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Volterra integral equations: Admissibility results and the generic property of uniqueness of solutions

by

Gregory Mark Dotseth

A Dissertation Submitted to the
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I. INTRODUCTION

In this dissertation we shall study results on two problems that involve Volterra integral equations. In Chapter II we study a problem involving uniqueness of solutions to an equation of the form

$$(1.1) \qquad x(t) = f(t) + \int_0^t g(t,s,x(s))\,ds.$$

It is known that there are continuous functions f and g such that (1.1) does not have a unique solution (see page 44 Miller [9]). A paper by Lasota and Yorke [6] shows that

$$(1.2) \qquad x(t) = x_o + \int_{t_o}^t g(s,x(s))\,ds$$

has a unique solution for "most" of the continuous g where $g : R \times U \to U$ and U is a Banach space. By "most" we mean that the set of g's that do not have unique solutions is of first category. It will be shown that a similar result is true for (1.1) where the f's can be taken from a fixed compact set of functions.

The second problem considered involves the concept of admissibility i.e. for a linear operator K and Frechet spaces F_1 and F_2 when does it follow that $KF_1 \subset F_2$. Corduneanu [2] presents admissibility results involving C_g and L_g^∞ spaces where g is a continuous positive valued function. These are results of C. Corduneanu, G. Bantas, N. Pavel, and others and use primarily the operator K defined by

$$(1.3) \qquad Kx = \int_0^t k(t,s)x(s)ds.$$

H. Gollwitzer [4] generalized the notions of C_g and L_g^∞ spaces to the case where $g(t)$ is a matrix valued function and was able to obtain admissibility results using these new spaces. In this thesis we prove new admissibility results using the C_g space where g is a matrix valued function and introduce a new space L_g^p with which an admissibility result is obtained. Some of these results are then applied to integral equations to show existence and uniqueness of solutions.

II. BACKGROUND

The space R^n is the real n-dimensional Euclidean space. When x is in R^n then $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ where $x_1, x_2, \ldots, x_n$ are real numbers and $\|x\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$. R_+ is the set of real numbers greater than or equal to zero.

In Lemma 2.1 we use the following result of J. Dugundji [3].

Theorem 2.1. Suppose X is a metrix space and L is a locally convex space. Let A be a closed subset of X. Then for any continuous function $f : A \to L$ there exists a continuous function $F : X \to L$ such that $f(a) = F(a)$ for all a in A.

In the proof of Theorem 3.1 we need the following definitions and results.

Definition 2.1. For any $t \geq 0$ the cross section $K(t)$ of equation (1.1) is defined as

(2.1) $K(t) = \{x(t) : x(s)$ is any continuous solution of (1.1) for $0 \leq s \leq t\}$

For any $T \geq 0$ define

$$K^*(T) = \cup \{K(t) : 0 \leq t \leq T\}. \tag{2.2}$$

When f and g are continuous in equation (1.1) there exists a number $\alpha_M > 0$ such that for $0 \leq \alpha < \alpha_M$ the set $K^*(\alpha)$ is compact as a subset of R^n. Also the number α_M can be picked maximal in the sense that either $\alpha_M = \infty$ or there exists a solution $x(t)$ of (1.1) such that $\limsup \|x(t)\| = \infty$ as $t \to \alpha^-$. (See Miller [7] Chapter II.)

<u>Lemma 2.1</u>. Let $f : R_+ \times R_+ \times R^n \to R^n$ be a continuous function. If A is a compact subset of $R_+ \times R_+ \times R^n$ and $a_o \in R_+ \times R_+ \times R^n$ then there exist a natural number N such that

$$\|f(a) - f(a_o)\| \leq N$$

for all a in A.

<u>Proof</u>: Since f is continuous, $f(A)$ is a compact set in R^n. Hence $f(A)$ is bounded, i.e. there exists an $M > 0$ such that

$$\|f(a)\| \leq M$$

for all a in A. Hence

$$\|f(a) - f(a_o)\| \leq \|f(a)\| + \|f(a_o)\|$$

$$\leq M + \|f(a_o)\|$$

Let N be a natural number larger than $M + \|f(a_o)\|$.

In Chapter IV we consider the space $C_c(I)$ of continuous functions from an interval I into R^n with the topology of uniform convergence on compact subsets of I. $C_c(I)$ is metrizable and complete. (See C. Corduneanu [2].) When g is a positive, continuous function on I then an element u of $C_c(I)$ is in $C_g(I)$ if

$$|U|_g \equiv \sup_I \frac{\|U(t)\|}{g(t)} \tag{2.3}$$

is finite. This definition of $C_g(I)$ is the one that Gollwitzer [4] generalizes. The space $C_g(I)$ with the norm given by (2.3) is a Banach subspace of $C_c(I)$ which is stronger than $C_c(I)$ in the sense that if a sequence $\{U_n\}$ converges to U_o in $C_g(I)$ then the sequence converges to U_o in $C_c(I)$. If U belongs to $C_g(I)$ then each component of $U(t)$ is bounded by $g(t)|U|_g$ on I. In some situations it might be more natural to weight the components of $U(t)$ differently. Also it is not natural in some situations to assume that $g(t)$ is positive for $t \geq 0$. The Gollwitzer generalization offers a natural way to handle these remarks.

The following lemmas are used in obtaining the results in Chapter IV.

Lemma 2.2. Let $k(t,s)$ and $k(s)$ be real valued functions defined on R^2 and R respectively. Suppose that $k(t,s) \to k(s)$ uniformly on compact sets and that there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$ as $n \to \infty$, such that

$$\int_{t_o}^{t_n} |k(t_n,s)|ds > 3d$$

and

$$\int_{t_o}^{t_n} |k(s)|ds < d$$

for some $d > 0$. Then there exists a subsequence $\{t_j\}$ of the given sequence $\{t_n\}$ such that

$$\int_{t_o}^{t_j} |k(t_{j+1},s)|ds < d.$$

Proof: Since $k(t,x) \to k(s)$ uniformly on compact sets

$$\lim_{t\to\infty} \int_{t_o}^{t_1} |k(t,s) - k(s)|ds = 0.$$

Hence there exists an $M > 0$ such that if $t > M$ then

$$\int_{t_o}^{t_1} |k(t,s) - k(s)|ds < d - \int_{t_o}^{t_1} |k(s)|ds.$$

Thus for $t > M$

$$\int_{t_o}^{t_1} |k(t,s)|ds - \int_{t_o}^{t_1} |k(s)|ds < d - \int_{t_o}^{t_1} |k(s)|ds$$

so

$$\int_{t_o}^{t_1} |k(t,s)|ds < d.$$

Since $t_n \to \infty$ there exists an n_2 such that $t_{n_2} > M$, so

$$\int_{t_o}^{t_1} |k(t_{n_2},s)|ds < d.$$

Now considering $\int_{t_o}^{t_{n_2}} |k(t,s)|ds$ it follows as above that there exists an $n_3 > n_2$ such that

$$\int_{t_o}^{t_{n_2}} |k(t_{n_3},s)|ds < d.$$

The desired result follows by mathematical induction.

Lemma 2.3. Let f be a measurable real valued function on the interval $[a,b]$. Suppose that

$$\int_a^b f(x)dx > k$$

and $|f(x)| \leq M < \infty$ for all x in $[a,b]$. Then there exists a continuous real valued function g defined on $[a,b]$ such that

$$\int_a^b g(x)\,dx > k.$$

Proof: It follows immediately from Lusin's Theorem that for each natural number n there exists a continuous function g_n such that $|g_n(x)| \leq M$ and the measure of the set $\{x \mid f(x) \neq g_n(x)\}$ is less than $\frac{1}{2nM(b-a)}$. Therefore

$$\int_a^b |f(x) - g_n(x)|\,dx < \frac{1}{n}.$$

So the sequence $\{g_n\}$ converges to f in the mean and hence a subsequence of $\{g_n\}$ converges to f a.e. (See Theorems 5-6 II and 5-7 I of Taylor [11].) Let $\{g_n\}$ represent the subsequence. The Lebesgue Dominated Convergence Theorem implies

$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = \int_a^b f(x)\,dx.$$

Hence there exists an integer N such that if $n > N$ then

$$\left| \int_a^b f(x)\,dx - \int_a^b g_n(x)\,dx \right| < \int_a^b f(x)\,dx - k.$$

Therefore

$$\int_a^b g_n(x)\,dx > k$$

and the conclusion follows.

