1974

Theory and applications of concomitants of order statistics

Martin James O'Connell
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Theory and applications of concomitants of order statistics

by

Martin James O'Connell

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

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In Charge of Major Work

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I. INTRODUCTION AND REVIEW OF THE LITERATURE

A. Statement of the Problem

Let \((X_i, Y_i)\) \(i = 1, \ldots, n\) be a random sample of \(n\) pairs drawn from a bivariate distribution. Arranging the \(X\)'s in ascending order

\[
X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}
\]

we denote the corresponding \(Y\)'s (not necessarily in ascending order) by

\[
Y_{[1:n]}, \ Y_{[2:n]}, \cdots, Y_{[n:n]}
\]

and call them concomitants of order statistics.

We assume that we can write

\[
Y_i = \mu_Y + \beta(X_i - \mu_X) + Z_i \quad (i = 1, \ldots, n)
\]

where the \(X\)'s and \(Z\)'s are \(2n\) mutually independent random variables; also \(\beta = \rho \sigma_Y / \sigma_X\), where \(\rho\) is the correlation coefficient of \(X\) and \(Y\); \(\mu_X, \mu_Y, \sigma_X^2\) and \(\sigma_Y^2\) are the respective means and variances of \(X\) and \(Y\). If the \((X_i, Y_i)\) \(i = 1, \ldots, n\), have a bivariate normal \(N(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)\) distribution, (1.1) and its associated assumptions follow immediately, with \(Z_i\) also normal.

Since the \(X\) and \(Z\) variates are independent, the \(X\)'s can be ordered without affecting the distributional properties of the \(Z\)'s and so we have
where $Z_{[r]}$ is that $Z_i$ accompanying $X_{r:n}$. These $Z_{[r]}$'s are also mutually independent and are independent of the $X_{r:n}$.

In this thesis we will be using the concomitants as represented by (1.2) in considering two problems. The first problem to be investigated will be that of finding the distribution of the concomitants and of their ranks. The second problem considered will be that of using the concomitants in the estimation of $\mu_y$ in a double sampling situation.

B. Distributional Properties

1. Introduction

The concomitants $Y_{[r:n]}$ are of primary interest in selection problems, where the selection is based on the ranks of the $X$'s. The $k$ ($< n$) highest $X$-scores may be chosen and we wish to know something about the concomitant $Y$-scores. David and Galambos (1974) suggest that the $X$'s may refer to a first test, and the $Y$'s to a later test, or the $X$'s might refer to a characteristic in a parent and the $Y$'s to the same characteristic in the offspring. Again, David and Galambos (1974) have suggested that the $Y_{[r:n]}$ can be of interest in an 'errors in variable' problem. That is, one may wish to select on the basis of true values $T$, but actually must select on the basis of observed values $V$, where $V = T + E$ and $T, E$ are independent variates. Here, with all variates
normally distributed, (1.1) applies if we let \( X = V \), \( Y = T \) and
\[
\rho = \rho(V, T) = \frac{\sigma_t}{(\sigma_t^2 + \sigma_e^2)^{1/2}}.
\]
A more complex situation, described by Gross (1973), illustrates an area in which concomitants play an important part. A college counselor develops a least squares regression model from a sample of currently enrolled students to predict the performance of individuals applying for future admission. Here the \( X \)-variate is the predictive score of a prospective student, and the \( Y \)-variate his score based on actual performance in college.

From these examples it is easily seen that the distribution of the concomitants is of interest. The ranks of the concomitants are also important. In the example described by Gross (1973) it is easily seen that the expected rank of future students would be of interest to the counselor, as would the probability of a student attaining a particular rank. In the 'errors in variable' problem, it might be the ranks of the true values which are of interest, while the actual ranking is done on the observed values.

In this dissertation we have confined ourselves to a finite sample, and have developed expressions for the exact distribution of the concomitants, and the exact distribution of the ranks. These distributions are shown in Chapter II.

2. Review

If (1.1) and its associated assumptions hold, then representing the concomitants in the form of (1.2) leads directly to the following
results given by Watterson (1959)

\[
\begin{align*}
E(Y_{[r:n]}) & = \mu_y + \rho \frac{\sigma_y}{\sigma_x} \mu_{r:n} \\
\text{Var}(Y_{[r:n]}) & = (1-\rho^2)\sigma_y^2 + \rho^2 \frac{\sigma_y^2}{\sigma_x^2} \sigma_{r:n} \\
\text{Cov}(X_{[r:n]}, Y_{[s:n]}) & = \rho \frac{\sigma_x}{\sigma_y} \sigma_{y,rs:n} \\
\text{Cov}(Y_{[r:n]}, Y_{[s:n]}) & = \rho^2 \frac{\sigma_y^2}{\sigma_x^2} \sigma_{rs:n}, \ r \neq s
\end{align*}
\]

(1.3)

Letting \( U = (X - \mu_x)/\sigma_x \), we define

\[
\begin{align*}
\mu_{r:n} & = E(U_{r:n}) \\
\sigma_{r:n}^2 & = \text{Var}(U_{r:n}) \\
\sigma_{rs:n} & = \text{Cov}(U_{r:n}, U_{s:n})
\end{align*}
\]

Watterson (1959) obtained these results for a bivariate normal population, using conditioning arguments; but it is easily shown from (1.2) that they hold whenever (1.1) and its associated assumptions are valid.

David and Galambos (1974) have found the asymptotic distribution of the concomitants and of their ranks in the case when the paired observations have a bivariate normal distribution. They have shown that the \( Y^*_{[r:n]} \) are independent, normally \( \text{N}(0, \frac{\sigma_y^2}{\sigma_x^2}(1-\rho^2)) \) distributed
random variables, where $Y_{[r:n]} = Y_{[r:n]} - E(Y_{[r:n]})$. More generally, if (1.2) holds and if $\lim_{n \to \infty} \sigma^2_{r:n} = 0$, they point out that the $Y_{[r:n]}$ will be asymptotically independent variates having the common distribution of $Z$.

Letting $R_{t,n}$ denote $\text{Rank}(Y_{[t:n]})$ and with $(X, Y)$ bivariate normal, David and Galambos (1974) have proved the following theorem concerning the ranks of the concomitants.

**Theorem:** Let $\{t\}$ be a sequence of integers such that, as $n \to \infty$, $t/n \to \lambda$, $0 < \lambda < 1$. Then for fixed $k \geq 1$

$$\lim_{n \to \infty} E[(R_{t,n}/n)^k] = \int_{-\infty}^{\infty} G^k(u) \, d\bar{F}(u),$$

where

$$G(u) = \int_0^1 \bar{F}[u + (1-\rho^2)^{-1/2} \rho[\Phi^{-1}(\lambda) - \Phi^{-1}(s)]] \, ds$$

and $\bar{F}(x)$ is the standard normal c.d.f. Or, equivalently

$$\lim_{n \to \infty} \Pr(R_{t,n} \leq nz) = \bar{F}(G^{-1}(z)).$$

In particular

$$\lim_{n \to \infty} E(R_{t,n}/n) = \int_0^1 \bar{F}[2(1-\rho^2)^{-1/2} \rho[\Phi^{-1}(\lambda) - \Phi^{-1}(s)]] \, ds. \tag{1.4}$$

David (1973) has evaluated this function for various values of $\rho$ and $\lambda$. In this dissertation we have made a comparison of the actual expectation with the asymptotic for various finite values of $n$. 
C. Concomitants as Estimators of Parameters

1. Introduction

Using concomitants can be quite useful in cases in which observations are taken on paired random variables, one of which is relatively cheap to measure, while the other is expensive. If \( Y \) is the expensive variable, \( X \) the cheap variable, and a random sample of size \( n \) is to be taken, it is natural in this double sampling situation to measure \( X \) on all members of the sample, but \( Y \) on only a relatively small number \( k \), and to try and use our knowledge of all the \( x \)'s to obtain an improved estimator of \( \mu_y \). In this vein we will develop a regression estimator of \( \mu_y \) using the concomitants. Further, we will develop estimators which only make use of a knowledge of the order relationship among the \( x \)'s. For example, if one wished to estimate the average height of a stand of trees, one might visually order the heights and then cut down a few trees to get an exact measurement. The problem then is to choose the trees to be felled so as to get the best possible estimate of the mean height.

2. Review

Cohen (1957) finds maximum likelihood estimators for a multivariate normal distribution subject to a general set of restrictions. As examples, he considers truncated and censored samples, and selected samples. That is, if \( N \) observations are available for one variable and \( n < N \) for the remaining variables, Cohen finds the maximum likelihood
estimators of the parameters. The bivariate situation, which is a special case of the above, is considered by Cohen (1955).

Concomitants of order statistics have been used previously in estimation of parameters by Watterson (1959). Watterson assumed a multivariate normal population and ordered one of the variables. He found estimators which were linear combinations of the variates in three situations,

a) when censoring was done on the ordered variate
b) when censoring was done on the concomitant variates
c) when censoring was done on both the ordered and concomitant variates.

From (1.3) we know some moments of the concomitants. In situations (b) and (c) Watterson used linear combinations of the concomitants to obtain estimators for $\mu_y$ and $\rho^2 \sigma_y^2$, (using the notation of (1.1)). He looked at the problem of finding the best estimator of $\mu_y$ when some values are missing. On the other hand, in this thesis we examine the problem of selecting the $k$ (< n) concomitants which, for various classes of estimators, will give the best estimator of $\mu_y$.

When $(X_i, Y_i)$ $i = 1, \ldots, n$ form a set of independent paired random samples and $E(Y|X) = \alpha + \beta x$, Barton and Casley (1958) made use of concomitants of order statistics to get a quick estimator of $\beta$. They defined

$$B' = \frac{\bar{Y}_{[k]} - \bar{Y}}{\bar{X}_{[k]} - \bar{X}}$$
where

\[ \overline{Y}_{(k)} = \frac{1}{k} \sum_{i=1}^{k} Y_{[n-i+1:n]} \]

\[ \overline{Y}_{(k)} = \frac{1}{k} \sum_{i=1}^{k} Y_{[i:n]} \]

\[ \overline{X}_{(k)} = \frac{1}{k} \sum_{i=1}^{k} X_{n-i+1:n} \]

They showed that $B'$ is an unbiased estimator of $\beta$ and that it has an efficiency of 75-80% when $(X, Y)$ are bivariate normal, provided $k$ is chosen as about $.27n$.

Tsukibayashi (1962) suggested an estimator for $\rho$ of the form

\[ \hat{\rho}' = B' \frac{(\overline{X}_{(k)} - \overline{Y}_{(k)})/c_{n,x}}{(\overline{Y}_{(k)} - \overline{Y}_{(k)})/c_{n,y}} = \frac{(\overline{Y}_{(k)} - \overline{Y}_{[k]})/c_{n,x}}{(\overline{Y}_{(k)} - \overline{Y}_{(k)})/c_{n,y}} \]

where $c_{n,x} = E [\overline{X}_{(k)} - \overline{X}_{(k)}/\sigma_x$, etc. When $X$ and $Y$ have the same marginal distribution then

\[ \hat{\rho}' = \frac{(\overline{Y}_{(k)} - \overline{Y}_{[k]})}{(\overline{Y}_{(k)} - \overline{Y}_{(k)})} \]

Concomitants are also used in the method of 'ranked set sampling'
due to McIntyre (1952) and further considered by Dell and Clutter (1972). In this method we assume that the sample size \( n = k^2 \) (or \( n = I k^2 \) where \( I \) is a positive integer). The sample is randomly subdivided into \( k \) subsamples of size \( k \). The \( X \)'s are ranked in each subsample and in the \( i^{th} \) (\( i = 1, \ldots, k \)) subsample only the concomitant \( Y_{[i:k]}^{(i)} \) of the \( i^{th} \) order statistic \( X_{[i:k]}^{(i)} \) is measured, where the superscript denotes the subsample. Their estimator of \( \mu_y \) is

\[
\hat{Y}_{[k]} = \frac{1}{k} \sum_{i=1}^{k} Y_{[i:k]}^{(i)},
\]

and \( \hat{Y}_{[k]} \) is always an unbiased estimator of \( \mu_y \). Comparisons will be made in Chapter III between this estimator and the estimators of \( \mu_y \) developed in this thesis.
II. DISTRIBUTION OF THE CONCOMITANTS AND OF THE RANKS OF CONCOMITANTS OF ORDER STATISTICS

A. Introduction

Let \((X_i, Y_i)\) for \(i = 1, \ldots, n\) be a paired random sample from some bivariate distribution. If we let \(X_{1:n}, \ldots, X_{n:n}\) be the set of ordered \(X\)-variates, then associated with each ordered variable \(X_{r:n}\) is a concomitant \(Y_{[r:n]}\). The main emphasis in this chapter is on obtaining the distribution of the ranks of these concomitants of the order statistics when \(X\) and \(Y\) are linearly related, apart from an error term. For the special case of paired random variables \((X_i, Y_i)\) from a bivariate normal population, some numerical results are given.

Preceding these main results some general expressions are found for the joint density of \(k\) unordered variates and one ordered random variable.

Under the assumptions made in this chapter, (1.3) holds and therefore we know the mean, variance and covariance of the concomitants without having to obtain the exact distribution function. Since the distribution function of \(Y_{[r:n]}\), and the joint distribution of \(k\) concomitants are of interest in themselves, an expression for them is found in Section E.

B. Joint Distribution of Ordered and Unordered Random Variables

In this section, we will develop an expression for the joint density function of unordered and ordered random variables, subject to certain
restrictions. This development will be of some use in the investigation of the distribution of the ranks of the concomitant random variables, but also has an intrinsic interest.

Let the set of random variables $X_1, \ldots, X_n$ be independent, identically distributed and absolutely continuous. Denote the density function of each by $p(x)$ and the cumulative distribution function by $P(x)$. Let $X_{1:n}, \ldots, X_{n:n}$ be the ordered set of these variables and define the random variable $T_i$ as

$$T_i = X_i - X_{r:n} \quad (i = 1, \ldots, n, \; 1 \leq r \leq n).$$

The density functions of primary interest are

a) $f_{X_1, X_{r:n} | T_n}(x_1, x | 0) \quad (1 \leq r \leq n)$

and

b) $f_{X_1, \ldots, X_{n-1}, X_{r:n} | T_n}(x_1, \ldots, x_{n-1}, x | 0) \quad (1 \leq r \leq n).$

The density noted in (a) is the joint density of $X_1$ and $X_{r:n}$ given that $x_n$ is the $r$\textsuperscript{th} largest value in the set $x_1, \ldots, x_n$. The density shown in (b) is the joint density of $n-1$ unordered random variables and one ordered variable given that the value of the remaining unordered variable is the $r$\textsuperscript{th} largest. Clearly the special role of subscripts 1 and $n$ is for convenience only since $X_1, \ldots, X_n$ are identically distributed. Therefore, the density in (a) is the same for an arbitrary choice of $X_i, X_j$ ($i \neq j$) in place of $X_1, X_n$; and the density function
in (b) remains the same for an arbitrary choice of \( X_i \) in place of \( X_n \).

As a further convenience we will define

\[
f_{j:k}(x_{j+1}, \ldots, x_k | x_1, \ldots, x_j, x, 0)
= f_{X_{j+1}, \ldots, X_k | X_1, \ldots, X_j, X_{r:n} \mid T_n}(x_{j+1}, \ldots, x_k | x_1, \ldots, x_j, x, 0)
\]

\[(k = 1, \ldots, n-1; \ j = 0, \ldots, k-1)\]

and

\[
f_k(x_1, \ldots, x_k, x | 0) = f_{X_{1}, \ldots, X_k, X_{r:n} \mid T_n}(x_1, \ldots, x_k, x | 0)
\]

\[(k = 1, \ldots, n-1)\]

To find \( f_1(x_1, x | 0) \) we condition further and express it as

\[
f_1(x_1, x | 0) = f_{01}(x_1 | x, 0) f_{X_{r:n} \mid T_n}(x | 0)
\]

As is well-known, e.g., David (1970, p. 8) we have

\[
f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!} \cdot p(x)^{r-1} (1-p(x))^{n-r} p(x) \quad -\infty < x < \infty
\]

(2.1)

The joint density of \( X_n, X_{r:n} \) can be written

\[
f_{X_{r:n}, X_n}(x, x_n) = f_{X_{r:n} \mid X_n}(x | x_n) f_{X_n}(x_n)
\].
It must be noted at this point that the joint distribution possesses a singularity in that

\[ Pr(X_n = X_{r:n}) = \frac{1}{n} \]

and

\[ Pr(X_{r:n} = X_n|X_n = x) = \left( \frac{n-1}{r-1} \right) P(x)^{r-1} (1-P(x))^{n-r} \]

Now

\[ f_{X_n}(x_n) = p(x_n) \]

and

\[ f_{X_{r:n}|X_n}(x|x_n) = f_{X_{r-1:n-1}}(x) \quad \text{if} \quad x > x_n \]

\[ = \left( \frac{n-1}{r-1} \right) P(x)^{r-1} (1-P(x))^{n-r} \quad x = x_n \]

\[ = f_{X_{r:n-1}}(x) \quad x < x_n . \]

Transforming the variables to \( T_n \) and \( X_{r:n} \) where \( T_n = X_n - X_{r:n} \) we see

\[ f_{X_{r:n}, T_n}(x, t_n) = f_{X_{r-1:n-1}}(x) p(x+t_n) \quad t_n < 0 \]

\[ = \left( \frac{n-1}{r-1} \right) P(x)^{r-1}(1-P(x))^{n-r} p(x+t_n) \quad t_n = 0 \]

\[ = f_{X_{r:n-1}}(x) p(x+t_n) \quad t_n > 0 . \]
Integrating over \( x \) we can see that, as expected since the distribution has a singularity at zero

\[
    f_{T_n}(0) = \Pr(T_n = 0) = \frac{1}{n}.
\]

Further

\[
    f_{X_{r:n}, T_n}(x, 0) = \binom{n-1}{r-1} P(x)^{r-1} (1-P(x))^{n-r} p(x).
\]

Since

\[
    f_{X_{r:n} | T_n} (x | 0) = \frac{f_{X_{r:n}, T_n}(x, 0)}{f_{T_n}(0)}
\]

it follows that

\[
    f_{X_{r:n} | T_n} (x | 0) = \frac{\binom{n-1}{r-1} P(x)^{r-1} (1-P(x))^{n-r} p(x)}{(r-1)! (n-r)!}.
\]

\[\text{\( -\infty < x < \infty \).} \quad (2.2)\]

We can see immediately that \( f_{X_{r:n} | T_n} (x | 0) = f_{X_{r:n}} (x) \). This is not surprising, since the density \( f_{X_{r:n}} (x) \) must be the same no matter which \( X_i = X_{r:n} \) (\( i = 1, \ldots, n \)).

