,\ n|e^{-i\alpha K_3}\,|\ell,\ \nu,\ j',\ n\rangle. \tag{3.23}$$

The explicit form of D^j_{mn} may be found in (18), $A^{jj'}_n$ will be calculated in chapter 5.

C. Representations of the Poincaré Group

The Poincaré group is not semisimple, since it is a semidirect product of the invariant Abelian subgroup T of translations and the Lorentz group. Representations of the Poincaré group may be constructed by using little group techniques (19).

We begin by partially labeling states of the Poincaré group by irreducible representations of the translation subgroup. Therefore we have a basis in which P^μ is diagonal i.e.

$$P^\mu |p\rangle = p^\mu |p\rangle \tag{3.24}$$

We next consider the effect of a Lorentz transformation on a fixed representation, $\tilde{p}^\mu$, of the translation group. Since the translation subgroup is invariant, we obtain either the same representation $\tilde{p}^\mu$ or a new irreducible representation p'^μ.

Those Lorentz transformations which leave $\tilde{p}^\mu$ unchanged are called the little group of $\tilde{p}^\mu$. Since $P_\mu P^\mu$ commutes with all generators of the Lorentz group, its value is invariant and characterizes the irreducible representation of the Poincaré group. Physically $P^\mu P_\mu = m^2$ is the square of the mass, and we consider the case of positive mass.

We choose a fixed $\tilde{p}^\mu$ in each representation and a fixed transformation $L(p, \tilde{p})$ for each p^μ in the representation which transforms $\tilde{p}^\mu$ into p^μ. We then have

$$\begin{aligned} |p^\mu\rangle &= L(p, \tilde{p}) \, |\tilde{p}^\mu\rangle \\ p^\mu p_\mu &= m^2 \ . \end{aligned} \tag{3.25}$$

The conventional choice for $\tilde{p}^{\mu}$ is

$$p^{\mu} = (m, o, o, o) . \tag{3.26}$$

The labeling is completed by specifying which irreducible representation of the little group is carried by $|\tilde{p}\rangle$. The little group of $\tilde{p}$ is the rotation group in three dimensions. Therefore the states are labeled by the values of J^2 and J_3 acting on $|\tilde{p}\rangle$. We define a general state by

$$|p^{\mu}, s, \lambda\rangle = L(p, \tilde{p}) | \tilde{p}, s, \lambda\rangle \tag{3.27}$$

where

$$J^2 \, | \, \tilde{p}, s, \lambda\rangle = s(s+1) \, | \, \tilde{p}, s, \lambda\rangle$$

$$J_3 \, | \, \tilde{p}, s, \lambda\rangle = \lambda \, | \, \tilde{p}, s, \lambda\rangle . \tag{3.28}$$

We consider two cases for $L(p, \tilde{p})$. In the first case we choose a pure Lorentz transformation, i.e.

$$L(p, \tilde{p}) = L(p) = e^{-i\vec{K}\cdot\hat{p}\alpha} \tag{3.29}$$

where

$$p^{\mu} = m(\cosh\alpha, \hat{p} \sinh\alpha) .$$

This basis will be called the canonical basis for the Poincaré group.

In the second case we choose $L(p, \tilde{p})$ to be the product of a rotation and an acceleration in the z direction.

$$H(p) = e^{-iJ_3\phi} e^{-iJ_2\theta} e^{-iK_3\alpha} \tag{3.30}$$

where

$$p^{\mu} = m(\cosh\alpha, \sinh\alpha \sin\theta \cos\phi, \sinh\alpha \sin\theta \sin\phi, \sinh\alpha \cos\theta) .$$

This basis is called the helicity representation, since $\vec{J} \cdot \vec{P}$ is diagonal and has the value $\lambda|\vec{p}|$.

The irreducible representations of the Poincaré group are therefore characterized by two numbers m, the mass, and s, the angular momentum of a state of zero three momentum. Physically s is the spin of a particle having this representation.

We assume that states in an irreducible representation are invariantly normalized

$$\langle p, s, \lambda | p', s, \lambda' \rangle = 2p_o \, \delta_{\lambda\lambda'} \, \delta^3 (\vec{p} - \vec{p}\,') \, . \tag{3.31}$$

If ϕ_λ is a wave function defined by

$$\phi_\lambda (p) = \langle p, s, \lambda | \phi \rangle \tag{3.32}$$

then the inner product of two wave functions is

$$(\phi, \phi') = \sum_\lambda \int \frac{d^3 p}{2P_o} \, \phi^*_\lambda (p) \, \phi'_\lambda (p) \, . \tag{3.33}$$

The transformation properties of the basis states under an arbitrary element of the Poincaré group may be derived by using the Cartan decomposition of the Lorentz group. If L is an arbitrary Lorentz transform we have

$$\begin{aligned} L \, L(p) &= L(p') \, R_C \\ L \, H(p) &= H(p') \, R_H \end{aligned} \tag{3.34}$$

where L(p) and H(p) are defined by 3.29 and 3.30.

Therefore the basis states transform as follows

$$\begin{aligned} e^{iP\cdot a} \, L|p, s, \lambda\rangle_C &= e^{ip'\cdot a} \sum_{\lambda'} D^s_{\lambda'\lambda} (R_C) \, |p', s, \lambda'\rangle_C \\ e^{iP\cdot a} \, L|p, s, \lambda\rangle_H &= e^{ip'\cdot a} \sum_{\lambda'} D^s_{\lambda'\lambda} (R_H) \, | p', s, \lambda'\rangle_H \, . \end{aligned} \tag{3.35}$$

Finally we consider the spin operator S^k. The spin operator is defined to be

$$S^k = L(p) \, J^k \, L(p)^{-1} \tag{3.36}$$

when acting on a basis state of momentum p.

It follows from 3.29 and 3.30 that S^3 is diagonal in the canonical basis and $\vec{P} \cdot \vec{S}$ is diagonal in the helicity basis.

$$S^3 |p, s, \lambda\rangle_C = \lambda |p, s, \lambda\rangle_C$$

$$\vec{P} \cdot \vec{S} |p, s, \lambda\rangle_H = |\vec{P}| \lambda |p, s, \lambda\rangle_H \tag{3.37}$$

With the aid of the operator identity

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \ldots . \tag{3.38}$$

we may express 3.36 as

$$m\vec{S} = P^o \vec{J} - \vec{P} \times \vec{K} - \frac{\vec{P}(\vec{P} \cdot \vec{J})}{m + P_o} . \tag{3.39}$$

The following commutation relations for S^k follow at once from 3.36.

$$[S^i, S^j] = i\epsilon^{ijk} S^k$$

$$[S^i, P^\mu] = 0 \tag{3.40}$$

IV. AN IWASAWA DECOMPOSITION OF THE LORENTZ GROUP

A. The Lie Algebra

The sets $\{J_i\}$ and $\{K_i\}$ defined by 3.15, form a Cartan decomposition of the Lorentz group. If we define

$$P = J_1 + K_2 \quad , \qquad Q = J_2 - K_1 \quad , \tag{4.1}$$

we see that the sets $\{J_i\}$, $\{K_3\}$ and $\{P, Q\}$ form an Iwasawa decomposition of the Lie algebra. Therefore every proper Lorentz transformation may be decomposed uniquely in the form

$$L = RAN \tag{4.2}$$

where R is a rotation, A is an acceleration in the z direction, and

$$N = e^{-i(aP + bQ)} . \tag{4.3}$$

In the new basis the commutation relations are

$$\begin{array}{lll} [P, Q] = 0 & [J_3, P] = iQ & [K_3 P] = iP \\ [J_3, K_3] = 0 & [J_3, Q] = -iP & [K_3, Q] = iQ \\ [J_1, P] = iK_3 & [J_2, P] = -iJ_3 & [J_1, K_3] = i(J_1 - P) \\ [J_1, Q] = iJ_3 & [J_2, Q] = iK_3 & [J_2, K_3] = i(J_2 - Q) \end{array}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad . \tag{4.4}$$

From 4.4 we see that $\{P, Q, J_3, K_3\}$ generate a subgroup. This subgroup is isomorphic with the conformal group in a plane. P,Q generate translations, J_3 generates rotations in the plane and K_3 generates scale changes. In this chapter we construct representations of the Lorentz group in a basis labeled by this subgroup.

B. Representations of the Conformal Group

In this section we construct representations of the group generated by $\{P, Q, J_3, K_3\}$.