Lemma 2.4. Let f be a function from R_+ to R such that

$$\int_0^\infty |f(s)|\,ds = K < \infty.$$

Let g be a continuous function from R_+ to R such that $\lim g(t) = 0$ as $t \to \infty$. Then the function

$$h(t) = \int_0^t g(t-s)f(s)\,ds$$

has zero as its limit as t goes to ∞.

Proof: Let $M = \max\{|g(t)| : t \text{ in } R_+\}$ and let $\epsilon > 0$. From the given conditions of f and g there exist numbers $T_1 = T_1(\epsilon)$ and $T_2 = T_2(\epsilon)$ such that

$$\int_{T_1}^\infty |f(s)|\,ds < \frac{\epsilon}{K+M}$$

and

$$|g(s)| < \frac{\epsilon}{K+M} \quad \text{when} \quad s > T_2.$$

Now set $T = \max\{T_1, T_2\}$ and suppose $t > 2T$. Then

$$\left| \int_0^t g(t-s)\,f(s)\,ds \right| \le \left| \int_0^T g(t-s)\,f(s)\,ds \right| + \left| \int_T^t g(t-s)\,f(s)\,ds \right|$$

$$\le \frac{\epsilon}{K+M}(K) + M\left(\frac{\epsilon}{K+M}\right)$$

$$= \epsilon.$$

The desired result follows.

III. GENERIC PROPERTY OF UNIQUENESS

A. Preliminaries

Consider a Volterra integral equation of the form

(3.1) $$x(t) = f(t) + \int_0^t g(t,s,x(s))\,ds$$

where f is a continuous function from R_+ to R^n and g is a continuous function from $R_+ \times R_+ \times R^n$ to R^n. Let

(3.2) $$X = \{g : R_+ \times R_+ \times R^n \to R^n : g \text{ is continuous}\}$$

and

(3.3) $$Y = \{f : R_+ \to R^n : f \text{ is continuous}\}$$

with X and Y both having the topology of uniform convergence where the convergence is uniform on the common domains of definition. For functions h and k with a common domain D and range in R^n we let

$$\|h - k\| = \sup\{\|h(x) - k(x)\| : x \text{ in } D\}$$

The solutions of (3.1) are continuous functions with domain in R_+ and range in R^n. A solution $x(t)$ of (3.1) through (t_o, x_o), where t_o is a non-negative real number and x_o is in R^n, is a continuous function such that $x(t)$ is continuous on $0 \leq t \leq t_o$, $x(t)$ satisfies (3.1) at $t = t_o$ and $x(t_o) = x_o$.

Definition 3.1. Let $x(t)$ be a solution of (3.1) through $(0, f(0))$ on $[0,a)$ where $a \leq \infty$. $x(t)$ is an unlimited solution of (3.1) on $[0,a)$ if and only if $(t, x(t))$ has no limit in $R_+ \times R^n$ as t approaches a^-.

Definition 3.2. Let B_1 and B_2 be Banach spaces and let $U_1 \subset B_1$. A function

$$G : U_1 \to B_2$$

is locally Lipschitz on U_1 if for each p in U_1 there is an open set O_p with $p \in O_p \subset U_1$ and a number $L_p > 0$ such that

$$\|G(x) - G(y)\|_2 \leq L_p \|x - y\|_1 \tag{3.4}$$

for all x, y in O_p where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the norms on B_1 and B_2 respectively.

Lemma 3.1. Let f be continuous on $R_+ \times R_+ \times E$ to E where E is a Banach space with norm $\|\cdot\|_E$ and let $\epsilon > 0$. Then there exists a locally Lipschitz function

$$G : R_+ \times R_+ \times E \to E$$

such that

$$\|f(x) - G(x)\|_E < \epsilon$$

for all x in $R_+ \times R_+ \times E$.

Proof: By Theorem 2.1 there exists a continuous function $F : R^1 \times R^1 \times E \to E$ such that $F(y) = f(y)$ for all y in $R_+ \times R_+ \times E$. So Lemma 1 of Lasota and Yorke [6] there exists a locally Lipschitz function

$$G : R^1 \times R^1 \times E \to E$$

such that

$$\|F(y) - G(y)\|_E < \epsilon$$

for all y in $R^1 \times R^1 \times E$. G restricted to $R_+ \times R_+ \times E$ is the desired function.

The next lemma follows from the results of Chapter II of R. K. Miller [9].

<u>Lemma 3.2</u>. Let g_o in X be a locally Lipschitz function. If $g = g_o$ and $f(t) \equiv x_o$ in equation (3.1) then there exists a unique unlimited solution $x_o(t)$ of (3.1) through $(0,x_o)$. Let $[0,a)$, $0 < a \leq \infty$ be the domain of $x_o(t)$. Let $\{g_m\}_{m=1}^{\infty}$ be a sequence contained in X such that g_m approaches g_o as $m \to \infty$. Let $\{x_m\}_{m=1}^{\infty}$ be a sequence in R^n such that x_m approaches x_o as $m \to \infty$. If there exist an unlimited solution $x_m(t)$ of (3.1) through $(0,x_m)$ when $g = g_m$ and $f(t) \equiv x_m$ then for each compact interval J in $[0,a)$, $x_m(t)$ is defined on J (for m sufficiently large) and

$$x_m(t) \to x_o(t)$$

as $m \to \infty$, uniformly for t in J.

B. Main Results

The next theorem is the main result of this chapter. The techniques and definitions introduced in the proof of this theorem are used as basic tools in proving the other results in this section.

<u>Theorem 3.1</u>. In equation (3.1) let $f(t) \equiv x_o$ be a fixed point in R^n and T be the set of g in X for which there does not exist a unique solution of (2.1) through $(0,x_o)$. Then T is a set of first category in X.

<u>Proof</u>: Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of closed bounded subsets of $U = R_+ \times R^n$ such that $(0,x_o)$ is in the interior of U_1, $\bigcup_{n=1}^{\infty} U_n = U$ and U_n is the interior of U_{n+1} for $n = 1,2,3,\ldots$. For g in X let

$$(3.5)\quad W_n(g,x_o) = \{(t,x) \in U_n : \|g(t,s,x) - g(0,0,x_o)\| \leq n \quad \text{for } 0 \leq s \leq t\}$$

where n is some fixed natural number. For a continuous function, $x_1(t)$, such that $x_1(0) = x_o$ let

$$(3.6)\quad \sigma_n(x_1(\cdot),g) = \sup\{t \geq 0 : x_1(t) \text{ is defined and } (s,x_1(s)) \text{ is in } W_n(g,x_o) \text{ for } 0 \leq s \leq t\}$$

If $x_1(\cdot)$ and $x_2(\cdot)$ are solutions of (3.1) through $(0,x_o)$ for some functions g_1 and g_2 in X, define their common interval in $W_n(g,x_o)$ by

$$(3.7)\quad J_n(x_1(\cdot),x_2(\cdot),g) = [0,\sigma_n(x_1(\cdot),g)] \cap [0,\sigma_n(x_2(\cdot),g)]$$

and define

$$\mu_n = \mu_n(x_1(\cdot),x_2(\cdot),g) = \sup\{\|x_1(t) - x_2(t)\| : t \text{ in } J_n(x_1(\cdot),x_2(\cdot),g)\}.$$

Now fix g in X and for $\delta > 0$ define

$$(3.8)\quad \mu_n(g;\delta) = \sup\{\mu_n(x_1(\cdot),x_2(\cdot),g) : \|g_1 - g\| \leq \delta \text{ and } \|g_2 - g\| \leq \delta\}$$

where the sup is taken over all solutions x_1 and x_2 of (3.1) associated with the functions g_1 and g_2

respectively. We next define

(3.9) $$V_n(g) = \limsup_{\delta \to 0} \mu_n(g;\delta)$$

Now suppose $V_n(g) = 0$ for some g in X and all $n = 1,2,\ldots$. Let x_1 and x_2 be solutions of (2.1) through $(0,x_o)$ and suppose both solutions are defined on $[0,t_1]$. Since the solutions are continuous the images of x_1 and x_2 on $[0.t]$ are compact. Thus there is some N such that $(s,x_1(s))$ and $(s,x_2(s))$ belong to $W_n(g,x_o)$ for all s in $[0,t_1]$. Since $\|g - g\| = 0 < \delta$ for all $\delta > 0$ it follows that

$$\mu_N(x_1,x_2,g) \leq \mu_N(g;\delta) \quad \text{for all} \quad \delta < 0.$$

Thus

$$\mu_N(x_1,x_2,g) \leq \limsup_{\delta \to 0} \mu_N(g;\delta)$$

$$= V_N(g) = 0$$

Therefore $x_1(s) = x_2(s)$ for s in $[0,t_1]$. Thus the solution of (1) when $V_n(g) = 0$ for all $n = 1,2,\ldots,$ is unique. Any function g in X gives rise to an unlimited solution through $(0.x_o)$ (see Miller [9], Theorem 2.2, page 95) and if g is locally Lipschitz then by Lemma 3.2 $V_n(g) = 0$ for $n = 1,2,\ldots$.

