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Ordered and partially-ordered variants of Ramsey's theorem

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Ordered and partially-ordered variants of Ramsey's theorem

by

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DEDICATION

To my favorite aunt, Loraine.

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While I am on my way to greener pastures, as Derrick puts it, I will treasure my time here at Iowa State and never forget the individuals who have made such an impact on my early development as a mathematician.

ABSTRACT

For a k-uniform hypergraph G with vertex set $\{1, \ldots, n\}$, the ordered Ramsey number $\mathrm{OR}_t^k(G)$ is the least integer N such that every t-coloring of the edges of the complete k-uniform graph on vertex set $\{1, \ldots, N\}$ contains a monochromatic copy of G whose vertices follow the prescribed order. Due to this added order restriction, the ordered Ramsey numbers can be much larger than the usual graph Ramsey numbers. We determine that the ordered Ramsey numbers of loose paths under a monotone order grows as a tower of height two less than the maximum degree in terms of the number of edges and as a tower of height one less than the maximum degree in terms of the number of colors. We also extend theorems of Conlon, Fox, Lee, and Sudakov on the ordered Ramsey numbers of 2-uniform matchings to provide upper bounds on the ordered Ramsey number of k-uniform matchings under certain orderings.

Beyond this, we introduce an extension of the ordered Ramsey number to consider graphs with only a partial ordering on their vertices. This extension also allows us to consider analogues of the Ramsey number where the host graph is constructed from an arbitrary poset. In particular, we focus on what we refer to as the *Boolean Ramsey number*, which illustrates the difficulty in this new direction in addition to demonstrating the connections to Turán-type problems in posets.

PRELIMINARIES AND NOTATION

This chapter will discuss terminology and notation that is familiar to those with at least an undergraduate understanding of graph theory. It should be used as a reference as necessary for those unfamiliar with this terminology.

A k-uniform hypergraph is a pair $(V(G), E(G))$ where $V(G)$ is some nonempty set, called the vertex set of G, and $E(G)$ is a collection of k-element subsets of $V(G)$, called the edge set of G. If the hypergraph G is unambiguous, we will simply use V and E as opposed to $V(G)$ and $E(G)$. An ordered hypergraph is a hypergraph in which $V = \{1, \ldots, n\}$ for some integer n, i.e. an ordered hypergraph is a hypergraph with a total order on its vertices. Often, the term "graph" will be used in the place of "hypergraph" or "ordered hypergraph." We will, for the most part, use k to denote the uniformity of a hypergraph.

For $k \geq 2$, the complete k-uniform (ordered) hypergraph with vertex set $\{1, \ldots, N\}$ is denoted K_N^k. The 2-uniform case is special, so K_N denotes K_N^2.

For k-uniform hypergraphs G and H, we say that H is a subgraph of G, or that G contains a copy of H, if there is an injection $\phi : V(H) \to V(G)$ such that $\{\phi(v_1), \ldots, \phi(v_k)\} \in E(G)$ whenever $\{v_1, \ldots, v_k\} \in E(H)$.

A *digraph* (directed graph) is, informally, a graph whose edges are tuples of vertices as opposed to unordered sets. Although we can define digraphs of any uniformity, we only require the notion of a 2-uniform digraph for the purposes of this paper. Formally, a 2-uniform digraph D is a pair $(V(D), E(D))$ where $V(D)$ is a vertex set and $E(D)$ is a subset of $V(D)^2 \setminus \{(x, x) : x \in V(D)\}$ where if $(x, y) \in E(D)$, then $(y, x) \notin E(D)$. We say that a digraph is *directed-acyclic* if it does not have any directed cycles, i.e. there is no set $x_1, \ldots, x_n \in V(D)$ such that all of $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)$ are edges of D.

We define subgraphs analogously to the undirected case by saying that for two digraphs D and R, D is a subgraph of R, or R contains a copy of D, if there is an injective function $\phi : V(D) \to V(R)$ such that $(\phi(x), \phi(y)) \in E(R)$ whenever $(x, y) \in E(D)$.

A partially-ordered set (poset) is a pair $(P, \leq)$ where P is some nonempty set and $\leq$ is a binary relation on P, called a partial-ordering of P, satisfying the following properties.

- Reflexivity: for all $x \in P$, $x \leq x$.
- Antisymmetry: for all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.
- Transitivity: for all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

If the partial-ordering is understood, we will simply use P instead of $(P, \leq)$. If for every $x, y \in P$, $x \leq y$ or $y \leq x$, then we say that P is a *chain*. On this other hand, if for every distinct $x, y \in P$, $x \not\leq y$ and $y \not\leq x$, we say that P is an *antichain*.

For a poset $(P, \leq)$, a *linear extension of* $\leq$ is another partial ordering $\leq_T$ on P where $(P, \leq_T)$ is a chain and $x \leq_T y$ whenever $x \leq y$. We refer to $(P, \leq_T)$ as a *linear extension* of $(P, \leq)$. It is a straightforward fact that if P is a finite poset, then it has at least one linear extension.

For integers $m \leq n$, let $[n] = \{1, \ldots, n\}$, $[m, n] = \{m, m+1, \ldots, n-1, n\}$, and let $\binom{[n]}{m}$ denote the set of m-element subsets of $[n]$. Also, we use $2^{[n]}$ to denote the set of all subsets of $[n]$.

We use $\lg n = \log_2 n$. We frequently use e the number of edges in a graph and rarely as the base of the natural logarithm. The *tower function of height* t, denoted by $\mathrm{tow}_t(n)$, is

$$\mathrm{tow}_0(n) = n, \quad \text{and} \quad \mathrm{tow}_t(n) = 2^{\mathrm{tow}_{t-1}(n)} \text{ for } t \geq 1.$$

We use standard notation for asymptotics. For two functions $f = f(n)$ and $g = g(n)$, we say that $f = O(g)$ if for n sufficiently large, $f(n) \leq c \cdot g(n)$ for some constant c.

Similarly, $f = \Omega(g)$ if for n sufficiently large, $f(n) \geq c \cdot g(n)$ for some constant c. If $f = O(g)$ and $f = \Omega(g)$, we say that $f = \Theta(g)$. Also, $f = o(g)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$; the most common use of this will be $o(1)$ which denotes a function that tends toward 0 as n tends toward infinity.

CHAPTER 1. OVERVIEW

Ramsey theory, very generally speaking, is the idea that every structure must contain very well-behaved substructures. In particular, Ramsey theory attempts to find conditions under which specific well-behaved substructures *must* occur. One of the most basic examples of a problem in Ramsey theory considers coloring the edges of K_6 red and blue. It is a straightforward exercise to observe that any red-blue coloring of the edges of K_6 must contain a monochromatic triangle. To see this, choose any vertex v_1, then by the pigeonhole principle, there must be at least three other vertices v_2, v_3, v_4 that are all connected to v_1 by the same color, say red. If any of the edges between v_2, v_3, v_4 are red, say v_2v_3, then $v_1v_2v_3$ forms a red triangle. Otherwise, all of the edges between v_2, v_3, v_4 are blue, so $v_2v_3v_4$ forms a blue triangle. Therefore, no matter how the colors are assigned to the edges of K_6, there will always be a monochromatic triangle. Furthermore, Figure 1.1 displays a red-blue coloring of the edges of K_5 that has no monochromatic triangle. Therefore, 6 is the least integer N such that every 2-coloring of the edges of K_N contains a monochromatic copy of K_3. The extension of this idea is the crux of Ramsey theory.

1.1 The Graph Ramsey Number

Define $\mathrm{R}_2(n)$ to be the least integer N such that every 2-coloring of K_N contains a copy of K_n whose edges are all the same color, which is called the *2-color diagonal Ramsey number of n*. The argument at the beginning of this chapter shows that $\mathrm{R}_2(3) = 6$. However, it is not obvious that $\mathrm{R}_2(n)$ is always defined, as it may be possible to color

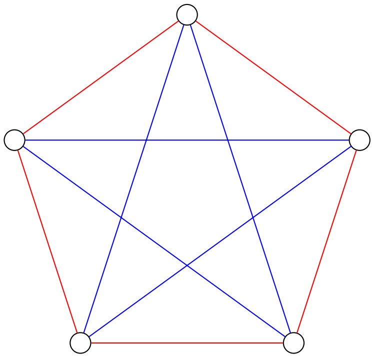

Figure 1.1 A 2-coloring of K_5 that avoids monochromatic copies of K_3.

the edges of K_N in a way that avoids monochromatic copies of K_n for any N. It turns out that $\mathrm{R}_2(n)$ exists for every n due to the following celebrated theorem of the logician Frank P. Ramsey.

Theorem 1.1 (Ramsey [23]). *For any positive integers n and t, there exists another positive integer N such that in any t-coloring of the edges of K_N, there must be a copy of K_n whose edges are all the same color.*

With this theorem in mind, we can actually define the *t-color diagonal Ramsey number*, denoted $\mathrm{R}_t(n)$ to be the least integer N such that any t-coloring of $E(K_N)$ admits a copy of K_n whose edges are all the same color. Immediately, we can then define $\mathrm{R}(n_1, \ldots, n_t)$, for positive integers $n_1, \ldots, n_t$, called the *off-diagonal Ramsey number of $n_1, \ldots, n_t$*, to be the least integer N such that any t-coloring of $E(K_N)$ contains a copy of K_{n_i} whose edges all have color i for some $i \in [t]$. The existence of $\mathrm{R}(n_1, \ldots, n_t)$ is seen from the observation that $\mathrm{R}(n_1, \ldots, n_t) \leq \mathrm{R}_t(\max\{n_1, \ldots, n_t\})$ as K_m is contained within K_n whenever $m \leq n$.

Although it was Ramsey who originally developed Ramsey theory, Erdős and Szekeres [14] brought this problem to the attention of mainstream mathematics. In their monumental 1935 paper [14], along with many other interesting results that we will discuss later, Erdős and Szekeres proved that

$$\mathrm{R}_2(n) \leq (1+o(1))\frac{4^{n-1}}{\sqrt{\pi n}}.$$

Furthermore, in 1947, Erdős [12] proved that

$$\mathrm{R}_2(n) \geq (1-o(1))\frac{n}{\sqrt{2}e}2^{n/2}$$

where e is the base of the natural logarithm. Surprisingly, although the methods used in each of these bounds are not overly complicated, these bounds, roughly $\Omega(2^{n/2}) \leq \mathrm{R}_2(K_n) \leq O(2^{2n})$, have remained largely unchanged despite significant effort. Currently, the best bounds on the 2-color diagonal Ramsey number are

$$(1-o(1))\frac{\sqrt{2}n}{e}2^{n/2} \leq \mathrm{R}_2(n) \leq n^{-O\left(\frac{\log n}{\log\log n}\right)}2^{2n}$$

due to Spencer [24] and Conlon [7] respectively. Shockingly, Spencer's lower bound is only an improvement over Erdős's original bound by a constant factor of 2.

Although most of the focus of Ramsey theory is on the 2-uniform case, there is no reason to restrict ourselves to this case as Ramsey also proved a version of Theorem 1.1 for k-uniform hypergraphs.

Theorem 1.2 (Ramsey [23]). *For any positive integers n and t, there exists another positive integer N such that in any t-coloring of the edges of K_N^k, there must be a copy of K_n^k whose edges are all the same color.*

Thus, we may extend the definition of the Ramsey number so that $\mathrm{R}_t^k(n)$, the least integer N such that any t-coloring of the edges of K_N^k contains a copy of K_n^k whose edges are all the same color, is well-defined. Furthermore, just like the 2-uniform case, we can also consider the off-diagonal case of $\mathrm{R}^k(n_1,\ldots,n_t)$. As the 2-uniform case is special, $\mathrm{R}^2(n_1,\ldots,n_t) = \mathrm{R}(n_1,\ldots,n_t)$.

The best bounds on the k-uniform 2-color diagonal Ramsey number are quite loose, especially in comparison to the bounds on the 2-uniform case. The best bounds on $\mathrm{R}_2^k(n)$

come from a 1965 paper of Erdős, Hajnal, and Rado [13] in which it is shown that

$$\mathrm{tow}_{k-2}(\Omega(n^2)) \leq \mathrm{R}_2^k(n) \leq \mathrm{tow}_{k-1}(O(n)).$$

In fact, it is conjectured that the upper bound is closer to the truth. Interestingly, if it could be shown that $2^{2^{\Omega(n)}} \leq \mathrm{R}_2^3(n)$, due to the "stepping up" argument used in [13], then it would automatically hold that $\mathrm{R}_2^k(n) = \mathrm{tow}_{k-1}(\Theta(n))$ for any k.

In fact, the notion of the Ramsey number naturally extends to any k-uniform hypergraph, as a k-uniform hypergraph on n vertices is a subgraph of K_n^k.

Formally, a t-coloring of the edges of a k-uniform hypergraph G is a function $c : E(G) \to [t]$. The *i-colored subgraph of G* is the subgraph of G induced by the edges in $c^{-1}(i)$. For another hypergraph H, we say that c contains an i-colored copy of H if H is a subgraph of the i-colored subgraph of G.

Definition 1.3. For k-uniform hypergraphs $G_1, \ldots, G_t$, the emphgraph Ramsey number of $G_1, \ldots, G_t$, denoted $\mathrm{R}^k(G_1, \ldots, G_t)$, is the least integer N such that for any t-coloring of the edges of K_N^k, there is some i for which there is an i-colored copy of G_i. If $G_1 = \cdots = G_t = G$, then we denote $\mathrm{R}^k(G_1, \ldots, G_t)$ by $\mathrm{R}_t^k(G)$ and refer to this as the *diagonal case.* If not all of the hypergraphs are the same, then we are in the *off-diagonal case.*

Again, the 2-uniform case is special, so $\mathrm{R}^2(G_1, \ldots, G_t) = \mathrm{R}(G_1, \ldots, G_t)$.

Notice that we can equivalently define $\mathrm{R}^k(G_1, \ldots, G_t)$ to be the largest integer N such that there exists a t-coloring of $E(K_{N-1}^k)$ that has no copy of G_i in color i for any $i \in [t]$. Ramsey theory, very basically, asks one question: for hypergraphs $G_1, \ldots, G_t$, what is $\mathrm{R}^k(G_1, \ldots, G_t)$? In order to answer this question, we must call upon both definitions of the Ramsey number.

If we want to show that $\mathrm{R}^k(G_1, \ldots, G_t) \leq N_1$, then we must show that any t-coloring of $E(K_{N_1}^k)$ contains an i-colored copy of G_i for some i. On this other hand, if we wish to show that $\mathrm{R}^k(G_1,\dots,G_t) \geq N_2$, we must demonstrate a t-coloring if $E(K^k_{N_2-1})$ that avoids i-colored copies of G_i for each i.

1.1.1 Arrow Notation

The original formulation of Ramsey theory is tied to coloring the complete graph and looking for monochromatic substructures. Because any vertex in the complete graph is indistinguishable from any other vertex, "naïve" techniques such as the pigeonhole principle can easily be applied to achieve bounds on the Ramsey numbers. However, it is natural to ask if we can define some analogue of Ramsey theory which instead considers coloring graphs other than the complete graph. The answer, of course, is yes and is done by defining what is known as *arrow notation.*

For k-uniform hypergraphs $H, G_1, \dots, G_t$, we say that $H \xrightarrow{k} (G_1,\dots,G_t)$ if any t-coloring of $E(H)$ admits an i-colored copy of G_i for some $i \in [t]$. We refer to H as the *host graph.* Using this notation, we can define the Ramsey number as follows:

$$\mathrm{R}^k(G_1,\dots,G_t) = \min\{|V(H)| : H \xrightarrow{k} (G_1,\dots,G_t)\}.$$

This is equivalent to the previous definition of the Ramsey number because if any t-coloring of $E(H)$ contains an i-colored copy of G_i for some i, then so does any t-coloring of $E(K^k_{|V(H)|})$.

Arrow notation can be used to define Ramsey-type numbers for many different parameters of the host graph other than just the number of vertices. An interesting Ramsey-type number which is defined through arrow notation is called the *size Ramsey number of* $G_1,\dots,G_t$, which is defined to be $\min\{|E(H)| : H \xrightarrow{k} (G_1,\dots,G_t)\}$. It is easy to observe that the size Ramsey number is bounded above by $\binom{\mathrm{R}^k(G_1,\dots,G_t)}{k}$ as this is the number of edges in the complete k-uniform graph of order $\mathrm{R}^k(G_1,\dots,G_t)$; however, in many cases, the size Ramsey number can be much smaller.

Using arrow notation, we can also define Ramsey-type numbers for different host families of graphs. For example, we could look at the family of hypercube graphs, Q_n,

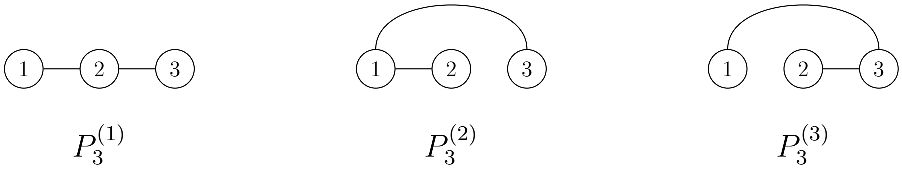

Figure 1.2 The three nonisomorphic labelings of P_3.

and given graphs $G_1, \ldots, G_t$ that can appear as subgraphs of a hypercube, we can try to determine the least integer N such that $Q_N \xrightarrow{2} (G_1, \ldots, G_t)$.

While we will not discuss the notion of size Ramsey numbers or Ramsey numbers on the hypercubes, we will return to arrow notation in Chapter 3 in order to define partially-ordered Ramsey numbers in full generality.

1.2 The Directed Ramsey Number

As Ramsey theory grew in popularity among discrete mathematicians, it was quickly realized that even seemingly simple questions were very challenging. Because of this, variants of Ramsey numbers were introduced both as possible stepping stones to these problems and as independently interesting concepts. In this paper, we focus on recent variants of Ramsey theory that consider graphs whose vertex sets are ordered in some fashion.

Consider the 2-uniform path on 3 vertices, P_3. By the pigeonhole principle, it is immediate to note that $\mathrm{R}_t(P_3) \leq t + 2$ as if c is a t-coloring of $E(K_N)$ that avoids monochromatic copies of P_3, then no vertex can be incident to two edges of the same color.

Now consider labeling on the vertices of P_3 with the set $\{1, 2, 3\}$ (see Figure 1.2). We can now ask the following question:

Fix an ordering of the vertices of K_N (i.e. consider K_N to have vertex set $[N]$), and color the edges; how large can N be so that I avoid monochromatic copies of a *particular*

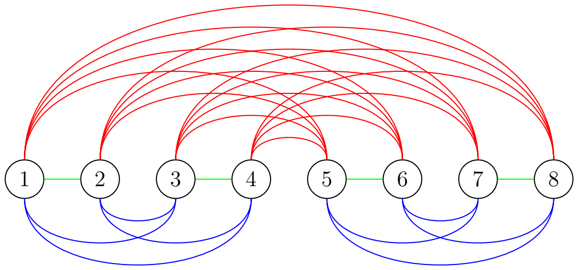

Figure 1.3 A 3-coloring of $E(K_8)$ that avoids monochromatic copies of $P_3^{(1)}$.

ordering of a given graph G? For example, Figure 1.3 displays a 3-coloring of $E(K_8)$ that avoids monochromatic copies of $P_3^{(1)}$; however, this coloring admits many monochromatic copies of $P_3^{(2)}$ and $P_3^{(3)}$. In fact, the pattern shown in Figure 1.3 can be repeated to show that there is a t-coloring of $E(K_{2^t})$ that does not have any monochromatic copies of $P_3^{(1)}$. On the other hand, any t-coloring of $E(K_{t+2})$ must admit monochromatic copies of both $P_3^{(2)}$ and $P_3^{(3)}$. Thus, we may be tempted to say that $\mathrm{R}_t(P_3^{(2)}) = \mathrm{R}_t(P_3^{(3)}) \leq t+2$ while $\mathrm{R}_t(P_3^{(1)}) > 2^t$.

