1972

Some problems of shear waves

Wen-chang Lin

Iowa State University

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Some problems of shear waves

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**LIST OF FREQUENTLY USED SYMBOLS**

(It is hoped that meanings of symbols not listed are obvious from context)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_i = (u,v,w)$</td>
<td>Displacement vector</td>
</tr>
<tr>
<td>$i,j,k = 1,2,3$</td>
<td>Cartesian tensor indices</td>
</tr>
<tr>
<td>$t_{ij} = t$</td>
<td>Spatial stress tensor</td>
</tr>
<tr>
<td>$\delta_{ij} = \delta$</td>
<td>Kronecker delta</td>
</tr>
<tr>
<td>$\lambda,\mu$</td>
<td>Lame's constants</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of the deformed medium</td>
</tr>
<tr>
<td>$n_i = (n_1, n_2, n_3)$</td>
<td>Unit normal vector</td>
</tr>
<tr>
<td>$\lambda^+$</td>
<td>Wave length</td>
</tr>
<tr>
<td>$k = \frac{2\pi}{\lambda^+}$</td>
<td>Wave number</td>
</tr>
<tr>
<td>$c$</td>
<td>Wave speed (or phase velocity)</td>
</tr>
<tr>
<td>$c_i = \sqrt{\frac{\nu_i}{\rho_i}}, i = 1,2$</td>
<td>Wave speed in medium 1, 2</td>
</tr>
<tr>
<td>$c_0$</td>
<td>First approximation of wave speed</td>
</tr>
<tr>
<td>$L_0$</td>
<td>Typical length</td>
</tr>
<tr>
<td>$Z = \frac{Z}{h}$</td>
<td>Stretched coordinate in the $z$ direction</td>
</tr>
<tr>
<td>$L, M$</td>
<td>Differential operators</td>
</tr>
<tr>
<td>$F, G, H$</td>
<td>Functions of time and a spatial coordinate</td>
</tr>
<tr>
<td>$X = x - c_0 t$</td>
<td>Coordinate moving with wave speed</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Small parameter for stretching</td>
</tr>
</tbody>
</table>
\( T = \alpha t \)  
\( \alpha \)  
\( \varepsilon \)  
\( \beta \)  
\( \kappa = \frac{a}{b} \)  
\( x_i = (x, y, z) \)  
\( T_{ij} = T \)  
\( \tau_{ij} = \tau \)  
\( e_{ij} = e \)  
\( d_{ij} = d \)  
\( \nu = \nu_i = (\nu_1, \nu_2, \nu_3) \)  
\( a = a_i = (a_1, a_2, a_3) \)  
\( U \)  
\( R = r - c_0 t \)  
\( \delta = \frac{\mu^*}{\rho_0 c_0 L_0} \)  
\( \lambda^*, \mu^* \)  
\( G_0 \)  
\( \varepsilon \)  
\( I, II, III \)
1, m, n, p

Elasticities

det. (e_{ij})

Determinant of e_{ij}

\beta_o

Coefficient of nonlinear term of the structure

\delta_o

Coefficient of dissipative term of the structure

ln

Natural logarithm

\rho_o

Density of the undeformed medium

\omega

Frequency of the forcing function

\omega_o

Frequency away from the natural frequency

\eta = \sqrt{\frac{\lambda + 2\mu}{\mu}}

Square root of the ratio of dilatational speed to shear speed
1. INTRODUCTION

Shear waves of isotropic elasticity have a number of properties peculiar to them; this makes them an attractive field of investigation, as is rotation in fluid dynamics. The objective of this work is to study a few of these aspects.

One interesting feature of shear waves is their existence on their own, at least in some geometries. Consider a longitudinal wave in a rod; since a stretch must involve a contraction, an axial displacement is always accompanied by a transverse displacement, though of a smaller order; the effective speed of propagation is not given by dilatational wave speed but is governed by Young's modulus. In contrast shear waves in a plate and torsional waves in a circular cylindrical rod travel with the same speed as shear waves in an infinite medium. Noncircular sections under torsion are warped due to axial displacement of a lower order; also, while warping vanishes for a circular cylinder in torsion, one has an infinite number of modes of propagation. However, the fundamental mode, which travels with shear wave speed and which is the only one surviving at large distances, turns out to be nondispersive.

In the second chapter, some problems of two-layered isotropic bounded media are considered. Only those cases, for which dispersion vanishes in the single-layered case, are studied here. These are: (1) plane waves in a two-layered plate; (2) cylindrical waves in a two-layered plate; (3) torsional waves in a two-layered circular cylinder. In the first and last cases, the frequency equations are also analyzed;
this is not done for the second case due to its complicated form. Then an asymptotic expansion is developed to bring out the dispersive nature of these waves. The comparison brings out the elegance and power of the method of asymptotic expansions.

Though a strict mathematical justification of the asymptotic expansion may remain unsatisfactory, depending on the demands of rigour, its validity remains unquestioned. The expansion elaborated here brings out not only the fundamental mode and dispersion, but it also contrasts the simplicity and directness with the analysis based on frequency equation.

Another peculiarity of shear waves has been their linear nature to a higher degree of approximation compared to the longitudinal wave. Amongst a number of complications created by nonlinearity, shock formation is one. A number of proofs exist to prove the formation of shocks. One approach is based on the construction of Riemann invariants; but this holds only for a reducible system. A systematized approach, valid for a nonreducible hyperbolic system, is through the use of the theory of singular surfaces. Such studies show that weak shear waves (strain derivatives discontinuous) do not grow to form shocks (strains discontinuous). But shear shocks do exist [1]. Thus, the conventional proof leaves much to be desired. Based on earlier studies from singular surface theory, Sedov and Nariboli [2] produced a linear equation that governs finite amplitude shear waves. In the present study, a different

---

1Numbers in square brackets refer to literature cited in the Bibliography.
expansion is proposed; this is based on shock wave studies; a nonlinear Burgers' equation, of a type novel to the literature, is the result. This illustrates that the nonlinear effects come into operation at a much larger time and at a greater distance from the source.

Guided by studies in water-wave theory, it is plausible to expect to obtain a new type of Korteweg-de-Vries equation for dispersive waves. However, attempts to obtain this have failed either for a single-layered or double-layered medium.

Resonance is an interesting phenomenon; here the solution becomes unbounded when forcing frequency coincides with natural frequency. This is unrealistic on physical grounds. Recent studies showed that shock solutions do exist at resonant frequencies; further, appearance of subharmonics was an interesting phenomenon pointed out in this work.

Collins [3] later made a more exhaustive study under a wider variety of boundary conditions. The general equation he considered consisted of acceleration, as second time derivative, equated to stress gradient. If the first nonlinear term in the expansion of stress in terms of strain is a term of even degree then shock waves are formed; however, if the first nonlinear term is of odd degree he proved that the shock waves are not formed. Of course, the first term in both cases is assumed to be a linear one, and these considerations hold at the resonant frequency of the linear theory. The first case corresponds to longitudinal waves; the second is supposed to correspond to shear waves.

However, in nonlinear theory, there is no pure shear wave; there is always a longitudinal strain accompanying the shear wave; it is only for
a traveling wave that one can assert that the longitudinal strain is proportional to the square of shear strain; it is difficult to see and does not seem to have been proved that the result holds for standing waves too.

Without this last result it is difficult to see that the case considered by Collins describes elastic shear waves. Thus, the consideration of resonant shear oscillations seems to remain open.

Nonlinear oscillations have another interesting property. Frequency of these oscillations depends on the amplitude. This aspect is considered in the fourth chapter.

In the case of standing waves nonlinearity leads to amplitude-dependent frequency. To bring this out, in the fourth chapter we study oscillatory shear waves for a single elastic layer by using Keller and Ting's ideas [4]. The layer is assumed to be finite in extent in one direction and infinite in the other two directions. Thus, specifically, these are shear oscillations in an infinite plate, the shear motion being parallel to the plane faces of the plate.

However, if, as indicated by a detailed study, we assume dispersion is negligible to a high order, it seems reasonable to assume that this describes the torsional oscillations of a finite circular cylinder and transverse oscillations of a plate strip sheared parallel to the parallel edges.
2. SHEAR WAVES IN TWO-LAYERED BOUNDED MEDIA-LINEAR THEORY

2.1 Introduction

Most boundary value problems of three-dimensional linear elasticity are governed by fourth-order differential equations and are less tractable. However, a large number of problems exist as in the case of rods and plates, which extend less in one or two directions in space. Until now, most of the existing literature on elastic waves in such bodies relies on three approaches. They are solved: (1) by equations which are more or less ad hoc; (2) by averaging the basic equations across the cross-section [5]; or (3) by introducing other field variables attached to each point of a continuum, such as directors, and define deformations from these [6]. The present study is based on the perturbation method. It regards theories of lower-dimensional bodies as asymptotic limits of the three-dimensional theory of a medium. Based on this view, this chapter presents a few studies of dispersive shear waves in two-layered media.

It must be noted that the fundamental mode of shear wave in a plate is nondispersive and this is the only one that is significant for large distances. To illustrate this view, consider the case of linear transverse waves in a semi-infinite plate. The nondimensionalized governing equation is

\[ v_{xx} + v_{zz} = v_{tt}, \quad 0 < x < \infty, \quad -h < z < h, \quad 0 < t < \infty. \]  

(2.1.1)

If the initial and boundary conditions for Equation (2.1.1) are
\[ v(x,z,0) = \frac{\partial v(x,z,0)}{\partial t} = 0, \]

\[ \frac{\partial v(x,h,t)}{\partial z} = \frac{\partial v(x,-h,t)}{\partial z} = 0, \quad (2.1.2) \]

\[ v(0,z,t) = \delta(t)g(z), g(-z) = g(z) \]

the solution is the Laplace inverse of

\[ \overline{v} = \frac{1}{2} A_0 e^{-sx} \]

\[ + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi z}{h} \right) \exp \left[ -(s^2 + \frac{n^2}{h^2}) \frac{1}{2} x \right], \quad (2.1.3) \]

with

\[ A_n = \frac{1}{h} \int_{-h}^{h} g(z) \cos \left( \frac{n\pi z}{h} \right) dz, \quad n = 0, 1, 2, \ldots. \]

Similarly for a cylinder, we can take a nondimensional equation

\[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial t^2}, \quad (2.1.4) \]

\[ 0 < r < 1, \quad 0 < z < \infty, \quad 0 < t < \infty. \]

If the initial and boundary conditions for Equation (2.1.4) are

\[ v(r,z,0) = \frac{\partial v}{\partial t} (r,z,0) = 0, \]
\frac{\partial}{\partial r} \left( \frac{\psi}{r} \right) \bigg|_{r=1} = 0,

\psi(r,0,t) = \delta(t)f(r),

the solution is the Laplace inverse of

\bar{\psi} = A_0 e^{-sz} + \sum_{n=1}^{\infty} A_n J_1(k_n r) \exp[-(k_n^2 + s^2)^{1/2} z],

with \( k_n \) determined by

\frac{\partial}{\partial r} \left[ \frac{J_1(kr)}{r} \right] \bigg|_{r=1} = 0.

