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# Modeling and controllability of a heat equation with a point mass

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**Modeling and controllability of a heat equation with a point mass**

by

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# DEDICATION

I would like to dedicate this work to my family and friends. In particular, to my beautiful wife Marie and my faithful furry friend Sheldon.

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## ABSTRACT

In this thesis, we propose a linear hybrid system describing heat flow on a medium composed by two rods connected by a point mass. We show that such a system can be obtained from a system describing heat flow of two rods connected by a thin wall of width $2\epsilon$ and density of $1/2\epsilon$. By passing to a weak limit, we obtain the desired system. We then show that the limiting system is null controllable with Dirichlet boundary control when the system's parameters satisfy a certain condition. Lastly, we consider simple parameters to show that the point mass system is null controllable with either Dirichlet or Neumann boundary control at one end.

# CHAPTER 1. INTRODUCTION

## 1.1 Control theory and historical highlights

Control theory is concerned with utilizing inputs or "feedbacks" to improve some aspect of the evolution of a system of dynamical equations. The dynamics are typically described by differential or partial differential equations, the latter being of main interest in this thesis.

Central to control theory are the concepts of controllability and feedback. Roughly speaking, a system is controllable if given any initial and terminal states $x_0$, $x_T$ of a system, there is a control $u$ that can be prescribed for which the solution $x(t)$ satisfies $x(0) = x_0$ and $x(T) = x_T$. A feedback control is a control $u(t)$ which can be written as a function of the state $x(t)$. That is $u(t) = \varphi(x(t))$. An important problem is the stabilization problem, which is to find a feedback that renders an unstable system stable. We are mainly concerned with the problem of controllability. Other lines of work in control related problems are control optimization and feedback optimization. One finds that all of these currents are interwoven.

In addition, mathematical modeling plays a key role in control theory in the sense that it is often the case that many models can be formulated that describe a particular physical system. In some cases, as in this thesis, a model can be obtained as a limiting model from a family of related models. In problems like this it is important to study how the corresponding solutions are related and ultimately how controls that may be applied to these systems are related.

The topic of control theory has a long history dating back to the ancient Romans who devised valves and shafts as controllers in structures known as aqueducts in order to adjust the direction and level of water flow. However, it is undoubtedly during the Industrial Revolution that control theory began to flourish. As machinery, electrical devices and engines were developed, so also were the first regulators and governors. Perhaps the most important example was the 1769 flyball governor of James Watt and its application to steam engines. The aim of this mechanism was to attempt to maintain a constant speed for the train as much as possible. Roughly speaking, the flyball governor was a mechanism in which the centrifugal force caused valves to open and close to regulate the amount of vapor escaping the engine. When vapor is released, the pressure is reduced inside the boiler and consequently the velocity begins to decrease. This is an example of a feedback control with the controller being the valve for the boiler, which is opened on the amount that is a function of the velocity of the train. The amount of feedback applied, called the *gain*, turned out to be a delicate issue. If the feedback gain was set to high, the feedback system would be unstable and act erratically.

It was not until 1840 when George Airy presented the first analysis of differential equations for Watt's governor. Nonetheless, it was the Scottish physicist James Clerk Maxwell who first published in (33) a complete mathematical analysis of the feedback system and explained what causes the instabilities. It is for this reason that he is widely consider the father of mathematical Control Theory.

The foundations of the modern approach toward control of PDE go back to Richard Kalman in the 1960s when he developed the theory of finite dimensional systems. Subsequently, mathematicians such as Lions, Russell, Fattorini, Littman, Komornik, Lagnese, Zuazua, Lasiecka and Triggiani generalized these ideas in the 1970s and 1980s to infinite dimensional systems where PDE are involved. While control theory is central to many areas of engineering, the theory needed to describe control theoretical properties of PDE involve advanced mathematical topics such as PDE theory, operator theory, functional analysis, harmonic analysis and advanced numerical methods. We refer to (27), (36) and (17) for a broader discussion of the history of Control Theory.

## 1.2 Summary of main results

To motivate this thesis, we refer to the work of Hansen and Zuazua in (25) where they consider a hybrid system of strings with an interior *point mass*. Let $L_1, L_2 > 0$ be given and consider two strings occupying the interval $(-L_1, L_2)$ connected by a point mass at $x = 0$. Let $z$ denote the position of the point mass, and let $u$ and $v$ denote the deformations of the strings to either side of $x = 0$. Then the equations modeling the system in the absence of controls are given by

$$\begin{cases} \rho_1 u_{tt} - \sigma_1 u_{xx} = 0, & t > 0,\ x \in (-L_1, 0) \\ \rho_1 v_{tt} - \sigma_2 v_{xx} = 0, & t > 0,\ x \in (0, L_2) \\ M z_{tt} = \sigma_2 v_x(t,0) - \sigma_1 u_x(t,0), & t > 0 \end{cases} \tag{1.1}$$

where $\rho_i$ and $\sigma_i$ denote the density and tension of the strings, and $M$ is the mass of the point mass. In addition, we suppose the strings satisfy Dirichlet boundary conditions and $u(0,t) = v(0,t) = z(t)$ for $t > 0$. It was observed in (25) that such a system is well-posed in an asymmetric space in which solutions have one degree more of regularity on one side of the point mass. This is due to a lack of spectral gap by the presence of the point mass. Moreover, by the method of characteristics, they prove boundary exact controllability on the asymmetric space.

The controllability and stabilization of hybrid systems such as (1.1) have been extensively studied in the context of strings and beams with interior point masses. In (30) Littman and Taylor use transform methods to prove boundary feedback stabilization of the string mass system. In (8) and (9), Castro and Zuazua applied methods of non-harmonic Fourier series to show boundary controllability of systems of either Rayleigh or Euler-Bernoulli beams with interior point masses. We refer to (29), (35), (10), (47), (20) and (19) for related results on control and stabilization of systems of beams with end masses.

Nonetheless, hybrid systems with point masses in the context of thermoelasticity had not been investigated. Indeed, in the absence of a point mass, a thermoelastic system may be given by

$$\begin{cases} u_t = u_{xx} - \gamma v_{xt}, & x \in (0,1),\ t > 0 \\ v_{tt} = v_{xx} - \gamma u_x, & x \in (0,1),\ t > 0 \end{cases} \tag{1.2}$$

where $\gamma$ is the coupling parameter. It is not obvious how (1.2) should be adjusted to include a point mass. In fact, one may ask what the meaning of a point mass would be in such a situation. Given that there exists extensive literature for the study of string and beams with point masses, it is natural to first consider an analogous system to (1.1) in the case of heat diffusion to understand the thermal component of (1.2). This serves as part of our motivation. That is, derive a system of heat equations with an interior point mass and then study the control-related properties of the system.

We propose the following set of equations for the problem of heat diffusion of two rods connected by a point mass are

$$\begin{cases} c_1\rho_1 u_t - k_1 u_{xx} = 0, & t > 0,\ x \in (-L_1, 0) \\ c_1\rho_1 v_t - k_2 v_{xx} = 0, & t > 0,\ x \in (0, L_2) \\ cz_t = k_2 v_x(t,0) - k_1 u_x(t,0), & t > 0 \end{cases} \tag{1.3}$$

where $c_i$, $\rho_i$ and $k_i$ denote the specific heat, density and conductivity of the rods to either side of the point mass, while $c$ denotes the specific heat of the point mass. It is worth mentioning that the first two equations describe the dynamics of the rod's temperature to each side of the point mass, and the third equation is reminiscent of Fick's law of diffusion. Intuitively, the rate of change in temperature of the point mass is given by the difference in heat flux across the interface $x = 0$.

In Chapter 3, we derive the equations (1.3) as weak limits of an *epsilon* dependent problem consisting of two rods connected by a thin wall of width $2\epsilon$ and density $1/2\epsilon$. Under appropriate assumptions, we may pass to a limit and obtain the desired system set of equations (1.3) with the corresponding boundary and continuity conditions. By doing so, we prove that solutions of the *epsilon* problem converge in a weak sense to solutions of the limiting system (1.3) of singular density. This approach was used by Castro in (6) to derive the point mass system for strings. Well-posedness results for the $\epsilon-$problem and limit problem (1.3) are proved by semigroup methods. A brief overview of semigroup theory is given in Chapter 2.

We conclude Chapter 3 by considering an additional approach in which a system isomorphic to (1.3), but with a limiting finite nonzero thickness of the middle layer, is obtained. Such a system is obtained by passing to a limit in an $\epsilon-$dependent problem of singular conductivity on the middle layer. Roughly speaking, this describes a system in which a constant steady state is reached instantaneously in the middle wall.

In Chapter 4, we study the null controllability of the limiting system obtained in Chapter 3 with Dirichlet control. In other words, given time $T > 0$, we investigate the property of steering the solution $(u, v, z)$ of (1.3) from an initial state $(u^0, v^0, z^0)$ to the zero state $(0, 0, 0)$ at time $T$ by means of a input control $f \in L^2(0, T)$ such that $v(t, L_2) = f(t)$. To achieve this, we apply the method of moments which was developed by Russell and Fattorini in (15) and (40). Refer to Chapter 2, for a brief overview of these techniques in the case of a simple equation with no point mass.

Going back to (1.3), we show under completely general assumptions on the coefficients that the problem of null controllability is equivalent to solving a moment problem of the form:

$$\frac{y_n^0 \; e^{\lambda_n T}}{b_n} = \int_0^T w(\tau) e^{\lambda_n \tau} \; d\tau. \tag{1.4}$$

To prove the existence of the solution $w(t)$ to the moment problem, we study the separability of the eigenvalues $\lambda_n$ of the system and obtain sufficient estimates of the Fourier coefficients $b_n$ of the control input element. One unique feature, is that we allow for different parameters of specific heat, density and conductivity on either side of the point mass. The eigenvalues depend in a complex way on the parameters of the system, and separate arguments are needed for the cases where they are rationally and irrationally related. In the end we are able to prove the necessary bounds to show separation of the eigenvalues for both cases, although the existence of the solution to the corresponding moment problem is shown only for the rationally related case.

In Chapter 5, we consider the case where the parameters are all equal to one and the two connected rods are of equal length. In addition to showing that the system is null controllable by means of Dirichlet control, we also consider the application of Neumann control by controlling the flux of temperature at $x = 1$ such that $v_x(t, 1) = f(t)$. The results are again based on a careful spectral analysis together with the moment method mentioned earlier. We are once again able to obtain the necessary bounds to show separation of the corresponding eigenvalues and subsequently show, in both Dirichlet and Neumann control, that the point mass system (1.3) is null controllable in time $T > 0$.

# CHAPTER 2. BASIC CONCEPTS

In this chapter, we discuss some definitions and introduce notation.

## 2.1 Semigroup theory

In this section we recall some basic results of semigroup theory. We refer to (32), (5) and (38) for a much broader discussion. Let $\mathcal{L}(X)$ denote the space of bounded linear operators from $X$ to itself where $X$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Let $\{T(t) \in \mathcal{L}(X) : 0 \leq t < \infty\}$ be a one-parameter family of bounded linear operators on $X$.

**Definition 2.1.0.1.** *We say $T(t)$ is a strongly continuous semigroup (or $C_0-$semigroup for short) on $X$ if*

*(i)* $T(0) = I$ *(Identity element),*

*(ii)* $T(t+s) = T(t)T(s)$ *for every* $t, s \geq 0$ *(semigroup property),*

*(iii)* $\lim_{t \downarrow 0} T(t)x = x$ *for all* $x \in X$ *(strong continuity).*

*In particular, a $C_0-$semigroup on $X$ is a contraction semigroup if $\|T(t)\| \leq 1$ for $t \geq 0$.*

**Definition 2.1.0.2.** *Let $T(t)$ be a $C_0-$semigroup on $X$. The operator $\mathcal{A} : D(\mathcal{A})(\subset X) \to X$ defined by*

$$\mathcal{A}x = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}$$

*for all $x \in D(\mathcal{A})$ with domain*

$$D(\mathcal{A}) = \left\{ x \in X \;\middle|\; \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

*is called the infinitesimal generator of the semigroup $T(t)$.*

**Definition 2.1.0.3.** *The unbounded operator* $\mathcal{A} : D(\mathcal{A}) \to X$ *is dissipative if*

$$\langle Ax, x\rangle \leq 0, \quad \forall x \in D(\mathcal{A}).$$

*The dissipative operator* $\mathcal{A}$ *is called* $m$*-dissipative if* $\mathcal{R}(\lambda_0 - \mathcal{A}) = X$ *for some* $\lambda_0 > 0$.

**Proposition 2.1.0.1.** *The linear operator* $\mathcal{A} : D(\mathcal{A}) \to X$ *is dissipative if and only if*

$$\|x\| \leq \|x - h\mathcal{A}x\|, \quad \forall x \in D(\mathcal{A}) \text{ and } h > 0.$$

**Theorem 2.1.0.1.** *Let* $A$ *be the infinitesimal generator of a* $C_0-$*semigroup* $T(t)$ *on* $X$ *satisfying* $\|T(t)\| \leq Me^{\omega t}$. *If* $B$ *is a bounded linear operator on* $X$, *then* $A+B$ *generates a* $C_0-$ *semigroup* $S(t)$ *on* $X$ *and*

$$\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$$

*for all* $t \geq 0$.

The following result is known as the Lümer-Phillips Theorem which gives necessary and sufficient conditions for an operator to be the infinitesimal generator of a $C_0-$semigroup of contractions.

**Theorem 2.1.0.2.** *The linear operator* $\mathcal{A} : D(\mathcal{A}) \to X$ *generates a* $C_0-$*semigroup of contractions if and only if*

*(i)* $\overline{D(\mathcal{A})} = X$

*(ii)* $\mathcal{A}$ *is* $m-$*dissipative*

The following consequence of the Lümer-Phillips Theorem provides an alternative characterization of contraction semigroups with respect to the operator and its adjoint operator without the need to check the $m-$dissipativity condition.

**Corollary 2.1.0.1.** *The linear operator* $\mathcal{A} : D(\mathcal{A}) \to X$ *generators a* $C_0-$*semigroup of contractions if and only if*

*(i)* $\mathcal{A}$ *is densely defined and closed,*

*(ii) both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative.*

We denote the resolvent operator $(\lambda I - \mathcal{A})^{-1}$ by $R(\lambda, \mathcal{A})$ where $\lambda \in \rho(\mathcal{A})$. The following theorem gives a useful expression for the resolvent operator of a $C_0-$semigroup.

**Theorem 2.1.0.3.** *Let $\omega_0$ be the growth rate defined of a the $C_0-$semigroup $T(t)$ by*

$$\omega_0 = \inf_{t>0} \frac{\log \|T(t)\|}{t}.$$

*Then if $\lambda \in \mathbb{C}$ satisfies $Re\lambda > \omega_0$, then $\lambda \in \rho(A)$ and for any $x \in X$ the following holds.*

$$R(\lambda, \mathcal{A})x = \int_0^\infty e^{-\lambda\tau} T(\tau)x \; d\tau.$$

We now introduce the notion of analytic semigroups.

**Definition 2.1.0.4.** *Let $\omega \in \mathbb{R}$ and $\theta_0 \in (\pi/2, \pi)$ be given. Define the sector $\mathbf{S}_{\omega,\theta}$ be defined as*

$$\mathbf{S}_{\omega,\theta} = \{z \in \mathbb{C} \mid z \neq \omega, \; |\arg(z - \omega)| < \theta_0\}.$$

**Theorem 2.1.0.4.** *Let $S(t)$ be a $C_0-$semigroup with $A$ as its infinitesimal generator. Furthermore, let $\omega \in \mathbb{R}$ and $M > 0$ be given such that*

$$\|S(t)\| \leq Me^{\omega t},$$

*for all $t \geq 0$. Then the following statement are equivalent.*

*(i) The operator $\mathcal{A}$ verifies the conditions*

$$\exists \theta_0 > \frac{\pi}{2}, \quad \mathbf{S}_{\omega,\theta_0} \subset \rho(\mathcal{A})$$

*and*

$$\exists M > 0, \; \forall \theta \in (0, \theta_0), \quad \|R(\lambda, \mathcal{A})\| \leq \frac{M}{|\lambda - \omega|}, \; \forall \lambda \in \mathbf{S}_{\omega,\theta_0}.$$

*(ii) The map $t \mapsto S(t) : [0, \infty) \to \mathcal{L}(X)$ belongs to $C^1(0, \infty; \mathcal{L}(X))$ and*

$$\exists N > 0, \; \forall t > 0, \quad \|(A - \omega I)S(t)e^{-\omega t}\| \leq \frac{N}{t}.$$

*(iii) The semigroup $S$ has an analytic extension in a sector $\mathbf{S}_{0,\theta'}$ with $\theta' \in (0, \pi/2)$ and $e^{-\omega t}S(t)$ is bounded in every closed sub sector of $\mathbf{S}_{0,\theta'}$.*

**Definition 2.1.0.5.** *A $C_0-$semigroup is said to be analytic if it verifies any one of the conditions of theorem 2.1.0.4.*

**Definition 2.1.0.6.** *A $C_0-$semigroup $T(t)$ on $X$ is called compact if $T(t)$ is a compact operator for each $t \geq 0$.*

The following theorem characterizes compact semigroups if we know a priori $A$ generates an analytic semigroup.

**Theorem 2.1.0.5.** *Let $T(t)$ be an analytic semigroup on $X$ and let $A$ be its infinitesimal generator. Then $T(t)$ is compact if and only if $R(\lambda, A)$ is compact for some $\lambda \in \rho(A)$.*

Now consider the following differential equation on $X$ with $x \in X$.

$$\frac{d}{dt}u(t) = \mathcal{A}u(t), \quad u(0) = x, \quad t > 0. \tag{2.1}$$

The above abstract Cauchy problem is to find a solution $u(t)$ of the above problem. We say $u(\cdot) : [0, \infty) \to X$ is a classical solution to the Cauchy problem if $u$ is continuous for $t \geq 0$, continuously differentiable and $u \in D(\mathcal{A})$ for $t > 0$ and satisfies (2.1). The next theorem shows that to obtain existence and uniqueness of a continuously differentiable solution $u \in D(\mathcal{A})$ differentiable on $[0, \infty)$ for (2.1) it is necessary and sufficient that $\mathcal{A}$ be the infinitesimal generator of a $C_0-$semigroup.

**Theorem 2.1.0.6.** *Let $\mathcal{A}$ be a densely defined linear operator with a nonempty resolvant $\rho(\mathcal{A})$. The abstract Cauchy problem (2.1) has a unique solution $u(t)$ which is continuously differentiable on $[0, \infty)$, for all $x \in D(\mathcal{A})$ if and only if $\mathcal{A}$ generates a $C_0-$semigroup $T(t)$ on $X$.*

## 2.2 Finite dimensional control problem

In this section we provide a standard formulation of the control problem in the finite-dimensional setting. As seen in the previous section, control theory began to be studied rigorously for systems whose solutions belong to spaces of finite dimension. These systems are generated by Ordinary Differential Equations (ODE). Given the longevity of the problem there exists an extensive literature devoted to the control of ODE. Here we give a brief description of the controllability of such systems in two ways. In the first one we show controllability is understood in terms of the rank of the so called controllability matrix. In the second, we show controllability is equivalent to the problem of observability. The second method is specially important since these ideas serve as a prototype to understand the machinery behind infinite-dimensional problems which are given by PDE model. Intuitively one can see that PDE are the infinite-dimensional version of a system of ODE. However, one must be careful when passing from finite to infinite dimensional spaces for not all control properties may be conserved. Nonetheless the study of finite-dimensional systems sheds light into the understanding of infinite-dimensional problems.

Let $n, m \in \mathbb{N}$ and $T > 0$ be a real constant. Consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \quad t \in (0,T) \\ x(0) = x^0 \end{cases} \tag{2.2}$$

where $A \in \mathcal{M}_{n\times n}$ and $B \in \mathcal{M}_{n\times m}$ are real constant matrices and $x^0$ is a vector in $\mathbb{R}^n$. The function $x : [0,T] \to \mathbb{R}^n$ represents the *state* of the system and $u : [0,T] \to \mathbb{R}^m$ is the $m$-dimensional *control*. The matrix $A$ describes the dynamics of the system and $B$ models the way in which we affect the state by means of the $m$ controls in $u$. This will become more clear in later examples. The solution to the controlled system (2.2) belongs to the finite dimensional space $\mathbb{R}^n$ and it is given by the variation of parameters formula as

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(t-s)\,ds. \tag{2.3}$$

However, we shall keep in mind that as we generalize to infinite dimensional systems, our solution space will be generated by a base of infinite dimension.

Ideally we want to have the least amount of controls as to affect the system as little as possible. That is, we would like $m$ to be as small as possible. This is because for larger values of $m$ there is a greater cost or simply the physics, chemistry or biology of the problem at hand does not allow for large values. Of course, the least amount of controls one can have is $m = 1$. We present an example to illustrate the problem.

**Example 2.2.0.1.** *Consider a cart of mass* 1 *moving along a horizontal track without friction. The state of the system* $x(t)$ *is the position of the cart at time* $t$. *If we can imagine that we can control the position of the cart by means of an external force* $u(t)$ *then the mechanics of the system are described by*

$$x''(t) = u(t), \qquad t > 0$$

*with initial position* $x(0) = x_0$ *and initial velocity* $x'(0) = v_0$. *Intuitively, the above says that we can determine the acceleration of the cart to be* $u$ *at time* $t$. *We can rewrite our problem in the form of* (2.2) *by letting* $x_1 = x$ *and* $x_2 = x'$. *Then the problem is described by*

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

*or*

$$\begin{cases} y'(t) = Ay(t) + Bu(t), & t \in (0, T) \\ y(0) = y_0 \end{cases}$$

*where*

$$y(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

*Then a control problem may seek to find a control function* $u$ *such that we can bring the cart to rest in time* $T$. *That is, find* $u$ *such that the terminal state* $y(T)$ *is* $\vec{0}$ *given some initial state* $y_0$. *The problem of finding the minimal time is known as the optimal control problem.*

We refer to (27) for other examples of control systems. The problem posed in Example 2.2.0.1 is vague in the sense that we do not ask much about the initial state of the system. For instance, we do not know if the cart can be brought to rest from any initial state or whether it can almost be brought to rest. Furthermore, we do not specify the space of control inputs. In general, we say the control system (2.2) is *controllable* in time $T$ from an initial state $x^0$ to a terminal state $x^T$ if there exists a suitable control $u$ such that $x(T) = x^T$. In the next section we discuss the different degrees of controllability often encountered in the literature.

### 2.2.1 Controllability

We say that the finite-dimensional control system (2.2) is *exactly controllable* in time $T > 0$ if given any initial data $x^0 \in \mathbb{R}^n$ and terminal state $x^T \in \mathbb{R}^n$, we can find $u \in L^2(0, T; \mathbb{R}^m)$ such that $x(T) = x^T$. In other words,

$$x^T = e^{AT} x^0 + \int_0^T e^{A(T-s)} Bu(T-s)\ ds.$$

We can show by an example that this may not always occur.