It follows from the commutation relations that $\exp(i2\pi J_3)$ is an invariant of the group. Only single and double valued representations occur in the Lorentz group, so we discuss the cases

$$e^{i2\pi J_3} = \epsilon = \pm 1 \ . \tag{4.5}$$

We take a basis in which J_3 and $P^2 + Q^2$ are diagonal. In this basis we have

$$J_3 \,|\, r, \lambda, \epsilon\rangle = \lambda \,|\, r, \lambda, \epsilon\rangle \tag{4.6}$$

where
$$\lambda = 0, \pm 1, \pm 2 \ldots \quad \text{for } \epsilon = 1$$
$$\lambda = \pm \frac{1}{2}, \pm \frac{3}{2} \ldots \quad \text{for } \epsilon = -1$$

and
$$(P^2 + Q^2) \,|\, r, \lambda, \epsilon\rangle = r^2 |\, r, \lambda, \epsilon\rangle$$
$$0 < r < \infty \ . \tag{4.7}$$

The commutation relations 4.4 and the identity

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots \tag{4.8}$$

imply
$$e^{-iK_3\alpha} P_{\pm} e^{iK_3\alpha} = e^{\alpha} P_{\pm}$$
$$[J_3, P_{\pm}] = \pm P_{\pm} \tag{4.9}$$

where
$$P_{\pm} = P \pm iQ \ .$$

We make a choice of phase for states in a representation by taking $|1, 0, 1\rangle$ and $|1, \frac{1}{2}, -1\rangle$ as standard states in the single and double valued representations and defining a general state by

$$|e^{\alpha}, \lambda, \epsilon\rangle = e^{i\alpha K_3}\,|1, \lambda, \epsilon\rangle$$

$$P_{\pm}\,|r, \lambda, \epsilon\rangle = r|r, \lambda \pm 1, \epsilon\rangle\,. \tag{4.10}$$

We assume states are invariantly normalized.

$$\langle r, \lambda, \epsilon\,|\,r', \lambda', \epsilon\rangle = r\,\delta(r - r')\,\delta_{\lambda\lambda'} \tag{4.11}$$

For wave functions $\phi_\lambda(r) = \langle r, \lambda\,|\,\phi\rangle$

the inner product is then

$$\langle\phi\,|\,\phi'\rangle = \sum_\lambda \int_0^\infty \frac{dr}{r}\,\phi_\lambda^*(r)\,\phi_\lambda'(r)\,. \tag{4.12}$$

We now calculate matrix elements of finite transformations. A general transformation may be written in the form

$$T(\theta, \alpha, a, b) = e^{-i\theta J_3}\,e^{-i\alpha K_3}\,e^{-i(aP + bQ)}. \tag{4.13}$$

The result of $\exp[-i(aP + bQ)]$ acting on a state is found by expanding the exponential.

$$e^{-i(aP + bQ)}|r, \lambda\rangle = e^{-\gamma P_+ + \gamma^* P_-}\,|r, \lambda\rangle$$

where

$$\gamma = \frac{b + ia}{2}$$

We use the fact that P_+ commutes with P_- to obtain

$$\sum_{n,m} \frac{(-\gamma P_+)^m}{m!}\frac{(\gamma^* P_-)^n}{n!}\,|r, \lambda\rangle\,. \tag{4.14}$$

We use 4.10 and define a new index of summation, $\lambda' = \lambda + m - n$ in order to write 4.14 in the form

$$\sum_{\lambda'=-\infty}^{\infty}\sum_{n=0}^{\infty}\left(\frac{-\gamma}{|\gamma|}\right)^{\lambda'-\lambda}\frac{(-1)^n(|\gamma|\,r)^{2n+\lambda'-\lambda}}{n!\,(n+\lambda'-\lambda)!}\,|r, \lambda'\rangle\,. \tag{4.15}$$

The final result follows from the definition of Bessel's function (20).

$$e^{-i(aP + bQ)} |r, \lambda\rangle =$$

$$\sum_{\lambda'} \left(\frac{-b-ia}{\sqrt{a^2 + b^2}}\right)^{\lambda'-\lambda} J_{\lambda'-\lambda} (\sqrt{a^2 + b^2}\ r)\ |r, \lambda'\rangle \qquad (4.16)$$

The matrix element of a general transformation follows from 4.6, 4.10 and 4.16.

$$\langle r', \lambda' \mid T(\theta, \alpha, a, b) \mid r, \lambda\rangle =$$

$$r'\delta(r' - e^{-\alpha} r)\ e^{-i\lambda'\theta} \left(\frac{-b-ia}{\sqrt{a^2 + b^2}}\right)^{\lambda'-\lambda} J_{\lambda'-\lambda} (\sqrt{a^2 + b^2}\ r) \qquad (4.17)$$

C. Representations of the Lorentz Group in a Conformal Group Basis

We now construct representations of the Lorentz group in a basis of the form

$$|\ell, \nu, r, \lambda\rangle$$

where r, λ label states in a representation. The representations are characterized by the two invariants

$$\ell\nu = \vec{J} \cdot \vec{K} \qquad \text{and} \qquad \ell^2 - \nu^2 - 1 = J^2 - K^2 . \qquad (4.18)$$

In terms of the basis $\{J_i, K_3, P, Q\}$ these are

$$(\ell \pm i\nu)^2 = (iK_3 \pm J_3 - 1)^2 - P^2 - Q^2 + 2J_\pm P_\mp \qquad (4.19)$$

where

$$J_\pm = J_1 \pm iJ_2 .$$

The effect of $J_\pm$ acting on a state can be related to K_3 by using 4.10 and 4.19.

$$J_\pm |\ell, \nu, r, \lambda\rangle =$$

$$\frac{1}{2r}\,[(\ell \pm i\nu)^2 + r^2 - (iK_3 \pm \lambda)^2] \quad |\ell, \nu, r, \lambda \pm 1\rangle \qquad (4.20)$$

Iwasawa's decomposition, 4.2, implies that any Lorentz transformation may be written in the form

$$L = e^{-i\Psi J_3}\, e^{-i\phi J_2} \quad T(\theta, \alpha, a, b) \qquad (4.21)$$

where T is defined by 4.13 and its matrix elements are given by 4.17.

We calculate the matrix elements of the rotation about the y axis by expanding in an angular momentum basis. The result is

$$\langle \ell, \nu, r', \lambda' \mid e^{-i\phi J_2} \mid \ell, \nu, r, \lambda\rangle =$$

$$\sum_{j=\max(\lambda,\lambda')}^{\infty} \langle \ell, \nu, r', \lambda' | \ell, \nu, j, \lambda'\rangle\, d^{j}_{\lambda'\lambda}(\phi)\, \langle \ell, \nu, j, \lambda | \ell, \nu, r, \lambda\rangle . \qquad (4.22)$$

The explicit form of the expansion coefficient $\langle \ell, \nu, r, \lambda | \ell, \nu, j, \lambda \rangle$ is given in the next section.

Since the left side of 4.22 is defined to be $\delta_{\lambda\lambda'}\, r\delta(r-r')$ when $\phi = 0$, the right side must be considered to be a distribution. In other words if $\Psi(r)$ is a square integrable function, we give the right side of 4.22 meaning by making the definition

$$\int_0^{\infty} \frac{dr}{r} \langle \ell, \nu, r', \lambda' \mid e^{-i\phi J_2} | \ell, \nu, r, \lambda\rangle\, \Psi(r) =$$

$$\sum_{j=\max(\lambda,\lambda')}^{\infty} \langle \ell, \nu, r', \lambda' \mid \ell, \nu, j, \lambda'\rangle \; d^{j}_{\lambda'\lambda}(\phi)$$

$$\int_0^{\infty} \frac{dr}{r} \langle \ell, \nu, j, \lambda | \ell, \nu, r, \lambda\rangle\, \Psi(r) . \qquad (4.23)$$

D. Expansion Coefficients

In this section we derive and solve differential equations for the expansion coefficients relating the conventional angular momentum basis to the conformal group basis. We introduce the notation

$$\langle \ell, \nu, j, m | \ell, \nu, r, m \rangle = f^j_m (r, \ell, \nu) \tag{4.24}$$

for the expansion coefficient.

It follows from 4.10 that

$$\frac{r}{i} \frac{\partial}{\partial r} \langle \ell, \nu, j, m | \ell, \nu, r, m\rangle = \langle \ell, \nu, j, m | K_3 | \ell, \nu, r, m\rangle. \tag{4.25}$$

Two differential relations for f^j_m may be derived by multiplying 4.20 on the left by an angular momentum basis state.

$$\langle \ell, \nu, j, m | J\pm | \ell, \nu, r, \lambda\rangle =$$

$$\langle \ell, \nu, j, m | \frac{1}{2r} [(\ell \pm i\nu)^2 + r^2 - (iK_3 \pm \lambda)^2] | \ell, \nu, r, \lambda \pm 1\rangle \tag{4.26}$$

With the aid of 4.24 and 4.25 we obtain

$$2r\sqrt{(j \pm m + 1)(j \mp m)} \quad f^j_{m\pm 1} =$$

$$[(\ell \mp i\nu)^2 + r^2 - (r \frac{\partial}{\partial r} \mp m - 1)^2] \; f^j_m \; . \tag{4.27}$$

In the cases m = j and m = -j 4.27 becomes a second order differential equation since

$$f^j_{j+1} = f^j_{-j-1} = 0 \, .$$

The solutions are

$$f^j_j (r, \ell, \nu) = C_j \; r^{j+1} \; Z_{\ell - i\nu}(r)$$

$$f^j_{-j} (r, \ell, \nu) = C'_j \; r^{j+1} \; Z_{\ell + i\nu} (r) \tag{4.28}$$

where Z_μ is a modified Bessel function. Since the angular momentum states are normalized, we have

$$\langle \ell, \nu, j, m | \ell, \nu, j', m'\rangle = \delta_{jj'} \delta_{mm'} =$$

$$\sum_\lambda \int_0^\infty \frac{dr}{r} \langle \ell, \nu, j, m | \ell, \nu, r, \lambda\rangle\langle \ell, \nu, r, \lambda | \ell, \nu, j', m'\rangle$$

$$= \delta_{mm'} \int_0^\infty \frac{dr}{r} f_m^j (r) f_m^{j'*}(r) . \qquad (4.29)$$

Therefore $r^{-1/2} f_m^j$ must be square integrable and Z_μ is a modified Bessel function of the third kind (20).

