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Limit theorems for persistent random walks in cookie environments

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Limit theorems for persistent random walks in cookie environments

by

Zachary D. Voller

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:
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Iowa State University

Ames, Iowa

2016

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DEDICATION

I would like to dedicate this thesis to my wife, Saira. You've always been my biggest supporter. This was truly a team effort, and none of this would have been possible without you. Thank you for giving me an opportunity to finish this journey.

I would also like to thank the rest of my family, especially my parents Larry and Karen and my sister Virginia, for always believing in me.

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ABSTRACT

Excited random walks (ERW) or random walks in a cookie environment is a modification of the nearest neighbor simple random walk such that in several first visits to each site of the integer lattice, the walk's jump kernel gives a preference to a certain direction and assigns equal probabilities to the remaining directions. If the current location of the random walk has been already visited more than a certain number of times, then the walk moves to one of its nearest neighbors with equal probabilities. The model was introduced by Benjamini and Wilson and extended by Martin Zerner. In the cookies jargon, upon first several visits to every site of the lattice, the walker consumes a cookie providing them a boost toward a distinguished direction in the next step. The excited random walk is a popular mainstream model of theoretical probability. An interesting application of this model to the motion of DNA molecular motors has been discovered by Antal and Krapivsky (Phys. Review E, 2007), see also the article of Mark Buchanan Attack of the cyberspider in Nature Physics, 2009.

Many basic asymptotic properties of excited random walk have their counterparts for random walk in random environment (RWRE). The major difference between two processes is that while the random (cookie) environment is dynamic and rapidly changes with time the environments considered in the RWRE process are stationary both in space and in time. The similarity between the asymptotic behaviors of these two classes of random walks can be explained using the fact that certain functionals (for instance, exit times and exit probabilities) of the local time (or occupation time, also referred to as the number of previous visits to a current location) process converge after a proper rescaling to diffusion processes with time-independent coefficients. Thus phenomenon, discovered by Kosygina and Mpountford, can be exploited for a heuristic explanation of the analogy between the role of the local drift of ERW

(bias created by the cookie environment) and a random potential which governs the behavior of RWRE.

In this thesis we consider an excited random walk on \mathbb{Z} with the jump kernel that depends not only on the number of cookies present at the current location of the walker, but also on direction from which the current location is entered. Random walks with the jump kernel that depends not only on the current location and possibly the history of the random walk at this location but also on the direction where the current location is visited from are usually referred to as persistent random walks. We therefore refer to our model as an persistent random walk in a cookie environment (PRWCE).

We prove recurrence and transience criteria and derive a necessary and sufficient condition for the asymptotic speed of the walk to be strictly positive. The law of large number in the transient case is complement by a central limit theorem for the position of the random walk. Surprisingly, it turns out that a transient PRWCE even in one dimension does not necessarily satisfy the usual 0 – 1 for the direction of the escape. More precisely, due to irreversibility of an associated with the cookie environment Markov process that governs the random motion, it is possible that a transient PRWCE on integers will escape to both negative and positive directions with non-zero probabilities. This is in the strike contrast to the usual ERW and to the one-dimensional persistent random walk in random environment where the associated Markov process (decisions of the walker modelled by a coin-tossing procedure) turns out to be a reversible Markov chain.

The investigation of the asymptotic behavior of a recurrent PRWCE and to a large extent of the transient walk in the case when the 0 – 1 law is violated remain a subject of the future investigation. Two additional interesting problems that are discussed in the thesis and remain unsolved are stable (non-Gaussian) limit theorems and the asymptotic behavior of the maximum local time. For all these open problems we state conjectures regarding the expected behavior of the random walk and indicate plausible strategies for proving this conjectures.

Our proof technique rely on a suitable extension of a Ray-Knight type theorem obtained

for usual excited random walks in dimension one by Kosygina and Zerner. The theorem establishes a relation between asymptotic behavior of the random walk and basic properties of certain branching-type processes. Informally speaking, the duality between branching processes and nearest-neighbor random walks describes excursions of a random walk as a branching structure: each jump from a site $n - 1$ to n creates an opportunity for jumps from n to $n + 1$ (children in the language of branching). The correspondence between occupation times of random walks and branching processes carries over to processes in random environment and supplies a powerful technique for investigation of the asymptotic behavior of, for instance, random walk in random environments and excited random walks on \mathbb{Z} .

As it was shown by Kosygina and Mountford, stable limit laws for excited random walks in dimension one are essentially equivalent to certain scaling properties of the branching processes associated with the Ray-Knight interpretation of the local times. Proving these scaling properties for the PRWCE is a subject of the ongoing investigation and remains beyond the scope of the thesis.

CHAPTER 1. INTRODUCTION

In this chapter we introduce our model and state our main results (Section 1.2) The chapter starts with the definition of the underlying simple random walk model (Section 1.1).

1.1 The simple random walk

Let $(X_n)_{n \geq 0}$ denote the **simple random walk** (SRW) on \mathbb{Z} with $X_0 = 0$ and transition probabilities

$$P(X_n = j \mid X_{n-1} = i) = \begin{cases} p & \text{if } j = i + 1 \\ q = 1 - p & \text{if } j = i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

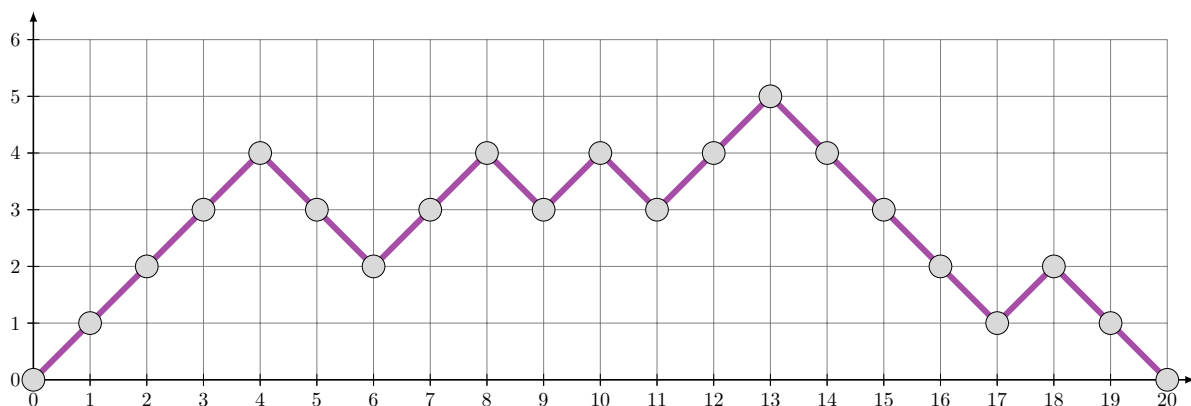


Figure 1.1 Realization of a random walk

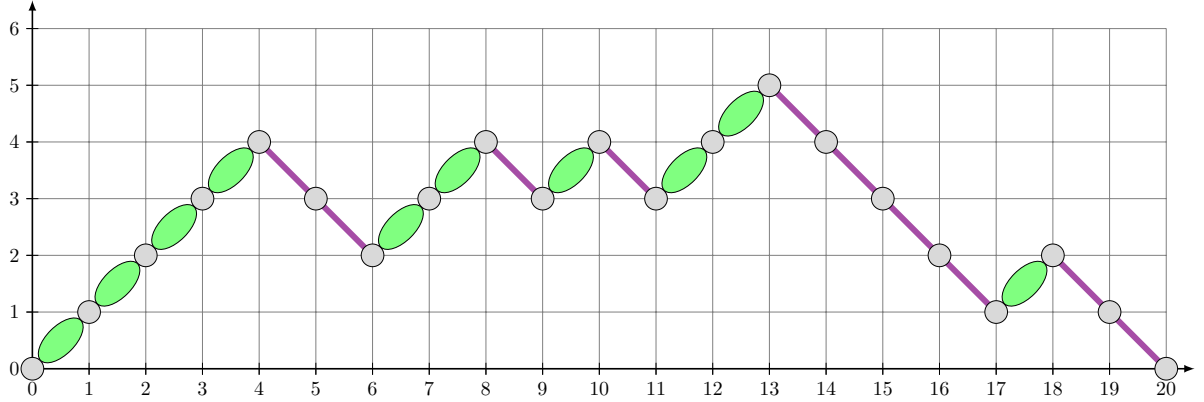


Figure 1.2 Realization of the random walk, with upcrossings marked by the leaves.

1.2 Introduction and statement of results

Let M be a natural number and assume that M cookies, labeled by $i = 1, \dots, M$, are placed at each vertex of the one-dimensional integer lattice. When the nearest-neighbor random walk visits site $k \in \mathbb{Z}$ for the i -th time, $i \leq M$, the walker consumes one cookie and then goes to the right with probability either $\lambda_{k,i}$ or $1 - \mu_{k,i}$, depending on whether it just entered site k from site $k - 1$ or $k + 1$. Here $\Lambda_k = (\lambda_{k,i})_{1 \leq i \leq M}$ and $\Delta_k = (\mu_{k,i})_{1 \leq i \leq M}$ are two fixed M -dimensional vectors with the components taking values in the interval $[0, 1]$. When the random walk visits a site at which there are no more cookies left, it jumps to one of the two neighbor sites with equal probabilities.

We assume that the sequence $\omega = (\Lambda_k, \Delta_k)_{k \in \mathbb{Z}}$ is a realization of an i.i.d. sequence of random pairs of vectors, and refer to it as the *random cookie environment*. We denote by P the (product) distribution of ω in the space of environments $\Omega := ([0, 1]^M \times [0, 1]^M)^{\mathbb{Z}}$ and let E_P denote the corresponding expectation operator. We assume the following non-degeneracy

condition:

$$E_P\left(\prod_{i=1}^M \lambda_{0,i}\right) \cdot E_P\left(\prod_{i=1}^M (1 - \lambda_{0,i})\right) \cdot E_P\left(\prod_{i=1}^M \mu_{0,i}\right) \cdot E_P\left(\prod_{i=1}^M (1 - \mu_{0,i})\right) > 0. \quad (1.1)$$

The random walk can be considered either under the *quenched* measure P_ω in a fixed environment ω , or under the *annealed* measure \mathbb{P} obtained by averaging P_ω over the set of all possible environments. We first define the quenched law P_ω . Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For $n \in \mathbb{Z}_+$, let X_n denote the location of the walker on \mathbb{Z} after n steps and let

$$\zeta_n = \#\{i \in [0, n] : X_i = X_n\}$$

be the number of visit by the random walk to its current location. Let $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ be the sigma algebra of “events occurred up to and including time n ”. Then, in a fixed environment ω , the quenched distribution P_ω of the nearest-neighbor discrete-time random walk $X := (X_n)_{n \geq 0}$ on the space of infinite paths $\mathbb{Z}^{\mathbb{Z}_+}$ (\mathbb{Z} for the state space and \mathbb{Z}_+ for the time coordinate) is defined by the following transition kernel:

$$\begin{aligned} & P_\omega(X_{n+1} = k + 1 | \mathcal{F}_n, X_n = k, \zeta_n = i, X_n - X_{n-1} = j) \\ &= P_\omega(X_{n+1} = k + 1 | X_n = k, \zeta_n = i, X_n - X_{n-1} = j) \\ &= \begin{cases} \lambda_{k,i} & \text{if } i \leq M \text{ and } j = 1 \\ 1 - \mu_{k,i} & \text{if } i \leq M \text{ and } j = -1 \\ 1/2 & \text{if } i > M. \end{cases} \end{aligned}$$

and

$$\begin{aligned} & P_\omega(X_{n+1} = k - 1 | \mathcal{F}_n, X_n = k, \zeta_n = i, X_n - X_{n-1} = j) \\ &= P_\omega(X_{n+1} = k - 1 | X_n = k, \zeta_n = i, X_n - X_{n-1} = j) \\ &= 1 - P_\omega(X_{n+1} = k + 1 | X_n = k, \zeta_n = i, X_n - X_{n-1} = j). \end{aligned}$$