**Example 2.2.1.1.** *Consider the following system similar to that of example 2.2.0.1:*

$$\begin{cases} \frac{dx_1}{dt} = -x_1 + u(t) \\ \frac{dx_2}{dt} = -x_2 \end{cases}$$

*with initial data* $(x_1, x_2)(0) = (x_1^0, x_2^0)$. *Note how the equations are not coupled and so one can see the solution is given by*

$$x_1(t) = e^{-t} x_1^0 + \int_0^t e^{\tau - t}\ u(\tau)\ d\tau$$

$$x_2(t) = e^{-t} x_2^0.$$

*Note that we cannot act on* $x_2$ *with* $u$. *So there is no way to choose* $u$ *to drive* $x_2$ *to a desired terminal state.*

Other weaker notions of controllability are that of approximate and null controllability. The problem of approximate controllability consists in determining if we can find a suitable control to approximate the desired state $x^T$ with $x(T)$ from any given initial state $x^0$. The problem of null controllability consists in determining whether we can find a suitable control such that $x(T) = 0$ from any given initial state. It is ofter useful to describe these ideas in terms of the reachable states. We define the set of reachable states from $x^0 \in \mathbb{R}^n$ in time $T > 0$ by

$$R(T; x^0) = \left\{ x(T)\ :\ x \text{ solves } (2.2),\ u \in L^2(0, T; \mathbb{R}^m) \right\}.$$

Whenever the reachable set $R(T; x^0)$ equals $\mathbb{R}^n$ for all $x^0 \in \mathbb{R}^n$ we say the system is exactly controllable. If the reachable set $R(T; x^0)$ is dense in $\mathbb{R}^n$ for all $x^0 \in \mathbb{R}^n$ then we say the system is *approximately controllable*. On the other hand, if the zero vector is in the set of reachable set $R(T;x^0)$ for any $x^0 \in \mathbb{R}^n$ then we say the system is *null controllable*. Surprisingly, all three degrees of controllability are equivalent in the finite-dimensional setting. Indeed, exact and approximate controllability are equivalent since the reachable set must be a close affine subspace and the only space with these characteristics is the whole space $\mathbb{R}^n$ itself. The reason why exact and null controllability are equivalent is more subtle. Let the system (2.2) be exactly controllable and let $x^0$ and $x^T$ be given. Now let $y \in \mathbb{R}^n$ be the solution to the uncontrolled problem with terminal condition $x^T$:

$$\begin{cases} y'(t) = Ay(t), \quad t \in (0,T) \\ y(T) = x^T \end{cases}.$$

Then by the linearity of $A$, the vector $z := x - y \in \mathbb{R}^n$ solves the same ODE as in (2.2) with initial state $x^0 - y(0)$. That is

$$\begin{cases} z'(t) = Az(t) + Bu(t), \quad t \in (0,T) \\ z(0) = x^0 - y(0) \end{cases}.$$

Moreover, we have that $z(T) = 0$. Since the vector $x^0 - y(0)$ is arbitrary in $\mathbb{R}^n$ it follows that the system (2.2) is null controllable. The reverse implication follows similarly.

The above discussion shows that in the finite-dimensional case, it is sufficient to address the question of null controllability to know if a system is controllable. However, we will later see that this equivalence is not guaranteed for infinite-dimensional systems.

Now we ask, under what conditions is a finite dimensional system exactly controllable? A well known necessary and sufficient condition is that of the *Kalman criterion*. This is a very useful tool since it is easy to verify.

**Theorem 2.2.1.1.** *The finite-dimensional control* (2.2) *is said to be exactly controllable if and only if the matrix*

$$C := [B, AB, \dots, A^{n-1}B]$$

*is full rank.*

The condition that rank $= n$ is known as the *Kalman condition* and the matrix $C \in \mathcal{M}_{n\times nm}$ is known as *Kalman's controllability matrix.*

*Proof.* Assume that the system is exactly controllable but that the matrix $C$ is rank deficient. By the definition of exactly controllable we have that $R(T; x^0) = \mathbb{R}^n$ for all $x^0$. In particular for $x^0 = \vec{0}$, we have that

$$R(T; 0) = \{0\}^{\perp}. \tag{2.4}$$

On the other hand, the rank deficiency of $C$ and Rank-nulllity theorem imply that the null space of $C$ is nontrivial. That is, one can find a nonzero row vector $v \in \mathbb{R}^n$ such that $vC = \vec{0}$. Hence

$$vB = vAB = \cdots = vA^{n-1}B = \vec{0} \tag{2.5}$$

Let the characteristic polynomial of $A$ be given by

$$p_A(t) = t^n - c_1t^{n-1} - c_2t^{n-2} - \cdots - c_{n-1}t - c_n$$

for some $\{c_i\}_{i=1}^n \in \mathbb{R}^n$. By the Cayley-Hamilton theorem we have that $p_A(A) = 0$. By left multiplying by $v$ and right multiplying by $B$ we see that (2.5) implies

$$vA^nB = c_1vA^{n-1}B + c_2vA^{n-2}B + \cdots + c_{n-1}vAB + c_nvB = \vec{0}$$

Using an inductive argument we obtain that $vA^{n+k}B = \vec{0}$ for all $k = 0, 1, 2, \ldots$. Combing the latter with (2.5) we have that $vA^kB = \vec{0}$ for all $k = 0, 1, 2, \ldots$. Applying the series expansion of the exponential matrix $e^{At}$ we have

$$ve^{As}B = v\left(I + As + \frac{A^2s^2}{2!} + \ldots\right)B = \vec{0} \tag{2.6}$$

for any real $s$. Recall that the variation of parameters formula (2.3) gives the state of the system at time $t$ and for $x^0 = \vec{0}$ we have that is given by

$$x(t) = \int_0^t e^{A(s)}Bu(t-s)\, ds.$$

By left multiplying by $v$ and using (2.6) we have that

$$vx(T) = \int_0^T ve^{A(s)}Bu(t-s)\ ds = 0$$

The above shows that the nonzero vector $v$ is orthogonal to reachable set from $\vec{0}$ which contradicts (2.4). Hence exact controllability implies that the Kalman condition is satisfied.

Now assume that the Kalman condition holds but that the system is not exactly controllable. Recall that exact and null controllability are equivalent. Since the system is not null controllable we can observe that

$$\bigcup_{t>0} \left\{ x(t) = \int_0^t e^{As}Bu(t-s)\ ds \ :\ u \text{ is piecewise continuous} \right\} = \mathbb{R}^n.$$

Since we have assumed by contradiction that the system is not exactly controllable, the above statement is false. In particular,

$$\left\{ x(1) = \int_0^1 e^{As}Bu(1-s)\ ds \ :\ u \text{ is piecewise continuous} \right\} \neq \mathbb{R}^n$$

holds or otherwise there exists a nonzero row vector $v \in \mathbb{R}^n$ such that

$$v\int_0^1 e^{As}Bu(1-s)\ ds = 0$$

for all $u$ piecewise continuous. We get

$$ve^{As}B = \vec{0}$$

and when $s = 1$ we have that $vB = \vec{0}$. By differentiating above with respect to $s$ and evaluating at 1 we have $vAB = \vec{0}$. By an inductive argument we see that

$$vB = vAB = \cdots = vA^{n-1}B = \vec{0}$$

which cannot occur since $C$ is full rank. Hence the system must be exactly controllable as needed. □

Note that Kalman's condition is independent of $T > 0$ since the matrices are constant. Hence, if the system is controllable for some time $T > 0$ then it is also controllable for any positive time.

### 2.2.2 Observability

Although, Kalman's condition is a useful tool due to its simplicity, it is not very enlightening when moving to the infinite-dimensional case. Fortunately, there is another characterization of the controllability of a system in terms of the adjoint problem. In this section we introduce this problem which is also known as the observability problem. Let $A^*$ be the adjoint matrix of $A$. That is, the matrix that satisfies

$$(Ax, y) = (x, A^* y)$$

for all $x, y \in \mathbb{R}^n$ where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^n$. Consider the adjoint problem

$$\begin{cases} -\varphi'(t) = A^* \varphi \\ \varphi(T) = \varphi^0 \end{cases} \tag{2.7}$$

This problem goes back in time. That is, it *starts* from terminal time $T$ of the original control problem. Just as in the problem of controllability, we must have a notion of *observability* of the observation problem (2.7). We say that the problem (2.7) is *observable* whenever there exists a constant $C > 0$ only dependent of the terminal time $T > 0$ such that

$$\left|\varphi^0\right|^2 \leq C \int_0^T |B^* \varphi|^2 \, dt. \tag{2.8}$$

The classical result is the equivalence between control and observation.

**Proposition 2.2.2.1.** *The control system* (2.2) *is controllable in time* $T > 0$ *if and only if the adjoint problem* (2.7) *is observable.*

*Proof.* Let the adjoint system (2.7) be observable. That is, the observability inequality (2.8) holds. For solutions $y$ and $\varphi$ of (2.2) and (2.7) we have

$$(\dot{y}, \varphi) = (Ay, \varphi) + (Bu, \varphi)$$
$$-(y, \dot{\varphi}) = (A^* \varphi, y)$$

and thus

$$\frac{d}{dt}(y, \varphi) = (Bu, \varphi) = (u, B^* \varphi).$$

Integrating with respect to $t$ over the interval $(0, T)$ we see that

$$(y(T), \varphi(T)) - (y(0), \varphi(0)) = \int_0^T (u, B^* \varphi)\ dt. \tag{2.9}$$

So, the above shows that

$$-(y(0), \varphi(0)) = \int_0^T (u, B^* \varphi)\ dt \tag{2.10}$$

is sufficient to steer the initial data $y^0$ to zero. The mapping $J : \mathbb{R}^n \to \mathbb{R}$ defined by

$$J(\varphi^0) = \frac{1}{2} \int_0^T |B^* \varphi|^2\ dt + (y^0, \varphi(0))$$

is clearly a functional and since we have assumed the system is observable, it follows that $J$ achieves its minimum. Let $\tilde{\varphi}^0$ be some minimizer of $J$ and so

$$\frac{d}{d\epsilon} J(\tilde{\varphi}^0 + \epsilon \varphi^0)\big|_{\epsilon=0} = 0$$

for all $\varphi^0 \in \mathbb{R}^n$. This along with (2.10) implies that the control $u$ to steer $y^0$ to the zero state is of the form $u = B^* \tilde{\varphi}$.

Now assume the system is not exactly observable. Then we can find terminal data $\varphi^0$ in $\mathbb{R}^n$ such that $B^* \varphi = 0$ for all $t \in (0, T)$. We can choose initial state $y^0 \in \mathbb{R}^n$ such that $(y^0, \varphi(0))$ not zero and so from (2.9), we have that

$$(y(T), \varphi(T)) = (\varphi(0), z(0)) \neq 0$$

such that the terminal state $y(T)$ cannot be zero.

This concludes the proof that the systems is controllable if and only if the dual system is observable. □

To this point we have most of what we need in the finite dimensional case. Many things do not translate to infinite dimensional cases although there are many analogies and hence it is worth studying. In the next section, we apply a particular method to answer the question of controllability in the context of partial differential equations.

## 2.3 Infinite dimensional control

In this section we study the null-controllability property of a simple heat equation with no point mass on a bounded domain. The purpose is to understand the necessary tools developed by R. D. Russell and H. O. Fattorini in (15) and (40), to reduce the problem of null-controllability to that of a problem of moments. In particular, in (15), these techniques are applied to general parabolic equations with boundary controls. In (1), Avdonin and Ivanov discuss this method extensively in a more general setting. We also refer to (48), (34) and (18) where the ideas presented here can also be found.

### 2.3.1 Well-posedness

Consider the heat diffusion problem of a rod occupying the interval $(0,1)$ of the $x$-axis:

$$\begin{cases} y_t = y_{xx}, & x \in (0,1), t \in (0,T) \\ y(t,0) = 0,\ y(t,1) = f(t), & t \in (0,T) \\ y(0,x) = y^0(x), x \in (0,1) \end{cases} \tag{2.11}$$

where we seek to find $f \in L^2(0,T)$ given $T > 0$ and initial data $y^0 \in L^2(0,1)$, such that $y(T,x) = 0$. That is, we wish to find a control function $f$ in $L^2(0,T)$ acting on the system at $x = 1$ to bring the temperature of the rod to zero in time $T$. The well-posedness of (2.11) is well understood as given by the following theorem.

**Theorem 2.3.1.1.** *Given any $f \in L^2(0,T)$ and $y^0 \in L^2(0,1)$, the system (2.11) has a unique weak solution $y \in C([0,T]; H^{-1}(0,1))$. Furthermore, the map $\{y^0, f\} \to \{u\}$ is linear and there exists $C = C(T) > 0$ such that*

$$\|y\|_{L^\infty(0,T;H^{-1}(0,1))} \leq C(\|y^0\|_{L^2(0,1)} + \|f\|_{L^2(0,T)}.$$

Details of the above theorem can be found in (31). Note that in the subsequent chapters, we will show the well-posedness of our result using semigroup techniques.

### 2.3.2 Controllability

Very often it is useful to study control problems in terms of the reachable state. Let $T > 0$ be given, and define the reachable state from the initial data $y^0 \in L^2(0,1)$ by

$$R(T; u^0) = \{y(T) \ : \ y \text{ solution of (2.11) with } f \in L^2(0,T)\}$$

In this fashion, we say that $R(T; u^0)$ is a reachable state of the control problem (2.11) in time $T > 0$ from $y^0$ with a control $f$. Next we define the various degrees of controllability.

**Definition 2.3.2.1.** *We say that system (2.11) is approximately controllable in time $T$, if for every initial data $y^0 \in L^2(0,1)$, the set of reachable states $R(T; y^0)$ is dense in $L^2(0,1)$.*

**Definition 2.3.2.2.** *We say that system (2.11) is exactly controllable in time $T$, if for every initial data $y^0 \in L^2(0,1)$, the set of reachable states $R(T; y^0)$ coincides with $L^2(0,1)$.*

**Definition 2.3.2.3.** *We say that system (2.11) is null controllable in time $T$, if for every initial data $y^0 \in L^2(0,1)$, the set of reachable states $R(T; y^0)$ contains the zero state.*

As we will see, the null-controllability property is the only relevant degree of controllability. The remaining two are either not attainable or trivial.

Since solutions of (2.11) are $C^\infty$ away from boundary at time $T$, the set of reachable sets at time $T$ consists entirely of $C^\infty$ functions and thus exact controllability is not necessarily attained. Hence, we do not discuss exact controllability.

Let $\mathbb{T}$ denote the semigroup generated by the heat operator. Note that for any $y^0, \varphi^0 \in L^2(0,1)$, the linearity of the system gives

$$R(T; y^0 - \varphi^0) = R(T; y^0) - \mathbb{T}\varphi^0.$$

If the zero state is reachable then from the above we have that $\mathbb{T}\varphi^0 \in R(T; y^0)$. Thus, the range of the semigroup operator is reachable from any initial state $y^0$. Furthermore, this implies that $\mathbb{T}(T)(L^2(0,1)) \subset R(T; y^0)$. Since $\mathbb{T}(T)(L^2(0,1))$ is dense in $L^2(0,1)$ we see that null controllability implies approximate controllability. The above discussion shows that null controllability is the most relevant type of controllability to study in the context of heat operators.

### 2.3.3 The method of moments

We now outline the process to convert the problem of null controllability to a problem of moments. We begin by introducing the dual or observation problem of (2.11):

$$\begin{cases} -\varphi_t = \varphi_{xx}, & x \in (0,1), t \in (0,T) \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T) \\ \varphi(T,x) = \varphi^T(x), x \in (0,1) \end{cases} \tag{2.12}$$

with terminal data $\varphi^T \in L^2(0,1)$.

**Lemma 2.3.3.1.** *System (2.11) is null controllable in time $T > 0$ if and only if, for any $u^0 \in L^2(0,1)$ there exists control $f \in L^2(0,T)$ such that*

$$\int_0^T f(t)\varphi_x(t,1)\ dt = \int_0^1 y^0(x)\varphi(0,x)\ dx \tag{2.13}$$

*for any $\varphi^T \in L^2(0,1)$, where $\varphi$ is the solution to the observation system (2.12).*

Remark that the above result is analogous to 5.2.1.1 and 5.2.2.1 when we add a point mass.

*Proof.* Let $f \in L^2(0,T)$ and $\varphi^T \in L^2(0,1)$ be given. For $y$ and $\varphi$ solutions to the control problem (2.11) and observation problem (2.12), integration by parts leads to

$$\int_0^1 y^0(x)\varphi(0,x)\ dx - \int_0^1 y(T,x)\varphi^T(x)\ dx = \int_0^T f(t)\varphi_x(t,1)\ dx.$$

If (2.13) holds then

$$\int_0^1 y(T,x)\varphi^T(x)\ dx = 0$$

for all $\varphi^T \in L^2(0,1)$, in which case $y(T,x) = 0$. Thus (2.11) is null controllable in time $T > 0$. Similarly, one sees that if the system is null controllable with control function $f \in L^2(0,T)$ then the identity (2.13) holds as needed. □

From here on, let $-\lambda_n$ denote the sequence of negative, decreasing eigenvalues of the heat operator where $\lambda_n = n^2\pi^2$ for $n \in \mathbb{N}$.

**Proposition 2.3.3.1.** *The system* (2.11) *is null-controllable in time $T > 0$ if and only if for any $y^0 \in L^2(0,1)$, there exists a control function $f \in L^2(0,T)$ such that,*

$$\frac{a_n}{b_n} e^{-\lambda_n T} = \int_0^T f(T-t) e^{-\lambda_n t}\, dt, \qquad n \in \mathbb{N} \tag{2.14}$$

*where $a_n$ are the Fourier coefficients of $y^0$ and $b_n = (-1)^n 2n\pi$.*

It is worth noting that $b_n$ are the Fourier coefficients of the corresponding observability operator acting on the control input $f$.

*Proof.* Note that the eigenfunctions of the heat operator are given by $\{\sin(\pi n x)\}_{n\in\mathbb{N}}$ and they form an orthogonal basis for $L^2(0,1)$. Hence, for $y^0 \in L^2(0,1)$ we have

$$y^0 = \sum_{m\in\mathbb{N}} a_m \sin(\pi m x).$$

One can write eigensolutions to the dual problem (2.12) as $\varphi(t,x) = e^{-\lambda_n (T-t)} \sin(\pi n x)$. Applying these solutions to identity (2.13) we have that

$$\begin{aligned}\int_0^T f(t) e^{-\lambda_n (T-t)} (-1)^n n\pi\, dt &= \int_0^1 \sum_{m\in\mathbb{N}} a_m \sin(\pi m x) e^{-\lambda_n T} \sin(\pi n x)\, dx \\ &= \frac{a_n}{2} e^{-\lambda_n T}\end{aligned}$$

which is equivalent to (2.14) after defining $b_n := (-1)^n 2n\pi$. □

From Proposition 2.3.3.1 we see that to show that there is $f \in L^2(0,T)$ such that (2.11) can be stirred to zero from any initial state $y^0 \in L^2(0,1)$, is equivalent to show there is $f \in L^2(0,T)$ satisfying the moment problem (2.14) for any terminal data $\varphi^T \in L^2(0,1)$ of the dual problem (2.12). One can express the solution to (2.14) as

$$f(T-t) = \sum_{m\in\mathbb{N}} \frac{a_m}{b_m} e^{-\lambda_m T} \theta_m(t), \tag{2.15}$$

where $\{\theta_m\}_{m\in\mathbb{N}}$ is a biorthogonal sequence to the family of real exponential functions $\{e^{-\lambda_m}\}_{m\in\mathbb{N}}$. That is

$$\int_0^T \theta_m(t) e^{-\lambda_n t}\, dt = \delta_{n,m} \tag{2.16}$$

for all $n, m \in \mathbb{N}$. The next result shows that once the existence of the biorthogonal sequence is obtained, the control function $f$ is in $L^2(0,T)$. Clearly, proving the existence of the biorthogonal sequence is sufficient to prove that (2.11) is null controllable.

**Proposition 2.3.3.2.** *The system* (2.11) *is null controllable in time* $T > 0$ *only if given* $T > 0$ *there exists a biorthogonal sequence* $\{\theta_m\}_{m\in\mathbb{N}}$ *as in* (2.16) *such that*

$$\|\theta_m\|_{L^2(0,T)} \leq M_1 e^{M_2 m} \tag{2.17}$$

*for all* $m \in \mathbb{N}$ *and some constants* $M_1, M_2 > 0$.

*Proof.* Assume (2.17). It is clear that

$$f(T-t) = \sum_{m\in\mathbb{N}} \frac{a_m}{b_m} e^{-\lambda_m T} \theta_m(t) \tag{2.18}$$

solves the moment problem (2.14) if it converges in the $L^2(0,T)$ sense. Note that using the orthogonality of the eigenvectors,

$$\begin{aligned}
\|f(T-t)\|_{L^2(0,T)} &\leq \sum_{m\in\mathbb{N}} \left\| \frac{a_m}{b_m} e^{-\lambda_m T} \theta_m(t) \right\|_{L^2(0,T)} \\
&\leq \sum_{m\in\mathbb{N}} \frac{a_m}{b_m} e^{-\lambda_m T} \left\|\theta_m(t)\right\|_{L^2(0,T)} \\
&\leq \sum_{m\in\mathbb{N}} \frac{|a_m|}{|b_m|} e^{-\lambda_m T} M_1 e^{M_2 m} \\
&\leq M_1 \sum_{m\in\mathbb{N}} \frac{|a_m|}{2\pi m} e^{-\lambda_m T + M_2 m}.
\end{aligned}$$

The above clearly converges and thus (2.18) defines a solution to the moment problem (2.14) as needed to show (2.11) is null controllable. □

The following section outlines the main tools to prove the existence of the biorthogonal sequence in (2.15). As mentioned before, this method was developed by R.D. Russell and H. O. Fattorini. See (15) and (40) for more details.

### 2.3.4 Biorthogonal sequence

The classical Müntz-Szász theorem for families of exponentials states that for an increasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive numbers, the family of exponentials $\{e^{-s_n t}\}_{n\in\mathbb{N}}$ is complete in

$L^2(0,T)$ if and only if

$$\sum_{n\in\mathbb{N}} \frac{1}{s_n} = +\infty.$$

Applying the above to $\lambda_n$ we see that $\Lambda$ is not complete in $L^2(0,T)$. For convenience, let us denote $\Lambda$ the family of exponentials $\{e^{-\lambda_n t}\}_{n\in\mathbb{N}}$. Furthermore, let us denote $E(\Lambda,T)$ and $E(m,\Lambda,T)$, the spaces generated by $\Lambda$ and $\Lambda\setminus\{e^{-\lambda_m t}\}$ in $L^2(0,T)$ respectively. Lastly, let $p_n(t) = e^{-\lambda_n t}$. The next theorem shows the existence of a biorthogonal sequence $\{\theta_m\}_{m\in\mathbb{N}}$ to $\Lambda$.

**Theorem 2.3.4.1.** *Given $T>0$, there exists a unique sequence $\{\theta_m(T,\cdot)\}_{m\in\mathbb{N}}$ of minimal $L^2(0,T)$ norm such that*

$$\{\theta_m(T,\cdot)\}_{m\in\mathbb{N}} \subset E(\Lambda,T).$$

*Proof.* By the Müntz-Szász theorem, we have that $\Lambda$ is not complete and thus it is minimal. That is, $\Lambda$ is independent and spans a proper closed subspace of $L^2(0,T)$. So $p_m \notin E(m,\Lambda,T)$ for all $m\in\mathbb{N}$. Let $q_m$ be the orthogonal projection $p_m$ over $E(m,\Lambda,T)$ and define

$$\theta_m(T,\cdot) = \frac{p_m - q_m}{\|p_m - q_m\|^2_{L^2(0,T)}}.$$

Then

$$\int_0^T \theta_m(T,t)p_n(t)\,dt = \delta_{nm}$$

and $\theta_m(T,\cdot)\in E(\Lambda,T)$ as needed to show $\{\theta_m(T,\cdot)$ gives a biorthogonal sequence to $\Lambda$ in $E(\Lambda,T)$. Let $\{\tilde{\theta}_m\}_{m\in\mathbb{N}}$ be another biorthogonal sequence and so $\theta_m-\tilde{\theta}_m\in E(\Lambda,T)$ is perpendicular to $p_n$ for all $n\in\mathbb{N}$. Since $\{p_n\}$ is dense in $E(\Lambda,T)$, it follows that $\theta_m-\tilde{\theta}_m=0$ as needed to show the sequence is unique. Note that for all $m\in\mathbb{N}$, there is a unique $q_m\in E(\Lambda,T)^\perp$ such that $\tilde{\theta}_m=\theta_m+q_m$. Then by the properties of projection,

$$\|\tilde{\theta}_m\| = \|\theta_m+q_m\| = \sqrt{\|\theta_m\|^2+\|q_m\|^2} \geq \|\theta_m\|$$

as needed to show the norm of $\{\theta_m\}$ is minimal in $L^2(0,T)$. □

Next we compute the $L^2(0,T)$ norm of the biorthogonal sequence in the case of $T=\infty$ and $T<\infty$.

Let $T=\infty$. First, recall the following linear algebra result.

**Lemma 2.3.4.1.** *If $C=\{c_{ij}\}_{1\le i,j\le n}$ is a $n\times n$-matrix with coefficients $c_{i,j}=\dfrac{1}{a_i+b_j}$, then*

$$det(C)=\frac{\prod\limits_{i\le i,j\le n}(a_i-a_j)(b_i-b_j)}{\prod\limits_{1\le i,j\le n}(a_i+b+j)}.$$

See (12) for more details.