The general expansion coefficient is found by using 4.27 as ladder operators on the functions f_j^j and f_{-j}^j. The result, which is proved by induction in appendix A, is

$$f_m^j(r, \ell, \nu) =$$

$$B(\ell, \nu, j, m) \sum_{n=o} \frac{(m-j, n)(\ell-j, n)(-i\nu-j, n)(-2)^n r^{j+1-n}}{(1, n)(-2j, n)} K_{\ell-i\nu-j+m+n}(r)$$

$$= B(\ell, \nu, j, m) \sum_{n=o} \frac{(-m-j, n)(\ell-j, n)(i\nu-j, n)(-2)^n r^{j+1-n}}{(1, n)(-2j, n)} K_{\ell+i\nu-j-m+n}(r)$$

where $(a, n) = a(a+1) \; . \; . \; . \; . \; (a+n-1)$

$$B(\ell, \nu, j, m) =$$

$$\left[\frac{2j!\ (2j+1)!}{\Gamma(j+i\nu+1)\Gamma(j-i\nu+1)(j+m)!(j-m)!(j+\ell)!(j-\ell)!\ 2^{2j-1}}\right]^{\frac{1}{2}} . \qquad (4.30)$$

We list several properties of the functions f_m^j. From 4.30 we note that

$$\begin{aligned} f_m^j(r, \ell, \nu) &= f_{-m}^j(r, \ell, -\nu) \\ &= f_\ell^j(r, m, \nu) \\ &= f_{-m}^{\ *}(r, \ell, \nu) \\ &= f_{-i\nu}^j(r, \ell, im) \quad . \qquad (4.31) \end{aligned}$$

The functions f_m^j have the following integral representation which is derived in appendix B.

$$f^j_m(r, \ell, \nu) =$$

$$C(\ell, \nu, j, m) r^{\ell-m+i\nu+1} \int_0^\infty dt\, e^{-t-r^2/4t}\, t^{m-i\nu-1}\, {}_2F_2 \left({j+\ell+1,\ \ell-j; \atop m+\ell+1,\ \ell-i\nu+1;}\ t \right)$$

where

$$C(\ell, \nu, j, m) = \frac{(-1)^{j-\ell} 2^{m-\ell-i\nu} \Gamma(j-i\nu+1)}{(\ell + m)!\ \Gamma(\ell-i\nu+1)}$$

$$\left[\frac{(j+m)!\ (j+\ell)!\ (2j+1)}{2\Gamma(j-i\nu+1)\Gamma(j+i\nu+1)(j-m)!(j-\ell)!}\right]^{\frac{1}{2}} \qquad (4.32)$$

and ${}_2F_2$ is a hypergeometric function (21).

According to 4.29 the expansion coefficients satisfy the orthogonality relation

$$\int_0^\infty \frac{dr}{r} f^{j\,*}_m(r, \ell, \nu)\, f^{j'}_m(r, \ell, \nu) = \delta_{jj'} \quad . \qquad (4.33)$$

Finally we list an additional orthogonality relation which is proven in chapter 6.

$$\sum_{m=-j}^{j} \int_0^\infty \frac{dr}{r^3}\ f^{j*}_m(r, \ell, \nu)\ f^j_m(r, \ell', \nu')$$

$$= \frac{\pi(2j+1)}{\ell^2 + \nu^2}\ [\delta_{\ell\ell'}\ \delta(\nu-\nu') + \delta_{\ell-\ell'}\ \delta(\nu+\nu')]$$

V. MATRIX ELEMENTS OF THE LORENTZ GROUP IN AN ANGULAR MOMENTUM BASIS

A. Matrix Elements of Pure Lorentz Transformations

The results of chapter 3 reduce the problem of finding matrix elements of an arbitrary Lorentz transformation to that of finding matrix elements of an acceleration in the z direction.

In this section we calculate matrix elements of pure Lorentz transformations in the z direction by two methods. The first method is simple and the resulting expression exhibits the index symmetries of the matrix element explicitly. The second method yields an expression which is useful when a state of low angular momentum is involved in the matrix element.

We calculate the matrix element

$$A_m^{jj'}(\alpha, \ell, \nu) = \langle \ell, \nu, j, m | e^{-i\alpha K_3} | \ell, \nu, j', m \rangle \qquad (5.1)$$

by transforming to a conformal group basis. The result, which follows from 4.10 is

$$A_m^{jj'}(\alpha, \ell, \nu) = \int_0^\infty \frac{dr}{r} f_m^j(r) f_m^{j'*}(yr) \qquad (5.2)$$

where $y = e^\alpha$.

It is clear from 5.2 that the index symmetries

$$f_m^j(r, \ell, \nu) = f_\ell^j(r, m, \nu) = f_{-m}^j(r, \ell, -\nu) = f_{-m}^{j*}(r, \ell, \nu) \qquad (5.3)$$

imply corresponding symmetries for $A_m^{jj'}$.

$$A_m^{jj'}(\alpha, \ell, \nu) = A_\ell^{jj'}(\alpha, m, \nu) = A_{-m}^{jj'}(\alpha, \ell, -\nu) = A_{-m}^{jj'}(\alpha, \ell, \nu)^* \qquad (5.4)$$

Two additional symmetries may be derived for $A_m^{jj'}$. If we make the change of variable $r' = yr$ in 5.2 we find

$$A_m^{jj'}(\alpha, \ell, \nu) = A_{-m}^{j'j}(-\alpha, \ell, \nu) \quad . \tag{5.5}$$

If we substitute the identity

$$e^{i\pi J_2}\, e^{-i\alpha K_3}\, e^{-i\pi J_2} = e^{i\alpha K_3} \tag{5.6}$$

in 5.1 we obtain

$$A_m^{jj'}(\alpha, \ell, \nu) = (-1)^{j-j'}\, A_{-m}^{jj'}(-\alpha, \ell, \nu). \tag{5.7}$$

We next substitute the explicit form of the expansion coefficient f_m^j in 5.2 and evaluate the integral. Since two expressions for f_m^j are given in chapter 4, the integral may be evaluated in four ways. However the symmetry

$$A_m^{jj'}(\alpha, \ell, \nu) = A_{-m}^{jj'}(\alpha, \ell, -\nu)$$

reduces this to two cases.

We choose these to be (a) f_m^j and $f_m^{j'}$ are both of the first form in 4.30 and (b) f_m^j is of the first form and $f_m^{j'}$ is of the second form. We evaluate the first case below. The second case is similar and we list the result.

We substitute 4.30 in 5.2 to obtain

$$A_m^{jj'}(\alpha, \ell, \nu) = B(\ell, \nu, j, m) B(\ell, \nu, j', m) \sum_{n=o} \sum_{p=o}$$

$$(-2)^{n+p} \frac{(m-j,n)(\ell-j,n)(-i\nu-j,n)(m-j',p)(\ell-j',p)(i\nu-j',p)}{(1,n)\quad(-2j,n)\quad(1,p)\quad(-2j',p)}$$

$$y^{j'+1-p} \int_o^\infty r^{j+j'+1-n-p}\, K_{\ell-i\nu-j+m+n}(r)\, K_{\ell+i\nu-j'+m+p}(yr)\, dr \quad . \tag{5.8}$$

The integral in 5.8 is evaluated with the aid of the formula (20)

$$\int_0^\infty t^{-\lambda} K_\mu(at)\, K_\nu(bt)\, dt =$$

$$(\tfrac{a}{2})^{\lambda-1} (\tfrac{b}{a})^\nu\, \Gamma(\tfrac{1+\nu+\mu-\lambda}{2})\, \Gamma(\tfrac{1+\nu-\mu-\lambda}{2})\, \Gamma(\tfrac{1-\nu+\mu-\lambda}{2})$$

$$\frac{\Gamma(\frac{1-\nu-\mu-\lambda}{2})}{8\Gamma(1-\lambda)}\ {}_2F_1\left(\frac{1+\nu+\mu-\lambda}{2}, \frac{1+\nu-\mu-\lambda}{2} ; 1-\lambda ; 1-\frac{b^2}{a^2}\right)$$

$$1-\mathrm{Re}\lambda-|\mathrm{Re}\mu|-|\mathrm{Re}\nu|>0. \qquad (5.9)$$

The final result is

$$A_m^{jj'}(\alpha, \ell, \nu) = N(\ell, \nu, j, j', m) \sum_{n=o}^{\min{j-\ell \atop j-m}} \sum_{p=o}^{\min{j'-\ell \atop j'-m}}$$

$$\frac{(-j-j'-1, n+p)(m-j, n)(\ell-j, n)(m-j', p)(\ell-j', p)}{(-j-j'+\ell+m, n+p)(1, n)(-2j, n)(1, p)(-2j', p)}$$

$$y^{\ell+i\nu+m+1}\ {}_2F_1(\ell+m+1, j+i\nu+1-n; j+j'+2-n-p; 1-y^2)$$

valid for $\ell + m \geq 0$ (5.10)

where

$$N(\ell, \nu, j, j', m) = \frac{(\ell+m)!\ (j+j'-\ell-m)!}{(j+j'+1)!\ \Gamma(j-i\nu+1)\ \Gamma(j'+i\nu+1)}$$

$$\left[\frac{2j!(2j+1)!\ \Gamma(j+i\nu+1)\ \Gamma(j-i\nu+1)\ 2j'!(2j'+1)!\ \Gamma(j'+i\nu+1)\Gamma(j'-i\nu+1)}{(j+m)!\ (j-m)!\ (j+\ell)!\ (j-\ell)!\ (j'+m)!(j'-m)!\ (j'-\ell)!\ (j'+\ell)!}\right]^{\frac{1}{2}}$$

The second case is evaluated in the same way. The result is

$$A_m^{jj'}(\alpha, \ell, \nu) = N'(\ell, \nu, j, j', m) \sum_{n=o}^{j-m} \sum_{p=o}^{j'+m}$$

$$\frac{(-j-j'-1,n+p)(\ell-j,n)(-i\nu-j,n)(\ell-j',p)(-i\nu-j',p)}{(-j-j'+\ell-i\nu,n+p)(1,n)(-2j,n)(1,p)(-2j',p)}$$

$$y^{\ell-m-i\nu+1}\,{}_2F_1\,(1+\ell-i\nu,\ j+1-m-n;\ j+j'+2-n-p;\ 1-y^2) \tag{5.11}$$

where

$$N'(\ell,\ \nu,\ j,\ j',\ m) = \frac{\Gamma(1+\ell-i\nu)\ \Gamma\ (j+j'+1-\ell+i\nu)}{(j+j'+1)!}$$

$$\left[\frac{(j-m)!\ 2j!\ (2j+1)! \qquad (j'+m)!\ 2j'!\ \ (2j'+1)!}{(j+m)!(j+\ell)!(j-\ell)!\Gamma(j+i\nu+1)\Gamma(j-i\nu+1)(j'-m)!(j'+\ell)!(j'-\ell)!\Gamma(j'+i\nu+1)\Gamma(j'-i\nu+1)}\right]^{\frac{1}{2}}$$

We note that while the above results were derived for the principal series they are also valid for the supplementary series. It is interesting to note that if the factorials are expressed as gamma functions, equation 5.11 may be obtained from 5.10 by the substitution

$$A_m^{jj'}(\alpha,\ \ell,\ \nu) = A_{-i\nu}^{jj'}\ (\alpha,\ \ell,\ im)\ .$$

Other forms may be obtained with the aid of symmetries expressed in 5.4, 5.5 and 5.7.