The above transition kernel is well-defined for $n \geq 1$. To extend the definition to $n = 0$, we will introduce an auxiliary integer-valued random variable X_{-1} and stick throughout the paper

to the convention that $X_0 - X_{-1} \in \{-1, 1\}$ and $P - a. s.$,

$$P_\omega(X_0 - X_{-1} = 1) = \begin{cases} 1 & \text{if } X_0 > 0 \\ 1/2 & \text{if } X_0 = 0 \\ 0 & \text{if } X_0 < 0. \end{cases}$$

The annealed distribution \mathbb{P} is defined by setting

$$\mathbb{P}(X \in A) = E_P(P_\omega(X \in A))$$

for Borel subsets A of the product space $\mathbb{Z}^{\mathbb{Z}_+}$. To emphasize the initial position of the random walk, we will use notations $P_{\omega,k}$ and \mathbb{P}_k to denote the corresponding probability measures for the random walk starting at $X_0 = k$. We will occasionally identify \mathbb{P} with \mathbb{P}_0 and P_ω with $P_{\omega,0}$.

Excited random walks on \mathbb{Z}^d , $d \geq 1$, (this class of models is often referred to as random walks in a cookie environment) were introduced by Benjamini and Wilson in Benjamini and Wilson (2003). The model was generalized and systematically studied in Zerner Zerner (2005, 2006), and since then has attracted attention of many researchers. The proofs in our paper rely on a variation of the reduction to a branching process with migration method, which was introduced by Basdevant and Singh in Basdevant and Singh (2007, 2008) and further developed by Kosygina and Zerner in Kosygina and Zerner (2008) and Kosygina and Mountford in Kosygina and Mountford (2011).

Persistent random walks in a random environment were considered by Szász and Tóth Szász and Tóth (1984), who also discussed applications in physics, in particular to the stochastic Lorentz gas. Encoding the path of the random walk into a suitable branching process has been proven as a powerful tool for study one-dimensional random walks in a random environment (for a remarkable example of the use of this method, see, for instance, Kesten et al. (1975)). The method was carried over to persistent random walks in random environment by

Alili in Alili (1999). We remark that persistent random walks described in Alili (1999); Szász and Tóth (1984) as well as the model discussed in the present paper can be viewed as random walks on the strip $\mathbb{Z} \times \{-1, 1\}$, where -1 and 1 correspond to, respectively, negative and positive current velocity of the random walk. Our work is stimulated by the successful application of the branching encoding techniques to the RWRE on strips accomplished in Alili (1999).

Our model is reduced to the model introduced in Kosygina and Zerner (2008) once $\lambda_{k,i} = 1 - \mu_{k,i}$ is assumed. Having in mind both, the basic version with $P(\lambda_{0,i} > 1/2, \mu_{0,i} > 1/2, 1 \leq i \leq M) = 1$ as well as the similarity with the random environment model described in Alili (1999); Szász and Tóth (1984), we call the above random walk directionally persistent or simply *persistent* cookie random walk on \mathbb{Z} . The main results of the paper are stated below in Theorem 1.2.1 (recurrence and transience criteria) and Theorem 1.2.2 (speed regimes). Limit theorems for the magnitude of fluctuations of a transient random walk in our model is the content of Theorem 1.2.3. For regular (“non-persistent”) excited random walks in dimension one such limit theorems were established in Basdevant and Singh (2008); Dolgopyat (2011); Kosygina and Mountford (2011), see also Dolgopyat () for a review and some recent developments. We remark that recurrence criteria and speed regimes for one-dimensional excited random walks were studied in Kosygina and Zerner (2008); Zerner (2005, 2006) and Basdevant and Singh (2007); Kosygina and Zerner (2008); Mountford et al. (2006); Zerner (2005), respectively. For some interesting variations of the one-dimensional model see, for instance, Raimond and Schapira (2010); Pinsky (2010).

We now turn to the statement of our main results. For $\omega \in \Omega$, $k \in \mathbb{Z}$, and $1 \leq i \leq M$ define the following stochastic 2×2 matrices (transition kernels):

$$H_{\omega,i}^{(k)} = \begin{bmatrix} 1 - \lambda_{k,i} & \lambda_{k,i} \\ \mu_{k,i} & 1 - \mu_{k,i} \end{bmatrix} \quad \text{and} \quad \mathbb{H}_{\omega,i}^{(k)} = \prod_{j=1}^i H_{\omega,j}^{(k)}. \quad (1.2)$$

In Section 3.2 we will associate with these transition kernels auxiliary Markov chains gov-

erning the jumps of the random walk in the presence of cookies. Those Markov chains are naturally defined in the state space $\{-1, 1\}$. Correspondingly, to denote the entries of the above matrices we will usually write them in the following form:

$$H_{\omega,i}^{(k)} = \begin{bmatrix} H_{\omega,i}^{(k)}(-1, -1) & H_{\omega,i}^{(k)}(-1, 1) \\ H_{\omega,i}^{(k)}(1, -1) & H_{\omega,i}^{(k)}(1, 1) \end{bmatrix} \quad \text{and} \quad \mathbb{H}_{\omega,i}^{(k)} = \begin{bmatrix} \mathbb{H}_{\omega,i}^{(k)}(-1, -1) & \mathbb{H}_{\omega,i}^{(k)}(-1, 1) \\ \mathbb{H}_{\omega,i}^{(k)}(1, -1) & \mathbb{H}_{\omega,i}^{(k)}(1, 1) \end{bmatrix}$$

It turns out that the asymptotic behavior of the persistent random walk is determined by the value of two parameters $\delta_p = M(2p - 1)$ and $\delta_q = M(2q - 1)$, where

$$p = \frac{1}{M} \sum_{i=1}^M E_P(\mathbb{H}_{\omega,i}^{(k)}(-1, 1)) \quad \text{and} \quad q = \frac{1}{M} \sum_{i=1}^M E_P(\mathbb{H}_{\omega,i}^{(k)}(1, -1)). \quad (1.3)$$

In particular, we have the following result.

Theorem 1.2.1 (recurrence and transience criteria). *Let*

$$A_1 = \left\{ \lim_{n \rightarrow \infty} X_n = +\infty \right\} \quad \text{and} \quad A_{-1} = \left\{ \lim_{n \rightarrow \infty} X_n = -\infty \right\}. \quad (1.4)$$

Then:

- (i) *If $\delta_p > 1$ and $\delta_q \leq 1$, the random walk is transient to the right. That is, $\mathbb{P}_0(A_1) = 1$.*
- (ii) *If $\delta_p \leq 1$ and $\delta_q \leq 1$, the walk is recurrent. That is, $\mathbb{P}_0(A - 1 \cup A_{-1}) = 0$.*
- (iii) *If $\delta_q > 1$ and $\delta_p \leq 1$, the walk is transient to the left. That is, $\mathbb{P}_0(A_{-1}) = 1$.*
- (iv) *If $\delta_p > 1$ and $\delta_q > 1$, the walk is transient. That is $(A_1 \cup A_{-1}) = 1$.*

Furthermore, in this case $\mathbb{P}_0(A_1) > 0$ and $\mathbb{P}_0(A_{-1}) > 0$.

The proof of Theorem 1.2.1 is given in Section 3.2. It will be seen from an alternative construction of the random walk X given in Section 3.2 that δ_p and δ_q represent the potential cumulative local drift contained in the cookies placed in a positive or a negative site, respectively. More precisely, let $b_k(\omega) \in \Omega$ denote the environment modified from $\omega \in \Omega$ by replacing ω_{k-1}

and ω_{k+1} with reflection barriers which return the walker arriving from site k back to this site with probability one. It will turn out (see in particular (3.7) below) that

$$\sum_{i=1}^{\infty} \mathbb{E}_k [E_{b_k(\omega)} [X_{T_{k,i+1}} - X_{T_{k,i}}]] = \begin{cases} \delta_p & \text{if } k > 0 \\ \delta_q & \text{if } k < 0, \end{cases}$$

where $T_{k,i}$ is the time of i -th visit to site $k \in \mathbb{Z}$. A nice heuristic argument elucidating the relevance and importance of the above expected value to the theory of excited random walks is given in Remark 1 at (Zerner, 2005, p. 102). In contrast to the regular excited random walks, in our model δ_p differs in general from $-\delta_q$. Technically, the reason for this description is the non-reversibility of the matrices \mathbb{H} and their products.

We next consider a classification of possible speed regimes of the random walk. By the asymptotic speed we mean $\lim_{n \rightarrow \infty} X_n/n$, provided that the latter limit exists. In contrast to the classical (Markovian) one-dimensional models which are typically *ballistic* (i. e., having non-zero asymptotic speed), random walks in random media typically have asymptotic zero speed within a certain range of parameters (so called *non-ballistic regime*). See, for instance, Alili (1999); Basdevant and Singh (2007); Kosygina and Zerner (2008); Pinsky (2010); Solomon (1975) for an illustration of this general phenomenon. In our case, we have the following:

Theorem 1.2.2 (asymptotic speed regimes). *There exists a constant $v \in \mathbb{R}$ such that*

$$\lim_{n \rightarrow \infty} X_n/n = v, \quad \mathbb{P}_0 - \text{a. s.}$$

Moreover,

$$v \begin{cases} > 0, & \delta_p > 2 \text{ and } |\delta_q| \leq 1 \\ < 0, & \delta_q < -2 \text{ and } |\delta_p| \leq 1 \\ = 0, & E_0 [\tau_2^{(l)} - \tau_1^{(l)}] = \infty \\ = (3.12), & \text{otherwise.} \end{cases}$$

The proof of Theorem 1.2.2 is included in Section 3.4.

The following is our reformulated version of Kosygina and Zerner (2008)[Theorem 3]:

Theorem 1.2.3 (Annealed central limit theorem). *Let v denote the velocity defined in Theorem 1.2.2, and define*

$$B_n := \frac{1}{\sqrt{n}} (X_n - nv), \quad \text{for } n \geq 0.$$

If either $\delta_p > 4$, with $|\delta_q| \leq 1$, or $\delta_q < -4$, with $\delta_p \leq 1$, then the process $(B_n)_{n \geq 0}$ converges in law \mathbb{P}_0 to a non-degenerate normal zero-mean random variable.

The proof of Theorem 1.2.3 is included in Section 3.5.