We see that the dominant far-field solution is given by \( \delta(t-x) \), the inverse of \( \exp(-sx) \), and this indeed is the fundamental mode. The frequency equations for these cases, for waves traveling in \( x \) or \( z \) direction with wave number \( k \), are

\sin k (c^2 - 1)^{1/2} h = 0,

and

\frac{\partial}{\partial r} \left\{ \frac{J_1[(c^2-1)^{1/2} r k]}{r} \right\} \bigg|_{r=1} = 0.

These have roots \( c = 1 \), which is the fundamental mode and is clearly nondispersive, due to its independence of wave number \( k \).
In this chapter shear waves in two-layered media made of two different elastic materials are studied. The linear equations of elasticity are used and solved by the wave front theory. A number of recent papers [7,8] demonstrate the asymptotic nature of the accepted theories of plates and shells. Based on these studies, the resulting equation is obtained as

$$\frac{\partial e}{\partial T} + \frac{1}{2T} e + \beta \frac{\partial^3 e}{\partial x^3} = 0, \quad (2.1.5)$$

where \(e\) is a shear strain, \(T\) is a large time, \(x\) is the coordinate describing the vicinity of the wave front, and \(\beta\) is a constant depending on the properties and dimensions of the two media. Equation (2.1.5) governs the nature of the wave fronts at large distance for a large time. It reveals the dispersive nature of the system and is the analog of Jeffrey's equation in fluid [9] but is more general.

The procedure is quite straightforward. With an appropriate similarity hypothesis and a reasonable expansion for the displacement, a sequence of wave equations can be obtained by taking terms of the same order. The terms of lowest order lead to a fundamental mode of the solution. The terms next to lowest in order give the first approximation of the phase speed. Finally, the terms next in order yield the resulting Equation (2.1.5); this reveals dispersive nature of shear waves in two-layered media. Further, it is terms of this order that give rise to the second approximation of the phase speed, as a side bonus; this term is identical to that obtained from frequency equation, for a long wavelength.
One can, of course, obtain the second term in the expansion of the phase speed from the frequency equation. This approximation is done on the assumption of long waves. Such procedure is carried out in this chapter for two of the three cases considered. It can be seen from the results that this procedure, which is quite common, is sometimes very much involved; in the case of the composite cylinder such analysis is rather lengthy. However, the present method brings out this dispersive term directly and elegantly. Besides giving this dispersive term it automatically leads to the equation that governs this far-field dispersive structure.

The differential equation (2.1.5) is new to the literature; it provides the description of the behavior of the dominant shear strain propagating with a linear wave velocity in two-layered media. To illustrate we must know more about the constant $\beta$ in Equation 2.1.5, which takes the following form:

$$\beta = (c_1^2 - c_2^2)^2 \beta_0,$$

where $\beta_0$, different in each case, depends on geometry, densities and the elasticities. It is clear that the constant $\beta$ vanishes when propagation speeds are equal.

The dependence of the problem on elasticities and densities is not necessarily in the combination as the speed; though differential equations have such a dependence, interfacial conditions depend only on the elasticities. Still, the dispersion depends only on such a combination. Thus, waves in a medium with different elasticities and densities continue
to remain nondispersive if the speeds are the same.

A few solutions of Equation 2.1.5 are discussed. For Jeffrey's equation one has the solution characterized by an Airy function describing the profile of a weak dispersive wave front. The self-similar solution is also obtained for the generalized Jeffrey equation, on the lines of earlier studies.

2.2 Interfacial Boundary Conditions

Shock relations are the best approach to obtain interfacial relations. It must be remembered that, though the usual derivation of these are aimed to obtain discontinuity conditions for the same medium, they continue to hold if the media on the two sides of the shock waves are different. For our purpose it is adequate to consider the first two shock relations implied by continuity equations and the equations of motion. These are [10]

\[ m_o = p(v_n - G) = \rho_o(v_{on} - G), \quad (2.2.1) \]

\[ m_o [v_i] = [t_{ij}] n_j. \quad (2.2.2) \]

\((\rho, \rho_o)\) and \((v_n, v_{on})\) are the values of density and normal velocity of the material particles lying on either side of the separating shock surface, \([F] = F - F_o\); \(m_o\) is clearly the mass flux crossing the shock-surface.

In the present study the materials on either side do not penetrate into each other. Thus \(m_o = 0, v_n = v_{on} = G\) and so \([t_{ij}]n_j = 0\) and force
vector is continuous.

These continue to hold even in the nonlinear problems; in such cases the interface deforms and the direction of the normal depends on the deformation. However, for the linear case, normal to the undeformed surface is used in these formulae. Then it is easy to see that the continuity of the displacement and the force vector across the undeformed interface is all but implied by this formulae. It is the last that is used in our work.

2.3 Plane Shear Waves in a Two-layered Plate

A. Frequency equation

Let x axis be in the mid-plane and z axis be normal to the plane faces, which are given by z = ± h. We designate as medium 1 occupying 0 < z < h and medium 2 -h < z < 0; both extend -∞ < x < ∞ in x direction. Further, the speeds of shear waves in these media are taken as c_1 and c_2, with c_1 < c_2.

In linear theory, the displacement u_i is taken as [0, v(x,z,t), 0]. Suffixes 1, 2 refer to the corresponding media. While those without suffixes hold in general. For the displacement, the stress strain relation and the equation of motion

\[ t_{ij} = \lambda u_k, k \delta_{ij} + u(u_i,j + u_j,i); \quad (2.3.1) \]

\[ t_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (2.3.2) \]

take the following form:
Thus, only the second equation of motion remains to be satisfied, the other two being identically satisfied. In linear theory of elasticity, we take the unit normal vector as

\[ n_i = (0, 0, 1), \]

and hence, the force vector is

\[ t_{ij} n_j = t_{11} n_1 + t_{12} n_2 + t_{13} n_3 = (0, 0, t_{yz}). \]

Since the surface force (stress) is continuous on the interface, from Equation 2.2.2 one has

\[ t_{1ij} n_j = t_{2ij} n_j', \]

\[ t_{1yz} = t_{2yz}, \]

\[ \mu_1 v_{1z} = \mu_2 v_{2z} \text{ on the interface } z = 0, \quad (2.3.5) \]
and on the free boundaries

\[ t_{\text{1yz}} = 0, \text{ or } v_{\text{1z}} = 0 \text{ on } z = h; \quad (2.3.6) \]

\[ t_{\text{2yz}} = 0, \text{ or } v_{\text{2z}} = 0 \text{ on } z = -h. \quad (2.3.7) \]

Lastly, since materials do not penetrate into each other.

\[ u_{\text{1i}} = u_{\text{2i}}, \]

or

\[ v_{\text{1}} = v_{\text{2}} \text{ on } z = 0. \quad (2.3.8) \]

Our interest here is to obtain frequency equation. We seek solution for a horizontal shear wave traveling with a speed \( c \) in the \( x \) direction. So we take

\[ v = V(z) \exp ik(ct-x), \quad (2.3.9) \]

where \( k \) is the wave number and \( c \) is the wave speed. Upon substitution of this displacement into the equation of motion (2.3.4), we have, for medium 1,

\[ \frac{d^2 v_{\text{1}}}{dz^2} + k^2 \rho_{\text{1}}^2 v_{\text{1}} = 0, \quad (2.3.10) \]

with

\[ \rho_{\text{1}}^2 = \frac{c_{\text{1}}^2}{c_{\text{2}}^2} - 1 \text{ and } c_{\text{1}}^2 = \frac{u_{\text{1}}}{\rho_{\text{1}}}; \quad (2.3.11) \]

and hence,
with constants $A_1$ and $B_1$, yet to be determined.

It is reasonable to assume, as is done in the following work, that for $c_1 < c < c_2$, relevant solutions exist. This yields

$$v_1 = [A_1 \exp ikp_1 z + B_1 \exp (-ikp_1 z)] \exp ik(ct-x), \quad (2.3.12)$$

with constants $A_1$ and $B_1$, yet to be determined.

It is reasonable to assume, as is done in the following work, that for $c_1 < c < c_2$, relevant solutions exist. This yields

$$\frac{d^2 v_2}{dz^2} - k^2 p_2^2 v_2 = 0, \quad (2.3.13)$$

with

$$p_2^2 = 1 - \frac{c^2}{c_2^2} \quad \text{and} \quad c_2^2 = \frac{\mu_2}{\rho_2}; \quad (2.3.14)$$

and hence,

$$v_2 = [A_2 \exp kp_2 z + B_2 \exp (-kp_2 z)] \exp ik(ct-x), \quad (2.3.15)$$

with constants $A_2$ and $B_2$, yet to be determined. The application of boundary conditions for $v_1$ and $v_2$ gives

$$A_1 + B_1 - A_2 - B_2 = 0,$$

$$\mu_1 ip_1 A_1 - \mu_1 ip_1 B_1 - \mu_2 p_2^2 A_2 + \mu_2 p_2^2 B_2 = 0,$$

$$A_1 \exp ikp_1 h - B_1 \exp (-ikp_1 h) = 0,$$

$$A_2 \exp (-kp_2 h) - B_2 \exp (kp_2 h) = 0.$$ 

(2.3.16)

This is a system of four homogeneous equations. A necessary and sufficient condition that the system of homogeneous equations (2.3.16) has a
solution, other than the trivial solution, is that the coefficients determinant must be zero. This yields the frequency equation as

\[ \mu_1 p_1 \tan k p_1 h = \mu_2 p_2 \tanh k p_2 h. \quad (2.3.17a) \]

For large wave length, the wave number \( k \to 0 \). Since

\[ \tan \Delta \to \Delta, \text{ as } \Delta \to 0, \]
\[ \tanh \Delta \to \Delta, \text{ as } \Delta \to 0, \]

we have, as the first approximation,

\[ \mu_1 p_1^2 = \mu_2 p_2^2. \quad (2.3.17b) \]

Substitution of (2.3.11) and (2.3.14) into (2.3.17b) gives

\[ c_o^2 = \frac{\mu_1 + \mu_2}{\rho_1 + \rho_2}. \quad (2.3.17c) \]

For the next approximation, we retain one more term in the series expansion of tangent and hyperbolic tangent as

\[ \tan \Delta = \Delta + \frac{\Delta^3}{3} + \ldots, \]
\[ \tanh \Delta = \Delta - \frac{\Delta^3}{3} + \ldots. \]

Substitution of the above expansions into (2.3.17a) yields

\[ \frac{p_1 (1 + \frac{1}{3} k^2 p_1^2 h^2 + \ldots)}{p_2 (1 - \frac{1}{3} k^2 p_2^2 h^2 + \ldots)} = \frac{\mu_2 p_2}{\mu_1 p_1}, \]

or
\[ \mu_1 p_1^2 - \mu_2 p_2^2 = -\frac{k\gamma^2}{3} (\mu_1 p_1^4 + \mu_2 p_2^4). \]  

(2.3.17d)

Left side of (2.3.17d) yields

\[ \mu_1 \left( \frac{c_2^2}{c_1^2} - 1 \right) - \mu_2 \left( 1 - \frac{c_2^2}{c_1^2} \right) = -\mu_1 - \mu_2 + \rho_1 c_1^2 + \rho_2 c_2^2 \]

\[ = c^2 (\rho_1 + \rho_2) - (\mu_1 + \mu_2). \]

This equation is a fourth degree in c and hence inconvenient to solve.