Next let $V_n(g) = 0$ for some g in X and a fixed n. Let $x(t)$ be a solution of (3.1) for g. The solution $x(t)$ is unique on $[0,\sigma_n(x,g)]$. Suppose the sequence $\{g_i\}$ converges to g in X. We will show next that $V_n(g_i) \to 0$ as $i \to \infty$. Suppose not, then there exists an $\eta > 0$ and a sequence $\{g_i\}_{i=1}^{\infty}$ such that $g_i \to g$ as $i \to \infty$ and $V_n(g_i) > \eta$. By the definition of $V_n(g_i)$ there exists g_{1i}, g_{2i} with solutions x_{1i}, x_{2i} through $(0,x_o)$ such that for each $i = 1,2,\ldots$. $\mu_n(x_{1i}, x_{2i}, g_i) > \eta$ and such that

$$\|g_i - g_{1i}\| \le 2^{-i} \quad \text{and} \quad \|g_i - g_{2i}\| \le 2^{-i}.$$

So

$$\|g - g_{1i}\| \to 0$$

and

$$\|g - g_{2i}\| \to 0 \quad \text{as} \quad i \to \infty.$$

Hence using Corollary 4.4, page 115, of Miller [9], $x_{1i}(t) \to x(t)$ and $x_{2i}(t) \to x(t)$ uniformly as $i \to \infty$ for t in $[0,\sigma_n(x,g)]$. Now suppose $x_{1i}(t) \to x(t)$ as $i \to \infty$ for some t in $\bigcap_{i=1}^{\infty} [0,\sigma_n(x_{1i},g_i)]$. Then

$$\|g(t,s,x(t)) - g(0,0,x_o)\|$$

$$\leq \|g(t,s,x(t)) - g(t,s,x_{1i}(t))\| + \|g(t,s,x_{1i}(t) - g_i(t,s,x_{1i}(t)\|$$

$$+ \|g_i(t,s,x_{1i}(t)) - g_i(0,0,x_o)\| + \|g_i(0,0,x_o) - g(0,0,x_o)\|$$

$$\leq n + \epsilon$$

where $\epsilon \to 0$ as $i \to \infty$. Because of the choice of t the third term on the right hand side of the above inequality is bounded by n. By continuity the other terms go to zero as $i \to \infty$. Thus

$$\|g(t,s,x(t)) - g(0,0,x_o)\| \leq n$$

so

$$(t,x(t)) \text{ belongs to } W_n(g,x_o).$$

If t belongs to $[0,\sigma_n(x,g)]$ then it follows in the same manner that

$$(t,x_{1i}(t)) \in W_n(g_i,x_o)$$

for i large enough. The last statement implies that $\liminf_{i\to\infty} \sigma_n(x_{1i},g_i) \geq \sigma_n(x,g)$. Suppose $\liminf_{i\to\infty} \sigma_n(x_{1i},g_i) = K > \sigma_n(x,g)$. Then there exists a subsequence of $\{i\}$, we will label it by $\{i\}$ again, and an $r > 0$ such that $\sigma_n(x_{1i},g_i) \geq \sigma_n(x,g) + r$. Now let α be the largest number such that the cross section set $K^*(t)$ of (3.1) with $f(t) \equiv x_o$ is compact for all t in the interval $0 \leq t < \alpha$. Since $\alpha > \sigma_n(x,g)$ there exists a β such that $\sigma_n(x,g) < \beta < \min\{\alpha,\sigma_n(x,g)+r\}$. Now by Theorem 4.2, page 108, of Miller [9] there exists a subsequence of $\{x_{1i}\}$ which converges uniformly to a solution $x(t)$ of (3.1) on $[0,\beta]$. Since $[0,\beta] \subset \bigcap_{i=1}^{\infty} [0,\sigma_n(x_{1i},g_i)]$ it follows from previous remarks that $(t, x(t))$ belongs to $W_n(x,g)$ for $t \in [0,\beta]$. This contradicts the choice of $\sigma_n(x,g)$. Hence $\liminf_{i\to\infty} \sigma_n(x_{1i},g_i) = \sigma_n(x,g)$. Similarly $\limsup_{i\to\infty} \sigma_n(x_{1i},g_i) = \sigma_n(x,g)$. Thus

$$\lim_{i\to\infty} \sigma_n(x_{1i},g_i) = \lim_{i\to\infty} \sigma_n(x_{2i},g_i) = \sigma_n(x,g).$$

Now since the solution $x(t)$ is unique on $[0,\sigma_n(x,g)]$

$$\|x_{2i}(t) - x_{1i}(t)\| \to 0$$

uniformly as $i \to \infty$ on $[0,\sigma_n(x,g)]$. By definition $W_n(g_i,x_o) \subset U_n$ so x_{1i} and x_{2i} are uniformly bounded on $[0,\sigma_n(x_{1i},g_i)] \cap [0,\sigma_n(x_{2i},g_i)]$ for all i. For t in $[0,\sigma_n(x_{1i},g_i)] \cap [0,\sigma_n(x_{2i},g_i)]$ and $t > \sigma_n(x,g)$ we have

$$\|x_{1i}(t) - x_{2i}(t)\| \leq \left\| \int_0^t [g_{1i}(t,s,x_{1i}(s)) - g_{2i}(t,s,x_{2i}(s))]ds \right\|$$

$$\leq \|x_{1i}(\sigma_n(x,g)) - x_{2i}(\sigma_n(x,g))\|$$

$$+ \left\| \int_{\sigma_n(x,g)}^t [g_{1i}(t,s,x_{1i}(s)) - g_{2i}(t,s,x_{2i}(s))]ds \right\|$$

$$\leq \|x_{1i}(\sigma_n(x,g)) - x_{2i}(\sigma_n(x,g))\|$$

$$+ \left\| \int_{\sigma_n(x,g)}^t [g_{1i}(t,s,x_{1i}(s)) - g(t,s,x_{1i}(s))]ds \right\|$$

$$+ \left\| \int_{\sigma_n(x,g)}^t [g(t,s,x_{1i}(s) - g(t,s,x_{2i}(s))]ds \right\|$$

$$+ \left\| \int_{\sigma_n(x,g)}^t [g(t,s,x_{2i}(s)) - g_{2i}(t,s,x_{2i}(s))]ds \right\|$$

$$\leq \|x_{1i}(\sigma_n(x,g)) - x_{2i}(\sigma_n(x,g))\|$$

$$+ [\sigma_n(x_{1i},g_i) - \sigma_n(x,g)]2^{-i}$$

$$+ [\min\{\sigma_n(x_{1i},g_i),\sigma_n(x_{2i},g_i)\} - \sigma_n(x,g)]2M$$

$$+ [\sigma_n(x_{2i},g_i) - \sigma_n(x,g)]2^{-i}$$

where $M = \sup\{\|g(t,s,x)\| : (t,x) \in U_n \text{ and } 0 \leq s \leq t\}$.

Thus $\mu_n(x_{1i},x_{2i},g_i)$ approaches zero as $i \to \infty$. This is a contradiction so $V_n(g_i) \to 0$ as $g_i \to g$.