**Lemma 2.3.4.2.** *There exists positive constant $M$ such that for any $m\in\mathbb{N}$ we have*

$$\prod_{\substack{m\in\mathbb{N}\\ m\ne k}}\frac{m^2+k^2}{|m^2-k^2|}\le e^{Mm}. \tag{2.19}$$

*Proof.* Consider

$$\begin{aligned}\prod_{\substack{m\in\mathbb{N}\\ m\ne k}}\frac{m^2+k^2}{|m^2-k^2|}&=\exp\left(\sum_{\substack{m\in\mathbb{N}\\ m\ne k}}\ln\left(\frac{m^2+k^2}{|m^2-k^2|}\right)\right)\\&\le\exp\left(\sum_{\substack{m\in\mathbb{N}\\ m\ne k}}\ln\left(1+\frac{2m^2}{|m^2-k^2|}\right)\right).\end{aligned}$$

Note,

$$\begin{aligned}&\sum_{\substack{m\in\mathbb{N}\\ m\ne k}}\ln\left(1+\frac{2m^2}{|m^2-k^2|}\right)\\&\le\int_1^m\ln\left(1+\frac{2m^2}{|m^2-x^2|}\right)+\int_m^{2m}\ln\left(1+\frac{2m^2}{|x^2-m^2|}\right)+\int_{2m}^{\infty}\ln\left(1+\frac{2m^2}{|x^2-m^2|}\right)\\&\le m\left(\int_0^1\ln\left(1+\frac{2}{|1-x^2|}\right)+\int_1^2\ln\left(1+\frac{2}{|x^2-1|}\right)+\int_2^{\infty}\ln\left(1+\frac{2}{|x^2-1|}\right)\right)\end{aligned}$$

After integrating, we see that each of the three integrals is convergent and thus

$$M_2=\int_0^1\ln\left(1+\frac{2}{|1-x^2|}\right)+\int_1^2\ln\left(1+\frac{2}{|x^2-1|}\right)+\int_2^{\infty}\ln\left(1+\frac{2}{|x^2-1|}\right)$$

as needed to show (2.19). □

Before proving the main result concerning the $L^2(0,\infty)$ norm of the biorthogonal sequence, define the following spaces

$$E^N(\Lambda,T) = \text{ subspace generated by } \Lambda^N = \{e^{-\lambda_n t}\}_{1\le n\le N}$$
$$E^N(m,\Lambda,T) = \text{ subspace generated by } \{e^{-\lambda_n t}\}_{1\le n\le N,\ n\neq m}$$

in $L^2(0,T)$.

**Theorem 2.3.4.2.** *There exists two positive constants $M_1$ and $M_2$ such that $\{\theta_m(\infty,\cdot)\}_{m\in\mathbb{N}}$ satisfies*

$$\|\theta_m(\infty,\cdot)\|_{L^2(0,\infty)} \le M_1 e^{M_2 m}$$

*for all $m \in \mathbb{N}$.*

*Proof.* Observe that both $E^N(\Lambda,T)$ and $E^N(m,\Lambda,T)$ are finite dimensional and

$$E(\Lambda,T) = \bigcup_{N\in\mathbb{N}} E^N(\Lambda,T), \quad E(m,\Lambda,T) = \bigcup_{N\in\mathbb{N}} E^N(m,\Lambda,T)$$

Then, for all $N\in\mathbb{N}$ we see there is a unique biorthogonal family $\{\theta_m^N\}_{1\le m\le N}$ in $E^N(\Lambda,T)$ to $\Lambda^N$. We may construct this sequence using the projection $q_m^N$ of $p_m$ over $E^N(m,\Lambda,T)$ as

$$\theta_m^N = \frac{p_m - q_m^N}{\|p_m - q_m^N\|_{L^2(0,T)}}. \tag{2.20}$$

By the finite dimensionality of $E^N(\Lambda,T)$, write $\theta_m^N$ as

$$\theta_m^N = \sum_{1\le k\le N} c_k^m p_k.$$

Taking the $L^2(0,T)$ inner product with $p_n$ we see that

$$\sum_{k\in\mathbb{N}} c_k^m \int_0^T p_k(t)p_n(t)\; dt = \delta_{mn}$$

and taking $L^2(0,T)$ norm of $\theta_m^N$ we see that

$$\|\theta_m^N\|_{L^2(0,T)} = c_m^m.$$

Denoting

$$g_k^j = \langle p_k, p_j \rangle_{L^2(0,T)} = \frac{1}{\lambda_j + \lambda_k},$$

we see that $G = \{g_k^j\}_{1 \le k,j \le N}$ is the Gramm matrix of $\Lambda^N$. Then, from the above identities we see that $c_k^m$ are the elements of $G^{-1}$ and by Cramer's rule

$$c_m^m = \frac{\det(G_m)}{\det(G)}$$

where $G_m$ is the matrix obtained by replacing the $m^{\text{th}}$ column of $G$ with the $m^{\text{th}}$ vector of the canonical basis. Thus so far,

$$\|\theta_m^N\|_{L^2(0,T)} = \sqrt{\frac{\det(G_m)}{\det(G)}}.$$

Applying Lemma 2.3.4.1, we see that

$$\|\theta_m^N\|_{L^2(0,T)} = \sqrt{2}\pi m \prod_{\substack{1 \le k \le N \\ k \ne m}} \frac{m^2 + k^2}{|m^2 - k^2|}.$$

The last step consists of passing to the limit as $N \to \infty$. Recall that

$$\lim_{N \to \infty} \prod_{\substack{1 \le k \le N \\ k \ne m}} \frac{m^2 + k^2}{|m^2 - k^2|} = \prod_{\substack{k \in \mathbb{N} \\ k \ne m}} \frac{m^2 + k^2}{|m^2 - k^2|}$$

if and only if

$$\sum_{\substack{k \in \mathbb{N} \\ k \ne m}} \ln\left(\frac{m^2 + k^2}{|m^2 - k^2|}\right)$$

converges. Indeed,

$$\begin{aligned}\sum_{\substack{k \in \mathbb{N} \\ k \ne m}} \ln\left(\frac{m^2 + k^2}{|m^2 - k^2|}\right) &\le \sum_{\substack{k \in \mathbb{N} \\ k \ne m}} \ln\left(1 + \frac{2m^2}{|m^2 - k^2|}\right) \\ &\le 2m^2 \sum_{\substack{k \in \mathbb{N} \\ k \ne m}} \frac{1}{|m^2 - k^2|}.\end{aligned}$$

Since the above converges, we see that

$$\lim_{N \to \infty} \|\theta_m^N\|_{L^2(0,T)} = \sqrt{2}\pi m \prod_{\substack{k \in \mathbb{N} \\ k \ne m}} \frac{m^2 + k^2}{|m^2 - k^2|}.$$

is convergent and by Lemma 2.3.4.2 there exists $M_1, M_2 > 0$ such that

$$\lim_{N\to\infty} \|\theta_m^N\|_{L^2(0,T)} \leq M_1 e^{M_2 m}.$$

The last step is to show that $\theta_m^N \to \theta_m$ in the $L^2(0,T)$ sense. Using the definition (2.20) of $\theta_m^N$, we have

$$\lim_{N\to\infty} \|\theta_m^N\|_{L^2(0,T)} = \lim_{N\to\infty} \frac{1}{\|p_m - q_m^N\|_{L^2(0,T)}}.$$

Let $\epsilon > 0$ be given. Note that $q_m \in E(m,\Lambda,T)$ can be approximated in $E^{(\epsilon)}(m,\Lambda,T)$. That is, we can find $N(\epsilon) \in \mathbb{N}$ and the corresponding $q_m^{N(\epsilon)} \in E^{(\epsilon)}(m,\Lambda,T)$ such that

$$\|q_m - q_m^{N(\epsilon)}\|_{L^2(0,T)} < \epsilon.$$

Recall that by the properties of projections, any $q_m \in E(m,\Lambda,T)$ satisfies

$$\|p_m(t) - q_m(t)\|_{L^2(0,T)} = \min_{q\in E(m,\Lambda,T)} \|p_m(t) - q(t)\|_{L^2(0,T)}$$

and thus

$$\|p_m - q_m\| = \min_{q\in E(m,\Lambda,T)} \|p_m(t) - q\| \leq \|p_m(t) - q_m^N\|.$$

Conversely,

$$\begin{aligned}
\|p_m(t) - q_m^N\| &= \min_{q\in E^N(m,\Lambda,T)} \|p_m(t) - q\| \\
&\leq \|p_m(t) - q_m^{N(\epsilon)}\| \\
&\leq \|p_m(t) - q_m\| + \|q_m - q_m^{N(\epsilon)}\| \\
&< \|p_m(t) - q_m\| + \epsilon.
\end{aligned}$$

Combing the two inequalities and passing to the limit, we see that

$$\begin{aligned}
\lim_{N\to\infty} \|\theta_m^N\|_{L^2(0,T)} &= \lim_{N\to\infty} \frac{1}{\|p_m - q_m^N\|_{L^2(0,T)}} \\
&= \frac{1}{\|p_m - q_m\|_{L^2(0,T)}} \\
&= \|\theta_m\|_{L^2(0,T)}
\end{aligned}$$

as needed to show

$$\|\theta_m\|_{L^2(0,T)} \leq M_1 e^{M_2 m}$$

for some $M_1, M_2 > 0$. $\square$

This concludes the convergence of the biorthogonal sequence in the case that $T = \infty$.

Finally, in this section we compute the $L^2(0,T)$ norm bound of the biorthogonal sequence with $T < \infty$. So let $T < \infty$. We first need the following result which can be found in (21), (34) or (15).

**Theorem 2.3.4.3.** *Given $T \in (0, \infty)$, the restriction operator to $\Lambda$ given by*

$$R_T \;:\; E(\Lambda, \infty) \to E(\Lambda, T), \quad R_T(y) = y\big|_{[0,T]}$$

*is invertible and there is $C = C(T) > 0$ such that $\|R_T^{-1}\| \leq C$.*

The following result gives the minimal $L^2(0,T)$ of the biorthogonal sequence. Let $\{\theta_m\}$ be the biorthogonal sequence as given in Theorem 2.3.4.1.

**Theorem 2.3.4.4.** *There exists two positive constants $M_1$ and $M_2$ such that $\{\theta_m(T,\cdot)\}_{m\in\mathbb{N}}$ satisfies*

$$\|\theta_m(T,\cdot)\|_{L^2(0,T)} \leq M_1 e^{M_2 m}$$

*for all $m \in \mathbb{N}$.*

*Proof.* From Theorem 2.3.4.3 we have that $R_T^{-1}$ is bounded and we let let ${R_T^{-1}}^* \;:\; E(\Lambda, \infty) \to E(\Lambda, T)$ be its adjoint. Hence, by the properties of projections and the adjoint operator,

$$\delta_{km} = \langle p_k, \theta_m(\infty, \cdot)\rangle_{L^2(0,\infty)} = \langle R_T(p_k), {R_T^{-1}}^*(\theta_m(\infty, \cdot))\rangle_{L^2(0,\infty)}.$$

By the uniqueness of the biorthogonal sequence and ${R_T^{-1}}^*(\theta_m(\infty, \cdot) \in E(\Lambda, T)$, we see that

$${R_T^{-1}}^*(\theta_m(\infty, \cdot)) = \theta_m(T, \cdot)$$

for all $m \in \mathbb{N}$ where we have used the uniqueness of $\theta_m(T,\cdot)$. Now

$$\begin{aligned}\|\theta_m(T,\cdot)\|_{L^2(0,T)} &= \|R_T^{-1*}(\theta_m(\infty,\cdot))\|_{L^2(0,T)}\\ &\leq \|R_T^{-1*}\|\|\theta_m(\infty,\cdot)\|_{L^2(0,\infty)}\\ &= \|R_T^{-1}\|\|\theta_m(\infty,\cdot)\|_{L^2(0,\infty)}.\end{aligned}$$

Recalling the bounds of the biorthogonal sequence $\theta_m(\infty,\cdot)$ in $L^2(0,\infty)$ in 2.3.4.2, and the bound of the inverse of the restriction operator, we can find $M_1, M_2 > 0$ such that

$$\|\theta_m(T,\cdot)\|_{L^2(0,T)} \leq M_1 e^{M_2 m}$$

for all $m \in \mathbb{N}$ as needed. □

## 2.4 Concluding remark

For a wider class of problems, one can obtain an analogous moment problem

$$\frac{a_n}{b_n} e^{-\lambda_n T} = \int_0^T f(T-t) e^{-\lambda_n t}\, dt, \qquad n \in \mathbb{N}$$

with the corresponding sequence of eigenvalues $\{\lambda_n\}$. Similarly to the case of $n^2\pi^2$, we may apply the Müntz-Szász Theorem to perturbations of $n^2$ and thus conclude that the family of exponentials $\{e^{-\lambda_n t}\}$ is not complete. Once again, this allows the construction of a biorthogonal sequence $\{\theta_m\}$ in $L^2(0,T)$. Once existence is established with the corresponding $L^2(0,T)$ bound, we set

$$f(T-t) = \sum_{m\in\mathbb{N}} \frac{a_m}{b_m} e^{-\lambda_m T} \theta_m(t),$$

which converges whenever $|b_m| \geq C m^{-\alpha}$ for some $C > 0$ and $\alpha > 0$.

# CHAPTER 3. MODELING OF A HEAT EQUATION WITH A DIRAC DENSITY

Consider a linear hybrid system consisting of two wires or rods connected by a thin wall of width $2\epsilon > 0$ and density $1/2\epsilon$. Assume the two rods occupy the intervals $\omega_{\epsilon,1} = (-L_1, -\epsilon)$ and $\omega_{\epsilon,2} = (\epsilon, L_2)$, and the wall occupies the interval $\omega_\epsilon = (-\epsilon, \epsilon)$. Correspondingly, let $u_\epsilon = u_\epsilon(t, x)$, $v_\epsilon = v_\epsilon(t, x)$ and $z_\epsilon = z_\epsilon(t, x)$ denote the temperature distribution on their respective domains $\omega_{\epsilon,1}$, $\omega_{\epsilon,2}$, and $\omega_\epsilon$. We suppose the temperature of the rods and wall satisfy the heat equation on their respective domains with Dirichlet boundary conditions at endpoints $x = -L_1, L_2$. The linear equation modeling heat flow of such a system is as follows:

$$\begin{cases} c_1\rho_1\dot{u_\epsilon} - k_1{u_\epsilon}'' = 0, & t > 0,\ x \in \omega_{\epsilon,1} \\ c_2\rho_2\dot{v_\epsilon} - k_2{v_\epsilon}'' = 0, & t > 0,\ x \in \omega_{\epsilon,2} \\ \frac{c}{2\epsilon}\dot{z_\epsilon} - k{z_\epsilon}'' = 0, & t > 0,\ x \in \omega_\epsilon \\ u_\epsilon(t, -\epsilon) = z_\epsilon(t, -\epsilon),\ z_\epsilon(t, \epsilon) = v_\epsilon(t, \epsilon), & t > 0 \\ k_1{u_\epsilon}'(t, -\epsilon) = k{z_\epsilon}'(t, -\epsilon),\ k{z_\epsilon}'(t, \epsilon) = k_2{v_\epsilon}'(t, \epsilon), & t > 0 \\ u_\epsilon(t, -L_1) = v_\epsilon(t, L_2) = 0, & t > 0. \end{cases} \tag{3.1}$$

Throughout this thesis, $'$ will denote spatial derivatives and $\dot{}$ will denote temporal derivatives. The parameters $c > 0$ and $k > 0$ in the third equation represent the specific heat and conductivity of the wall connecting the two rods. The parameters $c_i$, $\rho_i$ and $k_i$ in (3.1) are positive and represent the specific heat, density and thermal conductivity of the rod on the subdomain $\omega_{\epsilon,i}$. It will later be convenient to use the diffusivity coefficient $\alpha_i^2 = k_i/c_i\rho_i$ for $i = 1, 2$. The fourth equation guarantees continuity of the temperature across the interface $x = \pm\epsilon$ and the fifth equation represents the heat flux continuity condition at the interfaces (see (37, Chapter

8) ). We complete the system by adding the initial conditions

$$\{u_\epsilon^0(x), v_\epsilon^0(x), z_\epsilon^0(x)\} = \{u_\epsilon(0,x), v_\epsilon(0,x), z_\epsilon(0,x)\} \tag{3.2}$$

in an appropriately defined function space at time $t = 0$ so we may determine the solution of (3.1) uniquely.

We show in this chapter that with appropriate assumptions on the initial conditions, the solution $\{u_\epsilon, v_\epsilon, z_\epsilon\}$ of (3.1) with (3.2) converges in a weak sense to the solution of the following limiting hybrid system:

$$\begin{cases} c_1\rho_1\dot{u} - k_1u'' = 0, & t > 0,\ x \in \omega_1 := \omega_{0,1} \\ c_2\rho_2\dot{v} - k_2v'' = 0, & t > 0,\ x \in \omega_2 := \omega_{0,2} \\ c\ \dot{z} = k_2v'(t,0) - k_1u'(t,0), & t > 0 \\ u(t,0) = v(t,0) = z(t), & t > 0 \\ u(t,-L_1) = v(t,L_2) = 0, & t > 0, \end{cases} \tag{3.3}$$

with initial conditions of the form

$$\{u^0(x), v^0(x), z^0\} = \{u(0,x), v(0,x), z(0)\} \tag{3.4}$$

given in an appropriately defined function space at time $t = 0$. The third equation in (3.3) states that the rate of change in temperature of the point mass is proportional to the net heat flux into the point mass. This can be viewed as a form of Fick's law of diffusion.

Similar hybrid systems involving strings and beams with point masses have been studied in the context of controllability and stabilization theory. See for example (25), (30), (8), (9), (29), (35), (10), (47), (20) and (19). In particular, C. Castro showed in (6) that a system similar to (3.3) with strings can be obtained from a system similar to that in (3.1) and gave a detailed spectral analysis.

## 3.1 Well-posedness

We begin by proving well-posedness of the limit problem (3.3).

### 3.1.1 The limit problem

Given $u$, $v$ defined on $\omega_1$, $\omega_2$ and $z \in \mathbb{R}$, let $y = (u, v, z)^t$, where $^t$ denotes transposition and define

$$\mathcal{H} = L^2(\omega_1) \times L^2(\omega_2) \times \mathbb{R}$$

equipped with the norm

$$\|y\|_{\mathcal{H}}^2 = c_1\rho_1\|u\|_{\omega_1}^2 + c_2\rho_2\|v\|_{\omega_2}^2 + c\ |z|^2$$

where $\|\cdot\|_{\omega_i}$ is the usual norm in $L^2(\omega_i)$ for $i = 1, 2$. Define

$$\vartheta_{\omega_1} = \{u \in H^1(\omega_1) \mid u(-L_1) = 0\}$$

$$\vartheta_{\omega_2} = \{v \in H^1(\omega_2) \mid v(L_2) = 0\}$$

$$\vartheta = \{(u, v) \in \vartheta_{\omega_1} \times \vartheta_{\omega_2} \mid u(0) = v(0)\}$$

equipped with the norms

$$\|u\|_{\vartheta_{\omega_i}}^2 = k_i\|u'\|_{L^2(\omega_i)}^2, \quad \|(u, v)\|_{\vartheta}^2 = \|u\|_{\vartheta_{\omega_1}}^2 + \|v\|_{\vartheta_{\omega_2}}^2$$

for $i = 1, 2$. We can check that (see (25)) $\vartheta$ is algebraically and topologically equivalent to $H_0^1(\Omega)$, however one can think of $\vartheta$ as a subspace of $\vartheta_{\omega_1} \times \vartheta_{\omega_2}$. The space

$$\mathcal{W} = \{(u, v, z) \in \vartheta \times \mathbb{R} \mid u(0) = v(0) = z\}$$

is a closed subspace of $\vartheta \times \mathbb{R}$ with norm we may define as $\|y\|_{\mathcal{W}}^2 = \|(u, v)\|_{\vartheta}^2$. It is easy to see that the space $\mathcal{W}$ is densely and continuously embedded in the space $\mathcal{H}$. Define the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A} = \begin{pmatrix} \alpha_1^2 d^2 & 0 & 0 \\ 0 & \alpha_2^2 d^2 & 0 \\ -\frac{k_1}{c}\delta_0 d & \frac{k_2}{c}\delta_0 d & 0 \end{pmatrix} \tag{3.5}$$

where $d$ denotes the (distributional) derivative operator and $\delta_0$ denotes the Dirac delta function with mass at $x = 0$, and the domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{y \in \mathcal{W}\ :\ u \in H^2(\omega_1),\ v \in H^2(\omega_2)\}. \tag{3.6}$$

When $D(\mathcal{A})$ is endowed with the graph-norm topology

$$\|y\|_{D(\mathcal{A})}^2 = \|y\|_{\mathcal{H}}^2 + \|\mathcal{A}y\|_{\mathcal{H}}^2$$

it becomes a Hilbert space with continuous embedding in $\mathcal{H}$. We can therefore write the limit system (3.3) as

$$\dot{y}(t) = \mathcal{A}y(t), \quad y(0) = y^0, \quad t > 0 \tag{3.7}$$

where $y^0 = (u^0, v^0, z^0)$. Since $D(\mathcal{A})$ is dense in $\mathcal{W}$ and the latter is dense in $\mathcal{H}$, it follows that $\mathcal{A}$ is a densely defined operator.

**Lemma 3.1.1.1.** *The operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a bijection.*

*Proof.* Let $\vec{f} = (f, g, h) \in \mathcal{H}$ be arbitrary. Then the solution to $\mathcal{A}y = \vec{f}$ is given by

$$y = \begin{pmatrix} C_u(x + L_1) - F(x) \\ C_v(x - L_2) - G(x) \\ C_z \end{pmatrix} \tag{3.8}$$

where

$$\begin{aligned} F(x) &= \int_{-L_1}^{x} \int_{s}^{0} \alpha_1^{-2} f(r)\, drds, \\ G(x) &= \int_{x}^{L_2} \int_{0}^{s} \alpha_2^{-2} g(r)\, drds, \end{aligned} \qquad \begin{aligned} C_u &= \frac{-chL_2 + k_2(F(0) - G(0))}{k_2L_1 + k_1L_2} \\ C_v &= \frac{chL_1 + k_1(F(0) - G(0))}{k_2L_1 + k_1L_2} \\ C_z &= -\frac{chL_1L_2 + L_2k_1F(0) + L_1k_2G(0)}{k_2L_1 + k_1L_2}. \end{aligned} \tag{3.9}$$

Since $(u'', v'') = (\alpha_1^{-2} f, \alpha_2^{-2} g) \in L^2(\omega_1) \times L^2(\omega_2)$ it follows that $(u, v) \in H^2(\omega_1) \times H^2(\omega_2)$. Furthermore, one can check from (3.9) that $u(0) = v(0) = z$ and $u(-L_1) = v(L_2) = 0$ so that $y \in D(\mathcal{A})$. Thus $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is surjective.