In 2002, Choudum and Ponnusamy [3] introduced the first formalization of this idea through the concept of the *directed Ramsey number*.[1]

The *transitive tournament of order* n, denoted TT_n, is a directed-acyclic orientation of K_n. In other words, TT_n has the property that for every $\{x_1, x_2\} \in \binom{V(TT_n)}{2}$, either $(x_1, x_2) \in E(TT_n)$ or $(x_2, x_1) \in E(TT_n)$, and if $(x_1, x_2), (x_2, x_3) \in E(TT_n)$, then $(x_1, x_3) \in E(TT_n)$.

For directed-acyclic digraphs $D_1, \ldots, D_t$, the *directed Ramsey number* of $D_1, \ldots, D_t$, denoted $\mathrm{DR}(D_1, \ldots, D_t)$, is the least integer N such that any t-coloring of $E(TT_N)$ contains a copy of D_i in color i for some i. The fact that this number exists follows from the simple observation that $\mathrm{DR}_t(TT_n) = \mathrm{R}_t(n)$.

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[1] In [3], this number is actually referred to as the "ordered Ramsey number," but has since been renamed.

Choudum and Ponnusamy explored the directed Ramsey number for certain families of digraphs; most notably, directed paths and directed stars. We will make mention of these results in later sections and discuss their ties to ordered Ramsey numbers.

1.3 The Ordered Ramsey Numbers

An alternative formalization, called ordered Ramsey theory, has recently received significant attention [2, 5, 8, 10, 15, 20, 21]. In this variation, we again look for t-colorings of the complete graph that avoid monochromatic copies of a graph G, except that the *order* of the vertices of G in this monochromatic copy are very important.

Formally, *ordered k-uniform hypergraph* is a hypergraph G where the edge set $E(G)$ contains k-sets of vertices, and the vertex set $V(G)$ is totally ordered. An ordered hypergraph G is *contained* in an ordered hypergraph H if there is an injective, order-preserving map from the vertices of G to the vertices of H such that edges of G map to edges of H. Let K_N^k be the complete k-uniform hypergraph on the vertex set $\{1, \ldots, N\}$ and let $c : E(K_N^k) \to \{1, \ldots, t\}$ be a t-coloring of the edges in K_N^k. The i-colored subgraph of K_N^k is the ordered hypergraph given by the edges in $c^{-1}(i)$.

For ordered k-uniform hypergraphs $G_1, \ldots, G_t$, the *ordered Ramsey number* $\mathrm{OR}^k(G_1, \ldots, G_t)$ is the minimum N such that for every t-coloring of K_N^k there is some color i such that the i-colored subgraph contains G_i. This number is necessarily defined and finite, since there exists an n such that each G_i is a subgraph of K_n^k and hence $\mathrm{OR}^k(G_1, \ldots, G_t) \leq \mathrm{R}_t^k(n)$. If $G_1 = \cdots = G_t = G$, then we denote $\mathrm{OR}^k(G_1, \ldots, G_t)$ as $\mathrm{OR}_t^k(G)$ and refer to this as the *diagonal case*; otherwise it is the *off-diagonal case.*

Notice that each ordered graph gives rise to a directed graph in a natural way. If G is an ordered graph, form the digraph G' by letting $(x, y) \in E(G')$ whenever $\{x, y\} \in E(G)$ and $x < y$. Thus, it is easy to observe that $\mathrm{OR}_t(G) \geq \mathrm{DR}_t(G')$. However, the opposite inequality need not hold. This follows from the fact that for a given digraph G, there

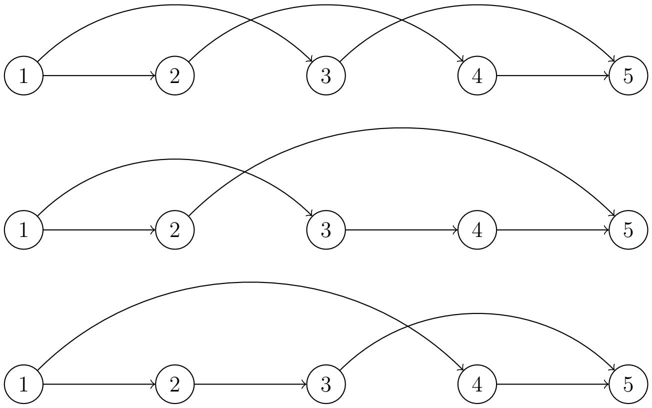

Figure 1.4 Nonisomorphic orderings of the same digraph.

may be multiple nonisomorphic orderings of the vertex set such that $x < y$ whenever $(x, y) \in E(G)$ (see Figure 1.4). Because of this, the ordered Ramsey number should be viewed as the "proper" way to extend the notion of Ramsey numbers to graphs with an order on their vertices.

If G is a 2-uniform path under the standard ordering, then the 2-color ordered Ramsey number of G is equal to the bound of the Erdős-Szekeres Theorem [14] (see [3, 20]), and if G is a tight 3-uniform path under the standard ordering, then the 2-color ordered Ramsey number of G is equal to the bound of the *happy ending problem* (see [15]). Due to these connections, much of the previous work has focused on the ordered Ramsey number of tight k-uniform paths under the standard ordering [15, 20, 21].

1.4 Applications of the Ordered Ramsey Number

Although the formal definition of ordered Ramsey numbers is fairly new, the idea has been around since the monumental 1935 paper by Erdős and Szekeres [14]. We briefly present the connections between Erdős-Szekeres type problems and the ordered Ramsey numbers of hyperpaths.

For positive integers k, ℓ, e such that $k > \ell$, the *(k, ℓ)-path on e edges*, denoted $P_e^{k,\ell}$, is the k-uniform ordered hypergraph on $e(k-\ell)+\ell$ vertices and e totally-ordered edges $A_1, A_2, \ldots, A_e$ where two consecutive edges A_i, A_{i+1} intersect exactly on the maximum ℓ vertices in A_i and the minimum ℓ vertices in A_{i+1}. The path $P_e^{k,k-1}$ is called the *tight k-uniform path* and otherwise $P_e^{k,\ell}$ is a *loose path*.

1.4.1 Erdős-Szekeres Type Problems

In 1935, Erdős and Szekeres [14] proved that any sequence of $(n-1)^2+1$ distinct real numbers must contain either an increasing or a decreasing subsequence of length n. The original proof of this fact had an inductive flavor, and there have since appeared very slick proofs that require only an elementary application of the pigeonhole principle. In addition to these, there is a very natural connection of this problem to the ordered Ramsey numbers of paths.

Let $(a_1, \ldots, a_N)$ be a sequence of distinct real numbers. Define a 2-coloring c of $E(K_N)$ as follows: for $i < j$, let $c(i,j) = 1$ if $a_i < a_j$ and $c(i,j) = 2$ if $a_i > a_j$. If $N \geq \mathrm{OR}_2(P_{n-1}^{2,1})$, then c must have a monochromatic copy of $P_{n-1}^{2,1}$. If this monochromatic copy lies in color 1, then the vertices of the $P_{n-1}^{2,1}$ correspond to an increasing subsequence of length n, and if the copy lies in color 2, then the vertices correspond to an increasing subsequence of length n. Therefore, if $f(n)$ is the least integer such that any sequence of $f(n)$ distinct real numbers contains either an increasing or a decreasing subsequence of length n, then $f(n) \leq \mathrm{OR}_2(P_{n-1}^{2,1})$. It turns out that $\mathrm{OR}_2(P_e^{2,1}) = e^2+1$ (we will discuss this in more generality in Chapter 2), so this connection to ordered Ramsey numbers provides yet another proof of the Erdős-Szekeres theorem.

Another problem discussed by Erdős and Szekeres in this paper is known as the *happy ending problem*. Their result states that for any positive integer n, there is another integer N such that for any N points in the plane in general position (i.e. no three on a line), there must be a collection of n of these points that form the vertices of a convex n-gon. The original proof of this fact requires an inductive argument, and there is again a nice connection to order Ramsey numbers. Let $(x_1, y_1), \ldots, (x_N, y_N)$ be a set of points in general position in the plane. Without loss of generality, we may assume that $x_1 < \cdots < x_N$ (as we may rotate the plane slightly if any two points lie on a vertical line). We can now define a 2-coloring of $E(K_N^3)$ as follows: for $i < j < k$, let $c(i, j, k) = 1$ if the quadratic curve passing through $\{(x_i, y_i), (x_j, y_j), (x_k, y_k)\}$ is concave up, and let $c(i, j, k) = 2$ if the quadratic curve passing through $\{(x_i, y_i), (x_j, y_j), (x_k, y_k)\}$ is concave down. If $N \geq \mathrm{OR}_2^3(P_{n-2}^{3,2})$, then c must admit a monochromatic copy of $P_{n-2}^{3,2}$. Whether this copy lies in color 1 or 2, the vertices correspond to the points of a convex n-gon.

1.4.2 Track Numbers

A graph G is said to be an *interval graph* if there is an assignment $I : V(G) \to 2^{\mathbb{R}}$ where $I(v)$ is an interval for each $v \in V(G)$ such that $\{u, v\} \in E(G)$ if and only if $I(v) \cap I(u) \neq \emptyset$. A *t-track representation* of G is a representation of G as the union of at most t interval graphs. The *track number* of G, denoted $\tau(G)$, is the least t such that G has a t-track representation.

It was conjectured that the track number of the line graph of K_n is unbounded, i.e. $\tau(L(K_n)) \to \infty$ as $n \to \infty$. This conjecture was resolved by Milans, Stolee, and West [20] through the following theorem.

Theorem 1.4 (Milans, Stolee, and West [20]). $\Omega\left(\frac{\lg\lg n}{\lg\lg\lg n}\right) \leq \tau(L(K_n)) \leq O(\lg\lg n)$.

This theorem was proved through the use of ordered Ramsey numbers by showing that if $t = \tau(L(K_n))$, then

$$\mathrm{OR}_{t-3}^3(P_2^{3,2}) \leq n < \mathrm{OR}_t^3(P')$$

where P' is a copy of $P_4^{3,2}$ on vertex set $\{1, \ldots, 6\}$ with the additional edges $\{1, 2, 5\}$ and $\{2, 5, 6\}$.

Due to the large amount of applications of the ordered Ramsey number of tight paths, in Chapter 2 we focus on determining the ordered Ramsey numbers for loose paths.

1.5 Partially-Ordered Ramsey Numbers

In Chapter 3, we explore a generalization of the ordered Ramsey number to graphs who have only a partial ordering on their vertices. Naturally following from this generalization is the urge to explore other host graphs arising from various posets as opposed to just the complete graph. We focus on using the Boolean lattice to create a host graph and explore its connections to the ordered Ramsey number. This direction exhibits the connections of the partially-ordered Ramsey number to popular questions in extremal combinatorics such as Turán-type problems about posets. In addition, looking at the Boolean lattice demonstrates why determining a partially-ordered Ramsey number may be more difficult than determining an ordered Ramsey number.

CHAPTER 2. ORDERED RAMSEY NUMBERS OF HYPERGRAPHS[1]

Recall that for positive integers k, ℓ, e such that $k > \ell$, the *(k,ℓ)-path on e edges*, denoted $P_e^{k,\ell}$, is the k-uniform ordered hypergraph on $e(k-\ell)+\ell$ vertices and e totally-ordered edges $A_1, A_2, \ldots, A_e$ where two consecutive edges A_i, A_{i+1} intersect exactly on the maximum ℓ vertices in A_i and the minimum ℓ vertices in A_{i+1}. The path $P_e^{k,k-1}$ is called the *tight k-uniform path* and otherwise $P_e^{k,\ell}$ is a *loose path*. For $\ell = 0$, we can extend the definition of $P_e^{k,\ell}$ by requiring that two consecutive edges A_i, A_{i+1} satisfy $\max A_i < \min A_{i+1}$, and hence the edges are disjoint, forming a *matching*. Note that when $k = 2$ the only possibilities are a tight path or a matching. We will primarily use the ordering given by this definition, and we will specify the special cases when we will consider a possibly different ordering on $P_e^{k,\ell}$.

Define the *intersection number*, $i(k,\ell)$, to be the maximum degree of a vertex in $P_e^{k,\ell}$ for all $e \geq k$. Observe that if $\ell > 0$, then $i(k,\ell)$ is the unique integer $m \geq 2$ that satisfies

$$\frac{m-2}{m-1} < \frac{\ell}{k} \leq \frac{m-1}{m}.$$

The tight paths $P_e^{k,k-1}$ have been investigated thoroughly. For 2-uniform tight paths, the ordered Ramsey number $\mathrm{OR}_t(P_e^{2,1})$ is determined by Choudum and Ponnusamy [3]. We provide a proof for completeness.

Theorem 2.1 (Choudum and Ponnusamy [3])**.** *For positive integers $e_1, \ldots, e_t$,*

$$\mathrm{OR}(P_{e_1}^{2,1}, \ldots, P_{e_t}^{2,1}) = \prod_{i=1}^{t} e_i + 1.$$

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[1] **The contents of this chapter have been submitted to *Discrete Mathematics* [10].**

Proof. Upper bound. Let $N = \prod_{i=1}^{t} e_i + 1$ and suppose that c is some t-coloring of $E(K_N)$ that avoids $P^{2,1}_{e_i}$ in color i for each $i \in [t]$. For each $x \in V(K_N)$ and $i \in [t]$, define $q_i(x)$ to be the largest integer such that there is an i-colored copy of $P^{2,1}_{q_i(x)}$ with x as its last vertex. As c avoids $P^{2,1}_{e_i}$ in color i for each $i \in [t]$, $0 \leq q_i(x) \leq e_i - 1$ for every i. Thus, if we let $q(x) = (q_1(x), \ldots, q_t(x))$, we see that there are at most $\prod_{i=1}^{t} e_i$ distinct values that q can attain. Thus, by the pigeonhole principle, there must be two vertices $x, y \in V(K_N)$ such that $q(x) = q(y)$. Suppose that $x < y$ and $c(x, y) = j$. In this case, if P is the set of edges in a j-colored copy of $P^{2,1}_{q_j(x)}$ ending in x, then $P \cup \{x, y\}$ is a set of edges in a j-colored copy of $P^{2,1}_{q_j(x)+1}$ ending in y. Hence, $q_j(x) < q_j(y)$; a contradiction to the fact that $q(x) = q(y)$, so c must admit a copy of $P^{2,1}_{e_i}$ for some i. We conclude that, $\mathrm{OR}(P^{2,1}_{e_1}, \ldots, P^{2,1}_{e_t}) \leq N$.

Lower bound. Let $N = \prod_{i=1}^{t} e_i$. We will construct a t-coloring of $E(K_N)$ that avoids $P^{2,1}_{e_i}$ in color i for each $i \in [t]$. To begin, let $g : V(K_N) \to \prod_{i=1}^{t} [e_i]$ be a bijection where if $g(x) = (x_1, \ldots, x_t)$ and $g(y) = (y_1, \ldots, y_t)$, then $x < y$ in $V(K_N)$ if and only if $x_i < y_i$ where i is the smallest index where $g(x)$ and $g(y)$ differ.

For $\{x, y\} \in E(K_N)$, let $c(x, y) = i$ whenever i is the smallest index where $g(x)$ and $g(y)$ differ. We claim that c avoids $P^{2,1}_{e_i}$ in color i for each $i \in [t]$. To see this, suppose not, then there is an i-colored copy of $P^{2,1}_{e_i}$ for some $i \in [t]$ with vertices $v^{(1)} < \cdots < v^{(e_i+1)}$ where $g(v^{(j)}) = (v^{(j)}_1, \ldots, v^{(j)}_t)$. As this copy of $P^{2,1}_{e_i}$ is monochromatic in color i, we see that $v^{(1)}_i < v^{(2)}_i < \cdots < v^{(e_i+1)}_i$. This, however, is impossible as there are only e_i distinct possible values for the ith coordinate of $g(x)$ for any x. Hence, c avoids $P^{2,1}_{e_i}$ in color i for each $i \in [t]$, so $\mathrm{OR}(P^{2,1}_{e_1}, \ldots, P^{2,1}_{e_t}) > N$. □

Fox, Pach, Sudakov, and Suk [15] determined the growth of $\mathrm{OR}^3_t(P^{3,2}_e)$ to be exponential in e and doubly-exponential in t, and Moshkovitz and Shapira [21] found that $\mathrm{OR}^k_t(P^{k,k-1}_e)$ grows as a tower of height $k-2$ in e and as a tower of height $k-1$ in t. In fact, Moshkovitz and Shapira determine $\mathrm{OR}^k_t(P^{k,k-1}_e)$ exactly in terms of high-dimensional integer partitions. Additionally, Duffus, Lefmann, and Rödl [11] implicitly studied $\mathrm{OR}_t^k(P_e^{k,k-1})$ (in the language of *shift graphs*) and determined a lower bound similar to that of Moshkovitz and Shapira and also showed that $\mathrm{OR}_2^k(P_2^{k,k-1}) \leq 2k+1$.

Using the bounds of Moshkovitz and Shapira on $\mathrm{OR}_t^k(P_e^{k,k-1})$, and a variation of their proof due to Milans, Stolee, and West [20], we prove the following bounds on the ordered Ramsey number of the monotone loose path.

Theorem 2.2. *For $k < 2\ell < 2k$, $t \geq 2$, e sufficiently large, and $\ell' = \ell-(k-\ell)(i(k,\ell)-1)$,*

$$(k-\ell)\,\mathrm{tow}_{i(k,\ell)-2}(e^{t-1}/2\sqrt{t}) + \ell' \leq \mathrm{OR}_t^k(P_e^{k,\ell}) \leq (k-\ell)\,\mathrm{tow}_{i(k,\ell)-2}(2e^{t-1}) + \ell'.$$

Therefore, the asymptotic growth of $\mathrm{OR}_t^k(P_e^{k,\ell})$ is a tower of height $i(k,\ell)-2$ in terms of e and a tower of height $i(k,\ell)-1$ in terms of t. In fact, when $2\ell \leq k$, or equivalently when $i(k,\ell) = 2$, we can exactly determine $\mathrm{OR}_t^k(P_e^{k,\ell})$.

Corollary 2.3. *For $0 < 2\ell \leq k$ and positive integers $e_1, \ldots, e_t$,*

$$\mathrm{OR}^k(P_{e_1}^{k,\ell}, \ldots, P_{e_t}^{k,\ell}) = (k-\ell)\prod_{i=1}^{t} e_i + \ell.$$

In Sections 2.1.2 and 2.1.3, we provide two proofs of Theorem 2.2, and in Section 2.1.1, we prove a more direct proof of slightly weaker bounds.

In Section 2.1.4, we present an upper bound on the t-color ordered Ramsey number $\mathrm{OR}_t(P_e^{2,1})$ for an arbitrarily-ordered copy of $P_e^{2,1}$ that nearly matches the upper bound on $\mathrm{OR}_t(M)$ for a 2-uniform matching M, which coincides with work of Cibulka, Gao, Krčál, Valla, and Valtr [5] on two colors.

Conlon, Fox, Lee, and Sudakov [8] and Balko, Cibulka, Král, and Kynčl [2] independently investigated how the ordered Ramsey number $\mathrm{OR}_t(G)$ differs among orderings of a 2-uniform graph G. In particular, they investigated upper bounds of $\mathrm{OR}_t(M)$ for a 2-uniform matching M, and found that these upper bounds are nearly sharp. In Section 2.2, we extend the methods in these papers to attain upper bounds on the ordered Ramsey numbers of k-uniform matchings under certain "controlled" orderings.

2.1 Ordered Ramsey Numbers of Loose Paths

We present two different methods of arriving proving Theorem 2.2. In addition, we also present an argument that achieves weaker bounds but is more direct.