Now let

$$(3.10) \qquad F_{n,m} = \{f \in X : V_n(f) > m^{-1}\}$$

where n,m are natural numbers. We show that $F_{n,m}$ is nowhere dense in the space X. Suppose some non-empty open set $S \subset X$ is in the closure, $\overline{F}_{n,m}$, of $F_{n,m}$. By Lemma 3.1 there exists a locally Lipschitz function f in S. Since f belongs to $\overline{F}_{n,m}$ and $V_n(f) = 0$, it follows from the preceding remarks that there is a sequence $\{f_i\}_{i=1}^{\infty}$ in $F_{n,m}$ such that $V_n(f_i) \to 0$ as $i \to \infty$. This contradicts the definition of $F_{n,m}$. Thus $F_{n,m}$ is nowhere dense in X. Clearly

$$\bigcup_{\substack{n=1\\m=1}}^{\infty} F_{n,m} = \{f \in X : V_p(f) \neq 0 \quad \text{for some} \quad p\}$$

So $T = \bigcup_{\substack{n=1\\m=1}}^{\infty} F_{n,m}$ is of first category in X. Since the set of f without unique solutions is contained in T it must also be of first category.

Corollary 3.1. Let $f(t)$ be a continuous function on R_+. Then the set of g in $X = \{g : R_+ \times R_+ \times R^n \to R^n \mid g$ is continuous$\}$ for which (3.1) does not have unique solutions is of first category.

Proof: Use the proof of Theorem 3.1 with $x_o = f(0)$.

Theorem 3.2. Let A be a compact set in R^n. Then the set of g in X such that (3.1) does not have unique solutions for

$$x(t) = x_o + \int_o^t g(t,s,x(s))ds$$

for all x_o in A is of first category.

Proof: For a fixed x_o in A let

(3.11) $$V_n(x_o,g) \equiv V_n(g)$$

where $V_n(g)$ is defined as in (3.9). Define

(3.12) $$V_n(A,g) = \sup\{V_n(x_o,g) \mid x_o \text{ in } A\}.$$

Existence and uniqueness of solutions for each x_o in A

follows as in the proof of Theorem 3.1. If g is locally Lipschitz then $V_n(x_o,g) = 0$ for each x_o in R^n. Thus $V_n(A,g) = 0$ for $n = 1,2,\ldots$.

Now suppose $V_n(A,g) = 0$ for some g and n. Let $\{g_i\}$ be a sequence of continuous functions in X converging to g such that $V_n(A,g_i)$ does not converge to zero. Then there exists a sequence $\{(a_j,g_{ij})\}$ and a number $\eta > 0$, where $\{g_{ij}\}$ is a subsequence of $\{g_i\}$ and $\{a_j\}$ is a sequence in A, such that $V_n(a_j,g_{ij}) > \eta$. Since A is compact we can assume $a_j \to a_o$ for some a_o in A. So there exists sequences $\{g_{1j}\}$ and $\{g_{2j}\}$ with associated solutions $\{x_{1j}\}$ and $\{x_{2j}\}$ respectively such that

$$\|g_{ij} - g_{1j}\| \to 0,$$

$$\|g_{ij} - g_{2j}\| \to 0$$

as $j \to \infty$ and $\mu_n(x_{1j},x_{2j},g_{ij}) > \eta$ in $W_n(g_{ij},a_j)$. It follows that

$$\|g - g_{1j}\| \to 0 \quad \text{and} \quad \|g - g_{2j}\| \to 0$$

as $j \to \infty$. Since $V_n(a_o,g) = 0$ the function $x(t)$ which satisfies

$$x(t) = a_o + \int_0^t g(t,s,x(s))\,ds$$

is unique on $[0,\sigma_n(x(\cdot),g)]$. So using Corollary 4.4, page 115 of Miller [9] and previous technique it follows that

$$\mu_n(x_{1j},x_{2j}) \to 0 \quad \text{as} \quad j \to \infty.$$

This contradicts the choice of x_{1j} and x_{2j}. So $V_n(A,g_i) \to 0$ as $g_i \to g$.

Now let

$$F_{n,m} = \{g \in X : V_n(A,g) > m^{-1}\} \tag{3.13}$$

where n,m are natural number. As in the proof of Theorem 3.1 it follows that

$$T = \bigcup_{\substack{n=1\\m=1}}^{\infty} F_{n,m}$$

is a set of first category in X containing all those g

without unique solutions for some initial value in A. Next consider

$$x(t) = h(t) + \int_0^t f(t,s,x(s))\,ds. \tag{3.14}$$

<u>Theorem 3.3</u>. Let H be a compact set in Y. Then the set of f in X for which (3.14) does not have a unique solution for each $h(t)$ belonging to H is of first category in X.

<u>Proof</u>: Define the sequence of sets U_n as before. For h in H and f in X define

$$W_n(h,f) = \{(t,x) \in U_n : \|f(t,s,x) - f(0,0,h(0))\| \leq n \quad \text{for } 0 \leq s \leq t\}$$

For a solution $x(t)$ of

$$x(t) = h(t) + \int_0^t g(t,s,x(s))\,ds$$

where g is some function in X define

$$(3.15)\quad \sigma_n(x(\cdot),h,g) = \sup\{t \geq 0 : x(t) \text{ is defined and } (s,x(s)) \in W_n(h,f) \text{ for } s \in [0,t]\}$$

If $x_1(\cdot)$ and $x_2(\cdot)$ are solutions of (3.14) for pairs of functions (h,g_1) and (h,g_2) respectively define their common interval in $W_n(h,g)$ by

$$(3.16)\quad J_n = J_n(x_1(\cdot),x_2(\cdot),g) = [0,\sigma_n(x_1(\cdot),g)] \cap [0,\sigma_n(x_2(\cdot),g)].$$

Let

$$(3.17)\quad \mu_n(x_1(\cdot),x_2(\cdot),g) = \sup\{\|x_1(t) - x_2(t)\| : t \text{ in } J_n\}$$

Now fixing h and g in H and X respectively let

$$(2.18)\quad \mu_n(h,g;\delta) = \sup\{\mu_n(x_1(\cdot),x_2(\cdot),g) : \|g - g_1\| \leq \delta \text{ and } \|g - g_2\| \leq \delta\}$$

We now define

$$V_n(h,g) = \limsup_{\delta \to 0} \mu_n(h,g;\delta) \tag{3.19}$$

and set

$$V_n(H,g) = \sup\{V_n(h,g) : h \text{ in } H\}.$$

With these definitions the proof of this result follows the same general line of reasoning as the proof of Theorem 2. We note $V_n(h,g)$ really depends only on $h(0)$ and for H compact in X it follows that $\{h(0) : h \text{ in } H\}$ is compact in R^n. Thus Theorems 3.1 and 3.2 are really corollaries of Theorem 3.3. The style of presentation was used to clarify the proofs.

IV. ADMISSIBILITY RESULTS AND APPLICATIONS

A. Preliminaries

Let R^n denote the usual space of n-vectors over the reals with the usual inner product. For a fixed $w \leq \infty$, set $I \equiv I(w) = \{t : 0 \leq t < w\}$. Let $C_c(I)$ be the space of continuous functions from I to R^n with the topology of uniform convergence. Let C_ℓ be the set of continuous functions defined on $I(\infty)$ which have limits at ∞. Let C_o be the set of f in C_ℓ such that $\lim f(t) = 0$ as $t \to \infty$. The symbol $|x|_\ell$ will denote the usual sup norm in C_ℓ.

Recall that a Frechet space is a complete metric space such that addition and scalar multiplication are continuous and the metric d is additively invariant, i.e. $d(x,y) = d(x-y,0)$ for all x and y in the space. The space $C_c(I)$ is a Frechet space.

Definition 4.1. Let F_1, F_2 be Frechet spaces and assume that T is a linear operator from F_1 to F_2. If E_i is a subspace of F_i, $i = 1,2$, then the pair (E_1,E_2) is said to be admissible with respect to T if TE_1 is a subset of E_2.