We now derive an alternative expression for $A_m^{jj'}$. This derivation, which is not as straight forward as the previous one, depends on the result

$$e^{iK_3\alpha}\ J_+e^{-iK_3\alpha} = J_+e^{\alpha} - P_+\ \sinh\alpha\ . \tag{5.12}$$

The proof of 5.12 follows from 4.4 and 4.8.

The matrix element A_m^{js} $(\alpha,\ \ell,\ \nu)$ may be written as

$$A_m^{js}\ (\alpha,\ell,\nu) = \left[\frac{(j+m)!\,(s-m)!}{(j-m)!\,(s+m)!\,2j!\,2s!}\right]^{\frac{1}{2}}\ \langle j,j\,|J_+^{j-m}\ e^{-i\alpha K_3}J_+^{\ s+m}\ |s,-s\rangle \tag{5.13}$$

by using the properties of J_+ and J_-.

At this point we use 5.12 to move the exponential in 5.13 to the left of the ladder operators. The matrix element in 5.13 then becomes

$$\langle j,j|J_+^{\,j-m}\, e^{-i\alpha K_3}\, J_+^{\,s+m}\,|s,\,-s\rangle =$$

$$\langle j,j|e^{-i\alpha K_3}\,(e^{\alpha}J_+ - P_+\sinh\alpha)^{j-m}\,J_+^{\,s+m}\,|s,\,-s\rangle\ . \tag{5.14}$$

Since J_+ commutes with P_+ 5.14 may be expressed as

$$\sum_{q=o}^{j-m}\binom{j-m}{q}\, e^{\alpha(j-m-q)}(-\sinh\alpha)^q\,\langle j,j|e^{-i\alpha K_3}\,P_+^{\,q}\,J_+^{\,j+s-q}\,|s,\,-s\rangle \tag{5.15}$$

where $\binom{j-m}{q}$ is a binomial coefficient.

We note that if $j > s$, the sum begins at $q = j - s$ since

$$J_+^{\,n}\,|s,\,-s\rangle = 0 \quad \text{for } n > 2s.$$

We change the index of summation to $p = q + s - j$ and obtain

$$\sum_{p=o}^{s-m}\binom{j-m}{j-s+p}\, e^{\alpha(s-m-p)}\,(-\sinh\alpha)^{j-s+p}\,\langle j,j|e^{-i\alpha K_3}\,P_+^{\,j-s+p}\,J_+^{\,2s-p}\,|s,\,-s\rangle . \tag{5.16}$$

5.16 is also valid for $s > j$ since the binomial coefficient is zero unless $j - s + p \geq 0$.

We use the result

$$J_+^{\,2s-p}\,|s,\,-s\rangle = \left[\frac{(2s-p)!\;2s!}{p!}\right]^{\frac{1}{2}}\,|s,\,s-p\rangle$$

and 5.13 to obtain the expression

$$A_m^{js}(\alpha,\,\ell,\,\nu) = \left[\frac{(j+m)!\,(s-m)!}{(s+m)!\,(j-m)!\,2j!}\right]^{\frac{1}{2}}\sum_{p=o}^{s-m}\binom{j-m}{j-s+p}$$
$$\left[\frac{(2s-p)!}{p!}\right]^{\frac{1}{2}}\, e^{\alpha(s-m-p)}(-\sinh\alpha)^{j-s+p}\,\langle j,j|e^{-i\alpha K_3}\,P_+^{\,j-s+p}\,|s,\,s-p\rangle . \tag{5.17}$$

We calculate the matrix element in 5.17 by inserting a complete set of conformal group states. The result is

$$\langle jj | e^{-iK_3\alpha} P_+^{j+p-s} | s, s-p \rangle =$$

$$e^{\alpha(j-s+p)} \int_0^\infty f_j^j(r)\, f_{s-p}^{s*}(yr)\, r^{j-s+p-1}\, dr$$

where $\quad y = e^{\alpha}$. (5.18)

We evaluate the integral in 5.18 by substituting the explicit form of f_m^j. The final result is

$$A_m^{js}(\alpha, \ell, \nu) = M(\ell, \nu, j, s, m) y^{2s+\ell+i\nu-m+1} (1-y^2)^{j-s}$$

$$\sum_{p=0}^{s-m} \sum_{n=0}^{p} \binom{j-m}{j-s+p} \frac{(\ell-s,n)(i\nu-s,n)(j+1-i\nu,\ p-n)(j-\ell+1,\ p-n)}{(1,n)(-2s,n)(1,p-n)(2j+2,\ p-n)}$$

$$\left(\frac{1-y^2}{2}\right)^p {}_2F_1 \left(j+\ell+1,\ j+i\nu+1;\ 2j+2+p-n;\ 1-y^2\right) \qquad (5.19)$$

where

$$M(\ell, \nu, j, s, m) =$$

$$\left[\frac{(j+m)!\,(s-m)!\ 2s!\ (2s+1)!\ (j+\ell)!\,(j-\ell)!\ \Gamma(j+i\nu+1)\ \Gamma(j-i\nu+1)}{(s+m)!\,(j-m)!\ 2j!\ (2j+1)!\ (s+\ell)!\,(s-\ell)!\ \Gamma(s+i\nu+1)\ \Gamma(s-i\nu+1)}\right]^{\frac{1}{2}} .$$

Several remarks concerning 5.19 are in order. We note that any matrix element can be related to the case $j \geq s > m \geq \ell > 0$ by using 5.4, 5.5 and 5.7. This results in the minimum number of terms in 5.19. For example if $s = m$ we have

$$A_s^{js}(\alpha, \ell, \nu) = M(\ell, \nu, j, s, s)\ y^{s+\ell+i\nu+1}\ (1-y^2)^{j-s}$$
$$ {}_2F_1(j+\ell+1,\ j+i\nu+1;\ 2j+2;\ 1-y^2) \ . \qquad (5.20)$$
$$j > s$$

It is clear that 5.19 correctly satisfies

$$A_m^{js}(0, \ell, \nu) = \delta_{js} \quad .$$

B. An Integral Representation

An integral representation for $A_m^{jj'}$ may be derived by substituting the integral representation of f_m^j in 5.2 and changing the order of integration. This integral representation will be useful in discovering the asymptotic behavior of $A_m^{jj'}$.

Upon substitution of 4.32 in 5.2 we obtain

$$A_m^{jj'}(\alpha, \ell, \nu) = C(\ell, \nu, j, m)\, C^*(\ell, \nu, j', m)\, y^{\ell-m-i\nu+1}$$

$$\int_0^\infty dr \int_0^\infty dt \int_0^\infty ds\; e^{-s-t-\frac{r^2}{4t} - \frac{y^2r^2}{4s}}\; r^{2\ell-2m+1}$$

$$t^{m-i\nu-1}\; {}_2F_2\left({j+\ell+1,\ -j+\ell\ ; \atop \ell+m+1,\ \ell-i\nu+1;}\ t\right)$$

$$s^{m+i\nu-1}\; {}_2F_2\left({j'+\ell+1,\ \ell-j'\ ; \atop \ell+m+1,\ \ell+i\nu+1;}\ s\right) \quad . \tag{5.21}$$

$\ell \geq m \geq -\ell\ ,\quad y > 0$

The order of integration may be changed and the r integration performed. The result is the integral representation

$$A_m^{jj'}(\alpha, \ell, \nu) = Q(\ell, \nu, j, j', m)\, y^{\ell-i\nu-m+1} \int_0^\infty dt \int_0^\infty ds$$

$$\frac{e^{-s-t}\, t^{\ell-i\nu}\, s^{\ell+i\nu}}{(s+y^2t)^{\ell-m+1}}\; {}_2F_2\left({j+\ell+1,\ \ell-j; \atop \ell+m+1,\ \ell-i\nu+1;}\ t\right)\; {}_2F_2\left({j'+\ell+1,\ \ell-j'; \atop \ell+m+1,\ \ell+i\nu+1;}\ s\right) \tag{5.22}$$

where

$Q(\ell, \nu, j, j', m) =$

$$\frac{\Gamma(j-i\nu+1)\ \Gamma(j'+i\nu+1)(\ell-m)!\ (-1)^{j-j'}}{\Gamma(\ell-i\nu+1)\ \Gamma(\ell+i\nu+1)\ [(\ell+m)!]^2}$$

$$\left[\frac{(2j+1)(2j'+1)(j+m)!\ (j+\ell)!(j'+m)!(j'+\ell)!}{(j-m)!(j-\ell)!(j'-m)!(j'-\ell)!\ \Gamma(j-i\nu+1)\ \Gamma(j+i\nu+1)\ \Gamma(j'-i\nu+1)\ \Gamma(j'+i\nu+1)}\right]^{\frac{1}{2}}$$

Valid for ν real, $y > 0$, $\ell \geq |m|$.