Finally, note that the null space of $\mathcal{A}$ is trivial since when $\vec{f} = (0, 0, 0)$ we see that $y$ is the trivial solution. Then $\mathcal{A}$ is injective and hence bijective. □

**Lemma 3.1.1.2.** *The operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is symmetric and dissipative.*

*Proof.* Consider $\varphi = (\mu, \nu, \zeta) \in D(\mathcal{A})$. Then

$$\begin{aligned}
\langle \mathcal{A}y, \varphi\rangle_{\mathcal{H}} &= k_1 u'\mu|_{-L_1}^{0} - k_1\langle u', \mu'\rangle_{\omega_1} + k_2 v'\nu|_0^{L_2} - k_2\langle v', \nu'\rangle_{\omega_2} + (k_2 v'(0) - k_1 u'(0))\zeta \\
&= -k_1 u\mu'|_{-L_1}^{0} + k_1\langle u, \mu''\rangle_{\omega_1} - k_2 v\nu'|_0^{L_2} + k_2\langle v, \nu''\rangle_{\omega_2} \\
&= c_1\rho_1\langle u, \alpha_1^2\mu''\rangle_{\omega_1} + c_2\rho_2\langle v, \alpha_2^2\nu''\rangle_{\omega_2} + cz\left(\frac{k_2}{c}\nu'(0) - \frac{k_2}{c}\mu'(0)\right) \\
&= \langle y, \mathcal{A}\varphi\rangle_{\mathcal{H}}
\end{aligned}$$

for all $y = (u, v, z) \in D(\mathcal{A})$. Hence $D(\mathcal{A}) \subset D(\mathcal{A}^*)$ and so $\mathcal{A}$ is a symmetric operator. In particular, when we choose $y = \varphi$ we see from the above computation that

$$\langle \mathcal{A}y, y\rangle_{\mathcal{H}} = -k_1\|u'\|_{\omega_1}^2 - k_2\|v'\|_{\omega_2}^2 = -\|y\|_{\mathcal{W}}^2.$$

Thus we have that $\langle \mathcal{A}y, y\rangle_{\mathcal{H}} \leq 0$ for any $y \in D(\mathcal{A})$ as needed to show $\mathcal{A}$ is dissipative. □

**Lemma 3.1.1.3.** *The operator $\mathcal{A}$ is closed, self-adjoint and its inverse is a compact operator in $\mathcal{H}$.*

*Proof.* As mentioned before, $\mathcal{A}$ is densely defined in $\mathcal{H}$ and from Lemmas 3.1.1.2 and 3.1.1.1 we have that it is symmetric and $R(\mathcal{A}) = \mathcal{H}$. It follows from Theorem 13.11 in (39), that $\mathcal{A}$ is self-adjoint and its inverse $\mathcal{A}^{-1}$ is bounded in $\mathcal{H}$. Furthermore, since the inverse is bounded, we have $0 \in \rho(\mathcal{A})$ and Theorem 13.9 in (39) implies that $\mathcal{A}$ is closed.

Next we claim that $K := \mathcal{A}^{-1}$ is compact. From formulas (3.8)-(3.9) we can decompose $K = K_1 + K_2$ where

$$K_1\vec{f} = \begin{pmatrix} -F(x) \\ -G(x) \\ 0 \end{pmatrix}, \quad K_2\vec{f} = \begin{pmatrix} C_u(x + L_1) \\ C_v(x - L_2) \\ C_z \end{pmatrix}.$$

Since the mappings $f \mapsto F(x)$ and $g \mapsto G(x)$ are Volterra-type operators, $K_1$ is compact. Since $0 \in \rho(\mathcal{A})$ and $K_1$ is compact, $K_2$ must be bounded. Since also $K_2$ has finite rank, it follows that $K_2$ is compact, and hence also $K$ is compact. □

**Proposition 3.1.1.1.** *The operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous $C_0$-semigroup of contractions which extends for $Re(t) > 0$ to an analytic semigroup.*

*Proof.* By Lemma 3.1.1.3, we have $\mathcal{A}$ is a closed, densely defined and self-adjoint operator and by Lemma 3.1.1.2 we see that both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative. Therefore, by the Lümer-Phillips theorem (see Luo et al (32)), we have that $\mathcal{A}$ generates a $C_0$-semigroup of contractions.

Furthermore, the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is contained in $(-\infty, 0)$ and by the computations shown in (5, page 55), we obtain $\|R(z,\mathcal{A})\| \leq \sec(\theta/2)/|z|$ for all $z = \rho e^{i\theta} \in \mathbb{C} \setminus (-\infty, 0]$ and $\theta \in (\pi/2, \pi)$. Rewriting the angle $\theta$ as $\pi/2 + \delta$ where $0 < \delta < \pi/2$ and letting $M = \sec(\theta/2)$ we see that

$$\|R(z,\mathcal{A})\| \leq \frac{M}{|z|} \text{ for all } z \in \mathbf{S}_\theta$$

where

$$\mathbf{S}_\theta = \{z \in \mathbb{C} \ : \ |\arg z| < \theta\}.$$

Note that $\mathbf{S}_\theta \cup \{0\}$ is contained in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. By Theorem 5.2. in (38) we have that $T$ can be extended to an analytic semigroup in a sector $\mathbf{S}_\delta$. If $\mathrm{Re}(z) > 0$ then for $\delta < \pi/2$ large enough, that $z \in \mathbf{S}_\delta$. Hence $A$ generates an analytic semigroup in the right half plane $\mathrm{Re}(z) > 0$. □

As a consequence of Proposition 3.1.1.1, we have the following result.

**Theorem 3.1.1.1.** *Given initial data $y^0 \in \mathcal{H}$, there exists a unique solution*

$$y \in C([0,\infty); \mathcal{H}) \tag{3.10}$$

*to the Cauchy problem (3.7). If in addition, $y^0 \in D(\mathcal{A})$, then $y \in C([0,\infty); D(\mathcal{A}))$.*

### 3.1.2 The approximate problem

Now consider functions $u_\epsilon$, $v_\epsilon$ and $z_\epsilon$ defined on $\omega_{\epsilon,1}$, $\omega_{\epsilon,2}$ and $\omega_\epsilon$ respectively and define $y_\epsilon = (u_\epsilon, v_\epsilon, z_\epsilon)^t$. Let

$$\mathcal{H}_\epsilon = L^2(\omega_{\epsilon,1}) \times L^2(\omega_{\epsilon,2}) \times L^2(\omega_\epsilon). \tag{3.11}$$

equipped with the norm

$$\|y_\epsilon\|^2_{\mathcal{H}_\epsilon} = c_1\rho_1\|u_\epsilon\|^2_{\omega_{\epsilon,1}} + c_2\rho_2\|v_\epsilon\|^2_{\omega_{\epsilon,2}} + \frac{c}{2\epsilon}\|z_\epsilon\|^2_{\omega_\epsilon}$$

where $\|\cdot\|_{\omega_{\epsilon,i}}$ is the usual norm in $L^2(\omega_{\epsilon,i})$ for $i = 1, 2$ and $\|\cdot\|_{\omega_\epsilon}$ is the usual norm in $L^2(\omega_\epsilon)$. Define

$$\vartheta_{\omega_{\epsilon,1}} = \{u_\epsilon \in H^1(\omega_{\epsilon,1}) \mid u_\epsilon(-L_1) = 0\}$$
$$\vartheta_{\omega_{\epsilon,2}} = \{v_\epsilon \in H^1(\omega_{\epsilon,2}) \mid v_\epsilon(L_2) = 0\}$$

equipped with the norms

$$\|u_\epsilon\|^2_{\vartheta_{\omega_{\epsilon,i}}} = k_i \|{u_\epsilon}'\|^2_{L^2(\omega_{\epsilon,i})},$$

for $i = 1, 2$. Next, consider the following subspace of $\vartheta_{\omega_{\epsilon,1}} \times \vartheta_{\omega_{\epsilon,2}} \times H^1(\omega_\epsilon)$:

$$\mathcal{W}_\epsilon = \{y_\epsilon \in \vartheta_{\omega_{\epsilon,1}} \times \vartheta_{\omega_{\epsilon,2}} \times H^1(\omega_\epsilon) \mid u_\epsilon(-\epsilon) = z_\epsilon(-\epsilon),\ z_\epsilon(\epsilon) = v_\epsilon(\epsilon)\} \tag{3.12}$$

with the norm

$$\|y_\epsilon\|^2_{\mathcal{W}_\epsilon} = k_1\|{u_\epsilon}'\|^2_{\omega_{\epsilon,1}} + k_2\|{v_\epsilon}'\|^2_{\omega_{\epsilon,2}} + k\|{z_\epsilon}'\|^2_{\omega_\epsilon}.$$

**Remark 3.1.2.1.** *It is easy to show that the spaces $\mathcal{W}_\epsilon$ are uniformly equivalent to $H_0^1(\Omega)$ in the sense that there exists some constant $C > 0$ such that*

$$\frac{1}{C}\|\varphi\|_{\mathcal{W}_\epsilon} \leq \|\varphi\|_{H_0^1(\Omega)} \leq C\|\varphi\|_{\mathcal{W}_\epsilon}$$

*where $C$ is independent of $\epsilon$ for all $0 < \epsilon < \epsilon_0$ with finite $\epsilon_0$. Furthermore, it is easy to see that the space $W_\epsilon$ is densely and continuously embedded in the space $\mathcal{H}_\epsilon$.*

We will also make use of the space $\mathcal{H}^2_\epsilon = H^2(\omega_{\epsilon,1}) \times H^2(\omega_{\epsilon,2}) \times H^2(\omega_\epsilon)$. Define the unbounded operators $\mathcal{A}_\epsilon : D(\mathcal{A}_\epsilon) \subset \mathcal{H}_\epsilon \to \mathcal{H}_\epsilon$ by

$$\mathcal{A}_\epsilon = \begin{pmatrix} \alpha_1^2 d^2 & 0 & 0 \\ 0 & \alpha_2^2 d^2 & 0 \\ 0 & 0 & \frac{2\epsilon k}{c} d^2 \end{pmatrix}, \tag{3.13}$$

with domain $D(\mathcal{A}_\epsilon)$ given by

$$D(\mathcal{A}_\epsilon) = \left\{ y_\epsilon \in \mathcal{W}_\epsilon \ : \ y_\epsilon \in \mathcal{H}^2_\epsilon,\ k_1{u_\epsilon}'(-\epsilon) = k{z_\epsilon}'(-\epsilon),\ k{z_\epsilon}'(\epsilon) = k_2{v_\epsilon}'(\epsilon) \right\}.$$

When $D(\mathcal{A}_\epsilon)$ is equipped with the graph-norm topology

$$\|y_\epsilon\|^2_{D(\mathcal{A}_\epsilon)} = \|y_\epsilon\|^2_{\mathcal{H}_\epsilon} + \|\mathcal{A}_\epsilon y_\epsilon\|^2_{\mathcal{H}_\epsilon},$$

it becomes a Hilbert space with continuous embedding in $\mathcal{H}_\epsilon$. We can now rewrite system (3.1) as a Cauchy problem:

$$\dot{y_\epsilon}(t) = \mathcal{A}_\epsilon y_\epsilon(t), \quad y_\epsilon(0) = y^0_\epsilon \in \mathcal{H}_\epsilon, \quad t > 0. \tag{3.14}$$

It is easy to see that $\mathcal{A}_\epsilon$ is densely defined on $\mathcal{H}_\epsilon$. As in Section 3.1.1 we have the following results.

**Lemma 3.1.2.1.** *The operator $\mathcal{A}_\epsilon : D(\mathcal{A}_\epsilon) \to \mathcal{H}_\epsilon$ is a bijective, dissipative, closed, self-adjoint operator with a compact inverse in $\mathcal{H}_\epsilon$.*

**Proposition 3.1.2.1.** *The operator $\mathcal{A}_\epsilon$ is the infinitesimal generator of a strongly continuous $C_0$-semigroup of contractions which extends for $Re(t) > 0$ to an analytic semigroup.*

The fact that $\mathcal{A}_\epsilon$ is the infinitesimal generator of an analytic $C_0$-semigroup implies that for all $y^0_\epsilon \in \mathcal{H}_\epsilon$ there exists a unique solution

$$y_\epsilon \in C([0,\infty); \mathcal{H}_\epsilon) \tag{3.15}$$

to (3.14). Moreover, if $y^0_\epsilon \in D(\mathcal{A}_\epsilon)$, then also $y_\epsilon \in C([0,\infty); D(\mathcal{A}_\epsilon))$.

## 3.2 Weak convergence

The energy functional of the hybrid system (3.3) is given by $E(t) = \|y\|^2_{\mathcal{H}}/2$. By taking test functions $\varphi \in C^1_0\big([0,\infty) \times \Omega\big)$, a weak form of the hybrid system (3.3) is given by

$$\begin{aligned}
\int\limits_{\omega_1} c_1\rho_1 u^0 \varphi(0,x)\ dx + \int\limits_{\omega_2} c_2\rho_2 v^0 \varphi(0,x)\ dx + c z^0 \varphi(0,0) \\
= -\int\limits_0^\infty \Big\{ \int\limits_{\omega_1} c_1\rho_1 u\dot{\varphi}\ dx + \int\limits_{\omega_2} c_2\rho_2 v\dot{\varphi}\ dx + cz\dot{\varphi}(t,0) \Big\}\ dt \\
+ \int\limits_0^\infty \Big\{ \int\limits_{\omega_1} k_1 u'\varphi'\ dx + \int\limits_{\omega_2} k_2 v'\varphi'\ dx \Big\}\ dt.
\end{aligned} \tag{3.16}$$

On the other hand, the energy functional for the $\epsilon$-dependent problem (3.1) is $E_\epsilon(t) = \|y_\epsilon\|^2_{\mathcal{H}_\epsilon}/2$ and by taking test functions $\varphi \in C^1_0\big([0,\infty)\times\Omega\big)$, a weak form is

$$\int\limits_{\omega_{\epsilon,1}} c_1\rho_1 u^0_\epsilon \varphi(0,x)\ dx + \int\limits_{\omega_{\epsilon,2}} c_2\rho_2 v^0_\epsilon \varphi(0,x)\ dx + \int\limits_{\omega_\epsilon} \frac{c}{2\epsilon} z^0_\epsilon \varphi(0,x)\ dx$$
$$= -\int\limits_0^\infty \Bigg\{ \int\limits_{\omega_{\epsilon,1}} c_1\rho_1 u_\epsilon \dot\varphi\ dx + \int\limits_{\omega_{\epsilon,2}} c_2\rho_2 v_\epsilon \dot\varphi\ dx + \int\limits_{\omega_\epsilon} \frac{c}{2\epsilon} z_\epsilon \dot\varphi\ dx \Bigg\}\ dt \tag{3.17}$$
$$+ \int\limits_0^\infty \Bigg\{ \int\limits_{\omega_{\epsilon,1}} k_1 {u_\epsilon}' \varphi'\ dx + \int\limits_{\omega_{\epsilon,2}} k_2 {v_\epsilon}' \varphi'\ dx + \int\limits_{\omega_\epsilon} k {z_\epsilon}' \varphi'\ dx \Bigg\}\ dt.$$

We give sufficient conditions such that we may pass to the limit in (3.17) to consequently obtain (3.16). Assume that $y^0_\epsilon \in D(\mathcal{A}_\epsilon)$ and furthermore, there exists $M_1 > 0$ such that

$$\|y^0_\epsilon\|_{\mathcal{H}_\epsilon} \le M_1, \tag{3.18}$$

for all $\epsilon > 0$. Then we obtain the following result.

**Lemma 3.2.0.2.** *The energy of system* (3.1) *is (uniformly) bounded by the initial energy in the sense that there exists constant $C > 0$ such that $E_\epsilon(t) \le C$ for all $\epsilon > 0$ whenever the initial data $y^0_\epsilon$ satisfies* (3.18)*.*

*Proof.* Note that if $y^0_\epsilon \in D(\mathcal{A}_\epsilon)$, then $y_\epsilon \in C\big([0,\infty); D(\mathcal{A}_\epsilon)\big)$ and the energy satisfies

$$\dot E_\epsilon(t) = k_1 {u_\epsilon}' u_\epsilon\big|^{-\epsilon}_{-L_1} + k_2 {v_\epsilon}' v_\epsilon\big|^{L_2}_{\epsilon} + k {z_\epsilon}' z_\epsilon\big|^{\epsilon}_{-\epsilon}$$
$$- \int\limits_{\omega_{\epsilon,1}} k_1 |{u_\epsilon}'|^2\ dx - \int\limits_{\omega_{\epsilon,2}} k_2 |{v_\epsilon}'|^2\ dx - \int\limits_{\omega_\epsilon} k |{z_\epsilon}'|^2\ dx \tag{3.19}$$
$$= -\|y_\epsilon\|^2_{\mathcal{W}_\epsilon},$$

which implies $\dot E_\epsilon(t) \le 0$, and thus $E_\epsilon(t) \le E_\epsilon(0)$. Hence by density, there exists $C = M_1^2/2$ for which $E_\epsilon(t) \le C$ for all $t > 0$ and initial data satisfying (3.18). Consequently, we find that $y_\epsilon \in L^\infty\big([0,\infty); \mathcal{H}_\epsilon\big)$ for all $\epsilon > 0$. □

Now assume there exists $M_2 > 0$ such that

$$\|y^0_\epsilon\|_{\mathcal{W}_\epsilon} \le M_2, \tag{3.20}$$

for all $\epsilon > 0$. Then we obtain the following result.

**Lemma 3.2.0.3.** *Assuming condition* (3.20) *holds, the sequence solutions* $\{y_\epsilon\}_{\epsilon>0}$ *to problem* (3.1) *is uniformly bounded in* $L^\infty\big([0,\infty); H_0^1(\Omega)\big)$.

*Proof.* Since $y_\epsilon^0 \in D(\mathcal{A}_\epsilon)$ we have that $y_\epsilon \in C\big([0,\infty); D(\mathcal{A}_\epsilon)\big)$ and

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|y_\epsilon\|^2_{\mathcal{W}_\epsilon} &= k_1 \dot{u}_\epsilon {u_\epsilon}'\big|_{-L_1}^{-\epsilon} + k_2 \dot{v}_\epsilon {v_\epsilon}'\big|_{\epsilon}^{L_2} + k\dot{z}_\epsilon {z_\epsilon}'\big|_{-\epsilon}^{\epsilon} \\ &\quad - k_1\langle \dot{u}_\epsilon, {u_\epsilon}''\rangle_{\epsilon,1}\, dx - k_2\langle \dot{v}_\epsilon {v_\epsilon}''\rangle_{\epsilon,2} - k\langle \dot{z}_\epsilon {z_\epsilon}''\rangle_\epsilon \\ &= -\|{y_\epsilon}'\|^2_{\mathcal{W}_\epsilon}.\end{aligned}$$

This shows that the sequence $\{\|y_\epsilon(t)\|^2_{\mathcal{W}_\epsilon}\}_{t>0}$ is monotone decreasing in $t$ and thus by density, $\|y_\epsilon(t)\|_{\mathcal{W}_\epsilon} \le \|y_\epsilon^0\|_{\mathcal{W}_\epsilon} \le M_2$ for all $t > 0$ as needed to show $y_\epsilon \in L^\infty(0,\infty;\mathcal{W}_\epsilon)$. From Remark 3.1.2.1 we see that the spaces $\mathcal{W}_\epsilon$ and $H_0^1(\Omega)$ are equivalent and thus there exists $K > 0$ independent of $\epsilon$ such that

$$\|y_\epsilon\|_{L^\infty([0,\infty);H_0^1(\Omega))} \le K$$

for all $\epsilon > 0$, as needed to show solutions to the $\epsilon$-dependent problem are uniformly bounded in $\mathcal{Y}$. □

Next, it is natural to assume the initial data convergences in a weak sense in $\mathcal{H}_\epsilon$. It is easy to see that $u_\epsilon^0$ being a sequence in $L^2(\omega_{\epsilon,1})$ implies that $\chi_{\omega_{\epsilon,1}} u_\epsilon^0$ is a sequence in $L^2(\omega_1)$. Likewise, $\chi_{\omega_{\epsilon,2}} v_\epsilon^0 \in L^2(\omega_1)$. Then we will assume that

$$\begin{cases} \chi_{\omega_{\epsilon,1}} u_\epsilon^0 \rightharpoonup u_0 \text{ weakly in } L^2(\omega_1) \text{ as } \epsilon \to 0 \\ \chi_{\omega_{\epsilon,2}} v_\epsilon^0 \rightharpoonup v_0 \text{ weakly in } L^2(\omega_2) \text{ as } \epsilon \to 0 \\ \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon^0 \, dx \to z_0 \text{ in } \mathbb{R} \text{ as } \epsilon \to 0. \end{cases} \tag{3.21}$$

The following theorem is our main result.

**Theorem 3.2.0.1.** *Let* $\{y_\epsilon\}_{\epsilon>0}$ *be the sequence of solutions to the* $\epsilon-$*dependent problem* (3.1) *with initial data* $y_\epsilon^0$. *Assuming* (3.18), (3.20) *and* (3.21), *the family* $\{y_\epsilon\}_{\epsilon>0}$ *of solutions to* (3.1) *problem satisfies*

$$y_\epsilon \overset{*}{\rightharpoonup} y \text{ in } L^\infty\big([0,\infty; H_0^1(\Omega)\big)$$

*as* $\epsilon \to 0$ *where* $y$ *is the weak solution to the limit problem* (3.3) *with initial* $y^0 \in \mathcal{H}$.

*Proof.* From Lemmas 3.2.0.2 and 3.2.0.3, the initial energy provides a uniform bound for the solutions $y_\epsilon \in L^\infty\big([0,\infty; H_0^1(\Omega)\big)$. We can then extract a subsequence of solutions (which is still denoted by the index $\epsilon$) such that

$$\chi_{\omega_{\epsilon,1}} u_\epsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\infty; L^2(\omega_1)) \cap L^\infty(0,\infty; \vartheta_{\omega_1}) \tag{3.22}$$

$$\chi_{\omega_{\epsilon,2}} v_\epsilon \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,\infty; L^2(\omega_2)) \cap L^\infty(0,\infty; \vartheta_{\omega_2}). \tag{3.23}$$

Next, observe that $g_\epsilon(t) := \frac{1}{2\epsilon}\langle 1, z_\epsilon(t)\rangle_\epsilon$ defines a function on $[0,\infty)$. Applying Holder's inequality we see that $|g_\epsilon(t)| \le \|z_\epsilon\|_\epsilon/\sqrt{2\epsilon}$. By condition (3.18) we have that $\{g_\epsilon\}_{\epsilon>0}$ is a uniformly bounded sequence in $L^\infty(0,\infty)$. Invoking the Banach-Alaoglu Theorem we can extract a subsequence of $z_\epsilon$ (still denoted with the index $\epsilon$) and find $z \in L^\infty(0,\infty)$ such that

$$g_\epsilon \overset{*}{\rightharpoonup} z \text{ in } L^\infty(0,\infty) \tag{3.24}$$

as $\epsilon \to 0$. We now pass to the limit in each of the nine terms in the characterization (3.17) of weak solutions of the $\epsilon$-problem with $\varphi \in C_0^1([0,\infty)\times\Omega)$. Since $\varphi(0,\cdot) \in C_0^1(\Omega)$ it follows from assumption (3.21) on the initial data $y_\epsilon^0$ that

$$\int_{\omega_{\epsilon,1}} u_\epsilon^0 \varphi(0,x)\ dx = \int_{\omega_1} \chi_{\omega_{\epsilon,1}} u_\epsilon^0 \varphi(0,x)\ dx \to \int_{\omega_1} u^0 \varphi(0,x)\ dx,$$
$$\int_{\omega_{\epsilon,2}} v_\epsilon^0 \varphi(0,x)\ dx = \int_{\omega_2} \chi_{\omega_{\epsilon,2}} v_\epsilon^0 \varphi(0,x)\ dx \to \int_{\omega_2} v^0 \varphi(0,x)\ dx.$$

Next we claim that $\int_{\omega_\epsilon} \frac{1}{2\epsilon} z_\epsilon^0 \varphi(0,x)\ dx \to z^0\varphi(0,0)$. By adding and subtracting the term $z^0\varphi(0,x)$ and applying the triangle inequality we find that

$$\left|\int_{\omega_\epsilon} \frac{1}{2\epsilon} z_\epsilon^0 \varphi(0,x)\ dx - z^0\varphi(0,0)\right| \le \max_{x\in\omega_\epsilon}|\varphi(0,x)| \left|\frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon^0 - z^0\ dx\right| + |z^0| \frac{1}{2\epsilon}\int_{\omega_\epsilon} |\varphi(0,x) - \varphi(0,0)|\ dx.$$

The first term in the right hand side above tends to zero from (3.21) and the boundedness of $\varphi(0,\cdot)$ on $\omega_\epsilon$. The second term tends to zero as well by the continuity of $\varphi(0,x)$.