To find \( f_{01}(x_1 | x, 0) \) we look first at the associated c.d.f. \( F_{01}(x_1 | x, 0) \). To evaluate \( F_{01}(x_1 | x, 0) = \Pr(X_1 \leq x_1 | X_n = X_{r:n} = x) \) we consider

\[
    \frac{\Pr(X_1 \leq x_1, X_n = X_{r:n}, x \leq X_n < x + \delta x)}{\Pr(X_n = X_{r:n}, x \leq X_n < x + \delta x)}.
\]
Now

\[ \Pr(X_n = X_{r:m}, x \leq X_n < x + \delta x) \]

\[ = \left[ \frac{n-1}{r-1} P(x)^{r-1} (1-P(x))^{n-r} + O(\delta x) \right] \Pr(x \leq X_n < x + \delta x) \]

where \( O(\delta x) \) refers to terms of order \( \delta x \), and if \( x_1 \leq x \)

\[ \Pr(X_1 < x_1, X_n = X_{r:m}, x \leq X_n < x + \delta x) \]

\[ = \Pr(X_1 \leq x_1) \Pr(r-2 \text{ of } X_2, \ldots, X_{n-1} \leq x) \Pr(x \leq X_n < x + \delta x) \]

\[ = \left[ P(x_1) \left( \frac{n-2}{r-2} \right) P(x)^{r-2} (1-P(x))^{n-r} + O(\delta x) \right] \Pr(x \leq X_n < x + \delta x) \]

Since \( \Pr(x \leq X_n < x + \delta x) \) cancels we have, as \( \delta x \to 0 \)

\[ F_{01}(x_1|x, 0) = \frac{P(x_1) \left( \frac{n-2}{r-2} \right) P(x)^{r-2} (1-P(x))^{n-r}}{\left( \frac{n-1}{r-1} \right) P(x)^{r-1} (1-P(x))^{n-r}} \]

\[ = \frac{r-1}{n-1} \frac{P(x_1)}{P(x)} . \]

Likewise, if \( x_1 > x \)
\[ F_{01}(x_1 | x, 0) = \frac{n-r}{n-1} \frac{P(x_1) - P(x)}{1 - F(x)} . \]

It follows then that

\[ f_{01}(x_1 | x, 0) = \frac{r-1}{n-1} \frac{p(x_1)}{F(x)} \quad x_1 \leq x \]

\[ = \frac{n-r}{n-1} \frac{p(x_1)}{1-F(x)} \quad x_1 > x \quad (2.3) \]

The joint distribution

\[ f_1(x_1, x | 0) = f_{01}(x_1 | x, 0) f_{x:n}(x) , \]

so it is clear from (2.1) and (2.3) that

\[ f_1(x_1, x | 0) = n \left( \frac{n-2}{r-2} \right) p(x)^{r-2}(1-F(x))^{n-r} p(x) p(x_1) \quad x_1 \leq x \]

\[ = n \left( \frac{n-2}{r-1} \right) p(x)^{r-1}(1-F(x))^{n-r-1} p(x) p(x_1) \quad x_1 > x \quad (2.4) \]

At this point, and throughout this dissertation we will adopt the convention that

\[ \binom{b}{a} = 0 \quad \text{if} \quad a < 0 \quad \text{or} \quad a > b . \]
Looking at the joint density of two unordered random variables and one ordered random variable we may express

\[ f_2(x_1, x_2, x | 0) = f_{12}(x_2 | x_1, x, 0) f_{01}(x_1 | x, 0) f_{x_{r:n}}(x). \]

Using similar arguments to those which led to (2.3) it can be shown that

\[ f_{12}(x_2 | x_1, x, 0) = \frac{r-2}{n-2} \frac{p(x_2)}{F(x)} \quad x_1 \leq x \]
\[ = \frac{r-1}{n-2} \frac{p(x_2)}{F(x)} \quad x_1 > x \]
\[ = \frac{n-r}{n-2} \frac{p(x_2)}{1-F(x)} \quad x_1 \leq x \]
\[ = \frac{n-r-1}{n-2} \frac{p(x_2)}{1-F(x)} \quad x_1 > x. \]

We may write

\[ f_{02}(x_1, x_2 | x, 0) = f_{12}(x_2 | x_1, x, 0) f_{01}(x_1 | x, 0), \]

so
This leads to the following general theorem.

**Theorem 2.1:** Let $X_1, \ldots, X_n$ be a set of absolutely continuous random variables with density function $p$ and c.d.f. $P$. Define the set of ordered random variables $X_1^{\leq} \leq X_2^{\leq} \leq \cdots \leq X_n^{\leq}$ and the random variable $T_n$ as $T_n = X_n - X_{r:n}$ ($i \leq r \leq n$). If the set $x_1, \ldots, x_k$ has exactly $j$ specific elements less than or equal to $x$ ($0 \leq j \leq k-1$, $k = 1, \ldots, n-1$), then

$$f_{X_1, \ldots, X_k | X_{r:n} T_n}(x_1, \ldots, x_k | x, 0) = \frac{(r-1)(n-r)}{(n-1)(n-2)} \frac{p(x_1) p(x_2)}{p(x)^2} \begin{cases} x_1 \leq x \\ x_2 \leq x \end{cases} \left\{ \begin{array}{ll} x_1 \leq x \text{ or } x_1 > x \\ x_2 > x \end{array} \right. \frac{p(x_1) p(x_2)}{[1 - P(x)]^2} \begin{cases} x_1 > x \\ x_2 > x \end{cases}$$

$$(2.5)$$

**Proof:** The proof will be by induction. Clearly the theorem holds for 1 and 2. We now assume it holds for $k-1$ and consider two
possibilities concerning $x_k$.

**Case I.** $x_k \leq x$.

If $x_k \leq x$ then $j-1$ of the $x_1, \ldots, x_{k-1}$ will be less than or equal to $x$.

We may write

$$f_{0k}(x_1, \ldots, x_k|x,0) = f_{k-1,k}(x_k|x_1, \ldots, x_{k-1}, x, 0)f_{0,k-1}(x_1, \ldots, x_{k-1}|x,0)$$

and

$$f_{0,k-1}(x_1, \ldots, x_{k-1}|x,0) = \frac{(r-1)(n-r)}{(j-1)(k-j)} \frac{k-1}{(k-1)} \frac{\prod_{i=1}^{k-1} p(x_i)}{p(x)^{j-1} (1-P(x))^{k-j}}.$$

By arguments similar to those leading to (2.3) we can show that for $x_k \leq x$

$$f_{k-1,k}(x_k|x_1, \ldots, x_{k-1}, x, 0) = \frac{r-j}{n-k} \frac{p(x_k)}{P(x)}.$$

Thus

$$f_{k-1,k}(x_1, \ldots, x_k|x,0) = \frac{r-j}{n-k} \frac{(r-1)(n-r)}{(j-1)(k-j)} \frac{k}{(k-1)} \frac{\prod_{i=1}^{k} p(x_i)}{p(x)^{j} (1-P(x))^{k-j}}$$

$$= \frac{(r-j)(r-r)}{(k-j)} \frac{k}{(k-1)} \frac{\prod_{i=1}^{k} p(x_i)}{p(x)^{j} (1-P(x))^{k-j}}.$$
which is the same form as (2.6).

Case II. \( x_k > x \).

If \( x_k > x \) and \( j \) of the \( x_1, \ldots, x_{k-1} \) are less than or equal to \( x \) and it follows that

\[
f_{0,k-1}(x_1, \ldots, x_{k-1}|x,0) = \left( \begin{array}{c} r-1 \cr j \end{array} \right) \frac{\prod_{i=1}^{k-1} p(x_i)}{n-1 \choose k-1} \frac{\prod_{i=1}^{k-1} p(x_i)}{P(x)^j (1-P(x))^{k-1-j}}.
\]

Once again using arguments similar to those which led to (2.3), for \( x_k > x \) we write

\[
f_{k-1,k}(x_k|x_1, \ldots, x_{k-1}, x,0) = \frac{n-r-(k-1-j)}{n-k} \frac{p(x_k)}{1-P(x)}
\]

and so

\[
f_{k-1,k}(x_1, \ldots, x_k|0) = \frac{n-r-(k-1-j)}{n-k} \left( \begin{array}{c} r-1 \cr j \end{array} \right) \frac{\prod_{i=1}^{k} p(x_i)}{k-1 \choose j} \frac{\prod_{i=1}^{k} p(x_i)}{P(x)^j (1-P(x))^{k-j}}
\]

This completes the proof.

Since the \( X_i \)'s are i.i.d. this theorem holds for any choice of \( k \) \( X_i \)'s with \( X_j = X_{r:n} \), \( j \neq i \).
As we stated at the beginning of this section, we are interested in

\[ f_{x_1, \ldots, x_{n-1}, x \mid r : n}(x_1, \ldots, x_{n-1}, x \mid 0) \cdot \]

This density is equivalent to

\[ f_{0, n-1}(x_1, \ldots, x_{n-1} \mid x, 0) f_{x_{r : n}}(x) \cdot \]

Looking at Theorem 2.1 and keeping in mind the convention assumed concerning combinatorial notation, we can see that for \( k = n-1 \) the only non-zero density function occurs when \( j = r-1 \). The density function is

\[ f_{0, n-1}(x_1, \ldots, x_{n-1} \mid x, 0) = \frac{(r-1)!(n-r)!}{(n-1)!} \prod_{i=1}^{n-1} \frac{p(x_i)}{P(x)^{r-1}(1-P(x))^{n-r}}. \]

Using the above equation and (2.1)

\[ f_{0, n-1}(x_1, \ldots, x_{n-1}, x \mid 0) = n \left[ \prod_{i=1}^{n-1} p(x_i) \right] p(x) \]

for a specific \( r-1 \) \( x_i \)'s \( \leq x \)

\( n-r \) \( x_i \)'s \( > x \).

Since there are \( \binom{n-1}{r-1} \) ways of arranging exactly \( r-1 \) of the \( n-1 \) \( x_i \)'s \( \leq x \) it follows that
\[
\mathfrak{f}_{x_1, \ldots, x_{n-1}, x} \equiv \frac{n!}{(r-1)!(n-r)!} \prod_{i=1}^{n-l} p(x_i) \prod_{i=l}^{n-r} p(x_i) 
\]
(2.7)

C. Distribution of the Ranks of Concomitants of Order Statistics

1. Probability distribution

a. Basic assumptions

Let \((X_i, Y_i)\) \(i = 1, \ldots, n\) be a set of independent paired random variables with some bivariate distribution, and let \(X_1, \ldots, X_n\) be a set of independent, identically distributed, absolutely continuous random variables, each having mean \(\mu_X\), variance \(\sigma^2_X\), standardized density function \(p(x)\) and c.d.f. \(P(x)\).

Further, let us assume that \(X\) and \(Y\) are linearly related, except for an error term, i.e., we have

\[
Y_i = \mu_Y + \beta(X_i - \mu_X) + Z_i \quad (i = 1, \ldots, n) \ ,
\]
(2.8)

where \(\beta = \rho \sigma_y / \sigma_x\), \(\rho\) being the correlation coefficient of \(X\) and \(Y\), \(\mu_y\) and \(\sigma^2_y\) being the mean and variance of the \(Y\)-variate respectively;
also \( Z = (Z_1, \ldots, Z_n) \), and \( X = (X_1, \ldots, X_n) \) are jointly independent and \( Z_1, \ldots, Z_n \) form a set of independent, identically distributed, absolutely continuous random variables, each with mean 0, variance \( \sigma^2_y(1-\rho^2) \), standardized density function \( q(z) \) and standardized c.d.f. \( Q(z) \).

The \( r \text{th} \) concomitant may therefore be written

\[
Y_{[r:n]} = \mu_Y + \frac{\rho \sigma_Y}{\sigma_X} (X_{r:n} - \mu_X) + Z_{[r]} \tag{2.9}
\]

where \( Z_{[r]} \) is the \( Z_i \) associated with \( X_{r:n} \). The \( Z_{[r]} \) are mutually independent, with the same distribution as the \( Z_i \) and are independent of \( X_{r:n} \).

Denote rank \( (Y_{[r:n]}) \) as \( R_{r,n} \) and define the indicator function \( I(Y_i, Y_{[r:n]}) \) as

\[
I(Y_i, Y_{[r:n]}) = 1 \text{ if } Y_i \leq Y_{[r:n]}
\]

\[
= 0 \text{ if } Y_i > Y_{[r:n]}
\]

Therefore

\[
R_{r,n} = \sum_{i=1}^{n} I(Y_i, Y_{[r:n]})
\]

From this representation it is clear that the distribution of \( R_{r,n} \) is independent of origin and scale of \( Y \), so we take \( \mu_Y = 0 \), \( \sigma^2_Y = 1 \); also from (2.8) and (2.9) we see we can, without loss of generality, take
\( \mu_X = 0 \) and \( \sigma_X^2 = 1 \). Thus, (2.8) and (2.9) may be rewritten as

\[
\begin{align*}
Y_i &= \rho X_i + Z_i \quad (i = 1, \ldots, n) \\
Y_{[r:n]} &= \rho X_{r:n} + Z_{[r]} \quad (r = 1, \ldots, n)
\end{align*}
\]

and

\[
\text{Var}(Z_i) = \text{Var}(Z_{[r]}) = 1 - \rho^2.
\]

b. **Probability distribution with positive correlation**

\[
\Pr(R_{r,n} = s+1) = \Pr(\text{exactly } s+1 \text{ of the } Y_i's \text{ are less than or equal to } Y_{[r:n]} \text{ with equality holding for at least one } Y_i).
\]

Since the \( Y_i's \) are i.i.d. random variables, the above probability remains the same no matter which \( Y_i's \leq Y_{[r:n]} \), so it follows that

\[
\Pr(R_{r,n} = s+1) = \binom{n-1}{s} \Pr\left(\begin{array}{c} Y_1 \leq Y_{[r:n]}, \ldots, Y_s \leq Y_{[r:n]}, \\
Y_{s+1} > Y_{[r:n]}, \ldots, Y_{n-1} > Y_{[r:n]} \\
x_n = X_{r:n} \end{array}\right)
\]

\[
= \binom{n-1}{s} P^*(s, r) \quad \text{say}.
\]

We worked with the conditional probability, to avoid the singularity which occurs when \( X_i = X_{r:n} \).
From (2.10) we may write

\[ \begin{align*}
P^*(s, r) = & \Pr \left( \begin{array}{c}
pX_1 + Z_1 \leq pX_{r:n} + Z_{[r]}, \ldots, pX_s + Z_s \leq pX_{r:n} + Z_{[r]}, \\
pX_{s+1} + Z_{s+1} > pX_{r:n} + Z_{[r]}, \ldots, pX_{n-1} + Z_{n-1} > pX_{r:n} + Z_{[r]} \\
\end{array} \right) \\
& \quad \left| X_{r:n} = x_n \right).
\end{align*} \] (2.12)

Assuming \( p > 0 \) and letting

\[ \begin{align*}
Z_i^* &= \frac{1}{\rho} (Z_{[r]} - Z_i) \\
T_i &= X_i - X_{r:n} \\
V_i &= \frac{1}{\rho} (Z_{[r]} - Z_i) - (X_i - X_{r:n}) \\
U &= \frac{Z_{[r]}}{\rho}
\end{align*} \] (2.13)

it is clear that

\[ P^*(s, r) = \Pr(V_1 \geq 0, \ldots, V_s \geq 0, V_{s+1} < 0, \ldots, V_{n-1} < 0 | T_n = 0). \] (2.14)

The conditioned random variables \( Z^*_i | Z_{[r]} = z_{[r]} \), \( i = 1, \ldots, n-1 \) are independent and identically distributed with mean \( z_{[r]} / \rho \) and variance \( (1-\rho^2)/\rho^2 \), when \( T_n = 0 \). Further, \( Z_1^*, \ldots, Z_{n-1}^* \), \( Z_{[r]} \)
are independent of $X_1, \ldots, X_{n-1}, X_{r:n}$.

It follows that

$$f_{Z^*_1, \ldots, Z^*_{n-1}, Z_[r]}(z_1, \ldots, z_{n-1}, z_[r], x_1, \ldots, x_{n-1}, x | 0)$$

$$= f_{X_1, \ldots, X_{n-1}, X_{r:n}}(x_1, \ldots, x_{n-1}, x | 0)$$

$$\quad \cdot \prod_{i=1}^{n-1} f_{Z^*_i | z_[r], T_n}(z_i | z_[r], 0) f_{Z_[r]}(z_[r], 0). \quad (2.15)$$

We have that

$$f_{Z^*_i | z_[r], T_n}(z_i | z_[r], 0) = c q(c(z_i - z_[r]/\rho))$$

$$(i = 1, \ldots, n-1)$$

where $c = |\rho/\sqrt{1-\rho^2}|$ and $q$ is the standardized density of $Z_i$.

Also

$$f_{Z_[r]} | T_n(z_[r] | 0) = \frac{1}{\sqrt{1-\rho^2}} q(z_[r]/\sqrt{1-\rho^2}) .$$

The joint p.d.f. of $X_{r:n}$ and $X_1, \ldots, X_{n-1}$ given $X_{r:n} = X_n$ is given by $(2.7)$
\[ f_{x_1, \ldots, x_{n-1}, x_{r:n}} (x_1, \ldots, x_{n-1}, x_0) = \frac{n!}{(r-1)! (n-r)!} p(x) \prod_{i=1}^{n-1} p(x_i) \]

\[ r-1 \text{ of the } x_i \leq x \]

\[ n-r \text{ of the } x_i > x. \]

This result can be obtained in a more straightforward manner by looking at the joint p.d.f. of the n order statistics. The joint p.d.f. of the n order statistics is given by

\[ \prod_{i=1}^{n} p(x_i) \to -\infty < x_1 < \ldots < x_n < \infty. \]

From this the joint p.d.f. of \( X_{r:n} \) and the remaining n-1 unordered variates is obtained on permitting all (equally likely) permutations of \( x_1, \ldots, x_{r-1} \) among themselves and of \( x_{r+1}, \ldots, x_n \) among themselves, and so may be written

\[ \frac{n!}{(r-1)! (n-r)!} \prod_{i=1}^{n} p(x_i) \to -\infty < x_1, \ldots, x_{r-1} < x_r < x_{r+1}, \ldots, x_n < \infty. \]

The right-hand side of (2.15) may be rewritten

\[ \frac{c^{n-1}}{\sqrt{1-\rho^2}} n \frac{n-1}{(r-1)} p(x) q \left( \frac{z_{[r]}}{\sqrt{1-\rho^2}} \right) \prod_{i=1}^{n-1} [q(c(z_i - z_{[r]}))p(x_i)] \]

\[ r-1 \quad x_i \leq x < n-r \quad x_i \]

\[ -\infty < x, x_i, z_{[r]} < \infty. \]

Making a transformation to the variables given in (2.13) it is clear that

\[ f_{v_1, \ldots, v_{n-1}, t_1, \ldots, t_{n-1}, x_{r:n}, u|T_n} (v_1, \ldots, v_{n-1}, t_1, \ldots, t_{n-1}, x, u|0) \]
\[ P(s, r) \text{ we must integrate the density in (2.16) over the ranges of } x, u \text{ and the } t_i \] for \( i = 1, \ldots, n-1 \) and then over the \( v_i \) from \(-\infty\) to 0 or from 0 to \( \infty \). Since the density given by (2.16) is positive, by Fubini's Theorem, the order of integration may be changed without affecting the result. If the expression in brackets in (2.16) is integrated over \( v_i \) and \( t_i \) first, then the resulting expression will be a function of \( x \) and \( u \) only, and due to the restricted number of ranges of integration of \( v_i \) and \( t_i \), there are only four possible forms for this expression:

1) \[ \int_{-\infty}^{0} \int_{0}^{\infty} c \, p(t+x) \, q(c(t+v-u)) \, dv \, dt = \int_{-\infty}^{0} [1 - Q(c(t-u))] \, p(t+x) \, dt . \]

Define
\[ \Theta_1(x,u) = \int_{-\infty}^{0} [1 - Q(c(t-u))] \, p(t+x) \, dt . \] (2.17)

2) \[ \int_{0}^{\infty} \int_{0}^{\infty} c \, p(t+x) \, q(c(t+v-u)) \, dv \, dt = \int_{0}^{\infty} [1 - Q(c(t-u))] \, p(t+x) \, dt . \]

Define
\[ \Theta_2(x,u) = \int_0^\infty [1 - Q(c(t-u))] p(t+x) \, dt \quad (2.18) \]

3) \[ \int_{-\infty}^0 \int_{-\infty}^0 c p(t+x) q(c(t+v-u)) \, dv \, dt = \int_0^\infty Q(c(t-u)) p(t+x) \, dt \]

Define

\[ \Theta_3(x,u) = \int_{-\infty}^0 Q(c(t-u)) p(t+x) \, dt \quad (2.19) \]

4) \[ \int_0^\infty \int_{-\infty}^0 c p(t+x) q(c(t+v-u)) \, dv \, dt = \int_0^\infty Q(c(t-u)) p(t+x) \, dt \]

Define

\[ \Theta_4(x,u) = \int_0^\infty Q(c(t-u)) p(t+x) \, dt \quad (2.20) \]

For a particular choice of \( r-1 \) \( t_i \)'s \( \leq 0 \), the number of times each of these four functions occurs can probably be best shown as a 2x2 table with fixed marginals.

\[
\begin{array}{ccc}
 & v \geq 0 & v < 0 \\
 t \leq 0 & s-k & r+k-s-1 & r-l \\
 & \Theta_2 & \Theta_3 & \\
 t > 0 & k & n-r-k & n-r \\
 & \Theta_2 & \Theta_4 & n-1 \\
 & s & n-s-1 & n-l \\
\end{array}
\]
Now there are \( \binom{n-1}{r-1} \) ways in which exactly \( r-1 \) \( t^i \)'s \( \leq 0 \). In light of the above table we may rewrite \( \binom{n-1}{r-1} \) as

\[
\binom{n-1}{r-1} = \sum_{k=0}^{s} \binom{s}{k} \binom{n-s-1}{n-r-k}.
\]

It follows then that

\[
P^*(s,r) = n \sum_{k=0}^{s} \binom{s}{k} \binom{n-s-1}{n-r-k} c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x) q(cu) \theta_{1}^{s-k}(x,u) \theta_{2}^{k}(x,u) \theta_{3}^{r+k-s-1}(x,u) \theta_{4}(x,u)^{n-r-k} du dx.
\]

Since

\[
Pr(R_{r,n} = s+1) = \binom{n}{s} P^*(s,r)
\]

and

\[
\binom{n-1}{s} \binom{s}{k} \binom{n-s-1}{n-r-k} = \binom{n-1}{r-1} \binom{r-1}{s-k} \binom{n-r}{k}
\]

we may express

\[
Pr(R_{r,n} = s+1) = c n \sum_{k=0}^{s} \binom{n-1}{r-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x) q(cu) \sum_{k=0}^{\infty} \binom{r-1}{s-k} \binom{n-r}{k} \theta_{1}^{s-k}(x,u) \theta_{2}^{k}(x,u) \theta_{3}^{r-1-s+k}(x,u) \theta_{4}(x,u)^{n-r-k} du dx \quad (2.22)
\]

\((s = 0, \ldots, n-1; r = l, \ldots, n)\)
where

\[ c = \left| \frac{\rho}{\sqrt{1-\rho^2}} \right| \]

and \( \theta_1, \theta_2, \theta_3, \theta_4 \) are defined by (2.17-2.20).