The case $\ell < |m|$ follows from the index symmetries 5.4.

C. Asymptotic Behavior of Matrix Elements

The asymptotic form of $A_m^{jj'}$ may be derived from 5.22 by noting that interchange of integration is also justifiable in the case $\ell > m > 0$, $y = 0$ and $y^{m+i\nu-\ell+1}\ A_m^{jj'}$ is continuous at $y = 0$. The result is

$$\lim_{y\to 0}\ y^{m+i\nu-\ell-1}\ A_m^{jj'}(\alpha,\ \ell,\ \nu)\ =$$

$$Q(\ell,\ \nu,\ j,\ j',\ m)\int_0^\infty e^{-t}t^{\ell-i\nu}\ {}_2F_2\left({j+\ell+1,\ \ell-j; \atop \ell+m+1,\ \ell-i\nu+1;}\ t\right)$$

$$\int_0^\infty e^{-s}\ s^{m+i\nu-1}\ {}_2F_2\left({j'+\ell+1,\ \ell-j'\ ; \atop \ell+m+1,\ \ell+i\nu+1\ ;}s\right) . \tag{5.23}$$

The integrals in 5.23 are Laplace transforms and may be evaluated (22). We obtain

$$Q(\ell,\ \nu,\ j,\ j',\ m)\ \Gamma(\ell-i\nu+1)\ \Gamma(m+i\nu)$$

$${}_2F_1(j+\ell+1,\ \ell-j;\ \ell+m+1;\ 1)$$

$${}_3F_2\left({j'+\ell+1,\ \ell-j',\ m+i\nu \atop \ell+m+1,\ \ell+i\nu+1}\ ;\ 1\right) . \tag{5.24}$$

The hypergeometric functions may be expressed as Γ functions (21).

The result is

$$\lim_{y\to 0} y^{m+i\nu-\ell-1} A^{jj'}_m(\alpha, \ell, \nu) =$$

$$\left[\frac{(j-m)!\ (j+\ell)!\ (j'-m)!\ (j'+\ell)!\ (2j+1)(2j'+1)}{(j-\ell)!\ (j+m)!\ (j'-\ell)!\ (j'+m)!}\right]^{\frac{1}{2}}$$

$$\frac{\Gamma(m+i\nu)\ \Gamma(j-i\nu+1)\ \Gamma(j'-i\nu+1)(-1)^{j'-\ell}}{\Gamma(\ell-i\nu+1)\ |\Gamma(j-i\nu+1)\ \Gamma(j'-i\nu+1)|(\ell-m)!} \quad . \qquad \ell \geq m > 0 \qquad (5.25)$$

Equation 5.25 may be extended with the aid of 5.4 and 5.5 to obtain

$$A^{jj'}_m(\alpha, \ell, \nu) \propto y^{|\ell-m|-i\nu+1} \qquad \ell, m > 0,\ y \ll 1$$

$$\propto y^{|\ell+m|+i\nu+1} \qquad \ell > 0,\ m < 0,\ y \ll 1$$

$$\propto y^{-|\ell-m|-i\nu-1} \qquad \ell, m > 0,\ y \gg 1$$

$$\propto y^{-|\ell+m|+i\nu-1} \qquad \ell > 0,\ m < 0,\ y \gg 1 \ . \qquad (5.26)$$

If $m = 0, A^{jj'}_0$ is real and from 5.10 we find

$$A^{jj'}_0(\alpha, \ell, \nu) \simeq \frac{(-1)^{j'-\ell}}{\ell!} \left[\frac{(j+\ell)!(j'+\ell)!(2j+1)(2j'+1)}{(j-\ell)!\ (j'-\ell)!}\right]^{\frac{1}{2}}$$

$$\left|\frac{\Gamma(i\nu)}{\Gamma(\ell+i\nu+1)}\right| 2y^{\ell+1} \cos(\nu\alpha+\phi) \qquad y \ll 1 \qquad (5.27)$$

where

$$e^{-i\phi} = \frac{\Gamma(i\nu)\ \Gamma(\ell+i\nu+1)\ \Gamma(j-i\nu+1)\ \Gamma(j'-i\nu+1)}{|\Gamma(i\nu)\ \Gamma(\ell+i\nu+1)\ \Gamma(j-i\nu+1)\ \Gamma(j'-i\nu+1)|} \quad .$$

The cases $\ell = 0$ and $y \gg 1$ follow from 5.4 and 5.5.

D. Orthogonality Relations

We consider the integral

$$I = \int_0^{4\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\Psi \int_0^{\infty} \sinh^2 \alpha \, d\alpha \int_0^{\pi} d\beta \int_0^{2\pi} d\gamma$$

$$D^{\ell\nu *}_{jm,kn}(L) \; D^{\ell'\nu'}_{j'm',k'n'}(L) \quad . \tag{5.28}$$

where $D^{\ell\nu}_{jm,kn}$ is defined by 3.22.

The integrations over rotation matrices may be performed (18) to obtain

$$I = \frac{(4\pi)^3}{(2j+1)(2k+1)} \quad \delta_{jj'} \, \delta_{kk'} \, \delta_{mm'} \, \delta_{nn'}$$

$$\sum_{\lambda} \int_0^{\infty} \sinh^2\alpha d\alpha \; A^{jk*}_{\lambda}(\alpha, \ell, \nu) \; A^{jk}_{\lambda}(\alpha, \ell' \nu'). \tag{5.29}$$

In view of the asymptotic behavior of A^{jk}_{λ} we may write 5.29 as

$$I = \frac{(4\pi)^3}{(2j+1)(2k+1)} \quad \delta_{jj'} \, \delta_{kk'} \, \delta_{mm'} \, \delta_{nn'}$$

$$\left[\frac{\pi(2j+1)(2k+1)}{2(\ell^2+\nu^2)} \quad \delta_{\ell\ell'} \; (\delta(\nu-\nu')+ \delta_{\ell 0} \, \delta(\nu+\nu')) + G(\ell, \ell', \nu, \nu', j, k)\right] \quad . \tag{5.30}$$

For fixed ℓ, ℓ', ν', j, and k, G is a continuous function of ν on any closed interval containing ν' since it has a uniformly convergent integral representation (23)

$$G(\nu) = \int_0^{\infty} g(\nu, \alpha) \, d\alpha \quad . \tag{5.31}$$

However 5.28 must be zero unless $\ell = \ell'$ and $\nu'^2 = \nu^2$ since for fixed k and n the functions

$$\Psi^{\ell\nu}_{jm}(L) \equiv D^{\ell\nu}_{jm,kn}(L) \tag{5.32}$$

carry a representation of the Lorentz group and 5.28 may be considered to be the inner product. If we take matrix elements of the differential operators (8) representing $J \cdot K$ and $J^2 - K^2$ we find

$$\begin{aligned} (\ell\nu - \ell'\nu')I &= 0 \\ (\ell^2 - \ell'^2 + \nu'^2 - \nu^2)I &= 0 \end{aligned} \tag{5.33}$$

It follows that G is identically zero and

$$\begin{aligned} I = \frac{2^5\pi^4}{\ell^2+\nu^2} \quad & \delta_{\ell\ell'}\, \delta_{jj'}\, \delta_{kk'}\, \delta_{mm'}\, \delta_{nn'} \\ & [\delta(\nu-\nu') + \delta_{\ell 0}\, \delta(\nu+\nu')] \\ & \ell \geq 0, \quad \nu \text{ real.} \end{aligned} \tag{5.34}$$

In the derivation of 5.34 ℓ was assumed to be nonnegative. In certain expansions it will be convenient to use the symmetry

$$D^{\ell\nu}_{jm,kn}(L) = D^{-\ell,-\nu}_{jm,kn}(L)$$

and assume $\nu \geq 0$, $\ell = 0, \pm\frac{1}{2}, \pm 1, \ldots$

In this case 5.34 becomes

$$\begin{aligned} I = \frac{2^5\pi^4}{\ell^2+\nu^2} \quad & \delta_{\ell\ell'}\, \delta(\nu-\nu')\, \delta_{jj'}\, \delta_{kk'}\, \delta_{nn'}\, \delta_{mm'} \\ & \nu \geq 0, \ \ell = 0, \pm\tfrac{1}{2}, \pm 1, \ldots \end{aligned} \tag{5.35}$$

A sum over all irreducible representations in the principal series will then have the form

$$\sum_{\ell=-\infty}^{\infty} \int_{0}^{\infty} d\nu \, .$$

This form has the advantage of including each representation once while the form

$$\sum_{\ell=0}^{\infty} \int_{-\infty}^{\infty} d\nu$$

contains the $\ell = 0$ representation twice.