However, it is not necessary to do so. Since it is reasonable to assume that

\[ c^2 = c_o^2 + k^2 c_1^2 + \ldots, \]

we substitute c by c_o for the right side of (2.3.17d). Noting that

\[ \alpha = \frac{\rho_1 (\mu_1 + \mu_2)}{\mu_1 (\rho_1 + \rho_2)} - 1 = \frac{\rho_1 \rho_2 (c_2^2 - c_1^2)}{\mu_1 (\rho_1 + \rho_2)} = \frac{\overline{\alpha}}{\mu_1}, \]

and similarly,

\[ \frac{\alpha^2}{\mu_2}, \]

we have

\[ \mu_1 p_1^4 + \mu_2 p_2^4 = \frac{\alpha^2}{\mu_1 \mu_2} \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) = \frac{\overline{\alpha}^2 (\mu_1 + \mu_2)}{\mu_1 \mu_2} \]

\[ = \frac{\rho_1^2 \rho_2^2 (c_2^2 - c_1^2)}{(\rho_1 + \rho_2)^2} \frac{(\rho_1 + \rho_2) c_o^2}{\mu_1 \mu_2}; \]
and hence (2.3.17d) becomes

\[ c^2 = c_o^2 - \frac{k^2h^2}{3} \frac{\rho_1\rho_2}{(\rho_1 + \rho_2)^2} \frac{(c_2^2 - c_1^2)^2}{c_1^2 c_2^2} \cdot c_o^2. \] (2.3.17e)

Thus, this gives the speed obtained from elementary theory, i.e.,

\[ c = c_o \text{ as } kh \to 0. \]

The physical meaning of \( c_o^2 \) as shown in Equation (2.3.17c) can be interpreted as the speed based on the mean. In fact, the speed for two-layered media with different thicknesses \( h_1 \) and \( h_2 \) can be shown as

\[ c_o^2 = \frac{\mu_1 h_1 + \mu_2 h_2}{\rho_1 h_1 + \rho_2 h_2}, \] (2.3.18)

which again shows the property of the weighted average.

Moreover, it is interesting to note that the next term in this case turns out to be

\[ k^2 \frac{\rho_1^2 \rho_2^2}{3} \frac{h_1^2 h_2^2}{u_1 u_2} \frac{(\mu_2 h_1 + \mu_1 h_2)}{(\rho_1 h_1 + \rho_2 h_2)^3} \frac{(c_2^2 - c_1^2)^2}{c_1^2 c_2^2}. \]

This again vanishes as \( c_1 \to c_2 \). Thus, the dispersion is unaffected not only by elasticities and densities but also by the thicknesses. The sole governing factor turns out to be the inequality of the speeds of propagation; so for convenience in subsequent work we assume the thicknesses of the two media are equal.
B. Asymptotic expansions

Equation 2.3.17e shows that wave speed, \( c \), is a function of wave number, \( k \). This assures that, unlike the shear waves in a homogeneous plate, which is nondispersive, the shear waves in a two-layered plate do have dispersive nature. Based on this result the object of this article and the following ones is to obtain explicitly the dispersive structure of the wave, which governs the nature of the wave front at large distance.

It is basic in any perturbation method to have a small parameter (or a large parameter). In the present and subsequent such studies, we assume that a typical length, \( L_0 \), exists in the \( x \) direction (of the direction of propagation). All lengths of the problems are based on the unit of this \( L_0 \). This length may be interpreted as a typical wave length. However, in a non-oscillatory motion in an unbounded medium, we need caution in the interpretation of the wave length. Without going to the detailed discussion of this viewpoint, we hope, it is adequate in the present work to understand this length to be the one across which significant changes in field variables take place.

We thus assume that the thickness \( h \) is a small parameter when measured in units of \( L_0 \). This gives physical import to the statement implied by "thin plate."

Based on earlier studies [11-13], we take the following form of asymptotic expansion:

\[
v = \varphi(x,z,t) + h^2 \psi(x,z,t) + h^4 \nu(x,z,t) + \ldots, \tag{2.3.19}
\]

with \( h \) as a typical thickness.
To derive the interior equations, the transformation in the $z$ direction must be introduced, namely

$$z = hZ. \quad \text{(2.3.20)}$$

Then, the equation of motion is given by

$$\frac{\mu}{\rho} \left( v_{xx} + \frac{1}{h^2} v_{zz} \right) = v_{tt}, \quad \text{(2.3.21)}$$

where a suffix indicates partial differentiation.

Boundary conditions for stress free surfaces and continuity of stress and displacement at common surfaces are given by

$$t_{1yZ} = 0 \quad \text{on } Z = 1,$$

$$t_{2yZ} = 0 \quad \text{on } Z = -1,$$

$$v_1 = v_2 \quad \text{on } Z = 0,$$

$$t_{1yZ} = t_{2yZ} \quad \text{on } Z = 0,$$

or in terms of the displacement $v$

$$v_{1Z} = 0 \quad \text{on } Z = 1,$$

$$v_{2Z} = 0 \quad \text{on } Z = -1,$$

$$v_1 = v_2 \quad \text{on } Z = 0,$$

$$\mu_1 v_{1Z} = \mu_2 v_{2Z} \quad \text{on } Z = 0.$$
Through the introduction of the transformation quoted in (2.3.20) and expansions given in (2.3.19) the equations of motion can now be written as

$$\frac{\partial^2}{\partial x^2} (\varphi_i + h^2 \varphi_i + \ldots) + \frac{1}{h^2} \frac{\partial^2}{\partial Z^2} (\varphi_i + h^2 \varphi_i + h^4 \varphi_i + \ldots)$$

$$= \frac{1}{c_i^2} \frac{\partial^2}{\partial t^2} (\varphi_i + h^2 \varphi_i + \ldots), \ i = 1,2 \quad (2.3.22)$$

where subscript $i = 1,2$ indicates medium 1, 2 and the dots in the equation denote terms of order $O(h^4)$ that need not be retained in the present study.

All field variables are now assumed to be functions of $(x,Z,t)$, and the sequence of problems of different orders must now be solved. The terms of order $1/h^2$ in (2.3.22) lead to

$$\varphi_1_{ZZ} = 0; \quad \varphi_2_{ZZ} = 0. \quad (2.3.23)$$

Solutions of (2.3.23) are

$$\varphi_1 = ZP_1(x,t) + F_1(x,t); \quad \varphi_2 = ZP_2(x,t) + F_2(x,t).$$

Surface condition gives

$$P_1 = P_2 = 0.$$

Displacement continuity on interface gives

$$F_1 = F_2 = F(x,t).$$
Therefore, 
\[ \varphi_1 = \varphi_2 = F(x,t), \] (2.3.24a)

which satisfies the condition of stress continuity automatically.

The terms of order unity in (2.3.22) result in the following:

\[ c_1^2 \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) = \frac{\partial^2 F}{\partial t^2}; \quad c_2^2 \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) = \frac{\partial^2 F}{\partial t^2}. \] (2.3.25a)

For convenience, let the differential operators \( L_1 \) and \( L_2 \) be defined as

\[ L_1 \equiv \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}, \quad L_2 \equiv \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}. \] (2.3.25b)

Then (2.3.25a) becomes

\[ \frac{\partial^2 \varphi_1}{\partial z^2} = L_1 F; \quad \frac{\partial^2 \varphi_2}{\partial z^2} = L_2 F. \] (2.3.25c)

Solutions of (2.3.25c) are

\[ \varphi_1 = G_1(x,t) + ZQ_1(x,t) + \frac{Z^2}{2} L_1 F; \]
\[ \varphi_2 = G_2(x,t) + ZQ_2(x,t) + \frac{Z^2}{2} L_2 F. \] (2.3.25d)

Stress free boundary conditions on \( Z = \pm 1 \) give

\[ Q_1(x,t) = -L_1 F; \quad Q_2(x,t) = L_2 F. \]
Continuity of displacement on \( Z = 0 \) yields

\[ G_1 = G_2 = G(x,t) ; \]

hence,

\[ \ddot{\psi}_1 = G(x,t) + (-Z + \frac{Z^2}{2}) L_1 F ; \]

\[ \ddot{\psi}_2 = G(x,t) + (Z + \frac{Z^2}{2}) L_2 F . \]

Lastly, continuity of stress on \( Z = 0 \) leads to

\[ -\nu_1 L_1 F = \nu_2 L_2 F , \]

or

\[ -\nu_1 \left( \frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} \right) = \nu_2 \left( \frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} \right) . \]

This clearly simplifies to

\[ \frac{2}{c_0} \frac{d^2 F}{dx^2} = \frac{d^2 F}{dt^2} , \]

This result brings out the elegance of the method of perturbation. Proceeding to higher approximation we obtain the dispersive nature of the waves.

The terms next in order (of \( \hbar^2 \)) in (2.3.22) lead to
\[
\frac{\partial^2 v_1}{\partial x^2} + \frac{2}{\partial z^2} \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{c_1^2} \frac{\partial^2 v_1}{\partial t^2};
\]

or

\[
\frac{\partial^2 v_2}{\partial x^2} + \frac{2}{\partial z^2} \frac{\partial^2 v_2}{\partial z^2} = \frac{1}{c_2^2} \frac{\partial^2 v_2}{\partial t^2},
\]

Solutions of (2.3.26b) are

\[
\begin{align*}
\frac{4}{v_1} &= \frac{L_1}{v_1} = L_1G + (-Z + \frac{Z^2}{2}) L_1^2F; \\
\frac{4}{v_2} &= \frac{L_2}{v_2} = L_2G + (Z + \frac{Z^2}{2}) L_2^2F.
\end{align*}
\]

Displacement continuity on \(Z = 0\) and stress free conditions on \(Z = \pm 1\) give

\[
\begin{align*}
H_1 &= H_2 = H(x,t), \\
R_1 + L_1G - \frac{1}{3} L_1^2F &= 0, \\
R_2 - L_2G + \frac{1}{3} L_2^2F &= 0,
\end{align*}
\]

so that
Lastly, continuity of stress on $Z = 0$ gives

$$\frac{\partial}{\partial z} \left( y^L + \frac{Z^2}{2} \right) L_1 G + \left( \frac{z}{3} - \frac{Z^3}{6} + \frac{Z^4}{24} \right) L_1^2 F ;$$  \hspace{1cm} (2.3.26d)

$$\frac{\partial}{\partial z} \left( y^L + \frac{Z^2}{2} \right) L_2 G + \left( - \frac{z}{3} + \frac{Z^3}{6} - \frac{Z^4}{24} \right) L_2^2 F .$$

The result (2.3.25g) gives the frequency equation obtained in (2.3.17a); while (2.3.26e) takes account of dispersive effect, which is equivalent to the correction of lateral inertia for a longitudinal wave.