In Section 2.1.1, we present a "stepping up" argument in order to bound the size of $\mathrm{OR}_t^k(P_e^{k,\ell})$ based on $\mathrm{OR}_t(P_e^{2,1})$. In particular, we prove that,

Theorem 2.4. *For integers* $0 < \ell < k$ *and* $e > 1$*, let* $k' = k - (k-\ell)(i(k,\ell) - 2)$ *and* $\ell' = \ell - (k-\ell)(i(k,\ell)-2)$*. Then,*

$$\ell' \cdot \left(\mathrm{tow}_{i(k,\ell)-1}(t - O(\lg t)) + 1\right) \leq \mathrm{OR}_t^k(P_e^{k,\ell}) \leq \ell' \cdot 2^{k'-2\ell'} \left(\mathrm{tow}_{i(k,\ell)-1}(t \lg e) + 1\right).$$

Notice that while Theorem 2.4 is weaker than Theorem 2.2, these bounds do in fact show that $\mathrm{OR}_t^k(P_e^{k,\ell})$ asymptotically grows as a tower of height $i(k,\ell) - 1$ in terms of t. In addition, the upper bound does show that $\mathrm{OR}_t^k(P_e^{k,\ell})$ grows at most as a tower of height $i(k,\ell) - 2$ in terms of e; however, the lower bound is not satisfying as it does not have any relation to e.

2.1.1 A "Stepping Up" Argument for Loose Paths

The goal of this "stepping up" argument is to somehow relate $P_e^{k,\ell}$ to $P_e^{2,1}$ given that Theorem 2.1 provides an exact formula for the ordered Ramsey number of latter. We begin by first relating $P_e^{k,\ell}$ to $P_e^{k',\ell'}$ for some k' and ℓ' with the property that $i(k',\ell') = 2$. After this, we then find a relationship between $P_e^{k',\ell'}$ to $P_e^{2,1}$. Putting together these arguments will then lead to the bounds in Theorem 2.4.

Theorem 2.5. *For* $k > \ell$, $\mathrm{OR}^k_{\binom{t}{\lfloor t/2 \rfloor}}(P_e^{k,\ell}) \leq \mathrm{OR}_t^{2k-\ell}(P_e^{2k-\ell,k}) \leq \mathrm{OR}^k_{e^t}(P_2^{k,\ell})$.

Proof. Upper Bound. Let $N = \mathrm{OR}_t^{2k-\ell}(P_e^{2k-\ell,\ell}) - 1$ and let c be a t-coloring of $E(K_N^{2k-\ell})$ that avoids $P_e^{2k-\ell,k}$. For $X \in \binom{[N]}{k}$ and $i \in [t]$, define $q_i(X)$ to be the largest integer so that there is a monochromatic copy of $P_{q_i(X)}^{2k-\ell,k}$ with X as its last k vertices. Define a coloring of $E(K_N^k)$ by $c'(X) = (q_1(X), \ldots, q_t(X))$. As c avoids $P_e^{2k-\ell,k}$, $q_i(X) \in \{0, \ldots, e-1\}$, so c' is an e^t-coloring of $E(K_N^k)$. Now suppose that c' admitted a monochromatic copy of $P_2^{k,\ell}$, then there are edges $J, J' \in \binom{[N]}{k}$ such that $c'(J) = c'(J')$ and $J \cap J'$ consists of the maximum ℓ vertices of J and the minimum ℓ vertices of J'. Thus, $J \cup J' \in \binom{[N]}{2k-\ell}$, so suppose that $c(J \cup J') = j$. Let P be a j-coloring copy of $P_{q_j(J)}^{2k-\ell,k}$ with J as its last vertices. Then $E(P) \cup J' \cong P_{q_j(J)+1}^{2k-\ell,k}$, so $q_j(J') > q_j(J)$; which is a contradiction to the fact that $c(J) = c(J')$. Thus, $\mathrm{OR}_{e^t}^k(P_2^{k,\ell}) > N$.

Lower Bound. Let $N = \mathrm{OR}^k_{\binom{t}{\lfloor t/2 \rfloor}}(P_e^{k,\ell}) - 1$ and let c be a $\binom{t}{\lfloor t/2 \rfloor}$-coloring of $E(K_N^k)$ that avoids $P_e^{k,\ell}$. Associate each color with an element of $\binom{[t]}{\lfloor t/2 \rfloor}$.

For $J \subseteq [N]$ let J^+ be the k largest elements of J and let J^- be the k smallest elements of J.

For $X \in \binom{[N]}{2k-\ell}$, X^- and X^+ form a copy of $P_2^{k,\ell}$. If $c(X^-) = c(X^+)$, let $c'(X)$ be any element of $c(X^-)$. If $c(X^-) \neq c(X^+)$, let $c'(X)$ be any element of $c(X^-) \setminus c(X^+)$.

Let Q be the vertex set of a copy of $P_e^{k,\ell}$ and let $\hat{Q} = Q \cup \{x_1, \ldots, x_{k-\ell}\}$ where $\max Q < x_1 < \cdots < x_{k-\ell}$. Notice that $\hat{Q}$ can be considered to be the vertex set of a copy of $P_e^{2k-\ell,k}$ and also as the vertex set of a copy of $P_{e+1}^{k,\ell}$. As c avoids monochromatic copies of $P_e^{k,\ell}$, there must be k-uniform edges J and J' coming from the copy of $P_e^{k,\ell}$ induced by Q that form a $P_2^{k,\ell}$ with $c(J) \neq c(J')$. Let J'' be the k-uniform edge coming from the copy of $P_{e+1}^{k,\ell}$ induced by $\hat{Q}$ such that J, J', and J'' form a copy of $P_3^{k,\ell}$. Thus, we observe that $c'(J \cup J') \in c(J) \setminus c(J')$ where $c'(J' \cup J'') \in c(J')$, so $c'(J \cup J') \neq c'(J' \cup J)$. Thus, $\hat{Q}$ does not induce a monochromatic copy of $P_e^{2k-\ell,k}$, so c' avoids monochromatic copies of $P_e^{2k-\ell,k}$ and $\mathrm{OR}_t^{2k-\ell}(P_e^{2k-\ell,k}) > N$.

$\square$

We can continue to apply this bound until $2\ell \leq k$. In particular, if $k' = k - (k-\ell)(i(k,\ell)-2)$ and $\ell' = \ell - (k-\ell)(i(k,\ell)-2)$, then $2\ell' \leq k'$, so we can relate $P_e^{k,\ell}$ to $P_e^{k',\ell'}$ through Theorem 2.5. In particular, define an analogue of the tower function for middle binomial coefficients by letting $b^{(0)}(x) = x$ and for $m \geq 1$, $b^{(m)}(x) = \binom{b^{(m-1)}(x)}{\lfloor b^{(m-1)}(x)/2 \rfloor}$. By iterating the bounds found in Theorem 2.5, we arrive at the following corollary.

Corollary 2.6. *For* $\ell < k < 2\ell$*, let* $k' = k - (k-\ell)(i(k,\ell)-2)$ *and* $\ell' = \ell - (k-\ell)(i(k,\ell)-2)$*. Then*

$$\mathrm{OR}^{k'}_{b^{(i(k,\ell)-2)}(t)}(P_e^{k',\ell'}) \leq \mathrm{OR}^{k}_{t}(P_e^{k,\ell}) \leq \mathrm{OR}^{k'}_{\mathrm{tow}_{i(k,\ell)-2}(t\lg e)}(P_2^{k',\ell'}).$$

Through this corollary, we have successfully reduced finding bounds on $\mathrm{OR}^k_t(P_e^{k,\ell})$ to finding bounds on $\mathrm{OR}^{k'}_t(P_e^{k',\ell'})$ where $2\ell' \leq k'$. All that remains to do is to relate $\mathrm{OR}^{k'}_t(P_e^{k',\ell'})$ to $\mathrm{OR}_t(P_e^{2,1})$. We do this through the next two theorems.

Theorem 2.7. *For* $p \in \mathbb{Z}^+$*,* $\mathrm{OR}^{pk}_t(P_e^{pk,p\ell}) = p \cdot \mathrm{OR}^k_t(P_e^{k,\ell})$*.*

Proof. Let $N = \mathrm{OR}^{pk}_t(P_e^{pk,p\ell})$ and $N' = \lfloor N/p \rfloor$. For some $x \in [N]$ define the blow-up of x to be $h(x) = \{p(x-1)+1, p(x-1)+2, \ldots, px\}$. For $\{x_1, \ldots, x_k\} \in \binom{N'}{k}$ extend h so that $h(x_1, \ldots, x_k) = \bigcup_{i=1}^k h(x_i)$, which is an element of $\binom{N}{pk}$. Further for a set $\{x_1, \ldots, x_{pk}\} \in \binom{N}{pk}$ define the reduction of this set to be $r(x_1, \ldots, x_{pk}) = \{\lceil x_1/p \rceil, \lceil x_p/p \rceil, \ldots, \lceil x_{p(k-1)}/p \rceil\}$, which is an element of $\binom{N'}{k}$.

Lower Bound. Let c be a t-coloring of $E(K^{pk}_{N-1})$ that avoids $P_e^{pk,p\ell}$. Let c' be a coloring of $E(K^k_{N'-1})$ defined by $c'(x_1, \ldots, x_k) = c(h(x_1, \ldots, x_k))$. If J and J' are the edges of a copy of $P_2^{k,\ell}$ in $K^k_{N'-1}$, then $h(J)$ and $h(J')$ form a copy of $P_2^{pk,p\ell}$ in K^{pk}_{N-1}. Therefore, if $J_1, \ldots, J_e$ are the edges of a monochromatic copy of $P_e^{k,\ell}$ under c', then $h(J_1), \ldots, h(J_e)$ are the edges of a monochromatic copy of $P_e^{pk,p\ell}$ under c; a contradiction. Therefore, c' avoids $P_e^{k,\ell}$, so $\mathrm{OR}^k_t(P_e^{k,\ell}) > N' - 1$.

Upper Bound. Let c be any t-coloring of $E(K^k_{N'})$. Let c' be a coloring of $E(K^{pk}_N)$ defined by $c'(x_1, \ldots, x_{pk}) = c(r(x_1, \ldots, x_{pk}))$. As $N = \mathrm{OR}^{pk}_t(P_e^{pk,p\ell})$, there must be edge $J_1, \ldots, J_e$ that form a monochromatic copy of $P_e^{pk,p\ell}$ under c'. Therefore, $r(J_1), \ldots, r(J_e)$ form a monochromatic copy of $P_e^{k,\ell}$ under c, so $\mathrm{OR}^k_t(P_e^{k,\ell}) \leq N'$. □

By Theorem 2.7, if we can find a relationship between $\mathrm{OR}^{k'}_t(P_e^{k',\ell'})$ and $\mathrm{OR}^{2\ell'}_t(P_e^{2\ell',\ell'})$, we are done as $\mathrm{OR}^{2\ell'}_t(P_e^{2\ell',\ell'}) = \ell' \cdot \mathrm{OR}_t(P_e^{2,1}) = \ell' \cdot (e^t + 1)$.

Theorem 2.8. *If $2\ell < k$ and $e > 1$, then*

$$\mathrm{OR}_t^{k-1}(P_e^{k-1,\ell}) \le \mathrm{OR}_t^{k}(P_e^{k,\ell}) \le 2 \cdot \mathrm{OR}_t^{k-1}(P_e^{k-1,\ell}).$$

Proof. Lower Bound. Let $N = \mathrm{OR}_t^{k-1}(P_e^{k-1,\ell}) - 1$ and let c be a t-coloring of $E(K_N^{k-1})$ that avoids monochromatic copies of $P_e^{k-1,\ell}$. For $X = \{x_1, \dots, x_k\} \in \binom{[N]}{k}$, define

$$\mathcal{L}(X) = \left\{ X' \in \binom{[N]}{k-1} : X' = X \setminus \{x_i\} \text{ for } \ell < i \le k - \ell \right\}.$$

As $2\ell < k$, $\mathcal{L}(X)$ is always nonempty. Define a t-coloring c' of $E(K_N^k)$ by letting $c'(X)$ to be any element of the set $\{c(X') : X' \in \mathcal{L}(X)\}$. Suppose that $X_1, \dots, X_e$ formed a copy of $P_e^{k,\ell}$ in K_N^k, then $X_1', \dots, X_e'$ form a copy of $P_e^{k-1,\ell}$ in K_N^{k-1} for any $X_i' \in \mathcal{L}(X_i)$. Thus, c' avoids monochromatic copies of $P_e^{k,\ell}$, so $\mathrm{OR}_t^k(P_e^{k,\ell}) > N$.

Upper Bound. Let $M = \mathrm{OR}_t^k(P_e^{k,\ell}) - 1$ and let c be a t-coloring of $E(K_N^k)$ that avoids monochromatic copies of $P_e^{k,\ell}$. Let $M = \lfloor N/2 \rfloor$. For $X = \{x_1, \dots, x_{k-1}\} \in \binom{M}{k-1}$, define

$$\mathcal{U}(X) = \left\{ \{y_1, \dots, y_k\} \in \binom{[N]}{k} : \{y_1, \dots, y_k\} \setminus \{y_i\} = \{2x_1, \dots, 2x_{k-1}\} \text{ for } \ell < i \le k - \ell \right\}.$$

Again, as $2\ell < k$, $\mathcal{U}(X)$ is always nonempty. Define a t-coloring c' of $E(K_M^{k-1})$ by letting $c'(X)$ be any element of $\{c(X') : X' \in \mathcal{U}(X)\}$. Suppose that $X_1, \dots, X_e$ formed a copy of $P_e^{k-1,\ell}$ in K_M^{k-1}, then $X_1', \dots, X_e'$ form a copy of $P_e^{k,\ell}$ in K_N^k for any $X_i' \in \mathcal{U}(X_i)$. Thus, c' must avoid monochromatic copies of $P_e^{k-1,\ell}$. We conclude that $\mathrm{OR}_t^{k-1}(P_e^{k-1,\ell}) > M$, so $2 \cdot \mathrm{OR}_t^{k-1}(P_e^{k-1,\ell}) > N$. □

Theorems 2.7 and 2.8 and the fact that $\mathrm{OR}_t(P_e^{2,1}) = e^t + 1$, directly imply the following corollary.

Corollary 2.9. *For $0 < 2\ell \le k$,*

$$\ell \cdot \left(e^t + 1\right) \le \mathrm{OR}_t^k(P_e^{k,\ell}) \le \ell \cdot 2^{k-2\ell} \cdot \left(e^t + 1\right).$$

Finally, putting together Corollaries 2.6 and 2.9, we arrive at the following.

Theorem 2.10. *For integers* $0 < \ell < k$ *and* $e > 1$*, let* $k' = k - (k-\ell)(i(k,\ell) - 2)$ *and* $\ell' = \ell - (k-\ell)(i(k,\ell)-2)$*. Then,*

$$\ell' \cdot \left(e^{b^{(i(k,\ell)-2)}(t)} + 1\right) \leq \mathrm{OR}^k_t(P^{k,\ell}_e) \leq \ell' \cdot 2^{k'-2\ell'} \left(\mathrm{tow}_{i(k,\ell)-1}(t \lg e) + 1\right)$$

Using the fact that $b^{(m)}(x) \geq \mathrm{tow}_m(x - O(\lg x))$ and $e^{\mathrm{tow}_m(x)} \geq \mathrm{tow}_{m+1}(x)$, we arrive at Theorem 2.4.

2.1.2 A Direct Relationship Between Loose and Tight Paths

In this section, we prove the bounds in Theorem 2.2 by finding a relationship between $\mathrm{OR}^k_t(P^{k,\ell}_e)$ and $\mathrm{OR}^{i(k,\ell)}_t(P^{i(k,\ell),i(k,\ell)-1}_e)$. This will directly imply Theorem 2.2 due to the following theorem of Moshkovitz and Shapira [21]

Theorem 2.11 (Moshkovitz and Shapira [21])**.** *For positive integers* k *and* t*, and* e *sufficiently large,*

$$\mathrm{tow}_{k-2}(e^{t-1}/2\sqrt{t}) \leq \mathrm{OR}^k_t(P^{k,k-1}_e) \leq \mathrm{tow}_{k-2}(2e^{t-1}).$$

We accomplish this through the following theorem.

Theorem 2.12. *For* $k > \ell \geq 1$*,* $i = i(k,\ell)$*, and positive integers* $e_1, \ldots, e_t$*,*

$$\mathrm{OR}^k(P^{k,\ell}_{e_1}, \ldots, P^{k,\ell}_{e_t}) = (k-\ell)\,\mathrm{OR}^i(P^{i,i-1}_{e_1}, \ldots, P^{i,i-1}_{e_t}) + \ell - (k-\ell)(i-1).$$

Proof. Let $i = i(k,\ell)$ and $\ell' = \ell - (k-\ell)(i-1)$. Let $N = \mathrm{OR}^i(P^{i,i-1}_{e_1}, \ldots, P^{i,i-1}_{e_t})$ and $N' = (k-\ell)N + \ell'$.

For a k-uniform edge $\{x_1, \ldots, x_k\}$, we define the *rational reduction*, denoted $\underline{r}(x_1, \ldots, x_k)$, to be the the i-uniform edge $\{\lceil x_1/(k-\ell)\rceil, \lceil x_{(k-\ell)+1}/(k-\ell)\rceil, \ldots, \lceil x_{(i-1)(k-\ell)+1}/(k-\ell)\rceil\}$. For an i-uniform edge $\{x_1, \ldots, x_i\}$, the *canonical preimage*, denoted $\underline{r}^{-1}(x_1, \ldots, x_i)$, is defined as

$$\underline{r}^{-1}(x_1, \ldots, x_i) = \left[\bigcup_{j=1}^{i-1}\bigcup_{a=1}^{k-\ell}\{(k-\ell)(x_j - 1) + a\}\right] \cup \left[\bigcup_{a=1}^{\ell'}\{(k-\ell)(x_i - 1) + a\}\right].$$

Observe that $(i-1)(k-\ell)+\ell' = k$ and hence $\underline{r}^{-1}(x_1,\ldots,x_i)$ has k ordered elements. Finally, note that $\underline{r}$ sends k-uniform edges from $K^k_{N'}$ to i-uniform edges in K^i_N and $\underline{r}^{-1}$ sends i-uniform edges from K^i_N to k-uniform edges in $K^k_{N'}$.

Lower Bound. There exists a t-coloring $c : E(K^i_{N-1}) \to [t]$ of K^i_{N-1} that avoids a j-colored copy of $P^{i,i-1}_{e_j}$ for each $j \in [t]$. Define a coloring $c' : E(K^k_{N'-1}) \to [t]$ by $c'(x_1,\ldots,x_k) = c(\underline{r}(x_1,\ldots,x_k))$. Suppose that there is a color j and a list $x_1 < \cdots < x_m$ of vertices such that there is a j-colored copy of $P^{k,\ell}_{e_j}$ in c' on the vertices $x_1,\ldots,x_m$. Then, for each k-uniform edge $\{x_p,\ldots,x_{p+k-1}\}$ in this copy of $P^{k,\ell}_{e_j}$, the edge $\underline{r}(x_p,\ldots,x_{p+k-1})$ has color j in c. Also, for two consecutive edges $\{x_p,\ldots,x_{p+k-1}\}$ and $\{x_{p+\ell},\ldots,x_{p+k+\ell-1}\}$ the rational reductions $\underline{r}(x_p,\ldots,x_{p+k-1})$ and $\underline{r}(x_{p+\ell},\ldots,x_{p+k+\ell-1})$ intersect in $i-1$ vertices. Thus, the e_j edges given by the rational reductions form a j-colored copy of $P^{i,i-1}_{e_j}$, a contradiction. Therefore, c' avoids a j-colored copy of $P^{k,\ell}_{e_j}$ and hence $\mathrm{OR}^k(P^{k,\ell}_{e_1},\ldots,P^{k,\ell}_{e_t}) \geq N'$.