VI. HARMONIC FUNCTIONS FOR THE POINCARÉ GROUP

A. An Angular Momentum Basis

We consider a basis for the Poincaré group in which states are labeled by irreducible representations of the Lorentz group. States in this basis have the form

$$|M, s; \ell, \nu, j, m\rangle, \ \nu > 0, \ \ell = 0, \pm\frac{1}{2}, \pm 1, \ldots. \qquad (6.1)$$

where M is the mass, s is the spin, and the numbers ℓ, ν, j, m specify a representation of the Lorentz group in an angular momentum basis. We assume states have the normalization

$$\langle M, s; \ell, \nu, j, m \,|\, M, s; \ell', \nu', j', m'\rangle =$$

$$\delta_{\ell\ell'}\, \delta(\nu-\nu')\, \delta_{jj'}\, \delta_{mm'} \, . \qquad (6.2)$$

If L is a Lorentz transformation, the basis 6.1 has the transformation law

$$L|M, s; \ell, \nu, j, m\rangle = \sum_{j',m'} D^{\ell\nu}_{j'm',jm}(L)\,|M, s; \ell, \nu, j', m'\rangle \qquad (6.3)$$

where the transformation matrices are defined by 3.27.

In terms of the basis 6.1, a wave function

$$\Psi(p, \lambda) = {}_H\langle p, s, \lambda|\Psi\rangle \qquad (6.4)$$

has an expansion

$$\Psi(p, \lambda) = \sum_{\ell,j,m} \int_0^\infty d\nu\, a^{\ell\nu}_{jm}\, Y^{\ell\nu}_{jm}(p, s, \lambda) \qquad (6.5)$$

where $|p, s, \lambda\rangle_H$ is a helicity state,

$$Y^{\ell\nu}_{jm}(p, s, \lambda) = {}_H\langle p, s, \lambda|M, s; \ell, \nu, j, m\rangle \qquad (6.6)$$

and

$$a^{\ell\nu}_{jm} = \langle M, s; \ell, \nu, j, m \mid \Psi\rangle . \qquad (6.7)$$

The explicit form of $Y^{\ell\nu}_{jm}$ follows from the definition of the helicity basis, 3.30, and the relation

$$\langle \tilde{p}, s, \lambda | M, s; \ell, \nu, j, m \rangle = N(M, s, \ell, \nu)\, \delta_{js}\, \delta_{m\lambda} . \qquad (6.8)$$

The result is

$$Y^{\ell\nu}_{jm}(p, s, \lambda) = {}_H\langle \tilde{p}, s, \lambda | e^{i\alpha K_3}\, e^{i\theta J_2} e^{i\phi J_3} | M, s; \ell, \nu, j, m \rangle$$

$$= N(M, s; \ell, \nu)\, A^{sj}_{\lambda}(-\alpha, \ell, \nu)\, D^{j}_{\lambda m}(0, -\theta, -\phi) \qquad (6.9)$$

where $D^{j}_{\lambda m}$ is a rotation matrix (18) and A^{sj}_{λ} is defined by 5.1.

The magnitude of N is fixed by 6.2. To see this we expand 6.2 in a helicity basis.

$$\langle M, s; \ell, \nu, j, m | M, s; \ell', \nu', j', m' \rangle = \sum_{\lambda} \int \frac{dp^3}{2p_o}\, Y^{\ell\nu}_{jm}(p, s, \lambda)^{*}\, Y^{\ell'\nu'}_{j'm'}(p, s, \lambda) \qquad (6.10)$$

Upon the change of variable

$$\frac{d^3p}{p_o} = M^2 \sinh^2\alpha\, d\alpha\, \sin\theta\, d\theta\, d\phi$$

6.10 becomes

$$\frac{M^2}{2} \sum_{\lambda} \int_0^{\infty} \sinh^2\alpha\, d\alpha \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$N(M, s, \ell, \nu)^{*}\, A^{sj}_{\lambda}(\alpha, \ell, \nu)^{*}\, D^{j}_{\lambda m}(0, -\theta, -0)^{*}$$

$$N(M, s, \ell', \nu')\, A^{sj'}(\alpha, \ell', \nu')\, D^{j'}_{\lambda m'}(0, -\theta, -\phi) \qquad (6.11)$$

$$= \frac{M^2 |N(M, s, \ell, \nu)|^2}{\ell^2 + \nu^2} (2s+1)\pi^2 \, \delta_{\ell\ell'} \, \delta(\nu-\nu') \, \delta_{jj'} \, \delta_{mm'} .$$

The last step follows from the orthogonality relation 5.34.

The final result is then

$$Y^{\ell\nu}_{jm}(p, s, \lambda) =$$

$$\frac{1}{\pi M} \left[\frac{\ell^2+\nu^2}{2s+1}\right]^{\frac{1}{2}} A^{sj}_{\lambda}(-\alpha, \ell, \nu) \, D^{j}_{\lambda m}(0, -\theta, -\phi) \qquad (6.12)$$

where a choice of phase has been made.

A representation (M, s) of the Poincaré group contains only values of ℓ satisfying $\ell^2 \leq s^2$. Therefore the Lorentz harmonics for low spin may be expressed in a simple form with the aid of 5.19. For example the cases s = 0 and s = 1/2 may be expressed in the form

$$Y^{s\nu}_{jm}(p, s, \lambda) = Y^{-s,-\nu}_{jm}(p, s, \lambda) =$$

$$Q(M, s, j, \lambda) \, D^{j}_{\lambda m}(0, -\theta, -\phi) \, y^{s-\lambda+i\nu+1}$$

$$(1-y^2)^{j-s} \, {}_2F_1(j-\lambda+1, j+i\nu+1, 2j+2, 1-y^2) .$$

where $y = e^{\alpha}$

and

$$Q(M, s, j, \lambda) =$$

$$\frac{1}{\pi M} \left[\frac{(j+s)! \; 2s! \; (j-\lambda)! \; (j+\lambda)! \; \Gamma(j+i\nu+1) \; \Gamma(j-i\nu+1)}{(j-s)! \; 2j! \; (2j+1)! \; (s+\lambda)! (s-\lambda)! \; \Gamma(s+i\nu) \; \Gamma(s-i\nu)}\right]^{\frac{1}{2}} . \qquad (6.13)$$

In the case of spin zero (6.13) becomes

$$Y^{0\nu}_{jm}(p, 0, 0) = \frac{1}{M(2j+1)} \left[\frac{2\Gamma(j+i\nu+1) \; \Gamma(j-i\nu+1)}{\Gamma(i\nu) \; \Gamma(-i\nu)}\right]^{\frac{1}{2}}$$

$$(\sinh\alpha)^{-1/2}\; P^{-1/2-j}_{-1/2\,-i\nu}\;(\cosh\alpha)\; Y^{j}_{m}\;(\theta,\;\phi) \qquad (6.14)$$

where Y^j_m is a spherical harmonic and P is a Legendre function. The Legendre functions with upper index -j-1/2 may be expressed as a finite number of terms (21). For example

$$P^{-1/2}_{-1/2-i\nu}(\cosh\alpha) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin\nu\alpha}{\nu(\sinh\alpha)^{1/2}} \quad .$$

The asymptotic behavior of the functions A^{sj}_{λ} implies that the functions $Y^{\ell\nu}_{jm}$ are square integrable with the measure $P_0^{-1}\, d^3p$ unless $\lambda = \pm\, \ell$.

Finally we note that the expansion coefficient 6.7 is given by

$$a^{\ell\nu}_{jm} = \sum_{\lambda} \int \frac{d^3p}{2p_o}\; Y^{\ell\nu\,*}_{jm}(p,\; s,\; \lambda)\; \Psi(p,\; \lambda). \qquad (6.15)$$

B. A Conformal Group Basis

We now consider harmonic functions of the form

$$X^{\ell\nu}_{rm}\;(p,\; s,\; \lambda) = {}_H\langle p,\; s,\; \lambda \mid M,\; s;\; \ell,\; \nu,\; r,\; m\rangle \qquad (6.16)$$

where $|\ell,\; \nu,\; r,\; m\rangle$ is a conformal group basis state for the Lorentz group. This set of functions may be obtained from the previous section with the aid of the expansion coefficients 4.24. The result is

$$X^{\ell\nu}_{rm}\;(p,\; s,\; \lambda) = \sum_{j=m}^{\infty} f^{j}_{m}\;(r,\; \ell,\; \nu)\; Y^{\ell\nu}_{jm}\;(p,\; s,\; \lambda). \qquad (6.17)$$

However a much simpler expression may be obtained with the aid of the Iwasawa decomposition.

According to the Iwasawa decomposition, the Lorentz transformation

$$H^{-1}(p)\; e^{-i\phi J_3} = e^{i\alpha K_3}\; e^{i\theta J_2} \qquad (6.18)$$

may be expressed in the form

$$e^{i\alpha K_3}\, e^{i\theta J_2} = RAN \quad . \tag{6.19}$$

The explicit form of RAN, which is derived in appendix C, is

$$RAN = e^{i\theta' J_2}\, e^{i\alpha' K_3}\, e^{ibQ}$$

where

$$\sin\theta' = \frac{\sin\theta}{\cosh\alpha - \cos\theta\sinh\alpha} \quad , \quad 0 \leq \theta' \leq \pi$$

$$e^{-\alpha'} = \cosh\alpha - \cos\theta\sinh\alpha$$

$$b = -\frac{\sinh\alpha\sin\theta}{\cosh\alpha - \cos\theta\sinh\alpha} \quad . \tag{6.20}$$

Therefore 6.16 may be written as

$$X^{\ell\nu}_{rm}(p, s, \lambda) =$$

$$_H\langle\tilde{p}, s, \lambda\,|\, e^{i\theta' J_2}\, e^{i\alpha' K_3}\, e^{ibQ}\, e^{iJ_3\phi}\, |M, s, \ell, \nu, r, m\rangle . \tag{6.21}$$

The final result, which follows from 4.17 and 6.8 is

$$X^{\ell\nu}_{rm}(p, s, \lambda) = \frac{1}{\pi M}\left[\frac{\ell^2+\nu^2}{2s+1}\right]^{\frac{1}{2}} e^{im\phi}$$

$$\sum_{n=-s}^{s} d^s_{\lambda n}(-\theta')\, f^s_n(e^{\alpha'}r, \ell, \nu)\, J_{n-m}(br) \tag{6.22}$$

where J is a Bessel function $d^s_{\lambda n}$ is a rotation matrix and f^s_n is defined by 4.24.