Definition 4.2. Suppose X is a subset of a Frechet space, $\mathfrak{F}$, and that X is a Banach space under the norm $\|\cdot\|_X$. The topology under the norm $\|\cdot\|_X$ is said to be a stronger topology than the topology on $\mathfrak{F}$ if for any sequence $\{x_n\}$ contained in X such that $\|x_n - x\|_X \to 0$ as $n \to \infty$ for some x in X then $x_n \to x$ in the topology of $\mathfrak{F}$.

The following notation and definition were developed by H. Gollwitzer [4]. For each t in $I(w)$, let $g(t)$ be a linear transformation on R^n such that

$$\|g(t)\| = \sup\{\|g(t)u\| : \|u\| = 1\} \tag{4.1}$$

is uniformly bounded on compact subsets of $I(w)$. Let $N(t)$ and $R(t)$ be the null space and range of $g(t)$, respectively and let $N^{\perp}(t)$ denote the orthogonal complement of $N(t)$. Also let $P(t)$ denote the orthogonal projection of R^n onto $R(t)$. Define $P(t)$ to be the zero operator if $R^{\perp}(t) = R^n$. Define $g_{-1}(t) : R(t) \to N^{\perp}(t)$ as the inverse of the restriction of $g(t)$ to $N^{\perp}(t)$. Extend $g_{-1}(t)$ to all of R^n by sending $R^{\perp}(t)$ into the zero vector and call the resulting transformation $g_{-1}(t)$.

<u>Definition 4.3</u>. A function u in $C_c(I(w))$ is said to be in $C_g(I(w))$ if $P(t)u(t) = u(t)$ for each t in $I(w)$ and

$$|u|_g \equiv \sup\{\|g_{-1}(t)u(t)\| : t \text{ in } I\}$$

if finite.

H. Gollwitzer [4] proves the following results.

<u>Theorem 4.1</u>. The space $C_g(I(w))$ is a subspace of $C_c(I(w)$ which is a Banach space under the norm, $|\cdot|_g$, C_g with the norm, $|\cdot|_g$, has a stronger topology than the topology on C_c.

<u>Proposition 4.1</u>. Let J be a measurable subset of the reals and let $g(t)$ be a linear transformation on R^n which is measurable on J. Let $g_{-1}(t)$ be as defined above. Then $g_{-1}(t)$ is measurable on J.

B. Admissibility of (C_g, C_ℓ)

In the following for an arbitrary $n \times n$ matrix $H = (H_{ij})_{n\times n}$, we set the norm of H equal to

$\sum_{i,j=1}^{n} |H_{ij}|$ i.e. $|H| = \sum_{i,j=1}^{n} |H_{ij}|$. Let $g(t) : E^n \to E^n$ be a continuous matrix valued function defined on $I(\infty)$ and let $k(t,s) : E^n \to E^n$ be a continuous matrix valued function for $0 \leq s \leq t < \infty$ with $k(t,s) = 0$ if $s > t$. Assume

$$\lim_{t\to\infty} k(t,s) = k(s) \tag{4.2}$$

where the convergence is uniform on compact subsets of $I(\infty)$. It follows that $k(s)$ is continuous. Define the operator K from $C_c(I)$ to $C_c(I)$ by

$$K\,x(t) = \int_0^t k(t,s)x(s)\,ds \tag{4.3}$$

The following theorem generalizes a result due to G. Bantas [1]. In this theorem we set $C_g \equiv C_g(I(\infty))$.

<u>Theorem 4.2</u>. The pair (C_g, C_ℓ) is admissible with respect to the operator K if and only if

(4.4) i) $\int_0^\infty |k(s)g(s)|\,ds < \infty$

$$(4.5) \quad \text{ii)} \quad \lim_{t\to\infty} \int_0^t |k(t,s)g(s)|ds = \int_0^\infty |k(s)g(s)|ds.$$

Proof: We first show that conditions i) and ii) are sufficient for admissibility, Let x belong to C_g. We show that $\lim_{t\to\infty} \int_0^t (k(t,s) - k(s))x(s)ds = 0$ in order to have the desired conclusion. Since x belongs to C_g then $x(t) = g(t)v(t)$ for some uniformly bounded $v(t)$. If we apply condition i), then

$$(4.6) \quad \left| \int_0^\infty k(s)x(s)ds \right| \leq \int_0^\infty |k(s)g(s)| \; \|v(s)\| ds < \infty$$

Thus $\int_0^\infty k(s)x(s)ds$ exists. We know that

(4.7)
$$\lim_{t\to\infty} \int_0^t [k(t,s) - k(s)]x(s)ds = \lim_{t\to\infty} \int_0^t [k(t,s) - k(s)]g(s)v(s)ds,$$

whenever either limit exists. Since $v(t)$ is uniformly bounded

$$(4.8) \qquad \lim_{t\to\infty} \int_0^t [k(t,s) - k(s)]x(s)ds = 0$$

if $\int_0^t |[k(t,s) - k(s)]g(s)|ds \to 0$ as t approaches ∞.

We use arguments similar to those found in Theorem 2.3.1 of C. Corduneanu [2] to show that $\int_0^t |[k(t,s) - k(s)]g(s)|ds \to 0$.

If $0 < \tau \leq t$, then

(4.9)

$$\int_0^t |[k(t,s) - k(s)]g(s)|ds \leq \int_0^\tau |[k(t,s) - k(s)]g(s)|ds$$

$$+ \int_\tau^t |k(t,s)g(s)|ds + \int_\tau^t |k(s)g(s)|ds$$

For a given $\epsilon > 0$ there exists $T_1 = T_1(\epsilon)$ such that for $\tau \geq T_1$

$$\int_\tau^t |k(s)g(s)|ds \leq \int_\tau^\infty |k(s)g(s)|ds < \epsilon$$

Condition ii) implies the existence of $T_2 = T_2(\epsilon)$ such that, for $t \geq T_2$

$$\int_0^t |k(t,s)g(s)|ds < \int_0^\infty |k(s)g(s)|ds + \epsilon$$

For a fixed τ, since g is uniformly bounded on $[0,\tau]$ then

$$\int_0^\tau |k(s)g(s)|ds - \int_0^\tau |k(t,s)g(s)|ds \leq \int_0^\tau |k(t,s) - k(s)||g(s)|ds$$

and $\int_0^\tau |k(t,s) - k(s)||g(s)|ds \to 0$ as t approaches ∞.

Thus there exists a $T(\tau,\epsilon) > 0$ such that $t > T(\tau,\epsilon)$ implies

$$-\int_0^\tau |k(t,s)g(s)|ds < -\int_0^\tau |k(s)g(s)|ds + \epsilon.$$

Now if we fix $\tau \geq \max\{T_1,T_2\}$ and let $t > T(\tau,\epsilon)$ then

$$\int_\tau^t |k(t,s)g(s)|ds = \int_0^t |k(t,s)g(s)|ds - \int_0^\tau |k(t,s)g(s)|ds$$

$$\leq \int_0^\infty |k(s)g(s)|ds + \epsilon - \int_0^\tau |k(s)g(s)|ds + \epsilon$$

$$\leq 3\epsilon.$$

For a fixed $\tau \geq \max\{T_1,T_2\}$ and $t \geq \max\{T_1,T_2,T(\tau,\epsilon)\}$

$$\int_0^t |[k(t,s) - k(s)]g(s)|ds \leq 5\epsilon \tag{4.10}$$

and (4.8) holds. Thus (C_g, C_ℓ) is admissible with respect to K.