Upper Bound[2]. Let $c' : E(K^k_{N'}) \to [t]$ be a t-coloring of $K^k_{N'}$. Define a t-coloring $c : E(K^i_N) \to [t]$ of K^i_N as $c(\{x_1,x_2,\ldots,x_i\}) = c'(\underline{r}^{-1}(x_1,\ldots,x_i))$. By the definition of N, there exists a j-colored copy of $P^{i,i-1}_{e_j}$ on vertices $x_1,\ldots,x_m$ for some $j \in [t]$. For each i-uniform edge $\{x_q,\ldots,x_{q+i-1}\}$ in this copy of $P^{i,i-1}_{e_j}$, the k-uniform edge $\underline{r}^{-1}(x_q,\ldots,x_{q+i-1})$ also has the color j with respect to c'. Further, for two consecutive i-uniform edges $\{x_q,\ldots,x_{q+i-1}\}$ and $\{x_{q+1},\ldots,x_{q+i}\}$ in this copy of $P^{i,i-1}_{e_j}$, the k-uniform edges $\underline{r}^{-1}(x_q,\ldots,x_{q+i-1})$ and $\underline{r}^{-1}(x_{q+1},\ldots,x_{q+i})$ intersect in exactly ℓ vertices. Therefore, there is a j-colored copy of $P^{k,\ell}_{e_j}$ with respect to the coloring c' and therefore $\mathrm{OR}^k(P^{k,\ell}_{e_1},\ldots,P^{k,\ell}_{e_t}) \leq N'$. □

2.1.3 An Approach via Posets

To study the ordered Ramsey number of loose paths, we first review the previous results on the ordered Ramsey number of tight paths. For a poset $P = (P, \subseteq)$, a *down-*

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[2] *The authors thank Josef Cibulka for providing the translation of colorings in this direction.*

set is a set $S \subseteq P$ such that if $y \in S$ and $x \subseteq y$, then $x \in S$. For a set $A \subseteq P$, let $\mathcal{D}(A)$ be the minimal down-set containing A; observe that $\mathcal{D}$ forms a bijection between antichains and down-sets of P. The poset $\mathcal{J}(P)$ consists of all down-sets in P, ordered by containment.

Let $m, e_1, \ldots, e_t$ be positive integers and $m \geq 1$. Define the poset $Q_m(e_1, \ldots, e_t)$ iteratively as follows: let $Q_1(e_1, \ldots, e_t)$ be a disjoint union of t chains of size $e_1-1, \ldots, e_t-1$, and $Q_{m+1}(e_1, \ldots, e_t) = \mathcal{J}(Q_m(e_1, \ldots, e_t))$. The size of $Q_k(e_1, \ldots, e_t)$ is equal to the largest N such that we can t-color K_N^k while avoiding ordered copies of $P_{e_1}^{k,k-1}, \ldots, P_{e_t}^{k,k-1}$.

Theorem 2.13 (Moshkovitz and Shapira [21]; Milans, Stolee, and West [20])**.** *Let $k, e_1, \ldots, e_t$ be positive integers and $k \geq 2$. Then,*

$$\mathrm{OR}^k(P_{e_1}^{k,k-1}, \ldots, P_{e_t}^{k,k-1}) = |Q_k(e_1, \ldots, e_t)| + 1.$$

We extend this result to loose paths by referring to the same poset definitions. In particular, the most important parameter affecting the asymptotic growth of $\mathrm{OR}_t^k(P_e^{k,\ell})$ is $i(k,\ell)$, and the value k contributes only to the leading constant.

Theorem 2.14. *If $k > \ell \geq 1$ and $e_1, \ldots, e_t$ are positive integers, then*

$$\mathrm{OR}^k(P_{e_1}^{k,\ell}, \ldots, P_{e_t}^{k,\ell}) = (k-\ell)|Q_{i(k,\ell)}(e_1, \ldots, e_t)| + \ell - (k-\ell)(i(k,\ell)-2).$$

Proof. Note that if $e_i = 1$ for any i, then any t-coloring avoiding an i-colored copy of $P_1^{k,\ell}$ will not use the color i; hence e_i can be removed from the list and we can consider $t-1$ coloring. Also note that $Q_1(e_1, \ldots, e_t)$ equals $Q_1(e'_1, \ldots, e'_{t'})$ where $e'_1, \ldots, e'_{t'}$ is the list of integers $e_j \geq 2$ for $j \in [t]$.

Let $i = i(k,\ell)$ and $\ell' = \ell - (k-\ell)(i-2)$. For $m \in [i]$, let $Q_m = Q_m(e_1, \ldots, e_t)$. Let $C_1 \cup \cdots \cup C_t$ be a partition of Q_1 into a disjoint union of t chains such that each C_j contains $e_j - 1$ elements.

Lower Bound. Let $A_1, \ldots, A_{k-\ell}$ be copies of Q_i and let $\pi : \bigcup_{j=1}^{k-\ell} A_j \to Q_i$ be the natural projection map. Also, let L be a chain of size $\ell'-1$. Define $Q_i^* = A_1 \cup \cdots \cup A_{k-\ell} \cup L$ to be a poset with the relation between two distinct elements $x, y \in Q_i^*$ defined as:

- If $x, y \in L$, keep the same relation as in L.
- If $x \in A_j$ and $y \in L$, let $x \subset y$.
- If $x \in A_j$ and $y \in A_{j'}$, where $\pi(x) \neq \pi(y)$, provide x and y with the same relationship as $\pi(x)$ and $\pi(y)$.
- If $x \in A_j$ and $y \in A_{j'}$, where $\pi(x) = \pi(y)$, let $x \subseteq y$ if $j \leq j'$.

We show that $\mathrm{OR}^k(P^{k,\ell}_{e_1}, \ldots, P^{k,\ell}_{e_t}) = |Q^*_i| + 1$.

Fix a linear extension of Q^*_i. We consider π to be a a projection from $Q^*_i \backslash L \to Q_i$. For a list $(x_1, \ldots, x_n)$ in $Q^*_i \setminus L$, we extend π so that $\pi(x_1, \ldots, x_n) = (\pi(x_1), \ldots, \pi(x_n))$. Further, given a list $(x_1, \ldots, x_n)$ in Q^*_i, we define the *reduction* of the list to be $r(x_1, \ldots, x_n) = (x_1, x_{(k-\ell)+1}, \ldots, x_{s(k-\ell)+1})$ where s is the largest integer such that $s(k-\ell)+1 \leq n$.

Notice first that $r(x_1, \ldots, x_{s(k-\ell)+\ell}) = (x_1, x_{(k-\ell)+1}, \ldots, x_{(k-\ell)(s+i-2)+1})$ and that $\ell' = (s(k-\ell)+\ell) - (k-\ell)(s+i-2)$. Hence, if $(x_1, \ldots, x_{s(k-\ell)+\ell})$ is a sublist of the linear extension of Q^*_i, then $r(x_1, \ldots, x_{s(k-\ell)+\ell})$ is a descent-free list in $Q^*_i \setminus L$.

Note that in this linear extension of Q^*_i, if $x \in A_j$ and $y \in A_{j+1}$ with $\pi(x) = \pi(y)$, then there is no $z \in Q^*_i$ such that $x < z < y$. Therefore, if $(x_1, \ldots, x_{s(k-\ell)+\ell})$ is a descent-free list in Q^*_i, then not only is $r(x_1, \ldots, x_{s(k-\ell)+\ell})$ a descent-free list in $Q^*_i \setminus L$, but $\pi(r(x_1, \ldots, x_{s(k-\ell)+\ell}))$ is a descent-free list with no repetition in Q_i.

Now, consider $2 \leq m \leq i$ and let $x, y \in Q_m$ with $x \not\supseteq y$. Let $f_m(x, y)$ be some element of the set $y \setminus x$ inside of Q_{m-1}. Further, we extend f_m so that if $(x_1, \ldots, x_n)$ is a descent-free list in Q_m, then $f_m(x_1, \ldots, x_n) = (f_m(x_1, x_2), \ldots, f_m(x_{n-1}, x_n))$. If $x \not\supseteq y$ and $y \not\supseteq z$, then $f_m(x, y) \in y \setminus x$ and $f_m(y, z) \in z \setminus y$, so $f_m(x, y) \not\supseteq f_m(y, z)$ as elements in Q_{m-1}. Hence, if $(x_1, \ldots, x_n)$ is a descent-free list in Q_m, then $f_m(x_1, \ldots, x_n)$ is a descent-free list of length $n-1$ in Q_{m-1}. For a decent-free list $(x_1, \ldots, x_n)$ in Q_i, define $f^{(0)}(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_n)$ and $f^{(h)}(x_1, \ldots, x_n) = f_{i-h}(f^{(h-1)}(x_1, \ldots, x_n))$. Observe that if $(x_1, \ldots, x_n)$ is a descent-free list of length n in Q_i, then $f^{(h)}(x_1, \ldots, x_n)$ is a descent-free list of length $n-h$ in Q_{i-h}.

For a descent-free list $(x_1, \ldots, x_k)$ in Q_i^*, let $(y_1, \ldots, y_i)$ be defined as

$$(y_1, \ldots, y_i) = (\pi(x_1), \pi(x_{(k-\ell)+1}), \ldots, \pi(x_{(k-\ell)(i-1)+1})) = \pi(r(x_1, \ldots, x_k)).$$

Observe that $(y_1, \ldots, y_i)$ is a descent-free list in Q_i, so $f^{(i-1)}(y_1, \ldots, y_i)$ is an element in Q_1.

For $N = |Q_i^*|$, define a t-coloring c on $E(K_N^k)$ as $c(x_1, \ldots, x_k) = j$ whenever $f^{(i-1)}(y_1, \ldots, y_i) \in C_j$, for $(y_1, \ldots, y_i) = \pi(r(x_1, \ldots, x_k))$. We now demonstrate that the coloring c avoids a j-colored $P_{e_j}^{k,\ell}$ for all colors $j \in [t]$.

Suppose that $(x_1, \ldots, x_{s(k-\ell)+\ell})$ is the vertex set of a j-colored copy of $P_s^{k,\ell}$ for some $s \geq 1$. Let

$$(y_1, \ldots, y_{s+i-1}) = (\pi(x_1), \ldots, \pi(x_{(k-\ell)(s+i-2)+1})) = \pi(r(x_1, \ldots, x_{s(k-\ell)+\ell})).$$

Notice that $(x_{(k-\ell)(r-1)+1}, \ldots, x_{(k-\ell)(r-1)+k})$ is an edge of $P_s^{k,\ell}$ for $r \in \{1, \ldots, s\}$, and

$$(y_r, y_{r+1}, \ldots, y_{r+i-1}) = \pi(r(x_{(k-\ell)(r-1)+1}, \ldots, x_{(k-\ell)(r-1)+k})).$$

Thus, $f^{(i-1)}(y_r, y_{r+1}, \ldots, y_{r+i-1})$ is an element of the chain C_j, so $f^{(i-1)}(y_1, \ldots, y_{s+i-1})$ is a descent-free list of length s in C_j. Because a descent-free list in a chain must be strictly increasing, $s \leq |C_j| = e_j - 1$. Thus, c avoids $P_{e_j}^{k,\ell}$ in color j for each $j \in [t]$.

Upper Bound. Let c be a t-coloring of $E(K_N^k)$ that avoids $P_{e_j}^{k,\ell}$ in color j for all $j \in [t]$. We will show that $N \leq (k-\ell)|Q_i| + \ell' - 1$.

For $Y \subseteq [N]$ with $|Y| = h > k - \ell$, let Y^+ denote the $h - (k-\ell)$ largest elements of Y and Y^- denote the $h - (k-\ell)$ smallest elements of Y. We will begin by iteratively defining a function $g_m : \binom{[N]}{k-(m-1)(k-\ell)} \to Q_m$ for $m \in [i]$ with the property that for all $Y \in \binom{[N]}{k-(m-2)(k-\ell)}$, $g_m(Y^-) \not\supseteq g_m(Y^+)$.

We start with the case $m = 1$. Suppose that $X \in \binom{[N]}{k}$ with $c(X) = j$. Let h be the largest integer such that there is an j-colored $P_h^{k,\ell}$ that has X as its maximum edge. Because c avoids $P_{e_j}^{k,\ell}$ in color j, $h \leq e_j - 1$. Supposing that $x_1 \subset \cdots \subset x_{e_j-1}$ are the elements of C_j in Q_1, let $g_1(X) = x_h$. For $Y \in \binom{[N]}{2k-\ell}$, if $c(Y^-) \neq c(Y^+)$, then $g_1(Y^-)$ and

$g_1(Y^+)$ are in different chains of Q_1, so they are not comparable. If $c(Y^-) = c(Y^+)$, then $g_1(Y^+) \supseteq g_1(Y^-)$ because Y^- and Y^+ form a $P_2^{k,\ell}$ in color $c(Y^-) = c(Y^+)$. Therefore $g_1(Y^-) \not\supseteq g_1(Y^+)$.

Let $1 < m \leq i$, and for $X \in \binom{[N]}{k-(m-1)(k-\ell)}$, define $g_m(X) = \mathcal{D}(\{g_{m-1}(Y) : Y^+ = X\})$. Because $Q_m = \mathcal{J}(Q_{m-1})$, $g_j(X) \in Q_j$. Suppose that $Y \in \binom{[N]}{k-(m-2)(k-\ell)}$ and note that $g_{m-1}(Y) \in g_m(Y^+)$. If also $g_{m-1}(Y) \in g_m(Y^-)$, then there is some $Z \in \binom{[N]}{k-(m-2)(k-\ell)}$ such that $Z^+ = Y^-$ and $g_{m-1}(Y) \subseteq g_{m-1}(Z)$. For $W = Y \cup Z$, it holds that $W^- = Z$ and $W^+ = Y$, so $g_{m-1}(W^-) \supseteq g_{m-1}(W^+)$; a contradiction. Therefore, $g_{m-1}(Y) \in g_m(Y^+) \setminus g_m(Y^-)$, so $g_m(Y^-) \not\supseteq g_m(Y^+)$.

Now that g_i is defined, and g_i maps $\binom{[N]}{\ell'}$ to Q_i, we construct a function $\phi : \{\ell', \ldots, N\} \to Q_i$. For $\ell' \leq x \leq n$, let $\phi(x) = g_i(\{x - \ell' + 1, \ldots, x\})$. We claim that for any $R \in Q_i$, $|\phi^{-1}(R)| \leq k - \ell$. If $\ell' \leq x_1 < \cdots < x_{k-\ell+1} \leq n$, then $\phi(x_1) = \cdots = \phi(x_{k-\ell+1})$. Let $W = \{x_{k-\ell+1} - \ell' + 1, \ldots, x_{k-\ell+1}\}$ and $Y = \{x_1 - \ell' + 1, \ldots, x_1\}$. Since $\phi(x_1) = \phi(x_{k-\ell+1})$ by assumption, we have $g_i(Y) = g_i(W)$. In particular, $g_i(Y) \supseteq g_i(W)$ as elements in Q_i. Realizing that $x_{k-\ell-\ell'+1} < \min W$, let $X = Y \cup \{x_1, \ldots, x_{k-\ell-\ell'+1}\} \cup W$. Note that $|X| = \ell' + k - \ell$ and that $X^- = Y$ while $X^+ = W$. However, $X \in \binom{[N]}{\ell'+k-\ell}$ and $g_i(X^-) \not\supseteq g_i(X^+)$, a contradiction.

Since $|\phi^{-1}(R)| \leq k - \ell$ for all $R \in Q_i$, $N - \ell' + 1 \leq (k - \ell)|Q_i|$, so $N \leq (k - \ell)|Q_i| + \ell' - 1$. □

Theorem 2.2 follows from Theorems 2.13 and 2.14. Corollary 2.3 follows from Theorem 2.14 after observing that $|Q_2(e_1, \ldots, e_t)| = \prod_{j=1}^{t} e_j$ because we can select a down-set of $Q_1(e_1, \ldots, e_t)$ by selecting at most one element from each chain to be a maximal element of the down-set.

For $m \geq 3$, the value of $|Q_m(e_1, \ldots, e_t)|$ is not known exactly, but note that $|Q_3(e_1, \ldots, e_t)|$ is the number of antichains in $Q_2(e_1, \ldots, e_t)$. When $e_1 = \cdots = e_t = 2$, the poset $Q_2(e_1, \ldots, e_t)$ is the t-dimensional boolean lattice, denoted $2^{[t]}$, and counting the number of antichains in $2^{[t]}$ is already a famous and difficult problem known as Dedekind's problem. Thus, we will use the bounds of Moshkovitz and Shapira on $\mathrm{OR}^k_t(P^{k,k-1}_e)$ [21, Corollary 3] to find the following corollary.

In [17], Gerencsér and Gyárfás showed that for $n \geq m \geq 1$,

$$\mathrm{R}(P^{2,1}_n, P^{2,1}_m) = n + \left\lfloor \frac{m}{2} \right\rfloor + 2.$$

Comparatively, $\mathrm{OR}(P^{2,1}_n, P^{2,1}_m) = nm + 1$, which shows a large discrepancy between the ordered and unordered variants of the Ramsey number in just the 2-uniform case. It should, however, be noted that over all orderings of a (k, ℓ)-path, the standard ordering on $P^{k,\ell}_e$ does not necessarily minimize the ordered Ramsey number. For example, it is easy to observe that there exists an ordering of $P^{k,k-1}_2$ such that $\mathrm{OR}^k_t(P^{k,k-1}_2) \leq k + t$.

The proof of Theorem 2.2 using Theorem 2.14 is valuable because it shows a direct connection between the poset $Q_i(e_1, \ldots, e_t)$ and the ordered Ramsey number $\mathrm{OR}^k(P^{k,\ell}_{e_1}, \ldots, P^{k,\ell}_{e_t})$ and the best asymptotic bounds on the ordered Ramsey numbers come from this poset perspective.

2.1.4 2-Uniform Paths

Now that we have determined the ordered Ramsey number for a particularly "nice" ordering of a (k, ℓ)-path, it is natural to ask for general bounds on $\mathrm{OR}^k_t(P^{k,\ell}_e)$ where the vertices of $P^{k,\ell}_e$ are ordered arbitrarily. In order to simplify that statement of the next lemma and theorem, we deviate slightly from our standard notation and use P_p instead of $P^{2,1}_{p-1}$ to denote the 2-uniform path on p vertices. The case for $t = 2$ was independently proven by Cibulka, Gao, Krčál, Valla, and Valtr [5, Theorem 6].

Lemma 2.15. *Let n and p be positive integers, and let P_{2^p} be any ordering of the 2-uniform ordered path on 2^p vertices. Then*

$$\mathrm{OR}(K_{2^n}, \overbrace{P_{2^p}, \ldots, P_{2^p}}^{t-1}) \leq 2^{\frac{1}{p}\left((p+1)^{t-1}(np-1)+1\right)}.$$

Proof. We prove by first showing that the theorem holds for all n when $t = 2$, and then continue by induction on t. For $n = 1$ and $t = 2$, we see that $\mathrm{OR}(K_2, P_{2^p}) = 2^p = 2^{\frac{1}{p}((p+1)(p-1)+1)}$.

Let $V(P_{2^p}) = \{v_1, \ldots, v_{2^p}\}$ with indices $i_1, \ldots, i_{2^p}$ defined such that the ordering on $V(P_{2^p})$ is $v_{i_1} < \cdots < v_{i_{2^p}}$.

Consider a 2-coloring c of $E(K_N)$ where $N = 2^{(p+1)n-1} = 2^p M$ with $M = 2^{(p+1)(p-1)}$. Let $V_1, \ldots, V_{2^p}$ be intervals partitioning $[N]$ with $|V_i| = M$ and $\max V_i < \min V_{i+1}$. As per the ordering of $V(P_{2^p})$, let $U_j = V_{i_j}$. Thus, any path $(u_1, \ldots, u_{2^p})$ with $u_j \in U_j$ is a copy of P_{2^p}.

For $j \in [2^p]$ define A_j to be the set of vertices v in U_j such that there exist $u_k \in U_k$ for $k \in [j-1]$ such that $c(u_1, u_2) = c(u_2, u_3) = \cdots = c(u_{j-1}, v) = 2$. Notice that $A_1 = U_1$ and $A_{2^p} = \emptyset$ by the assumption that c avoids P_{2^p} in color 2. Let I be the largest integer such that $|A_I| \geq M/2$; thus, let $A = A_I$ and $B = U_{I+1} \setminus A_{I+1}$. Note that $|B| \geq M/2$ and the bipartite graph induced by (A, B) has no edges of color 2.