We note that the simplest set of harmonic functions is obtained in the basis

$$|p, s, \lambda\rangle_X = L(p, \tilde{p})\,|\,\tilde{p}, s, \lambda\rangle$$

where

$$L(p, \tilde{p}) = e^{-iJ_3\phi}\, e^{-ibQ}\, e^{-i\alpha' K_3} \quad . \qquad (6.23)$$

In this basis the harmonic functions are

$$\tilde{X}^{\ell\nu}_{rm}(p, s, \lambda) = {}_X\langle p, s, \lambda \mid M, s; \ell, \nu, r, m\rangle =$$

$$\frac{1}{\pi M}\left[\frac{\ell^2+\nu^2}{2s+1}\right]^{\frac{1}{2}} e^{im\phi}\, f^s_\lambda(re^{\alpha'}, \ell, \nu)\, J_{\lambda-m}(br). \qquad (6.24)$$

b, $e^{\alpha'}$ and $e^{i\phi}$ may be expressed in terms of p, The result is

$$b = \frac{-\sqrt{p_x^2 + p_y^2}}{p_o - p_z}$$

$$e^{-\alpha'} = \frac{p_o - p_z}{M}$$

$$e^{i\phi} = \frac{p_x + ip_y}{\sqrt{p_x^2 + p_y^2}} \quad . \qquad (6.25)$$

The orthogonality relations

$$\sum_\lambda \int \frac{d^3p}{2p_o}\, X^{\ell\nu}_{rm}(p, s, \lambda)^*\, X^{\ell'\nu'}_{r'm'}(p, s, \lambda) =$$

$$\delta_{\ell\ell'}\, \delta(\nu-\nu')\, r\delta(r-r')\, \delta_{mm'} \qquad (6.26)$$

follow at once from 6.17 and 6.2. The orthogonality relations may also be calculated from 6.22. If we make the change of variable

$$\frac{d^3p}{p_o} = M^2 \frac{dy}{y^3}\, b\,db\, d\phi \qquad (6.27)$$

where $e^{\alpha'} = y$,

we find

$$\sum_{\lambda} \int \frac{d\vec{p}^3}{2p_o} \, X^{\ell\nu}_{rm}(p, s, \lambda)^* \, X^{\ell'\nu'}_{r'm'}(p, s, \lambda) =$$

$$\frac{[(\ell^2+\nu^2)(\ell'^2+\nu'^2)]^{\frac{1}{2}}}{2\pi^2 \quad (2s+1)}$$

$$\sum_{\lambda} \int_0^{\infty} \frac{dy}{y^3} \, f^{s*}_{\lambda}(ry, \ell, \nu) \, f^{s}_{\lambda}(r'y, \ell', \nu')$$

$$\int_0^{\infty} bdb \, J_{m-\lambda}(br) \, J_{m'-\lambda}(br') \int_0^{2\pi} e^{-i(m-m')\phi} \, d\phi$$

$$= \frac{[(\ell^2+\nu^2)(\ell'^2+\nu'^2)]^{\frac{1}{2}}}{\pi(2s+1)} \, r\delta(r-r') \, \delta_{mm'}$$

$$\sum_{\lambda} \int_0^{\infty} \frac{dy}{y^3} \, f^{s\,*}_{\lambda}(y, \ell, \nu) \, f^{s}_{\lambda}(y, \ell', \nu'). \qquad (6.28)$$

Therefore the expansion coefficients f^s_λ satisfy the orthogonality relation

$$\sum_{\lambda=-s}^{s} \int_0^{\infty} \frac{dy}{y^3} \, f^{s*}_{\lambda}(y, \ell, \nu) \, f^{s}_{\lambda}(y, \ell', \nu') =$$

$$\frac{\pi(2s+1)}{\ell^2+\nu^2} \, \delta_{\ell\ell'} \, \delta(\nu-\nu') \quad . \qquad (6.29)$$

As in the previous section, any normalizable function may be expanded in terms of the harmonic functions $X^{\ell\nu}_{rm}$.

$$\Psi(p, s, \lambda) = \sum_{\ell, m} \int_0^{\infty} d\nu \int_0^{\infty} \frac{dr}{r} \, a^{\ell\nu}_{rm} \, X^{\ell\nu}_{rm}(p, s, \lambda)$$

where

$$a^{\ell\nu}_{rm} = \sum_{\lambda} \int \frac{d^3p}{2p_o} \; X^{\ell\nu}_{rm} (p, \; s, \; \lambda)^{*} \, \Psi(p, \; s, \; \lambda) \; . \tag{6.30}$$

Finally we list the harmonic functions $\tilde{X}^{\ell\nu}_{rm}$ for the cases s = o and $s = \frac{1}{2}$.

$$\tilde{X}^{o\nu}_{rm} (p, \; o, \; o) = \frac{1}{\pi M} \left[\frac{2}{\Gamma(i\nu)\Gamma(-i\nu)}\right]^{\frac{1}{2}} e^{im\phi} \, J_m(br)$$

$$r e^{\alpha'} \, K_{i\nu}(r e^{\alpha'}) \; .$$

$$\tilde{X}^{1/2\nu}_{rm} (p, \; \tfrac{1}{2}, \; \tfrac{1}{2}) = \frac{1}{\pi M} \left[\frac{1}{\Gamma(1/2+i\nu)\Gamma(1/2-i\nu)}\right]^{\frac{1}{2}} e^{im\phi}$$

$$J_{m-1/2} \, (br)[e^{\alpha'} r]^{\frac{3}{2}} \, K_{1/2-i\nu}(r e^{\alpha'})$$

$$\tilde{X}^{1/2\nu}_{rm} (p, \; \tfrac{1}{2}, \; -\tfrac{1}{2}) = \frac{1}{\pi M} \left[\frac{1}{\Gamma(1/2+i\nu)\Gamma(1/2-i\nu)}\right]^{\frac{1}{2}} e^{im\phi}$$

$$J_{m+1/2}(br) \, [e^{\alpha'} r]^{\frac{3}{2}} \, K_{1/2+i\nu}(r e^{\alpha'})$$

Here J and K are Bessel functions.

VII. APPENDIX

A. The General Expansion Coefficient

In this appendix we prove 4.30. We consider the case of lowering operators acting on f_j^j. The proof for raising operators is similar.

We prove 4.30 by induction on m. For m = j 4.30 reduces to 4.28 and

$$\int_0^\infty r^{2j+1} K_{\ell-i\nu}(r)\, K_{\ell+i\nu}(r)\, dr = [B(\ell, \nu, j, j)]^{-2} \quad . \tag{7.1}$$

Next we prove that the step down operator acting on f_m^j yields f_{m-1}^j. For convenience we write 4.30 as

$$f_m^j(r, \ell, \nu) = B(m) \sum_n a_n(m)\, r^{j+1-n}\, K_{\ell-i\nu-j+m+n}(r) \quad . \tag{7.2}$$

We obtain an expression for f_{m-1}^j from 7.2 by operating with the step down operator 4.27.

$$f_{m-1}^j(r, \ell, \nu) = \frac{B(m-1)}{2r(j+m)}$$

$$[(\ell+i\nu)^2 + r^2 - (\theta+m-1)^2] \sum_n a_n(m)\, r^{j+1-n}\, K_{\ell-i\nu-j+m+n}(r) \tag{7.3}$$

where

$$\theta = r\frac{d}{dr} \quad .$$

We use the relations

$$\theta\, K_\mu(r) = -\mu\, K_\mu(r) - rK_{\mu-1}(r)$$

$$= \mu\, K_\mu(r) - rK_{\mu+1}(r) \tag{7.4}$$

to obtain

$$[(\ell+i\nu)^2+r^2-(\theta+m-1)^2]\; r^{j+1-n}\, K_{\ell-i\nu+m+n-j} =$$

$$r^{j+1-n}\,[4(\ell-j+n)(i\nu+j-n)\;K_{\ell-i\nu+m+n-j}$$

$$+2(m+j-n)r\;K_{\ell-i\nu+m-1+n-j}\;]\;. \tag{7.5}$$

A recurrence relation for $a_n(m)$ follows from 7.2, 7.3, and 7.5. The result is

$$a_n(m-1) \;=\; \frac{1}{j+m}\,[\,(j+m-n)\;a_n(m)$$

$$+2(\ell-j+n)(i\nu+j-n)\;a_{n-1}(m)] \tag{7.6}$$

Upon substitution of the explicit form of $a_n(m)$ in 7.6 and use of the properties of (x,n),

$$(x,n)(x+n) = (x,\;n+1),\quad (x,n) \;=\; x(x+1,\;n-1) \tag{7.7}$$

we find

$$a_n(m-1) \;=\; \frac{(-2)^n(m-1-j,n)(\ell-j,n)(-i\nu-j,n)}{(1,n)\quad(-2j,\;n)} \tag{7.8}$$

which completes the proof.