CHAPTER 2. BACKGROUND

In this chapter we present two cousin processes of the PRWCE. In Section 2.1 we discuss persistent random walk in random environment on \mathbb{Z} and in Section 2.2 we present the classical one-dimensional excited random walk. We consider a generalized version of ERW with “positive and negative” cookies introduced by Kosygina and Zerner in Kosygina and Zerner (2008).

2.1 Persistent random walks in random environments

As stated previously, persistent random walks in a random environment (PRWRE) were introduced by Szász and Tóth Szász and Tóth (1984). Their construction was tailored to the application of stochastic collisions of gas particles, and their random environment was defined based on the notion that these gas particles would collide with “random scatters” placed on each node of the integer lattice \mathbb{Z} . Thus the random environment Ω is defined as sequence of i.i.d. random variables $\{(\lambda_j, \mu_j) : j \in \mathbb{Z}\}$ in $(0, 1) \times (0, 1)$. Then given a realization ω of the environment, a *persistent random walk* $(X_n)_{n \in \mathbb{N}}$ is a second order homogeneous Markov chain with state space \mathbb{Z} and transition probabilities

$$P_\omega(X_{n+1} = j + 1 \mid X_{n-1} = j - 1, X_n = j) = \lambda_j$$

$$P_\omega(X_{n+1} = j - 1 \mid X_{n-1} = j - 1, X_n = j) = 1 - \lambda_j$$

$$P_\omega(X_{n+1} = j - 1 \mid X_{n-1} = j + 1, X_n = j) = \mu_j$$

$$P_\omega(X_{n+1} = j + 1 \mid X_{n-1} = j + 1, X_n = j) = 1 - \mu_j,$$

where the above definition becomes the usual random walk in random environment when $1 - \lambda_j = \mu_j$ for all $j \in \mathbb{Z}$.

Szász and Tóth (1984) utilized traditional techniques to show the following result; however, their result left questions regarding transience and recurrence criteria and the validity of the law of large numbers for PRWRE unresolved. The following version of the CLT for the one-dimensional PRWRE is adopted from Alili (1999). Under the usual conditions on the environment, there exists a unique $\kappa > 0$ such that $E_P[m_0^\kappa] = 1$, where E_P is the expectation with respect to the law of the environment and $m_0 = \mu_0/\lambda_0$.

Theorem 2.1.1. *Suppose that the environment $(\mu_n, \lambda_n)_{n \in \mathbb{Z}}$ is an i.i.d process, $E[\log m_0] < 0$ (which together with mild technical assumptions guarantees the existence of κ), then for suitably chosen $\nu > 0$ (the speed of the process) and $\sigma > 0$*

$$\frac{X_n - n\nu}{\sqrt{n}}, \quad n \geq 0,$$

converges in distribution to a standard normal random variable under the annealed \mathbb{P}_0 law.

Encoding the path of the random walk into a suitable branching process has been proven as a powerful tool for study one-dimensional random walks in a random environment (for a remarkable example of the use of this method, see, for instance, Kesten et al. (1975)). Alili (1999) was the first to extend this methodology to PRWRE, and our work is stimulated by the successful application of the branching encoding techniques to the RWRE accomplished in Alili (1999). Through the application, of this now standard technique, he was able to show:

Theorem 2.1.2 (Transience and recurrence). *Assume*

$$\mathbb{E}[\log \lambda_0] > -\infty \quad \text{and} \quad \mathbb{E}[\log \mu_0] > -\infty,$$

where $\mathbb{E}[\cdot] = E_P[P_\omega(\cdot)]$. Notice that this implies $\mathbb{E}[|\log(\mu_0/\lambda_0)|] < \infty$. Then the limiting behavior of X_n is characterized as follows:

(i) If $E_P [\log(\mu_0/\lambda_0)] < 0$, then

$$\lim_{n \rightarrow \infty} X_n = \infty \quad \mathbb{P}_0 - \text{a. s.}$$

(ii) If $E_P [\log(\mu_0/\lambda_0)] > 0$, then

$$\lim_{n \rightarrow \infty} X_n = -\infty \quad \mathbb{P}_0 - \text{a. s.}$$

(iii) If $E_P [\log(\mu_0/\lambda_0)] = 0$, then

$$\liminf_{n \rightarrow \infty} X_n = -\infty < \limsup_{\limsup \rightarrow \infty} X_n = \infty \quad \mathbb{P}_0 - \text{a. s.}$$

Additionally, we also have the following result. Let $(T_n)_{n \in \mathbb{Z}}$ be the hitting time of the random walk. Namely,

$$T_0 = 0, \quad T_n = \inf\{k > T_{n-e_n} : X_k = n\},$$

where e_n is the sign of n and the standard convention that $\inf \emptyset = +\infty$ is employed.

Theorem 2.1.3 (Law of large numbers type results). *Define the quantities:*

$$m_0 = \frac{\mu_0}{\lambda_0}, \quad r_0 = \frac{1 - \lambda_0}{\lambda_0}, \quad \text{and} \quad s_0 = \frac{1 - \mu_0}{\mu_0}$$

(i) If $E_P [m_0] < 1$ then

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = \nu \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{T_n}{n} = \nu^{-1} \quad \mathbb{P}_0 - \text{a. s.}$$

where $\nu^{-1} = 1 + (2E_P [r_0]) / (1 - E_P [m_0])$.

(ii) If $E_P [m_0^{-1}] < 1$ then

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = -\nu' \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{T_{-n}}{n} = \nu'^{-1} \quad \mathbb{P}_0 - \text{a. s.}$$

where $(\nu')^{-1} = 1 + (2E_P [s_0]) / (1 - E_P [m_0^{-1}])$.

(iii) If $E_P [m_0]^{-1} < \leq 1 \leq E_P [m_0^{-1}]$ then

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0 \quad \mathbb{P}_0 - \text{a. s.}$$

and

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \lim_{n \rightarrow \infty} \frac{T_{-n}}{n} = \infty \quad \mathbb{P}_0 - \text{a. s.}$$

2.2 Excited random walks

Excited random walks, also called random walks in a cookie environment, were introduced by Benjamini and Wilson in Benjamini and Wilson (2003) and extended by Zerner in Zerner (2005) and Zerner (2006). The model is a modification of the simple random walk where the walker receives a boost in one direction from cookies encountered during its first few visits to every site on the integer lattice \mathbb{Z} . If the site the walker jumps toward has been previously visited more than the number of cookies originally placed on the site, then the walker jumps toward one of its neighboring site with equal probability. Closely related models have also been investigated in Amir et al. (2007); Antal and Redner (2005); Antal and Krapivsky (2007); Holmes (2012); Raimond and Schapira (2010).

Let $M \in \mathbb{N}$ denote the number of cookies initially placed at each site $x \in \mathbb{Z}$, and define the *cookie environment* as follows:

$$\Omega_M := \begin{cases} \omega(x, i)_{x \in \mathbb{Z}, i \in \mathbb{N}} \in [0, 1] & 1 \leq i \leq M, \\ \omega(x, i) = 1/2 & i > M. \end{cases}$$

In this setting $\omega(x, i)$ represents the (biased for $i \leq M$) coin the walker uses to determine the probability that it continues toward the next site, $x + 1$, during its i^{th} visit to site $x \in \mathbb{Z}$. As one would expect, the probability that the walker falls back to site $x - 1$ during its i^{th} visit to x is given by the complementary probability $1 - \omega(x, i)$.

For a fixed $x \in \mathbb{Z}$, $(\omega(x, i))_{1 \leq i \leq M}$ denotes the pile of M “cookies” placed at site x , where for each $1 \leq i \leq m$, $\omega(x, i)$ is referred to as the *strength of the i -th cookie* in the pile.

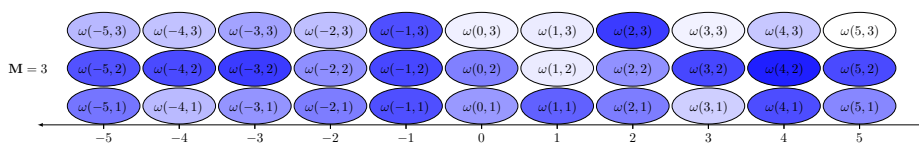


Figure 2.1 A cookie environment with $M = 3$. Imagine each shade corresponds to a different probability.

For convenience, as the indices leave little room for confusion, we will often label the cookie using its “strength.” Formally, the random walk in a cookie environment $\omega \in \Omega_M$ is defined as follows. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let $\Sigma = \mathbb{Z}^{\mathbb{Z}_+}$ (\mathbb{Z} for the state space and \mathbb{Z}_+ for the time coordinate) be the space of the infinite paths of a discrete-time random walk on \mathbb{Z} , and denote by $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ the corresponding Borel sigma algebra of “events occurred up to and including time n .” Then, given a fixed environment $\omega \in \Omega_M$ and an initial position $X_0 = x$, the *quenched distribution* $P_{\omega,x}$ of the excited random walk (ERW) $X := (X_n)_{n \geq 0}$ in the probability space $(\Sigma, \mathcal{F}, P_{\omega,x})$ is defined by the transition kernel

$$P_{\omega,x}(X_0 = x) = 1$$

$$P_{\omega,x}(X_{n+1} = X_n + 1 \mid (X_i)_{0 \leq i \leq n}) = \omega(X_n, \eta_n)$$

$$P_{\omega,x}(X_{n+1} = X_n - 1 \mid (X_i)_{0 \leq i \leq n}) = 1 - \omega(X_n, \eta_n),$$

where $\eta_n := \#\{0 \leq i \leq n : X_i = X_n\}$, and $(X_n, \eta_n)_{n \geq 0}$ is a Markov chain with respect to $P_{x,\omega}$. Now, let \mathbb{P} be a probability measure on Ω_M , where we assume that $(\omega(x, \cdot))_{x \in \mathbb{Z}}$ is an i.i.d. sequence under \mathbb{P} . Then the (associated with \mathbb{P}) *annealed* (average) law P_x of the ERW on (Σ, \mathcal{F}) is defined by setting $P_x(\cdot) = \mathbb{E}[P_{\omega,x}(\cdot)]$, where \mathbb{E} is the expectation induced by the probability law \mathbb{P} .

Importantly, the consumption of cookies $(\omega(x, i))_{x \in \mathbb{Z}, 1 \leq i \leq M}$ causes a bias in the (conditional on the history \mathcal{F}_n) expectation of walker’s displacement, which can be measured. Thus, given a fixed environment $\omega \in \Omega$ and the walker’s initial position $x \in \mathbb{Z}$, we define the *local drift* to be the quantity $E_{\omega,x}[X_{n+1} - X_n \mid X_n = x, \eta_n = i] = 2\omega(x, i) - 1$, and we denoted

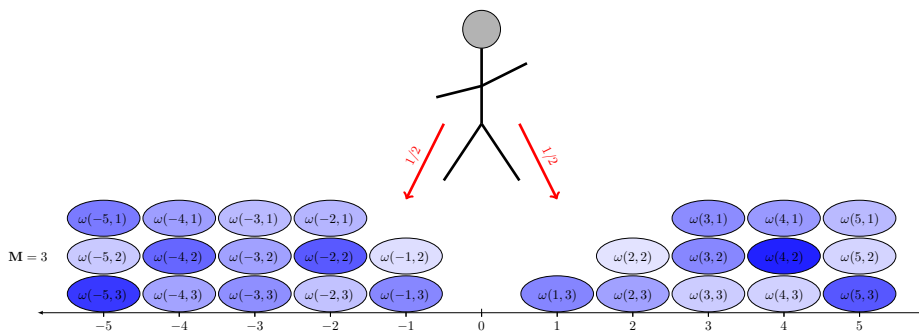


Figure 2.2 Transition probabilities for site $x = 0$ after its been visited more than $M = 3$ times.