Observe that $M/2 = 2^{(e+1)(n-1)-1} \geq \mathrm{OR}(K_{2^{n-1}}, P_{2^p})$ by the induction hypothesis on n. Therefore, A or B has a P_{2^p} in color 2 or both have a copy of $K_{2^{n-1}}$ in color 1. If the former is true, we are done, so suppose the latter holds. Therefore, $A \cup B$ has a K_{2^n} in color 1, so $\mathrm{OR}(K_{2^n}, P_{2^p}) \leq 2^{(p+1)n-1}$.

Now, suppose that $t > 2$ and consider a t-coloring, c, of $E(K_N)$ for $N = 2^{\frac{1}{p}\left((p+1)^{t-1}(np-1)+1\right)}$. Realizing that $\frac{(p+1)^{t-1}(np-1)+1}{p} = (p+1)\frac{(p+1)^{t-2}(np-1)+1}{p} - 1$, we find through the $t = 2$ case that

$$N \geq \mathrm{OR}(K_{2^{\frac{1}{p}((p+1)^{t-2}(np-1)+1)}}, P_{2^p}).$$

Thus, c either has a P_{2^p} in color t or a $K_{2^{\frac{1}{p}((p+1)^{t-2}(np-1)+1)}}$ which is void of color t. If the former holds, then we are done, so suppose the latter holds. By the induction hypothesis on t,

$$2^{\frac{1}{p}\left((p+1)^{t-2}(np-1)+1\right)} \geq \mathrm{OR}(K_{2^n}, \overbrace{P_{2^p}, \ldots, P_{2^p}}^{t-2});$$

therefore, we either have a K_{2^n} in color 1 or a P_{2^p} in some color $j \in \{2, \dots, t-1\}$. $\square$

Lemma 2.15 immediately implies the following theorem.

Theorem 2.16. *Let P_p be any ordered 2-uniform path on p vertices, then*

$$\mathrm{OR}_t(P_p) \leq 2^{\frac{1}{\lceil \lg p \rceil}\left((\lceil \lg p \rceil+1)^{t-1}(\lceil \lg p \rceil^2-1)+1\right)} = 2^{O\left(\lg^t p\right)}.$$

As a means to a lower bound on this value, Conlon, Fox, Lee and Sudakov [8] provided the following lower bound on the ordered Ramsey number of a randomly-ordered 2-uniform matching, which was also proved in a weaker form by Balko, Cibulka, Král and Kynčl [**?**].

Theorem 2.17 (Conlon, Fox, Lee and Sudakov [8, Theorems 2.3])**.** *There exists a positive constant c, such that if M is a randomly-ordered matching on e edges, then asymptotically almost surely,*

$$\mathrm{OR}_2(M) \geq (2e)^{c \lg(2e)/\lg\lg(2e)}.$$

Since P_p contains a matching of size $\lfloor p/2 \rfloor$, we see that almost every ordering of P_p yields $\mathrm{OR}_2(P_p) \geq 2^{\Omega(\lg^2 p/\lg\lg p)}$. Hence, Theorem 2.16 is fairly tight when $t = 2$. Therefore, for almost every ordering of P_p, $\mathrm{OR}_t(P_p)$ grows as a quasi-polynomial in p for a fixed t and possibly double-exponentially in t for a fixed p. Comparatively, for the standard ordering of P_p, $\mathrm{OR}_t(P_p)$ grows polynomially in p and exponentially in t.

2.2 Ordered Ramsey Numbers of k-Uniform Matchings

Recall that the ordered path $P_e^{k,0}$ has disjoint edges, and therefore is a matching. The proof of Theorem 2.14 holds for $\ell = 0$, but instead we will consider a more general class of ordered matchings.

For a fixed $0 \leq r \leq k$ and positive integer e, the *(k,r)-nested matching on e edges* is the ordered graph $M_e^{k,r}$ defined iteratively as: $E(M_1^{k,r})$ consists of one edge $A_1 = [k]$, and $E(M_{e+1}^{k,r})$ consists of the edges in $E(M_e^{k,r})$ and an edge A_{e+1} consisting of the r least integers greater than $\max V(M_e^{k,r})$ and the $k-r$ greatest integers less than $\min V(M_e^{k,r})$. We say (k,r) is the *nesting pattern* of $M_e^{k,r}$. Note that $M_e^{k,r}$ is isomorphic to $M_e^{k,k-r}$ when the ordering is reversed, and $M_e^{k,0} \cong M_e^{k,k} \cong P_e^{k,0}$.

In [1], Alon, Frankl and Lovász show that for integers $e_1 \geq \cdots \geq e_t$, if M_i is a k-uniform matching on e_i edges, then

$$\mathrm{R}^k(M_1, \ldots, M_t) = ke_1 + \sum_{i=2}^{t}(e_i - 1).$$

This value is not far from the value of the ordered Ramsey number for k-uniform nested matchings. The following lemma presents a lower bound on the ordered Ramsey number of t k-uniform nested matchings, even if the nesting patterns differ among the matchings.

Lemma 2.18. *For positive integers $e_1, \ldots, e_t$ and $r_1, \ldots, r_t \in \{0, \ldots, k\}$,*

$$\mathrm{OR}^k(M_{e_1}^{k,r_1}, \ldots, M_{e_t}^{k,r_t}) \geq k\left(1 + \sum_{i=1}^{t}(e_i - 1)\right).$$

Proof. Let $N = k\left(1 + \sum_{i=1}^{t}(e_i - 1)\right) - 1$. Let $L_1, \ldots, L_t, R_1, \ldots, R_t$ be intervals partitioning $[N]$, with $L_1 = R_1$, such that for $i \in \{1, \ldots, t-1\}$, $\max L_{i+1} < \min L_i$ and $\max R_i < \min R_{i+1}$. Further, let $|L_1| = ke_1 - 1$, and for $i \in \{2, \ldots, t\}$ let $|L_i| = (k - r_i)(e_i - 1)$ and $|R_i| = r_i(e_i - 1)$. For an edge $X \in \binom{[n]}{k}$, let $c(X) = \max\{i : X \cap (L_i \cup R_i) \neq \emptyset\}$. The interval L_1 is too small for c to contain a copy of $M_{e_1}^{k,r_1}$ in color 1.

Suppose that c contained a copy of $M_{e_i}^{k,r_i}$ in color i for some $i \in \{2, \ldots, t\}$. If $r_i = k$, then $L_i = \emptyset$ and $|R_i| = k(e_i - 1)$; therefore some edge of $M_{e_i}^{k,r_i}$ does not intersect R_i and hence does not have color i. The case $r_i = 0$ is similar, except $|L_i| = k(e_i - 1)$ and $R_i = \emptyset$.

Now suppose $1 \leq r_i < k$. Let $p_1, \ldots, p_{e_i}$ be the minimum vertices of the edges of $M_{e_i}^{k,r_i}$ and $q_1, \ldots, q_{e_i}$ be the set of maximum vertices, hence $p_1 < p_2 < \cdots < p_{e_i} < q_{e_i} < \cdots < q_1$. In fact, $p_m + k - r_i < p_{m+1}$ and $q_m - r_i > q_{m+1}$ for $m = 1, \ldots, e_i - 1$. Since each edge receives color i, either $p_m \in L_i$ or $q_m \in R_i$ for all m.

However, because $|L_i| = (k-r_i)(e_i-1)$ and $|R_i| = r_i(e_i-1)$, it must be the case that $p_{e_i} \notin L_i$ and $q_{e_i} \notin R_i$. To see this, suppose that $p_{e_i} \in L_i$, then $p_{e_i} - p_1 = (p_{e_i} - p_{e_i-1}) + \cdots + (p_2 - p_1) > (e_i - 1)(k - r_i)$. This, of course, implies that $p_1 \in L_{i'}$ for some $i' > i$, so the color of edge 1 of the copy of $M_{e_i}^{k,r_i}$ would not receive color i; a contradiction, so $p_{e_i} \notin L_i$. Similarly, $q_{e_i} \notin R_i$.

Thus, the color of edge e_i in the copy of $M_{e_i}^{k,r_i}$ does not receive color i; a contradiction. Therefore, c avoids $M_{e_i}^{k,r_i}$ for all i. □

When all nesting patterns are the same, the bound from Lemma 2.18 is sharp.

Theorem 2.19. *For positive integers $e_1, \ldots, e_t$, and $0 \le r \le k$,*

$$\mathrm{OR}^k(M_{e_1}^{k,r}, \ldots, M_{e_t}^{k,r}) = k\left(1 + \sum_{i=1}^{t}(e_i - 1)\right).$$

Proof. The lower bound follows from Lemma 2.18. We prove the upper bound by induction on $\sum_{i=1}^t e_i$. If $\sum_{i=1}^t e_i = t$, then $e_i = 1$ for all i, so $\mathrm{OR}^k(M_{e_1}^{k,r}, \ldots, M_{e_t}^{k,r}) = k$, and the claim holds.

Suppose that $\sum_{i=1}^t e_i > t$ and let c be a t-coloring of $E(K_N^k)$ where $N = k\left(1 + \sum_{i=1}^t (e_i - 1)\right)$. Suppose that $c(\{1, \ldots, r\} \cup \{N-k+r+1, \ldots, N\}) = j$ for some $j \in [t]$. Let G be the graph given by deleting the vertices in $\{1, \ldots, r\} \cup \{N-k+r+1, \ldots, N\}$ from K_N^k. Let $e'_j = e_j - 1$ and $e'_i = e_i$ for $i \neq j$. Notice that $G \cong K_{N-k}^k$ and $N - k = k\left(1 + \sum_{i=1}^t (e'_i - 1)\right)$. Therefore, since $\sum_{i=1}^t e'_i = \sum_{i=1}^t e_i - 1$, the induction hypothesis implies that G contains an i-colored copy of $M_{e'_i}^{k,r_i}$ for some i. Since $e'_i = e_i$ when $i \neq j$, we have $i = j$. Then the j-colored copy of $M_{e_j-1}^{k,r_j}$ along with the edge $\{1, \ldots, r\} \cup \{N-k+r+1, \ldots, N\}$ is a j-colored copy of $M_{e_j}^{k,r_j}$. □

Notice that the $r = 0$ and $r = k$ case of Theorem 2.19 agrees with the bound in Theorem 2.14 using $\ell = 0$. Interestingly, as opposed to the large discrepancy between the ordered and ordinary Ramsey numbers of paths, we see that $\mathrm{OR}_t^k(M_e^{k,r}) \le k \cdot \mathrm{R}_t^k(M_e^{k,r})$. However, this trend does not continue when the ordering of the matching is not nested as in $M_e^{k,r}$. Likely $M_e^{k,r}$ minimizes the ordered Ramsey number $\mathrm{OR}_t^k(M)$ among all orderings of k-uniform matchings M on e edges, though we make no formal conjecture here.

Conlon, Fox, Lee and Sudakov [8] explore the ordered Ramsey numbers of 2-uniform matchings.

Theorem 2.20 (Conlon, Fox, Lee and Sudakov [8])**.** *Let* $M_2, \ldots, M_t$ *be ordered* 2*-uniform matchings, and let* $p \geq 2$*. Then* $\mathrm{OR}(K_p, M_2, \ldots, M_t) \leq \mathrm{OR}(M_2, \ldots, M_t)^{\lceil \lg p \rceil}$*. Therefore, for an ordered* 2*-uniform matching* M *with* e *edges,* $\mathrm{OR}_t(M) \leq (2e)^{\lceil \lg(2e) \rceil^{t-1}} \leq 2^{\lceil \lg(2e) \rceil^t}$.

Compare the upper bound here with the lower bound from Theorem 2.17, showing that this upper bound is nearly tight. In terms of e, the bound above is quasi-polynomial, but in terms of t the bound is doubly-exponential.

Define the k-uniform graph G_s^k iteratively on s as follows: let G_0^k consist of a single vertex, and for $s \geq 1$, let G_s^k consist of k disjoint, consecutive copies of G_{s-1}^k, and introduce every k-uniform edge consisting of exactly one vertex from each copy. Notice that $G_s^2 = K_{2^s}$.

Using the graph G_s^k, we attain a bound on the t-color ordered Ramsey numbers of certain "nice" orderings of k-uniform matchings. This bound is a generalization of Theorem 2.20, where G_s^k replaces the complete graph.

Lemma 2.21. *Let* $M_2, \ldots, M_t$ *be any* k*-uniform ordered matchings and* $s \geq 0$*. Then*

$$\mathrm{OR}^k(G_s^k, M_2, \ldots, M_t) \leq \mathrm{OR}^k(M_2, \ldots, M_t)^s.$$

Proof. We prove by induction on s. When $s = 0$, the graph G_0^k consists of a single vertex, and hence every coloring of K_1^k contains a copy of G_s^k in every color.

Suppose that $s > 0$ and let $r = \mathrm{OR}^k(M_2, \ldots, M_t)$. Suppose, for the sake of contradiction, that c is a t-coloring of $K_{r^s}^k$ that avoids a j-colored copy of M_j for each $j \in \{2, \ldots, t\}$ and avoids a 1-colored copy of G_s^k. Let $V_1, \ldots, V_r$ be equal-sized intervals partitioning $[r^s]$ such that $\max V_i < \min V_{i+1}$ for $i \in [r-1]$. By the induction hypothesis, restricting c to V_i yields either a copy of G^k_{s-1} in color 1 or a j-colored copy of M_j for some $j \in \{2, \ldots, t\}$. Since c contains no j-colored copy of M_j, each V_i contains a copy of $G^k_{(s-1)}$. Since c avoids G^k_s, then for any indices $1 \leq i_1 < \cdots < i_k \leq r$ there must be $x_{i_j} \in V_{i_j}$ such that $c(x_{i_1}, \ldots, x_{i_k}) \neq 1$. Define a coloring of $E(K^k_r)$ by letting $c'(v_{i_1}, \ldots, v_{i_k})$ be any color in $\{c(x_{i_1}, \ldots, x_{i_k}) : x_{i_j} \in V_{i_j}\} \setminus \{1\}$. By the definition of r, c' contains an j-colored copy of M_j for some $j \in \{2, \ldots, t\}$ and therefore c also contains a j-colored copy of M_j; a contradiction. $\square$

Let M be an ordered k-uniform matching on vertex set $[ke]$. We say that M is *k-nestable* if there exist disjoint intervals $I_1, \ldots, I_k$, some of which may be empty or degenerate, spanning $[ke]$ such that $1 \in I_1, ke \in I_k$, where each edge in M either is contained in some interval I_j or intersects all intervals $I_1, \ldots, I_k$, and for each $j \in [k]$ the edges contained within I_j form a matching, denoted M_j, that is either k-nestable or empty. A set of intervals $I_1, \ldots, I_k$ satisfying these properties is a *k-nesting* of M. Notice that every matching contained as a subgraph of G^k_s for some s must be k-nestable; in particular, every 2-uniform matching is 2-nestable as $G^2_s \cong K_{2^s}$. The following lemma provides the converse to this observation.

Lemma 2.22. *If M is a k-uniform hypergraph consisting of a k-nestable matching on e edges and v additional isolated vertices, then M can be embedded into $G^k_{e+\lceil \log_k(e+v) \rceil}$.*

Proof. We prove by induction on e. If $e = 0$, then the claim holds immediately through the fact that G^k_s has k^s vertices.

Now suppose that $e \geq 1$. Let $I_1, \ldots, I_k$ be a k-nesting of M and let M_j be graph with vertex set I_j and edge set $E(M) \cap \binom{I_j}{k}$. Also let $M' = M - \bigcup_j M_j$. In other words, M_j is the matching induced on interval I_j along with all other vertices contained in I_j, and M' is the set of edges that intersect every interval. Notice that some of the M_j's may be empty or only consist of isolated vertices and that M' may be empty as well. Let $e' = |E(M')|$, $e_j = |E(M_j)|$ and v_j be the the number of isolated vertices of M_j.

Let $r = \max_j(e_j + \lceil \log_k(e_j + v_j)\rceil)$, then because $e_j < e$ for all j, M_j can be embedded into G^k_r by the inductive hypothesis. Thus, by embedding M_j into the j'th copy of G^k_r in G^k_{r+1}, we attain an embedding of $\bigcup_j M_j$ into G^k_{r+1}. Finally, it is easy to add the edges of M' into this embedding because the j'th vertex in an edge of M' has been embedded into the j'th copy of G^k_r in G^k_{r+1} due to the original k-nesting of M. Hence, we have an embedding of M into G^k_{r+1}.

Notice that $e_j \leq \min\{e - e', e - 1\}$ for all j and that $v_j \leq v + e'$ because e' new isolated vertices were added to each interval upon ignoring the edges of M'. Therefore, $e_j + 1 \leq e$ and $e_j + v_j \leq e + v$, so $r + 1 \leq e + \lceil \log_k(e + v)\rceil$. We conclude that M embeds into $G^k_{e+\lceil \log_k(e+v)\rceil}$. □

Notice that, Lemma 2.22 implies that a k-nestable matching on e edges embeds into $G^k_{e+\lceil \log_k e\rceil}$. In many cases, Lemma 2.22 will not be tight as $\max_j(e_j + \lceil \log_k(e_j + v_j)\rceil)$ may be substantially smaller than $e + \lceil \log_k(e + v)\rceil$; however, there are k-nestable matchings which come close to showing the tightness of the lemma. It is easy to observe that for $1 \leq r \leq k - 1$, $M^{k,r}_e$ embeds into G^k_{e+1} but not into G^k_e whenever $e \geq 2$. Thus, if $e \leq k$, $M^{k,r}_e$ embeds into $G^k_{e+\lceil \log_k e\rceil}$ but not into G^k_s for any $s < e + \lceil \log_k e\rceil$.

The following theorem follows from Lemmas 2.21 and 2.22 and the fact that $\mathrm{OR}^k_1(M) = ek$ if M is a k-uniform ordered matching with e edges.

Theorem 2.23. *Let $k \geq 3$ and $e \geq 2$. If M is a k-nestable ordered matching with e edges, then $\mathrm{OR}^k_t(M) \leq (ek)^{\lceil e+\log_k e\rceil^{t-1}} = k^{\lceil e+\log_k e\rceil^{t-1}(1+\log_k e)}$.*

This extends the previous bound on 2-uniform matchings [**?**]. While the bound remains doubly-exponential in terms of t, the bound has increased from quasi-polynomial to exponential in terms of e.

Notice that for these "nice" orderings of a k-uniform matching on e edges, the bound on the ordered Ramsey number $\mathrm{OR}_t^k(M)$ is only slightly larger than the ordered Ramsey number $\mathrm{OR}_t^k(P_e^{k,\ell})$ of the naturally-ordered (k,ℓ)-path on e edges when $i(k,\ell)=3$.

We say that a k-uniform ordered matching M is *simply interlacing* if for any pair of distinct edges A, B in M, where $A=\{a_1<a_2<\cdots<a_k\}$ and $B=\{b_1<b_2<\cdots<b_k\}$ either a_i and b_i are consecutive in $A\cup B$ for each i or there is some i where $a_i<b_1<b_k<a_{i+1}$ (where $a_0=-\infty$ and $a_{k+1}=+\infty$). If the former holds, we say that A and B *interlace*, and if the latter holds, we say that A and B *nest*. Notice that every 2-uniform matching is simply interlacing.

Corollary 2.24. *If $k\geq 3$, $e\geq 2$, and M is a simply-interlacing k-uniform ordered matching with e edges, then M is k-nestable; hence* $\mathrm{OR}_t^k(M)\leq k^{\lceil e+\log_k e\rceil^{t-1}(1+\log_k e)}$.