It is to be noted that the approximation described by (2.3.26e) is invalid for large times (and distances). Indeed if we take $F$ as proportional to $\sin k(x - c_0 t)$, the solution for $G$ will be of the form $t \cos k(x - c_0 t)$. Then this perturbation scheme breaks down for large times (and distances). So in the next section, using a stretched time describing the vicinity of the wave front, a uniform approximation is obtained.

C. A uniformly valid perturbation solution

Since interest is in the far-field solution, a change of variables is made as follows:

$$X = x - c_0 t \quad \text{and} \quad T = \alpha(h) t,$$

where $c_0$ is the linear wave speed, so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X} ; \quad \frac{\partial}{\partial t} = -c_0 \frac{\partial}{\partial X} + \alpha \frac{\partial}{\partial T} .$$  \hspace{1cm} (2.3.27b)
The above transformations introduce a new parameter, $\alpha$, which will be determined later.

With the aforecited transformations, all field variables are now assumed to be functions of $(X,Z,T)$. Selecting an appropriate similarity hypothesis that sets $\alpha = h^2$ yields a sequence of boundary value problems that must be solved.

The equation of motion takes the following form:

$$
\frac{\partial^2}{\partial X^2} (\Phi_i + h^2 \frac{\partial \Phi_i}{\partial X} + \ldots) + \frac{1}{h^2} \frac{\partial^2}{\partial Z^2} (\Phi_i + h^2 \frac{\partial \Phi_i}{\partial Z} + h^4 \frac{\partial^4 \Phi_i}{\partial Z^4} + \ldots) = \left( c_s^2 - 2\alpha \frac{\partial^2}{\partial X \partial T} \right) (\Phi_i + h^2 \frac{\partial \Phi_i}{\partial X} + \ldots), \quad i = 1, 2 \quad (2.3.28)
$$

where subscript $i = 1, 2$ indicates medium 1, 2 and again the dots in the equation note terms of order $O(h^4)$ that need not be retained.

The terms of order $1/h^2$ in (2.3.28) lead to

$$
\Phi_{1ZZ} = 0; \quad \Phi_{2ZZ} = 0. \quad (2.3.29a)
$$

Solutions of (2.3.29a) are

$$
\Phi_1 = \Phi_2 = F(X,T), \quad (2.3.29b)
$$

which satisfies all the boundary conditions.

The terms of order unity in (2.3.28) result in the following:
\[ \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2}{\partial z^2} = \frac{c^2}{c_1} \frac{\partial^2 F}{\partial x^2} ; \quad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2}{\partial z^2} = \frac{c^2}{c_2} \frac{\partial^2 F}{\partial x^2}, \]

or

\[ \frac{\partial^2 v_{1Z}}{\partial z^2} = \left( \frac{c_2^2}{c_1^2} - 1 \right) F'' = \gamma_1 F'' ; \tag{2.3.30a} \]

\[ \frac{\partial^2 v_{2Z}}{\partial z^2} = \left( \frac{c_2^2}{c_2^2} - 1 \right) F'' = \gamma_2 F'' , \]

where \((^{(i)} = \frac{\partial}{\partial x} ( ) \).  

Solutions of (2.3.30a) are

\[ \frac{\partial^2 v_1}{\partial z^2} = G_1 (x, T) + ZQ_1 (x, T) + \frac{z^2}{2} \gamma_1 F'' ; \]

\[ \frac{\partial^2 v_2}{\partial z^2} = G_2 (x, T) + ZQ_2 (x, T) + \frac{z^2}{2} \gamma_2 F'' . \]

Stress free boundary conditions on \( Z = \pm 1 \) give

\[ Q_1 (x, T) = -\gamma_1 F'' ; \]

\[ Q_2 (x, T) = \gamma_2 F'' . \]

Continuity of displacement on \( Z = 0 \) yields

\[ G_1 = G_2 = G(x, T) ; \]
thus

\[ V_1 = G(X,T) + (-Z + \frac{Z^2}{2}) \gamma_1 F''; \]

\[ V_2 = G(X,T) + (Z + \frac{Z^2}{2}) \gamma_2 F''. \]

Lastly, continuity of stress on \( Z = 0 \) leads to

\[-\mu_1 \gamma_1 F'' = \mu_2 \gamma_2 F''; \]

hence,

\[-\mu_1 \gamma_1 = \mu_2 \gamma_2, \text{ (2.3.30b)} \]

or

\[ \frac{\mu_1 \rho}{2} = \frac{\mu_2 \rho}{2}, \text{ (2.3.30c)} \]

assuming \( F'' \neq 0 \).

Again note that (2.3.30c), (2.3.25g) and (2.3.17a) are all identical.

The terms next in order (of \( h^2 \)) in (2.3.28) result in the following:

\[ \frac{2}{2} \gamma_1 v_{1XX} + \gamma_1 v_{1ZZ} = \frac{1}{2} \left( \frac{c^2}{c_1^2} \gamma_1 v_{1XX} - 2\alpha_{o o} \gamma_{o o} v_{o o} \right). \]

or

\[ \gamma_1 v_{1ZZ} = \left( \frac{c^2}{c_1^2} - 1 \right) \left[ \eta'' + \left( -Z + \frac{Z^2}{2} \right) \gamma_1 F'IV \right] - \frac{2\alpha_{o o} \gamma_{o o}}{c_1^2} \hat{F}' \]

or

\[ \gamma_1 v_{1ZZ} = - \frac{2\alpha_{o o} \gamma_{o o}}{c_1^2} F' + \gamma_1 \gamma' + \left( -Z + \frac{Z^2}{2} \right) \gamma_1 F'IV. \text{ (2.3.31a)} \]

Similarly,
\[ v_{2ZZ} = - \frac{2\alpha_0 c_0}{c_1^2} \dot{F} + \gamma_2 G'' + \left( Z + \frac{Z^2}{2} \right) \gamma_2 F_{IV}, \]  
(2.3.31b)

where \( \gamma_1 = \frac{c_2^2}{c_1^2} - 1 \); \( \gamma_2 = -1 + \frac{c_2^2}{c_1^2} \).

Solutions of (2.3.31) are

\[ v_1 = H_1(X,T) + Z R_1(X,T) + \frac{Z^2}{2} \left( - \frac{2\alpha_0 c_0}{c_1^2} \dot{F} + \gamma_1 G'' \right) \]

\[ + \left( - \frac{Z^3}{6} + \frac{Z^4}{24} \right) \gamma_1^2 F_{IV} ; \]

\[ v_2 = H_2(X,T) + Z R_2(X,T) + \frac{Z^2}{2} \left( - \frac{2\alpha_0 c_0}{c_2^2} \dot{F} + \gamma_2 G'' \right) \]

\[ + \left( - \frac{Z^3}{6} + \frac{Z^4}{24} \right) \gamma_2^2 F_{IV} . \]

Displacement continuity on \( Z = 0 \) and stress free conditions on \( Z = \pm 1 \) give

\[ H_1 = H_2 = H(X,T), \]

\[ R_1 - \frac{2\alpha_0 c_0}{c_1^2} \dot{F} + \gamma_1 G'' - \frac{1}{3} \gamma_1^2 F_{IV} = 0, \]

\[ R_2 + \frac{2\alpha_0 c_0}{c_2^2} \dot{F} - \gamma_2 G'' + \frac{1}{3} \gamma_2^2 F_{IV} = 0; \]
hence,
\[
\begin{align*}
\dot{v}_1 &= H(X,T) + (Z - \frac{Z^2}{2}) (- \frac{2\alpha c_o}{c_1^2} \ddot{\hat{F}} + \gamma_1 \dddot{G}) \\
&\quad + \left(\frac{Z}{3} - \frac{Z^3}{6} + \frac{Z^4}{24}\right) \gamma_1 F^{IV} ; \\
\dot{v}_2 &= H(X,T) + (Z + \frac{Z^2}{2}) \left(- \frac{2\alpha c_o}{c_2^2} \ddot{\hat{F}} + \gamma_2 \dddot{G}\right) \\
&\quad + \left(-\frac{Z}{3} - \frac{Z^3}{6} + \frac{Z^4}{24}\right) \gamma_2 F^{IV} .
\end{align*}
\]

Lastly, continuity of stress on $Z = 0$ yields
\[
\begin{align*}
- \nu_1 (\gamma_1 \dddot{G}'') - \frac{2\alpha c_o}{c_1^2} \ddot{\hat{F}} - \frac{\gamma_1}{3} F^{IV} \\
&= \nu_2 (\gamma_2 \dddot{F}'') - \frac{2\alpha c_o}{c_2^2} \ddot{\hat{F}} - \frac{\gamma_2}{3} F^{IV} ,
\end{align*}
\]

or
\[
2\alpha c_o \left(\frac{\nu_1}{c_1^2} + \frac{\nu_2}{c_2^2}\right) \ddot{\hat{F}} = -\frac{1}{3} (\nu_1 \gamma_1^2 + \nu_2 \gamma_2^2) F^{IV}
\]

\[+ (\nu_1 \gamma_1 + \nu_2 \gamma_2) \dddot{G}'' .
\]

Coefficient of $G''$ vanishes because of (2.3.30b). Thus
\[
2\alpha c_o \left(\rho_1 + \rho_2\right) \ddot{\hat{F}} = -\frac{1}{3} \frac{\rho_1 \rho_2}{\rho_1 \rho_2} \left(\frac{c_2^2 - c_1^2}{c_1^2} \frac{c_2^2}{c_2^2} \right) c_0 F^{IV} .
\]
Without loss of generality, we let $\alpha_0 = 1$. Denoting shear strain $e = \frac{\partial F}{\partial x}$ yields Jeffrey's wave equation [9].

\[
\frac{\partial e}{\partial t} + \beta_1 \frac{\partial^3 e}{\partial x^3} = 0, \quad (2.3.31c)
\]

where

\[
\beta_1 = \frac{1}{6} \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} \frac{(c_2^2 - c_1^2)^2}{c_1 c_2^2} c_0. \quad (2.3.31d)
\]

In terms of displacement, (2.3.31c) takes the form

\[
\frac{\partial^2 F}{\partial x \partial t} + \beta_1 \frac{\partial^4 F}{\partial x^4} = 0. \quad (2.3.31e)
\]

Equation (2.3.31e) is the linear equation governing the propagation of shear strain in a plate made of two different media each of thickness $h$.

The description is of far-field and is written in reference to a moving coordinate system located at the wave front.

It is to be noted that the solution (2.3.31c) does not have a steady solution.

2.4. Cylindrical Shear Waves in a Two-layered Plate

Here the uniformly valid asymptotic solution of cylindrical shear waves is studied. The derivation of the governing equation is quite
analogous to that of Art. 2.3.C. The important difference consists in the extension of Jeffrey's equation to more than one dimension. The wave front here is a cylindrical one; thus the coordinate system introduced later to get a uniform approximation for large times does not form a Galilean frame.

Another rather interesting feature of this problem must be noted. In order to write the frequency equation, we need functions which describe outgoing cylindrical wave solutions and are appropriate to the present differential operator. Hankel functions are possibly the best candidates for this purpose. However, the frequency equation will be quite involved as can be seen from the work for the torsional wave in the cylinder (see Art. 2.5.). The asymptotic method disposes of the need of all such work and directly leads one to the needed physical results. Thus, this brings out the additional merit of this approach.