Looking at the form of the \( \theta \)'s it is easily shown

\[ \theta_1(x,u) + \theta_3(x,u) = P(x) \]

\[ \theta_2(x,u) + \theta_4(x,u) = 1 - P(x) \]

and

\[ \theta_1(x,u) + \theta_2(x,u) + \theta_3(x,u) + \theta_4(x,u) = 1. \]

When

\[ \rho = 0 \quad \text{Pr}(R_{x,n} = s+1) = \frac{1}{n} \quad s = 0, \ldots, n-1 \]
\[ r = 1, \ldots, n \]

and when

\[ \rho = 1 \quad \text{Pr}(R_{x,n} = s+1) = 1 \quad \text{if} \quad s+1 = r, \quad r = 1, \ldots, n \]
\[ = 0 \quad \text{otherwise}. \]

c. **Probability distribution with negative correlation**

Throughout this development, we have assumed \( \rho > 0 \). If we now assume a correlation of \( -\rho \) between \( X \) and \( Y \), then (2.14) becomes

\[ P^*(s,r) = \text{Pr}(V_1 < 0, \ldots, V_s < 0, V_{s+1} \geq 0, \ldots, V_{n-1} \geq 0 | T_n = 0). \]
Since the $V_i$'s $i = 1, \ldots, n-1$ are identically distributed, we may change subscripts and write

$$P^*(s,r) = Pr(V_1 \geq 0, \ldots, V_{n-s-1} \geq 0, V_{n-s} < 0, \ldots, V_{n-1} < 0| T_n = 0)$$

and it follows that

$$Pr(R_{r,n} = s+1 | -\rho) = Pr(R_{r,n} = n-s | \rho). \quad (2.23)$$

d. Symmetry considerations

If we add the assumptions that $X$ and $Z$ are symmetric about zero to those made at the beginning of the section then

- $X$ has the same distribution as $X$
- $Z$ has the same distribution as $Z$
- $Y$ has the same distribution as $Y$.

Further

$$(X_{1:n'}, \ldots, X_{n:n}) \text{ has the same joint distribution as } \ (-X_{n:n'}, \ldots, -X_{1:n})$$

and

$$(Z_{[1]}, \ldots, Z_{[n]}) \text{ has the same joint distribution as } \ (-Z_{[n]}, \ldots, -Z_{[1]})$$

so that
\[ (Y_{[n:n]}, \ldots, Y_{[1:n]} \) has the same joint distribution as \]
\[ (-Y_{[n:n]}, \ldots, -Y_{[1:n]} \) \]

Now

\[ \Pr(R_{r,n} = s+1) = \Pr(s + 1 Y' \leq Y_{[r:n]}) \]
\[ = \Pr(s+1 (-Y' \leq -Y_{[n+1-r:n]}) \) \text{ by (A)} \]
\[ = \Pr(s+1 Y' \geq Y_{[n+1-r:n]}) \]
\[ = \Pr(n-s-1 Y' < Y_{[n+1-r:n]}) \]
\[ = \Pr(R_{n+1-r,n} = n-s) \]

so

\[ \Pr(R_{r,j} = s+1) = \Pr(R_{n+1-r,n} = n-s) . \quad (2.24) \]

Further, under a fairly general set of conditions we are able to show

\[ \Pr(R_{t,n} = s) = \Pr(R_{s,n} = t) . \]

To see this we look at \( \Pr(\text{rank } Y_i = s, \text{ rank } X_i = t) \) for an arbitrary \( (X_i, Y_i) \). If there exist rank preserving transformations from \( X_i \) and \( Y_i \) to \( X_i' \) and \( Y_i' \) (say), such that the marginal distributions of \( X_i' \) and \( Y_i' \) are identical then
\[ \text{Pr}(\text{rank } X_i = s, \text{rank } Y_i = t) = \text{Pr}(\text{rank } X_i = t, \text{rank } Y_i = s). \]

Since the transformation is rank preserving, we have

\[ \text{Pr}(\text{rank } X_i = s, \text{rank } Y_i = t) = \frac{1}{n} \text{Pr}(\text{rank } Y_i = t|\text{rank } X_i = s) \]

\[ = \frac{1}{n} \text{Pr}(R_{t,n} = s). \]

Similarly

\[ \text{Pr}(\text{rank } X_i = t, \text{rank } Y_i = s) = \frac{1}{n} \text{Pr}(R_{s,n} = t) \]

and so

\[ \text{Pr}(R_{t,n} = s) = \text{Pr}(R_{s,n} = t). \tag{2.24a} \]

In the bivariate normal case this result is seen to hold if the variables are standardized.

e. Checks

As a partial check of (2.22), we will show \[ \sum_{s=0}^{n-1} \text{Pr}(R_{r,n} = s+1) = 1. \]

We have,

\[ n-1 \sum_{s=0}^{n-1} \text{Pr}(R_{r,n} = s+1) = \sum_{s=0}^{n-1} \sum_{k=0}^{r-1} c_{s-k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n^{-1} (r-1) (s-k) (n-r-k) p(x) q(cu) du dx. \]
Now
\[ \sum_{s=0}^{n-1} \sum_{k=0}^{s-k} \binom{r-1}{s-k} \binom{n-r}{k} = \sum_{j=0}^{r-1} \sum_{k=0}^{j} \binom{r-1}{j} \binom{n-r}{k}, \quad j = s-k. \]

Therefore,
\[ \sum_{s=0}^{n-1} \Pr(R_{r,n} = s+1) \]

\[ = c \sum_{s=0}^{n-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=0}^{r-1} \sum_{k=0}^{j} \binom{r-1}{j} \binom{n-r}{k} \phi_j \phi_k \phi^{n-1-j} \phi^{n-r-k} \]
\[ q(\mu) \, du \, dx \]
\[ = \frac{n!}{(n-1)! (n-r)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \sum_{j=0}^{r-1} \sum_{k=0}^{j} \binom{r-1}{j} \phi_j \phi_k \phi^{n-1-j} \phi^{n-r-k} \right] q(\mu) \phi(x) \, du \, dx. \]

Since
\[ \sum_{j=0}^{r-1} \binom{r-1}{j} \phi_1 \phi^{r-1-j} = (\phi_1 + \phi_3)^{r-1} = F(x)^{r-1} \quad (2.25) \]

and
\[ \sum_{k=0}^{n-r} \binom{n-r}{k} \phi_2 \phi^{n-r-k} = (\phi_2 + \phi_4)^{n-r} = [1 - F(x)]^{n-r}, \quad (2.26) \]

it follows that
\[ r-1 \sum_{s=0}^{\infty} \Pr(R_{r,n} = s+1) \]

\[ = \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x)^{r-1} (1-P(x))^{n-r} p(x) q(cu) \, du \, dx \]

\[ = [c \int_{-\infty}^{\infty} q(cu) \, du] [\frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} P(x)^{r-1} (1-P(x))^{n-r} p(x) \, dx] \]

\[ = 1. \quad (2.27) \]

2. Moments of the ranks of concomitants

We shall develop the first and second moments of \( R_{r,n} \) and suggest a method for finding higher moments.

a. Expected rank of a concomitant

By definition

\[ E(R_{r,n}) = \sum_{s=0}^{n-1} (s+1) \Pr(R_{r,n} = s+1) \]

so by (2.19)

\[ E(R_{r,n}) = 1 + \sum_{s=0}^{n-1} s \Pr(R_{r,n} = s+1). \quad (2.28) \]

Once again letting \( j = s-k \) we may write
\[\sum_{s=0}^{n-1} \Pr(R_r, n = s + 1)\]

\[= \sum_{j=0}^{r-1} \sum_{k=0}^{n-r} c_{(j+k)} n^{(n-1)} j^{(r-1)} k^{(n-r)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1^j \phi_2^k \phi_3^{r-1-j} \phi_4^{n-r-k} \]

\[= n c n^{(n-1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \sum_{j=0}^{r-1} j^{(r-1)} \phi_1^j \phi_3^{r-1-j} \right] \left[ \sum_{k=0}^{n-r} k^{(n-r)} \phi_2^k \phi_4^{n-r-k} \right] p(x) q(cu) du \, dx\]

\[+ n c n^{(n-1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \sum_{j=0}^{r-1} j^{(r-1)} \phi_1^j \phi_3^{r-1-j} \right] \left[ \sum_{k=0}^{n-r} k^{(n-r)} \phi_2^k \phi_4^{n-r-k} \right] p(x) q(cu) du \, dx\].

Now

\[\sum_{j=0}^{r-1} j^{(r-1)} \phi_1^j \phi_3^{r-1-j} = (r-1)\phi_1 \sum_{j'=0}^{r-2} (j')^{(r-2)} \phi_3^{r-2-j'}\]

\[j' = j-1\]

\[= (r-1)\phi_1 P(x)^{r-2}. \quad (2.29)\]

Similarly

\[\sum_{k=0}^{n-r} k^{(n-r)} \phi_2^k \phi_4^{n-r-k} = (n-r)\phi_2 (1-P(x))^{n-r-1}. \quad (2.30)\]

Clearly,
\[ \sum_{s=0}^{n-1} \Pr(R_{r,n} = s+1) = n(n-1) \left( \sum_{r=2}^{n-2} \int_{-\infty}^{\infty} p(x)^{r-2} (1-p(x))^{n-r} p(x) \left[ c \int_{-\infty}^{\infty} \Theta_1(x,u) q(cu) du \right] dx \right) \]

\[ + n(n-1) \left( \int_{-\infty}^{\infty} p(x)^{r-1} (1-p(x))^{n-r-1} p(x) \left[ c \int_{-\infty}^{\infty} \Theta_2(x,u) q(cu) du \right] dx \right) . \]

(2.31)

Let

\[ \Theta_1(x, \cdot) = c \int_{-\infty}^{\infty} \int_{0}^{\infty} (1 - Q(c(t-u))) p(t+x) q(cu) dt du \] (2.32)

and

\[ \Theta_2(x, \cdot) = c \int_{-\infty}^{\infty} \int_{0}^{\infty} (1 - Q(c(t-u))) p(t+x) q(cu) dt du . \] (2.33)

From (2.28) and (2.31) it follows that

\[ E(R_{r,n}) = 1 + n(n-1) \left( \sum_{r=2}^{n-2} \int_{-\infty}^{\infty} p(x)^{r-2} (1-p(x))^{n-r} \Theta_1(x, \cdot) p(x) dx \right) \]

\[ + n(n-1) \left( \int_{-\infty}^{\infty} p(x)^{r-1} (1-p(x))^{n-r-1} \Theta_2(x, \cdot) p(x) dx \right) . \]

(2.34)

\[ r = 1, 2, \ldots, n . \]
b. Alternative method for finding $E(R_{r,n})$

Once again using

$$I(Y_i, Y_{[r:n]}) = 1 \quad \text{if} \quad Y_i \leq Y_{[r:n]}$$
$$= 0 \quad \text{otherwise}$$

we have

$$R_{r,n} = \sum_{i=1}^{n} I(Y_i, Y_{[r:n]})$$

and

$$E(R_{r,n}) = \sum_{i=1}^{n} \Pr(Y_i \leq Y_{[r:n]})$$

The set of random variables $Y_1$, $\ldots$, $Y_n$ are i.i.d., so $\Pr(Y_i \leq Y_{[r:n]})$ remains the same for all $i = 1, \ldots, n$. Therefore, we may take

$$E(R_{r,n}) = n \Pr(Y_1 \leq Y_{[r:n]})$$

To find $\Pr(Y_i \leq Y_{[r:n]})$ we use the fact that

$$\Pr(Y_1 \leq Y_{[r:n]}) = \sum_{j=1}^{n} \Pr(Y_1 \leq Y_{[r:n]}|X_j = X_{r:n}) \Pr(X_j = X_{r:n})$$

so we can consider

$$\Pr(Y_i \leq Y_{[r:n]}) = \frac{1}{n} [\Pr(Y_1 \leq Y_{[r:n]}|X_1 = X_{r:n}) + \sum_{j=2}^{n} \Pr(Y_1 \leq Y_{[r:n]}|X_j = X_{r:n})].$$
By the definition of a concomitant we know

\[ \Pr(Y_1 \leq Y_{[r:n]} | X_1 = X_{r:n}) = 1 \]

and by the fact that \( Y_j \) are i.i.d. \( \Pr(Y_1 \leq Y_{[r:n]} | X_j = X_{r:n}) \) is the same for \( j = 2, 3, \ldots, n \), we can express

\[ \Pr(Y_1 \leq Y_{[r:n]}) = \frac{1}{n} + \frac{(n-1)}{n} \Pr(Y_1 \leq Y_{[r:n]} | X_n = X_{r:n}) \quad (2.35) \]

\[ r = 1, \ldots, n \ . \]

Now we have

\[ \Pr(Y_1 \leq Y_{[r:n]} | X_n = X_{r:n}) = \Pr(\frac{1}{\rho} X_1 + Z_1 \leq \rho X_{r:n} + Z_{[r]} | X_n = X_{r:n}) \]

\[ = \Pr(\frac{1}{\rho} (Z_{[r]} - Z_1) - (X_1 - X_{r:n}) \geq 0 | X_n = X_{r:n}) \]

if \( \rho > 0 \). Letting

\[
\begin{align*}
Z^*_1 &= \frac{1}{\rho} (Z_{[r]} - Z_1) \\
T_1 &= X_1 - X_{r:n} \\
T_n &= X_n - X_{r:n} \\
V_1 &= \frac{1}{\rho} (Z_{[r]} - Z_1) - (X_1 - X_{r:n})
\end{align*}
\]

we can see

(2.36)
\[ \Pr(Y_1 \leq Y_{r:n} | X_n = X_{r:n}) = \Pr(V_1 \geq 0 | T_n = 0) \]

If \( X_n = X_{r:n} \) then \( Z_1 \) and \( Z_{[r]} \) are independent. \( Z_1^* \) is independent of \( X_1 \) and \( X_{r:n} \). Now

\[ f_{Z_1, Z_{[r]} | T_n}(z_1', z_{[r]} | 0) = q(\frac{z_1}{\sqrt{1-\rho^2}}) q(\frac{z_{[r]}}{\sqrt{1-\rho^2}}) \]

Letting

\[ U = \frac{Z_{[r]}}{\rho} \]

\[ Z_1^* = \frac{1}{\rho} (Z_{[r]} - z_1) \]

and making the appropriate transformation we have

\[ f_{Z_1^*, V | T_n}(z_1, u | 0) = c^2 q(c(z_1-u)) q(cu) \quad -\infty < z_1 < \infty \]
\[ -\infty < u < \infty \]

and

\[ c = |\rho/\sqrt{1-\rho^2}| \]

Thus

\[ f_{Z_1^* | T_n}(z_1 | 0) = \int_{-\infty}^{\infty} c^2 q(c(z_1-u)) q(cu) \, du \quad (2.37) \]

By the independence of \( Z_1^* \) and \( X_1, X_{r:n} \) conditioned on \( T_n \) we have
From (2.4) we know

\[ f_{X_1, X_r : |T_n} (x_1, x|0) = n(\begin{array}{c} \frac{n-2}{r-2} \\
\end{array}) P(x)^{r-2} (1-P(x))^{n-r} p(x) p(x_1) \]

\[ x_1 \leq x \]

\[ = n(\begin{array}{c} \frac{n-2}{r-1} \\
\end{array}) P(x)^{r-1} (1-P(x))^{n-r-1} p(x) p(x_1) \]

\[ x_1 > x . \]

Making the transformation suggested by (2.36) we see that we may write

\[ f_{V_1, T_1, X_r : |T_n} (v_1, t_1, x|0) \]

\[ = n(\begin{array}{c} \frac{n-2}{r-2} \\
\end{array}) P(x)^{r-2} (1-P(x))^{n-r} p(x) p(t_1+x) \]

\[ [c^2 \int_{-\infty}^{\infty} q(c(t_1+v_1-u)) q(\text{cu}) \text{ du}] . \quad t_1 \leq 0 \]

\[ = n(\begin{array}{c} \frac{n-2}{r-1} \\
\end{array}) P(x)^{r-1} (1-P(x))^{n-r-1} p(x) p(t_1+x) \]

\[ [c \int_{-\infty}^{\infty} q(c(t_1+v_1-u)) q(\text{cu}) \text{ du}] . \quad t_1 > 0 . \]

To find \( \text{Pr}(V_1 > 0|T_n = 0) \) we must integrate the above density over the ranges of \( t_1, x \) and non-negative \( v_1 \). Therefore,
\[
\Pr(V_1 \geq 0 | T_n = 0) = c^2 \int_0^\infty \int_0^\infty \int_0^\infty n(\frac{n-2}{r-2}) P(x)^{r-2} (1-P(x))^{n-r} \\
p(x) p(t_1^+x) q(c(t_1^+v_1-u)) q(cu) du dt_1 dx dv_1 \\
+ c^2 \int_0^\infty \int_0^\infty \int_0^\infty n(\frac{n-2}{r-1}) P(x)^{r-1} (1-P(x))^{n-r-1} \\
p(x) p(t_1^+x) q(c(t_1^+v_1-u)) q(cu) du dt_1 dx dv_1.
\]

Since each of the integrands is positive, we can, by Fubini's Theorem, change the order of integration and integrate over \( v_1 \) and \( t_1 \) first.