B. An Integral Representation

In this appendix we drive the integral representation 4.32. We begin with 4.30.

$$f_m^j(r,\;\ell,\;\nu) \;=\; B(\ell,\;\nu,\;j,\;m)\;\sum_n$$

$$\frac{(-2)^n(-m-j,\;n)(\ell-j,\;n)(i\nu-j,\;n)\;r^{j+1-n}}{(1,n)\;\;(-2j,\;n)}\;K_{j+m-n-\ell-i\nu}(r)$$

$B(\ell,\;\nu,\;j,\;m) \;=$

$$\left[\frac{2j\,!\;(2j+1)!}{2^{2j-1}\,(j+m)!\,(j-m)!\,(j+\ell)!\,(j-\ell)!\,\Gamma(j+i\nu+1)\,\Gamma(j-i\nu+1)}\right]^{\frac{1}{2}}$$

We assume $\ell + m \geq 0$ and define a new index of summation

$$n' = j - \ell - n \quad . \tag{7.9}$$

The symbols (a,n) appearing in 4.30 may be written in the form (a', n'). The result is

$$(\ell-j,\ n) = \frac{(-1)^n\,\Gamma\,(j-\ell+1)}{(1,\ n')}$$

$$(-m-j,\ n) = \frac{(-1)^n\,\Gamma(j+m+1)}{\Gamma(m+\ell+1)\,(m+\ell+1,n')}$$

$$(i\nu-j,n) = \frac{(-1)^n\,\Gamma(j-i\nu+1)}{\Gamma(\ell-i\nu+1)\,(\ell-i\nu+1,\ n')}$$

$$(-2j,\ n) = \frac{(-1)^n\,\Gamma(2j+1)}{\Gamma(j+\ell+1)\,(\ell+j+1,\ n')}$$

$$(1,\ n) = \frac{(-1)^{n'}\,\Gamma(j-\ell+1)}{(\ell-j,\ n')} \tag{7.10}$$

With the aid of 7.10, 4.30 may be written in the form

$$f^j_m(r,\ \ell,\ \nu) = \frac{\Gamma(j+m+1)\,\Gamma(j-i\nu+1)\ \ \Gamma(j+\ell+1)}{\Gamma(\ell-i\nu+1)\,\Gamma(\ell+m+1)\,\Gamma(2j+1)}\ B(\ell,\ \nu,\ j,\ m)$$

$$\sum_{n'} \frac{(j+\ell+1,\ n')(\ell-j,\ n')(-2)^{j-\ell-n'}(-1)^{n'}\ r^{\ell+1+n'}}{(1,\ n')(m+\ell+1,\ n')(\ell-i\nu+1,\ n')}\,K^{(r)}_{-(n'+m-i\nu)}\quad . \tag{7.11}$$

Finally we substitute the integral representation (20)

$$K_\mu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\mu \int_0^\infty e^{-t-\frac{z^2}{4t}}\, t^{-\mu-1} \qquad |\arg z| < \frac{\pi}{4} \tag{7.12}$$

for K to obtain

$$f^j_m(r, \ell, \nu) = C(\ell, \nu, j, m)\, r^{\ell+1-m+i\nu}$$

$$\int_0^\infty e^{-t-\frac{r^2}{4t}}\, t^{m-i\nu-1}\ {}_2F_2\left({j+\ell+1,\ \ell-j; \atop m+\ell+1,\ \ell-i\nu+1;}\ t\right)\ dt\,. \tag{7.13}$$

where

$$C(\ell, \nu, j, m) = \frac{\Gamma(j-i\nu+1)\, 2^{m-\ell-i\nu}\, (-1)^{j-\ell}}{\Gamma(\ell-i\nu+1)\ (\ell+m)!}$$

$$\left[\frac{(j+m)!\ (j+\ell)!\ (2j+1)}{2\Gamma(j+i\nu+1)\ \Gamma(j-i\nu+1)\ (j-m)!\ (j-\ell)!}\right]^{\frac{1}{2}}$$

C. A Special Case of the Iwasawa Decomposition

We prove 6.19 by choosing the following representation for the Lie algebra.

$$J_1 = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad J_2 = \frac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} \qquad J_3 = \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$K_1 = \frac{1}{2}\begin{pmatrix}0&i\\i&0\end{pmatrix} \qquad K_2 = \frac{1}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix} \qquad K_3 = \frac{1}{2}\begin{pmatrix}i&0\\0&-i\end{pmatrix}$$

$$P = \begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad Q = \begin{pmatrix}0&-i\\0&0\end{pmatrix} \tag{7.14}$$

The corresponding representations of one parameter subgroups are

$$e^{i\phi J_3} = \begin{pmatrix}e^{i\phi/2} & 0\\ 0 & e^{-i\phi/2}\end{pmatrix}, \qquad e^{i\theta J_2} = \begin{pmatrix}\cos\theta/2 & \sin\theta/2\\ -\sin\theta/2 & \cos\theta/2\end{pmatrix}$$

$$e^{iaP} = \begin{pmatrix} 1 & ia \\ 0 & 1 \end{pmatrix} , \qquad e^{ibQ} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$e^{i\alpha K_3} = \begin{pmatrix} e^{-\alpha/2} & 0 \\ 0 & e^{\alpha/2} \end{pmatrix} \quad . \tag{7.15}$$

Therefore a general Lorentz transformation has the following representation in the Iwasawa decomposition.

$$\mathrm{RAN} = e^{i\phi J_3}\, e^{i\theta' J_2}\, e^{i\Psi J_3}\, e^{i\alpha' K_3}\, e^{iaP}\, e^{ibQ} =$$

$$\begin{pmatrix} [e^{-\frac{\alpha'}{2} + i\frac{\phi+\Psi}{2}} \cos\frac{\theta'}{2}] & [e^{\frac{\alpha'}{2} + i\frac{\phi-\Psi}{2}} \sin\frac{\theta'}{2} + (b+ia)\, e^{-\frac{\alpha'}{2} + i\frac{\phi+\Psi}{2}} \cos\frac{\theta'}{2}] \\ [-e^{-\frac{\alpha'}{2} + i\frac{\Psi-\phi}{2}} \sin\frac{\theta'}{2}] & [e^{\frac{\alpha'}{2} - i\frac{\phi+\Psi}{2}} \cos\frac{\theta'}{2} - (b+ia)\, e^{-\frac{\alpha'}{2} + i\frac{\Psi-\phi}{2}} \sin\frac{\theta'}{2}] \end{pmatrix} \tag{7.16}$$

The left side of 6.19 has the representation

$$e^{i\alpha K_3}\, e^{i\theta J_2} = \begin{pmatrix} e^{-\frac{\alpha}{2}} \cos\frac{\theta}{2} & e^{-\frac{\alpha}{2}} \sin\frac{\theta}{2} \\ -e^{\frac{\alpha}{2}} \sin\frac{\theta}{2} & e^{\frac{\alpha}{2}} \cos\frac{\theta}{2} \end{pmatrix} \quad . \tag{7.17}$$

Upon equating 7.16 and 7.17 we find

$$a = \phi = \Psi = 0$$

$$e^{-\gamma'} = \cosh\alpha - \cos\theta \sinh\alpha$$

$$\sin\theta' = \frac{\sin\theta}{\cosh\alpha - \cos\theta \sinh\alpha} \qquad 0 < \theta' < \pi$$

$$b = \frac{-\sin\theta \;\; \sinh\alpha}{\cosh\alpha - \cos\theta \; \sinh\alpha} \tag{7.18}$$

which completes the proof.

VIII. LITERATURE CITED

1. Dirac, P. A. M., Roy. Soc. Proc. A183, 284 (1945).
2. Chandra, H., Roy. Soc. Proc. A189, 372 (1947).
3. Gel'fand, I. M. and Naimark, M. A., Izv. Akad. Nauk S.S.S.R. Ser. Mat. 11, 411 (1947).
4. Naimark, M. A., Usp. Mat. Nauk 9, 19 (1954).
5. Dolginov, A. Z., Soviet Phys. JETP 3, 589 (1956).
6. Dolginov, A. Z. and Toptygin, I. N., Soviet Phys. JETP 10, 1022 (1960).
7. Dolginov, A. Z. and Moskalev, A. N., Soviet Phys. JETP 10, 1202 (1960).
8. Ström, S., Ark, Fys. (Sweden) 29, 467 (1965).
9. Ström, S., Ark. Fys. (Sweden) 34, 215 (1967).
10. Chang, S. and O'Raifeartaigh, L., Unitary Representations of SL(2, C) in an E(2) Basis, Preprint, Institute for Advanced Study, Princeton, New Jersey, 1968.
11. Hussain, M., Lorentz Harmonics, Unpublished Ph.D. thesis, Glasgow, Scotland, Library, University of Glasgow, 1965.
12. Fischer, J., Niederle, J. and Raczka, R., J. Math. Phys. 7, 816 (1966).
13. Zmuidzinas, J. S., J. Math. Phys. 7, 764 (1966).
14. Pontryagin, L. S., Topological Groups, Gordon and Breach, New York, 1966.
15. Hermann, R., Lie Groups for Physicists, W. A. Benjamin, Inc., New York, 1966.
16. Iwasawa, K., Japan, J. Math. 19, 405 (1948).
17. Naimark, M. A., Linear Representations of the Lorentz Group, Pergamon Press, New York, 1964.
18. Rose, M. E., Elementary Theory of Angular Momentum, John Wiley and Sons, Inc., New York, 1957.
19. Wigner, E., Ann. Math. 40, 149 (1939).
20. Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Higher Transcendental Functions, Vol. 2, McGraw-Hill Book Company, New York, 1953.

21. Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Higher Transcendental Functions, Vol. 1, McGraw-Hill Book Company, New York, 1953.

22. Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Tables of Integral Transforms, Vol. 1, McGraw-Hill Book Company, New York, 1954.

23. Titchmarsh, E. C., The Theory of Functions, 2nd ed., Oxford Press, London, 1939.

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