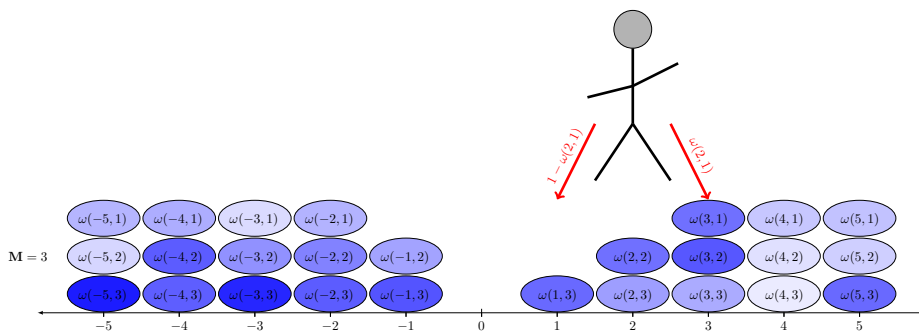


Figure 2.3 Transition probabilities for site $x = 2$ during the first time it's visited.

the expected *total drift at x* , averaged over all environments, by

$$\delta = \mathbb{E} \left[\sum_{i=1}^M (2\omega(x, i) - 1) \right],$$

where we allow both “positive” and “negative” cookies. Additionally, the assumption that ω_x i. i. d implies δ is independent of x .

There has been a great deal of headway made toward understanding the properties of ERW since their introduction by Benjamini and Wilson (2003), and several important aspects of the asymptotic behavior of ERW on \mathbb{Z} have been characterized. It turns out (see Kosygina and Mountford (2011); Kosygina and Zerner (2008)) that the asymptotic behavior of an one-

dimensional excited random walk is largely determined by the value of the parameter δ . In particular, under a mild non-degeneracy assumption on the cookie environment, we have:

- **Recurrence and transience** Kosygina and Zerner (2008): $\delta \in [-1, 1]$ implies the walk is recurrent (i.e. for $\mathbb{P} - a.a.$ environments ω it returns $P_{0,\omega} - a. s.$ infinitely many times to its starting). If $\delta > 1$ ($\delta < -1$) then the walk is transient to the right (transient to the left, receptively). That is, $X_n \rightarrow \infty$ ($-\infty$) as $n \rightarrow \infty$.

- **Law of large numbers and ballisticity** Kosygina and Zerner (2008): There is a deterministic speed v such that the ERW satisfies

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = v = \begin{cases} < 0, & \delta < -2, \\ = 0, & \delta \in [-2, 2] \\ > 0, & \delta > 2, \end{cases} \quad P_{\omega,0} - a. s.$$

or $\mathbb{P} - a.a.$ environments ω .

- **Annealed central limit theorem** Kosygina and Zerner (2008): For $|\delta| > 4$ with

$$B_t^n := \frac{1}{\sqrt{n}} (X_{[tn]} - [tn]v), \quad \text{for } t \geq 0,$$

then $(B_t^n)_{t \geq 0}$ converges in P_0 law to a non-degenerate Brownian motion w.r.t. the Skorohod topology on the space of cadlag functions.

- Basdevant and Singh (2008): If $\delta \in (1, 2)$, then as $n \rightarrow \infty$,

$$\frac{X_n}{n^{\delta/2}} \Rightarrow G_{\delta/2}.$$

- Basdevant and Singh (2008): If $\delta = 2$, then as $n \rightarrow \infty$, $(X_n \log n)/n$ converges in probability to some constant $c > 0$.

- Kosygina and Mountford (2011): If $\delta \in (2, 4)$, then as $n \rightarrow \infty$,

$$\frac{X_n - \nu n}{n^{2/\delta}} \Rightarrow \tilde{G}_{\delta/2}.$$

- Rastegar (2012)[Theorem 2.0.1] Suppose that $(\omega(x, i))_{i \in \mathbb{N}}$ are i.i.d. under \mathbb{P} and

$$\mathbb{E} \left[\prod_{i=1}^M \omega(0, i) \right] \cdot \mathbb{E} \left[\prod_{i=1}^M (1 - \omega(0, i)) \right] > 0$$

holds with $\delta \in (1, 2)$. Then,

$$\limsup_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/2}} > 0, \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/2}} < \infty, \quad P_0 - \text{a. s.}$$

Furthermore, for any $\alpha > \frac{1}{\delta}$ with $\delta \in (1, 2]$:

$$\lim_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/2}(\log n)^\alpha} = 0 \quad \text{while} \quad \lim_{n \rightarrow \infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/2}} = \infty, \quad P_0 - \text{a. s.}$$

The above result implies in particular that, unlike RWRE, a non-ballistic transient ERW does not spend a positive fraction of time at its favorite sites. While the asymptotic behavior of (where stands for the indicator of the event) for transient RWRE seems to be determined by the so called traps created by a random potential, and is radically different from that of the simple unbiased random walk, the limsup asymptotic of ξ_n^* for a non-ballistic transient ERW turns out to be rather similar to its counterpart for a simple non-biased random walk.

- Rastegar (2012)[Theorem 2.0.2] Suppose that $(\omega(x, i))_{i \in \mathbb{N}}$ are i.i.d. under \mathbb{P} and

$$\mathbb{E} \left[\prod_{i=1}^M \omega(0, i) \right] \cdot \mathbb{E} \left[\prod_{i=1}^M (1 - \omega(0, i)) \right] > 0$$

holds with $\delta > 2$ then the following holds for any $\alpha > \frac{1}{\delta}$ then

$$\lim_{n \rightarrow \infty} \frac{\xi_n^*}{n^{1/\delta}(\log n)^\alpha} = 0 \quad \text{while} \quad \lim_{n \rightarrow \infty} \frac{(\log n)^\alpha \xi_n^*}{n^{1/\delta}} = \infty, \quad P_0 - \text{a. s.}$$

Theorem 2.2.1 (limit theorems for the transient case). *In (i) and (iii) below G_α denotes the distribution function of a strictly stable law of index α , supported on the positive half line.*

(i) If $\delta \in (1, 2)$, then

$$\lim_{n \rightarrow \infty} P \left(\frac{T_n}{n^{\delta/2}} \leq x \right) = G_{\delta/2}(x), \quad \text{and} \quad \lim_{n \rightarrow \infty} P \left(\frac{X_n}{n^{\delta/2}} \leq x \right) = 1 - G_{\delta/2}(x^{-2/\delta}).$$

(ii) If $\delta = 2$, then both $n/(T_n \log n)$ and $(X_n \log n)/n$ converge in probability to a positive constant as $n \rightarrow \infty$.

(iii) If $\delta \in (2, 4)$, then

$$\lim_{n \rightarrow \infty} P \frac{T_n - v^{-1}n}{n^{2/\delta}} \leq x = G_{2/\delta}(x), \quad \text{and} \quad \lim_{n \rightarrow \infty} P \left(\frac{X_n - vn}{n^{2/\delta}} \leq x \right) = 1 - G_{2/\delta}(-xv^{-(1+2/\delta)}).$$

(iv) If $\delta = 4$, then there exists a constant $B > 0$ such that

$$\lim_{n \rightarrow \infty} P \left(\frac{T_n - v^{-1}n}{B\sqrt{n \log n}} \leq x \right) = \Psi(x)$$

and

$$\lim_{n \rightarrow \infty} P \left(\frac{X_n - vn}{v^{3/2}B\sqrt{n \log n}} \leq x \right) = \Psi(x),$$

where $\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$ is the standard normal distribution function.

(v) Let

$$X_t^n = \frac{1}{\sqrt{n}}(X_{[tn]} - [tn]v), \quad n \geq 0, t \geq 0.$$

If $|\delta| > 4$, then the process $X^n = (X_t^n)_{t \geq 0}$ converges weakly to a non-degenerate Brownian motion with respect to the Skorohod topology on the space of cad-lag functions.

CHAPTER 3. PROOFS

This chapter includes proofs of our main results stated in the Introduction. In Section 3.1 we introduce a suitable branching process which is used later on to describe the structure of local times of the PRWCE. This structure is used in Section 3.2 to derive transience and recurrence criteria for the PRWCE. In Section 3.3 we discuss a few examples when the asymptotic regime of the random walk is understood using an explicit computation of the parameters δ_p and δ_q . In Sections 3.4 and 3.5 we prove, respectively, the law of large numbers and the central limit theorem for the location of the random walk in the transient regime. The proofs of both the limit theorems rely on standard renewal structure arguments and ultimately reduce to (correspondingly, first and second) moments estimates for the duration of the inter-renewal times. The renewal structure for the transient random walk is formed by the sites in the lattice where the random walk never turns backward (in the case of the transience to the right). This renewal structure corresponds to a so-called regeneration structure for the branching process associated with edge upcrossings of the random walk and formed by the regeneration times where the auxiliary branching process with migration dies out and starts afresh due to the inflow of the immigrants.

3.1 Branching structure for PRWCE

In this section we verify that our formulation of the branching structure of the PRWCE process satisfies the conditions stated in Kosygina and Zerner (2008).

Definition 3.1.1. *Let μ be probability measures on $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and ν a probability measure*

on \mathbb{Z} such that

$$\nu(\mathbb{N}) > 0 \quad \text{and} \quad \nu(\{k \in \mathbb{Z} : k \geq -M\}) = 1. \quad (3.1)$$

Additionally, let $\{\xi_i^{(j)} : i, j \geq 1\}$ and $\{\eta_k : k \geq 0\}$ be independent random variables with distributions μ and ν , respectively. Then the process $(Z_k)_{k \geq 0}$ recursively defined by

$$Z_0 := 0, \quad Z_{k+1} := \xi_1^{(k+1)} + \dots + \xi_{Z_k + \eta_k}^{(k+1)}, \quad k \geq 0,$$

is said to be a (μ, ν) -branching process with offspring distribution μ , which will be *Geom*(1/2) throughout the remainder of this thesis, and migration distribution ν , where we adopt the convention that

$$\xi_1^{(k+1)} + \dots + \xi_i^{(k+1)} = 0 \quad \text{if} \quad i \leq 0.$$

These branching processes offer several properties that we will utilize in the proofs of our main results. First, (μ, ν) -branching processes are time homogeneous Markov chains. Furthermore, they enable us to emigrate and immigrate individuals in a single step. Specifically, if the size of the population is Z_k at time k then

- (1) If $\eta_k \geq 0$ the η_k immigrate into the system. Conversely, if $\eta_k < 0$ then $\min\{Z_k, |\eta_k|\}$ emigrate out of the system.
- (2) The remaining $(Z_k + \eta_k)_+$ individuals independently reproduce obeying the offspring distribution μ . The number of their progeny then determines the size of the next generation, denoted Z_{k+1} .