We have then

\[
\Pr(V_1 \geq 0 | T_n = 0) = \int_0^\infty n(\frac{n-2}{r-2}) P(x)^{r-2} (1-P(x))^{n-r} p(x) \\
+ \int_0^\infty n(\frac{n-2}{r-1}) P(x)^{r-1} (1-P(x))^{n-r} p(x) \\
[ c \int_0^\infty \int_0^\infty (1-Q(c(t_1-u)))q(cu) p(t+x) dt du ] dx ,
\]

which may be rewritten
\[ P(V_1 \geq 0 | T_n = 0) = n \int_{-\infty}^{\infty} P(x)^{r-2} (1-P(x))^{n-r} p(x) \theta_1(x, \cdot) \, dx \]
\[ + n(n-1) \int_{-\infty}^{\infty} P(x)^{r-1} (1-P(x))^{n-r-1} p(x) \theta_2(x, \cdot) \, dx \]
\( - \infty < x < \infty, \)

where \( \theta_1(x, \cdot) \) and \( \theta_2(x, \cdot) \) are given by (2.32) and (2.33). Now

\[ E(R_{r,n}) = 1 + (n-1) Pr(V_1 \geq 0 | T_n = 0) \]

so

\[ E(R_{r,n}) = 1 + n(n-1) \int_{-\infty}^{\infty} P(x)^{r-2} (1-P(x))^{n-r} p(x) \theta_1(x, \cdot) \, dx \]
\[ + n(n-1) \int_{-\infty}^{\infty} P(x)^{r-1} (1-P(x))^{n-r-1} p(x) \theta_2(x, \cdot) \, dx \]
\[ r = 1, 2, \ldots, n. \]

As can be seen, this is the same result as shown in (2.34).

c. Further properties of the expectation

\[ E(R_{r,n} | -\rho) = \sum_{s=0}^{n-1} (s+1) \Pr(R_{r,n} = s+1 | -\rho) \]
\[ = \sum_{s=0}^{n-1} (s+1) \Pr(R_{r,n} = n-s | \rho) \]
\[ = \sum_{s=0}^{n-1} (n+1) \Pr(R_{r,n} = n-s | \rho) - \sum_{s=0}^{n-1} (n-s) \Pr(R_{r,n} = n-s | \rho) \]
That is

\[ E(R_{r,n} | -\rho) = n+1 - E(R_{r,n} | \rho) . \]

If \( X \) and \( Z \) are symmetric about zero, it is easily shown that

\[ E(R_{n-r+1,n}) = n+1 - E(R_{r,n}) . \]

From Section C we know

\[ \Pr(R_{n-r+1:n} = n-s) = \Pr(R_{r:n} = s+1) . \]

So

\[
E(R_{n-r+1:n}) = \sum_{s=0}^{n-1} (s+1) \Pr(R_{n-r+1:n} = s+1)
\]

\[
= \sum_{s=0}^{n-1} [n+1 - (n-s)] \Pr(R_{r:n} = n-s)
\]

\[
= n+1 - \sum_{s=0}^{n-1} (n-s) \Pr(R_{r:n} = n-s)
\]

\[
= n+1 - E(R_{r,n}) . \quad (2.38)
\]

d. **Variance of the rank of a concomitant**

The second non-central moment of the rank is
\[ E(R_{r,n}^2) = \sum_{s=0}^{n-1} (s+1)^2 \Pr(R_{r,n} = s+1) \]

\[ = 1 + 2 \sum_{s=0}^{n-1} s \Pr(R_{r,n} = s+1) + \sum_{s=0}^{n-1} s^2 \Pr(R_{r,n} = s+1). \quad (2.39) \]

The second term on the right-hand side of (2.39) is given in (2.31), so it remains to find

\[ \sum_{s=0}^{n-1} s^2 \Pr(R_{r,n} = s+1). \]

Here again letting \( j = s-k \), we have

\[ \sum_{s=0}^{n-1} s^2 \Pr(R_{r,n} = s+1) = n \left( \sum_{j=0}^{r-1} \sum_{k=0}^{n-r} c(j+k)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{r-1}{j} \right) \left( \frac{n-r}{k} \right) \theta_l^j \theta_2^k \rho_r^{r-1-j} \rho_4^{n-r-k} p(x) q(\mu) \, dx \right). \quad (2.40) \]

Writing

\[ j^2 = j(j-1) + j \]

and

\[ k^2 = k(k-1) + k \]

(2.40) may be rewritten

\[ \sum_{s=0}^{n-1} s^2 \Pr(R_{r,n} = s+1) = c \, n \left( \sum_{j=0}^{r-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \sum_{j=0}^{r-1} j(j-1)(r-l) \theta_l^j \theta_3^{r-l-j} \right] \rho_r^{r-l-j} \rho_4^{n-r-k} p(x) q(\mu) \, dx \right). \]
\[
\begin{align*}
&= \sum_{k=0}^{n-r} \left[ \binom{n-r}{k} \theta_2^k \theta_4^{n-r-k} \right] p(x) q(cu) du dx \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{r-l} \left[ \sum_{j=0}^{n-r} \binom{n-r}{j} \theta_1^j \theta_3^{r-l-j} \right] \left[ \sum_{k=0}^{n-r} \binom{n-r}{k} \theta_2^k \theta_4^{n-r-k} \right] p(x) q(cu) du dx \\
&+ \sum_{s=0}^{r-l} \Pr(R_{r,n} = s+1).
\end{align*}
\]

(2.41)

Now

\[
\begin{align*}
&\sum_{j=0}^{n-1} \binom{n-1}{j} \binom{r-l}{j} \theta_1^j \theta_3^{r-l-j} \\
&= (r-l)(r-2) \sum_{j''=0}^{r-3} \binom{r-3}{j''} \theta_1^{j''} \theta_3^{r-3-j''} \\
&= (r-l)(r-2)(\theta_1^2)(\theta_1 + \theta_3)^{r-3} = (r-l)(r-2)\theta_1^2 p(x)^{r-3} \quad (2.42)
\end{align*}
\]

and

\[
\begin{align*}
&\sum_{k=0}^{n-r} \binom{n-r}{k} \theta_2^k \theta_4^{n-r-k} = (n-r)(n-r-1) \theta_2^2 (1-P(x))^{n-r-2}.
\end{align*}
\]

(2.43)

So from (2.25), (2.26), (2.29), (2.30), (2.42), and (2.43) we have
\[ \sum_{s=0}^{n-1} s^2 \Pr(R_{r,n} = s+1) = c \frac{n!}{(n-3)!} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (n-3) \theta_1(x,u) \theta_2(x,u) P(x)^{r-3} (1-P(x))^{n-r} \right. \
\left. + (n-3) \theta_1(x,u) \theta_2(x,u) P(x)^{r-2} (1-P(x))^{n-r-1} \right] \right. \
\left. + (n-3) \theta_2(x,u) P(x)^{r-1} (1-P(x))^{n-r-2} \left] p(x) q(cu) \right. du dx \right\} \\
+ \sum_{s=0}^{n-1} s \Pr(R_{r,n} = s+1) . \\
\]

Going back to (2.39) we have

\[ E(R_{r,n}^2) = 3E(R_{r,n}) - 2 + c \frac{n!}{(n-3)!} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (n-3) \theta_1(x,u) \theta_2(x,u) P(x)^{r-3} (1-P(x))^{n-r} \right. \
\left. + (n-3) \theta_1(x,u) \theta_2(x,u) P(x)^{r-2} (1-P(x))^{n-r-1} \right] \right. \
\left. + (n-3) \theta_2(x,u) P(x)^{r-1} (1-P(x))^{n-r-2} \left] p(x) q(cu) \right. du dx \right\} . \]

(2.44)

The variance of \( R_{r,n} \) follows directly from (2.44) and (2.34). Higher moments can be computed by finding the factorial moments and working back.
D. Numerical Results

1. Probability

In the special case of bivariate normality, we have calculated the probabilities of $R_{r,n}$ for $n = 3, 5, 9$ and $\rho = .1(.1).9, .95$ in Table 2.1. When $\rho = 0$ we know $\Pr(R_{r,n} = t) = 1/n$ $(r, t = 1, \ldots, n)$ and, when $\rho = 1$ $\Pr(R_{r,n} = r) = 1$. Looking at Table 2.1 we see that the numerical values tend to support this result; i.e., when $\rho$ is small the probabilities tend to be near $1/n$ and as $\rho$ increases $\Pr(R_{r,n} = r)$ also increases. The rate of increase however is smaller as $r$ goes toward the median; i.e., $\Pr(R_{1,n} = 1) > \Pr(R_{2,n} = 2) > \ldots > \Pr(R_{k,n} = k)$ for all $0 < \rho < 1$ where $k = (n+1)/2$. It is also interesting to note that, for $n = 5, 9$ and moderate values of $\rho$, $\Pr(R_{r,n} = r) < \Pr(R_{r,n} = r-1)$ for $r < (n+1)/2$. These numerical values were obtained by using the computer to do a triple Gaussian quadrature.

2. Expectation

In the special case of bivariate normality, we may rewrite (2.34) as

$$E(R_{r,n}) = 1 + n(n-1) \left[ \left( \frac{n-2}{r-2} \right) \int_{-\infty}^{\infty} \hat{g}(x)^{r-2} (1-\hat{g}(x))^{n-r} \theta_1(x, \cdot) \varphi(x) \, dx \right.$$

$$+ \left. \left( \frac{n-2}{r-1} \right) \int_{-\infty}^{\infty} \hat{g}(x)^{r-1} (1-\hat{g}(x))^{n-r-1} \theta_2(x, \cdot) \varphi(x) \, dx \right] , \quad (2.45)$$

and (2.32), (2.33) as

$$\theta_1(x, \cdot) = \int_{-\infty}^{0} \left[ 1 - \hat{g}(at) \right] \varphi(t+x) \, dt$$
\[ \theta_2(x, \cdot) = \int_0^\infty [1 - \Phi(at)] \varphi(t+x) \, dt \]

where \( \Phi(x) \) is the c.d.f. of a standard normal distribution, \( \varphi(x) \) is its density and

\[
a = \left| \frac{\rho}{\sqrt{2(1-\rho^2)}} \right|.\]

Using (1.4), David (1973) has evaluated the \( \lim_{n \to \infty} E(\hat{R}_{r,n}/n) \) as \( \lambda_r \). We have reproduced these values for \( \lambda_r = .55(.05,.95) \) and for \( \rho = .05(.05,.95), .99 \) in Table 2.2.

We have evaluated (2.45) using a double Gaussian quadrature, for \( n = 9, 19 \) and \( \rho = .05(.05,.95), .99 \) and for \( n = 39 \) and \( \rho = .5(.05,.95), .99 \). In Table 2.2, letting \( \lambda_r = r/(n+1) \), we have given the difference between \( E(\hat{R}_{r,n}/n+1) \) and \( \lim_{n \to \infty} E(\hat{R}_{r,n}/n) \).

Looking at Table 2.2, we can see that even for moderate sized samples the agreement between the finite results and the asymptotic results is quite good. Consequently, if one wished a rough idea of the expectation for particular values of \( \rho \) and \( n \), using Table 2.2 will be helpful.

E. Distribution of Concomitants of Order Statistics

Let \((X_i, Y_i) \ i = 1, \ldots, n\) be a set of independent paired random variables with a bivariate distribution. We assume that \( X \) and \( Y \) are linearly related except for an error term. That is, we may express

\[
Y_i = \mu_y + \frac{\rho \sigma_y}{\sigma_x} (X_i - \mu_x) + Z_i \quad i = 1, \ldots, n.
\]
We further assume that the distributional assumptions and independence properties of $X$ and $Z$ described in Section C of this chapter hold. We know that under these assumptions (1.3) holds and thus $E(Y_{[r:n]})$, $\text{Var}(Y_{[r:n]})$ and $\text{Cov}(Y_{[r:n]}, Y_{[s:n]})$ ($r = 1, \ldots, n$; $s \neq r$) are known.

From (2.9) we have

$$Y_{[r:n]} = \mu_Y + \frac{\rho \sigma_Y}{\sigma_X} (X_{r:n} - \mu_X) + Z_{[r]} \quad (r = 1, \ldots, n).$$

Thus

$$\text{Pr}(Y_{[r:n]} \leq y) = \text{Pr}(\mu_Y + \frac{\rho \sigma_Y}{\sigma_X} (X_{r:n} - \mu_X) + Z_{[r]} \leq y).$$

Conditioning on $X_{r:n}$ we may express

$$\text{Pr}(Y_{[r:n]} \leq y) = \int_{-\infty}^{\infty} \text{Pr}(\frac{Z_{[r]}}{\sigma_Y} \leq y' - \rho x') f_{X_{r:n}}(x) \, dx$$

where

$$y' = \frac{y - \mu_Y}{\sigma_Y},$$

$$x' = \frac{x - \mu_X}{\sigma_X}$$

and

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)! (n-r)!} P(x)^{r-1} (1-P(x))^{n-r} p(x) \, dx.$$
Letting $U_r = \sigma_y \sqrt{1-\rho^2} Z_{[r]}$, $U_r$ has mean zero, unit variance and c.d.f. $Q$. So

$$
\Pr(Y_{[r:n]} \leq y) = \int_{-\infty}^{\infty} \Pr(U_r \leq \frac{y' - \rho x'}{\sqrt{1-\rho^2}}) f_{X_{[r:n]}}(x) \, dx
$$

$$
= \int_{-\infty}^{\infty} Q\left(\frac{y' - \rho x'}{\sqrt{1-\rho^2}}\right) f_{X_{[r:n]}}(x) \, dx
$$

$$
= \frac{n!}{(r-1)!(n-r)!} \int_{-\infty}^{\infty} Q\left(\frac{y' - \rho x'}{\sqrt{1-\rho^2}}\right) P(x)^{r-1}(1-P(x))^{n-r} p(x) \, dx .
$$

(2.46)

Using similar arguments we can show that

$$
\Pr(Y_{[r_1:n]} \leq y_1, \ldots, Y_{[r_k:n]} \leq y_k)
$$

$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_k} f_{r_1, \ldots, r_k}(x_1, \ldots, x_k) \prod_{i=1}^{k} Q\left(\frac{y_i' - \rho x_i'}{\sqrt{1-\rho^2}}\right) \prod_{i=1}^{k} dx_i
$$

(2.47)

where $x_i' = \frac{x_i - \mu}{\sigma_x}$ (i = 1, ..., k),
\[ y_i' = \frac{(y_i - \mu_y)}{\sigma_y} \quad (i = 1, \ldots, k) \]

\[ 1 \leq r_1 < r_2 < \ldots < r_k \leq n, \quad x_1 \leq x_2 < \ldots < x_k \]

and \( f_{r_1, \ldots, r_k}(x_1, \ldots, x_k) \) is the joint distribution of \( k \) order statistics with ranks \( r_1, \ldots, r_k \).

David (1970, p. 9) shows

\[
\begin{align*}
 f_{r_1, \ldots, r_k}(x_1, \ldots, x_k) &= \frac{n^k}{k!} \left[ \prod_{i=1}^{k} p(x_i) \right] \prod_{i=1}^{k} \left\{ \frac{[P(x_{i+1})-P(x_i)]^{r_{i+1}-r_i-1}}{(r_{i+1}-r_i-1)!} \right\} \\
\end{align*}
\]

(2.48)

where \( x_0 = -\infty \), \( x_{k+1} = +\infty \), \( r_0 = 0 \), \( r_{k+1} = n+1 \).

F. Summary

In this chapter we have found the probability distribution and the first two moments of the concomitants of order statistics, under the assumptions of the existence of a linear relationship between \( X \) and \( Y \) (i.e., \( Y = \mu_y + \beta X + Z \)) and independence of \( X \) and \( Z \). The results obtained are highly dependent on these two assumptions, and to weaken either of them might make the methods used in this chapter inapplicable.

The numerical results tend to confirm what we might expect, i.e., that the behavior of the \( Y_{[r:n]} \) is heavily dependent on \( \rho \). When \( \rho \) is small using the \( Y_{[r:n]} \) differs little from using the random \( Y_1 \)'s, while for large \( \rho \) the \( Y_{[r:n]} \) are quite useful.
Under the assumption of bivariate normality, we have compared the asymptotic expectations of the ranks with the expectations in various finite cases, and found the agreement quite good. Therefore, these asymptotic values can be used to give an approximation to the exact expectation.

We have not worked to any great extent with the distribution function and the joint distribution of the concomitants in this chapter. From (2.46) and (2.47) we can see that they are rather awkward to use, and the moments of the $X_{[r:n]}$ can be more easily obtained from (2.9).
Table 2.1. Probability distribution of \( R_{r,n} \); \( \Pr(R_{r,n} = t) \)

\[
\Pr(R_{n-r+1,n} = t) = \Pr(R_{r,n} = n-t+1)
\]

\[
\Pr(R_{r,n} = t \mid \rho) = \Pr(R_{r,n} = n-t+1 \mid \rho)
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$$E(R_{n,n}/(n+1)|\rho = 0) = E(R_{n,n}/(n+1)|\rho = .5) = .5$$

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III. CONCOMITANTS IN PARAMETER ESTIMATION

A. Introduction

Cases arise in which measurements can be taken on paired random variables \((X, Y)\), one of which is relatively inexpensive to measure while the other is quite costly to measure. Let \(Y\) represent the costly variable and \(X\) the inexpensive variable. For a random sample size \(n\) it seems reasonable to measure \(X\) on all members of the sample, but to measure \(Y\) on only a relatively small number \(k\) \((k < n)\). We will attempt to take advantage of our knowledge of all the \(x\)'s in order to find estimators for \(\mu_x\), the mean of the random variable \(X\).

Following Cochran (1963, p. 334) we will assume that the population is infinite and that the relationship between \(X_i\) and \(Y_i\) \((i = 1, \ldots, n)\) is linear in the sense that

\[
Y_i = \mu_y + \beta(X_i - \mu_x) + Z_i \quad (i = 1, \ldots, n) .
\]

Here \(Z_i\) is an 'error variate', independent of \(X_i\) and having mean \(0\) and variance \(\sigma^2(1-\rho^2)\), where \(\beta = \rho \frac{\sigma_y}{\sigma_x}\), \(\rho\) being the correlation coefficient between the variates \(X\) and \(Y\) which have respective standard deviations \(\sigma_x\) and \(\sigma_y\).

Making use of the above linear relationship and a knowledge of the \(x\)'s, we will investigate an estimator of \(\mu_y\) similar to the regression estimator studied by Cochran (1963, p. 334). We will also investigate a more general class of estimators of \(\mu_y\) which are linear combinations...
of the concomitants and only make use of a knowledge of the order relationship among the x's.

B. Regression Estimators of \( \mu_y \)

1. Maximum likelihood results

Let \((X, Y)\) have a bivariate normal \( N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho) \) distribution. Cohen (1955) has found the maximum likelihood estimators of the parameters in the situation in which a random sample of \( k \) pairs of observations \((X_i, Y_i), \ldots, (X_k, Y_k)\) is drawn and a further random sample of \( n-k \) observations, \( X_{k+1}, \ldots, X_n \), is drawn on the X-variate.

We will consider the maximum likelihood estimator for \( \mu_y \) when:

a) all parameters except \( \mu_y \) are known, and

b) no parameters are known.