Next we show that admissibility implies conditions i) and ii). If x is in C_g and Kx belongs to C_ℓ then Kx is a bounded continuous function and H. Gollwitzer [4] has shown

$$(4.11) \quad \int_0^t |k(t,s)g(s)|ds < A \quad \text{for all} \quad t \geq 0$$

where A is some fixed positive constant. Since $k(t,s) = 0$ if $s > t$, then

$$(4.12) \quad \int_0^\infty |k(t,s)g(s)|ds \leq A \quad \text{for all} \quad t \geq 0.$$

Hence, by Fatou's lemma,

$$(4.13) \quad \int_0^\infty |k(s)g(s)|ds \leq \underline{\lim}_{t\to\infty} \int_0^\infty |k(t,s)g(s)|ds \leq A,$$

and condition i) follows

Now consider $k(t,s)g(s)e_j$ where e_j is a member of the usual orthonormal basis for R^n. By Fatou's lemma and $k(t,s) = 0$ if $s > t$, it follows that

$$L_1 = \int_0^\infty |k(s)g(s)e_j|ds \leq \underline{\lim}_{t\to\infty} \int_0^\infty |k(t,s)g(s)e_j|ds$$

$$= \underline{\lim}_{t\to\infty} \int_0^t |k(t,s)g(s)e_j|ds$$

$$\leq \overline{\lim}_{t\to\infty} \int_0^t |k(t,s)g(s)e_j|ds.$$

Let $L^1 = \overline{\lim}_{t\to\infty} \int_0^t |k(t,s)g(s)e_j|ds$. If for each j, $L_1 = L^1$ then condition ii) obviously holds. Suppose, for a contradiction, that $L_1 < L^1$. Then for some i

$$L_3 = \int_0^\infty |(k(s)g(s))_{ij}|ds < \overline{\lim}_{t\to\infty} \int_0^t |(k(t,s)g(s))_{ij}|ds = L_2$$

We now show that there exists a function y in C_g such that Ky is not in C_ℓ. First note that there exists a $t_0 \geq 0$ such that for $0 < d < \frac{L_2 - L_3}{3}$ and $t \geq t_0$

$$\int_{t_0}^t |(k(s)g(s))_{ij}|ds < d.$$

Since

$$\overline{\lim_{t\to\infty}} \int_{t_o}^{t} |(k(t,s)g(s))_{ij}|ds + \overline{\lim_{t\to\infty}} \int_{o}^{t_o} |(k(t,s)g(s))_{ij}|ds$$

$$\geq \overline{\lim_{t\to\infty}} \int_{o}^{t} |(k(t,s)g(s))_{ij}|ds = L_2$$

then

$$\overline{\lim_{t\to\infty}} \int_{t_o}^{t} |(k(t,s)g(s))_{ij}|ds \geq L_2 - \overline{\lim_{t\to\infty}} \int_{o}^{t_o} |(k(t,s)g(s))_{ij}|ds$$

$$= L_2 - \int_{o}^{t_o} |(k(s)g(s))_{ij}|ds$$

$$\geq L_2 - L_3 > 3d.$$

Therefore there exists an increasing sequence $\{t_n\}$, t_n approaches ∞, such that

$$\int_{t_o}^{t_n} |(k(t_n,s)g(s))_{ij}|ds > 3d, \quad n \geq 1$$

Since

$$\int_{t_o}^{t_n} |(k(s)g(s))_{ij}|ds < d \quad \text{for} \quad n \geq 1$$

we can assume w. o. ℓ. g that the sequence $\{t_n\}$ is chosen so that

$$\int_{t_o}^{t_n} |(k(t_{n+1},s)g(s))_{ij}|ds < d, \quad n \geq 1.$$

(See Lemma 2.2 above.) For each $n \geq 1$ let $x_n(t) = (-1)^{n-1}\mathrm{sgn}(k(t_n,t)g(t))_{ij}$ on $[t_{n-1},t_n)$. Then $x_n(t)$ is measurable and $|x_n(t)| \leq 1$ for t in $[t_{n-1},t_n)$. Hence

$$(-1)^{n-1}\int_{t_{n-1}}^{t_n} (k(t_n,s)g(s))_{ij}x_n(s)ds =$$

$$\int_{t_o}^{t_n} |(k(t_n,s)g(s))_{ij}|ds - \int_{t_o}^{t_{n-1}} |(k(t_n,s)g(s))_{ij}ds > 2d.$$

Using Lusin's Theorem on the structure of measurable functions and the Lebesgue Dominated Convergence Theorem, we can replace each $x_n(t)$ by a continuous function $\tilde{x}_n(t)$ such that $|\tilde{x}_n(t)| \leq 1$ on $[t_{n-1},t_n)$, $\lim_{t\to t_n^-}\tilde{x}_n(t) = \tilde{x}_{n+1}(t_n)$ and

$$(4.14) \qquad (-1)^{n-1} \int_{t_{n-1}}^{t_n} (k(t_n,s)g(s))_{ij}\tilde{x}_n(s)\,ds > 2d.$$

Define $x(t)$ on $[0,\infty)$ by $x(t) = \tilde{x}_1(t_o)$ for $0 \leq t \leq t_o$ and $x(t) = \tilde{x}_n(t)$ for t belonging to $[t_{n-1}, t_n)$. Since $x(t)$ is continuous and uniformly bounded then $g(t)(x(t)e_j)$ belongs to C_g.

Now consider

$$\int_o^t k(t,s)g(s)\,(x(s)e_j)\,ds$$
$$= \int_o^{t_o} k(t,s)g(s)\,(x(s)e_j)\,ds + \int_{t_o}^{t} k(t,s)g(s)\,(x(s)e_j)\,ds.$$

Since $k(t,s) \to k(s)$ as $t \to \infty$ uniformly on compact intervals then

$$\lim_{t\to\infty} \int_o^{t_o} k(t,s)g(s)\,(x(s)e_j)\,ds = \int_o^{t_o} k(s)g(s)\,(x(s)e_j)\,ds.$$

Using (4.14) we have

$$(-1)^{n-1}\int_{t_o}^{t_n}(k(t_n,s)g(s))_{ij}x(s)\,ds=(-1)^{n-1}\int_{t_o}^{t_{n-1}}(k(t_n,s)g(s))_{ij}x(s)\,ds$$

$$+\ (-1)^{n-1}\int_{t_{n-1}}^{t_n}(k(t_n,s)g(s))_{ij}x(s)\,ds > d.$$

Hence

$$\int_{t_o}^{t_n}(k(t_n,s)g(s))_{ij}x(s)\,ds$$

is greater than d for $n = 2p + 1$ and is less than $-d$ for $n = 2p$. So

$$\lim_{t\to\infty}\int_{t_o}^{t}k(t,s)g(s)(x(s)e_j)\,ds$$

doesn't exist. This is a contradiction. Thus $L_2 = L_3$, $L_1 = L^1$ and condition ii) follows.

<u>Corollary 4.1</u>. (C_g, C_o) is admissible with respect to the operator K if and only if

$$\lim_{t\to\infty}\int_{o}^{t}|k(t,s)g(s)|\,ds = 0 \tag{4.15}$$

<u>Proof</u>: Since $C_o \subset C_\ell$, assuming admissibility it follows that conditions i) and ii) of Theorem 4.2 are true. Hence from the proof of Theorem 4.2 if $x(s)$ is in C_g then

$$(4.16) \qquad \lim_{t\to\infty} \int_o^t k(t,s)x(s)\,ds = \int_o^\infty k(s)x(s)\,ds$$

$$= \int_o^\infty k(s)g(s)v(s)\,ds$$

$$= 0$$

Since v can be any continuous bounded function it follows that $k(s)g(s) = 0$ for all non-negative s. Thus by condition ii) of Theorem 4.2

$$\lim_{t\to\infty} \int_o^t |k(t,s)g(s)|\,ds = 0.$$

Now assume (4.15) is true. For a fixed $t_1 > 0$

$$\lim_{t\to\infty} \int_o^t |k(t,s)g(s)|\,ds$$

$$= \lim_{t\to\infty}\left(\int_o^{t_1} |k(t,s)g(s)|\,ds + \int_{t_1}^t |k(t,s)g(s)|\,ds\right) = 0$$

Thus

$$\lim_{t\to\infty} \int_0^{t_1} |k(t,s)g(s)|ds = \int_0^{t_1} |k(s)g(s)|ds$$

$$= 0$$

Since t_1 is an arbitrary fixed value.

$$\int_0^{\infty} |k(s)g(s)|ds = 0.$$

So from Theorem 4.2, K is admissible with respect to (C_g, C_ℓ). Since

$$\lim_{t\to\infty} |Kx(t)| \leq \lim_{t\to\infty} \int_0^{t} |k(t,s)g(s)| \; \|v(s)\| ds$$

$$= 0$$

it follows that K is admissible with respect to (C_g, C_o).