Because the propagation has cylindrical symmetry, it is reasonable to study the problem in the cylindrical polar coordinate system. The mid-plane of the plate is given by $z = 0$ and the undeformed plane faces by $z = \pm h$. For the type of waves considered here (shear waves) the problem has symmetry about the mid-plane. Let $u_{ij}(0, v(r, z, t), 0)$ be the displacement vector.

We can, as in the previous cases, proceed to obtain here succeeding approximations. The highest order equation will again be governed by

$$c_o^2 L F(r, t) - F_{tt} = 0 , \quad (2.4.1)$$

where

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} .$$
The next approximation will be governed by

\[ c_o^2 L(G(r,t)) - \frac{\partial^2 G}{\partial t^2} = MF, \quad (2.4.2) \]

where M is some other differential operator easily written down. This is the nonuniform approximation which has to be modified as before. Since this work involves no novel features we do not present it here; but instead we proceed to obtain only the uniformly valid approximation to bring out only the distinctive feature.

The perturbation is performed in a reference frame, traveling with the velocity of the linear wave. Because the study attempts to look at the wave in a far-field sense, a further time-stretch transformation must then be introduced. The following transformations accomplish both goals.

\[ R = r - c_o t \text{ and } T = \tau(h) t, \quad (2.4.3a) \]

where \( c_o \) is the linear wave speed, so that

\[ \frac{\partial}{\partial r} = \frac{\partial}{\partial R}; \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial c_o R} + \alpha \frac{\partial}{\partial T}. \quad (2.4.3b) \]

As in the previous article a small parameter \( \alpha \) and the geometric parameter \( h^2 \) completely define the physical problem.

An important consequence of the transformation (2.4.3a) is

\[ \frac{1}{r} = \frac{\alpha}{c_o T} + O(\alpha^2), \quad \frac{1}{r^2} = O(\alpha^2). \quad (2.4.3c) \]

With transformations (2.4.3a) and \( z = h^2 \), all field variables are now
assumed to be functions of \((R, Z, T)\). Selecting an appropriate similarity hypothesis that sets \(\alpha = \alpha_0 h^2 \) yields a sequence of boundary value problems to be solved.

The choice of this \(\alpha\) here, as in other cases, is dictated by the requirement that the terms in the equation with order less than unity must balance. Another interesting feature in the cylindrical case is worth noting. The powers of \(r^n\) for \(n <-1\) do not affect the problem at all to this approximation. This can be seen from (2.4.3c), i.e.,

\[
\frac{\delta^2}{\delta r^2} + \frac{1}{r} \frac{\delta}{\delta r} - \frac{1}{r^2} = \frac{\delta^2}{\delta R^2} + \frac{\alpha_0 h^2}{c_0 T} \frac{\delta}{\delta R} + O(h^4) .
\]

Hence, the equations of motion take the following form:

\[
\frac{\delta^2}{h^2 z^2} \left( \frac{\sigma}{m} + h^2 \frac{\sigma}{m} + h^4 \frac{\sigma}{m} + ... \right) + \left( \frac{\delta^2}{\delta R^2} + \frac{\alpha_0 h^2}{c_0 T} \frac{\delta}{\delta R} + ... \right) \left( \frac{\sigma}{m} + h^2 \frac{\sigma}{m} + ... \right),
\]

\[
= \frac{1}{c_i^2} \left( c_o^2 \frac{\delta^2}{\delta R^2} - 2 c_o \alpha_0 h^2 \frac{\delta^2}{\delta R \delta T} \right) \left( \frac{\sigma}{m} + h^2 \frac{\sigma}{m} + ... \right), \tag{2.4.4}
\]

where \(m = 1, 2\) refers to medium 1,2.

The terms of order \(1/h^2\) in (2.4.4) lead to

\[
\frac{\sigma}{1z z} = 0; \quad \frac{\sigma}{2z z} = 0 . \tag{2.4.5a}
\]

Solutions of (2.4.5a) are

\[
\frac{\sigma}{1} = \frac{\sigma}{2} = f (R, T), \tag{2.4.5b}
\]
which satisfy all the boundary conditions.

The terms of order unity in (2.4.4) yield

\[
\frac{\partial^2 v_1}{\partial z^2} = \frac{c_1^2}{c_1^2 - 1} \frac{\partial^2 \phi}{\partial r^2} = \gamma_1 F'' ;
\]

\[
\frac{\partial^2 v_2}{\partial z^2} = \frac{c_2^2}{c_2^2 - 1} \frac{\partial^2 \phi}{\partial r^2} = \gamma_2 F'' ,
\]

where \( (') = \frac{\partial}{\partial r} \). The solutions of (2.4.6a) are

\[
2 v_1 = G_1(R,T) + ZQ_1(R,T) + \frac{Z^2}{2} \gamma_1 F'' ;
\]

\[
2 v_2 = G_2(R,T) + ZQ_2(R,T) + \frac{Z^2}{2} \gamma_2 F'' .
\]

Conditions of displacement on \( Z = 0 \) and free stresses on \( Z = \pm 1 \) give

\[
G_1 = G_2 = G(R,T) ,
\]

\[
Q_1(R,T) = -\gamma_1 F'' ,
\]

\[
Q_2(R,T) = \gamma_2 F'' .
\]

Hence,

\[
2 v_1 = G(R,T) + (Z + \frac{Z^2}{2}) \gamma_1 F'' ;
\]

\[
2 v_2 = G(R,T) + (Z + \frac{Z^2}{2}) \gamma_2 F'' .
\]
Lastly, from continuity of stress on \(Z = 0\) we find \(-\mu_1 \gamma_1 F'' = \mu_2 \gamma_2 F''\),

thus

\[
-\mu_1 \gamma_1 = \mu_2 \gamma_2,
\]

or

\[
\mu_1 p_1^2 = \mu_2 p_2^2,
\]

assuming \(F'' \neq 0\).

The terms next in order \([\text{of } O(h^2)]\) in (2.4.4) result in the following:

\[
\frac{\partial^2 v_1}{\partial R^2} + \frac{\alpha_o}{c_o T} \frac{\partial\phi_1}{\partial R} + \frac{\alpha^2 v_1}{\partial Z^2}
\]

\[
= \frac{c^2}{c_1^2} \frac{\partial^2 v_1}{\partial R^2} - 2 \frac{\alpha_o c_o}{c_1} \frac{\partial v_1}{\partial R} T,
\]

or upon substitution of (2.4.6c) into (2.4.7a), we may write

\[
4 v_{1ZZ} = \gamma_1 G(R,T) - \frac{\alpha_o}{c_o T} F' - 2 \frac{\alpha_o c_o}{c_1^2} \phi' + \gamma_1 \frac{Z^2}{2} - Z F' \; (2.4.7b)
\]

\[
4 v_{2ZZ} = \gamma_2 G(R,T) - \frac{\alpha_o}{c_o T} F' - 2 \frac{\alpha_o c_o}{c_2^2} \phi' + \gamma_2 \frac{(\frac{Z^2}{2} + Z)}{2} F' \; (2.4.7b)
\]

where \((\cdot)' = \frac{\partial}{\partial T} (\cdot)\).

The solutions of (2.4.7b) are
\[ v_1 = H_1 (R,T) + Z R_1 (R,T) + \frac{Z^2}{2} \left( \gamma_1 G - \frac{2\alpha_o c_o}{c_1^2} \ddot{F}' - \frac{\alpha_o}{c_o T} F' \right) \]
\[ + \gamma_1^2 \left( \frac{Z^4}{24} - \frac{Z^3}{6} \right) F_{IV}; \]
\[ v_2 = H_2 (R,T) + Z R_2 (R,T) + \frac{Z^2}{2} \left( \gamma_1 G - \frac{2\alpha_o c_o}{c_2^2} \ddot{F}' - \frac{\alpha_o}{c_o T} F' \right) \]
\[ + \gamma_2^2 \left( \frac{Z^4}{24} + \frac{Z^3}{6} \right) F_{IV}. \]

Conditions of displacement on \( Z = 0 \) and stress free conditions on \( Z \neq 0 \) give

\[ H_1 = H_2 = H(R,T), \]
\[ R_1 + \gamma_1 G - 2 \frac{\alpha_o c_o}{c_1^2} \ddot{F}' - \frac{\alpha_o}{c_o T} F' - \frac{\gamma_1^2}{3} F_{IV} = 0, \]
\[ R_2 + \gamma_2 G - 2 \frac{\alpha_o c_o}{c_2^2} \ddot{F}' - \frac{\alpha_o}{c_o T} F' + \frac{\gamma_2^2}{3} F_{IV} = 0, \]

so that

\[ v_1 = H (R,T) + \left( \frac{Z^2}{2} - Z \right) \left( \gamma_1 G - \frac{2\alpha_o}{c_1^2} \ddot{F}' - \frac{\alpha_o}{c_o T} F' \right) + \left( \frac{Z}{24} - \frac{Z^3}{6} + \frac{Z}{3} \right) \gamma_1^2 F_{IV}; \]
\[ v_2 = H(R,T) + \left( \frac{Z^2}{2} + Z \right) \left( \gamma_2 G - \frac{2\alpha_o}{c_2^2} \ddot{F}' - \frac{\alpha_o}{c_o T} F' \right) + \left( \frac{Z^4}{24} - \frac{Z^3}{6} - \frac{Z}{3} \right) \gamma_2^2 F_{IV}. \]

Lastly, stress continuity on \( Z = 0 \) leads to
\[
2\alpha_0 c_0 (\rho_1 + \rho_2) \ddot{F}^i + \frac{\alpha_0}{c_0^T} (\mu_1 + \mu_2) F^i + \frac{1}{3} (\mu_1 \gamma_1^2 + \mu_2 \gamma_2^2) F^{IV} = (\mu_1 \gamma_1 + \mu_2 \gamma_2) G. \tag{2.4.7c}
\]

The coefficient of \( G \) in (2.4.7c) vanishes on account of (2.4.6d). And on account of (2.4.6e)

\[
(\mu_1 + \mu_2)/(\rho_1 + \rho_2) = c_0^2.
\]

Thus, by denoting shear strain \( e = \frac{\partial F}{\partial R} \), we have

\[
\alpha_0 \left( \frac{\partial e}{\partial T} + \frac{1}{2T} e \right) + \beta_1 \frac{\partial^3 e}{\partial R^3} = 0, \tag{2.4.8a}
\]

where

\[
\beta_1 = \frac{1}{6} \frac{\mu_1 \rho_1^4 + \mu_2 \rho_2^4}{\rho_1 + \rho_2} = \frac{1}{6} \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)^2} \frac{(c_2^2 - c_1^2)^2}{c_1^2 c_2^2} c_0,
\]

which is the same as that for planar waves. By letting \( \alpha_0 = 1 \), without loss of generality, (2.4.8a) results in the following:

\[
\frac{\partial e}{\partial T} + \frac{1}{2T} e + \beta_1 \frac{\partial^3 e}{\partial R^3} = 0, \tag{2.4.8b}
\]

or

\[
\frac{\partial F}{\partial R \partial T} + \frac{\partial F}{\partial R} + \beta_1 \frac{\partial^4 F}{\partial R^4} = 0. \tag{2.4.8c}
\]
The final result given by Equation (2.4.8c) describes cylindrical Jeffrey waves. It appears to be novel to the literature and first noted by Nariboli and Tsai [12]. In their work it describes longitudinal cylindrical waves in a plate. It is worthwhile stressing again that $\alpha$ multiplies the first two terms of (2.4.8c); they are thus both retained or they both drop out. The first, of course, describes the unsteadiness while the second gives the effect of curvature. Again the solution of Airy integral of this equation exists. They are given by [12]

$$u = \frac{(2/kh^2 t)^{n/2 + 1/3}}{(3kh^2 t/2)^{1/3}} \text{Ai}(\xi) \times \text{constant},$$

where

$$\xi = \frac{(r - t)}{(3kh^2 t/2)^{1/3}}.$$

The power of the present approach becomes more obvious here if we try to solve it by standard methods. Proof of such a result by the use of stationary phase method is bound to be quite involved.