From (3.22) and (3.23) we have that in particular for $\dot\varphi \in C\big([0,\infty)\times\Omega\big)$ that

$$\int_0^\infty \int_{\omega_{\epsilon,1}} u_\epsilon \dot\varphi\ dx\ dt \to \int_0^\infty \int_{\omega_1} u\dot\varphi(t,x)\ dx\ dt,$$
$$\int_0^\infty \int_{\omega_{\epsilon,2}} v_\epsilon \dot\varphi\ dx\ dt \to \int_0^\infty \int_{\omega_2} v\dot\varphi(t,x)\ dx\ dt.$$

Next we want to show that

$$\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(t,x)\dot{\varphi}(t,x)\ dx\ dt \to \int_0^\infty z(t)\dot{\varphi}(t,0)\ dt. \tag{3.25}$$

Observe that

$$\begin{aligned}
&\left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(t,x)\dot{\varphi}(t,x)\ dx\ dt - \int_0^\infty z(t)\dot{\varphi}(t,0)\ dx\ dt\right| \\
&\qquad = \left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon\dot{\varphi}(t,x) - z\dot{\varphi}(t,0)\ dx\ dt\right| \\
&\qquad \le \left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(\dot{\varphi}(t,x) - \dot{\varphi}(t,0))\ dx\ dt\right| + \left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} (z_\epsilon - z)\dot{\varphi}(t,0)\ dx\ dt\right|.
\end{aligned}$$

The last term tends to zero by (3.24). Regarding the first term, observe that applying Hölder's inequality we have

$$\begin{aligned}
\left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(\dot{\varphi}(t,x) - \dot{\varphi}(t,0))\ dx\ dt\right| &\le \int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} |z_\epsilon|\,|\dot{\varphi}(t,x) - \dot{\varphi}(t,0)|\ dx\ dt \\
&\le \int_0^\infty \frac{1}{\sqrt{2\epsilon}}\|z_\epsilon\|_\epsilon \frac{1}{\sqrt{2\epsilon}}\sqrt{\int_{\omega_\epsilon} |\dot{\varphi}(t,x) - \dot{\varphi}(t,0)|\ dx}\ dt \\
&\le \int_0^\infty \frac{1}{\sqrt{2\epsilon}}\|z_\epsilon\|_\epsilon \sqrt{\frac{1}{2\epsilon}\int_{\omega_\epsilon} |\dot{\varphi}(t,x) - \dot{\varphi}(t,0)|\ dx}\ dt.
\end{aligned}$$

Note that $\|z_\epsilon\|_\epsilon/\sqrt{2\epsilon}$ is bounded by condition (3.18) and by the continuity of $\dot{\varphi}$, we have that the above tends to zero as $\epsilon \to 0$. Since $\varphi' \in C([0,\infty)\times\Omega)$ we have from (3.22) and (3.23) that

$$\begin{aligned}
\int_0^\infty \int_{\omega_{\epsilon,1}} {u_\epsilon}'\varphi'\ dx\ dt &= \int_0^\infty \int_{\omega_1} \chi_{\omega_{\epsilon,1}}{u_\epsilon}'\varphi'\ dx\ dt \to \int_0^\infty \int_{\omega_1} u'\varphi'\ dx\ dt \\
\int_0^\infty \int_{\omega_{\epsilon,2}} {v_\epsilon}'\varphi'\ dx\ dt &= \int_0^\infty \int_{\omega_2} \chi_{\omega_{\epsilon,2}}{v_\epsilon}'\varphi'\ dx\ dt \to \int_0^\infty \int_{\omega_2} v'\varphi'\ dx\ dt.
\end{aligned}$$

Finally, the term $\int_0^\infty \int_{\omega_\epsilon} k{z_\epsilon}'\varphi'\ dxdt$ tends to zero as $\epsilon \to 0$ since $z_\epsilon(t,\cdot) \in H^1(\omega_\epsilon)$ and $\varphi' \in C([0,\infty)\times\Omega)$.

From the above discussion we now have that there exists a convergent subsequence of $y_\epsilon$ in the weak star sense, whose limit satisfies the equation (3.16). Since the limiting system has a unique weak solution, it follows that the convergence holds for the whole sequence $\{y_\epsilon\}_{\epsilon>0}$. Therefore, we have shown that the limiting system (3.3) can be approximated with the sequence of $\epsilon$ dependent problems (3.1) as needed. □

## 3.3 A heat equation with singular conductivity

In this section we propose an alternate approach. The main idea is to introduce a change of variables in the weak form to obtain a system with singular conductivity and finite density. Furthermore, the middle wall remains fixed as we pass to the limit.

Recall the weak form of the approximate system, (3.17). Let $h > 0$ be a fixed positive constant and introduce the following change of variables:

$$\begin{aligned} \xi &= x - (h-\epsilon), \qquad x \in \omega_{\epsilon,1} \\ \xi &= x + (h-\epsilon), \qquad x \in \omega_{\epsilon,2} \\ \xi &= \tfrac{hx}{\epsilon}, \qquad\qquad\quad x \in \omega_{\epsilon} \end{aligned} \tag{3.26}$$

By defining

$$\begin{aligned} u_h(x,t) &= u_\epsilon(x + (h-\epsilon), t) \\ v_h(x,t) &= v_\epsilon(x - (h-\epsilon), t) \\ z_h(x,t) &= z_\epsilon\left(\frac{\epsilon\xi}{h}, t\right) \end{aligned}$$

and

$$\psi(x,t) = \begin{cases} \varphi(x + (h-\epsilon), t) & x \in \big( - L_1 - (h-\epsilon), -h\big) \\ \varphi(x - (h-\epsilon), t) & x \in \big(L_2 + (h-\epsilon), -h\big) \\ \varphi\left(\frac{\epsilon\, x}{h}, t\right) & x \in \big( - h, -h\big) \end{cases}$$

we obtain the following weak form

$$\begin{aligned} &\int_{-L_1-(h-\epsilon)}^{-h} c_1\rho_1 u_h^0 \psi(0,x)\, dx + \int_{h}^{L_2+(h-\epsilon)} c_2\rho_2 v_h^0 \psi(0,x)\, dx + \int_{-h}^{h} \frac{c}{2h} z_h^0 \psi(0,x)\, dx \\ &\quad = -\int_0^\infty \left\{ \int_{-L_1-(h-\epsilon)}^{-h} c_1\rho_1 u_h \dot\psi\, dx + \int_{h}^{L_2+(h-\epsilon)} c_2\rho_2 v_h \dot\psi\, dx + \int_{-h}^{h} \frac{c}{2h} z_h \dot\psi\, dx \right\} dt \\ &\quad\quad + \int_0^\infty \left\{ \int_{-L_1-(h-\epsilon)}^{-h} k_1 u_h' \psi'\, dx + \int_{h}^{L_2+(h-\epsilon)} k_2 v_h' \psi'\, dx + \int_{-h}^{h} \frac{kh}{\epsilon} z_h' \psi'\, dx \right\} dt. \end{aligned} \tag{3.27}$$

One can observe from the above that taking $h=\epsilon$ yields the weak formulation (3.17). This corresponds to the weak form of

$$\begin{cases} c_1\rho_1\dot{u_h} - k_1{u_h}'' = 0, & t>0,\ x\in\big(-1-(h-\epsilon),-h\big) \\ c_2\rho_2\dot{v_h} - k_2{v_h}'' = 0, & t>0,\ x\in\big(1+(h-\epsilon),-h\big) \\ \frac{c}{2h}\dot{z_h} - \frac{hk}{\epsilon}{z_h}'' = 0, & t>0,\ x\in\big(-h,-h\big) \\ u_h(t,-h) = z_h(t,-h),\ z_h(t,h) = v_h(t,h), & t>0 \\ k_1{u_h}'(t,-h) = \frac{kh}{\epsilon}{z_h}'(t,-h),\ \frac{kh}{\epsilon}{z_h}'(t,h) = k_2{v_h}'(t,h), & t>0 \\ u_h\big(t,-1-(h-\epsilon)\big) = v_h\big(t,1+(h-\epsilon)\big) = 0, & t>0. \end{cases} \tag{3.28}$$

Apply the change of variables (3.26) to the norms $\|\cdot\|_{\mathcal{H}_\epsilon}$ and $\|\cdot\|_{\mathcal{W}_\epsilon}$. Then from the assumptions

$$\|y_\epsilon^0\|_{\mathcal{H}_\epsilon} \le M_1, \qquad \|y_\epsilon^0\|_{\mathcal{W}_\epsilon} \le M_2,$$

we can obtain an analogous results to that in Lemma 3.2.0.3. In particular, we see from the above that there is some constant $C>0$ such that

$$\frac{1}{\epsilon}\int_{-h}^{h} kh|{z_h}'|^2\,dx \le C$$

for all $\epsilon>0$. Hence we see that $\|{z_h}'\|_{L^2(-h,h)} = 0$. Hence, it is easy to see that when passing to the limit in (3.27) we obtain,

$$\begin{aligned} &\int_{-L_1-h}^{-h} c_1\rho_1 u^0\psi(0,x)\,dx + \int_{h}^{L_2+h} c_2\rho_2 v^0\psi(0,x)\,dx + \int_{-h}^{h}\frac{c}{2h}z^0\psi(0,x)\,dx \\ &\qquad = -\int_0^\infty\left\{\int_{-L_1-h}^{-h} c_1\rho_1 u\dot{\psi}\,dx + \int_{h}^{L_2+h} c_2\rho_2 v\dot{\psi}\,dx + \int_{-h}^{h}\frac{c}{2h}z\dot{\psi}\,dx\right\}dt \\ &\qquad\quad + \int_0^\infty\left\{\int_{-L_1-h}^{-h} k_1 u'\psi'\,dx + \int_{h}^{L_2+h} k_2 v'\psi'\,dx\right\}dt. \end{aligned} \tag{3.29}$$

with $\|{z_h}'\|_{L^2(-h,h)} = 0$. Now propose the above weak formulation with the class of test functions

$$\{\psi\in C_0^1([0,\infty)\times(-L_1-h,L_2+h))\ :\ \psi(t,x)=\psi(t),\ x\in(-h,h),t>0\}$$

Then (3.29) defines the weak for of the problem

$$
\begin{cases}
c_1\rho_1\dot{u} - k_1 u'' = 0, & t > 0,\ x \in (-L_1 - h, -h) \\
c_2\rho_2\dot{v} - k_2 v'' = 0, & t > 0,\ x \in (h, L_2 + h) \\
c\ \dot{z} = k_2 v'(t,h) - k_1 u'(t,-h), & t > 0, x \in (-h, h) \\
z' = 0, & t > 0, x \in (-h, h) \\
u(t,-h) = v(t,h) = z, & t > 0, x \in (-h, h) \\
u(t,-L_1) = v(t,L_2) = 0, & t > 0,
\end{cases}
\tag{3.30}
$$

We note the similarities between the above system of equations and that in (3.3). It is worth mentioning that we interpret the above as a system in which the middle region reaches the steady state instantaneously due to the singular conductivity. Since the solution is continuous at the interface, the middle temperature is constant in the space variable. We have the following result.

**Theorem 3.3.0.2.** *Let $\{y_h\}$ be the sequence of solutions to (3.28) with initial data $y_h^0$. Assuming (3.18), (3.20) and (3.21), the family $\{y_h\}$ of solutions to (3.28) problem satisfies*

$$y_h \overset{*}{\rightharpoonup} y \text{ in } L^\infty\big([0,\infty; H_0^1(\Omega)\big)$$

*as $\epsilon \to 0$ where $y$ is the weak solution to the limit problem (3.30) with initial $y^0 \in \mathcal{H}$.*

A similar system is described by (13) and (2) through different techniques and in higher dimension.

# CHAPTER 4. NULL BOUNDARY CONTROLLABILITY OF A POINT MASS 1-DIMENSIONAL HEAT EQUATION WITH GENERAL PARAMETERS

In this chapter we consider a linear hybrid system consisting of two wires or rods connected by a point mass. The temperature of two rods connected by a point mass occupy the interval $\Omega$ which is partitioned into $\omega_1 = (-L_1, 0)$ and $\omega_2 = (0, L_2)$. We say $u = u(t, x)$ and $v = v(t, x)$ denote the temperature distribution on their respective domains $\omega_1$ and $\omega_2$. On the other hand, $z = z(t)$ denotes the temperature of the point mass at $x = 0$. The system is described by the following linear equations:

$$\begin{cases} c_1\rho_1\dot{u} - k_1u'' = 0, & x \in \omega_1, \quad t > 0 \\ c_2\rho_2\dot{v} - k_2v'' = 0, & x \in \omega_2, \quad t > 0 \\ c\dot{z} = k_2v'(t,0) - k_1u'(t,0), & t > 0 \\ u(t,0) = v(t,0) = z(t), & t > 0 \\ u(t,-L_1) = 0, & t > 0 \end{cases} . \tag{4.1}$$

with initial conditions at time $t = 0$ given by

$$\begin{cases} u^0(x) = u(0,x), & x \in \omega_1 \\ v^0(x) = v(0,x), & x \in \omega_2 \\ z^0 = z(0), \end{cases}$$

where $\{u^0, v^0, z^0\}$ will be given in an appropriately defined function space. Throughout this paper, $'$ will denote spatial derivatives and $\dot{}$ will denote temporal derivatives. The parameters $c > 0$ and $k > 0$ in the third equation represent the specific hear and conductivity of the wall connecting the two rods. The parameters $c_i$, $k_i$ and $\rho_i$ in (4.1) are positive and represent het specific hear, density and thermal conductivity of the rod on the subdomain $\omega_i$. It is worth noting that the third differential equation states the the rate of change in temperature of the point mass is proportional to the net heat flux into the point mass. In (22) the authors have shown that under appropriate assumptions, solutions of (4.1) with homogeneous boundary condition

$$v(t, L_2) = 0, \qquad t > 0 \tag{4.2}$$

are the weak limit of the solutions of a system with a "thin" wall instead of a point mass satisfying its own heat equation.

The main purpose is to show boundary null controllability of temperature of (4.1) with Dirichlet control

$$v(t, L_2) = f(t), \qquad t > 0 \tag{4.3}$$

The authors showed in (23), that system (4.1) is null controllable in time $T > 0$ with Dirchlet and Neumann control whenever we choose $\Omega = (-1, 1)$ and all parameters are equal to one. In this chapter we show that the same result follows in an arbitrary interval with arbitrary parameters. Hybrid systems such as (4.1), have been extensively studied in the context of strings and beams with interior point masses. In (25), Hansen and Zuazua used the method of characteristics to prove the boundary null controllability of an analogous string system with an interior point mass. In (30) Littman and Taylor use transform methods to prove boundary feedback stabilization of the string mass system. In (8) and (9), Castro and Zuazua applied methods of non-harmonic Fourier series to show boundary controllability of systems of either Rayleigh or Euler-Bernoulli beams with interior point masses. We refer to (29), (35), (10), (47), (20) and (19) for related results on control and stabilization of systems of beams with end masses.

Our main result is given by the following proposition.

**Proposition 4.0.0.2.** *System (4.1) with Dirichlet control (4.3) is null controllable in any time $T > 0$. More precisely, given $T > 0$ there is a control $f \in L^2(0, T)$ such that given initial data $\{u^0, v^0, z^0\} \in L^2(\omega_1) \times L^2(\omega_2) \times \mathbb{R}$ we have that $\{u(T, x), v(T, x), z(T)\} = \{0, 0, 0\}$.*

In the next section we will see that solutions in Proposition 4.0.0.2, are defined by transposition in the spaces $C(0,T;X_{-1/2})$ for the case of Dirichlet control.

Our general approach is to reduce the control problem to a moment problem.

## 4.1 Preliminaries

We briefly discuss the well-posedness of the system (4.1) with homogeneous Dirichlet boundary condition (4.2). Given $u$, $v$ and $z$ defined on $\omega_1$, $\omega_2$ and $\mathbb{R}$ respectively, define $y=(u,v,z,)^t$ where $^t$ denotes transposition. Let

$$\mathcal{H}=L^2(\omega_1)\times L^2(\omega_2)\times\mathbb{R}$$

equipped with the norm

$$\|y\|^2_{\mathcal{H}}=c_1\rho_1\|u\|^2_{\omega_1}+c_2\rho_2\|v\|^2_{\omega_2}+c|z|^2$$

where $\|\cdot\|_{\omega_i}$ is the usual norm in $L^2(\omega_i)$ for $i=1,2$. Let

$$\begin{aligned}
&\vartheta_{\omega_1}=\{u\in H^1(\omega_1)\mid u(-L_1)=0\}\\
&\vartheta_{\omega_2}=\{v\in H^1(\omega_2)\mid v(L_2)=0\}\\
&\vartheta=\{(u,v)\in\vartheta_1\times\vartheta_2\mid u(0)=v(0)\}\\
&\mathcal{W}=\{(u,v,z)\in\vartheta\times\mathbb{R}\mid u(0)=v(0)=z\}.
\end{aligned}\tag{4.4}$$

equipped with the norms

$$\begin{aligned}
&\|u\|^2_{\vartheta_{\omega_i}}=k_i\|u'\|^2_{L^2(\omega_i)},\qquad i=1,2\\
&\|(u,v)\|^2_{\vartheta}=\|u\|^2_{\vartheta_{\omega_1}}+\|v\|^2_{\vartheta_{\omega_2}}
\end{aligned}$$

Note that $\vartheta$ is algebraically and topologically equivalent to $H^1_0(\Omega)$ and $\mathcal{W}$ is a closed subspace of $\vartheta\times\mathbb{R}$ with norm $\|(u,v,z)\|^2_{\mathcal{W}}=\|(u,v)\|^2_{\vartheta}$. It is easy to show (see (22)) that the space $\mathcal{W}$ is densely and continuously embedded in the space $\mathcal{H}$. Define the operator $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$

by

$$\mathcal{A} = \begin{pmatrix} \frac{k_1}{c_1\rho_1} d^2 & 0 & 0 \\ 0 & \frac{k_2}{c_2\rho_2} d^2 & 0 \\ -\frac{k_1}{c}\delta_0 d & \frac{k_2}{c}\delta_0 d & 0 \end{pmatrix} \tag{4.5}$$

where $d$ denotes the (distributional) derivative operator, $\delta_0$ denotes the Dirac delta function with mass at $x = 0$, and the domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{y \in \mathcal{W} \ : \ u \in H^2(\omega_1), \ v \in H^2(\omega_2)\}. \tag{4.6}$$

with the graph-norm topology

$$\|y\|^2_{D(\mathcal{A})} = \|y\|^2_{\mathcal{H}} + \|\mathcal{A}y\|^2_{\mathcal{H}}$$

so that it becomes a Hilbert space with continuous embedding in $\mathcal{H}$. We can therefore write the homogeneous point-mass systems (4.1), (4.2) as

$$\dot{y}(t) = \mathcal{A}y(t), \quad y(0) = y^0, \quad t > 0 \tag{4.7}$$

where $y^0 = (u^0, v^0, z^0)^t$ is the initial data.

**Proposition 4.1.0.3.** *The unbounded operator $\mathcal{A}$ given by (4.5) in domain $D(\mathcal{A})$ as in (4.6) is a bijective, self-adjoint and dissipative operator with a compact inverse. Furthermore, $\mathcal{A}$ is the infinitesimal generator of a strongly continuous, compact and analytic semigroup $(\mathbb{T}_t)_{t\geq 0}$.*

The proof of Proposition 4.1.0.3 can be found in (22). As a consequence of the above proposition, given initial data $y^0 \in \mathcal{H}$ there exists a unique solution

$$y \in C([0,\infty);\mathcal{H})$$

to the Cauchy problem (4.7). If in addition, $y^0 \in D(\mathcal{A})$ then $y \in C([0,\infty), D(\mathcal{A}))$.

As the authors have shown in (23) that $\mathcal{A}$ has only negative eigenvalues and so$-\mathcal{A}$ is positive, self-adjoint and provides an isomorphism: $D(\mathcal{A}) \to \mathcal{H}$. Moreover, fractional powers of $-\mathcal{A}$ are well-defined. Let $X_1 = D(\mathcal{A})$ and for $\alpha \in [0,1]$, define $X_\alpha = D((-\mathcal{A})^\alpha)$ and $X_{-\alpha} = X'_\alpha$, the dual space relative to the pivot space $\mathcal{H} = X_0$ of $X_\alpha$. Correspondingly, the semigroup $\mathbb{T}$ remains an analytic semigroup on the invariant subspaces $X_\alpha$, $0 \le \alpha \le 1$, and extends continuously to an analytic semigroup on spaces $X_\alpha$, $-1 \le \alpha \le 0$; see e.g., (44) for full explanation. The norm on $X_\alpha$ is given by $\|y\|_\alpha^2 = \langle (-\mathcal{A})^\alpha y, (-\mathcal{A})^\alpha y\rangle_{\mathcal{H}}$. In particular, $X_{1/2}$ is the completion of $X_1$ with respect to the norm

$$\|y\|_{1/2}^2 = \langle -\mathcal{A}y, y\rangle_0.$$

Integration by parts gives

$$\|y\|_{1/2}^2 = \langle y, y\rangle_{\mathcal{W}}.$$

Thus, $X_{1/2}$ is topologically equivalent to $H_0^1(\Omega)$.

It will be convenient to perform a change of variables. Let

$$\xi = \frac{1}{L_1}x, \qquad \tau = \frac{k_1}{c_1\rho_1 L_1^2}\, t,$$

and define the following constants

$$\alpha^{-2} = \frac{c_1\rho_1\ k_2}{c_2\rho_2 k_1} \qquad\qquad \gamma = \frac{c_1\rho_1 L_1}{c}$$
$$\beta = \frac{c_1\rho_1 L_1 k_2}{c\ k_1}, \qquad\qquad L = \frac{L_2}{L_1},$$

In this manner, we obtain the equivalent system,

$$\begin{cases} \dot{u} - u'' = 0, & x \in (-1,0), \quad t > 0 \\ \dot{v} - \alpha^{-2}v'' = 0, & x \in (0,L), \quad t > 0 \\ \dot{z} = \beta\ v'(t,0) - \gamma\ u'(t,0), & t > 0 \\ u(t,0) = v(t,0) = z(t), & t > 0 \\ u(t,-1) = 0, & t > 0 \end{cases} . \tag{4.8}$$

with homogeneous boudary condition

$$v(t,L) = 0, \qquad t > 0 \tag{4.9}$$

and initial data given by

$$\begin{cases} u(x,0) = u^0(x), & x \in (-1,0) \\ v(x,0) = v^0(x), & x \in (0,L) \\ z(0) = z^0, \end{cases}$$

The operator $\mathcal{A}$ defined in (4.5) simplifies as

$$\mathcal{A} = \begin{pmatrix} d^2 & 0 & 0 \\ 0 & \alpha^{-2}d^2 & 0 \\ -\gamma\ \delta_0 d & \beta\ \delta_0 d & 0 \end{pmatrix}. \tag{4.10}$$

**Remark 4.1.0.2.** *The constant $\alpha$ is particularly important since the spectral analysis will be different in the case that $\alpha$ is rational or irrational.*

## 4.2 Spectral analysis

By Proposition 4.1.0.3, the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is contained in the negative real axis and consists of eigenvalues $\{\lambda_n\}$ tending to negative infinity with corresponding eigenvectors $\{\varphi_n\}_{n\in\mathbb{N}}$ forming an orthogonal system for $\mathcal{H}$.