Following the construction in Kosygina and Zerner (2008), we too consider the case where the number of emigrants is bounded by the number of cookies and the number of immigrants fluctuates. In addition to equation (3.1), we also adopt the same assumptions regarding the distributions μ and ν . Thus defining the average migration

$$\lambda := \sum_{k \geq -M} k \nu(\{k\}), \quad (3.2)$$

and the moment generating function for the offspring distribution

$$f(s) := \sum_{k \geq 0} \mu(\{k\}) s^k, \quad s \in [0, 1]$$

where we require

Assumption 3.1.2.

$$f(0) > 0, \quad f'(1) = 1, \quad b := f''(1)/2 < \infty, \quad \lambda < \infty \quad (3.3)$$

$$\sum_{k \geq 1} \mu(\{k\}) k^2 \ln(k) < \infty. \quad (3.4)$$

The next theorem provides the relationship between the survival of the (μ, ν) -branching process $(Z_k)_{k \geq 0}$ and the parameter

$$\theta := \frac{\lambda}{b} = \frac{\sum_{k \geq -M} k \nu(\{k\})}{\frac{1}{2} \sum_{k \geq 0} k(k-1) \mu(\{k\})}.$$

Additionally, define the *lifetime* of the process $(Z_k)_{k \geq 0}$

$$N^{(Z)} := \inf\{k \geq 1 : Z_k = 0\} \quad \text{and} \quad \widetilde{Z}_k := Z_k \mathbb{1}\{k < N^{(Z)}\},$$

called the *stopped process* as it remains extinct (i.e. $\widetilde{Z}_k = 0$ is an absorbing state). Then Formanov and Yasin (1989); Formanov et al. (1990) determined the conditions that specify the branching process' survival and quantify the magnitude of the expected number of total offspring, if any, over the total lifetime of the process.

Theorem A (Kosygina and Zerner (2008) - Theorem A). *Let $(Z_k)_{k \geq 0}$ be a (μ, ν) -branching process satisfying (3.1), (3.3), and (3.4). Denote the tail of the distribution of $N^{(Z)}$ by*

$$u_n := P(N^{(Z)} > n) = P(\widetilde{Z}_n > 0) \quad n \in \mathbb{N},$$

and let the expected total number of offspring for the stopped process $(\widetilde{Z}_k)_{k \geq 0}$ up to time $\mathbb{N}_0 \cup \{\infty\}$ be given by

$$v_n := E \left[\sum_{m=0}^n \widetilde{Z}_m \right].$$

Then the parameter θ specifies the behavior of the stopped process.

- (i) If $\theta > 1$ then $\lim_{n \rightarrow \infty} u_n = c_1 \in (0, 1)$, and $P\left(\widetilde{Z}_k > 0 : \forall k \in \mathbb{N}_0\right) = c_1 > 0$.
- (ii) If $\theta = 1$ then $\lim_{n \rightarrow \infty} u_n \ln(n) = c_2 \in (0, 1)$, and $\widetilde{Z}_k \rightarrow 0$ a. s. as $k \rightarrow \infty$.
- (iii) If $\theta = -1$ then $\lim_{n \rightarrow \infty} v_n \ln(n)^{-1} = c_3 \in (0, 1)$, and the expected total of offspring for the process $(\widetilde{Z}_k)_{k \geq 0}$ is infinite, that is $v_\infty = \infty$.
- (iv) If $\theta < -1$ and

$$\sum_{k \geq 1} k^{1+|\theta|} \mu(\{k\}) < \infty$$

then $\lim_{n \rightarrow \infty} n^{1+|\theta|} u_n = c_4 \in (0, 1)$. Furthermore, we have the expected total of offspring for the process $(\widetilde{Z}_k)_{k \geq 0}$ is finite, that is $\lim_{n \rightarrow \infty} v_n = c_5 \in (0, \infty)$.

Then main result from Kosygina and Zerner (2008)[Section 2] relating the total number of offspring and θ follows, and can be reproduced for our setting using an identical argument.

Theorem 3.1.3 (Corollary 4 - Kosygina and Zerner (2008)). *Let $(Z_k)_{k \geq 0}$ be a (μ, ν) -branching process satisfying (3.1), (3.3), and (3.4). Then $(\widetilde{Z}_k)_{k \geq 0}$ goes extinct a. s. iff $\theta \leq 1$. Furthermore, the expected total number of children, v_∞ , for the process $(\widetilde{Z}_k)_{k \geq 0}$ iff $\theta < -1$.*

3.1.1 PRWCE as branching process with migration

Recall the the branching process U_{k+1} for ERW from section 2.2. The construction of the branching process for PRWCE, with M cookies per site, proceeds in a similar fashion. First notice that the dependence of transition probabilities on the previous step can be replicated by placing two biased coins on each site per cookie, as depicted in figure 3.1. Suppose $\omega \in \Omega_M$ fixed, and consider a walker visiting a site $k \in \mathbb{Z}$ for the first time from the left. In this case, the walker would pick up the top two coins (the cookie), select the biased λ coin, and flip the coin to determine the next site it visits, as depicted in figure 3.2. Similarly, a walker visiting any site from the right for under the M^{th} would select the top μ coin prior to deciding the next site it visits, depicted in figure 3.3.

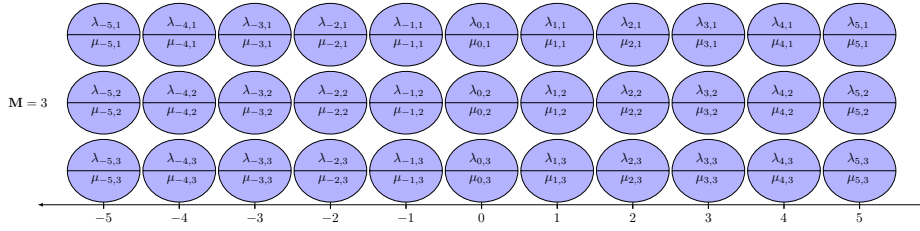


Figure 3.1 A cookie environment for the PRWCE with $M = 3$.

Finally, a walker visiting a site for more than the M^{th} time would select a placebo cookie,

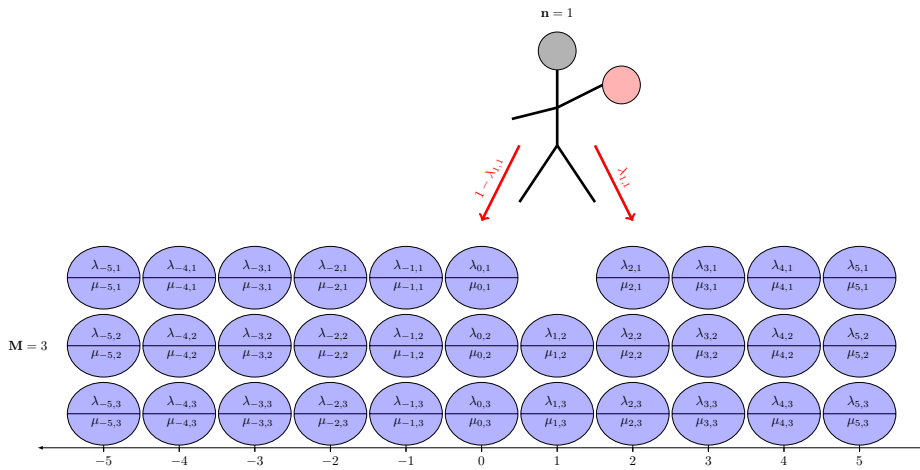


Figure 3.2 Visiting a site for the first time from the left.

and move to either of the neighboring sites with probability $1/2$, as depicted in figure 3.4.

Thus, exactly as specified for the ERW, every particle of the k^{th} generation will execute at least one coin toss (more if they reproduce), and at most the first M particles of that generation will flip biased coins. These are the first $(U_k \wedge M)$. The remaining particles reproduce independently using fair coins, just as before, or according to a $Geom(1/2)$ distribution. Lastly the $\eta_{U_K \wedge M}^{(k+1)}$ offspring of the emigrants return adding there numbers to the next generation. Hence,

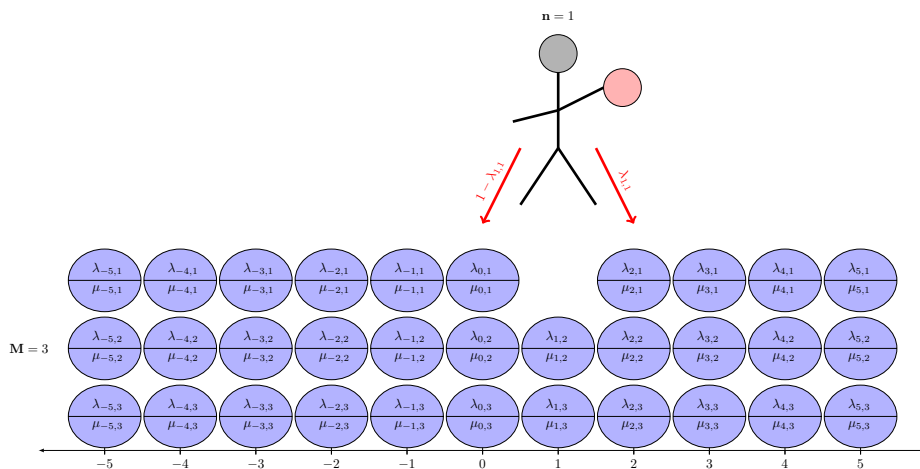


Figure 3.3 Visiting a site with cookies remaining from the right.

we can right the size of the $(k + 1)^{st}$ generation as

$$U_{k+1} := \xi_1^{(k+1)} + \dots + \xi_{U_k - M}^{(k+1)} + \eta_{U_K \wedge M}^{(k+1)},$$

where $\xi_1^{(j)}$ and $\eta_l^{(k)}$ are independent random variables for all $\{i, j, k \geq 1, 1 \leq l \leq M\}$ and each $\xi_i^{(j)}$ is a $Geom(1/2)$ random variable.

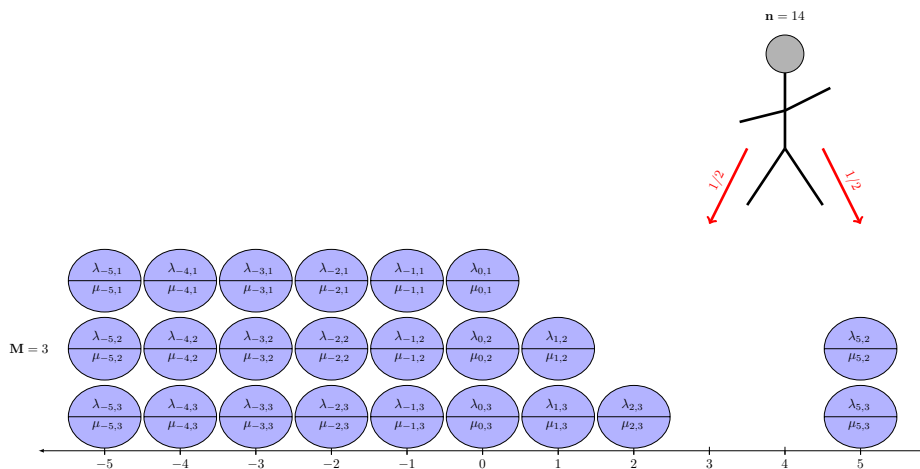


Figure 3.4 Visiting a for more than M times.