2.5. Torsional Waves in a Two-layered Circular Cylinder

A. Basic equations

Here we consider an infinite solid circular cylinder made of two elastic materials. The region 1 bounded by a cylinder $0 < r < a$, where $r$ is the radial distance in a cylindrical coordinate system, $r$, $\theta$, $z$, and is made of a material with shear modulus $\mu_1$ and density $\rho_1$. The annular region $2$, $a < r < b$ is made of another elastic material with shear modulus $\mu_2$ and density $\rho_2$. The cylindrical surface of the outer one is
assumed to be stress free. Study is again limited to that of torsional wave motion.

A frequency equation is first sought. The long wave approximation gives the speed and then the dispersive term successively. As will be seen the frequency equation will have a logarithmic term. Fortunately, to the approximation we have continued in the present work, this term drops out. However, we have not been able to prove that it will not appear at all at any approximation. Indeed, if it does appear, the perturbation expansion will have to be modified from that stage on. In such a case, there is a very strong analogy with the low Reynolds number flow. However, in this study, we do not proceed to study this extremely interesting feature.

From studies, well established by now, it has proven convenient to subdivide waves in rods into three types: the longitudinal waves, the flexural waves and the torsional waves. Such a study, originated by Pochhammer and Chree, now forms part of any book of elastic propagation [14,15]. As elaborated in the Introduction (Art. 2.1.), torsional waves in a circular cylinder are the only one of this group, which have a non-dispersive mode. From experience, the asymptotic method always leads only to this fundamental mode. However, this is no longer true for the two layer case. The torsional waves in this case do turn out to be dispersive. Again in this case, too, though the frequency equation is analyzed, because of its intrinsic interest in the development of the perturbation method, successive approximations are not written down. Only the uniformly valid asymptotic solution is presented.
The problem is clearly governed by the following system:

\[
\rho_m \frac{\partial^2 v_m}{\partial t^2} = \mu_m \left( \frac{\partial^2 v_m}{\partial r^2} + \frac{1}{r} \frac{\partial v_m}{\partial r} - \frac{v_m}{r^2} + \frac{\partial^2 v_m}{\partial z^2} \right) \quad (2.5.1)
\]

where \( m = 1, 2 \) refers to medium 1, 2 and

\( v_1(o,z,t) \) finite,

\[ v_1(a,z,t) = v_2(a,z,t), \quad (2.5.2) \]

\[ \frac{\partial}{\partial r} \left( \frac{v_2}{r} \right) = 0, \quad (2.5.3) \]

\[ \frac{\partial}{\partial r} \left( \frac{v_1}{r} \right) = 0, \quad (2.5.4) \]

B. Frequency equation

To obtain frequency equation, we seek a solution for a wave traveling with a velocity \( c \) in \( z \) direction. So we take

\[ v_1 = V_1(r) \exp ik(ct - z). \quad (2.5.6a) \]

Substitution of (2.5.6a) into (2.5.1) yields

\[ V_1'' + \frac{1}{r} V_1' + \left( k^2 p^2 - \frac{1}{r^2} \right) V_1 = 0, \quad (2.5.6b) \]

with \( (') = \frac{\partial}{\partial r} (\cdot) \) and \( p^2 = \frac{c_2^2}{c_1^2} - 1 \), where it is again assumed
Equation (2.5.6b) is of the form of Bessel equation of order \( n \)
\[
\frac{1}{x} x'' - \frac{1}{x} x' + \left( \alpha^2 - \frac{n^2}{x^2} \right) x = 0,
\]
which has a general solution
\[
x = AJ_n (\alpha x) + BY_n (\alpha x),
\]
where \( J_n (\alpha x) \) and \( Y_n (\alpha x) \) are Bessel functions of the first and second kind of order \( n \). The latter is one which is singular at \( x = 0 \).

Therefore, solution of (2.5.6b) yields \( V_1 = A_1 J_1 (kpr) \) on account of the finiteness of displacement at the origin. This yields
\[
V_1 = A_1 J_1 (kpr) \exp ik(ct - z). \tag{2.5.6c}
\]

For medium 2, the displacement takes the form
\[
V_2 = V_2(r) \exp ik(ct - z). \tag{2.5.7a}
\]

By applying (2.5.7a) to (2.5.1), we have
\[
V_2'' + \frac{1}{r} V_2' - \left( \frac{k^2 q^2}{r^2} + \frac{1}{r^2} \right) V_2 = 0, \tag{2.5.7b}
\]

with
\[
q^2 = 1 - \frac{c_1^2}{c_2^2}.
\]

The solution of (2.5.7b) gives
\[
V_2 = A_2 I_1 (kqr) + B_2 K_1 (kqr),
\]
with $I_1$ and $K_1$ as modified Bessel functions of order 1 of the first and second kind.

As is done in Art. 2.3.A, the next step is to get the frequency equation by setting the coefficient determinant of the three homogeneous equations, obtained by applying three boundary conditions, in three unknowns $A_1$, $A_2$ and $B_2$ equal to zero. Instead of doing that we proceed as follows.

Let

$$V_2 = A \frac{I_1(kqr)}{I_1(kqa)} + B \frac{K_1(kqr)}{K_1(kqa)}; \quad (2.5.8a)$$

then by the condition of displacement continuity at $r = a$, (2.5.2) yields

$$V_1 = (A + B) \frac{J_1(kpr)}{J_1(kpa)}. \quad (2.5.8b)$$

By stress free condition, (2.5.3) gives

$$A \left\{ \frac{kqI_0(kqb)}{I_1(kqa)} - \frac{2}{b} \frac{l_1(kqb)}{l_1(kqa)} \right\} - B \left\{ \frac{kqK_0(kqb)}{K_1(kqa)} + \frac{2}{b} \frac{K_1(kqb)}{K_1(kqa)} \right\} = 0. \quad (2.5.9a)$$

Lastly, the stress continuity at $r = a$ (2.5.4) yields

$$A \left\{ \mu_1 \left( \frac{kpJ_0(kpa)}{J_1(kpa)} - \frac{2}{a} \right) - \mu_2 \left( \frac{kqI_1(kqa)}{I_1(kqa)} \right) \frac{2}{a} \right\} + B \left\{ \mu_1 \left( \frac{kpJ_0(kpa)}{J_1(kpa)} - \frac{2}{a} \right) + \mu_2 \left( \frac{kqK_0(kqa)}{K_1(kqa)} + \frac{2}{a} \right) \right\} = 0. \quad (2.5.9b)$$
In preceding work the following formulae are used [16]:

\[
\frac{d}{dr} \left[ J_1(kpr) \right] = kp \left[ J_0(kpr) - \frac{1}{kpr} J_1(kpr) \right],
\]

\[
\frac{d}{dr} \left[ I_1(kqr) \right] = kq \left[ I_0(kqr) - \frac{1}{kqr} I_1(kqr) \right],
\]

\[
\frac{d}{dr} \left[ K_1(kqr) \right] = -kq \left[ K_0(kqr) + \frac{1}{kqr} K_1(kqr) \right],
\]

so that

\[
\frac{d}{dr} \left\{ \frac{J_1(kpr)}{r} \right\} = - \frac{1}{r^2} J_1(kpr) + \frac{1}{r} \frac{dJ_1(kpr)}{dr} = \frac{kp}{r} J_0(kpr) - \frac{2}{r^2} J_1(kpr),
\]

\[
\frac{d}{dr} \left\{ \frac{I_1(kqr)}{r} \right\} = \frac{kq}{r} I_0(kqr) - \frac{2}{r^2} I_1(kqr),
\]

\[
\frac{d}{dr} \left\{ \frac{K_1(kqr)}{r} \right\} = - \frac{kq}{r} K_0(kqr) - \frac{2}{r^2} K_1(kqr).
\]

A necessary and sufficient condition that (2.5.9a) and (2.5.9b) have a solution other than a trivial solution is the coefficient determinant equals zero. It is convenient to introduce the following
notations. Let \( k_p = x_1 \); \( k_q = x_2 \) and \( k_q = \gamma_2 \), and let

\[
A_1 = \frac{x_1 J_0(x_1)}{J_1(x_1)} - 2, \quad (2.5.10a)
\]

\[
A_2 = \frac{x_2 J_0(x_2)}{J_1(x_2)} - 2, \quad (2.5.10b)
\]

\[
A_3 = \frac{\gamma_2 J_0(\gamma_2)}{J_1(\gamma_2)} - 21_1(\gamma_2), \quad (2.5.10c)
\]

\[
B_1(\gamma_2) = \gamma_2 K_0(\gamma_2) + 2K_1(\gamma_2), \quad (2.5.10d)
\]

\[
B_1(x_2) = x_2 K_0(x_2) + 2K_1(x_2), \quad (2.5.10e)
\]

hence, (2.5.9a) and (2.5.9b) become

\[
\{ A_3 K_1(x_2) \} A - \{ B_1(\gamma_2) \} B = 0, \quad (2.5.11a)
\]

\[
K_1(x_2) \{ \mu_1 A_1 - \mu_2 A_2 \} A + \{ \mu_1 A_1 K_1(x_2) + \mu_2 B_1(x_2) \} B = 0 . \quad (2.5.11b)
\]

The vanishing of the coefficient determinant of (2.5.11a) and (2.5.11b) yields

\[
\mu_1 A_1 \{ B_1(\gamma_2) + A_3 K_1(x_2) \} = \mu_2 \{ A_2 B_1(\gamma_2) - A_3 B_1(x_2) \}. \quad (2.5.12)
\]

The long wavelength approximation now leads to (see Appendix 8.1)
\[ u_1 x_1^2 + u_2 \left( x_2^2 - \frac{y_2^2}{x_2^2} \right) = \mu_1 \left\{ \frac{2y_1^2}{x_2^4} + \frac{x_1^4}{24} - \frac{x_1^2 y_2^4}{8x_2^2} \right\} \]

\[ + \mu_2 \left\{ \frac{x_2^4}{24} + \frac{x_2 y_2^2}{4} - \frac{3 y_2^4}{8} + \frac{y_2^2 x_2^2}{12} \right\}, \quad (2.5.13) \]

with \( x_1 = k_p a, \ x_2 = k_q a, \ y_2 = k_q b, \ p^2 = \frac{c^2}{c_1^2} - 1 \) and \( q^2 = 1 - \frac{c^2}{c_2^2} \).