### 4.2.0.1 Spectral analysis for $L\alpha \notin \mathbb{Q}$

**Proposition 4.2.0.4.** *The eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $\mathcal{A}$ are given by $\lambda_n = -\mu_n^2$ where $\{\mu_n\}_{n\in\mathbb{N}}$ are the roots of the characteristic equation*

$$\mu = \gamma\cot(\mu) + \alpha\beta\cot(L\alpha\mu). \tag{4.11}$$

*The corresponding eigenvectors are given by*

$$\varphi_{2k}(x) = \begin{pmatrix} \sin\big((1+x)\mu_{2k}\big) \\ \frac{\sin(\mu_{2k})}{\sin(L\alpha\mu_{2k})}\sin\big((L-x)\alpha\mu_{2k}\big) \\ \sin(\mu_{2k}) \end{pmatrix}, \quad \varphi_{2k-1}(x) = \begin{pmatrix} \frac{\sin(L\alpha\mu_{2k-1})}{\sin(\mu_{2k-1})}\sin\big((1+x)\mu_{2k-1}\big) \\ \sin\big((L-x)\alpha\mu_{2k-1}\big) \\ \sin(\mu_{2k-1}) \end{pmatrix} \tag{4.12}$$

*and $\varphi_n \in D(\mathcal{A})$ for all $n \in \mathbb{N}$.*

*Proof.* Look for nontrivial functions $\varphi_n = (U_n, V_n, Z_n)^t \in D(\mathcal{A})$ such that $\mathcal{A}\varphi_n = \lambda_n\varphi_n$. The eigensystem reduces to the problem of finding $(U_n, V_n)$ such that

$$\begin{cases} U_n''(x) = \lambda_n U_n(x), & x \in \omega_1 \\ V_n''(x) = \alpha^2 \lambda_n V_n(x), & x \in \omega_2 \\ \beta V_n'(0) - \gamma U_n'(0) = \lambda_n Z_n \\ U_n(0) = V_n(0) = Z_n \\ U_n(-1) = V_n'(L) = 0. \end{cases}$$

One can observe that $Z_n = 0$ leads to

$$U_n(0) = \sin(\mu) = 0$$
$$V_n(0) = C\sin(L\alpha\mu) = 0$$

for some $C \in \mathbb{R}$. Because $L\alpha \notin \mathbb{Q}$, it follows that there is no $\mu$ satisfying the above. Hence, $Z_n \neq 0$ for all $n \in \mathbb{N}$. We obtain from the continuity condition that

$$Z_n = \sin(\mu_n) = C\sin(L\alpha\mu_n).$$

Since we have assumed that $Z_n \neq 0$ we may solve for $C$ such that

$$C = \frac{\sin(\mu)}{\sin(L\alpha\mu)}$$

is bounded. From the third equation we obtain

$$Z_n = \frac{\sin(\mu_n)}{\mu_n}\left(\alpha\beta\cot(L\alpha\mu_n) + \gamma\cot(\mu_n)\right)$$

and once again applying the continuity condition we have that

$$\mu_n = \gamma\cot(\mu_n) + \alpha\beta\cot(L\alpha\mu_n)$$

Hence the solution to the eigensystem is

$$\varphi_n(x) = \begin{pmatrix} \sin\big((1+x)\mu_n\big) \\ \frac{\sin(\mu_n)}{\sin(L\alpha\mu_n)}\sin\big((L-x)\alpha\mu_n\big) \\ \sin(\mu_n) \end{pmatrix}$$

which is equivalent to (4.12) by separating the index into even and odd, and multiplying by the appropriate factor. □

**Proposition 4.2.0.5.** *The sequence* $\{\mu_n\}_{n\in\mathbb{N}}$ *of roots of the characteristic equation* (4.11) *satisfies the asymptotic estimate*

$$\mu_n = \begin{cases} n\pi + \frac{\gamma}{n\pi} + \mathcal{O}\left(\frac{1}{n^2}\right), & n \text{ is odd} \\ an\pi + \frac{\alpha\beta}{n\pi} + \mathcal{O}\left(\frac{1}{n^2}\right), & n \text{ is even.} \end{cases} \tag{4.13}$$

*In particular, there exists a constant* $M > 0$ *such that consecutive eigenvalues of* $\mathcal{A}$ *in* (4.7) *satisfy the gap condition:*

$$|\lambda_{n+1} - \lambda_n| \geq \frac{M}{n}. \tag{4.14}$$

*Proof.* Next, we need to study how far apart are the roots of (4.11). It will be convenient to define the following constants

$$a = \frac{1}{\alpha\ L},\ A = \frac{\gamma}{\pi},\ B = \frac{\alpha\beta}{\pi},$$

and apply the change of variables $\mu = \pi x$. Then (4.11) becomes

$$x = A\cot(\pi x) + B\cot\left(\frac{\pi x}{a}\right) \tag{4.15}$$

Note that if $0 < a < 1$ then we may let $x = ya$. Then

$$y = \frac{A}{a}\cot\left(\frac{\pi y}{1/a}\right) + \frac{B}{a}\cot(\pi y)$$

Clearly this is an equivalent relation to that of (4.15) with $1/a > 1$. So without loss of generality assume that $a \geq 1$ and we focus on studying the separability of the roots of (4.15). Let $a$ be irrational. Note that in particular we see that for any $k \in \mathbb{N}$

$$\lim_{x\to k^-}\cot(\pi x) = -\infty, \qquad \lim_{x\to(ak)^-}\cot\left(\frac{\pi x}{a}\right) = -\infty$$
$$\lim_{x\to k^+}\cot(\pi x) = \infty, \qquad \lim_{x\to(ak)^+}\cot\left(\frac{\pi x}{a}\right) = \infty$$

So the vertical asymptotes of (4.15) occur at $x = k$ and $x = ak$ for $k \in \mathbb{N}$ and these do not repeat. Since

$$\frac{d}{dx}\left(A\cot(\pi x) + B\cot\left(\frac{\pi x}{a}\right) - x\right) = -\pi A\csc^2(\pi x) - \frac{\pi}{a}\ B\csc^2\left(\frac{\pi x}{a}\right) - 1 < 0$$

we have that there is exactly one root between each singularity. This implies that we can enumerate the roots of the characteristic equation and as a consequence, we may also enumerate the eigenvalues of $\mathcal{A}$. Let $\{\lambda_k\}_{k=1}^{\infty}$ denote the set of eigenvalues of $\mathcal{A}$. The eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfy $\lambda_k = -\mu_k^2$ where $\mu_k$ is the $k^{\text{th}}$ positive root of (4.15). We denote $R = \{\mu_k\}_{k=1}^{\infty}$, $\Sigma = \{\sigma_k\}_{k=1}^{\infty}$ and $S = \{s_k\}_{k=1}^{\infty}$ where

$$\sigma_k = \min\{\mu_j : \mu_j > k\}$$
$$s_k = \min\{\mu_j : \mu_j > ak\}$$

Since $a$ is irrational, we have that $\Sigma \cap S = \emptyset$ and $R = \Sigma \cup S$. Then we want to show that there exists some $M > 0$ such that for all integers $k$,

$$\mid \mu_k - \mu_{k+1} \mid \geq \frac{M}{k} \tag{4.16}$$

Assume (4.16) is false. Then there is a sequence of consecutive roots $(\mu_{k_n}, \mu_{k_n+1})$ that tend to zero. In other words

$$k_n \mid \mu_{k_n+1} - \mu_{k_n} \mid < \frac{1}{n} \tag{4.17}$$

for all $n \in \mathbb{N}$. Among other things, (4.17) implies that

$$\lim_{n\to\infty} \mid \mu_{k_n+1} - \mu_{k_n} \mid = 0 \tag{4.18}$$

We have four cases to consider:

I.- $(\mu_{k_n}, \mu_{k_n+1}) \in \Sigma \times \Sigma$, III.- $(\mu_{k_n}, \mu_{k_n+1}) \in S \times \Sigma$

II.- $(\mu_{k_n}, \mu_{k_n+1}) \in \Sigma \times S$, IV.- $(\mu_{k_n}, \mu_{k_n+1}) \in S \times S$

We can readily discard case IV. The reason is that since $a > 1$, given any $n \in \mathbb{N}$ there is $j_n \in \mathbb{N}$ such that $a_n < j_n < a_n(n+1)$. Hence there is a root in $\Sigma$ between the corresponding roots $s_n$ and $s_{n+1}$ of $S$. If follows that at least one of cases (I-III) occur infinity often. For such a case, there exists a subsequence of $\{k_n\}$, (which we still call $\{k_n\}$), such that $(\mu_{k_n}, \mu_{k_n+1})$ is in the relevant case (I-III) for all $n \in \mathbb{N}$. We show in each case that this is impossible to occur.

I.- Consider the case that there exists a infinite sequence of positive roots $(\mu_{k_n}, \mu_{k_{n+1}})$ in $\Sigma\times\Sigma$. Thus there is a corresponding subsequence on integers $\{l_n\}$ for which $(\mu_{k_n}, \mu_{k_{n+1}}) = (\sigma_{l_n}, \sigma_{l_n+1})$. Furthermore, from (4.18) we have that

$$\lim_{n\to\infty} \mid \sigma_{l_n+1} - \sigma_{l_n} \mid = 0$$

Since $\sigma_{l_n} < l_n + 1 < \sigma_{l_n+1}$, we see that in particular,

$$[\sigma_{l_n}]_1 \nearrow 1 \tag{4.19}$$

where we say the sequence $\sigma_{l_n}$ approaches 1 from below in modulo 1. So there exists $M \in \mathbb{N}$ such that $n \geq M$ implies $\mid \sigma_{l_n} - (l_n + 1) \mid < \frac{1}{2}$. We observe that for any $w$ such that $w < l_n$, we have that

$$\mid \sigma_{l_n} - w \mid > \frac{1}{2}$$

for all $n \geq M$. Let $w = j_n$ be the maximum integer such that $a\ j_n < l_n$. Hence, $\sigma_{l_n} > 1/2 + a\ j_n$ and the monotonicity of $\cot(x)$ give

$$\cot\left(\frac{\pi\sigma_{l_n}}{a}\right) < \cot\left(\frac{\pi}{a}(1/2 + a\ j_n)\right) = \cot\left(\frac{\pi}{2\ a}\right) \tag{4.20}$$

Because $a > 1$ we know that $\cot\left(\frac{\pi}{2\ a}\right)$ is a finite number and thus by (4.20), the sequence $\{\cot\left(\frac{\pi\sigma_{l_n}}{a}\right)\}$ is bounded. However, $\sigma_{l_n}$ is a root of the characteristic equation. In other words, it satisfies

$$\sigma_{l_n} = A\cot(\pi\sigma_{l_n}) + B\cot\left(\frac{\pi\sigma_{l_n}}{a}\right) \tag{4.21}$$

As mentioned before, as $n \to \infty$, we have $\sigma_{l_n} \to +\infty$ and by (4.19) we see that $\cot(\pi\sigma_{l_n}) \to -\infty$. We deduce that the remaining sequence $\{\cot\left(\frac{\pi\sigma_{l_n}}{a}\right)\}$ tends to infinity. This is however a contradiction since we had concluded in (4.20) that this is not bounded.

II.- The next case to consider is when there exists an infinite sequence of consecutive positive roots $(\mu_{k_n}, \mu_{k_n+1})$ in $\Sigma \times S$ for $n \in \mathbb{N}$. For convenience, we refer to $\mu_{k_n} \in \Sigma$ by $\sigma_{l_n}$ and similarly, we refer to $\mu_{k_n+1}$ by $s_{j_n}$. So from (4.18) we have that

$$\lim_{n\to\infty} \mid \sigma_{l_n} - s_{j_n} \mid = 0 \tag{4.22}$$

By the definition of $\Sigma$ and $S$, we know that $\sigma_{l_n} < a\ j_n < s_{j_n}$ and thus the above implies that

$$[\sigma_{l_n}]_a \nearrow a \tag{4.23}$$

By (4.23) we know that the term $\cot(\frac{\pi\ \sigma_{l_n}}{a})$ tends to negative infinity. Since $\sigma_{l_n}$ satisfies the characteristic equation (4.15), we have that $\cot(\pi\sigma_{l_n})$ goes to infinity as $n \to \infty$. We deduce that

$$[\sigma_{l_n}]_1 \searrow 0 \tag{4.24}$$

For the moment let us consider the function $\phi(x) = B\cot\left(\frac{\pi x}{a}\right) - x$, and let $s^*_{j_n}$ denote the $j_n^{\text{th}}$ positive root of $\phi$. Taking $n \to \infty$ we see that $\cot\left(\frac{\pi s^*_{j_n}}{a}\right)$ tends to infinity and thus

$$[s^*_{j_n}]_a \searrow 0 \tag{4.25}$$

Consider

$$s^*_{j_n} - l_n = \left(s^*_{j_n} - aj_n\right) + \left(aj_n - \sigma_{l_n}\right) + \left(\sigma_{l_n} - l_n\right)$$

and note that as $n \to \infty$, each term above is arbitrarily small by (4.25), (4.23) and (4.24) respectively. This implies that $|s^*_{j_n} - l_n|$ tends to zero and so $[s^*_{j_n}]_1 \searrow 0$. In particular, there exists $n$ such that $s^*_{j_n} - l_n < 1/2$. Hence for a sufficiently large $n$ we have

$$l_n < a\ j_n < s^*_{j_n} < l_n + 1/2$$

and thus $(a\ j_n, s^*_{j_n}) \subset (l_n, l_n + 1/2)$. Observe that $\cot(\pi x) > 0$ for $x \in (l_n, l_n + 1/2)$ and so $A\cot(\pi x)$ is also positive on the smaller interval $(a\ j_n, s^*_{j_n})$. Moreover, the function $\phi$ is monotone decreasing and thus $\phi(x) > 0$ for $x \in (a\ j_n, s^*_{j_n})$. Hence

$$\phi(x) + A\cot(\pi x) > 0$$

for $x \in (a\ j_n, s^*_{j_n})$. This implies that the root $s_{j_n}$ of the characteristic equation (4.15) satisfies $s^*_{j_n} < s_{j_n}$. By Lemma __ , we have that there exists $M > 0$ such that

$$|s^*_{j_n} - a\ j_n| \geq \frac{M}{j_n} \tag{4.26}$$

By (4.26) and $\sigma_{l_n} < a\ j_n < s_{j_n}$, we have that

$$|\sigma_{l_n} - s_{j_n}| \geq |a\ j_n - s_{j_n}| > |a\ j_n - s^*_{j_n}| \geq \frac{M}{j_n} \tag{4.27}$$

Recall that $\mu_{k_n} = \sigma_{l_n}$ and $\mu_{k_n+1} = s_{j_n}$. Note that in particular, for sufficiently large $n$, $k_n + 1 > j_n$. Thus we may say that $k_n \geq j_n$. From the latter and (4.27) we obtain

$$|\mu_{k_n} - \mu_{k_n+1}| \geq \frac{M}{k_n}$$

The above is however, a contradiction to our assumption (4.18). Hence no such subsequence $(\sigma_{l_n}, s_{j_n}) \in \Sigma \times S$ can exists.

III.- The next case to consider is when there exists an infinite sequence of consecutive positive roots $(\mu_{k_n}, \mu_{k_{n+1}})$ in $S \times \Sigma$ for $n \in \mathbb{N}$. For convenience, we refer to $\mu_{k_n} \in S$ by $s_{j_n}$ and similarly, we refer to $\mu_{k_n+1} \in \Sigma$ by $\sigma_{l_n}$. Once again, from (4.18) we have that

$$\lim_{n\to\infty} |\ \sigma_{k_n} - s_{j_n}\ | = 0 \tag{4.28}$$

The argument in case III will be very similar to that in case II. Knowing that in this case $s_{j_n} < l_n < \sigma_{l_n}$, the above implies that

$$[s_{j_n}]_1 \nearrow 1 \tag{4.29}$$

By (4.29) we know that the term $\cot(\pi\ s_{j_n})$ tends to negative infinity. Since $s_{j_n}$ satisfies the characteristic equation (4.15), we have that $\cot(\frac{\pi s_{j_n}}{a})$ goes to infinity as $n \to \infty$. We deduce that

$$[s_{j_n}]_a \searrow 0 \tag{4.30}$$

Now we consider the function $\psi(x) = A\cot(\pi x) - x$, and let $\sigma^*_{l_n}$ denote the $l_k^{\text{th}}$ positive root of $\psi$. Taking $n \to \infty$ we see that $\cot(\pi\sigma^*_{l_n})$ tends to infinity and thus

$$[\sigma^*_{l_n}]_1 \searrow 0 \tag{4.31}$$

Consider

$$\sigma^*_{l_n} - a\ j_n = \left(\sigma^*_{j_n} - l_n\right) + \left(l_n - s_{j_n}\right) + \left(s_{j_n} - a\ j_n\right)$$

and note that as $n \to \infty$, each term above is arbitrarily small by (4.31), (4.29) and (4.30) respectively. This implies that $|\sigma^*_{l_n} - a\ j_n|$ tends to zero and so $[\sigma^*_{l_n}]_a \searrow 0$. In particular, there exists $n$ such that $\sigma^*_{l_n} - a\ j_n < 1/2$. Hence for a sufficiently large $n$ we have

$$a\ j_n < l_n < \sigma^*_{j_n} < a\ j_n + 1/2$$

and thus $(l_n, \sigma^*_{l_n}) \subset (a\ j_n, a\ j_n + 1/2)$. Observe that $\cot(\frac{\pi x}{a}) > 0$ for $x \in (a\ j_n, a\ j_n + 1/2)$ and so $B\cot(\frac{\pi x}{a})$ is also positive on the smaller interval $(l_n, \sigma^*_{l_n})$. Moreover, the function $\psi$ is monotone decreasing and thus $\psi(x) > 0$ for $x \in (l_n, \sigma^*_{l_n})$. Hence

$$\phi(x) + B\cot(\frac{\pi x}{a}) > 0$$

for $x \in (l_n, \sigma^*_{l_n})$. This implies that the root $\sigma_{j_n}$ of the characteristic equation (4.15) satisfies $\sigma^*_{l_n} < \sigma_{l_n}$. We have that there exists $M > 0$ such that

$$|\sigma^*_{l_n} - l_n| \geq \frac{M}{l_n} \tag{4.32}$$

By (4.32) and $s_{j_n} < l_n < \sigma^*_{l_n} < \sigma_{l_n}$, we have that

$$|\sigma_{l_n} - s_{j_n}| \geq |\sigma_{l_n} - l_n| > |\sigma^*_{l_n} - l_n| \geq \frac{M}{l_n} \tag{4.33}$$

Recall that $\mu_{k_n} = s_{j_n}$ and $\mu_{k_n+1} = \sigma_{l_n}$. Note that in particular, for sufficiently large $n$ we have $k_n + 1 > l_n$. Thus we may say that $k_n \geq l_n$. From the latter and (4.33) we obtain

$$|\mu_{k_n} - \mu_{k_n+1}| \geq \frac{M}{k_n}$$

The above is however, a contradiction to our assumption (4.18). Hence no such subsequence $(s_{j_n}, \sigma_{l_n}) \in S \times \Sigma$ can exists.

We have shown that in any of the four cases (I-IV), there cannot exists a sequence of consecutive roots and therefore (4.16) must hold for all integers $k$ when $a$ is irrational. □

#### 4.2.0.2 Spectral analysis for $L\alpha \in \mathbb{Q}$

Now suppose that $a \geq 1$ is rational. So there are $p, q \in \mathbb{N}$ relatively prime such that $a = p/q$.

**Proposition 4.2.0.6.** *The eigenvalues* $\{\lambda_n\}_{n\in\mathbb{N}}$ *of* $\mathcal{A}$ *are given by* $\lambda_n = -\mu_n^2$ *where* $\{\mu_n\}_{n\in\mathbb{N}}$ *are the roots of the characteristic equation*

$$\mu = \gamma\cot(\mu) + \alpha\beta\cot(L\alpha\mu). \tag{4.34}$$

*and* $\lambda_{0,n} = -(np\pi)^2$. *The corresponding eigenvectors are given by*

$$\varphi_{0,n}(x) = \begin{pmatrix} \sin(np\pi x) \\ \frac{\gamma}{\beta\alpha}\sin(\alpha np\pi x) \\ 0 \end{pmatrix},$$

$$\varphi_{2n-1}(x) = \begin{pmatrix} \sin\Big((1+x)\mu_{2n-1}\Big) \\ \frac{\sin(\mu_{2n-1})}{\sin(L\alpha\mu_{2n-1})}\sin\Big((L-x)\alpha\mu_{2n-1}\Big) \\ \sin(\mu_{2n-1}) \end{pmatrix}, \tag{4.35}$$

$$\varphi_{2n}(x) = \begin{pmatrix} \frac{\sin(L\alpha\mu_{2n})}{\sin(\mu_{2n})}\sin\Big((1+x)\mu_{2n}\Big) \\ \sin\Big((L-x)\alpha\mu_{2n}\Big) \\ \sin(L\alpha\mu_{2n}) \end{pmatrix}$$

*and* $\varphi_n \in D(\mathcal{A})$ *for all* $n \in \mathbb{N}$.

*Proof.* Following the ideas of the proof of Proposition 4.2.0.4, one can obtain the eigenvalues $\lambda_n = -\mu_n^2$. The difference here is that $Z_n = 0$ does yield a branch of eigenvalues. Looking for nontrivial functions $\varphi_n = (U_n, V_n, Z_n)^t \in D(\mathcal{A})$ such that $\mathcal{A}\varphi_n = \lambda_n\varphi_n$ with $Z_0 = 0$ we see that the eigensystem reduces to the problem of finding $(U_n, V_n)$ such that

$$\begin{cases} U_n''(x) = \lambda_n U_n(x), & x \in \omega_1 \\ V_n''(x) = \alpha^2\lambda_n V_n(x), & x \in \omega_2 \\ \beta V_n'(0) - \gamma U_n'(0) = 0 \\ U_n(0) = V_n(0) = 0 \\ U_n(-1) = V_n'(L) = 0. \end{cases}$$

The continuity conditions gives

$$U_n(0) = \sin(\mu) = 0$$

$$V_n(0) = C\sin(L\alpha\mu) = 0$$

for some $C \in \mathbb{R}$. Using the fact that $L\alpha = q/p \in \mathbb{Q}$ we have that $\mu_n = pn\pi$ and so

$$U_n(x) = (-1)^{pn} \sin(\mu_n x)$$
$$V_n(x) = C(-1)^{qn+1} \sin(\alpha \mu_n x)$$

Now we want to determine the value of $C$. From the condition

$$\beta V_x(0) - \gamma U_x(0) = 0$$

we find that

$$\beta C(-1)^{qn+1} \alpha \mu_n - \gamma(-1)^{pn} \mu_n = 0$$

$$C = (-1)^{pn-qn-1} \frac{\gamma}{\beta\alpha}$$

Note that with the above value of $C$, $U_n$ and $V_n$ have the term $(-1)^{pn}$ in common. We can ignore this term and so

$$\begin{pmatrix} \sin(\mu_n x) \\ \frac{\gamma}{\beta\alpha} \sin(\alpha \mu_n x) \\ 0 \end{pmatrix},$$

with $n \in \mathbb{N}$ is the corresponding eigenpair to the eigenvector to the sequence of eigenvalues $-\mu_n^2 = -(np\pi)^2$. We use the subscript 0 in $\phi_{0,n}$ and $\lambda_{0,n}$ to know that this is the eigenfunction in the case that $z = 0$.