Proof. By Theorem 2.23, it suffices to show that M is k-nestable. Define a relation on the edges of M by $A\preceq B$ if $A=B$ or if $b_i<a_1<a_k<b_{i+1}$ for some $0\leq i\leq k-1$, where $A=\{a_1<\cdots<a_k\}$ and $B=\{b_1<\cdots<b_k\}$ (again under the convention that $b_0=-\infty$). We observe that $\preceq$ is not quite a partial ordering. Suppose that $A=\{a_1,\dots,a_k\}$, $B=\{b_1,\dots,b_k\}$ and $C=\{c_1,\dots,c_k\}$ where $a_k<b_1$, $b_k<c_k$, and $a_i<c_i<a_{i+1}$ for all $1\leq i\leq k-1$. Thus, $A\preceq B$ and $B\preceq C$, but $A\not\preceq C$. Thus, $\preceq$ is not a transitive relation. However, $\preceq$ is reflexive and antisymmetric, so $\preceq$ admits "maximal" elements in the sense that A is maximal if there is no $B\neq A$ such that $A\preceq B$. Let $A_1,\dots,A_p$ be the edges of M that are either maximal with respect to $\preceq$ or interlace with some maximal edge. Therefore, it must be the case that A_i and $A_{i'}$ interlace. We refer to these edges as *spanning edges.*

For each $i\in[p]$, label the vertices in A_i as $A_i=\{a_{i,1}<\cdots<a_{i,k}\}$; also let $a_{i,0}=-\infty$ and $a_{i,k+1}=+\infty$. Observe that for each $j\in[k-1]$, we have $\max_{i\in[p]} a_{i,j}<\min_{i\in[p]} a_{i,j+1}$, as otherwise there is a pair of edge A_i and $A_{i'}$ where $a_{i,j}>a_{i',j+1}$ and hence $a_{i,j}$ and $a_{i',j}$ are not consecutive in $A_i\cup A_{i'}$. Therefore, we can define disjoint intervals $I_1,\dots,I_k$ such that $I_j = [\min_{i\in[p]} a_{i,j}, \max_{i\in[p]} a_{i,j}]$. These intervals do not necessarily span $V(M)$, but we will expand them to include vertices not in $A_1,\dots,A_p$.

For a non-spanning edge B in M, there is at least one edge A_i where $B \prec A_i$. Therefore, there exists a $j \in \{0,\dots,k-1\}$ such that $a_{i,j} < \min B < \max B < a_{i,j+1}$. Observe that since $k \geq 3$, for any $i' \in [p]$ the edge B is comparable to $A_{i'}$ since there is some $a_{i',j'}$ not in the interval $[a_{i,j}, a_{i,j+1}]$. While it may not be the case that $B \prec A_{i'}$, it is true that for every $i' \in [p]$ and $a_{i',j+c_{i'}} < \min B < \max B < a_{i',j+c_{i'}+1}$ for some $c_{i'} \in \{-1,0,+1\}$, as $A_{i'} \prec B$ only when $a_{i',k} < \min B$. Therefore, let j_B be the minimum integer satisfying $j_B \geq 1$ and $j_B \geq j + c_{i'}$ for each $i' \in [p]$.

If B, B' are two non-spanning edges in M and $j_B < j_{B'}$, then $\max B < a_{i,j_B+1}$ for all $i \in [p]$ and $a_{i',j_{B'}} < \min B'$ for some $i' \in [p]$. Then $\max B < a_{i',j_B+1} < \min B'$. Therefore, if for every non-spanning edge B in M we minimally extend the interval I_{j_B} to contain the edge B, the intervals $I_1,\dots,I_k$ will always be disjoint.

Note that the matching M_j given by the edges entirely within the interval I_j is a simply-interlacing k-uniform ordered matching and hence is k-nestable by an inductive argument. Therefore, the intervals $I_1,\dots,I_k$ form a k-nesting of M. □

We conclude by noting that Lemma 2.21 will not apply to most ordered k-uniform matchings for $k \geq 3$. For $k \geq 4$, let A and B be defined as

$$A = \{1,\dots,\lfloor k/2\rfloor\} \cup \{k+1,\dots,k+\lceil k/2\rceil\}, \quad B = \{\lfloor k/2\rfloor+1,\dots,k\} \cup \{k+\lceil k/2\rceil,\dots,2k\}.$$

Observe that the ordered matching with edges A and B is not k-nestable. While every ordered 3-uniform matching on two edges is 3-nestable, there exists an ordered 3-uniform matching that is not 3-nestable. A randomly-ordered matching contains these configurations with high probability, so the bound of Theorem 2.23 does not apply to most ordered matchings.

CHAPTER 3. PARTIALLY-ORDERED RAMSEY NUMBERS

In this chapter, we provide a generalization of ordered Ramsey theory to graphs with a partial ordering on their vertex sets. We begin by describing the theory in its full generality, but then will focus on a particular case which demonstrates the difficulty of the problem.

In order to define partially-ordered Ramsey numbers, we first must understand what it means to have containment between posets.

Suppose that $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets. A *poset homomorphism* is a map $\phi : P \to Q$ such that $\phi(x) \leq_Q \phi(y)$ whenever $x \leq_P y$. We say that P is a *subposet* of Q, or that Q contains a copy of P, if there is an injective poset homomorphism from P to Q. We will often slightly abuse notation and say that $P \subseteq Q$ if Q contains a copy of P.

Beyond this we will often refer to the following concepts in our exploration of partially-ordered Ramsey numbers.

Recall from Section 2.1.3 that for a poset P and a subset $S \subseteq P$, the minimal *downset* containing S is $\mathcal{D}(S) = \{x \in P : x \leq y \text{ for some } y \in S\}$. We will also need a similar notion called an *upset*, defined by $\mathcal{U}(S) = \{x \in P : y \leq x \text{ for some } y \in S\}$.

The *height* of a poset P is defined to be the length of the longest chain contained in P. Along these lines, we can define the ith level of P to be the set of $x \in P$ such that $\mathcal{D}(x)$ has height i. Notice that a poset of height h has exactly h different levels and that each level forms an antichain. The *width* of a poset is the maximum size of an antichain.

Additionally, for a poset $(P, \leq)$, the *dual of* P is the poset $(P, \leq')$ where $x \leq' y$ if and only if $y \leq x$. In other words, the dual of a poset is formed by reversing the original relations.

3.1 The Foundation of Partially-Ordered Ramsey Numbers

Let $P, Q_1, \ldots, Q_t$ be posets. We say that $P \xrightarrow{1} (Q_1, \ldots, Q_t)$ if any t-coloring of P contains a copy of Q_i in color i for some i. If $\mathcal{P} = \{P_n : n \geq 1\}$ and $\mathcal{Q}$ are both families of posets, we say that $\mathcal{P}$ is a *Ramsey host family for* $\mathcal{Q}$ if for any integer t and any $(Q_1, \ldots, Q_t) \in \mathcal{Q}^t$, there is some integer N such that $P_n \xrightarrow{1} (Q_1, \ldots, Q_t)$ for every $n \geq N$. From this, if $\mathcal{P} = \{P_n : n \geq 1\}$ is a Ramsey host family for $\{Q_1, \ldots, Q_t\}$, we can define the 1*-uniform* $\mathcal{P}$*-Ramsey number* of $Q_1, \ldots, Q_t$, denoted $\mathcal{P}\mathrm{R}^1(Q_1, \ldots, Q_t)$, to be the least integer N such that $P_n \xrightarrow{1} (Q_1, \ldots, Q_t)$ for all $n \geq N$. If $Q_1 = \cdots = Q_t = Q$, then we abbreviate $\mathcal{P}\mathrm{R}^1(Q_1, \ldots, Q_t)$ by $\mathcal{P}\mathrm{R}^1_t(Q)$. Notice that if $\mathcal{Q}$ is the class of all finite posets, then $\mathcal{P} = \{P_n : n \geq 1\}$ is a Ramsey host family for $\mathcal{Q}$ if and only if for any positive integer d, there exists another positive integer N such that $[d] \subseteq P_n$ for all $n \geq N$. If $\mathcal{P}$ is a Ramsey host family for any finite poset and $P_n \subseteq P_{n+1}$ for all n, then we say that $\mathcal{P}$ is a *universal Ramsey host family.*

In Section 3.2.1, we explore the connections between the partially-ordered Ramsey number and Turán-type problems in posets.

After looking at ordered graphs, we would like to generalize to graphs that do not have a total ordering on their vertex sets.

Definition 3.1. A *poset-graph* G is a triple $(V(G), E(G), \leq_G)$ where $V(G)$ is a set of vertices, $(V(G), \leq_G)$ is a poset, and $E(G)$ is a subset of the comparable pairs of $(V(G), \leq)$.

By this definition, an ordered graph is a poset-graph $(V(G), E(G), \leq_G)$ where $(V(G), \leq_G)$ is a chain. We will refer to $(V(G), \leq_G)$ as the *underlying poset of* G and will often denote this simply by $V(G)$ when the partial ordering is understood.

For a poset $(P, \leq_P)$, we define the *comparability graph of* P, denoted $\mathcal{G}(P)$, to be the poset-graph with $(V(\mathcal{G}(P)), \leq_{\mathcal{G}(P)}) = (P, \leq_P)$, where $\{x, y\} \in E(\mathcal{G}(P))$ if and only if x and y are comparable in P.

For poset-graphs G and H, we say that H contains a copy of G or that G is a subgraph of H if there is an injective map $\phi : V(G) \to V(H)$ that is both a poset homomorphism and a graph homomorphism. In other words, H contains a copy of G if there is an injective map $\phi : V(G) \to V(H)$ such that $\phi(x) \leq_H \phi(y)$ whenever $x \leq_G y$, and $\{\phi(x), \phi(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$.

The *dual* of a poset graph $(V(G), E(G), \leq_G)$ is simply the poset-graph $(V(G), E(G), \leq'_G)$ where $(V(G), \leq'_G)$ is the dual of $(V(G), \leq_G)$.

Let P be a poset and let $G_1, \ldots, G_t$ be poset-graphs. We say that $P \xrightarrow{2} (G_1, \ldots, G_t)$ if every t coloring of $E(\mathcal{G}(P))$ contains a copy of G_i in color i for some i. If $\mathcal{P} = \{P_n : n \geq 1\}$ has the property that there is an N such that $P_n \xrightarrow{2} (G_1, \ldots, G_t)$ for all $n \geq N$ we can define the 2*-uniform* $\mathcal{P}$*-Ramsey number* of $G_1, \ldots, G_t$, denoted $\mathcal{P}\mathrm{R}(G_1, \ldots, G_t)$, to be the least integer N such that $P_n \xrightarrow{2} (G_1, \ldots, G_t)$ for all $n \geq N$.

We can more generally define a k-uniform poset-hypergraph G to be a triple $(V(G), E(G), \leq_G)$ where $(V(G), \leq_G)$ is a poset and $E(G)$ is a subset of the k-chains of $(V(G), \leq_G)$. As with the 2-uniform poset-graphs, we will refer to $V(G)$ as the underlying poset of G.

We further extend the definition of the comparability graph of a poset P to define $\mathcal{G}^k(P)$ to be the k-uniform poset graph with P as its underlying poset and whose edges consist of every k-chain of P. Note that $\mathcal{G}^2(P) = \mathcal{G}(P)$.

For a poset P and k-uniform poset-graphs $G_1, \ldots, G_t$, we can define $P \xrightarrow{k} (G_1, \ldots, G_t)$ analogously to the $k = 2$ case. We also can define $\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t)$ as expected whenever $\mathcal{P}$ is a Ramsey host family for $G_1, \ldots, G_t$.

In the 1-uniform case, we defined a *universal Ramsey host family* to be a family $\{P_n : n \geq 1\}$ where $P_n \subseteq P_{n+1}$ and for every integer d, there is an N such that $[d] \subseteq P_N$. It is easy to observe that if $\mathcal{P}$ is a universal Ramsey host family, then $\mathcal{P}\mathrm{R}^1(Q_1, \ldots, Q_t)$ exists for any finite posets $Q_1, \ldots, Q_t$. Not surprisingly, it turns out that being a universal Ramsey host family guarantees that $\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t)$ exists for any k-uniform poset-graph $G_1, \ldots, G_t$. While this fact is practically immediate, we include a proof for the sake of completeness.

Proposition 3.2. *Let $\mathcal{P} = \{P_n : n \geq 1\}$ is a universal Ramsey host family, then for any k-uniform poset-graphs $G_1, \ldots, G_t$, $\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t)$ exists.*

Proof. For $i \in [t]$, let G_i' be any linear extension of G_i and let $R = \mathrm{OR}^k(G_1', \ldots, G_t')$. As $\mathcal{P}$ is a universal Ramsey host family, let d be such that $[R] \subseteq P_d$. Thus, any t-coloring of $\mathcal{G}^k(P_d)$ defines a t-coloring of K_R^k. By the definition of R, this t-coloring must admit a copy of G_i' in color i for some $i \in [t]$. As a copy of G_i' is also a copy of G_i, we see that $\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t) \leq d$. □

When constructing bounds on the 2-uniform partially-ordered Ramsey number, we will often need the following definition. for a t-coloring c of $E(\mathcal{G}(P))$ and $v \in P$, we define $\mathcal{D}_i(v) = \{x \in \mathcal{D}(v) : c(x, v) = i\}$ and similarly define $\mathcal{U}_i(v) = \{x \in \mathcal{U}(v) : c(x, v) = i\}$.

3.1.1 Comments on the Chain Ramsey Number

In Ramsey theory, we are used to coloring the edges of the complete graph, so we begin by establishing a few simple observations that relate the partially-ordered Ramsey numbers under any universal Ramsey host family to the partially-ordered Ramsey numbers under the family of chains.

If $\mathcal{P} = \{[n] : n \geq 1\}$, then we will denote $\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t)$ by $\mathrm{CR}^k(G_1, \ldots, G_t)$ and refer to this number as the *chain Ramsey number.* We focus on this particular host family due to its tight connections with the ordered Ramsey number.

To begin, notice that if $G_1, \ldots, G_t$ are ordered graphs (i.e. $(V(G_i), \leq_{G_i})$ is a chain), then it is immediate to observe that $\mathrm{CR}^k(G_1, \ldots, G_t) = \mathrm{OR}^k(G_1, \ldots, G_t)$. Let $(V(G), E(G), \leq_G)$ be a poset graph and suppose that for any two linear extentions $\leq$ and $\leq'$ of $\leq_G$, $(V(G), E(G), \leq)$ is isomorphic to $(V(G), E(G), \leq')$. In this case,

$$\mathrm{CR}^k_t((V(G), E(G), \leq_G)) = \mathrm{OR}^k_t((V(G), E(G), \leq)).$$

Next, suppose that $G_1, \ldots, G_t$ are comparability graphs of some posets (i.e. $G_i = \mathcal{G}^k(P_i)$ for some poset P_i). Define the graph G_i' to be the digraph formed by letting $V(G_i') = V(G_i)$ and $(x_1, \ldots, x_k) \in E(G_i')$ if and only if $x_1 \leq_{G_i} \cdots \leq_{G_i} x_k$. In this case, $\mathrm{CR}^k(G_1, \ldots, G_t) = \mathrm{DR}^k(G_1', \ldots, G_t')$, where $\mathrm{DR}^k(G_1', \ldots, G_t')$ refers to the natural extension of the 2-uniform directed Ramsey number to higher uniformities. This observation follows from the fact that the directed Ramsey number only cares about the ordering of the vertices if they are contained in a edge, which is the same as the chain Ramsey number in the case when the graphs are comparability graphs.

The next proposition is fairly straightforward, but will be very important in our exploration of partially-ordered Ramsey numbers when the host family consists of the Boolean lattices.

Proposition 3.3. *Let $G_1, \ldots, G_t$ be k-uniform poset-graphs, let $\mathcal{P} = \{P_n : n \geq 1\}$ be a universal host family. If $p(n) = |P_n|$, then*

$$\mathrm{CR}^k(G_1, \ldots, G_t) \leq p(\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t)),$$

and if $N = \mathrm{CR}^k(G_1, \ldots, G_t)$, then

$$\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t) \leq \min\{M : [N] \subseteq P_M\}.$$

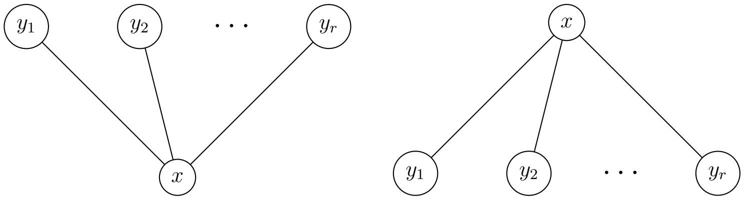

Figure 3.1 The poset-graphs $\vee_r$ and $\wedge_r$ respectively.

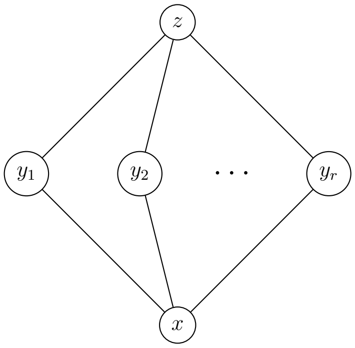

Figure 3.2 The r-diamond poset-graph D_r.

Proof. Let $R = \mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t)$ and let P' be any linear extension of P_R. By identifying P' with $[p(n)]$, it is immediate that any t-coloring c of $E(K^k_{p(R)})$ induces a coloring of $E(\mathcal{G}^k(P_R))$, so c must admit a copy of G_i in color i for some i. Thus, $\mathrm{CR}^k(G_1, \ldots, G_t) \leq p(R)$.

If $N = \mathrm{CR}^k(G_1, \ldots, G_t)$, and $[N] \subseteq P_M$, then any t-coloring of $E(\mathcal{G}^k(P_M))$ induces a coloring of $E(K^k_N)$, which must admit a copy of G_i in color i for some i. Thus, $\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t) \leq M$. □

Turning our attention solely to the chain Ramsey number, we define the poset-graph $\vee_r$, called the r-cup, by letting $V(\vee_r) = \{x, y_1, \ldots, y_r\}$ where $x \leq_{\vee_r} y_i$ and $\{x, y_i\} \in E(\vee_r)$ for each i. We define $\wedge_r$, called the r-cap, to be the dual of $\vee_r$ (see Figure 3.1).

We also define the r-diamond poset-graph to be the graph formed by identifying the maximal elements of $\vee_r$ with the minimal elements of $\wedge_r$ (see Figure 3.2).

In [3], Choudum and Ponnusamy proved the following theorem (although not in the language of partially-ordered Ramsey numbers).

Theorem 3.4 (Choudum and Ponnusamy [3])**.** *For $r, s \geq 2$,*

$$\mathrm{CR}(\vee_r, \wedge_s) = \left\lfloor \frac{\sqrt{1+8(r-1)(s-1)}-1}{2} \right\rfloor + r + s$$

Using this fact, Balko, Cibulka, Král, and Kynčl [2] argued that $11 \leq \mathrm{OR}_2(D_2) \leq 13$ and show that the lower bound is tight with computer assistance. We apply their technique that yields an upper bound of 13 to attain an general upper bound for the chain Ramsey number of D_r.

Theorem 3.5. *Let $r \geq 2$, then*

$$\mathrm{CR}_2(D_r) \leq 2 \cdot \left\lfloor \frac{\sqrt{1+8(r-1)^2}-1}{2} \right\rfloor + 6r - 1$$

Proof. Let $N = 2 \cdot \left\lfloor \frac{\sqrt{1+8(r-1)^2}-1}{2} \right\rfloor + 6r - 1$ and suppose that c is a 2-coloring of $E(K_N)$ that avoids monochromatic copies of D_r. Therefore, $|\mathcal{U}_i(1) \cap \mathcal{D}_i(N)| \leq r-1$ for $i = 1, 2$. Hence, $|\mathcal{U}_1(1) \cap \mathcal{D}_2(N)| + |\mathcal{U}_2(1) \cap \mathcal{D}_1(N)| = N - 2r$. By the pigeonhole principle, without loss of generality, $|\mathcal{U}_1(1) \cap \mathcal{D}_2(N)| \geq \lceil (N-2r)/2 \rceil = \mathrm{CR}(\vee_r, \wedge_r)$. Thus, c restricted to $\mathcal{U}_1(1) \cap \mathcal{D}_2(N)$ must admit either a $\wedge_r$ in color 1 or a $\vee_r$ in color 2. Both of these imply the existence of a monochromatic D_r; a contradiction. □

At this point, it should be noted that a bound on $\mathcal{P}\mathrm{R}(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m})$ can be attained in terms of a 2-color Ramsey number for any universal host family $\mathcal{P}$.