Cohen has shown the maximum likelihood estimator in case (a) to be

\[
\hat{\mu}_y = \bar{Y}_k - \frac{\rho \sigma_y}{\sigma_x} (\bar{X}_k - \mu_x) \tag{3.2}
\]

where

\[
\bar{Y}_k = \frac{1}{k} \sum_{i=1}^{k} Y_i \tag{3.3}
\]

and

\[
\bar{X}_k = \frac{1}{k} \sum_{i=1}^{k} X_i \tag{3.4}
\]

In the bivariate normal case \( Y_i \) can be expressed in the form of (3.1)
with the additional requirement that both \( X_i \) and \( Z_i \) are normal. This representation shows that \( \hat{\mu}_y \) is unbiased and

\[
\text{Var}(\hat{\mu}_y) = \frac{(1-\rho^2)}{k} \sigma_y^2.
\]

(3.5)

In the case where no parameters are known, Cohen found the MLE \( \tilde{Y}_k \) of \( \mu_y \) to be

\[
\tilde{Y}_k = \overline{Y}_k - b(\overline{X}_k - \overline{X}_n)
\]

(3.6)

where

\[
\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i
\]

(3.7a)

and

\[
b = \frac{k}{\sum_{i=1}^{k} \frac{(X_i - \overline{X}_k)(Y_i - \overline{Y}_k)}{(X_i - \overline{X}_k)^2}}
\]

(3.7b)

2. Cochran's results

Under the assumption that (3.1) and its associated conditions hold, Cochran (1963, p. 334) studied a regression estimator for which the estimator given by (3.6) is a special case. The estimator is of the form

\[
\tilde{Y}_k = \overline{Y}_k - b(\overline{X}_k - \overline{X}_n)
\]

(3.8)
where $\bar{X}_n$ and $b$ are given by (3.7a) and (3.7b) respectively. Cochran makes no assumptions about the distribution of the $X$-variate, nor does he assume that the sample taken is necessarily random. Letting $X = (X_1, \ldots, X_n)$ Cochran gives

$$E(Y_k | X) = \mu_y + \bar{Z}_k - (\bar{X}_k - \bar{X}_n) \sum_{i=1}^{k} \frac{(X_i - \bar{X}_k)z_i}{\sum_{i=1}^{k} (X_i - \bar{X}_k)^2}$$

$$+ \frac{\rho \sigma_y}{\sigma_x} (\bar{X}_n - \mu_x) \quad (3.9)$$

and

$$\text{Var}(Y_k | X) = \rho^2 \frac{\sigma_y^2}{\sigma_x^2} \frac{(\bar{X}_n - \mu_x)^2}{\sigma_x^2}$$

$$+ \frac{\sigma_y^2(1-\rho^2)}{\sigma_x^2} \left[ \frac{1}{k} + \frac{\sum_{i=1}^{k} (X_i - \bar{X}_n)^2}{\sum_{i=1}^{k} (X_i - \bar{X}_k)^2} \right]. \quad (3.10)$$

Under the assumption of bivariate normality the estimator $\tilde{Y}_k$ is unbiased, and

$$\text{Var}(\tilde{Y}_k) = \frac{\rho^2 \sigma_y^2}{n} + \sigma_y^2(1-\rho^2) \left[ \frac{1}{k} + \left( \frac{1}{k} - \frac{1}{n} \right) \left( \frac{1}{k-3} \right) \right] \quad (3.11)$$

since the numerator and denominator of the expression in brackets in (3.10) are independent.
3. Regression estimator using concomitants of order statistics

Representing the ordered X-variates as

\[ X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \]

and the concomitant Y-variates as

\[ Y_{[1:n]}, \ Y_{[2:n]}, \ \cdots, \ Y_{[n:n]} \]

under the assumptions of (3.1) and its associated conditions we investigate a regression estimator of the form

\[ \tilde{Y}_{[k]} = \bar{Y}_{[k]} - b^*(\bar{X}(k) - \bar{X}_n) \]  

Where

\[ \bar{Y}_{[k]} = \frac{1}{k} \sum_{i=1}^{k} Y_{[r_i:n]} \]  

\[ \bar{X}(k) = \frac{1}{k} \sum_{i=1}^{k} X_{[r_i:n]} \]  

and

\[ b^* = \frac{k}{\sum_{i=1}^{k} \frac{(X_{[r_i:n]} - \bar{X}(k))(Y_{[r_i:n]} - \bar{Y}_{[k]})}{\sum_{i=1}^{k} (X_{[r_i:n]} - \bar{X}(k))^2}} \]

Our problem is to choose the k ranks \( r_1, \ldots, r_k \) to make \( \tilde{Y}_{[k]} \) an unbiased estimator of \( \mu_y \) and to minimize \( \text{Var}(\tilde{Y}_{[k]}) \).
Since we may write

\[ Y_{[r_1:n]} = \mu_y + \frac{\rho \sigma_Y}{\sigma_X} (X_{[r_1:n] - \mu_X}) + Z_{[r_1]} \]  

(3.13)

\[ Z_{[r_1]} \text{ and } X_{[r_1:n]} \text{ independent and } E(Z_{[r_1]}) = 0 \text{ for all } r_1, \]

from (3.9) we see that

\[ E(\tilde{Y}_{[k]}) = \mu_y \]

and so \( \tilde{Y}_{[k]} \) is unbiased no matter which \( k \) ranks are chosen. Similarly from (3.10)

\[ \text{Var}(\tilde{Y}_{[k]}) = \frac{\rho^2 \sigma_Y^2}{n} + \sigma_Y^2 (1 - \rho^2) \left[ \frac{1}{k} + E \left( \frac{(\bar{X}(k) - \bar{X}_n)^2}{\sum_{i=1}^{k} (X_{[r_1:n]} - \bar{X}(k))^2} \right) \right]. \]

(3.14)

Letting

\[ W = \frac{(\bar{X}(k) - \bar{X}_n)^2}{\sum_{i=1}^{k} (X_{[r_1:n]} - \bar{X}(k))^2} \]

(3.15)

we see that the set of \( r_i \) (\( i = 1, \ldots, k \)) minimizing \( \text{Var}(\tilde{Y}_{[k]}) \) is the same as the set minimizing \( E(W) \) and does not depend on \( \rho \). Since the numerator and denominator of \( W \) are not independent, the problem of evaluating \( E(W) \) is quite difficult.

Assuming now that \( X \) has a normal distribution, we will use a
Taylor expansion to obtain an approximation to $E(W)$. Since $W$ is location and scale invariant, without loss of generality we can assume $X_i \sim N(0, 1) \ (i = 1, \ldots, n)$.

Letting

$$g(X_{(k)}) = \left[ \sum_{i=1}^{k} (X_{r_i:n} - \bar{X}_{(k)})^2 \right]^{-1}$$

where $\bar{X}_{(k)} = (X_{r_1:n}, \ldots, X_{r_k:n})$ we will take the Taylor expansion about the point $\mu = (\mu_{r_1:n}, \ldots, \mu_{r_k:n})$, where $\mu_{r_i:n}$ is the mean of the $r_i$th order statistic from a sample of size $n$. So

$$E(W) = E \left[ (\bar{X}_{(k)} - \bar{X}_{n})^2 g(X_{(k)}) \right].$$

The first three terms of the Taylor expansion have been worked out in the appendix and $E(W)$ found for each.

The first approximation to $E(W)$ is

$$E_1 = \gamma (E(\bar{X}_{(k)})^2 - \frac{1}{n}) \quad (3.16)$$

where

$$\gamma = \left( \sum_{i=1}^{k} \mu_{r_i:n}^2 \right)^{-1}$$

and

$$\mu_{r_i:n} = \mu_{r_i:n} - \bar{\mu}$$
The second approximation to $E(W)$ is

$$E_2 = 3E_1 + \frac{2\nu}{n} - \frac{2\nu^2}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \mu_{r_i:n} \mu_{r_j:n} E(X_{r_i:n} X_{r_j:n} X_{r_k:n}). \quad (3.17)$$

The third approximation to $E(W)$ is

$$E_3 = 4E_2 - 6E_1 - \frac{3\nu}{n} - \frac{3\nu^3}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \mu_{r_i:n} \mu_{r_j:n} \sigma_{r_i r_j:n}$$

$$+ \frac{\nu^2}{n} \left( \sum_{i=1}^{k} \sigma_{r_i:n}^2 - \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k} \sigma_{r_i r_j:n} \right)$$

$$- \frac{\nu^2}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{\ell=1}^{k} E(X_{r_i:n} X_{r_j:n} X_{r_\ell:n})$$

$$+ \frac{\nu^2}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{\ell=1}^{k} \sum_{m=1}^{k} \left( \frac{4\nu}{k} \mu_{r_i:n} \mu_{r_j:n} \mu_{r_\ell:n} + \frac{1}{k} \right)$$

$$\cdot E(X_{r_i:n} X_{r_j:n} X_{r_\ell:n} X_{r_m:n}) \quad (3.18)$$

where $\sigma_{r_i r_j:n} = \text{Cov}(X_{r_i:n}, X_{r_j:n})$.

It is easily seen from the form of the above approximations that they involve second, third and fourth order moments and cross product moments of order statistics taken from a standard normal population.
Since we are more interested in the choice of ranks which will minimize \( E(W) \) than in the exact value of this expression, it would help if using the first approximation \( E_1 \) would lead to an optimal choice of ranks.

Extensive numerical work has been done and for \( k=2 \) \( E_1 \), \( E_2 \) and \( E_3 \) were calculated for all choices of symmetric ranks (i.e., if rank \( r_1 \) is selected, so is \( n-r_1+1 \)) in samples of size \( n = 4(1)50 \) and \( n = 60(10)100 \). The choices of \( r_1 \) and \( r_2 \) which minimized \( E_1 \), \( E_2 \) and \( E_3 \) were the same in almost all cases.

For \( k=3 \) and \( n = 9(10)19 \) both \( E_1 \) and \( E_2 \) were calculated for all choices of symmetric ranks (\( r_2 = \frac{1}{2} (n+1) \)) and once again \( E_1 \) and \( E_2 \) led to the same choices of \( r_1 \) and \( r_3 \) in almost all cases.

Further, for \( k=2 \), with those ranks which gave a minimum, \( E_2 < E_1 \) and \( E_2 < E_3 \); also for \( n > 20 \) we found \( E_3 < E_1 \). For \( k=3 \) we found \( E_2 < E_1 \) at those ranks which minimized the approximation. From this it appears that the Taylor expansion gives an alternating sequence and so \( E_1 \) would not be a bad approximation to \( E(W) \). Consequently, only this approximation was calculated for \( k = 4(1)10 \) and \( n = 19(10)49 \). The optimal choice of ranks to minimize \( \text{Var}(\tilde{y}_{[k]}) \) is given for \( k=2 \), \( n = 4(1)50(10)100 \) and \( k = 3(1)10 \), \( n = 19(10)49 \) in Table 3.1.

C. Linear Functions of Concomitants as Estimators of \( \mu_y \)

1. Best linear function

Assuming the validity of (3.1) and its associated conditions, we turn now to a simple class of estimators of \( \mu_y \), namely
where both \( a_i \) and \( r_i \) (i = 1, ..., k) are to be determined so as to make \( Y^* \) unbiased and of minimum variance.

Since

\[
Y[r_i:n] = \mu_y + \frac{\rho \sigma_y}{\sigma_x} (X_{r_i:n} - \mu_x) + Z[r_i] \quad (i = 1, ..., k)
\]

it follows that

\[
E(Y^*) = \mu_y \sum_{i=1}^{k} a_i + \rho \sigma_y \sum_{i=1}^{k} a_i \mu_{r_i:n}
\]

(3.19)

where

\[
\mu_{r_i:n} = E\left( \frac{X_{r_i:n} - \mu_x}{\sigma_x} \right)
\]

so that if \( Y^* \) is to be unbiased we must have

\[
\sum_{i=1}^{k} a_i = 1 \quad \text{and} \quad \sum_{i=1}^{k} a_i \mu_{r_i:n} = 0.
\]

Therefore, a set of sufficient conditions for unbiasedness when \( X \) has a symmetric distribution is
1) the ranks to be chosen symmetrically

\[ a_{k-i+1} = a_i \]

\[ \sum_{i=1}^{k} a_i = 1 \]

From (1.3) we find that

\[ \text{Var}(Y^*) = \sigma^2 \left[ \frac{1}{\gamma} \left( 1 - \rho^2 \right) \sum_{i=1}^{k} a_i^2 + \rho^2 \text{Var}(X^*) \right] \]

where

\[ X^* = \sum_{i=1}^{k} a_i X_{r_i:n} \]

The problem we now consider is the choice of \( a_i, r_i \) (\( i = 1, \ldots, k \)) under the conditions of (3.20) so as to minimize \( \text{Var}(Y^*) \).

We consider first the special case \( \rho = 1 \) when minimization of \( \text{Var}(Y^*) \) is equivalent to minimization of \( \text{Var}(X^*) \). If the distribution of \( X \) is symmetric, then a symmetric spacing of the ranks is used. The optimal choice of ranks \( r_i \) and coefficients \( a_i \) was studied by Ogawa (1951) in the asymptotic case when \( r_i/n \rightarrow \lambda_i \) as \( n \rightarrow \infty \) with \( 0 < \lambda_1 < \ldots < \lambda_k < 1 \). Higuchi (1956) proved that the optimal spacing was necessarily symmetric when \( X \) has a normal distribution.

When \( |\rho| < 1 \) the situation changes greatly, since under a wide set of conditions (including the normal case) \( \text{Var}(X^*) \rightarrow 0 \) as \( n \rightarrow \infty \). In this situation \( \text{Var}(Y^*) \) is minimized for \( a_i = 1/k \) (\( i = 1, \ldots, k \)).
and any choice of ranks $r_i$ which satisfy the conditions of (3.20). That is, if $X$ has a symmetric distribution, then any choice of symmetrically spaced ranks is appropriate. In this case $Y^* = \bar{Y}[k]$ and for all $n$ we have

$$\text{Var}(\bar{Y}[k]) = \frac{\sigma_y^2}{n} \left( 1 - \rho^2 \right) + \rho^2 \sigma_y^2 \text{Var} \bar{X}(k),$$

(3.22)

and we can see that the asymptotic efficiency of $\bar{Y}[k]$ to $\bar{Y}_k$, the mean of $k$ randomly chosen $Y$'s, is $1/(1-\rho^2)$.

From this point on we will assume that $n$ is finite and $X$ symmetrically distributed and so we will confine consideration to symmetric spacing of the ranks, i.e.,

$$r_{k-i+1} = n-r_i+1 \quad i = 1, \ldots, \frac{1}{2}k \quad (k \text{ even})$$

$$i = 1, \ldots, \frac{1}{2} (k+1) \quad (k \text{ odd}).$$

**Theorem:** For any given symmetric spacing of the $r_i$ the minimum variance of

$$Y^* = \sum_{i=1}^{k} a_i Y[r_i;n]$$

occurs when

$$a_{k+l-i} = a_i \quad i = 1, \ldots, IP(\frac{1}{2}(k+1))$$

where $IP(x)$ denotes the integer part of $x$. 
Proof: From (3.21) it is easily seen that the result needs only be established for $X^*$ since if $a_{k-i+1} \neq a_i$ then $\sum_{i=1}^{k} a_i^2$ will be decreased on replacing both $a_{k-i+1}$ and $a_i$ by $\frac{1}{2} (a_{k-i+1} + a_i)$.

The result is obvious for $k=1$ and easily shown for $k=2$. Suppose that $X'$ is of the required form for $k$ ranks $r_i$ $(i=1, \ldots, k)$ and consider, for $i < j$,

$$X^* = b_1 X_{r_i:n} + (b-b_1) X_{r_j:n} + (1-b) X',$$

where $r_i + r_j = n + 1$ and $\text{Var}(X') = c \sigma_x^2$ (c a constant). We will show that, for fixed $b (<1)$, $\text{Var}(X^*)$ is minimized when $b_1 = \frac{1}{2} b$, so that $X^*$ is of the required form for $k+2$ values of the ranks.

Since $r_j = n+1-r_i$ we have

$$\text{Cov}(X_{r_i:n}, X') = \text{Cov}(X_{r_i:n}, X')$$

and thus

$$\text{Var}(X^*) = \sigma_x^2 \left[ (2b_1^2 - 2bb_1 + b^2) \sigma_{r_i:n}^2 + (1-b)^2 \sigma^2 + 2b_1 (b-b_1) \sigma_{r_i:n} + 2b(1-b) \text{Cov}(X_{r_i:n}, X') \right]$$

$$= 2(b_1 - \frac{1}{2} b)^2 \left( \sigma_{r_i:n}^2 - \sigma_{r_i r_j:n} \right) + G(b)$$

where $G(b)$ is a function of $b$ only. Since $\sigma_{r_i:n}^2 > \sigma_{r_i r_j:n}$ the
minimum of $\text{Var}(X^*)$ occurs when $b_1 = \frac{1}{2} b$ and the main result, being true for $k = 1, 2$, has thus been proved by induction from $k$ to $k+2$.

The coefficients $a_i$ ($i = 1, \ldots, k$) which minimize $\text{Var}(Y^*)$ are generally functions of $\rho$, and of the choice of ranks. We will find expressions for the coefficients for $k = 2, 3, 4$ and $5$.

For $k = 2$, $a_1 = a_2 = \frac{1}{2}$ and

$$Y^* = \overline{Y}_{[2]} = \frac{1}{2} \left( \overline{Y}_{[1:n]} + \overline{Y}_{[n-1:n]} \right).$$

From (3.22) we see that the ranks which minimize $\text{Var}(\overline{Y}_{[2]})$ are those which minimize $\text{Var}(\overline{Y}_{(2)})$. In the case where $X$ is normally distributed the optimal ranks were tabulated by Dixon (1957) for $n \leq 20$. In Table 3.1 the optimal ranks are given for $n = 4(1)50(1)100$. Asymptotically it is well-known that $r_1 = \text{IP}(0.27n+1)$.

For $k=3$,

$$a_1 = a_3$$
$$a_2 = 1 - 2a_1.$$

To find $a_1$ we rewrite (3.21) as a function of $a_1$, and set the derivative with respect to $a_1$ equal to zero. We find the value of $a_1$ minimizing $\text{Var} Y^*$ for a given value of $r_1$ is

$$a_1 = \frac{(1-\rho^2) + \rho^2 \left( \frac{\sigma^2_{r_2:n} - \sigma r_1 r_2:n}{r_2:n} \right)}{3(1-\rho^2) + \rho^2 \left( \frac{\sigma^2_{r_1:n} - 4\sigma r_1 r_2:n + 2\sigma^2_{r_2:n} + \sigma r_1 r_3:n}{r_1:n} \right)}.$$ 

(3.23)
Here $r_2$ is the median rank.

For $k=4$

$$a_1 = a_4$$

$$a_2 = a_3 = (\cdot 5 - a_1),$$

and the minimizing $a_1$ is

$$a_1 = \frac{(1 - \rho^2) + \rho^2 (\sigma^2_{r_2:n} + \sigma^2_{r_3:n} - \sigma_{r_1r_2:n} - \sigma_{r_1r_3:n})}{4(1 - \rho^2) + 2\rho^2 (\sigma^2_{r_1:n} + \sigma^2_{r_2:n} + \sigma_{r_1r_4:n} + \sigma_{r_2r_3:n} - 2\sigma_{r_1r_2:n} - 2\sigma_{r_1r_3:n})}.$$  

(3.24)

For $k=5$

$$a_1 = a_5$$

$$a_2 = a_4$$

$$a_3 = 1 - 2a_1 - 2a_2,$$

and the $a_1$ and $a_2$ which minimize $\text{Var}(Y^*)$ for a given choice of ranks are the $a_1$, $a_2$ which are solutions to

$$\begin{align*}
  c_{11} a_1 + c_{12} a_2 &= d_1 \\
  c_{21} a_1 + c_{22} a_2 &= d_2
\end{align*}$$

(3.25)

where
For \( k = 3, 4, 5 \) and \( |\rho| < 1 \) it is easily seen from (3.23-3.25) that when \( \sigma_{r_i^1 r_j^1} \to 0 \) as \( n \to \infty \) \((i,j = 1, \ldots, 5)\) the coefficients \( a_i \to 1/k \).