□

**Proposition 4.2.0.7.** *The sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of roots of the characteristic equation (4.34) satisfies the asymptotic estimate*

$$\mu_n = \begin{cases} n\pi + \frac{\gamma}{n\pi} + \mathcal{O}\left(\frac{1}{n^2}\right), & n \text{ is odd} \\ an\pi + \frac{\alpha\beta}{n\pi} + \mathcal{O}\left(\frac{1}{n^2}\right), & n \text{ is even.} \end{cases} \tag{4.36}$$

*In particular, there exists a constant $M > 0$ such that consecutive eigenvalues of $\mathcal{A}$ in (4.7) satisfy the gap condition:*

$$|\lambda_{n+1} - \lambda_n| \geq \frac{M}{n}. \tag{4.37}$$

*Proof.* Without loss of generality, suppose that $a \geq 1$. Furthermore, note that the case that $a = 1$ leads to a situation where all the singularities are the integers. This case is shown in Chapter 5. So we may now consider the case $a > 1 \in \mathbb{Q}$ in which case, $p > q$. Since $\alpha \in \mathbb{Q}$, we observe that in this case

$$f(x) := A\cot(\pi x) + B\cot\left(\frac{\pi x}{a}\right) = A\cot(\pi x) + B\cot\left(\frac{q\pi x}{p}\right)$$

is $p$-periodic since

$$f(x+p) = A\cot(\pi x + p\pi) + B\cot\left(\frac{\pi x}{a}\right)$$

Note that $f$ is monotonically decreasing between each pair singularities since

$$f'(x) = -A\pi\sec^2(\pi x) - B\frac{q\pi}{p}\sec^2\left(\frac{q\pi x}{p}\right) < 0$$

The singularities are $\{k\}_{k=1}^{\infty} \cup \{ak\}_{k=1}^{\infty}$. By the above properties of $f$ note that there is exactly one solution to $x = f(x)$ between each singularity. However, the fact that $\{k\}_{k=1}^{\infty}$ and $\{ak\}_{k=1}^{\infty}$ share some elements creates some issues if we are to prove separation the same way as in the irrationally related case. To this end define a sequence $\mu$ containing the singularities of $f$ as follows:

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \alpha_{1,i} & \dots & \alpha_{1,p+q-1} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \dots & \alpha_{2,i} & \dots & \alpha_{2,p+q-1} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \dots & \alpha_{3,i} & \dots & \alpha_{3,p+q-1} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{j,1} & \alpha_{j,2} & \alpha_{j,3} & \dots & \alpha_{j,i} & \dots & \alpha_{j,p+q-1} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & \dots & \alpha_{1,i} & \dots & \alpha_{1,p+q-1} \\ 0+p & 1+p & \dots & \alpha_{1,i}+p & \dots & \alpha_{1,p+q-1}+p \\ 0+2p & 1+2p & \dots & \alpha_{1,i}+2p & \dots & \alpha_{1,p+q-1}+2p \\ \vdots & \vdots & & \vdots & & \vdots \\ 0+(j-1)p & 1+(j-1)p & \dots & \alpha_{1,i}+(j-1)p & \dots & \alpha_{1,p+q-1}+(j-1)p \\ \vdots & \vdots & & \vdots & & \vdots \end{pmatrix}$$

So let us define

$$\alpha := \{\alpha_{j,i}\}_{j\in\mathbb{N}, i\in\{1,\dots,p+q-1\}}$$

and let $\alpha_j$ to be the $j^{\text{th}}$ column of $\alpha$. Note that

$$\alpha_{j,i} = \alpha_{1,i} + (j-1)p$$

by periodicity of $f$. Furthermore

$$\alpha_{1,i} = \{0, 1, \alpha_{1,3}, \dots, \alpha_{1,p+q-1}\}$$

It is worth noting that since $a > 1$, it follows that $\alpha_{j,1} \in \mathbb{Q}$ implies $\alpha_{j,p+q-1} \in \mathbb{Q}$. This ensures that there is a minimum gap between the sequences $\alpha_j$ are all separated. Observe in particular, that $\mathbb{N} \cap (a\mathbb{N}) = \{\alpha_{j,1}\}$ and that the sequence $\alpha$ is $p$ periodic. Define the minimum gap of a sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ by

$$\Delta x = \inf \{ \ |x_{n+1} - x_n| \ : \ n \in \mathbb{N} \ \}$$

In this sense, the periodicity of the sequence $\alpha$ implies that

$$\Delta\alpha = \Delta\{\alpha_{1,i}\}_{i\in\{1,\dots,p+q-1\}}$$

It is key to note that the sequence $\{\alpha_{1,i}\}$ has a positive minimum gap and from the above, $\alpha$ inherits a positive gap. That is $\Delta\alpha > 0$. Fixing $i \in \{1, \dots, p+q-1\}$ and using the monotonicity and periodicity of $f$ we find that

$$|\alpha_{j,i} - \mu_{j,i}| > |\alpha_{j+1,i} - \mu_{j+1,i}|$$

This argument is true for all $i \in \{1, \dots, p+q-1\}$. Because there are finitely many values for $i$, we can state the following uniform estimate. Given $\epsilon > 0$ we can find $J \in \mathbb{N}$ such that $j \geq J$ implies

$$|\mu_{j,i} - \alpha_{j,i}| < \epsilon$$

In particular, choose $\epsilon = \Delta\alpha/4$ and so

$$-|\mu_{j,i} - \alpha_{j,i}| > -\frac{\Delta\alpha}{4}$$

for all $i \in \{1, \dots, p+q-1\}$. For convenience define the indexing set

$$I = \{ \ (j,i) \ : \ j \in \mathbb{N}, \ \ i \in \{1, \dots, p+q-1\} \ \}$$

Choose for each $n \in I$ and so

$$\begin{aligned}
|\mu_{n+1} - \mu_n| &= \left|\left(\mu_{n+1} - \alpha_{n+1} - \mu_n + \alpha_n\right) + \left(\alpha_{n+1} - \alpha_n\right)\right| \\
&\geq |\alpha_{n+1} - \alpha_n| - |(\mu_{n+1} - \alpha_{n+1}) - (\mu_n - \alpha_n)| \\
&\geq |\alpha_{n+1} - \alpha_n| - |\mu_{n+1} - \alpha_{n+1}| - |\mu_n - \alpha_n| \\
&\geq \Delta\alpha - \frac{\Delta\alpha}{4} - \frac{\Delta\alpha}{4} \\
&= \frac{\Delta\alpha}{2}
\end{aligned}$$

This shows that $\Delta\alpha/2$ is a lower bound for the set $\{ \ |\mu_{n+1} - \mu_n| \ : \ n \in \mathbb{N} \ \}$ since only finitely many remain. Taking the infimum over $n$ we have that

$$\Delta\mu \geq \frac{\Delta\alpha}{2} = \frac{\Delta\{\alpha_{1,i}\}}{2}$$

The above shows that $\Delta\mu > 0$. □

## 4.3 Proof of controllability results for rationally related parameters

Following the techniques described in Chapters 2 and 5, we can reduce the problem of controllability to that of a problem of moments. Naturally, we have a different moment problem depending on the coefficient $\alpha$ being rational or irrational. We discuss the main points for the case that $L\alpha \in \mathbb{Q}$. We can derive an analogous result to that of Lemma 5.2.1.1 and 5.2.2.1 in Chapter 5.

**Lemma 4.3.0.4.** *The control system is null controllable in time $T$ if and only if, for any initial data $y^0 \in \mathcal{H}$ there is some control $f \in L^2(0,T)$ such that*

$$\begin{aligned}\gamma \int_{-1}^{0} u^0(x)\tilde{u}(0,x)dx + \beta\alpha^2 \int_0^L v^0(x)\tilde{v}(0,x)dx + z^0\tilde{z}(0) \\ = \beta \int_0^T f(t)\tilde{v}_x(t,L)dt\end{aligned}$$

*holds for all $\tilde{y}^T \in \mathcal{H}$.*

Applying Lemma to solutions of eigenfunctions we obtain the Moment problem:

$$\begin{aligned}\frac{a^0_{0,n}}{np\pi} e^{-(np\pi)^2 T} &= \int_0^T f(T-\tau)e^{-(np\pi)^2\tau}\, d\tau, \qquad n \in \mathbb{N} \\ \frac{a^0_{1,n}}{\mu_n(\mu_n \sin\mu_n - \gamma\cos\mu_n)} e^{-\mu_n^2 T} &= \int_0^T f(T-\tau)e^{-\mu_n^2\tau}\, d\tau, \qquad n \in \mathbb{N} \\ \frac{a^0_{2,n}}{\beta\alpha\eta_n\cos(L\alpha\eta_n)} e^{-\eta_n^2 T} &= \int_0^T f(T-\tau)e^{-\eta_n^2\tau}\, d\tau, \qquad n \in \mathbb{N}\setminus q\mathbb{N}\end{aligned}$$

Here $a^0_{0,n},\ a^0_{1,n},\ a^0_{3,n}$ contain the initial data $y^0$. In general, the problem is of the form

$$c_n e^{\lambda_n T} = \int_0^T w(\tau)e^{\lambda_n\tau}\, d\tau$$

where

$$\begin{cases} c_n \sim 1, & p \nmid n \\ c_n \sim n^{-1}, & p|n \text{ and if } z = 0 \end{cases}$$

by using the result of Proposition 4.2.0.7. The above there is $K, \delta > 0$ such that

$$|c_n e^{\lambda_n T}| \leq K e^{-\delta n^2}$$

The separation of the eigenvalues gives the existence of a biorthogonal sequence

$$\int_0^T \theta_j(\tau) e^{\lambda_n \tau}\, d\tau = \delta_{jn}$$

By the method of Russell and Fattorini described in Chapter 2 we have there are $M_1, M_2 > 0$ such that

$$\|\theta_j\| \leq M_1 e^{M_2 j}$$

This implies the convergence of the solution

$$w(\tau) = \sum_j c_j e^{\lambda_j T} \theta_j(t)$$

to the moment problem. Thus we obtain the following result.

**Theorem 4.3.0.3.** *The general thermal point mass system* (4.1) *is null-controllable in time* $T > 0$ *by means of Dirichlet control* (4.3) *in the rationally related case.*

# CHAPTER 5. NULL BOUNDARY CONTROLLABILITY OF A ONE DIMENSIONAL HEAT EQUATION WITH AN INTERNAL POINT MASS

In this chapter we prove the boundary null controllability of the temperature of a linear hybrid system consisting of two wires or rods connected by a point mass. More precisely, we consider the following system:

$$\begin{cases} \dot{u} - u'' = 0, & t > 0,\ x \in \omega_1 = (-1, 0) \\ \dot{v} - v'' = 0, & t > 0,\ x \in \omega_2 = (0, 1) \\ \dot{z} = v'(t, 0) - u'(t, 0), & t > 0 \\ u(t, 0) = v(t, 0) = z(t), & t > 0 \\ u(t, -1) = 0, & \end{cases} \tag{5.1}$$

with either Dirichlet control

$$v(t, 1) = f(t), \qquad t > 0 \tag{5.2}$$

or Neumann control

$$v'(t, 1) = f(t), \qquad t > 0. \tag{5.3}$$

In the above and throughout this article, $'$ denotes spatial derivatives and $\dot{}$ denotes temporal derivatives. In addition, $u = u(t, x)$ and $v = v(t, x)$ denote the temperature on $\omega_1$ and $\omega_2$, and $z = z(t)$ denotes the temperature of the point mass. The initial conditions at time $t = 0$ are given by

$$\begin{cases} u^0(x) = u(0,x), & x \in \omega_1 \\ v^0(x) = v(0,x), & x \in \omega_2 \\ z^0 = z(0), \end{cases}$$

where the triple $\{u^0, v^0, z^0\}$ will be given in an appropriately defined function space.

System (5.1) with the homogenous boundary condition

$$v(t,1) = 0, \qquad t > 0 \tag{5.4}$$

can be viewed as the limit of the following "epsilon" system with unit density on $(-1,1)\backslash(-\epsilon,\epsilon)$ and with density $1/2\epsilon$ on $(-\epsilon,\epsilon)$:

$$\begin{cases} \dot{u_\epsilon} - u_\epsilon'' = 0, & t > 0,\ x \in (-1,-\epsilon) \\ \dot{v_\epsilon} - v_\epsilon'' = 0, & t > 0,\ x \in (\epsilon,1) \\ \frac{1}{2\epsilon}\dot{z}_\epsilon - z_\epsilon'' = 0, & t > 0,\ x \in (-\epsilon,\epsilon) \end{cases} \tag{5.5}$$

where $u_\epsilon$, $v_\epsilon$ and $z_\epsilon$ satisfy the conditions

$$\begin{aligned} &u_\epsilon(t,-\epsilon) = z_\epsilon(t,-\epsilon),\ z_\epsilon(t,\epsilon) = v_\epsilon(t,\epsilon), \\ &u_\epsilon'(t,-\epsilon) = z_\epsilon'(t,-\epsilon),\ z_\epsilon'(t,\epsilon) = v_\epsilon'(t,\epsilon), \\ &u_\epsilon(t,-1) = v_\epsilon(t,1) = 0, \end{aligned} \tag{5.6}$$

for $t > 0$. In fact, in (22) the authors have shown that under appropriate assumptions of the initial data, solutions of (5.5) with (5.6) converge weakly to solutions of (5.1) and (5.4).

The hybrid system (5.1) is a variant of previously studied hybrid models for systems of strings and beams with interior point masses. Hansen and Zuazua used the method of characteristics in (25) to prove the boundary null controllability of an analogous string system with an interior point mass. In (30) Littman and Taylor use transform methods to prove boundary feedback stabilization of the string mass system. In (8) and (9), Castro and Zuazua used method of non-harmonic Fourier series to prove boundary controllability of systems of either Rayleigh or Euler-Bernoulli beams with interior point masses. We refer to (29), (35), (10), (47), (20) and (19) for related results on control and stabilization of systems of beams with end masses.

Our main results are the following.

**Theorem 5.0.0.4.** *System* (5.1) *with either Dirichlet control* (5.2) *or Neumann control* (5.3) *is null controllable in any time* $T > 0$. *More precisely, given* $T > 0$ *there is a control* $f \in L^2(0,T)$ *such that given initial data* $\{u^0, v^0, z^0\} \in L^2(\omega_1) \times L^2(\omega_2) \times \mathbb{R}$ *we have that* $\{u(T,x), v(T,x), z(T)\} = \{0,0,0\}$.

The solutions in Theorem 5.0.0.4, are defined by transposition in the spaces $C(0,T;X_{-1/2})$ for the case of Dirichlet control and $C(0,T;\mathcal{H})$ for the case of Neumann control; see Section 5.2.

Our general approach is to reduce the control problem to a moment problem. We consider the case of Dirichlet control and Neumann control separately in Section 5.2.

## 5.1 Preliminaries

We begin with a discussion of well-posedness of the system (5.1) with either homogeneous Dirichlet boundary condition (5.4) or Neumann boundary condition

$$v'(t,1) = 0, \qquad t > 0. \tag{5.7}$$

Given $u$, $v$ and $z$ defined on $\omega_1$, $\omega_2$ and $\mathbb{R}$ respectively, define $y = (u,v,z,)^t$ where $^t$ denotes transposition. Let

$$\mathcal{H} = L^2(\omega_1) \times L^2(\omega_2) \times \mathbb{R}$$

equipped with the norm

$$\|y\|_{\mathcal{H}}^2 = \|(u,v,z)\|_{\mathcal{H}}^2 = \|u\|_{\omega_1}^2 + \|v\|_{\omega_2}^2 + |z|^2$$

where $\|\cdot\|_{\omega_i}$ is the usual norm in $L^2(\omega_i)$ for $i = 1,2$. In the Dirichlet case (5.4), let

$$\begin{aligned} \vartheta_{\omega_1} &= \{u \in H^1(\omega_1) \mid u(-1) = 0\} \\ \vartheta_{\omega_2} &= \{v \in H^1(\omega_2) \mid v(1) = 0\} \\ \vartheta &= \{(u,v) \in \vartheta_1 \times \vartheta_2 \mid u(0) = v(0)\} \end{aligned} \tag{5.8}$$

equipped with the norms

$$\|u\|^2_{\vartheta_{\omega_i}} = \|u'\|^2_{L^2(\omega_i)}, \qquad i = 1, 2$$
$$\|(u, v)\|^2_{\vartheta} = \|u\|^2_{\vartheta_{\omega_1}} + \|v\|^2_{\vartheta_{\omega_2}}.$$

One can see that $\vartheta$ is algebraically and topologically equivalent to $H^1_0(\Omega)$ although it will be more convenient to think of $\vartheta$ as a subspace of $\vartheta_1 \times \vartheta_2$. The space

$$\mathcal{W} = \{(u, v, z) \in \vartheta \times \mathbb{R} \mid u(0) = v(0) = z\}$$

is a closed subspace of $\vartheta \times \mathbb{R}$ with norm $\|(u, v, z)\|^2_{\mathcal{W}} = \|(u, v)\|^2_{\vartheta}$. In the Neumann case (5.7), replace the definition of $\vartheta_{\omega_2}$ in (5.8) by

$$\vartheta_{\omega_2} = H^1(\omega_2), \tag{5.9}$$

and otherwise the space $\mathcal{W}$ is defined the same way. In either case, it is easy to show (see (22)) that the space $\mathcal{W}$ is densely and continuously embedded in the space $\mathcal{H}$. Define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A} = \begin{pmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ -\delta_0 d & \delta_0 d & 0 \end{pmatrix} \tag{5.10}$$

where $d$ denotes the (distributional) derivative operator, $\delta_0$ denotes the Dirac delta function with mass at $x = 0$, and the domain $D(\mathcal{A})$ of $\mathcal{A}$ is given in the Dirichlet case (5.4) by

$$D(\mathcal{A}) = \{y \in \mathcal{W} \ : \ u \in H^2(\omega_1), \ v \in H^2(\omega_2)\}. \tag{5.11}$$

and in the Neumann case (5.7) by

$$D(\mathcal{A}) = \{y \in \mathcal{W} \ : \ u \in H^2(\omega_1), \ v \in H^2(\omega_2), v'(1) = 0\}. \tag{5.12}$$

When $D(\mathcal{A})$ is endowed with the graph-norm topology

$$\|y\|^2_{D(\mathcal{A})} = \|y\|^2_{\mathcal{H}} + \|\mathcal{A}y\|^2_{\mathcal{H}}$$

it becomes a Hilbert space with continuous embedding in $\mathcal{H}$. We can therefore write the homogeneous point-mass systems (5.1), (5.4) and (5.1), (5.7) as

$$\dot{y}(t) = \mathcal{A}y(t), \quad y(0) = y^0, \quad t > 0 \tag{5.13}$$

where $y^0 = (u^0, v^0, z^0)$.

**Proposition 5.1.0.8.** *The unbounded operator $\mathcal{A}$ given by (5.10) in domain $D(\mathcal{A})$ as in (5.11) is a bijective, self-adjoint and dissipative operator with a compact inverse. Furthermore, $\mathcal{A}$ is the infinitesimal generator of a strongly continuous, compact and analytic semigroup $(\mathbb{T}_t)_{t\geq 0}$.*

Refer to (22) for a detailed proof of the above proposition for the Dirichlet case (5.1), (5.4) . As a consequence of Proposition 5.1.0.8, given initial data $y^0 \in \mathcal{H}$ there exists a unique solution

$$y \in C([0, \infty); \mathcal{H})$$

to the Cauchy problem (5.13). If in addition, $y^0 \in D(\mathcal{A})$ then $y \in C([0, \infty), D(\mathcal{A}))$.

In the next subsection it is shown that $\mathcal{A}$ has only negative eigenvalues, hence $-\mathcal{A}$ is positive, self-adjoint it provides an isomorphism: $D(\mathcal{A}) \to \mathcal{H}$. Moreover, fractional powers of $-\mathcal{A}$ are well-defined. Let $X_1 = D(\mathcal{A})$ and for $\alpha \in [0, 1]$, define $X_\alpha = D((-\mathcal{A})^\alpha)$ and $X_{-\alpha} = X'_\alpha$, the dual space relative to the pivot space $\mathcal{H} = X_0$ of $X_\alpha$. Correspondingly, the semigroup $\mathbb{T}$ remains an analytic semigroup on the invariant subspaces $X_\alpha$, $0 \leq \alpha \leq 1$, and extends continuously to an analytic semigroup on spaces $X_\alpha$, $-1 \leq \alpha \leq 0$; see e.g., (44) for full explanation. The norm on $X_\alpha$ is given by $\|y\|^2_\alpha = \langle (-\mathcal{A})^\alpha y, (-\mathcal{A})^\alpha y\rangle_{\mathcal{H}}$. In particular, $X_{1/2}$ is the completion of $X_1$ with respect to the norm

$$\|y\|^2_{1/2} = \langle -\mathcal{A}y, y\rangle_0.$$

Integration by parts gives

$$\|y\|^2_{1/2} = \langle y, y\rangle_{\mathcal{W}}.$$

Thus, $X_{1/2}$ is topologically equivalent to $H^1_0(\Omega)$ in the Dirichlet case (5.4) and $\{f \in H^1(\Omega) : f(-1) = 0\}$ in the Neumann case (5.7).

### 5.1.1 Spectral analysis for Dirichlet case (5.1), (5.4)

By Proposition 5.1.0.8, the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is contained in the negative real axis and consists of eigenvalues $\{\lambda_n\}$ tending to negative infinity with corresponding eigenvectors $\{\varphi_n\}_{n\in\mathbb{N}}$ forming an orthogonal system for $\mathcal{H}$.

**Proposition 5.1.1.1.** *The eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $\mathcal{A}$ in the Dirichlet case (5.4) are distinct and given by*

$$\lambda_{2k} = -(k\pi)^2, \quad \lambda_{2k-1} = -\mu_k^2 \ \textit{ for } k \in \mathbb{N}$$

*where $\mu_k$ is the $k$-th positive root of the characteristic equation*

$$\mu = 2\cot\mu. \tag{5.14}$$

*The corresponding eigenvectors are given by*

$$\varphi_{2k}(x) = \begin{pmatrix} \sin(k\pi x) \\ \sin(k\pi x) \\ 0 \end{pmatrix}, \quad \varphi_{2k-1}(x) = \begin{pmatrix} \sin((1+x)\mu_k) \\ \sin((1-x)\mu_k) \\ \sin(\mu_k) \end{pmatrix}$$

*and $\varphi_n \in D(\mathcal{A})$ for all $n \in \mathbb{N}$.*

*Proof.* Look for nontrivial functions $\varphi_n = (U_n, V_n, Z_n)^t \in D(\mathcal{A})$ such that $\mathcal{A}\varphi_n = \lambda_n\varphi_n$. We use an even index in the case that $Z_n = 0$ and an odd index when $Z_n \neq 0$. The eigensystem corresponding to $Z_{2k} = 0$ reduces to the problem of finding $(U_{2k}, V_{2k})$ such that

$$\begin{cases} U''_{2k}(x) = \lambda_{2k} U_{2k}(x), & x \in \omega_1 \\ V''_{2k}(x) = \lambda_{2k} V_{2k}(x), & x \in \omega_2 \\ U'_{2k}(0) = V'_{2k}(0) \\ U_{2k}(0) = V_{2k}(0) = 0 \\ U_{2k}(-1) = V_{2k}(1) = 0. \end{cases}$$

It is easy to check that $\varphi_{2k}$ satisfies the above with $\lambda_{2k} = -(k\pi)^2$.

Now consider the case that $Z_{2k-1} \neq 0$. The eigenvalue problem reduces to the problem of finding functions $(U_{2k-1}, V_{2k-1})$, and real value $Z_{2k-1}$ such that

$$\begin{cases} U''_{2k-1}(x) = -\mu_k^2 U_{2k-1}(x), & x \in \omega_1 \\ V''_{2k-1}(x) = -\mu_k^2 V_{2k-1}(x), & x \in \omega_2 \\ V'_{2k-1}(0) - U'_{2k-1}(0) = -\mu_k^2 Z_{2k-1} \\ U_{2k-1}(0) = V_{2k-1}(0) = Z_{2k-1} \\ U_{2k-1}(-1) = V_{2k-1}(1) = 0. \end{cases} \tag{5.15}$$

From the boundary condition $U_{2k-1}(-1) = V_{2k-1}(1) = 0$, we have that the solution is of the form

$$U_{2k-1}(x) = \sin\big((x+1)\mu_k\big)$$
$$V_{2k-1}(x) = C\sin\big((x-1)\mu_k\big)$$

for some constant $C$ to be determined. The continuity condition $U_{2k-1}(0) = V_{2k-1}(0) = Z_{2k-1}$ gives

$$Z_{2k-1} = \sin(\mu_k) = -C\sin(\mu_k).$$

Since $Z_{2k-1}$ is nonzero we have that $\mu_k$ is not a multiple of $\pi$. Furthermore, we find that $C = -1$. Then from the third equation in (5.15) we see that

$$2\cot(\mu_k) = \mu_k. \tag{5.16}$$

Hence the solution to the eigensystem (5.15) is

$$\begin{pmatrix} U_{2k-1}(x) \\ V_{2k-1}(x) \\ Z_{2k-1} \end{pmatrix} = \begin{pmatrix} \sin((1+x)\mu_k) \\ \sin((1-x)\mu_k) \\ \sin(\mu_k) \end{pmatrix}.$$

Finally, note that since the function $F(\mu) = 2\cot\mu - \mu$ decreases monotonically from $+\infty$ to $-\infty$ over the interval $((k-1)\pi, k\pi)$ for all $k \in \mathbb{N}$, there is exactly one root of $F$ in each interval $((k-1)\pi, k\pi)$ for all $k \in \mathbb{N}$. Hence the eigenvalues

$$\{-(k\pi)^2\}_{k\in\mathbb{N}} \cup \{-\mu_k^2\}_{k\in\mathbb{N}}$$

are distinct. □

**Proposition 5.1.1.2.** *The sequence $\{\mu_k\}$ in the Dirichlet case (5.4) satisfies the asymptotic estimate*

$$\mu_k = (k-1)\pi + \frac{2}{k\pi} + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{5.17}$$

*Consequently, consecutive eigenvalues of $\mathcal{A}$ in (5.13) satisfy the gap condition:*

$$|\lambda_{n+1} - \lambda_n| \geq 4 + \mathcal{O}\left(\frac{1}{n}\right). \tag{5.18}$$

*Moreover, the eigenfunctions are asymptotically normalized in the sense that*

$$\lim_{n\to\infty} \|\varphi_n\| = 1.$$

*Proof.* From the end of the previous proof, $\mu_k = (k-1)\pi + \epsilon_k$, where $0 < \epsilon_k < \pi$. The characteristic equation (5.14) can be rewritten as

$$\frac{(k-1)\pi + \epsilon_k}{2} = \cot \epsilon_k$$

and thus by monotonicity,

$$(k-1)\pi/2 < \cot \epsilon_k < k\pi/2.$$

Taking inverse cotangent of each term gives

$$\arctan \frac{2}{k\pi} < \epsilon_k < \arctan \frac{2}{(k-1)\pi}.$$

Hence by Taylor's formula we obtain (5.17).