Proposition 3.6. *Let $R = 1 + \sum_{i=1}^{n}(r_i - 1)$ and $S = 1 + \sum_{i=1}^{m}(s_i - 1)$, then if $\mathcal{P}$ is any universal host family,*

$$\mathcal{P}\mathrm{R}(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) = \mathcal{P}\mathrm{R}(\vee_R, \wedge_S).$$

Proof. Upper Bound. Let $L = \mathcal{P}\mathrm{R}(\vee_R, \wedge_S)$, and let c be any $(n+m)$-coloring of $E(\mathcal{G}(P_L))$. Let c' be a 2-coloring of $E(\mathcal{G}(P_L))$ formed by $c'(x, y) = 1$ if $c(x, y) \in [n]$

and $c'(x,y) = 2$ if $c(x,y) \in [n+1, n+m]$. By the definition of L, c' must admit either a vee_R in color 1 or a $\wedge_S$ in color 2. Suppose that c' admits a copy of $\vee_R$ in color 1, then, by the pigeonhole principle, there is some $i \in [n]$ for which c restricted to this copy of $\vee_R$ contains r_i edges in color i. Hence, c admits a copy of $\vee_{r_i}$ in color i. A similar conclusion holds if c' admits a $\wedge_S$ in color 2. Hence, $\mathcal{P}\mathrm{R}(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) \leq L$.

Lower Bound. Let c be a 2-coloring of $E(\mathcal{G}(P_{L-1}))$ that avoids copies of $\vee_R$ in color 1 and copies of $\wedge_S$ in color 2. For any $v \in P_{L-1}$, we can easily color the edges between v and $\mathcal{U}_1(v)$ with the colors $\{1, \ldots, n\}$ and the edges between v and $\mathcal{D}_2(v)$ with the color $\{n+1, \ldots, n+m\}$ without creating monochromatic copies of $\vee_{r_i}$ or $\wedge_{s_j}$. Hence, this new coloring shows that $\mathcal{P}\mathrm{R}(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) > L - 1$. □

Additionally, it is immediate to observe that if $\mathcal{P} = \{P_n\}$ is a universal host family, then $\mathcal{P}\mathrm{R}(\vee_{r_1}, \ldots, \vee_{r_t}) = N$ where N is the least integer such that $\max_{x \in P_N} |\mathcal{U}(x)| \geq 2 + \sum_{i=1}^{t}(r_i - 1)$. Similarly, $\mathcal{P}\mathrm{R}(\wedge_{s_1}, \ldots, \wedge_{s_t}) = M$ where M is the least integer such that $\max_{x \in P_N} |\mathcal{D}(x)| \geq 2 + \sum_{i=1}^{t}(s_i - 1)$.

Corollary 3.7. *If $R = 1 + \sum_{i=1}^{n}(r_i - 1)$ and $S = 1 + \sum_{i=1}^{n}(s_i - 1)$, then*

$$\mathrm{CR}(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) = \left\lfloor \frac{\sqrt{1 + 8(R-1)(S-1)} - 1}{2} \right\rfloor + R + S$$

3.2 Boolean Ramsey Numbers

If $\mathcal{P} = \{2^{[n]} : n \geq 1\}$, we will denote $\mathcal{P}\mathrm{R}^k(G_1, \ldots, G_t)$ by $\mathrm{BR}^k(G_1, \ldots, G_t)$ and refer to this as the *Boolean Ramsey number*. In addition, let $\mathcal{B}_n^k = \mathcal{G}^k(2^{[n]})$, so we can explicitly define $\mathrm{BR}^k(G_1, \ldots, G_t)$ to be the least integer N such that any t-coloring of $E(\mathcal{B}_N^k)$ contains a copy of G_i in color i for some i. We focus on the Boolean Ramsey number due to its connection to Turán-type questions on the Boolean lattice and due to the fact that a good deal is known about the structure of this lattice.

3.2.1 1-Uniform Boolean Ramsey Numbers

In graph theory, a Turán-type problem is a question of the following form: for a graph H, if $\mathcal{H}_n$ is the family of graphs on n vertices that do not contain H as a subgraph, what is $\max_{G\in\mathcal{H}_n}|E(G)|$? In other words, how many edges can a graph on n vertices have and still avoid having H as a subgraph. One of the first results in this direction is a theorem by Mantel which states that a triangle-free graph on n vertices can have at most $n^2/4$ edges. Extending this, Turán showed that if a graph on n vertices that does not contain a copy of K_r can have at most $\frac{r-2}{r-1}\frac{n^2}{2}$ edges.

In many regards, Ramsey-type problems are an extension of Turán-type problems in the sense that both are attempting to avoid certain subgraphs. The main difference is that in a Ramsey-type problem, we are partitioning the host graph and considering each piece, where a Turán-type problem only cares about one specific piece of the partition.

Turán-type problems can also be asked about posets, in particular the Boolean lattice. Most have been phrased in the following way: for a poset P, what is the size of the largest subset of $2^{[n]}$ that does not contain P as a subposet. In this direction, the 1-uniform Boolean Ramsey number is an extension of this Turán-type question.

Most results toward determining the largest size of a P-free family of $2^{[n]}$ for some poset P use a special function known as the *Lubell function.* For a subset $\mathcal{F}\subseteq 2^{[n]}$, the Lubell function of $\mathcal{F}$ is defined to be

$$\mathrm{lu}_n(\mathcal{F}) = \sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1}.$$

A simple but key observation is that for any subset $\mathcal{F}$ of $2^{[n]}$,

$$|\mathcal{F}| \le \mathrm{lu}_n(\mathcal{F})\binom{n}{\lfloor n/2\rfloor}.$$

Therefore, if for a P-free family $\mathcal{F}\subseteq 2^{[n]}$, we can determine $\mathrm{lu}_n(\mathcal{F})$, then we can bound the size of $\mathcal{F}$ from above. We can also use the Lubell function to attain Ramsey-type results by noticing that for subsets $\mathcal{F},\mathcal{G}\subseteq 2^{[n]}$ with $\mathcal{F}$ and $\mathcal{G}$ disjoint,

$\mathrm{lu}_n(\mathcal{F} \cup \mathcal{G}) = \mathrm{lu}_n(\mathcal{F}) + \mathrm{lu}_n(\mathcal{G})$. The following proposition should be considered a formalization of an idea used by Johnston, Lu and Milans [18] to explore a very specific case of the 1-uniform Boolean Ramsey number.

Proposition 3.8. *Let P be a poset and let $L_n(P) = \max\{\mathrm{lu}_n(\mathcal{F}) : \mathcal{F} \text{is } P\text{-free}\}$. If $t < \frac{n+1}{L_n(P)}$, then $\mathrm{BR}^1_t(P) \leq n$.*

Proof. Notice that if $\mathcal{F} \subseteq 2^{[n]}$ has $\mathrm{lu}_n(\mathcal{F}) > L_n(P)$, then $\mathcal{F}$ must contain a copy of P. Therefore, let c be any t-coloring of $2^{[n]}$ and for $i \in [t]$, let $\mathcal{F}_i = c^{-1}(i)$. By the linearity of the Lubell function

$$n+1 = \mathrm{lu}_n(2^{[n]}) = \mathrm{lu}_n\left(\bigcup_{i=1}^{t} \mathcal{F}_i\right) = \sum_{i=1}^{t} \mathrm{lu}_n(\mathcal{F}_i).$$

Therefore, there is some i for which $\mathrm{lu}_n(\mathcal{F}_i) \geq \frac{n+1}{t}$. As $t < \frac{n+1}{L_n(P)}$, $\mathrm{lu}_n(\mathcal{F}_i) > L_n(P)$, so c admits a copy of P in color i. Thus, $\mathrm{BR}^1_t(P) \leq n$. ☐

Determining the Lubell function of a P-free family is generally a difficult task and we often can only attain asymptotic results. Note that Proposition 3.8 states that if $L_n(P) = \ell + o(1)$ for some constant ℓ, then asymptotically in t, $\mathrm{BR}^1_t(P) \leq (\ell + o(1))t$.

On the other hand, notice that $2^{[ht-1]}$ has exactly ht levels, so consider coloring levels $(i-1)h+1$ to ih with color i. Thus, in this t-coloring of $2^{[ht-1]}$, we have avoided monochromatic copies of any poset of height $h+1$ as there is no monochromatic chain of length $h+1$. Using this idea, we arrive at the following straightforward fact.

Proposition 3.9. *If $P_1, \ldots, P_t$ are posets where P_i has height h_i, then $\mathrm{BR}^1(P_1, \ldots, P_t) \geq \sum_{i=1}^{t}(h_i - 1)$.*

By putting together Propositions 3.8 and 3.9, we see that if $L_n(P) = \ell + o(1)$ and P has height h, then

$$(h-1)t \leq \mathrm{BR}^1_t(P) \leq (\ell + o(1))t. \tag{3.1}$$

Along these lines, Kramer, Martin, and Young [19] determined that $L_n(2^{[2]}) = 2.25 + o(1)$. Thus, by applying the bound in (3.1), we find that

$$2t \leq \mathrm{BR}^1_t(2^{[2]}) \leq (2.25 + o(1))t.$$

Beyond this, we provide the following result on off-diagonal Boolean Ramsey numbers to support our belief that the lower bound in (3.1) is tight in general.

Theorem 3.10. $\mathrm{BR}^1(2^{[n_1]}, [n_2], \ldots, [n_t]) = n_1 + \sum_{i=2}^{t}(n_i - 1)$.

Proof. The lower bound follows from Proposition 3.9, so we need only show the upper bound.

We first prove that $\mathrm{BR}^1(2^{[n]}, [m]) \leq n + m - 1$ by induction on m. For $m = 1$, the result is trivial, so suppose that $m \geq 2$ and let $N = n + m - 1$. Let c be any 2-coloring of $2^{[N]}$ and suppose that c avoids copies of $[m]$ in color 2; we will show that c must admit a copy of $2^{[n]}$ in color 1. Let $L = \{X \in 2^{[N]} : N \notin X\}$. As L is a copy of $2^{[N-1]}$, the induction hypothesis states that c restricted to L must admit either a copy of $2^{[n]}$ in color 1 or a copy of $[m - 1]$ in color 2. If the former holds, then we are done. Otherwise, c restricted to L admits a copy of $[m-1]$ in color 2. Suppose that $X_1, \ldots, X_s$ are the copies of $[m - 1]$ in color 2 contained in L, then $\bigcup_{i=1}^{s} \max X_i$ must form an antichain (although some of the $\max X_i$'s may be the same). Because c avoids copies of $[m]$ in color 2, we see that $\mathcal{U}\left(\bigcup_{i=1}^{s} \max X_i\right) \setminus \bigcup_{i=1}^{s} \max X_i$ must be void of color 2. Let $U = \mathcal{U}\left(\bigcup_{i=1}^{s} \max X_i\right) \cap L$ and let $U' = \{Y \cup \{N\} : Y \in U\}$. Notice that $U' \subseteq \mathcal{U}\left(\bigcup_{i=1}^{s} \max X_i\right) \setminus \bigcup_{i=1}^{s} \max X_i$, so U' has no elements of color 2. Furthermore, as U is an upset restricted to L, $2^{[N-1]}$ embeds into $(L \setminus U) \cup U'$. However, c restricted to $(L \setminus U) \cup U'$ does not contain any copies of $[m - 1]$ in color 2, so by the induction hypothesis, it must admit a copy of $2^{[n]}$ in color 1 as needed. We conclude that $\mathrm{BR}^1(2^{[n]}, [m]) \leq N$.

Now that we have proved that $\mathrm{BR}^1(2^{[n]}, [m]) = n+m-1$, the t-color version follows by induction on t. For $t \geq 3$, let c be a t-coloring of $2^{[N]}$ where $N = n_1 + \sum_{i=2}^{t}(n_i - 1)$. Letting $N' = n_1 + \sum_{i=2}^{t-1}(n_i - 1)$, by the 2-color case, $\mathrm{BR}^1(2^{[N']}, [n_t]) \leq N' + n_t - 1$, so either c admits a copy of $[n_t]$ in color t or c admits a copy of $2^{[N']}$ which is void of color t. We are done if the former happens, so suppose the latter holds. Then by the induction hypothesis, c restricted to this copy of $2^{[N']}$ must admit a copy of $2^{[n_1]}$ in color 1 or a copy of $[n_i]$ in color i for some $2 \leq i \leq t-2$. Therefore, $\mathrm{BR}^1(2^{[n_1]}, [n_2], \ldots, [n_t]) \leq N' + n_t - 1 = N$. $\square$

Based on this result, we present the following conjecture.

Conjecture 3.11. *Let $P_1, \ldots, P_t$ be posets such that P_i has height h_i and P_i is contained in $2^{[h_i-1]}$, then $\mathrm{BR}^1(P_1, \ldots, P_t) = \sum_{i=1}^{t}(h_i - 1)$.*

Notice that to confirm this conjecture, it suffices to show that $\mathrm{BR}^1(2^{[n]}, 2^{[m]}) = n+m$.

3.2.2 2-Uniform Boolean Ramsey Numbers

We now turn our attention to 2-uniform Boolean Ramsey numbers.

From Proposition 3.3, we immediate arrive at the following observation.

Proposition 3.12. *Let $G_1, \ldots, G_t$ be k-uniform poset-graphs, then*

$$\lceil \lg \mathrm{CR}^k(G_1, \ldots, G_t) \rceil \leq \mathrm{BR}^k(G_1, \ldots, G_t) \leq \mathrm{CR}^k(G_1, \ldots, G_t) - 1.$$

We can use Proposition 3.12 to attain an easy lower bound in the Boolean Ramsey number when the chain Ramsey number has already been determined. The upper bound is tight in the case where $G_1, \ldots, G_t$ are ordered graphs, but we expect it to be far from the truth when when the poset-graphs contain large antichains. Our next couple results support this expectation showing that for certain classes of poset-graphs G, $\mathrm{BR}_t(G) = \Theta(\lg \mathrm{CR}_t(G))$.

We begin by looking at matchings where we do not require any relations between elements unless they are connected by an edge.

Theorem 3.13. *Let $m_1 \geq \cdots \geq m_t$. If $M_i = \mathcal{G}\left(\bigcup_{j=1}^{m_i}[2]\right)$, then*

$$\left\lceil \lg\left(m_1 + 1 + \sum_{i=1}^{t}(m_i - 1)\right)\right\rceil \leq \mathrm{BR}(M_1, \ldots, M_t) \leq \left\lceil \lg\left(1 + \sum_{i=1}^{t}(m_i - 1)\right)\right\rceil + 1.$$

Proof. Lower bound. The lower bound is found by noting that $\mathrm{CR}(M_1,\ldots,M_t) = \mathrm{R}(M_1,\ldots,M_t) = m_1 + 1 + \sum_{i=1}^{t}(m_i - 1)$ and applying Proposition 3.12.

Upper bound. Let $N = \left\lceil \lg\left(1 + \sum_{i=1}^{t}(m_i - 1)\right)\right\rceil$ and let c be a t-coloring of $E(\mathcal{B}_{N+1})$. Let $X = \{S \in 2^{[N+1]} : N+1 \notin S\}$. For $x \in X$, define $c'(x) = c(x, x \cup \{N+1\})$, so c' is a t-coloring of X. Notice that $|X| = 2^N \geq 1 + \sum_{i=1}^{t}(m_i - 1)$, so if $T_i = \{x \in X : c'(x) = i\}$, then by the pigeonhole principle, $|T_i| \geq m_i$ for some i. Thus, $T_i \cup \{x \cup \{N+1\} : x \in T_i\}$ contains a copy of M_i in color i, so $\mathrm{BR}(M_1,\ldots,M_t) \leq N + 1$. □

Notice here that the upper and lower bounds differ by at most 1. We believe that the upper bound is always the truth.

The next result shows that for any $r, s \geq 2$, $\mathrm{BR}(\vee_r, \wedge_s) = \Theta(\lg(r+s))$.

Theorem 3.14. *For integers $r, s \geq 2$,*

$$\left\lceil \lg\left(\left\lfloor \frac{\sqrt{1 + 8(r-1)(s-1)} - 1}{2} \right\rfloor + r + s\right)\right\rceil \leq \mathrm{BR}(\vee_r, \wedge_s) \leq \left\lceil \frac{\lg(r+s-1)}{\lg(3/2)} \right\rceil.$$

Proof. Lower Bound. The lower bound follows from Theorem 3.4 and applying Proposition 3.12.

Upper Bound. Let $n = \lceil \lg(r+s-1)/\lg(3/2)\rceil$ and suppose that c is a 2-coloring of $E(\mathcal{B}_n)$ that avoids copies of $\vee_r$ in color 1 and avoids copies of $\wedge_s$ in color 2. Thus, for any $v \in 2^{[n]}$, $|\mathcal{U}_1(v)| \leq r - 1$ and $|\mathcal{D}_2(v)| \leq s - 1$. In particular, this implies that $|\mathcal{D}_1(v)| = |\mathcal{D}(v)| - 1 - |\mathcal{D}_2(v)| \geq 2^{|v|} - s$.

Let $W = 2^{[n]} \setminus \{[n]\}$ and let $T = \{v \in W : |\mathcal{U}_2(v) \cap W| = r - 1\}$. As c avoids copies of $\vee_r$ in color 1, for any $v \in T$, $c(v, [n]) = 2$. Hence, $|T| \leq s - 1$ because c avoids copies of $\wedge_s$ in color 2.

Let R be the number of edges of color 1 that have both vertices in W, then

$$\begin{aligned} R &= \sum_{v\in W} |\mathcal{D}_1(v)| \geq \sum_{v \in W}(2^{|v|} - s) \\ &= \sum_{i=0}^{n-1}\binom{n}{i}2^i - s(2^n - 1) \\ &= 3^n - 2^n(s+1) + s. \end{aligned}$$

On the other hand,

$$\begin{aligned}
R &= \sum_{v\in W} |\mathcal{U}_1(v)\cap W| = \sum_{v\in T}(r-1) + \sum_{v\in W\setminus T} |\mathcal{U}_1(v)\cap W| \\
&\leq |T|(r-1) + (2^n - 1 - |T|)(r-2) \\
&= |T| + (2^n-1)(r-2) \leq s - 1 + (2^n-1)(r-2).
\end{aligned}$$

Therefore, $3^n - 2^n(s+1) + s \leq R \leq s - 1 + (2^n-1)(r-2)$, so

$$\left(\frac{3}{2}\right)^n \leq r + s - 1 - (r-1)2^{-n}.$$

However, $n = \lceil \lg(r+s-1)/\lg(3/2) \rceil$, so

$$\left(\frac{3}{2}\right)^n \geq \left(\frac{3}{2}\right)^{\log_{3/2}(r+s-1)} = r + s - 1,$$

which is a contradiction. Thus, $\mathrm{BR}(\vee_r, \wedge_s) \leq n$. □

By applying Proposition 3.6, we arrive at the following corollary.

Corollary 3.15. *If* $R = 1 + \sum_{i=1}^n (r_i - 1)$ *and* $S = 1 + \sum_{i=1}^m (s_i - 1)$, *then*

$$\left\lceil \lg\left(\left\lfloor \frac{\sqrt{1+8(R-1)(S-1)}-1}{2} \right\rfloor + R + S \right) \right\rceil \\
\leq \mathrm{BR}(\vee_{r_1}, \ldots, \vee_{r_n}, \wedge_{s_1}, \ldots, \wedge_{s_m}) \leq \left\lceil \frac{\lg(R+S-1)}{\lg(3/2)} \right\rceil.$$

In some regards, these results state that if we were to color a linear extension of $\mathcal{B}_n$, then we guarantee a monochromatic copy of our poset-graph when n is approximately the logarithm of the chain Ramsey number. In other words, in determining the chain Ramsey number, many of the edges are unimportant.