3.2 Recurrence and transience: Proof of theorem 1.2.1

The proof relies on a variation of the reduction to a branching process with migration method, which was introduced by Basdevant and Singh and further developed by Kosygina and Zerner. Standard arguments for one-dimensional nearest-neighbor random walks (see for instance Lemma 5 in Kosygina and Zerner (2008)) show that

$$\mathbb{P}_0 \left(\limsup_{n \rightarrow \infty} X_n \in \{-\infty, +\infty\} \right) = \mathbb{P}_0 \left(\liminf_{n \rightarrow \infty} X_n \in \{-\infty, +\infty\} \right) = 1.$$

In words: the probability that the random walk will stay forever in a finite box around the origin is zero. Indeed, since there is a finite supply of cookies in any finite box, the above claim follows from its counterpart for the simple nearest-neighbor random walk on the line and the strong Markov property.

Consider the first excursion of the random walk to the right, away from zero. Formally, let $X_0 = 1$ be the starting point of the random walk, let $T_0 = \inf\{n \geq 1 : X_n = 0\}$ be the first hitting time of zero, and consider the (infinite if $T_0 = \infty$) sequence $(X_n)_{0 \leq n \leq T_0}$. It is shown in Section 5 of Kosygina and Zerner (2008) (the proof goes through verbatim for our model) that $\mathbb{P}_1(T_0 < \infty) \in \{0, 1\}$, and moreover $\mathbb{P}_0(\liminf_{n \rightarrow \infty} X_n = -\infty)$ is equal to one if $\mathbb{P}_1(T_0 < \infty) = 1$ and is equal to zero otherwise. Similar results holds for the first excursion to the left: $\mathbb{P}_{-1}(T_0 < \infty) \in \{0, 1\}$, and moreover $\mathbb{P}_0(\limsup_{n \rightarrow \infty} X_n = \infty)$ is equal to one if $\mathbb{P}_{-1}(T_0 < \infty) = 1$ and is equal to zero otherwise. This yields the following criteria for transience and recurrence in terms of the duration T_0 of the first excursion away from zero.

Lemma 3.2.1. *We have:*

- (i) *X is transient to the left if and only if $\mathbb{P}_1(T_0 < \infty) = 1$ and $\mathbb{P}_{-1}(T_0 < \infty) < 1$.*
- (ii) *X is recurrent if and only if $\mathbb{P}_1(T_0 < \infty) = 1$ and $\mathbb{P}_{-1}(T_0 < \infty) = 1$.*
- (iii) *X is transient to the right if and only if $\mathbb{P}_1(T_0 < \infty) < 1$ and $\mathbb{P}_{-1}(T_0 < \infty) = 1$.*

Furthermore, in keeping with the result shown in Section 3.3 example 3.3.4, we allow for the case where $\mathbb{P}_1(T_0 < \infty) < 1$ and $\mathbb{P}_{-1}(T_0 < \infty) < 1$ at the same time.

We next study the first excursion of the random walk away from zero to the right in more detail. Set $U_0 = 1$ and let

$$U_k = \#\{n \geq 0 : n < T_0, X_n = k, X_{n+1} = k + 1\}, \quad k \in \mathbb{N}. \quad (3.5)$$

Thus U_k is the number of *upcrossings from site k* , alternatively the number of upcrossings on the edge connecting site k with site $k + 1$, before a return to zero. Notice that if $X_0 = 1$, then the following two events coincide modulo a null set of measure \mathbb{P}_1 :

$$\{T_0 = \infty\} = \{U_k > 0, \forall k \in \mathbb{N}\}, \quad \mathbb{P}_1 - \text{a. s.}$$

In view of Lemma 3.2.1, in order to prove Theorem 1.2.1, it suffices to show that

$$\mathbb{P}_1(T_0 = \infty) = \mathbb{P}_1(U_k > 0, \forall k \in \mathbb{N}) > 0 \quad \text{iff} \quad \delta_p > 1. \quad (3.6)$$

To show that (3.6) holds, we will first construct a convenient representation of the jump sequence $(X_n - X_{n-1})_{n \geq 0}$ in an appropriate probability space. Let $Y_n = X_n - X_{n-1}$. Thus Y_n take values -1 or 1 , and

$$X_n = X_{n-1} + Y_n, \quad n \geq 0.$$

Notice that once the random walk exits a site k in a certain direction, we know with probability one from which direction it will re-enter the site next time, if it returns at all. Thus, similar to constructions for simple random walks, random walks in random environments, and regular excited random walks, the “entire sequences of decisions” of X_n at different sites of \mathbb{Z} are independent each of other. Therefore, rather than working with Y_n , it is more convenient to operate with an infinite collection of independent sequences $(Y_i^{(k)})_{i \in \mathbb{N}}$, $k \in \mathbb{Z}$, where $Y_i^{(k)} \in \{-1, +1\}$ based upon the next move of the walker after its i -th visit to site $k \in \mathbb{Z}$ (provided that such a visit actually takes place). If the walker moves on to site $k + 1$, then

$Y_i^{(k)} = +1$. Conversely, $Y_i^{(k)} = -1$ in the event that the walker visits site $k - 1$ next.

Formally, given a cookie environment $\omega \in \Omega_M$, let

$$G_{\omega,k} := (Y_i^{(k)})_{0 \leq i \leq M}, \quad k \in \mathbb{Z},$$

be a collection of independent *non-homogeneous* two-state Markov chains, each one defined on the state space $\{-1, 1\}$ with initial distribution

$$P_\omega \left(Y_0^{(k)} = 1 \right) = \begin{cases} 0 & \text{if } k > 0 \\ 1/2 & \text{if } k = 0, \\ 1 & \text{if } k < 0 \end{cases}$$

$$P_\omega \left(Y_0^{(k)} = -1 \right) = 1 - P_\omega \left(Y_0^{(k)} = 1 \right),$$

and transition kernel for $1 \leq i \leq M$ given by

$$\begin{bmatrix} P_\omega \left(Y_i^{(k)} = -1 \mid Y_{i-1}^{(k)} = -1 \right) & P_\omega \left(Y_i^{(k)} = 1 \mid Y_{i-1}^{(k)} = -1 \right) \\ P_\omega \left(Y_i^{(k)} = -1 \mid Y_{i-1}^{(k)} = 1 \right) & P_\omega \left(Y_i^{(k)} = 1 \mid Y_{i-1}^{(k)} = 1 \right) \end{bmatrix} = H_{\omega,i}^{(k)},$$

where matrices $H_{\omega,i}^{(k)}$ are defined in (1.2).

Further, let $(Y_i^{(k)})_{k \in \mathbb{Z}, i > M}$ be a double-indexed sequence of i.i.d. Bernoulli variables independent of $(Y_i^{(k)})_{k \in \mathbb{Z}, 0 \leq i \leq M}$ with

$$P_\omega \left(Y_i^{(k)} = 1 \right) = P_\omega \left(Y_i^{(k)} = -1 \right) = 1/2.$$

Recall $\zeta_n = \#\{i \in [0, n] : X_i = X_n\}$ and set

$$X_{n+1} = X_n + Y_{\zeta_n}^{(X_n)}. \quad (3.7)$$

This representation of X_n is consistent with the definition of the random walk given in Section 1.2, and might serve as an alternative (equivalent) definition of random walk X . Then setting

$$T_{k,i} := \inf\{n > 0 : X_n \text{ visits site } k \text{ for the } i^{\text{th}} \text{ time}\} \in \mathbb{N} \cup \{\infty\},$$

one observes that the sign of the velocity $X_{T_{k,i+1}} - X_{T_{k,i}}$ alternates in a *deterministic* manner during successive visits $i = 1, 2, \dots$ to a site $k \in \mathbb{Z}$. The realization (3.7) of the random walk by means of the sequence $(Y_i^{(k)})_{k \in \mathbb{Z}, i \geq 0}$ is used throughout the remainder of this paper.

Following Kosygina and Zerner (2008), in the remainder of this paper we will refer to the event $\{Y_i^{(k)} = 1\}$ as success and to the event $\{Y_i^{(k)} = -1\}$ as failure. Set $S_0^{(k)} = 0$ and let

$$S_m^{(k)} = \text{the number of successes in } (Y_i^{(k)})_{i \geq 0} \text{ prior to the } m\text{-th failure.}$$

Let $V_0 = 1$, $W_0 = 0$, and recursively define the processes V_k and W_k :

$$V_{k+1} = S_{V_k}^{(k)} \quad \text{with} \quad W_{k+1} = S_{W_k \vee M}^{(k)}, \quad k \geq 0.$$

Since $(S_i^{(k)})_{i \geq 0}$, $k \in \mathbb{N}$, are independent and identically distributed sequences, both processes $(V_k)_{k \geq 0}$ and $(W_k)_{k \geq 0}$ are homogeneous Markov chains on \mathbb{Z}_+ .

Recall U_k from (3.5). The following results are immediate adaptations to our model of their direct analogues proved in (Kosygina and Zerner, 2008, Section 4).

Lemma 3.2.2. (i) $U_k = V_k$ for all $k \geq 0$ on the event $\{T_0 < \infty\}$.

(ii) $U_k \leq V_k$ for all $k \geq 0$ on the event $\{T_0 = \infty\}$.

(iii) $\mathbb{P}_1(U_k > 0, \forall k \in \mathbb{N}) > 0 \iff \mathbb{P}_1(V_k \rightarrow_{k \rightarrow \infty} \infty) > 0 \iff \mathbb{P}_1(W_k \rightarrow_{k \rightarrow \infty} \infty) > 0$.

The first two claims are a consequence of the branching structure of $(U_k)_{k \geq 0}$. To see the legitimacy of the first claim, notice that the value of V_k is just the number of downcrossing from site k since $\{T_0 < \infty\}$ implies there must be a matching downcrossing for every upcrossing. However, if $\{T_0 = \infty\}$ then there must be sites without corresponding downcrossings. Thus, the value of V_k is U_k plus the value of a geometric random variable. Finally, the last properties is an implication of the strong Markov property for $(V_k)_{k \geq 0}$ and of the observation that transition kernels of the irreducible Markov chains $(V_k)_{k \geq 0}$ and $(W_k)_{k \geq 0}$ differ only on the first

$M + 1$ states.

The key observation made in Kosygina and Zerner (2008) is that the study of the asymptotic behavior of W_k , and hence of V_k , can be reduced to the study of a *branching process with migration* $(Z_k)_{k \geq 0}$, which is defined in the following way. For $k \geq 0$, let $Z_k = W_{k+1} - S_M^{(k)}$ and $\eta_k = S_M^{(k)} - M$. We will use the notation $x \vee y$ to denote $\max\{x, y\}$ for $x, y \in \mathbb{R}$. Set $Z_0 = 0$ and define recursively

$$Z_{k+1} = W_{k+2} - S_M^{(k+1)} = S_{W_{k+1} \vee M}^{(k+1)} - S_M^{(k+1)} = \sum_{j=1}^{(W_{k+1} - M) \vee 0} \xi_j^{(k+1)},$$

where for $k \in \mathbb{Z}$

$\xi_j^{(k)}$ = the number of successes between $(M + j - 1)^{th}$ and $(M + j)^{th}$ failure at k ,

and the empty sum $\sum_{j=1}^0 \xi_j^{(k+1)}$ should be understood as zero.