Looking at (3.23-3.25) we see that the coefficients which minimize Var(\( Y^\star \)) are functions of \( \rho \) and of the choice of ranks. When \( X \) is normally distributed Table 3.2 gives the values of these coefficients for the optimal choice of ranks when \( n = 19 \) and \( k = 3, 4, 5 \). These numerical results indicate that the terms involving \( (1-\rho^2) \) in (3.23-3.25) are dominant and consequently the coefficients lie very close to \( 1/k \) for most values of \( \rho \). Since the minimizing coefficients change for each choice of ranks, the optimal choice of ranks may not be the same for all values of \( \rho \). However, in the numerical work done the optimal choice of ranks remained the same for all values of \( \rho \) used.
2. **Equally weighted linear function**

Since the coefficients of $Y^*_0$, the optimal $Y^*$, are functions of $\rho$ and the choice of ranks, they will almost always be difficult to obtain and the estimator $Y^*_0$ will be hard to use in practice. Estimators of the form of $Y^*$ are unbiased, whatever the value of $\rho$ and the effect of using an incorrect value of $\rho$ in finding the coefficients will just be to produce an estimator which is not quite of minimum variance.

Further, from the numerical work done, it appears that the coefficients of $Y^*_0$ lie quite close to $1/k$, so it would be of interest to look at an estimator of the form $\overline{Y}_{[k]}$, where

$$\overline{Y}_{[k]} = \frac{1}{k} \sum_{i=1}^{k} Y[r_{i:n}] .$$

This estimator has the advantage that it is easy to use and its variance should not be much above the minimum.

If $X$ has a symmetric distribution, then choosing the ranks $r_i$ ($i = 1, \ldots, k$) will make $\overline{Y}_{[k]}$ an unbiased estimator of $\mu_Y$, since $a_i = 1/k$ $i = 1, \ldots, k$. From (3.22) we see that the choice of ranks minimizing $\text{Var}(\overline{Y}_{[k]})$ are those which minimize $\text{Var}(\overline{X}_{(k)})$ and $\overline{X}_{(k)}$ is given by (3.12b). The optimal choice of ranks is given in Table 3.1 for $k = 2$, $n = 4(1)50(10)100$ and $k = 3(1)10$, $n = 19(10)49$, when $X$ is normally distributed. Asymptotically, when $X$ is normal, extensive work has been done by Mosteller (1946). Mosteller assumed that as
\[ n \to \infty , \frac{r_i}{n} \to \lambda_i \quad (i = 1, \ldots, k) \] and found the optimal choice of the \( \lambda_i \) \((i = 1, \ldots, k)\) for \( k = 3 \). He also gave a general form of the equations which had to be solved to find the choice of \( \lambda_i \) to minimize \( \text{Var}(\bar{X}(k)) \). We have found these optimal values of \( \lambda_i \) for \( k = 4(1)10 \) and Table 3.3 gives the optimal choice of \( \lambda_i \) for \( k = 2(1)10 \).

Considering these \( \lambda_i \) and letting \( r_i = IP(n\lambda_i + 1) \) it is easily seen that this choice of ranks will be approximately the same as those given in Table 3.1 under \( \bar{Y}[k] \). Clearly then, in an experimental situation, Table 3.3 can be used in making the choice of which \( k \) ranks to use. Looking at Table 3.3 we can see that the \( \lambda_i \) are roughly evenly spaced, so a choice of ranks \( r_i = IP(n\lambda_i' + 1) \) where \( \lambda_i' = (i-1/2)/k \) \((i = 1, \ldots, k)\), would probably lead to a good estimator of \( \mu_y \).

D. Comparisons of Estimators

The estimators to be compared are

1) \( \bar{Y}_k = \frac{1}{k} \sum_{i=1}^{k} Y_i \) - the average of a random choice of \( k \) \( Y \)-variates

2) \( \bar{Y}_k = \bar{Y}_k - b(\bar{X}_k - \bar{X}_n) \) - the regression estimator with a random choice of concomitants

3) \( \bar{Y}[k] = \frac{1}{k} \sum_{i=1}^{k} Y_i[r_i:n] \) - the equally weighted linear function of concomitants
4) \( \tilde{Y}_{[k]} = \bar{Y}_{[k]} - b^*(X(k) - \bar{X}_n) \) - the regression estimator with an optimal choice of concomitants.

When (3.1) and its associated assumptions hold, and \( X \) is normally distributed, we see from (3.11), (3.14) and (3.22)

1) \( \text{Var}(\bar{Y}_k) = \frac{\sigma^2}{y^2} / k \)

2) \( \text{Var}(\bar{Y}_k) = \frac{\sigma^2 (1-\rho^2)}{y^2} k + \frac{\rho^2 \sigma^2}{n^2} + \frac{\sigma^2 (1-\rho^2)}{y^2} \left( \frac{1}{k} - \frac{1}{n^2} \right) \quad k > 3 \)

3) \( \text{Var}(\bar{Y}[k]) = \frac{\sigma^2 (1-\rho^2)}{y^2} k + \rho^2 \frac{\sigma^2}{y^2} \text{Var}(\bar{X}(k)) \)

4) \( \text{Var}(\tilde{Y}[k]) = \frac{\sigma^2 (1-\rho^2)}{y^2} k + \frac{\rho^2 \sigma^2}{n^2} + \sigma^2 (1-\rho^2) \text{E} \left[ \frac{(\bar{X}(k) - \bar{X}_n)^2}{k} \frac{1}{\Sigma_{i=1} (X_{r_i:n} - \bar{X}(k))^2} \right] \)

Letting

\[
\text{n}(X(k)) = \frac{(\bar{X}(k) - \bar{X}_n)^2}{\Sigma_{i=1} (X_{r_i:n} - \bar{X}(h))^2}
\]

where

\( \bar{X}(k) = (X_{r_1:n}, \ldots, X_{r_k:n}) \),

if a Taylor expansion is taken
\[ E(h(X)) = h(\mu) + \ldots \]

where \( \mu = E(X) \) and asymptotically \( h(\mu) = 0 \). So that as \( n \to \infty \)

\[ E(W) = E \left[ \frac{(\overline{X}(k) - \overline{X}_n)^2}{\Sigma \sum (X_{r_i:n} - \overline{X}(k))^2} \right] \to 0 ; \]

also, since \( \overline{X}(k) \) is a linear function of the order statistics, \( \text{Var}(\overline{X}(k)) \to 0 \) as \( n \to \infty \). It follows that, asymptotically

1) \( \text{Var}(\overline{Y}_k) = \frac{\sigma_Y^2}{k} \)

2) \( \text{Var}(\widetilde{Y}_k) = \frac{\sigma_Y^2(1-p^2)}{k} \left( \frac{k-2}{k-3} \right) \)

3) \( \text{Var}(\overline{Y}_{[k]}) = \frac{(1-p^2)}{k} \sigma_Y^2 \)

4) \( \text{Var}(\widetilde{Y}_{[k]}) = \frac{(1-p^2)}{k} \sigma_Y^2 \).

So, for \( |p| > 0 \), the estimators \( \overline{Y}_{[k]} \) and \( \widetilde{Y}_{[k]} \) which involve ordering the X's are asymptotically more efficient than \( \overline{Y}_k \) and \( \widetilde{Y}_k \).

Comparing \( \text{Var}(\overline{Y}_{[k]}) \) and \( \text{Var}(\widetilde{Y}_{[k]}) \) for finite \( n \), we see that the simple estimator \( \overline{Y}_{[k]} \) actually has smaller variance than \( \widetilde{Y}_{[k]} \) as long as
\[ \rho^2 < \rho^2_c = \frac{E(W)}{\text{Var}(\tilde{X}(k)) - \frac{1}{n} + E(W)} . \]

For \( n \) in the range 19-49 it was found that \( \rho_c \) is almost independent of \( n \) but decreases as \( k \) increases, average values being

\begin{center}
\begin{tabular}{cccccccccc}
\( k \) & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\( \rho_c \) & .65 & .57 & .50 & .44 & .41 & .38 & .36 & .33 & .31 \\
\end{tabular}
\end{center}

However, \( \tilde{Y}_k \) is never much inferior and with \( \rho \) close to 1 it becomes an appreciably better estimator than \( \tilde{Y}_k \). This is illustrated by Table 3.4 which gives the variances of \( \tilde{Y}_k \), \( \tilde{Y}_k \) and their asymptotic variance for \( k = 2, 3 \) and \( n = 9(10)^{49} \), under the assumption that \( \sigma^2 = 1 \). Table 3.4 also lists the asymptotic variance, and the variances of \( \tilde{Y}_k \), \( \tilde{Y}_k \) and \( \tilde{Y}_k \) for \( k = 4(10)^{10} \) and \( n = 19(10)^{49} \). At this point, we should note that \( \text{Var}(\tilde{Y}_k) \) given in Table 3.4 is an approximation, as is \( \text{Var}(\tilde{Y}_k) \) for \( n > 20 \). The approximations used are shown in the appendix.

It is not surprising that \( \text{Var}(\tilde{Y}_k) = 1/k \) comes out very poorly compared to the estimators in Table 3.4 unless \( \rho \) is very small. What is somewhat surprising is the relatively poor performance of \( \tilde{Y}_k \) except when \( \rho \) is close to 1. However, this estimator improves as \( k \) increases. It is noteworthy that the estimators \( \tilde{Y}_k \) and \( \tilde{Y}_k \) have variances which are quite close to the asymptotic variance. This seems to indicate that it would be difficult to improve on these estimators.
to any great extent. Clearly, from Table 3.4, the estimators \( \bar{Y}_{[k]} \) and \( \tilde{Y}_{[k]} \) are to be preferred over \( \bar{y}_k \) and \( \tilde{y}_k \). If it comes down to a choice between using \( \bar{Y}_{[k]} \) or \( \tilde{Y}_{[k]} \) then, unless we have reason to suspect \( \rho \) is close to 1, \( \bar{Y}_{[k]} \) recommends itself because of its simplicity.

E. Comparison with Ranked Set Sampling

The concept of 'ranked set sampling' was explained in the introduction. In this section we shall compare \( \hat{Y}_{[k]} \), the 'ranked set' estimator with \( \bar{Y}_{[k]} \). From (1.5) we know

\[
\hat{Y}_{[k]} = \frac{1}{k} \sum_{i=1}^{k} Y_{[i:k]}(i)
\]

where \( Y_{[i:k]}(i) \) is the concomitant of the \( i \)th order statistic in the \( i \)th subsample.

If (3.1) holds

\[
\hat{Y}_{[k]} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} \sum_{i=1}^{k} \left( \frac{X_{i:k}^{(i)} - \mu_X}{\sigma_X} \right) + \bar{Z},
\]

where \( X_{i:k}^{(i)} \) is the \( i \)th ordered random variable in the \( i \)th subsample.

Now

\[
E(\hat{Y}_{[k]}) = \mu_Y
\]

and, since \( X_{i:k}^{(i)} \) (\( i = 1, \ldots, k \)) are mutually independent,
\begin{equation}
\text{Var}(\hat{Y}_{[k]}) = \frac{(1-\rho^2)\sigma^2}{k} + \frac{\rho^2 \sigma^2}{k^2} \sum_{i=1}^{k} \sigma^2_{i:k} .
\end{equation}

It is to be noted that (3.26) and (3.27) do not require \( X \) to have a symmetric distribution. Although breaking the sample size \( n \) into \( k \) subsamples of size \( k \) is arbitrary, this procedure has the advantage that it requires rankings to be made in small groups of \( k \) rather than in the large sample \( n \), thus reducing the probability of errors in ranking in some situations. For \( X \) normally distributed, the relative efficiency of \( \overline{Y}_{[k]} \) to \( \hat{Y}_{[k]} \) was found for \( k = 3, 7 \) and \( \rho = .05(.05), .95, .99 \), in Table 3.5. From Table 3.5 we see that \( \overline{Y}_{[k]} \) is considerably more efficient than \( \hat{Y}_{[k]} \) unless \( \rho \) is quite small.

F. Summary

In this chapter we have investigated three estimators of \( \mu_y \) using concomitants of order statistics. We have shown that these estimators \( \hat{Y}^* \), \( \overline{Y}_{[k]} \) and \( \tilde{Y}_{[k]} \) compare favorably to using the average of a random choice of \( k \) \( Y \)-variates \( \overline{Y}_k \) and to a regression estimator \( (\overline{Y}_k) \) using a random choice of \( Y \)-variates. It is easily seen that, while the best linear combination of concomitants \( Y_0^* \) is a better estimator of \( \mu_y \) than \( \overline{Y}_{[k]} \), the difficulties in determining the coefficients of \( Y_0^* \) make it impractical to use; also from the numerical results obtained it appears that the coefficients of \( Y_0^* \) are quite close to \( 1/k \) except for \( \rho \) close to 1, so the loss in efficiency from using \( \overline{Y}_{[k]} \) will be quite small.
Comparing $\tilde{Y}[k]$ to $Y[k]$ we see that their variances are almost equivalent, with $Y[k]$ being more efficient for smaller values of $p$ and $\tilde{Y}[k]$ being more efficient as $p$ increases. Looking at Table 3.1 which gives the optimal choice of ranks for $\tilde{Y}[k]$ and $Y[k]$ we see that using the regression estimator leads to a choice of ranks more in the tails than the choice of ranks obtained using $Y[k]$.

We have discovered that if the estimator $Y[k]$ is to be used, an optimal or near optimal choice of ranks can be made by using Table 3.3, which gives the asymptotic quantiles first studied by Mosteller (1946). That is, letting $r_i = \text{IP}(n\lambda_i+1)$, $(i = 1, \ldots, k)$ will it appears, lead to an optimal or near optimal choice of ranks. Since the values in Table 3.3 are roughly equidistant it would appear that choosing ranks which are equidistant will lead to a good estimator of $\mu_Y$.

As we can see $Y[k]$ has a number of advantages in practical use. It is quite easy to find, reasonably efficient as compared to the other estimators investigated and, since it leads to choosing ranks which are roughly equidistant it would appear to be fairly robust for symmetric distributions. In addition, it has applicability in a wider range of situations than $Y[k]$, since it depends only on a knowledge of the ranks of the $X$-variate and not on a knowledge of the actual values.
Table 3.1. Optimal ranks of concomitants for the estimators $\tilde{Y}_{[k]}$ and $\bar{Y}_{[k]}$

$n = \text{sample size}$  
$k = \text{no. of concomitants}$

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Table 3.4 (continued)

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Table 3.5. Relative efficiency of $\hat{Y}_k$ to $Y_\beta$

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IV. BIBLIOGRAPHY


V. ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. H. A. David for suggesting this area of research, and for his many helpful suggestions in the course of the research.

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VI. APPENDIX

A. General Form of Approximations to \( E(W) \)

In this section, we will use a Taylor expansion to obtain approximations to \( E(W) \), where

\[
W = \frac{(\bar{X}(k) - \bar{X}_n)^2}{\sum_{i=1}^{k} (X_{r_i:n} - \bar{X}(k))^2}
\]

\[
\bar{X}(k) = \frac{1}{k} \sum_{i=1}^{k} X_{r_i:n} \quad \text{and} \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]

under the assumption that \( X_1, \ldots, X_n \) are independent, normally distributed random variables. Since as we have stated previously \( W \) is location and scale invariant, we assume \( X_i \sim N(0, 1) \), \( i = 1, \ldots, n \).

Letting

\[
ge(X(k)) = \left[ \sum_{i=1}^{k} (X_{r_i:n} - \bar{X}(k))^2 \right]^{-1},
\]

with \( X(k) = (X_{r_1:n}, \ldots, X_{r_k:n}) \), we will take a Taylor expansion of \( g(X(k)) \) about the point \( \mu = (\mu_{r_1:n}, \ldots, \mu_{r_k:n}) \).

The first term of the expansion

\[
ge_1(X(k)) = g(\mu) = \gamma,
\]
where
\[ \gamma = \left( \sum_{i=1}^{k} \mu_{r_i:n}^2 \right)^{-1}, \tag{6.1} \]

\[ \mu_{r_i:n} = \mu_{r_i:n} - \bar{\mu}, \]

and
\[ \bar{\mu} = \frac{1}{k} \sum_{i=1}^{k} \mu_{r_i:n}. \]

So
\[ E(W) = E(\bar{X}_{(k)} - \bar{X}_n)^2 \cdot s_1(\bar{X}_{(k)}) = E_1 \text{ say}. \]

\[ E_1 = \gamma E(\bar{X}_{(k)} - \bar{X}_n)^2 = \gamma E(\bar{X}_{(k)}^2 - 2\bar{X}_{(k)} \bar{X}_n + \bar{X}_n^2). \]

Now \( E(\bar{X}_n^2) = \frac{1}{n} \) and since, from David (1970, p. 31), we know
\[ E(X_{i:n} \bar{X}_n) = \frac{1}{n} \quad (i = 1, \ldots, n), \]

we see
\[ E(\bar{X}_{(k)} \bar{X}_n) = \frac{1}{k} \sum_{i=1}^{k} E(X_{r_i:n} \bar{X}_n) = \frac{1}{n}, \]

and we may write
\[ E_1 = \gamma \left[ E(\bar{X}_{(k)}^2) - \frac{1}{n} \right]. \tag{6.2} \]

In the normal case the second and third approximations can be
Lemma 6.1: If $X_1, X_2, \ldots, X_n$ are independent $N(\mu, 1)$ variates, then

$$E \left[ (\overline{X}_n - \overline{X})^2 f(X'_n) \right] = E \left[ \overline{X}_n^2 f(X'_n) \right] - \frac{1}{n} E(f(X'_n)) \quad (6.3)$$

where $f$ is any function such that the expectations exist and $X'_n = (X_1 - \overline{X}_n, \ldots, X_n - \overline{X}_n)$.

Proof: From inspection we see that the left-hand side of (6.3) is location free, so without loss of generality, we assume $\mu = 0$.

Expanding, we write

$$E \left[ (\overline{X}_n - \overline{X})^2 f(X'_n) \right] = E \left[ (\overline{X}_n^2 - 2\overline{X}_n \overline{X} + \overline{X}^2) f(X'_n) \right]$$

and

$$E(\overline{X}_n \overline{X}_n f(X'_n)) = E \overline{X}_n (\overline{X}_n f(X'_n)) + E \overline{X}^2 f(X'_n) \quad (6.4)$$

Since by a well-known property of the normal distribution we know that $\overline{X}_n$ and functions of $X_i - \overline{X}_n$ are independent, and since $(\overline{X}_n - \overline{X}) f(X'_n)$ is like $f(X'_n)$ a function of $X'_n$ we have that

$$E \overline{X}_n (\overline{X}_n - \overline{X}) f(X'_n) = E(\overline{X}_n) E(\overline{X}_n - \overline{X}) f(X'_n) = 0$$

since $E(\overline{X}_n) = 0$. So
\[
E(\overline{x}_n f(x_n')) = E(\overline{x}_n^2 f(x_n')) = \frac{1}{n} E f(x_n')
\] (6.5)

and we have

\[
E \left[ (\overline{x}_n - \overline{x}_n^2 f(x_n')) \right] = E \left[ \overline{x}_n^2 f(x_n') - \frac{1}{n} E(f(x_n')) \right],
\]

which is the desired result.