Further, for later convenience let \( \gamma_1 = p \) and \( \gamma_2 = q \). Then, to terms of order \( k^2 \), (2.5.13) leads to

\[ u_1 \gamma_1^2 a^2 + u_2 \left( \gamma_2^2 a^2 - \gamma_2^2 b^2 \right) = \mu_1 \left( \gamma_1 y_2^2 a b^2 - \frac{y_1 y_2^4 a^2 b^2}{4} + \frac{y_1^2 y_2^4}{24} \right) \]

\[ + \mu_2 \left( \frac{y_2^4 a^2}{24} + \frac{y_2^2 a^2 b^2}{4} - \frac{3 y_2^4 a^2}{8} + \frac{y_2^2 b^2}{12 a^3} \right) k^2. \quad (2.5.14) \]

As \( c \to c_0 \) for long wavelength, \( k = 0 \) and thus the right side of the above equation vanishes. Therefore the first approximation gives

\[ a^4 \left( \rho_1 c_0^2 - \mu_1 \right) + \left( \mu_2 - \rho_2 c_0^2 \right) \left( a^4 - b^4 \right) = 0, \]

or

\[ c_0^2 \left[ \rho_1 a^4 + \rho_2 \left( b^4 - a^4 \right) \right] = \mu_1 a^4 + \mu_2 \left( b^4 - a^4 \right). \quad (2.5.15) \]

The terms of order \( k^2 \) in (2.5.13) yield the right side of (2.5.14) with \( c \) replaced by \( c_0 \), the first approximation shown in (2.5.15).
Therefore, $\gamma_1$ and $\gamma_2$ now become

\[
\gamma_1 = \rho_2 = \frac{c_0}{c_1} - 1 = \frac{\rho_1}{\mu_1} \frac{\mu_1 a^4 + \mu_2 (b^4 - a^4)}{\rho_1 a^4 + \rho_2 (b^4 - a^4)} - 1
\]

\[
= D \frac{b^4 - a^4}{\mu_1},
\]

\[
\gamma_2 = q^2 = 1 - \frac{c_0}{c_2} = D \frac{a^4}{\mu_2},
\]

(2.5.16)

with

\[
D = \frac{\rho_1 \mu_2 - \rho_2 \mu_1}{\rho_1 a^4 + \rho_2 (b^4 - a^4)}.
\]

(2.5.17)

Substitution of the above $\gamma_1$ and $\gamma_2$ into the right side of (2.5.14) gives

\[
\frac{D^2}{\mu_1} \left[ \frac{a^4}{24} (b^4 - a^4) \right] + \frac{D^2}{\mu_2} \left[ \frac{a^4}{24} (a^8 - 6a^4 b^4 + 8a^2 b^6 - 3b^8) \right] k^2
\]

\[
= \frac{a^4}{24} \left( \frac{1}{\mu_1} (b^4 - a^4) + \frac{1}{\mu_2} (b^2 - a^2)^3 (3b^2 + a^2) \right) D^2 k^2.
\]

Then (2.5.14), after multiplied by $a^2$, becomes

\[
a^4 (\rho_1 c^2 - \mu_1) + (b^4 - a^4) (\rho_2 c^2 - \mu_2) = - a D^2 k^2
\]

(2.5.18a)

with $D$ as shown in (2.5.17) and

\[
\alpha = - \frac{a^6 (b^2 - a^2)^2}{24} \left\{ \frac{(a^2 + b^2)^2}{\mu_1} + \frac{(b^2 - a^2)(3b^2 + a^2)}{\mu_2} \right\}.
\]

(2.5.18b)
Equation (2.5.18a) gives the speed obtained from elementary theory, i.e., \( c_0^2 \) as \( k \to 0 \).

It must be noted that dispersion again vanishes if \( c_1 = c_2 \) though the properties of media are different.

C. Elementary derivation

It may be interesting to see how the wave speed is obtained by the elementary derivation and compare it with the result obtained from the frequency equation.

Suppose that \( \ell m \) and \( \ell' m' \) are two cross sections separated by a distance \( dz \), as shown in Figure 2.1. For the section \( \ell' m' \) to rotate relative to the section \( \ell m \) through an angle \( d\theta \), a torque \( T \) must be applied to it.

Suppose that, under the action of the torque that is applied to this section, the end \( B \) of the generator \( BA \) is displaced through an extremely small distance

\[
BB = r d\theta . \tag{2.5.19}
\]

Let \( \tau \) be the shear stress, caused by the displacement of the generator \( BA \) to the position \( BA \). Then by Hook's law

\[
\tau = \mu \psi ,
\]

where \( \psi \) is the angle \( BAB \) and \( \mu \) is the shear modulus.

It follows from this that force applied to the cross section \( dA \) is expressed by the product

\[
\tau dA = \mu \psi dA . \tag{2.5.20}
\]

Furthermore, because of the very small dimensions of the triangle \( BAB \),
Figure 2.1 Twist of a two-layered solid cylinder
one may assume that

$$BB = \psi dz,$$  \hspace{1cm} (2.5.21)

and by comparing (2.5.19) and (2.5.21) one has

$$\psi = r \frac{\partial \theta}{\partial z}.$$

Consequently,

$$\tau dA = \mu \frac{\partial \theta}{\partial z} r dA,$$

$$dT = r \tau dA = \mu \frac{\partial \theta}{\partial z} r^2 dA,$$

or

$$T = \mu_1 \frac{\partial \theta_1}{\partial z} \int_a^b r^2 2\pi r dr + \mu_2 \frac{\partial \theta_2}{\partial z} \int_a^b r^2 2\pi r dr$$

$$= \frac{\pi}{2} \left[ \mu_1 a^4 \frac{\partial \theta_1}{\partial z} + \mu_2 (b^4 - a^4) \frac{\partial \theta_2}{\partial z} \right].$$

And since $$\theta_1 = \theta_2 = 0$$ for the first approximation as shown in the previous article,

$$T = \frac{\pi}{2} \left[ \mu_1 a^4 + \mu_2 (b^4 - a^4) \right] \frac{\partial \phi}{\partial z} = \overline{\mu} \frac{\partial \phi}{\partial z}. \hspace{1cm} (2.5.22)$$

The torque for the section whose abscissa is $$z$$ is equal to $$\overline{\mu} \frac{\partial \phi}{\partial z}$$; the torque for the section with abscissa $$z + dz$$ is equal to
\[ \bar{\mu} \frac{3\bar{\theta}}{3z} + \bar{\mu} \frac{3^2\bar{\theta}}{3z^2} \, dz, \]

where
\[ \bar{\mu} = \frac{\pi}{2} \left[ \mu_1 a^4 + \mu_2 (b^4 - a^4) \right]. \quad (2.5.23) \]

To obtain the equation for the torsional waves, one needs to equate the resultant torque \( \bar{\mu}(3^2\bar{\theta}/3z^2)dz \) with the product of the angular acceleration \( 3^2\bar{\theta}/3t^2 \) and the moment of inertia of the element \( lmm_1l_1 \) about the axis of the rod. Thus

\[ \bar{\mu} \frac{3^2\bar{\theta}}{3t^2} \, dz = \frac{3^2\bar{\theta}}{3t^2} \rho \, dz, \quad (2.5.24) \]

where \( \rho \) denotes the moment of inertia of a unit length of the rod and is

\[ \rho = \int \rho \, d\rho = \int \rho r^2 dA = \int \rho r^2 2\pi rdr \]

\[ = 2\pi \rho_1 \int_a^b r^3 dr + 2\pi \rho_2 \int_a^b r^3 dr \]

\[ = \frac{\pi}{2} \left[ \rho_1 a^4 + \rho_2 (b^4 - a^4) \right]. \quad (2.5.25) \]

Therefore, on account of (2.5.23) and (2.5.25), (2.5.24) becomes

\[ \frac{3^2\bar{\theta}}{3t^2} = c_o^2 \frac{3^2\bar{\theta}}{3z^2}, \quad (2.5.26a) \]
with

\[
    c^2_0 = \frac{\bar{\mu}}{\rho^*} = \frac{\mu_1 a^4 + \mu_2 (b^4 - a^4)}{\rho_1 a^4 + \rho_2 (b^4 - a^4)}.
\]  

(2.5.26a)

This result and that obtained by the frequency equation (2.5.15) are identical.

Such a derivation is possible for the other two cases considered here. However, it is omitted because of its elementary nature.

Though the fundamental wave equation is thus obtainable from elementary consideration, its adoption for a higher power approximation is obviously going to be quite involved.

D. A uniformly valid perturbation solution

The first approximation of the phase speed can be obtained by both the frequency equation (for long wavelength) and the elementary derivation as is done previously. The higher approximation of the phase speed, however, is difficult to obtain by elementary derivation. Here it is obtained by asymptotic expansion and the result is compared with that obtained by the frequency equation.

The governing system remains the same, and hence (2.5.1-4) still hold in this problem.

Since interest is in the far-field solution, a change of variable is made.

\[
    Z = z - c_0 t \quad \text{and} \quad T = \alpha(b) t,
\]  

(2.5.27a)

where \( c_0 \) is the linear wave speed, so that

\[
    \frac{\partial}{\partial z} = \frac{\partial}{\partial Z} ; \quad \frac{\partial}{\partial t} = -c_0 \frac{\partial}{\partial Z} + \alpha \frac{\partial}{\partial T}
\]  

(2.5.27b)
To derive the interior equations, the transformation in the $r$
coordinate must be introduced, namely

$$r = bR,$$  \hspace{1cm} (2.5.28a)

so that

$$\frac{\partial}{\partial r} = \frac{1}{b} \frac{\partial}{\partial R},$$  \hspace{1cm} (2.5.28b)

and the boundary conditions as

$$v_{2R} - v_2 = 0 \quad \text{on } R = 1,$$  \hspace{1cm} (2.5.29a)

$$v_1 = v_2 \quad \text{on } R = \kappa,$$  \hspace{1cm} (2.5.29b)

$$\mu_1 \left( \frac{1}{b} v_{1R} - \frac{1}{a} v_1 \right) = \mu_2 \left( \frac{1}{b} v_{2R} - \frac{1}{a} v_2 \right) \quad \text{on } R = \kappa,$$  \hspace{1cm} (2.5.29c)

with $\kappa$ as the ratio of inner radius $a$ to outer radius $b$ of medium 2.

Note that here $b$ and $a$ are the ratios of outer and inner radii $b$ and $a$
to the typical length.

The above transformations introduce a new parameter $\alpha$ that with
the geometric parameter $h^2$, completely define the physical problem of
the torsional waves in a rod.

With the aforementioned transformations, all field variables are
now assumed to be functions of $(R,Z,T)$.