Notice that $W_{k+1} - M = Z_k + S_M^{(k)} - M = Z_k + \eta_k$. Furthermore, $(\xi_j^{(k)})_{k \in \mathbb{Z}, j > M}$ are i.i.d. random variables, each one distributed according to the $\text{Geom}(1/2)$ law, while the sequence $(S_M^{(k)})_{k \in \mathbb{Z}}$ is formed by independent random variables $S_M^{(k)}$ and is independent of $(\xi_j^{(k)})_{k \in \mathbb{Z}, j > M}$. By $\text{Geom}(1/2)$ we mean the probability function which assigns probability 2^{-n-1} to $n \in \mathbb{Z}_+$. With a slight abuse of notation, we will sometime use the same notation $\text{Geom}(1/2)$ to denote a random variable with this distribution.

Random variable Z_k can be interpreted as the size of the population of a branching process at time k . The dynamics of the population in this process can be described as follows: at time $k \in \mathbb{Z}_+$ either η_k particles immigrate or $\min\{Z_k, |\eta_k|\}$ particles emigrate according to whether $\eta_k \geq 0$ or $\eta_k < 0$. Then $(Z_k + \eta_k) \vee 0$ particles present in the system reproduce independently according to the $\text{Geom}(1/2)$ distribution, and their children form the population at time $k + 1$. Notice that $(Z_k)_{k \geq 0}$ is a time homogeneous Markov chain.

Since $(\xi_j^{(k)})_{k \in \mathbb{Z}, j > M}$ are i.i.d. geometrically distributed random variables, the identity

$$Z_{k+1} = \sum_{j=1}^{(W_{k+1}-M) \vee 0} \xi_j^{(k+1)}$$

implies that the following two events coincide modulo a null set of measure \mathbb{P} :

$$\{W_k \rightarrow_{k \rightarrow \infty} \infty\} = \{Z_k \rightarrow_{k \rightarrow \infty} \infty\}, \quad \mathbb{P} - \text{ a. s.}$$

Furthermore, since $(Z_k)_{k \geq 0}$ is an irreducible Markov chain under \mathbb{P} , the event $\{Z_k \rightarrow_{k \rightarrow \infty} \infty\}$ is equivalent to the non-extinction of the branching process:

$$\{Z_k \rightarrow_{k \rightarrow \infty} \infty\} = \{Z_k \rightarrow_{k \rightarrow \infty} 0\}^c, \quad \mathbb{P} - \text{ a. s.}$$

where we use the notation A^c to denote the complement of an event A .

To complete the proof of (3.6), and hence of Theorem 1.2.1, we will use the following general fact about branching processes with migration (see, for instance, Theorem A in (Kosygina and Zerner, 2008, p. 1957)):

Lemma A. *The extinction probability $\mathbb{P}(\lim_{k \rightarrow \infty} Z_k = 0)$ of the branching process $(Z_k)_{k \geq 0}$ is less than one iff $\mathbb{E}[\eta_k] = \mathbb{E}[S_M^{(1)} - M] > 1$.*

To conclude the proof of (3.6) it therefore suffices to show that $\mathbb{E}[S_M^{(1)} - M] = \delta_p$, where δ_p is introduced in (1.3) through the relation $\delta_p = M(2p - 1)$. Let

$$F = \text{the number of failures among the first } M \text{ trials in } (Y_i^{(1)})_{i \geq 0}.$$

It is not hard to show (see the last three lines in the proof of Lemma 9 in Kosygina and Zerner (2008)) that

$$\mathbb{E}[S_M^{(1)} | F] = \underbrace{M - F}_{\substack{\text{number of successes} \\ \text{in the first } M \text{ trials}}} + \underbrace{(M - F) \cdot \mathbb{E}[\text{Geom}(1/2)]}_{\substack{\text{expected number of successes in the} \\ \text{remaining } M - F \text{ geometric trials}}},$$

and hence, since $\mathbb{E}[\text{Geom}(1/2)] = 1$,

$$\mathbb{E}\left[S_M^{(1)}\right] - M = M - 2\mathbb{E}[F]. \quad (3.8)$$

In terms of the Markov chain $(Y_i^{(1)})_{0 \leq i \leq M}$ and transition matrices $\mathbb{H}_{\omega,j}^{(1)}$ defined in (1.2), we have

$$\begin{aligned} \mathbb{E}[F] &= \mathbb{E}_P \left[\mathbb{E}_\omega \left[\sum_{j=1}^M \mathbb{1} \{Y_j^{(1)} = -1\} \right] \right] = \sum_{j=1}^M \mathbb{E}_P \left[\mathbb{H}_{\omega,j}^{(1)}(-1, -1) \right] \\ &= M - \sum_{j=1}^M \mathbb{E}_P \left[\mathbb{H}_{\omega,j}^{(1)}(-1, 1) \right], \end{aligned}$$

which implies (3.6) by virtue of (3.8). The proof of Theorem 1.2.1 is therefore complete. \square

3.3 Examples

We will conclude this section with a few examples. First, we notice that in the particular case $\lambda_k := \lambda_{k,1} = \dots = \lambda_{k,M}$ and $\mu_k := \mu_{k,1} = \dots = \mu_{k,M}$, the definition (1.3) yields (see for instance (Norris, 1998, p. 5))

$$p = E \left[\frac{\lambda_0}{\lambda_0 + \mu_0} - \frac{\lambda_0(1 - \lambda_0 - \mu_0)}{M(\lambda_0 + \mu_0)^2} (1 - (1 - \lambda_0 - \mu_0)^M) \right] \quad (3.9)$$

and

$$q = E \left[\frac{\mu_0}{\lambda_0 + \mu_0} - \frac{\mu_0(1 - \lambda_0 - \mu_0)}{M(\lambda_0 + \mu_0)^2} (1 - (1 - \lambda_0 - \mu_0)^M) \right]. \quad (3.10)$$

Lemma 3.3.1. *In this case $\delta_p > 1$ and $\delta_q > 1$ is impossible.*

Example 3.3.2. *Fix a constant $\alpha \in (0, 1)$. Suppose $P(\lambda_0 = \mu_0 = \alpha) = 1$, and let $\gamma = 1 - 2\alpha$.*

Then, by (3.9) and (3.10),

$$p = q = \frac{1}{2} - \frac{\gamma}{2M} \frac{1 - \gamma^M}{1 - \gamma}, \quad \text{which implies} \quad \delta_p = \delta_q = -\gamma \frac{1 - \gamma^M}{1 - \gamma} < 1.$$

In particular, the walk is recurrent.

Example 3.3.3. We notice that, in contrast to the model considered in Kosygina and Zerner (2008), permuting the cookies within the cookie pile can in general change the asymptotic behavior of the random walk (since the matrices in the right-hand side of (1.2) do not necessarily commute). For instance, let $M = 2$ and consider the following two cases:

(i) The following holds $P - a. s.$ (in particular, cookie environment ω is a deterministic sequence):

$$H_{\omega,0}^{(k)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \quad \text{and} \quad H_{\omega,1}^{(k)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}.$$

Then (1.3) yields $p = 0.1$ and $q = 0.46$. Correspondingly, $\delta_p = -1.6$ and $\delta_q = -0.16$, and hence the walk is recurrent.

(ii) The following holds $P - a. s.$ (in particular, cookie environment ω is a deterministic sequence):

$$H_{\omega,0}^{(k)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix} \quad \text{and} \quad H_{\omega,1}^{(k)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

Then (1.3) yields $p = 0.14$ and $q = 0.86$. Correspondingly, $\delta_p = -\delta_q = -1.44$, and hence the walk is transient to the left.

Example 3.3.4. Further, the PRWCE differs from the model considered in Kosygina and Zerner (2008), in one particularly interesting and unexpected way. Let $M = 2$ and consider the following result, which holds $P - a. s.$ (since the cookie environment ω is a deterministic sequence):

$$H_{\omega,0}^{(k)} = \begin{bmatrix} 0.1 & 0.9 \\ 0.8 & 0.2 \end{bmatrix} \quad \text{and} \quad H_{\omega,1}^{(k)} = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}.$$

Then (1.3) implies $p = 0.815$ and $q = 0.78$, which results in the values $\delta_p = 1.26 > 1$ and $\delta_q = 1.12 > 1$. Thus, the walker has a positive probability of never returning to zero from both the left and the right.

Example 3.3.5. Finally, consider a general persistent cookie random walk in a simple deterministic environment with $M = 2$. Namely, assume that $P(\lambda_{0,i} = \lambda) = 1$ and $P(\mu_{0,i} = \mu) =$

1 for $i = 1, 2$ and some constants $\lambda, \mu \in (0, 1)$. Then $p = \frac{\lambda}{2}(3 - \lambda - \mu)$. Consequently, $\delta_p > 1$ if and only if $\mu < 3 - \lambda - \frac{3}{2\lambda}$. In particular, $\delta_p > 1$ implies $\lambda > \frac{3-\sqrt{3}}{2}$ and $\mu < \frac{1}{2}$.

3.4 Speed regimes: Proof of theorem 1.2.2

Enforcing the same assumptions as before, let $\delta_p > 1$ and $\delta_q \leq 1$, which implies that $X_n \rightarrow \infty$ \mathbb{P}_0 a. s. and the walk is not recurrent from the right. Hence for $D := \inf\{n \geq 1 : X_n < X_0\}$ we have $\mathbb{P}_0(D = \infty) > 0$. Thus, there must exist, \mathbb{P}_0 – a. s., infinitely many *renewal times* $(\tau_k)_{k \geq 1}$, corresponding to edges that never see a downcrossing, such that, for any $k \in \mathbb{N}$

$$X_n < X_{\tau_k}, \text{ for all } 0 \leq n < \tau_k \quad \text{and} \quad X_n \geq X_{\tau_k}, \text{ for all } n > \tau_k.$$

Furthermore, the sequence $(X_{\tau_k})_{k \geq 1}$ has independent increments. This implies that the sequence $(X_{\tau_1}, \tau_1), \{(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)\}_{k \geq 1}$ is independent with respect to \mathbb{P}_0 . Additionally, any renewal corresponds to the walker being exposed to an “untouched” environment (i.e. all of the sites in front of it contain M cookies). Hence, we also have that the sequence $\{(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)\}_{k \geq 1}$ is identically distributed under \mathbb{P}_0 . While the above may not be the usual renewal structure, critically, we do have a renewal structure in the sense that there is a positive probability that our walker, upon visiting a site for the first time, will never backtrack below this site. Therefore, by Kac’s lemma Durrett (2010)[Theorem 7.3.3], we have

$$E_0[X_{\tau_2} - X_{\tau_1}] = 1/\mathbb{P}_0(D = \infty) < \infty. \quad (3.11)$$

We will need to following two lemmas prior to our discussion of the proof for theorem 1.2.2.