The two term Taylor expansion of \(g(X(k))\),

\[
g_2(X(k)) = g(\mu) + \sum_{i=1}^{k} \left. \frac{dg}{dx_{r_i:n}} \right|_{x_{r_i:n} = \mu} (x_{r_i:n} - \mu_{r_i:n}).
\]

\[
\frac{dg}{dx_{r_i:n}} = - (\sum_{i=1}^{k} (x_{r_i:n} - \overline{x}(k))^2)^{-2} (x_{r_i:n} - \overline{x}(k))
\]

so

\[
\left. \frac{dg}{dx_{r_i:n}} \right|_{x_{r_i:n} = \mu} = - 2\gamma^2 \mu_{r_i:n}.
\]

and

\[
g_2(X(k)) = \gamma - 2\gamma^2 \sum_{i=1}^{k} \mu_{r_i:n} (x_{r_i:n} - \mu_{r_i:n}).
\]

Letting

\[
E(\bar{\omega}) = E(\overline{x}_n - \overline{x}_n^2) g_2(X(k)) = E_2 \quad \text{(say)},
\]
we have

\[ E_2 = E_1 - 2\gamma^2 \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot E[(X_{r_i:n} - \mu_{r_i:n})(\bar{X}_{n}(k) - \bar{X}_{n})^2] \]

\[ = E_1 - 2\gamma^2 \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot E[X_{r_i:n}(\bar{X}_{n}(k) - \bar{X}_{n})^2] + 2\gamma^2 \gamma^{-1} E(\bar{X}_{n}(k) - \bar{X}_{n})^2 \]

\[ = 3E_1 - 2\gamma^2 \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot E[X_{r_i:n}(\bar{X}_{n}(k) - \bar{X}_{n})^2] \] .

(6.6)

Now

\[ \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot X_{r_i:n} = \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot (X_{r_i:n} - \bar{X}_{n}) \]

so Lemma 6.1 can be applied to (6.6) and we have

\[ E_2 = 3E_1 - 2\gamma^2 \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot E(X_{r_i:n}(\bar{X}_{n})^2) + \frac{2\gamma^2}{n} \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot E(X_{r_i:n}) \] .

(6.7)

We know

\[ \sum_{i=1}^{k} \mu_{r_i:n}^i \cdot E(X_{r_i:n}) = \sum_{i=1}^{k} \mu_{r_i:n}^{12} = \gamma^{-1} \]

and

\[ E(X_{r_i:n}(\bar{X}_{n}(k))^2) = \frac{1}{k^2} \sum_{j=1}^{k} E(X_{r_i:n}X_{r_j:n}X_{r_\ell:n}) \].
Therefore (6.7) may be rewritten as

$$E_2 = 3E_1 + \frac{2\gamma}{n} - \frac{2\gamma^2}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{\ell=1}^{k} \mu_{r_i:n} \mu_{r_j:n} \mu_{r_\ell:n} E(X_{r_i:n} X_{r_j:n} X_{r_\ell:n}) \cdot (6.8)$$

Looking now at the three term Taylor expansion of $g(X_{(k)})$

$$g_3(X_{(k)}) = g_2(X_{(k)}) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} \left[ \frac{\partial^2 g}{\partial x_{r_i:n} \partial x_{r_j:n}} \bigg|_{x_{(k)} = \mu} \right] \times (x_{r_i:n} - \mu_{r_i:n}) (x_{r_j:n} - \mu_{r_j:n}) .$$

We have

$$\frac{\partial^2 g}{\partial x_{r_i:n}^2} = 4 \left( \sum_{i=1}^{k} (x_{r_i:n} - \bar{x}_{(k)})^2 \right)^{-3} (x_{r_i:n} - \bar{x}_{(k)})$$

$$- 2 \left( \sum_{i=1}^{k} (x_{r_i:n} - \bar{x}_{(k)})^2 \right)^{-2} (1 - \frac{1}{k}) (i = 1, \ldots, n)$$

and

$$i \neq j \quad \frac{\partial^2 g}{\partial x_{r_i:n} \partial x_{r_j:n}} = 2 \left[ 4 \left( \sum_{i=1}^{k} (x_{r_i:n} - \bar{x}_{(k)})^2 \right)^{-3} (x_{r_i:n} - \bar{x}_{(k)}) \right]$$

$$\times (x_{r_j:n} - \bar{x}_{(k)}) + \frac{1}{k} \left( \sum_{i=1}^{k} (x_{r_i:n} - \bar{x}_{(k)})^2 \right)^{-2} \right] .$$
Evaluating at $X(k) = \mu$, we write

$$\frac{d^2g}{dx_{i:n}^2} \bigg|_{X(k) = \mu} = 8\gamma^3 \mu_{i:n}^\prime \mu_{i:n}^\prime - 2\gamma^2 (1 - \frac{1}{k}) \quad i = 1, \ldots, n$$

and

$$\frac{d^2g}{dx_{i:n}^2 dx_{j:n}^2} \bigg|_{X(k) = \mu} = 8\gamma^3 \mu_{i:n}^\prime \mu_{j:n}^\prime + \frac{2\gamma^2}{k} \quad i, j = 1, \ldots, n, \quad i \neq j .$$

The third approximation to $E(W)$ is

$$E(W) = E[(\overline{X}_k - \overline{X}_n)^2 g_3(X(k))] = E_3 \quad (\text{say}) ,$$

where

$$E_3 = E_2 + E \left[ 4 \gamma^3 \left[ \sum_{i=1}^{k} (X_{i:n} - \mu_{i:n})^2 (\overline{X}_k - \overline{X}_n)^2 \right] \right]$$

$$- \gamma^2 \sum_{i=1}^{k} (X_{i:n} - \mu_{i:n})^2 (\overline{X}_k - \overline{X}_n)^2 + \frac{\gamma^2}{k} \left[ \sum_{i=1}^{k} (X_{i:n} - \mu_{i:n})^2 \right]^2$$

$$\times (\overline{X}_k - \overline{X}_n)^2 . \quad (6.9)$$
Letting

\[ X'_{ri:n} = X_{ri:n} - \mu_{ri:n} \]

and

\[ \bar{X}'(k) = \frac{1}{k} \sum_{i=1}^{k} X'_{ri:n} \]  \hspace{1cm} (i = 1, \ldots, k) \tag{6.10} \]

we may express (6.9) as

\[
E_3 = E_2 + 4\gamma^3 \left[ (\bar{X}(k) - \bar{X}_n)^2 \left[ \sum_{i=1}^{k} \mu_{ri:n} X_{ri:n} - \gamma^{-1} \right] \right. \\
- \left. \gamma^2 \sum_{i=1}^{k} (X'_{ri:n} - \bar{X}'(k))^2 \right]. \tag{6.11} 
\]

Expanding and rewriting (6.11) we have

\[
E_3 = E_2 + E \left[ (\bar{X}(k) - \bar{X}_n)^2 \left[ 4\gamma^3 \sum_{i=1}^{k} \mu_{ri:n} X_{ri:n}^2 - \gamma^2 \sum_{i=1}^{k} (X'_{ri:n} - \bar{X}'(k))^2 \right] \right. \\
+ \left. 4\gamma E(\bar{X}(k) - \bar{X}_n)^2 - 8\gamma^2 \sum_{i=1}^{k} \mu_{ri:n} X_{ri:n} \right]. \tag{6.12} 
\]

Now

\[ 4\gamma E(\bar{X}(k) - \bar{X}_n)^2 = 4E_1 \]

and from (6.8) we can see
- \( 2\gamma^2 \mathbb{E}[(\bar{X}(k) - \bar{X}_n)^2] \sum_{i=1}^{k} \mu_{r_i:n} X_{r_i:n} ] = E_2 - 3E_1. \)

Consequently

\[
4\gamma^3 \mathbb{E}(\bar{X}(k) - \bar{X}_n)^2 - 8\gamma^2 \mathbb{E}[(\bar{X}(k) - \bar{X}_n)^2] \sum_{i=1}^{k} \mu_{r_i:n} X_{r_i:n} ] = 4E_2 - 8E_1. \\
(6.13)
\]

Since we may write

\[
\sum_{i=1}^{k} \mu_{r_i:n} X_{r_i:n} = \sum_{i=1}^{k} \mu_{r_i:n} (X_{r_i:n} - \bar{X}_n)
\]

and

\[
X_{r_i:n} - \bar{X}(k) = (X_{r_i:n} - \bar{X}_n) - (\bar{X}(k) - \bar{X}_n)
\]

we may apply Lemma 6.1 and get

\[
\mathbb{E} \left\{ (\bar{X}(k) - \bar{X}_n)^2 \left[ 4\gamma^3 \left( \sum_{i=1}^{k} \mu_{r_i:n} X_{r_i:n} \right)^2 - \gamma^2 \sum_{i=1}^{k} \left( X_{r_i:n} - \bar{X}(k) \right)^2 \right] \right\}
\]

\[
= \mathbb{E} \left[ \bar{X}_n^2 \left[ 4\gamma^3 \left( \sum_{i=1}^{k} \mu_{r_i:n} X_{r_i:n} \right)^2 - \gamma^2 \sum_{i=1}^{k} \left( X_{r_i:n} - \bar{X}(k) \right)^2 \right] \right] \\
- \frac{1}{n} \left[ 4\gamma^3 \mathbb{E} \left( \sum_{i=1}^{k} \mu_{r_i:n} X_{r_i:n} \right)^2 - \gamma^2 \sum_{i=1}^{k} \mathbb{E} \left( X_{r_i:n} - \bar{X}(k) \right)^2 \right].
\]

Now
Substituting in (6.12) we have

\[ E_3 = 5\varepsilon_2 - 8\varepsilon_1 + 4\gamma^3 E \left[ \frac{\vec{x}^2}{(\kappa)} \left( \sum_{i=1}^{k} \mu_{r_i}^i n \cdot \vec{x}_{r_i} n \right)^2 \right] \]

\[ - \frac{4\gamma^3}{n} E (\sum_{i=1}^{k} \mu_{r_i}^i n \cdot \vec{x}_{r_i} n)^2 - \gamma^2 E \left[ \left( \frac{\vec{x}^2}{(\kappa)} \right) \left( \sum_{i=1}^{k} \mu_{r_i}^i n \cdot \vec{x}_{r_i} n \right)^2 \right] \]

\[ + \frac{\gamma^2}{n} \sum_{i=1}^{k} E (\vec{x}_{r_i} n - \vec{r}_i n)^2 + 2\gamma^2 E \left( \frac{\vec{x}^2}{(\kappa)} \left( \sum_{i=1}^{k} \mu_{r_i}^i n \cdot \vec{x}_{r_i} n \right) \right) \]

\[ - \gamma E (\frac{\vec{x}^2}{(\kappa)}) \cdot \]  

(6.14)

We can see

\[ 2\gamma^2 E \left[ \left( \frac{\vec{x}^2}{(\kappa)} \right) \left( \sum_{i=1}^{k} \mu_{r_i}^i n \cdot \vec{x}_{r_i} n \right) \right] = 3\varepsilon_1 + \frac{2\gamma}{n} - \varepsilon_2 \]

\[ \gamma E (\frac{\vec{x}^2}{(\kappa)}) = E_1 + \frac{\gamma}{n} \]

and
\[ \frac{4\gamma^3}{n} \mathbb{E} \left( \sum_{i=1}^{k} \mu_{r_i}^{i \cdot n} X_{r_i}^{i \cdot n} \right)^2 = \frac{4\gamma^3}{n} \sum_{i=1}^{k} \sum_{j=1}^{k} \mu_{r_i}^{i \cdot n} \mu_{r_j}^{j \cdot n} \sigma_{r_i r_j}^{i \cdot n} + \frac{4\gamma}{n} \]

where \( \sigma_{r_i r_j}^{i \cdot n} \) is the covariance of the \( r_i^{th} \) and \( r_j^{th} \) order statistics in a sample of size \( n \) from a standard normal parent population.

We can therefore reduce (6.14) to

\[
E_3 = 4E_2 - 6E_1 - \frac{3\gamma}{n} - \frac{4\gamma^3}{n} \sum_{i=1}^{k} \sum_{j=1}^{k} \mu_{r_i}^{i \cdot n} \mu_{r_j}^{j \cdot n} \sigma_{r_i r_j}^{i \cdot n} \\
+ \frac{\gamma^2}{n} \left( \sum_{i=1}^{k} \sigma_{r_i}^{i \cdot n} - \frac{1}{k} \sum_{i=1}^{k} \sigma_{r_i}^{i \cdot n} \right) \\
+ 4\gamma^3 \mathbb{E} \left[ \frac{\chi^2}{(k)} \sum_{i=1}^{k} \sum_{j=1}^{k} \mu_{r_i}^{i \cdot n} \mu_{r_j}^{j \cdot n} X_{r_i}^{i \cdot n} X_{r_j}^{j \cdot n} - \frac{\gamma^2}{k} \sum_{i=1}^{k} \left( X_{r_i}^{i \cdot n} - \bar{X}(k) \right)^2 \right].
\]

(6.15)

It then follows that

\[
E_3 = 4E_2 - 6E_1 - \frac{3\gamma}{n} - \frac{4\gamma^3}{n} \sum_{i=1}^{k} \sum_{j=1}^{k} \mu_{r_i}^{i \cdot n} \mu_{r_j}^{j \cdot n} \sigma_{r_i r_j}^{i \cdot n} \\
+ \frac{\gamma^2}{n} \left( \sum_{i=1}^{k} \sigma_{r_i}^{i \cdot n} - \frac{1}{k} \sum_{i=1}^{k} \sigma_{r_i}^{i \cdot n} \right) - \frac{\gamma^2}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{l=1}^{k} \mu_{r_i}^{i \cdot n} \mu_{r_j}^{j \cdot n} \mu_{r_l}^{l \cdot n} \sigma_{r_i r_j r_l}^{i \cdot n \cdot j \cdot n \cdot l \cdot n}.
\]
B. Special Cases

Under the assumptions that \( X \) is normally distributed and that a symmetric choice of ranks is made, we shall get slightly simpler expressions for \( E_1 \), \( E_2 \) and \( E_3 \) when \( k = 2 \) and for \( E_1 \) and \( E_2 \) when \( k = 3 \).

For \( k = 2 \) \( r_2 = n - r_1 + 1 \) and the following equalities hold,

\[
\begin{align*}
\mu_{r_1:n} &= -\mu_{r_2:n} \\
\sigma_{r_1:n}^2 &= \sigma_{r_2:n}^2 \\
E(X_{r_1:n}^a X_{r_2:n}^b) &= (-1)^{a+b} E(X_{r_1:n}^b X_{r_2:n}^a)
\end{align*}
\]

(6.17)

where \( a \) and \( b \) are non-negative integers.

When the ranks are chosen symmetrically

\[ E(\overline{X}^2_{(k)}) = \text{Var} \overline{X}_{(k)}. \]

From (6.1) and (6.17) we see that for \( k = 2 \)
\[ \gamma^{-1} = 2\mu^2 r_2 : n \]

also

\[ E(\bar{x}^2(2)) = \frac{1}{4} \sum_{i=1}^{2} \sigma_{r_i r_j}^2 : n = \frac{1}{2} \left( \sigma_{r_2}^2 : n + \sigma_{r_1 r_2}^2 : n \right) \]

so

\[ E_1 = \frac{\frac{1}{2}(\sigma_{r_2}^2 : n + \sigma_{r_1 r_2}^2 : n) - \frac{1}{n}}{2\mu^2 r_2 : n} \]  \hspace{1cm} (6.18)

\[ E_2 = 3E_1 + \frac{1}{n \mu^2 r_2 : n} - \frac{1}{8 \mu r_2 : n} E(\mu_{r_1} x_{r_1}^3 : n + 2\mu_{r_1} x_{r_1}^2 : n x_{r_2} : n) \]

\[ + \mu_{r_1} x_{r_1} x_{r_2} : n + \mu_{r_2} x_{r_2}^2 : n + 2\mu_{r_2} x_{r_2} : n x_{r_2}^2 : n + 2\mu_{r_2} x_{r_2} x_{r_1} : n \]

Using (6.17) this reduces to

\[ E_2 = 3E_1 + \frac{1}{n \mu^2 r_2 : n} - \frac{1}{4 \mu^3 r_2 : n} \left( E x_{r_2}^3 : n + E x_{r_2}^2 : n x_{r_1} : n \right) \]  \hspace{1cm} (6.19)

To obtain an expression for \( E_3 \) we look at (6.16) term by term.

For the first three terms we have
\[ 4E_2 - 6E_1 - \frac{3Y}{n} = 4E_2 - 6E_1 - \frac{3}{2n \mu_{r_2}^2} \cdot \quad (6.20) \]

For the fourth term

\[
\frac{4y^3}{n} \sum_{i=1}^{2} \sum_{j=1}^{2} \mu_{r_1}^i \mu_{r_2}^j \mu_{r_1}^j \mu_{r_2}^i \sigma_{r_1} \sigma_{r_2} \mu_{r_2}^i \\
= \frac{1}{2n \mu_{r_2}^i} (\mu_{r_1}^i \mu_{r_1}^j + 2\mu_{r_2}^i \mu_{r_1}^j \mu_{r_2}^j + \mu_{r_2}^i \mu_{r_2}^j \mu_{r_2}^j) \\
= \frac{1}{n \mu_{r_2}^i} (\sigma_{r_2}^2 - \sigma_{r_1} \sigma_{r_2} \mu_{r_2}^j) \quad (6.21)
\]

The fifth term is

\[
\frac{y^2}{n} \left( \frac{1}{n} \sum_{i=1}^{k} \sigma_{r_1}^2 - \frac{1}{n} \sum_{i=1}^{k} \sigma_{r_1} \sigma_{r_2} \right) = \frac{1}{4n \mu_{r_2}^2} (\sigma_{r_2}^2 - \sigma_{r_1} \sigma_{r_2}) \cdot \quad (6.22)
\]

The term

\[
\frac{y^2}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{l=1}^{k} E(X_{r_1}^2 X_{r_j}^2 X_{r_l}^2) \\
= \frac{1}{16 \mu_{r_2}^4} E \left[ x_{r_1}^4 + 2x_{r_1}^3 x_{r_2} + x_{r_1}^2 x_{r_2}^2 + x_{r_1}^2 x_{r_2} + x_{r_2}^2 x_{r_1}^2 \right] \]

\[+ 2x_{r_2}^3 x_{r_1} + x_{r_2}^4 \]
\[ \frac{1}{8 \mu_{r_2:n}^4} \left[ E\left( X_{r_2:n}^{h} \right) + E\left( X_{r_1:n}^{2} X_{r_2:n}^{2} \right) \right]. \quad (6.23) \]

The term

\[ \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{\ell=1}^{k} \sum_{m=1}^{k} \mu_{r_1:n}^{i} \mu_{r_2:n}^{j} E(X_{r_1:n} X_{r_j:n} X_{r_{\ell}:n} X_{r_m:n}) \]

\[ = \frac{1}{8 \mu_{r_2:n}^4} \left[ E\left( \mu_{r_1:n}^{2} X_{r_1:n}^{4} \right) + 2 \mu_{r_1:n}^{2} X_{r_1:n}^{2} X_{r_2:n}^{2} + \mu_{r_1:n}^{2} X_{r_1:n}^{2} X_{r_2:n}^{2} + 2 \mu_{r_1:n}^{2} X_{r_1:n}^{2} X_{r_2:n}^{2} \right] \]

\[ + 2 \mu_{r_1:n}^{2} X_{r_1:n}^{2} X_{r_2:n}^{2} + \mu_{r_2:n}^{2} X_{r_1:n}^{2} X_{r_2:n}^{2} + 2 \mu_{r_2:n}^{2} X_{r_1:n}^{2} X_{r_2:n}^{2} \]

\[ = \frac{1}{4 \mu_{r_2:n}^4} \left[ E\left( X_{r_2:n}^{4} \right) - E\left( X_{r_1:n}^{2} X_{r_2:n}^{2} \right) \right]. \quad (6.24) \]