C. Admissibility of (L_g^p, C_G)

Now let $g(t)$ be a measurable $n \times n$ matrix valued function defined on R. Define $P(t)$, $g_{-1}(t)$ as in the definition of C_g and let $p \geq 1$. We let $L^p \equiv L^p(R,R^n)$ $p \geq 1$, stand for the set of measurable functions $h : R \to R^n$ such that

$$\|h\|_p = \left(\int_R |h(t)|^p dt \right)^{1/p} < \infty.$$

As usual functions are identified if they are equal a.c.

Definition 4.4. We say a function x from R to R^n belongs to L_g^p if and only if

i) x is measurable on R;

ii) $P(t)x(t) = x(t)$ a.e. on R;

iii) $g_{-1}(t)x(t)$ belongs to $L^p(R,R^n)$

Lemma 4.1. L_g^p is a Banach space under the norm

$$(4.17) \qquad \|x(t)\|_g \equiv \left(\int_R |g_{-1}(t)x(t)|^p dt \right)^{1/p}.$$

Proof: Gollwitzer [4] has shown that $g_{-1}(t)x(t)$ is measurable when g and x are measurable. If x belongs to L_g^p and $\|x(t)\|_g = 0$ then $g_{-1}(t)x(t) = 0$ a.e. Since $x(t)$ belongs to the range of $g(t)$ a.e. it follows that $x(t) = 0$ a.e. Clearly

$$\|x(t) + y(t)\|_g \leq \|x(t)\|_g + \|y(t)\|_g$$

and

$$\|\alpha\, x(t)\|_g \leq |\alpha| \;\|x(t)\|_g$$

for x and y in L_g^p and for any α belonging to R. Thus $\|\cdot\|_g$ is a norm on L_g^p.

Let $\{x_n\}$ be a Cauchy sequence in L_g^p. Then $v_n(t) = g_{-1}(t)x_n(t)$ is a Cauchy sequence in L^p. Thus there exists a $v(t)$ in L^p such that $\{v_n(t)\}$ has $v(t)$ as a limit in L^p. We will show that $x_n(t)$ has $g(t)v(t)$ as a limit in L_g^p. Note that $v_n(t)$ belongs to $N^{\perp}(t)$ a.e. for $n \geq 1$. Since $N^{\perp}(t)$ is closed $v(t)$ belongs to $N^{\perp}(t)$ a.e. Thus $g(t)v(t)$ belongs to L_g^p. Now $\|g_{-1}(t)(x_n(t) - g(t)v(t))\|_p = \|v_n(t) - v(t)\|_p$, which has zero as its limit as n approaches ∞. Therefore

$\lim\|x_n(t) - g(t)v(t)\|_g = 0$ as n approaches ∞. Therefore L_g^p is a Banach space.

In the next theorem we assume that $k(t,s)$ is a measurable $n \times n$ matrix valued function on R^2 and that the operator K, defined by

$$(4.18) \qquad Kx(t) = \int_R k(t,s)x(s)ds,$$

has the property defined in the following:

<u>Definition 4.5</u>. The operator K is said to have the property $B_{p,g}$, iff for any bounded set S in L_g^p, the set KS is equicontinuous at any t_o in R.

<u>Theorem 4.3</u>. Let $g(t)$ be a measurable $n \times n$ matrix valued function defined on R which is invertible for each t belonging to R and choose p, q such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then (L_g^p, C_G) is admissible with respect to K, the property $B_{p,g}$ holds, and K is a continuous operator on L_g^p if and only if for all t

i) $P(t)k(t,s)g(s) = k(t,s)g(s)$ a.e. in s,

ii) $\int_R |G_{-1}(t)k(t,s)g(s)|^q ds < A$ for all t in R;

where A depends on G, k, and g;

iii) $\lim_{t \to t_o} \int_R |[k(t,s) - k(t_o,s)]g(s)|^q ds = 0$ for each t_o in R.

(Note: In i) $P(t)$ is the projection operator associated with $G(t)$.)

Proof: First we show that conditions i), ii) and iii) are sufficient. Let x belong to L_g^p and set $v(t) = g_{-1}(t)x(t)$. Using Holder's inequality and condition ii) it follows that

$$(4.19) \qquad \|G_{-1}(t) \int_R k(t,s)x(s)ds\| \leq A^{1/q}\|x(s)\|_g$$

for each t in R. Also

$$\int_R |[k(t,s) - k(t_o,s)]x(s)|ds \leq$$

$$\left(\int_R |[k(t,s) - k(t_o,s)]g(s)|^q ds\right)^{1/q} \|x\|_g$$

so using condition iii) it follows that the function $Kx(t)$ is continuous. Now

$$P(t) \int_R k(t,s)x(s)\,ds = \int_R P(t)k(t,s)g(s)v(s)\,ds$$

$$= \int_R k(t,s)g(s)v(s)\,ds$$

$$= \int_R k(t,s)x(s)\,ds.$$

The above comments show that K is a continuous operator from L_g^p to C_G.

If S is a bounded set in L_g^p (i.e. there exists an $r > 0$ such that if x is in S then $\|x\|_g < r$), then

(4.20)

$$\int_R |[k(t,s)-k(t_o,s)]x(s)|\,ds \leq \left(\int_R |[k(t,s)-k(t_o,s)]g(s)|^q\right)^{1/q} r$$

so KS is equicontinuous at t_o.

We now show that the conditions are necessary. Let u be in L_g^p and set $v(t) = g_{-1}(t)u(t)$. Then Ku belongs to C_G, so

$$(I-P(t)) \int_R k(t,s)g(s)v(s)\,ds = 0 \quad \text{a.e.}$$

where I is the identity matrix. Since u is arbitrary in L_g^p, $(I - P(t))k(t,s) = 0$ a.e. for a fixed t. Thus i) holds.

To show that

$$\int_R |G_{-1}(t)k(t,s)g(s)|^q ds < A$$

for all t in R and some positive A, we show that each entry in the matrix representation of $G_{-1}(t)k(t,s)g(s)$ is in L^q. Let y be in $L^p(R,R)$, then $g(t)(y(t)e_j)$ is in $L_g^p(R,R^n)$ and $Kg(t)(y(t)e_j)$ is in C_G. Thus

$$\sup\{|\int_R (G_{-1}(t)k(t,s)g(s))_{ij}y(s)\,ds| : t \text{ in } R\} < B_y$$

for $i = 1,2,\ldots,n$ and some constant B_y depending on G, k, g and y. For a fixed t define

$$(4.21) \quad L(y(\cdot),t) = \int_R (G_{-1}(t)k(t,s)g(s))_{ij}y(s)\,ds.$$

Then for each y in $L^p(R,R)$, $|L(y(\cdot);t)| < B_y$ for all t in R. Since K is a continuous operator, $L(\cdot,t)$ is a continuous functional on $L^p(R,R)$ for each t. If we set

$\|L(\cdot,t)\| = \sup\{|L(y,t)| : y \in L^p(R,R,), \|y\| = 1\}$ then by the Uniform Boundedness Theorem there exists an $M > 0$ such that

$$\sup\{\|L(\cdot,t)\| : t \text{ in } R\} \leq M.$$

It follows by the Riesz Representation Theorem (see page 213 of H. L. Royden [10]) that

$$\int_R |(G_{-1}(t)k(t,s)g(s))_{ij}|^q ds \leq M^q$$

for all t in R. Hence condition ii) holds.

To show that condition iii) holds, we let S be the unit sphere in $L^p(R,R^n)$. Then $g(t)S$ is the unit sphere in L_g^p. For a fixed $\epsilon > 0$ and t_o in R, property $B_{p,q}$ implies the existence of a $\delta = \delta(\epsilon,t_o)$ such that

$$|t - t_o| < \delta \quad \text{implies} \quad \left|\int_R [(k(t,s) - k(t_o,s)]g(s)x(s)ds\right| < \epsilon$$

for all x belonging to S. Hence for an x in $L^p(R,R^n)$ and for a fixed t such that $|t - t_o| < \delta$, we have

$$\left|\int_R [k(t,s) - k(t_o,s)]g(s)x(s)ds\right| < \epsilon\|x\|_p$$

Thus for t, t_o fixed as above

$$(4.22)\quad T(t,t_o)x(\cdot) \equiv \int_R [k(t,s) - k(t_o,s)]g(s)x(s)\,ds$$

defines a continuous operator whose norm satisfies the relation

$$\|T(t,t_o)\| = \left(\int_R |[k(t,s) - k(t_o,s)]g(s)|^q ds\right)^{1/q} \leq \epsilon.$$

Condition iii) follows from this result.