Question. What families of poset-graphs have the property that $\mathrm{BR}_t(G) = \Theta(\lg \mathrm{CR}_t(G))$? In particular, is there some relationship between height and width of the underlying poset that guarantees this property?

We now turn our attention to trying to determine the 2-color Boolean Ramsey number of the diamond.

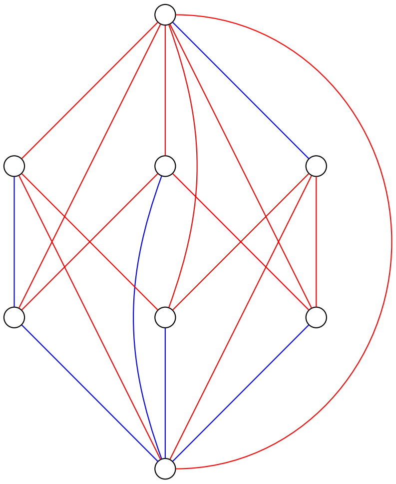

Figure 3.3 A 2-coloring of $E(\mathcal{B}_3)$ that avoids D_2 in red and $\wedge_2$ in blue.

Lemma 3.16. $\mathrm{BR}(D_2, \wedge_2) = 4$.

Proof. The lower bound is established by Figure 3.3, so we need only verify the upper bound. For the sake of contradiction, suppose that c is a 2-coloring of $E(\mathcal{B}_4)$ that avoids D_2 in color 1 and $\wedge_2$ in color 2. Thus, $|\mathcal{D}_2([4])| \leq 1$. If $c([4], \emptyset) = 2$, then let $X = \{S \subseteq [4] : 4 \notin S\}$ and if $c([4], \emptyset) = 1$, then let X be the copy of $2^{[3]}$ that does not contain $[4]$ and also does not contain the element x with the property that $c([4], x) = 2$. Thus, c restricted to the edges induces by X must admit either a $\vee_2$ in color 1 or a $\wedge_2$ in color 2. By the assumption on c, it must admit a $\vee_2$ in color 1, which implies a copy of D_2 in color 1 as the two maximal elements of the $\vee_2$ must be in $\mathcal{D}_1([4])$; a contradiction. □

Using this lemma, we can provide bounds on the 2-color Boolean Ramsey number of D_2.

Theorem 3.17. *$5 \leq \mathrm{BR}_2(D_2) \leq 7$.*

Proof. The lower bound is established through Figure 3.4, so we need only show the upper bound. Suppose that c is a 2-coloring of $E(\mathcal{B}_7)$ that avoids monochromatic diamonds.

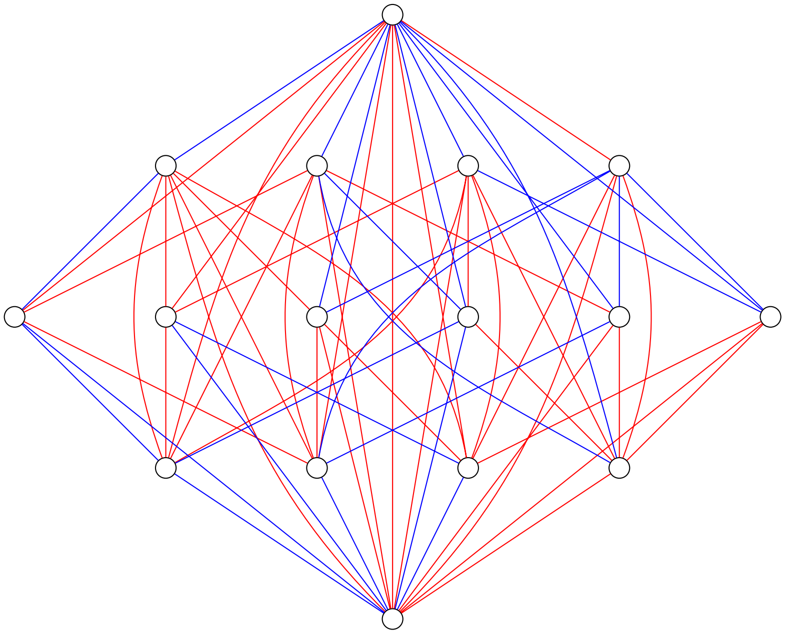

Figure 3.4 A 2-coloring of $E(\mathcal{B}_4)$ that avoids monochromatic copies of D_2.

Without loss of generality, we may suppose that $c(\emptyset, \{1\}) = c(\emptyset, \{2\}) = c(\emptyset, \{3\}) = 1$. Let $X = \mathcal{U}(\{1\}) \cap \mathcal{U}(\{2\}) \cap \mathcal{U}(\{3\}) = \mathcal{U}([3])$ and notice that X is isomorphic to $2^{[4]}$. By Lemma 3.16, we see that c restricted to the edges induced by X must admit a copy of $\wedge_2$ in both color 1 and color 2 as c avoids monochromatic copies of D_2. In particular, there are $v_1, v_2, v_3 \in X$ where $v_1v_2v_3$ forms a copy of $\wedge_2$ in color 2. Therefore, the structure in Figure 3.5 must appear where the red edges signify color 1, blue edges signify color 2, and the dotted edges represent edges whose color we have yet to determine. Figure 3.6 shows that there is no way to 2-color the dotted edges of this structure and avoid monochromatic copies of D_2. $\square$

In fact, using computer assistance, we arrive at the following fact

Proposition 3.18. $\mathrm{BR}(D_2, \vee_3) = 4$

By applying this proposition, we can tighten the bound the on Boolean Ramsey number of D_2.

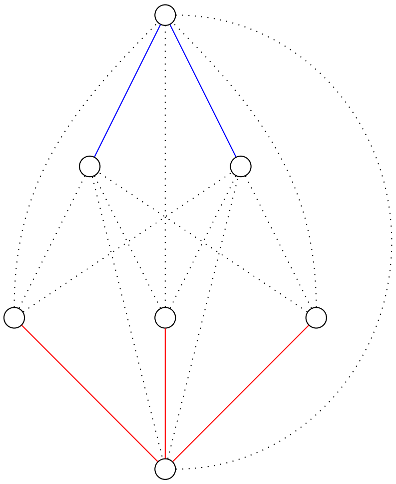

Figure 3.5 The structure used to force diamonds in Theorem 3.17.

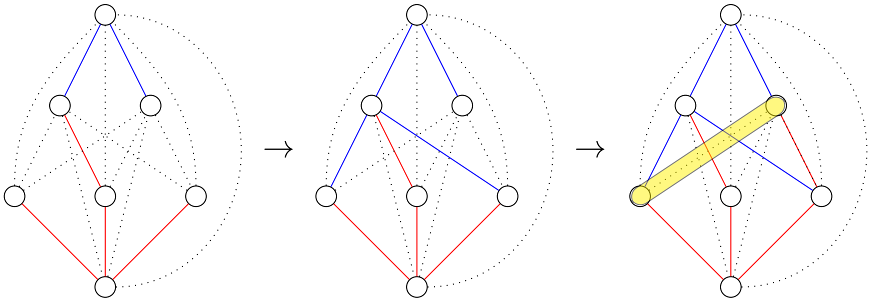

Figure 3.6 We cannot extend the partial coloring of Figure 3.5 without creating monochromatic copies of D_2.

Theorem 3.19. *$5 \leq \mathrm{BR}_2(D_2) \leq 6$.*

Proof. We need only verify the upper bound. Suppose that c is a 2-coloring of $E(\mathcal{B}_6)$ that avoids monochromatic copies of D_2, then without loss of generality $c([6] \setminus \{1\}, [6]) = c([6] \setminus \{2\}, [6]) = 1$. Let $X = \mathcal{D}([6] \setminus \{1\}) \cap \mathcal{D}([6] \setminus \{2\}) = \mathcal{D}([6] \setminus \{1, 2\})$ and notice that X is isomorphic to $2^{[4]}$. Hence, by Proposition 3.18, c restricted to the edges in X must admit a copy of $\vee_3$ in both color 1 and color 2. In particular, there are $v_1, v_2, v_3, v_4 \in X$ such that $v_1 v_2 v_3 v_4$ forms a copy of $\vee_3$ in color 2. Thus, we again arrive at the structure shown in Figure 3.5, where blue represents color 1 and red represents color 2. However, this is a contradiction by the same argument used in Theorem 3.17. $\square$

In fact, we conjecture that $\mathrm{BR}_2(D_2) = 5$, but a proof is not immediate and a computer search is currently intractable with our current implementation.

CHAPTER 4. SUMMARY AND DISCUSSION

4.1 Ordered Ramsey Numbers

Our investigation into arbitrarily-ordered k-uniform matchings provides upper bounds that are similar to the previous bounds in the 2-uniform case. Extending the techniques from 2-uniform matchings comes at the cost that it does not apply to all k-uniform ordered matchings, but they do provide bounds that are exponential and not a tower. However, our methods do not allude to lower bounds, and hence it is unclear whether our upper bounds are tight.

The largest question left open from our study of ordered Ramsey numbers is related to arbitrary orderings of (k, ℓ)-paths. While we found upper bounds on $\mathrm{OR}_t(P_e^{2,1})$, our techniques did not easily extend to higher uniformities. Upper bounds on $\mathrm{OR}_t^k(P_e^{k,\ell})$ for arbitrary orderings of $P_e^{k,\ell}$ would be very interesting and would significantly extend our current techniques. Noticing that $\mathrm{tow}_{k-2}(\Omega(n^2)) \leq \mathrm{R}_2^k(n) \leq \mathrm{tow}_{k-1}(O(n))$ (see [13]), the bound for $\mathrm{OR}_t^k(P_e^{k,k-1})$ for the natural ordering cannot be far off a general bound for $\mathrm{OR}_t^k(P_e^{k,k-1})$ for an arbitrary ordering. However, $\mathrm{OR}_t^k(P_e^{k,\ell})$ for the natural ordering grows as a tower of height $i(k,\ell) - 1$, so the upper bound for $\mathrm{OR}_t^k(P_e^{k,\ell})$ for an arbitrary ordering may be much larger, especially if $i(k,\ell) = 2$. Thus, bounds on tight paths may not lead to bounds on loose paths in the same way that Theorem 2.2 draws this connection for monotone paths.

4.2 Partially-Ordered Ramsey Numbers

In our formalization and exploration of partially-ordered Ramsey numbers, we came across a multitude of interesting and difficult problems. The biggest challenge that arises when working with a host graph defined by a poset other than a chain is that one cannot rely heavily on the pigeonhole principle, as one often does when exploring graph Ramsey numbers. Due to this, new techniques need to be developed in order to approach these questions. We now present a number of questions and conjectures that have arisen from our study of partially-ordered Ramsey numbers.

Conjecture 4.1. *If P_i is a poset of height h_i and is contained in $2^{[h_i-1]}$, then*

$$\mathrm{BR}^1(P_1,\ldots,P_t)=\sum_{i=1}^{t}(h_i-1).$$

Again, we comment that in order to confirm this conjecture, it suffices to show that $\mathrm{BR}^1(2^{[n]},2^{[m]})=n+m$.

Conjecture 4.2. $\mathrm{BR}_2(D_2)=5$

Question 4.3. *What are bounds on $\mathrm{BR}_t(D_r)$?*

Notice that if A_n is an antichain with n elements and $R=\mathrm{BR}(\vee_r,\wedge_r)$, then $\mathrm{BR}_2(D_r)\leq \mathrm{BR}^1(2^{[R]},2^{[R]},A_r,A_r)+2$. However, determining the 1-uniform Boolean Ramsey number is difficult on its own.

Question 4.4. *What families of poset-graphs have the property that $\mathrm{BR}_t^k(G)=\Theta(\mathrm{CR}_t^k(G))$?*

Question 4.5. *Are there nontrivial functions $f_k(h,w)$ and $g_k(h,w)$ such that if G is a k-uniform poset-graph whose underlying poset has height h and width w, then $g_k(h,w)\leq \mathrm{BR}_t^k(G)\leq f_k(h,w)$?*

Conjecture 4.6. $\mathrm{BR}_t(\mathcal{B}_n)=2^{\Theta(n)}$

Notice that by applying the bounds on $\mathrm{R}_2(n)$, we attain

$$\Omega(2^{n/2}) \leq \mathrm{BR}_2(\mathcal{B}_n) \leq O(2^{2^{2n}}).$$

Question 4.7. *Let* H_n *be the* 2*-uniform poset-graph formed formed from the Hasse diagram of* $2^{[n]}$*. What is* $\mathrm{BR}_t(H_n)$*?*

Note that $P_n^{2,1}$ is a subgraph of H_n, so $\mathrm{BR}_t(H_n) \geq n^t$. We predict that this is essentially the correct growth, i.e. $\mathrm{BR}_t(H_n) = \Theta(n^t)$.

APPENDIX A. COMPUTATIONAL RESULTS

We present a table of Boolean Ramsey numbers of small graphs that have been determined through computation. These numbers were found by an algorithm written by Derrick Stolee.

We quickly define the poset graphs referenced in Table A.1.

- The *crown of order* n, denoted cr_n, is the poset graph with $V(cr_n) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, where $x_i \leq_{cr_n} y_i$ and $x_i \leq_{cr_n} y_{i+1 \pmod n}$, and $\{x_i, y_i\}, \{x_i, y_{i+1 \pmod n}\} \in E(cr_n)$. Figure A.1 displays a picture of cr_5 for clarity.

- As in Theorem 3.13, we let $M_n = \mathcal{G}(\bigcup_{i=1}^{n}[2])$.

- The *standard poset-graph of order* n, denoted sd_n, is the poset-graph with $V(sd_n) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, where $x_i \leq_{sd_n} y_j$ for all $j \neq i$ and $\{x_i, y_j\} \in E(sd_n)$ for all $i \neq j$. Figure A.2 displays a picture of sd_5 for clarity.

- The (r, s)-star poset-graph, denoted $S_{r,s}$, is formed by identifying the maximal element of $\wedge_s$ with the minimal element of $\vee_r$. See Figure A.3 for clarity.

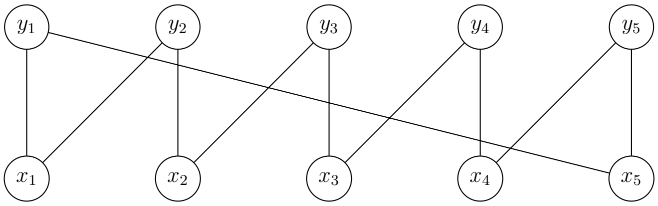

Figure A.1 The crown poset-graph of order 5, cr_5.

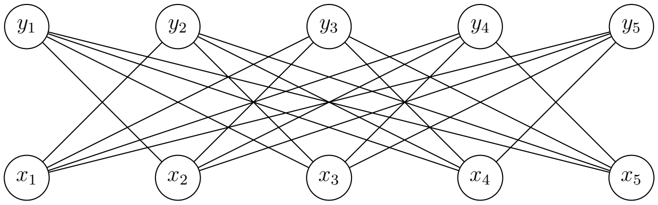

Figure A.2 The standard poset-graph of order 5, sd_5.

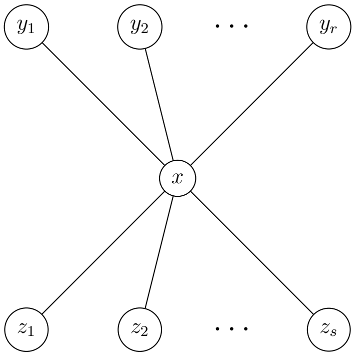

Figure A.3 The (r, s)-star poset-graph, $S_{r,s}$.

Table A.1 Computed Boolean Ramsey Numbers

	$\mathcal{B}_2$	$\wedge_2$	$\wedge_3$	$\wedge_4$	$\wedge_5$	$\wedge_6$	[2]	[3]	[4]	[5]	cr_2	cr_3	cr_4	cr_5	cr_6	$\vee_2$	$\vee_3$	$\vee_4$	$\vee_5$	$\vee_6$	D_2	D_3	D_4	D_5	D_6	M_2	M_3	M_4	M_5	M_6	sd_3	sd_4	$S_{1,1}$	$S_{1,2}$	$S_{1,3}$	$S_{1,4}$	$S_{2,1}$	$S_{2,2}$	$S_{2,3}$	$S_{3,1}$	$S_{3,2}$	$S_{4,1}$
$\mathcal{B}_2$		4	4				2	4								4	4									3	4	4	4				4									
$\wedge_2$	4	2	3	3	3	3	2	3	4	5	4	4	4	4	4	3	3	4	4	4	4	4	4	4		3	3	4	4	4	4	4	3	4	4		3	4	4	4		4
$\wedge_3$	4	3	3	3	3	4		3	4		4	4	4	4		3	4	4			4					3	4	4	4	4	4		3	4			4			4		4
$\wedge_4$		3	3	3	4	4		4			4					4	4									4	4	4	4				4				4			4		4
$\wedge_5$		3	3	4	4	4										4										4	4	4	4													
$\wedge_6$		3	4	4	4	4										4										4	4	4														
[2]	2	2									3					2					2					2																
[3]	4	3	3	4				4			4					3	3	4			4					3	4	4	4	4			4	5	5		5			5		
[4]		4	4													4	4									4	5															
[5]		5														5										5																
cr_2		4	4	4			3	4								4	4	4								3	4	4	4	4			4									
cr_3		4	4													4	4									3	4	4	4	4												
cr_4		4	4													4	4									4	4	4	4													
cr_5		4	4													4	4									4	4	4	4													
cr_6		4														4										4	4	4														
$\vee_2$	4	3	3	4	4	4	2	3	4	5	4	4	4	4	4	2	3	3	3	3	4	4	4	4		3	3	4	4	4	4	4	3	3	4	4	4	4		4	4	
$\vee_3$	4	3	4	4				3	4		4	4	4	4		3	3	3	3	4	4					3	4	4	4	4	4		3	4	4	4	4					
$\vee_4$		4	4					4			4					3	3	3	4	4						4	4	4	4				4	4	4	4						
$\vee_5$		4														3	3	4	4	4						4	4	4	4													
$\vee_6$		4														3	4	4	4	4						4	4	4														
D_2		4	4				2	4								4	4									3	4	4	4				4									
D_3		4														4										4																
D_4		4														4										4																
D_5		4														4										4																
D_6																										4																
M_2	3	3	3	4	4	4	2	3	4	5	3	3	4	4	4	3	3	4	4	4	3	4	4	4	4	3	3	4	4	4	3	4	3	4	4		4	4		4		
M_3	4	3	4	4	4	4		4	5		4	4	4	4	4	3	4	4	4	4	4					3	3	4	4	4	4	4	4	4			4	4				
M_4	4	4	4	4	4	4		4			4	4	4	4	4	4	4	4	4	4	4					4	4	4	4	4	4		4	4			4	4				
M_5	4	4	4	4	4			4			4	4	4	4		4	4	4	4		4					4	4	4	4	4	4		4	4			4					
M_6		4	4					4			4	4				4	4									4	4	4	4		4		4									
sd_3		4	4													4	4									3	4	4	4	4												
sd_4		4														4										4	4															
$S_{1,1}$	4	3	3	4				4			4					3	3	4			4					3	4	4	4	4			4	5	5		5			5		
$S_{1,2}$		4	4					5								3	4	4								4	4	4	4				5	5								
$S_{1,3}$		4						5								4	4	4								4							5									
$S_{1,4}$																4	4	4																								
$S_{2,1}$		3	4	4				5								4	4									4	4	4	4				5				5					
$S_{2,2}$		4														4										4	4	4														
$S_{2,3}$		4																																								
$S_{3,1}$		4	4	4				5								4										4							5									
$S_{3,2}$																4																										
$S_{4,1}$		4	4	4																																						

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