The last term

\[ \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{\ell=1}^{k} \sum_{m=1}^{k} E(X_{r_1:n} X_{r_j:n} X_{r_{\ell}:n} X_{r_m:n}) \]
Using (6.20-6.25) in (6.16) it follows that

$$E_3 = 4E_2 - 6E_1 - \frac{3}{2n} \mu_{r_2:n}^2 - \frac{3}{4n} (\sigma_{r_2:n}^2 - \sigma_{r_1 r_2:n})$$

$$+ \frac{3}{16} \mu_{r_2:n}^4 [E(\mathbf{x}_{r_2:n}^4) - E(\mathbf{x}_{r_1:n}^2 \mathbf{x}_{r_2:n}^2)].$$

For $k = 3$, $r_3 = n-r_1+1$, $r_2$ is the median and the following equalities hold.
\[ \begin{align*}
\mu_{r_2} &= 0 \\
\mu_{r_1} &= -\mu_{r_3} \\
\sigma^2_{r_1} &= \sigma^2_{r_3} \\
\sigma_{r_1 r_2} &= \sigma_{r_2 r_3} \\
E(x^3_{r_1}) &= -E(x^3_{r_3}) \\
E(x^2_{r_1} x_{r_2}) &= -E(x^2_{r_3} x_{r_2}) \\
E(x^2_{r_1} x_{r_3}) &= -E(x^2_{r_1} x^2_{r_3}) \\
E(x^2_{r_1} x^2_{r_2}) &= -E(x^2_{r_3} x^2_{r_2}) \\
\end{align*} \] (6.27)

In this case

\[ \gamma = \frac{1}{2\mu_{r_3}^2} \]

and

\[ E(\overline{X}^2_3) = \frac{1}{9} [2\sigma^2_{r_3} + \sigma^2_{r_2} + 2\sigma_{r_1 r_3} + 4\sigma_{r_2 r_3}] \]

so
\[ E_1 = \frac{1}{\mu_{r_3}^2} \left[ \frac{1}{9} \left( \sigma_{r_3}^2 : n + 5 \sigma_{r_2}^2 : n + \sigma_{r_1 r_3}^2 : n + 4 \sigma_{r_2 r_3}^2 : n \right) - \frac{1}{2n} \right] \quad (6.28) \]

From (6.8) we see that

\[ E_2 = 3E_1 + \frac{1}{n \mu_{r_3}^2} - \frac{2}{36 \mu_{r_3}^3} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{\ell=1}^{3} \mu_{r_i}^n E(x_i : n x_j : n x_{\ell} : n) \]

(6.29)

After expanding (6.29), substituting identities from (6.27) and cancelling, we have

\[ 3 \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{\ell=1}^{3} \mu_{r_i}^n E(x_i : n x_j : n x_{\ell} : n) \]

\[ = \mu_{r_3}^n \left[ 2E(x_3^2 : n) + 2E(x_{r_1} : n x_3^2 : n) + 4E(x_{r_3}^2 : n x_{r_2} : n) + 2E(x_{r_3}^2 : n x_{r_3} : n) \right] \]

and therefore

\[ E_2 = 3E_1 + \frac{1}{n \mu_{r_3}^2} - \frac{1}{9 \mu_{r_3}^3} \left[ E(x_3^2 : n + x_{r_1}^2 : n x_{r_1} : n) \right] \]

\[ \quad - \frac{1}{9 \mu_{r_3}^3} \left[ E(x_{r_2}^2 : n x_{r_3} : n + 2x_{r_2}^2 : n x_{r_3}^2 : n) \right] \quad (6.30) \]
As is easily seen, the approximations developed for $E(W)$ involve higher order moments and cross product moments of order statistics. To find $E_3$ for $k = 3$ or $E_2$, $E_3$ for $k > 3$ will involve product moments of three or more ordered variables and so these approximations will not be considered at the present time since, from the numerical results, it appears $E_1$ is an adequate approximation.

When $X$ has a standard normal distribution, Pearson and Hartley (1972, p. 205) give $\mu_{r_1:n}(r_1 = 1, \ldots, n)$ for $n \leq 100$ and $\sigma_{r_1 r_j:n}(r_1, r_j = 1, \ldots, n)$ for $n \leq 20$. In Section C we will give approximations for $\sigma_{r_1 r_j:n}$ and higher moments of order statistics.

C. Joint Moments of Order Statistics

1. Moments in terms of cumulants

Cook (1951) sets out the joint moments of two random variables

$$\mu_{ab} = E(x_i^a x_j^b)$$

in terms of their cumulants

$$K_{ab} = K(x_i^a x_j^b).$$

The following are the joint moments which will be used

$$\mu_{10} = K_{10}$$

$$\mu_{01} = K_{01}$$
\[ \mu_{12} = K_{12} + K_{02}K_{10} + 2K_{11}K_{01} + K_{10}^2K_{01} \]

\[ \mu_{13} = K_{13} + K_{03}K_{10} + 3K_{12}K_{01} + 3K_{11}K_{01} + 3K_{01}K_{10} + 3K_{11}^2K_{01} + K_{10}^3K_{01} \]

\[ \mu_{14} = K_{14} + 4K_{04}K_{10} + 3K_{02}K_{10} + 6K_{02}K_{01}K_{10} + K_{10}^4K_{01} \]

\[ \mu_{22} = K_{22} + 2K_{21}K_{01} + K_{20}K_{10} + K_{20}K_{02} + 2K_{12}K_{10} + 2K_{11}K_{10}K_{01} + K_{10}^2K_{01} + K_{10}K_{02} \]

2. **Joint cumulants of order statistics**

Looking at two ordered random variables \( X_{r_i:n}, X_{r_j:n} \) (\( i \neq j \)), David and Johnson (1954) have developed approximations for the first four cumulants and cross cumulants up to order \( (n+2)^{-3} \). We shall just consider terms up to order \( (n+2)^{-2} \).
Denote the c.d.f. of the random variables by $F(x)$ and let

$$p_{r_i} = \frac{r_i}{n+1}$$

$$q_{r_i} = 1 - p_{r_i}$$

and

$$X_{r_i} = F^{-1}(p_{r_i}) \quad i = 1, \ldots, k.$$ Further, let

$$X'_{r_i} = \frac{d X_{r_i}}{d p_{r_i}}$$

$$X''_{r_i} = \frac{d^2 X_{r_i}}{d p_{r_i}^2}, \quad \text{etc.}$$

Then

$$K_{01} = E(X_{r_j:n})$$

$$K_{10} = E(X_{r_i:n})$$

$$K_{20} = \text{Var}(X_{r_i:n}) = \sigma^2_{r_i:n}$$

$$K_{02} = \text{Var}(X_{r_j:n}) = \sigma^2_{r_j:n}$$

$$K_{11} = \text{Cov}(X_{r_i:n}, X_{r_j:n}) = \sigma_{r_i r_j:n}.$$ Now $K_{ba}$ has the same form as $K_{ab}$ with $r_i$ and $r_j$ interchanged,
so from David and Johnson (1954) we have

\[ K_{02} = \frac{p_{r_j} q_{r_j}}{n+2} x_j^2 + \frac{p_{r_j} q_{r_j}}{(n+2)^2} [2(q_{r_j} - p_{r_j}) x_j x_{r_j} + p_{r_j} q_{r_j} (x_j x_{r_j} + \frac{1}{2} x_{r_j}^2)] + \ldots \] (6.31a)

\[ K_{11} = \frac{p_{r_i} q_{r_i}}{n+2} x_i x_{r_i} + \frac{p_{r_i} q_{r_i}}{(n+2)^2} [(q_{r_i} - p_{r_i}) x_{r_i} x_{r_j} + (q_{r_j} - p_{r_j}) x_i x_{r_j} + \frac{1}{2} q_{r_i} x_{r_j} x_{r_i}] + \ldots \] (6.31b)

\[ K_{12} = \frac{p_{r_j} q_{r_i}}{(n+2)^2} [2(q_{r_j} - p_{r_j}) x_j x_{r_i} + 2p_{r_j} q_{r_j} x_j x_{r_j} x_{r_i} + p_{r_j} q_{r_j} x_{r_j} x_{r_i} x_{r_i}] + \ldots \] (6.31c)

\[ K_{03} = \frac{p_{r_j} q_{r_j}}{(n+2)^2} [2(q_{r_j} - p_{r_j}) x_j^3 + 3p_{r_j} q_{r_j} x_j x_{r_j} x_{r_j}] + \ldots \] (6.31d)

\[ K_{04} = \frac{1}{(n+2)^3} \] (6.31d)

D. Specific Results

For a standard normal population with c.d.f. \( F(x) \) and density function \( \varphi(x) \), we define
\[
Q_{r_i} = \Phi^{-1}(p_{r_i})
\]

where

\[
p_{r_i} = r_i/(n+1) \quad i = 1, \ldots, k.
\]

and

\[
q_{r_i} = 1 - p_{r_i}
\]

From David (1970, p. 65) we have

\[
Q'_{r_i} = \frac{1}{\varphi(Q_{r_i})}
\]

\[
Q''_{r_i} = \frac{Q_{r_i}}{\varphi^2(Q_{r_i})}
\]

\[
Q'''_{r_i} = \frac{Q_{r_i} (7 + 6Q_{r_i}^2)}{\varphi^3(Q_{r_i})}
\]

\[
Q''''_{r_i} = \frac{Q_{r_i} (7 + 6Q_{r_i}^2)}{\varphi^4(Q_{r_i})}
\]

As an approximation to \( \sigma^2_{r_i:n} \) we have

\[
\sigma^2_{r_i:n} = \kappa_{20} = \frac{p_{r_i} q_{r_i}}{(n+2) \varphi^2(Q_{r_i})} + \frac{p_{r_i} q_{r_i}}{(n+2)^2} [2(q_{r_i} - p_{r_i}) \frac{Q_{r_i}}{\varphi^3(Q_{r_i})}] 
\]
\[ + p_{r_i} q_{r_i} \left( \frac{1 + 2q_{r_i}^2}{\varphi(Q_{r_i})} + \frac{1}{2} \frac{Q_{r_i}^2}{\varphi^2(Q_{r_i})} \right) \], \quad (6.33) \]

and

\[ \sigma_{r_i r_j} = \frac{p_{r_i} q_{r_j}}{(n+2) \varphi(Q_{r_i}) \varphi(Q_{r_j})} + \frac{p_{r_i} q_{r_j}}{(n+2)^2} \left( \frac{q_{r_i} - p_{r_i}}{\varphi(Q_{r_i}) \varphi^2(Q_{r_j})} \right) \]

\[ + \frac{(q_{r_j} - p_{r_j}) Q_{r_j}}{\varphi(Q_{r_j}) \varphi^2(Q_{r_j})} + \frac{1}{2} p_{r_i} q_{r_i} \frac{1 + 2q_{r_i}^2}{\varphi^3(Q_{r_i}) \varphi(Q_{r_j})} \]

\[ + \frac{1}{2} \frac{p_{r_j} q_{r_j} (1 + 2q_{r_j}^2)}{\varphi^3(Q_{r_j}) \varphi(Q_{r_i})} + \frac{1}{2} p_{r_i} q_{r_j} \frac{Q_{r_i} Q_{r_j}}{\varphi^2(Q_{r_i}) \varphi^2(Q_{r_j})} \]. \quad (6.34) \]

When \( r_j = n-r_i+1 \) we have

\[
\begin{align*}
Q_{r_j} &= -Q_{r_i} = Q \quad \text{(say)} \\
p_{r_j} &= 1 - p_{r_j} = q_{r_i} \\
q_{r_j} &= p_{r_i} \\
\varphi(-Q) &= \varphi(Q)
\end{align*}
\]

Consequently, in this case
\[ \sigma_{r_i r_j : n} = \frac{q_{r_{ij}}^2}{(n+2) \varphi^2(q)} + \frac{q_{r_{ij}}^2}{(n+2)^2} \left[ \frac{2(q_{r_{ij}} - p_{r_{ij}})q}{\varphi^3(q)} + \frac{p_{r_{ij}} q_{r_{ij}} (1 + 2q^2)}{\varphi^4(q)} \right] - \frac{1}{2} \frac{q_{r_{ij}}^2 q_{r_{ij}}^2}{\varphi(q)} \]  

(6.36)

For \( k = 2, 3 \), \( E_1 \) is easily approximated using (6.33), (6.34) and (6.36) when \( n > 20 \).

To find \( E_2 \) for \( k = 2 \) we see from (6.8) that we must find an expression for

\[ E(X_{r_2 : n}^3 + E(X_{r_2 : n}^2 X_{r_1 : n}) \].

When \( r_j = n-r_i + 1 \)

\[ E(X_{r_2 : n}^3 + X_{r_2 : n}^2 X_{r_1 : n}) = \mu_{03}^1 + \mu_{12}^1 \]

\[ = K_{03} + K_{12} + 2K_{01} (K_{02} + K_{11}) \].

In particular

\[ E(X_{r_2 : n}^3 + X_{r_1 : n}^2 X_{r_2 : n}) = K_{03} + K_{12} + \mu_{r_2 : n}^1 (r_{r_2 : n}^2 + q_{r_1 r_2 : n}) \]  

(6.37)

and we may rewrite (6.8) as
\[ E_2 = 3E_1 + \frac{1}{n \mu_{r_2 \cdot n}} - \frac{1}{2 \mu_{r_2 \cdot n}} \left( \sigma_{r_2 \cdot n} \sigma_{r_1 r_2 \cdot n} \right) - \frac{1}{4 \mu_{r_2 \cdot n}} (K_{03} + K_{12}) , \]  

(6.38)

where

\[ K_{03} = \frac{p_{r_2} q_{r_2}}{(n+2)^2} \left[ \frac{2(q_{r_2} - p_{r_2})}{\phi^3(Q)} + \frac{3 p_{r_2} q_{r_2} Q}{\phi^4(Q)} \right] \]  

(6.39)

and

\[ K_{12} = \frac{p_{r_2}^2}{(n+2)^2} \left[ \frac{2(q_{r_2} - p_{r_2})}{\phi^3(Q)} + \frac{2 p_{r_2} q_{r_2} Q}{\phi^4(Q)} - \frac{p_{r_2}^2 Q}{\phi^4(Q)} \right] \]  

(6.40)

In general we have

\[ K_{03} = \frac{p_{r_j} q_{r_j}}{(n+2)^2} \left[ \frac{2(q_{r_j} - p_{r_j})}{\phi^3(Q_{r_j})} + \frac{3 p_{r_j} q_{r_j} Q_{r_j}}{\phi^4(Q_{r_j})} \right] \]  

(6.41)

and when \( r_j = n-r_i+1 \) and \( Q = Q_{r_j} = -Q_{r_i} \)

\[ K_{12} = \frac{p_{r_j}^2}{(n+2)^2} \left[ \frac{2(q_{r_j} - p_{r_j})}{\phi^3(Q_{r_j})} + \frac{2 p_{r_j} q_{r_j} Q_{r_j}}{\phi^4(Q_{r_j})} - \frac{p_{r_j}^2 Q_{r_j}}{\phi^4(Q_{r_j})} \right] . \]  

(6.42)

To find \( E_3 \) we must from (6.26) obtain an expression for
\[ E \left[ X_{r_2:n}^{1} - X_{r_1:n}^{2} - X_{r_2:n}^{2} \right] = \mu_{04}^{'} - \mu_{22}^{'} . \]

Now

\[ \mu_{04}^{'} - \mu_{22}^{'} = K_{04} - K_{22} + 4(K_{03} + K_{12})K_{01} + 2(K_{02}^{2} - K_{11}^{2}) \]

\[ + 4K_{01}(K_{02} + K_{11}) . \]

We ignore \( K_{04} \) and \( K_{22} \) since each is of order \( (n+2)^{-3} \) and higher. So we see we may rewrite (6.26) as

\[ E_3 = 4E_2 - 6E_1 - \frac{3}{2n \mu_{r_2:n}^2} - \frac{3}{4n \mu_{r_2:n}^4} (\sigma_{r_2:n}^2 - \sigma_{r_1r_2:n}^2) \]

\[ + \frac{3}{4 \mu_{r_2:n}^3} (\sigma_{r_2:n}^2 + \sigma_{r_1r_2:n}^2) + \frac{3}{8 \mu_{r_2:n}^4} (\sigma_{r_2:n}^4 - \sigma_{r_1r_2:n}^2) \]

\[ + \frac{3}{4 \mu_{r_2:n}^3} (K_{03} + K_{12}) , \tag{6.43} \]

where \( \sigma_{r_2:n}^2 \), \( \sigma_{r_1r_2:n}^2 \), \( K_{03} \) and \( K_{12} \) can be found from (6.34), (6.36), (6.39) and (6.40) respectively.

For \( k = 3 \), to find \( E_2 \) we must find expressions for

\[ E(X_{r_3:n}^{3} + X_{r_3:n}^{2} X_{r_1:n}^{1}) \text{ and } E(X_{r_2:n}^{2} X_{r_3:n}^{2} + 2X_{r_2:n}^{2} X_{r_3:n}^{1}) . \]

Now, \( E(X_{r_3:n}^{3} + X_{r_3:n}^{2} X_{r_1:n}^{1}) \) has the same form as (6.37) with \( r_3 \) replacing \( r_2 \).
When writing of variables $X_{r_1:n}$, $X_{r_3:n}$ we will use $K$ to denote the cumulant, and when writing of $X_{r_2:n}$ $X_{r_3:n}$ we will use $K^*$. We can see

$$E(X_{r_2:n}^2 X_{r_3:n} + 2X_{r_2:n} X_{r_3:n}^3)$$

$$= K_{21}^* + 2K_{12}^* + \mu_{r_3:n} (\sigma_{r_2:n}^2 + 4\sigma_{r_2:r_3:n}^2), \quad (6.44)$$

and since $Q_{r_2} = 0$, we may write

$$\sigma_{r_2:n}^2 = \frac{1}{4(n+2) \varphi^2(0)} + \frac{1}{16(n+2)^2 \varphi^3(0)} \quad (6.45)$$

$$\sigma_{r_2 r_3:n}^2 = \frac{q_{r_3}}{2(n+2) \varphi(0) \varphi(Q_{r_3})} + \frac{q_{r_3}}{2(n+2)^2} + \frac{1}{8(\varphi(0) \varphi(Q_{r_3}))} \quad (6.46)$$
\[
K_{12}^* = \frac{p r_3}{2(n+2)^2} \left[ \frac{2(p r_3 - r_3)}{\varphi^2(Q r_3 \varphi(0))} + \frac{2 p r_3 Q r_3 Q r_3}{\varphi^3(Q r_3 \varphi(0))} \right]
\] (6.47)

\[
K_{12}^* = \frac{q r_3}{2(n+2)^2} \left[ \frac{q r_3 Q r_3}{2\varphi^2(0) \varphi^2(Q r_3)} \right]
\] (6.48)

So from (6.30) we see that for \( k = 3 \)

\[
E_2 = 3E_1 + \frac{1}{n \mu_{r_3}^2 n} - \frac{1}{9\mu_{r_3}^3 n} (K_{03} + K_{12} + K_{21}^* + 2K_{12}^*)
\]

\[
- \frac{1}{9\mu_{r_3}^2 n} (\sigma_{r_3 n}^2 + \sigma_{r_1 r_3 n}^2 + \sigma_{r_2 r_3 n}^2 + 4\sigma_{r_2 r_3 n}^2)
\] (6.49)

where \( K_{03} \) and \( K_{12} \) are given by (6.41) and (6.42) respectively with \( i = 1 \) and \( j = 3 \).

The values of \( E_2 \) and \( E_3 \) used in Chapter III for \( k = 2 \) were obtained using (6.38) and (6.43) respectively. For \( k = 3 \), (6.49) was used to find \( E_2 \).