Lemma 3.4.1. *Let $(\tau_k)_{k \geq 1}$ be a sequence of renewal times associated with a cookie environment Ω_M such that $\delta_p > 1$, $|\delta_q| \leq 1$ and consider the i.i.d. sequence $\{(X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k)\}_{k \geq 1}$ associated with the process $(X_n)_{n \geq 0}$. Then we have the following:*

(1) $v := \lim_{n \rightarrow \infty} X_n/n$ exists. Furthermore

$$v = \lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{E_0[X_{\tau_2} - X_{\tau_1}]}{E_0[\tau_2 - \tau_1]} \quad \mathbb{P}_0 - \text{ a. s.}$$

where $v > 0$ iff $E_0[\tau_2 - \tau_1] < \infty$.

(2) $E_0[\tau_2 - \tau_1] < \infty$ implies $E_0[X_{\tau_2} - X_{\tau_1}] < \infty$;

(3) $E_0[\tau_2 - \tau_1] < \infty$.

The first property of the lemma follows from the argument preceding the lemma, equation (3.11), and an application of the strong law of large numbers. The second property is just a trivial consequence of the fact that the process can move at most m sites to the right in m steps. Finally, the third property results from an identical argument as that in Kosygina and Zerner (2008) which, similar to the construction of the proof for theorem 1.2.1, utilizes a sequence of auxiliary process. For $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, define the processes

$$D_k := \#\{n : \tau_1 < n < \tau_2, X_n = X_{\tau_2} - k, X_{n+1} = X_{\tau_2} - k - 1\},$$

which denote the number of downcrossings from site $X_{\tau_2} - k$ between times τ_1 and τ_2 ,

$$F_0^{(k)} := 0, \quad F_m^{(k)} := \# \text{ of failures in } (Y_i^{(k)})_{i \geq 1} \text{ prior to } m^{\text{th}} \text{ success};$$

$$V_0 := 0, \quad V_{k+1} := F_{V_{k+1}}^{(k)}, \quad k \geq 0;$$

$$\widetilde{V}_k := V_k \cdot \mathbb{1}\{k < N^{(V)}\}, \quad \text{where } N^{(V)} := \inf\{k \geq 1 : V_k = 0\};$$

$$(Z_k)_{k \geq 0} := (\text{Geom}(1/2), \nu) - \text{branching process} \quad \nu \text{ distribution of } \eta_k := F_M^{(k)} - M + 1;$$

$$\widetilde{Z}_k := Z_k \cdot \mathbb{1}\{k < N^{(Z)}\}, \quad \text{where } N^{(Z)} := \inf\{k \geq 1 : Z_k = 0\}.$$

Then the argument proceeds via

$$\begin{aligned}
v > 0 &\iff E_0[\tau_2 - \tau_1] < \infty && \because 1^{st} \text{ property in lemma;} \\
&\iff E_0\left[\sum_{k \geq 1} D_k\right] < \infty && \because \text{Kosygina and Zerner (2008)[Lemma 11];} \\
&\iff E_0\left[\sum_{k \geq 0} \widetilde{V}_k\right] < \infty && \because \text{Kosygina and Zerner (2008)[Lemma 12];} \\
&\iff E_0\left[\sum_{k \geq 0} \widetilde{Z}_k\right] < \infty && \because \text{Kosygina and Zerner (2008)[Lemma 14];} \\
&\iff \theta < -1 && \because \text{Theorem 3.1.3} \\
&\iff \delta_p > 2, \text{ since } \theta = 1 - \delta_p && \because \text{Kosygina and Zerner (2008)[Lemma 17].}
\end{aligned}$$

Clearly, by symmetry, the case where $\delta_q < -2$ and $|\delta_p| \leq 1$ is exactly the same except for $v < 0$.

In the event that $\delta_p > 1$ and $\delta_q < -1$, any excursion to either the right or the left has a positive probability of never returning. Once this happens, then the above discussed renewal times will be generated, and the above lemma becomes

Lemma 3.4.2. *Set $l \in \{L, R\}$ according to whether the walker is transient to the right ($l = R$) or to the left ($l = L$), and let $(\tau_k^{(l)})_{k \geq 1}$ be a sequence of renewal times associated with the transient excursion in cookie environment Ω_M satisfying $\delta_p > 1$ and $\delta_q < -1$. Then, the i.i.d. sequence $\left\{ (X_{\tau_{k+1}^{(l)}} - X_{\tau_k^{(l)}}, \tau_{k+1}^{(l)} - \tau_k^{(l)}) \right\}_{k \geq 1}$ associated with the process $(X_n)_{n \geq 0}$ still satisfy the conditions in the discussion preceding Lemma 3.4.1, and the following, revised, conclusions still hold:*

(1) $v := \lim_{n \rightarrow \infty} X_n/n$ exists. Furthermore, there exists a constant $\alpha \in (0, 1)$ such that

$$v = \lim_{n \rightarrow \infty} \frac{X_n}{n} = \alpha \frac{E_0[X_{\tau_2^{(R)}} - X_{\tau_1^{(R)}}]}{E_0[\tau_2^{(R)} - \tau_1^{(R)}]} + (1 - \alpha) \frac{E_0[X_{\tau_2^{(L)}} - X_{\tau_1^{(L)}}]}{E_0[\tau_2^{(L)} - \tau_1^{(L)}]} \quad \mathbb{P}_0 - \text{a. s.} \quad (3.12)$$

where $v \neq 0$ iff $E_0[\tau_2^{(l)} - \tau_1^{(l)}] < \infty$.

$$(2) E_0 \left[\tau_2^{(l)} - \tau_1^{(l)} \right] < \infty \text{ implies } E_0 \left[\left| X_{\tau_2^{(l)}} - X_{\tau_1^{(l)}} \right| \right] < \infty.$$

$$(3) E_0 \left[\tau_2^{(l)} - \tau_1^{(l)} \right] < \infty.$$

It is worth noting that if either $\delta_p > 2$, with $-2 < \delta_q < -1$, or $1 < \delta_p < 2$, with $\delta_q < -2$) then Lemma 3.4.2 still applies, but one half of the speed equation must be zero. Finally, the recurrent case directly follows using the same argument as Kosygina and Zerner (2008)[Proposition 13], yielding the final requirement to satisfy Theorem 1.2.2. \square

3.5 Proof of CLT: transient case

Enforcing the same assumptions as before, let $\delta_p > 4$ and $|\delta_q| \leq 1$. The the CLT is a consequence of the following lemma:

Lemma 3.5.1. *Assume $\delta_p > 4$, $|\delta_q| \leq 1$, and let v be the velocity define in theorem 1.2.2 then*

(1) X_n satisfies the CLT, that is

$$\lim_{n \rightarrow \infty} \frac{X_n - nv}{\sigma \sqrt{n}} \Rightarrow \mathcal{N}(0, 1)$$

where

$$\sigma^2 = \frac{E \left[(X_{\tau_2} - X_{\tau_1}) - v(\tau_2 - \tau_1) \right]^2}{E \left[\tau_2 - \tau_1 \right]} > 0.$$

$$(2) E \left[(\tau_2 - \tau_1)^2 \right] < \infty \text{ implies } E \left[(X_{\tau_2} - X_{\tau_1})^2 \right] < \infty.$$

$$(3) E \left[(\tau_2 - \tau_1)^2 \right] < \infty.$$

The first property in the lemma clearly holds, provided that the third property can be shown. The second property follows for the exact same reason as the second property in Lemma 3.4.1. Finally, for the third property, consider the axillary processes defined in Lemma 3.4.1. The the

argument proceeds as follows

$$\begin{aligned}
\delta_p > 4 &\iff E_0 \left[\left(\sum_{k \geq 0} \tilde{V}_k \right)^2 \right] < \infty && \because \text{Kosygina and Zerner (2008)[Lemma 18];} \\
&\iff E_0 \left[\left(\sum_{k \geq 1} D_k \right)^2 \right] < \infty && \because \text{Kosygina and Zerner (2008)[Lemma 12];} \\
&\iff E_0 \left[(\tau_2 - \tau_1)^2 \right] < \infty && \because \text{Kosygina and Zerner (2008)[Lemma 11] for } p = 2.
\end{aligned}$$

Similarly as above, if $\delta_p > 4$ and $\delta_q < -4$, then any excursion to either the right or the left has a positive probability of never returning. Once this happens, then the above discussed renewal times will be generated, and the above lemma becomes

Lemma 3.5.2. *Assume $\delta_p > 4$, and let v be the velocity define in theorem 3.4.2 then for some $l \in \{L, R\}$*

(1) X_n satisfies the CLT, that is

$$\lim_{n \rightarrow \infty} \frac{X_n - nv}{\sigma \sqrt{n}} \Rightarrow \mathcal{N}(0, 1)$$

where

$$\sigma^2 = \frac{E \left[\left(X_{\tau_2}^{(l)} - X_{\tau_1}^{(l)} \right) - v \left(\tau_2^{(l)} - \tau_1^{(l)} \right)^2 \right]}{E \left[\tau_2^{(l)} - \tau_1^{(l)} \right]} > 0.$$

$$(2) E \left[\left(\tau_2^{(l)} - \tau_1^{(l)} \right)^2 \right] < \infty \text{ implies } E \left[\left(X_{\tau_2}^{(l)} - X_{\tau_1}^{(l)} \right)^2 \right] < \infty.$$

(3) *The restrictions for the 2nd moments become*

$$E \left[\left(\tau_2^{(l)} - \tau_1^{(l)} \right)^2 \right] < \infty \iff \begin{cases} \delta_p > 4, & \text{if } l = L \\ \delta_q < -4 & \text{if } l = R. \end{cases}$$

CHAPTER 4. CONCLUSIONS AND FUTURE DIRECTIONS

In this thesis, we obtained basic limit results concerning the asymptotic behavior of the one-dimensional persistent random walk in a cookie environment. There several directly related interesting problems one could investigate. One direction for future work is to extend the central limit theorem in a transient regime in the following three directions:

1. To obtain limit theorems for the recurrent PRWCE. For the classical recurrent ERW it is done in Dolgopyat (2011); Dolgopyat and Kosygina (2012)
2. To obtain stable (non-gaussian) limit laws in the transient case. We conjecture that limit theorems in the “standard” transient and recurrent regimes described in parts Theorem 1.4 are similar to the stable limit theorems for the regular ERW, RWRE, and PRWRE in dimension one.
3. To cover to a full extent the transient case when the 0 – 1 law for the direction of transience is violated (the regime described in the conclusions of part (iv) of Theorem 1.4). The obvious conjecture is that in the case when δ_p and δ_q correspond to stable laws with different indexes in the classical ERW case, the resulting limit theorem for the PRWCRE is a stable limit law with the lower index.
4. It would be interesting to extend the results of Rastegar (2012) for the maximum local time to the PRWCRE. It seems plausible that the proof methods of Rastegar (2012) which rely on the branching-random walk duality can be carried over to the model considered in this thesis. This would require having in hand delicate estimates for exit times and exit probabilities, similar to those obtained for the classical ERW in Kosygina and

Mountford (2011). These estimates are could serve as a cornerstone of the proof of the stable limit laws in the transient case (see also Kosygina and Zerner ()).

5. Finally, a natural questions that is left unanswered in this thesis is the exact description of the possible range of the pair (δ_p, δ_q) in the transient regime considered in part (iv) of Theorem 1.4. We conjecture, but were unable to prove this, that there is no actual restriction on this range.

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