The estimate (5.18) can be obtained from

$$\begin{aligned} |\lambda_{2k+1} - \lambda_{2k}| &= (\mu_{k+1} + k\pi)(\mu_{k+1} - k\pi) \\ &= \left(2k\pi + \mathcal{O}\left(\frac{1}{k}\right)\right)\left(\frac{2}{k\pi} + \mathcal{O}\left(\frac{1}{k^2}\right)\right) \\ &= 4 + \mathcal{O}\left(\frac{1}{k}\right). \end{aligned}$$

Finally, it is easy to check that $\|\varphi_{2k}\| = 1$ for all $k \in \mathbb{N}$ and using estimate (5.17) that $\|\varphi_{2k-1}\|^2 = 1 + \mathcal{O}(k^{-2})$. □

### 5.1.2 Spectral analysis for Neumann case (5.1), (5.7)

As in Subsection 5.1.1, the eigenvalues of $\mathcal{A}$ (denoted $\lambda_n$) form a discrete sequence of negative numbers tending to negative infinity with corresponding eigenvectors $\varphi_n$ which form an orthogonal system for $\mathcal{H}$.

**Proposition 5.1.2.1.** *The eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ of $\mathcal{A}$ in the Neumann case (5.7) are given by $\lambda_n = -\mu_n^2$ where $\{\mu_n\}_{n\in\mathbb{N}}$ are the roots of the characteristic equation*

$$\mu = 2\cot 2\mu. \tag{5.19}$$

*The corresponding eigenvectors are given by*

$$\begin{aligned} \varphi_{2k-1}(x) &= \sqrt{2}\begin{pmatrix} \sin(\mu_{2k-1}(x+1)) \\ \tan\mu_{2k-1}\cos(\mu_{2k-1}(x-1)) \\ \sin\mu_{2k-1} \end{pmatrix} \\ \varphi_{2k}(x) &= \sqrt{2}\begin{pmatrix} \cot\mu_{2k}\sin(\mu_{2k}(x+1)) \\ \cos(\mu_{2k}(x-1)) \\ \cos\mu_{2k} \end{pmatrix} \end{aligned} \tag{5.20}$$

*and $\varphi_n \in D(\mathcal{A})$ for all $n \in \mathbb{N}$.*

*Proof.* The eigenvalue problem $\mathcal{A}\varphi_n = \lambda_n\varphi_n$ with $\varphi_n = (U_n, V_n, Z_n)^t \in D(\mathcal{A})$ is the following system:

$$\begin{cases} U_n''(x) = \lambda_n U_n(x), & x \in \omega_1 \\ V_n''(x) = \lambda_n V_n(x), & x \in \omega_2 \\ V_n'(0) - U_n'(0) = \lambda_n Z_n \\ U_n(0) = V_n(0) = Z_n \\ U_n(-1) = V_n'(1) = 0. \end{cases} \tag{5.21}$$

First note that the possibility of $Z_n = 0$ leads to the trivial solution. Hence $Z_n \neq 0$ for all $n \in \mathbb{N}$. Then from the first two equations and the boundary conditions we find that

$$U_n(x) = \sin(\mu_n(x+1))$$

$$V_n(x) = C\cos(\mu_n(x-1))$$

for some nonzero constant $C$ to be determined. The continuity condition $U_n(0) = V_n(0)$ gives

$$\sin \mu_n = C \cos \mu_n$$

and since $Z_n$ is nonzero for all $n \in \mathbb{N}$ we have that $C = \tan \mu_n$. Then from the third equation in (5.21) we see that

$$\mu_n = -\tan \mu_n + \cot \mu_n$$

which is equivalent to the *characteristic equation* (5.19). Hence the corresponding sequence of eigenvectors is

$$\varphi_n(x) = \begin{pmatrix} \sin(\mu_n(1+x)) \\ \tan \mu_n \cos(\mu_n(x-1)) \\ \sin \mu_n \end{pmatrix}$$

which agrees with (5.20) after multiplying by normalizing factors $\sqrt{2}$ for $n = 2k-1$ and $\sqrt{2} \cot \mu_n$ for $n = 2k$. □

Following the ideas of Proposition 5.1.1.2, one can prove the following result.

**Proposition 5.1.2.2.** *The sequence $\{\mu_k\}$ in the Neumann case (5.7) satisfies the asymptotic estimate*

$$\mu_k = \frac{(k-1)\pi}{2} + \frac{1}{k\pi} + \mathcal{O}\left(k^{-2}\right). \tag{5.22}$$

*Consequently, consecutive eigenvalues of $\mathcal{A}$ in (5.13) satisfy the gap condition:*

$$|\lambda_{n+1} - \lambda_n| \geq \frac{n\pi^2}{2} + \mathcal{O}(1). \tag{5.23}$$

*Moreover, the eigenfunctions are asymptotically normalized in the sense that*

$$\lim_{n\to\infty} \|\varphi_n\| = 1.$$

## 5.2 Proof of Controllability results

We begin with the case of Neumann control: (5.1), (5.3).

### 5.2.1 Neumann control

The dual observation problem to (5.1), (5.3) is

$$\begin{cases} -\dot{\tilde{u}} - \tilde{u}'' = 0, & t > 0,\ x \in \omega_1 \\ -\dot{\tilde{v}} - \tilde{v}'' = 0, & t > 0,\ x \in \omega_2 \\ -\dot{\tilde{z}} = \tilde{v}'(t,0) - \tilde{u}'(t,0), & t > 0 \\ \tilde{u}(t,0) = \tilde{v}(t,0) = \tilde{z}(t), & t > 0 \\ \tilde{u}(t,-1) = \tilde{v}'(t,1) = 0, & t > 0 \end{cases}$$

with terminal data at $t = T$ given by

$$\begin{cases} \tilde{u}^T(x) = \tilde{u}(T,x), & x \in \omega_1 \\ \tilde{v}^T(x) = \tilde{v}(T,x), & x \in \omega_2 \\ \tilde{z}^T = z(T). \end{cases}$$

By letting $\tilde{y} = (\tilde{u}, \tilde{v}, \tilde{z})^t$, the above problem can be written as

$$-\dot{\tilde{y}} = \mathcal{A}\tilde{y}, \quad \tilde{y}(T) = \tilde{y}^T \in \mathcal{H}, \quad t > 0. \tag{5.24}$$

Then $\tilde{y} \in C([0,T], \mathcal{H})$ and is given by

$$\tilde{y}(t) = \mathbb{T}(T-t)\tilde{y}^T; \qquad 0 \le t \le T. \tag{5.25}$$

Let $y$ be a smooth solution of the control problem with smooth $f \in L^2(0,T)$. Formal integration by parts then shows

$$\begin{aligned} 0 &= \int_0^T \int_{-1}^0 (\dot{u} - u'')\tilde{u}\ dxdt + \int_0^T \int_0^1 (\dot{v} - v'')\tilde{v}\ dxdt \\ &= \langle y(T), \tilde{y}^T \rangle_{\mathcal{H}} - \langle y^0, \tilde{y}(0) \rangle_{\mathcal{H}} - \int_0^T f(t)\tilde{v}(t,1)\ dt. \end{aligned}$$

Equivalently,

$$\langle y(T), \tilde{y}^T \rangle_{\mathcal{H}} = \langle y^0, \mathbb{T}_T \tilde{y}^T \rangle_{\mathcal{H}} + \int_0^T f(t)\tilde{v}(t,1)\ dt. \tag{5.26}$$

Since the functional $\ell(\tilde{y}) := \tilde{v}(1)$ is continuous on $X_{1/2} = \mathcal{W}$ it follows from Propositions 5.1.3 and 10.2.1 in (44) that for solutions of (5.24) there exists $C > 0$ for which

$$\|\tilde{v}(t,1)\|_{L^2(0,T)} \le C \|\tilde{y}^T\|_{\mathcal{H}} \qquad \forall\ \tilde{y}^T \in \mathcal{H}. \tag{5.27}$$

Hence, equation (5.26) uniquely defines $y(T)$ as an element of $\mathcal{H}$. Applying this definition for $s \in [0,T]$ we see

$$y \in C([0,T],\mathcal{H}) \tag{5.28}$$

and there exists $C > 0$ for which

$$\|y\|_{L^\infty(0,T;\mathcal{H})} \leq C(\|y^0\|_{\mathcal{H}} + \|f\|_{L^2(0,T)}). \tag{5.29}$$

As before we have the following lemma.

**Lemma 5.2.1.1.** *The control problem (5.1), (5.3) is null controllable in time $T > 0$ if and only if, for any $y^0 \in \mathcal{H}$ there is $f \in L^2(0,T)$ such that*

$$\langle y^0, \mathbb{T}_T \tilde{y}^T \rangle_{\mathcal{H}} = -\int_0^T f(t)\tilde{v}(t,1)dt \tag{5.30}$$

*holds for all $\tilde{y}^T \in \mathcal{H}$, where $\tilde{y}$ is the solution to the observation problem (5.24).*

*Proof.* First assume that (5.30) holds for all $\tilde{y}^T \in \mathcal{H}$. Then by (5.26), $y(T) = 0$. Conversely, if $f$ is a control for which $y(T) = 0$, then (5.30) follows from equation (5.26). □

We are now ready to reduce the control problem (5.1), (5.3) to a moment problem. Any initial data $y^0 = (u^0, v^0, z^0)^t$ in $\mathcal{H}$ for the control problem can be expressed in terms of the eigenfunctions as

$$y^0 = \sum_{n\in\mathbb{N}} y_n^0 \varphi_n \tag{5.31}$$

where the Fourier coefficients $\{y_n^0\}_{n\in\mathbb{N}}$ belong to $\ell^2$. Let $\tilde{y}_n = (\tilde{u}_n, \tilde{v}_n, \tilde{z}_n)^t$ be the eigensolution of (5.24) given by

$$\tilde{y}_n(t,x) = e^{\lambda_n(T-t)}\varphi_n(x). \tag{5.32}$$

In particular, note that

$$\tilde{v}_n(t,1) = \begin{cases} \sqrt{2}e^{\lambda_n(T-t)}\tan\mu_n, & n \text{ odd} \\ \sqrt{2}e^{\lambda_n(T-t)}, & n \text{ even.} \end{cases}$$

Applying these solutions to equation (5.30) we obtain the following moment problem:

$$\frac{a_n}{b_n}e^{\lambda_n T} = \int_0^T f(T-\tau)e^{\lambda_n \tau}d\tau, \quad n \in \mathbb{N} \tag{5.33}$$

where

$$b_n = \begin{cases} -\tan\mu_n & n \text{ is odd} \\ -1, & n \text{ is even} \end{cases} \tag{5.34}$$

and by Proposition 5.1.2.2, $a_n = \|\varphi_n\|^2 y_n^0 \in \ell^2$. In particular note that for $n = 2k-1$

$$\tan\mu_{2k-1} = \tan\left(\frac{1}{k\pi} + \mathcal{O}(k^{-2})\right) = \frac{1}{k\pi} + \mathcal{O}(k^{-2})$$

and furthermore since $\tan\mu_{2k-1} \neq 0$ for all $k \in \mathbb{N}$, there exists $\epsilon > 0$ such that

$$|b_n| \geq \frac{\epsilon}{n}, \qquad \forall n \in \mathbb{N}.$$

From our estimates of $\mu_n$, $\lambda_n$, $b_n$ and $a_n$, it is easy to show that there are constants $K, \delta > 0$ such that

$$\left|\frac{a_n}{b_n}e^{\lambda_n T}\right| \leq Ke^{-\delta n^2}, \qquad n \in \mathbb{N}. \tag{5.35}$$

From equations (5.22) and (5.23) we see that the series $\sum 1/\lambda_n$ converges, and that there exists a constant $\rho > 0$ such that $|\lambda_{k+1} - \lambda_k| > \rho$ for all $k \in \mathbb{N}$. This implies the existence of a biorthogonal sequence $\{\theta_j(\tau)\}_{j\in\mathbb{N}}$ (see (40), (16)) such that

$$\int_0^T \theta_j(\tau)e^{\lambda_n\tau}\, d\tau = \delta_{j,n} = \begin{cases} 1, & j = n \\ 0, & j \neq n. \end{cases} \tag{5.36}$$

By the method of Russell and Fattorini in (40) we have that there are $M_1, M_2 > 0$ such that

$$\|\theta_j\| \leq M_1 e^{M_2 j}.$$

It is easy to see that the above implies the convergence of

$$f(T-\tau) = \sum_{j\in\mathbb{N}} \frac{a_j}{b_j}e^{\lambda_j T}\theta_j(\tau)$$

which provides a solution to the moment problem (5.47). The proof of Theorem 5.0.0.4 for the case of Neuman control (5.3), is a direct consequence of Lemma 5.2.1.1 and the existence of the biorthogonal sequence $\{\theta_j(\tau)\}_{j\in\mathbb{N}}$.

### 5.2.2 Dirichlet Control

The dual observation problem to (5.1), (5.2) is

$$\begin{cases} -\dot{\tilde{u}} - \tilde{u}'' = 0, & t > 0,\ x \in \omega_1 \\ -\dot{\tilde{v}} - \tilde{v}'' = 0, & t > 0,\ x \in \omega_2 \\ -\dot{\tilde{z}} = \tilde{v}'(t,0) - \tilde{u}'(t,0), & t > 0 \\ \tilde{u}(t,0) = \tilde{v}(t,0) = \tilde{z}(t), & t > 0 \\ \tilde{u}(t,-1) = \tilde{v}(t,1) = 0, & t > 0 \end{cases} \tag{5.37}$$

with terminal data at $t = T$ given by

$$\begin{cases} \tilde{u}^T(x) = \tilde{u}(T,x), & x \in \omega_1 \\ \tilde{v}^T(x) = \tilde{v}(T,x), & x \in \omega_2 \\ \tilde{z}^T = z(T), \end{cases} \tag{5.38}$$

and observation $Y(t) = \tilde{v}'(t,1)$. By letting $\tilde{y} = (\tilde{u}, \tilde{v}, \tilde{z})^t$, the above problem can be written as a Cauchy problem as

$$-\dot{\tilde{y}} = \mathcal{A}\tilde{y}, \quad \tilde{y}(T) = \tilde{y}^T, \quad t > 0. \tag{5.39}$$

If $\tilde{y}^T \in X_{1/2} = \mathcal{W}$ then $\tilde{y} \in C([0,T], X_{1/2})$ is given by

$$\tilde{y}(t) = \mathbb{T}(T-t)\tilde{y}^T; \qquad 0 \le t \le T. \tag{5.40}$$

Let $y$ be a smooth solution of the control problem with smooth $f \in L^2(0,T)$ and let $\tilde{y}$ be solution of the dual problem (5.39). Integration by parts as earlier results in the identity

$$\langle y(T), \tilde{y}^T \rangle = \langle y^0, \mathbb{T}_T \tilde{y}^T \rangle_{\mathcal{H}} - \int_0^T f(t)\tilde{v}'(t,1)\, dt \tag{5.41}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $X_{-1/2} \times X_{1/2}$.

In the case of the heat equation

$$\begin{cases} \dot{q} = q'' & 0 < x < 1,\ t > 0 \\ q(t,0) = q(t,1) = 0 & t > 0 \\ q(0,x) = q^0 \in H_0^1(0,1) & 0 < x < 1 \end{cases}$$

it is well known (e.g. (14)) that for each $T > 0$ there exists $C > 0$ for which

$$\|q'(\cdot, 1)\|_{L^2(0,T)} \le C\|q^0\|_{H_0^1(0,1)}.$$

One can verify that the same estimate holds for solutions of (5.39) in the sense that there exists $C > 0$ for which

$$\|\tilde{v}'(t, 1)\|_{L^2(0,T)} \le C\|\tilde{y}^T\|_{1/2} \qquad \forall\ \tilde{y}^T \in X_{1/2}. \tag{5.42}$$

Since the semigroup $\mathbb{T}$ is strongly continuous on $X_{1/2} = H_0^1(\Omega)$ it follows that the identity (5.41) defines the value $y(s)$ for all $s \in [0, T]$ as an element of $X_{-1/2}$ for which there exists $C > 0$ such that

$$\|y\|_{L^\infty(0,T;X_{-1/2})} \le C(\|y^0\|_{\mathcal{H}} + \|f\|_{L^2(0,T)})$$

and moreover

$$y \in C([0, T], X_{-1/2}). \tag{5.43}$$

The above estimate (5.43) is sometimes referred to as admissibility of the boundary control operator corresponding to Dirichlet control, and can also be derived in the framework of "well posed boundary control systems"; see of (44, Prop. 10.7.1).

Analogous to Lemma 5.2.1.1 the following lemma characterizes the problem of null controllability of (5.1), (5.2) in terms of the solution $\tilde{y}$ of the observation problem (5.39).

**Lemma 5.2.2.1.** *The control problem (5.1), (5.2) is null controllable in time $T > 0$ if and only if, for any $y^0 \in \mathcal{H}$ there is $f \in L^2(0, T)$ such that*

$$\langle y^0, \mathbb{T}_T\tilde{y}^T\rangle_{\mathcal{H}} = \int_0^T f(t)\tilde{v}'(t, 1)dt \tag{5.44}$$

*holds for all $\tilde{y}^T \in \mathcal{H}$, where $\tilde{y} = (\tilde{u}, \tilde{v}, \tilde{z})^t$ is a solution of (5.39).*

We are now ready to reduce the control problem (5.1), (5.2) to a moment problem. Any initial data $y^0 = (u^0, v^0, z^0)^t$ in $\mathcal{H}$ for the control problem can be expressed in terms of the eigenfunctions as

$$y^0 = \sum_{n\in\mathbb{N}} y_n^0\varphi_n \tag{5.45}$$

where the Fourier coefficients $\{y_n^0\}_{n\in\mathbb{N}}$ belong to $\ell^2$. Let $\tilde{y}_n = (\tilde{u}_n, \tilde{v}_n, \tilde{z}_n)^t$ be the eigensolution of (5.39) given by

$$\tilde{y}_n(t,x) = e^{\lambda_n(T-t)}\varphi_n(x). \tag{5.46}$$

In particular, note that

$$\tilde{v}_n'(t,1) = \begin{cases} e^{\lambda_{2k}(T-t)}k\pi(-1)^k, & n = 2k \\ -e^{\lambda_{2k-1}(T-t)}\mu_k, & n = 2k-1. \end{cases}$$

We plug these solutions into equation (5.44) to obtain the corresponding moment problem

$$a_n e^{\lambda_n T} = b_n \int_0^T f(T-\tau)e^{\lambda_n \tau}d\tau \tag{5.47}$$

for all $n \in \mathbb{N}$ where

$$b_n = \tilde{v}_n'(T,1) = \begin{cases} (-1)^k k\pi, & n = 2k \\ -\mu_k, & n = 2k-1 \end{cases} \tag{5.48}$$

and by Proposition 5.1.1.2, $a_n = \|\varphi_n\|^2 y_n^0 \in \ell^2$. Again, it is easy to show that there exists constants $K, \delta > 0$ such that (5.35) holds. From equations (5.17) and (5.18) we see that the series $\sum 1/\lambda_n$ converges, and that there exists a constant $\rho > 0$ such that $|\lambda_{k+1} - \lambda_k| > \rho$ for all $k \in \mathbb{N}$. This implies the existence of a biorthogonal sequence $\{\theta_j(\tau)\}_{j\in\mathbb{N}}$ such that there are constants $M_1, M_2 > 0$ such that

$$\|\theta_j\| \le M_1 e^{M_2 j}.$$

Hence, as earlier,

$$f(T-\tau) = \sum_{j\in\mathbb{N}} \frac{a_j}{b_j} e^{\lambda_j T}\theta_j(\tau)$$

converges and provides a solution to the moment problem (5.47). The proof of Theorem 5.0.0.4 for the case of Dirichlet control (5.2), is a direct consequence of Lemma 5.2.2.1.

**Remark 5.2.2.1.** *The numbers $b_k$ in (5.48) and (5.34), are called control input coefficients and can be viewed as Fourier coefficients of an element $b$ of $X_{-1}$ for which the control problem (5.1) with either (5.2) or (5.3) can be formulated as*

$$\dot{y} = \mathcal{A}y + bf, \quad y(0) = y^0.$$

*In the Neumann case, the input element is admissible on the state space* $X_0 = \mathcal{H}$*, or equivalently that* (5.28) *and* (5.29) *hold. In the Dirichlet case,* $b$ *is admissible on the state space* $X_{-1/2}$*. Both of these spaces are slightly suboptimal in the sense that the Carleson measure criterion due to Ho and Russell (26) and Weiss (45) can be used as in (24) to show admissibility holds in the spaces* $X_{1/4}$ *and* $X_{-1/4}$ *respectively for the Neumann and Dirichlet control problems.*

# CHAPTER 6. CONCLUSION

In this thesis, we have achieved two main objectives. Firstly, we gave a physical justification of the thermal point mass system as a limit of a system of heat equations. Secondly, we proved null controllability in many situations.

In Chapter 3 have shown how the idea of a point mass can be extended from the context of strings and beams (see (25) and (8)) to the problem of heat diffusion in the one dimensional case. The main well-posedness result for the limiting thermal point mass system (3.3) was given in Theorem 3.1.1.1 by semigroup techniques. Then, by applying two different ideas, we derived the correct set of equations. In the first approach, we applied the ideas of Castro in (6), and considered the problem of heat diffusion of a three layered composite medium with a middle layer of width $2\epsilon$ and density of $1/2\epsilon$. As stated in Theorem 3.2.0.1, solutions to the *epsilon* dependent problem converged to the solution of the thermal point mass system (3.3) whenever we assumed the initial data $y^0_\epsilon \in D(\mathcal{A}_\epsilon)$ was uniformly bounded in the sense of the spaces $\mathcal{H}_\epsilon$ and $\mathcal{W}_\epsilon$ as defined in (3.11) and (3.12) respectively.

In the second approach, we instead considered a system isomorphic to (3.3), but with a limiting finite nonzero thickness of the middle layer. The limiting system is given in (3.30) and it was obtained by passing to a limit in the $\epsilon-$dependent problem (3.28) of singular conductivity on the middle layer. Refer to Theorem 3.3.0.2 for a statement of this result.

We then discussed in Chapter 4 the controllability of the limiting system (3.3) mentioned above. Whereas previous work on point masses for beams and strings considered system parameters of specific heat, density and conductivity equal to one, we allowed for parameters to be arbitrary real numbers. We gave a precise description of the spectrum of the operator $\mathcal{A}$ associated with the point mass system (4.1) and showed that consecutive eigenvalues satisfy a minimum gap condition. One aspect of this analysis was of great importance. One must consider the cases where the parameters are rationally and irrationally related. In the end, our eigenvalue estimates in Propositions 4.2.0.5 and 4.2.0.7 were only sufficient to prove null controllability of the system in the case of rationally related coefficients. This is stated in Theorem 4.3.0.3 and the proof consists in proving the existence of the solution of an equivalent moment problem.

In Chapter 5 we then took the system's parameters equal to one on a symmetric region $(-1, 1)$, to consider the problem of null controllability when applying Dirichlet and Neumann controls. In both cases, we proved Propositions 5.1.1.2 and 5.1.2.2 to show that consecutive eigenvalues satisfy a minimum gap condition for both the case of Dirichlet and Neumann control. Our main result was given in Theorem 5.0.0.4. In the proof, we derived the corresponding moment problems and showed that the bounds obtained are sufficient to show the existence of a solution to the moment problems. Therefore, the point mass system is null controllable with Dirichlet and Neumann control.

As future work is concerned, there is still work to be done to complete the proof of the existence of a moment problem solution in the case where the system's parameters are irrationally related. Secondly, we have begun preliminary work in the case of several point masses. In the cases of simple parameters, one can show separability of the eigenvalues by simply applying Taylor series. However, in the case of general parameters, the proof used in Chapter 4 seems to fail. More work is required. Lastly, once the irrationally related case is complete, we can apply these ideas to both the thermoelastic system and a system of Schödinger equations with a point mass.

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