Remark: When $p = 1$ the theorem is true with $\text{ess sup}\{|h(t,s)| : s \text{ in } R\}$ used in place of $\int_R |h(t,s)|^q ds$. The proof of this result is essentially the same as the proof given above.

Definition 4.6. Let $G(t)$ be a linear transformation on R^n which is measurable on R. Let $P(t)$ be defined as the orthogonal projection of R^n onto the range of $G(t)$. A measurable function from R to R^n is said to be in $L_G^\infty(R,R^n)$ $(= L_G^\infty)$ if $P(t)u(t) = u(t)$ a.e. on R and

$$(4.23)\quad \|u\|_{G\infty} = \text{ess sup}\{\|G_{-1}(t)u(t)\| : t \text{ in } R\} \text{ is finite.}$$

The following result is proved in the same manner as the corresponding theorem in Corduneanu [2] (Theorem 2.6.1).

Theorem 4.4. Let $k(t,s)$ be an $n \times n$ measurable matrix valued function on R^2. The pair (L_g^∞, L_G^∞) is admissible with respect to the operator $Kx(t) = \int_R k(t,s)x(s)ds$ if and only if

$$\int_R |G_{-1}(t)k(t,s)g(s)|ds \leq M$$

a.e. on R for some fixed M depending on G, k, and g.

D. Applications

The following two theorems are generalizations of results found in Corduneanu [2]. We consider the equation

$$(4.24) \qquad x(t) = h(t) + \int_0^t k(t,s)f(s,x(\cdot))ds$$

where $k(t,s)$ is a continuous $n \times n$ matrix valued function for $0 \leq s \leq t < \infty$ and $k(t,s) = 0$ for $s > t$. Also assume that $\lim k(t,s) = k(s)$ as t approaches ∞, where the convergence is uniform on compact sets of $[0,\infty)$.

Theorem 4.5. Consider (4.24) and assume

i) $h(t)$ belongs to C_ℓ

ii) (C_g, C_ℓ) is admissible with respect to the operator K which is defined by

$$Kx(t) = \int_0^t k(t,s)x(s)\,ds$$

iii) $f(t,x(\cdot))\ (= fx)$ defines a mapping from C_ℓ to C_g such that $|fx - fy|_g \leq \lambda |x-y|_\ell$ for any x, y in C_ℓ.

Then there exists a unique solution of (4.1) in C_ℓ whenever λ is sufficiently small.

Proof: The operator

$$Tx(t) = h(t) + \int_0^t k(t,s)f(s,x(\cdot))\,ds$$

maps C_ℓ to C_ℓ and the theorem follows by noting that T is a contraction mapping when λ is small enough.

In the next theorem, we assume that $k(t,s)$ of (4.24) is a continuous matrix valued function and that $P(t)$ is the orthogonal projection of R^n onto the range of $G(t)$ for each t.

<u>Theorem 4.6</u>. Consider (4.24) and assume

i) $h(t)$ belongs to C_G,

ii) $\int_0^t |G_{-1}(t)k(t,s)g(s)|ds < A$ for all t in R_+ where A is some fixed positive number depending on G_{-1}, k and g,

iii) $P(t)k(t,s)g(s) = k(t,s)g(s)$ for $0 \leq s \leq t < \infty$,

iv) $f(t,x(\cdot))$ $(= f\,x)$ is a mapping from C_G to C_ℓ such that $|fx - fy|_g \leq \lambda|x - y|_G$ for all x, y in C_G.

Then, if λ is sufficiently small, there exists a unique solution of (4.1) in C_G.

<u>Proof</u>: Gollwitzer [4] shows that condition ii) and iii) imply that (C_g, C_G) is admissible with respect to the operator K as defined by

$$K\,x(t) = \int_0^t k(t,s)\,x(s)\,ds.$$

Thus if x belongs to C_G then $T\,x$ defined by

$$Tx(t) = h(t) + \int_0^t k(t,s) f(s,x(\cdot))\,ds$$

is a function from C_G to C_G. Now for x and y in C_G.

$$|Tx - Ty|_G \leq A|fx - fy|_g$$

$$\leq A\lambda|x - y|_G.$$

If we set $\lambda < A^{-1}$ then T is a contraction map and the theorem follows.

Next we consider

$$(4.25)\quad x'(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds + f(t,x(\cdot)),$$

$$x(0) = x_o$$

where A is a constant $n \times n$ matrix, f is a map from $LL^1(R_+)$ to $LL^1(R_+)$, $B(t)$ is a matrix valued function which is L^1 on R_+ and

$$\det(sI - A - B^*(s)) \neq 0 \quad \text{for} \quad \text{Re } s \geq 0.$$

Here $B^*(s)$ is the Laplace transform of B and $LL^1(R_+)$

is the set of measurable functions $h : R_+ \to R^n$ such that for each compact subset I of R_+

$$\int_I |h(t)|\,dt < \infty$$

Definition 4.7. The resolvent $R(t)$ associated with (4.25) is the unique solution of the matrix equation

$$R^1(t) = A\,R(t) + \int_0^t B(t-s)R(s)\,ds, \quad R(0) = I$$

where I is the identity matrix. It was shown by Grossman and Miller [5] that the solution of (4.25) can be written in the form

$$x(t) = R(t)x_o + \int_0^t R(t-s)f(s,x(\cdot))\,ds$$

where $R(t)$ is the resolvent defined above. Under the given conditions on A and $B(t)$, Miller [7] shows that

$$R(t) \to 0 \quad \text{as} \quad t \to \infty$$

and that $R(t)$ belongs to $L^p(0,\infty)$ for all p in $[1,\infty)$.

Suppose the function f in (4.25) is of the form

$$f(t,x(\cdot)) = g(t)x(t) \tag{4.26}$$

where $g(t)$ is a continuous matrix valued function which is in $L^1(0,\infty)$. If $x(t)$ is continuous and uniformly bounded on R_+ then $g(t)x(t)$ is in C_g. On applying Corollary 5.2, page 167, of Miller [9], we show that

$$h(t) = \int_0^t |R(t-s)g(s)|ds$$

is in $LL^1(R_+)$ and that for any $t > 0$

$$\int_0^T |h(s)|ds \leq \left(\int_0^\infty |R(s)|ds\right)\left(\int_0^\infty |g(s)|ds\right) < \infty.$$

Lemma 2.4 implies that $\lim h(t) = 0$ at $t \to \infty$. Thus by Corollary 4.1 the operator K defined by

$$Kx(t) = R(t)x_o + \int_0^t R(t-s)g(s)x(s)ds$$

maps C_ℓ into $C_o \subset C_\ell$. Now for any pair x, y in C_ℓ

$$|Kx - Ky| = \sup\left\{\left|\int_0^t R(t-s)g(s)[x(s)-y(s)]ds\right| : t \text{ in } R_+\right\}$$

$$\leq \sup\left\{\int_0^t | (t-s)g(s)[x(s)-y(s)]|ds : t \text{ in } R_+\right\}$$

$$\leq \sup\left\{\int_0^t |R(t-s)|ds : t \text{ in } R_+\right\} \|g(t)\| \; |x-y|_\ell$$

$$\leq \left(\int_0^\infty |R(u)|du\right) \|g\| \; |x-y|_\ell$$

Thus if $\|g\| = \sup\{|g(t)| : t \text{ in } R_+\}$ is small enough, K is a contraction map.

From the preceding remarks we have the following theorem.

Theorem 4.7. Let A be a constant $n \times n$ matrix, $B(t)$ a matrix valued function which is in $L^1(0,\infty)$ and

$$\det(sI - A - B^*(s)) \neq 0 \quad \text{for} \quad \text{Re } s \geq 0.$$

Let $g(t)$ be a continuous matrix valued function in $L^1(0,\infty)$ and in (4.25) let $f(t,x(\cdot)) = g(t)x(t)$. Then (4.25) has a unique solution in C_o if $\|g\|$ is small enough.

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