Selecting an appropriate similarity hypothesis that sets
$\alpha = \alpha_0 h^2$ as before yields a sequence of boundary value problems to be solved.

The equation of motion is
\( L_r v + v_{zz} = \frac{\rho}{\mu} v_{tt} \), \hspace{1cm} (2.5.30a)

where

\[
L_r \equiv \frac{\partial}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}
\]

(2.5.30b)

in the original coordinate. So that in terms of the transformed coordinate, it takes the following form:

\[
\frac{1}{b^2} \left( \frac{\partial}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} - \frac{1}{R^2} \right) \left( \varphi_L + b^2 \varphi_L + \cdots \right) + \frac{\partial^2}{\partial Z^2} \left( \varphi_L + b^2 \varphi_L + \cdots \right)
\]

(2.5.30c)

where \( \ell = 1, 2 \) refers to medium 1, 2.

The terms of order \( 1/b^2 \) in (2.5.30c) lead to

\[
L \varphi_1 = L \varphi_2 = 0 , \hspace{1cm} (2.5.31a)
\]

with \( L = \frac{\partial}{\partial R} \left( \frac{\partial}{\partial R} + \frac{1}{R} \right) \).

(2.5.31b)

The solutions of the first order linear equations (2.5.31a) are

\[
\varphi_1 = \frac{1}{R} F_1(Z,T) + \frac{R}{2} H_1(Z,T) ;
\]

\[
\varphi_2 = \frac{1}{R} F_2(Z,T) + \frac{R}{2} H_2(Z,T) .
\]

\( F_1 \) equals zero because of the finiteness of the displacement at the
origin. Equation (2.5.29a) gives \( F_2 = 0 \) and (2.5.29b) gives \( H_1 = H_2 = H \), so that

\[
\varphi_1 = \varphi_2 = RH(Z,T),
\]

(2.5.31c)

which satisfies boundary conditions (2.5.29c).

The terms of order unity in (2.5.30c) give

\[
\begin{align*}
2 L v_1 &= \left( \frac{c_0}{c_1} - 1 \right) \frac{\partial^2 \varphi_1}{\partial z^2} = \gamma_1 R^H' ; \\
2 L v_2 &= \left( \frac{c_0}{c_2} - 1 \right) \frac{\partial^2 \varphi_2}{\partial z^2} = \gamma_2 R^H' ,
\end{align*}
\]

(2.5.32a)

where (') = \( \frac{\partial}{\partial Z} (\cdot) \). The solutions of (2.5.32a) are

\[
\begin{align*}
v_1 &= \frac{N_1}{R} + \frac{R}{2} G_1(Z,T) + \frac{R^3}{8} H''(Z,T) ; \\
v_2 &= \frac{N_2}{R} + \frac{R}{2} G_2(Z,T) + \frac{R^3}{8} H''(Z,T) .
\end{align*}
\]

(2.5.32b)

\( N_1 = 0 \) because of the finiteness of displacement at the origin.

\( N_2 = \frac{1}{8} H'' \) from (2.5.29a), hence

\[
\begin{align*}
v_1 &= RG_1(Z,T) + \frac{R^3}{8} \gamma_1 H'' (Z,T) , \\
v_1 &= RG_2 (Z,T) + \frac{1}{8} \left( R^3 + \frac{1}{R} \right) \gamma_2 H'' (Z,T) .
\end{align*}
\]
From the displacement continuity (2.5.29b) we have

\[ \frac{2}{v_1} = RG + \frac{\gamma_1}{8} \left( R^3 - R\kappa^2 \right) H'' ; \] 

\[ \frac{2}{v_2} = RG + \frac{\gamma_2}{8} \left( R^3 + \frac{1}{R} - R\kappa^2 - \frac{R}{\kappa^2} \right) H'' . \]

Lastly, the stress continuity (2.5.29c) yields

\[ \gamma_1 H'' = \gamma_2 \left( 1 - \frac{1}{\kappa^4} \right) H'' , \]

so that

\[ \gamma_1 \mu_1^2 = \gamma_2 \mu_2^2 \left( \frac{1}{\kappa^4} - 1 \right) , \]

or

\[ b^4 \gamma_1 \mu_1^2 = \gamma_2 \mu_2^2 (a^4 - b^4) \]

assuming \( H'' \neq 0 \).

To obtain \( \gamma_1 \) and \( \gamma_2 \) in terms of \( \rho \), \( \mu \) and \( \kappa \) for later use, the following algebraic computation is done:

\[ \gamma_1 \left( \frac{c_2^2}{c_1^2} - 1 \right) = \gamma_2 \left( 1 - \frac{c_2^2}{c_1^2} \right) \left( \frac{1}{\kappa^4} - 1 \right) , \]

or

\[ -\mu_1 \kappa^4 - \mu_2 (1 - \kappa^4) + \rho_1 \kappa^4 c_0^2 + \rho_2 c_0^2 (1 - \kappa^4) = 0 , \]

or

\[ \frac{c_2^2}{c_0} = \frac{\mu_1 \kappa^4 + \mu_2 (1 - \kappa^4)}{\rho_1 \kappa^4 + \rho_2 (1 - \kappa^4)} , \]
which checks with that obtained by frequency equation (2.5.15), and

\[
\gamma_1 = \frac{c_0^2}{c_1^2} - 1 = \frac{\rho_1 \mu_2 - \rho_2 \mu_1}{\bar{\rho}} \frac{1 - \kappa^4}{\mu_1}, \quad (2.5.32h)
\]

where \( \bar{\rho} = \rho_1 \kappa^4 + \rho_2 (1 - \kappa^4) \). \( (2.5.32i) \)

Substituting this \( \gamma_1 \) back into (2.5.32f), we have

\[
\gamma_2 = -\frac{\rho_1 \mu_2 - \rho_2 \mu_1}{\bar{\rho}} \frac{\kappa^4}{\mu_2}, \quad (2.5.32j)
\]

The terms next in order [of \( O(b^2) \)] in (2.5.30c) lead to

\[
L \nu_1 = \gamma_1 \frac{\partial^2 \nu_1}{\partial z^2} - \frac{2c_0 \alpha_0}{c_1^2} \frac{\partial^2 \varphi_1}{\partial \tilde{z} \partial \tilde{T}}
\]

\[
= \gamma_1 \kappa \gamma + \frac{\gamma_1}{\kappa^2} (R^3 - R \kappa^2) \frac{\partial \nu}{\partial \tilde{T}} - \frac{2c_0 \alpha_0}{c_1^2} \frac{\partial \varphi_1}{\partial \tilde{z}} \tilde{R} \tilde{H}; \quad (2.5.33a)
\]

\[
L \nu_2 = \gamma_2 \kappa \gamma + \frac{\gamma_2}{8} \left( R^3 + \frac{1}{\kappa^2} - R \kappa^4 - \frac{R}{\kappa^2} \right) - \frac{2c_0 \alpha_0}{c_1^2} \frac{\partial \varphi_1}{\partial \tilde{z}} \tilde{R} \tilde{H},
\]

where \( (\cdot)' = \frac{\partial}{\partial \tilde{t}} (\cdot) \). The solutions of the first order linear equation, by making use of finiteness of displacement at the origin and boundary conditions (2.5.29a) and (2.5.29b), are
\[ v_1 = - \frac{c_0 a_H H'}{4c_1^2} (R^3 - R\kappa^2) + \frac{\gamma_1 G''}{8} (R^3 - R\kappa^2) + \frac{\gamma_1 H'''}{192} (R^5 - 3R^3\kappa^2 + 2R\kappa^4) ; \]

\[ v_2 = - \frac{c_0 a_H H'}{4c_2^2} (R^3 + \frac{1}{R} - R\kappa - \frac{R}{\kappa^2}) + \frac{\gamma_2 G''}{8} (R^3 + \frac{1}{R} - R\kappa - \frac{R}{\kappa^2}) \]

\[ + \frac{\gamma_2 H'''}{192} \{ R^5 + 2R\kappa^4 + 12(R \ln R - R \ln \kappa) - 6R + \frac{8}{R} \]

\[ - 3(\kappa^2 + \frac{1}{\kappa^2}) (R^3 + \frac{1}{R}) + R(12 - \frac{8}{\kappa^2} + \frac{3}{\kappa^4}) \} . \]

Lastly, by the condition of stress continuity (2.5.29c), we have

\[ \mu_1 \{ - \frac{\kappa^2 c_0 a_H H'}{2c_1^2} + \frac{\gamma_1 \kappa^2 G''}{4} - \frac{4\gamma_1 H'''}{96} \} \]

\[ = \mu_2 \{ - \frac{c_0 a_H H'}{2c_2^2} (\kappa^2 - \frac{1}{\kappa^2}) + \frac{\gamma_2 G''}{4} (\kappa^2 - \frac{1}{\kappa^2}) \]

\[ + \frac{\gamma_2 H'''}{96} (- \kappa^4 + 6 - \frac{8}{\kappa^2} + \frac{3}{\kappa^4}) , \]

or
\[
[k^2(\rho_2 - \rho_1) - \frac{\rho_2}{k^2}] c_o \dot{H} + \frac{\kappa g'''}{4} \left\{ \mu_1 \gamma_1 - \mu_2 \gamma_2 \left( 1 - \frac{1}{k^4} \right) \right\}
\]

\[
= \frac{H^{IV}}{48} \left\{ \mu_1 \gamma_1^2 k^4 + \mu_2 \gamma_2^2 \left( \frac{3}{k^4} - \frac{8}{k^2} + 6 - k^4 \right) \right\}.
\]

The coefficient of \( G'' \) vanishes in view of (2.5.32e). By making use of (2.5.32h-j), we find

\[
-c_o \frac{\dot{r}}{\kappa^2} \dot{H}^{IV} = \frac{H^{IV}}{48} \left\{ \mu_1 \gamma_1^2 k^4 + \frac{\mu_2 \gamma_2^2}{k^4} \left( 3 + k^2 \right) \left( 1 - k^2 \right)^3 \right\}
\]

\[
= \frac{H^{IV}}{48} \left( \frac{\rho_1 \mu_2 + \rho_2 \mu_1}{\rho} \right)^2 \frac{k^4}{\mu_1} \left( 1 - k^2 \right)^2 + \frac{8}{\mu_2} \frac{k^8}{\kappa^4} \frac{(3 + k^2)(1 - k^2)^3}{\kappa^4}
\]

\[
= \frac{H^{IV}}{48} \left( \frac{\rho_1 \mu_2 - \rho_2 \mu_1}{\rho} \right)^2 \frac{k^4}{\mu_1} \left( 1 - k^2 \right)^2 \left\{ \frac{(1 + k^2)^2}{\mu_1} + \frac{(3 + k^2)(1 - k^2)}{\mu_2} \right\},
\]

or

\[
\frac{\partial^2 H}{\partial z \partial T} + \beta_1 \frac{\partial^4 H}{\partial z^4} = 0,
\]

(2.5.34)

where

\[
\beta_1 = \frac{k^6(1 - k^2)^2}{48 c_o \rho} \left( \frac{\rho_1 \mu_2 - \rho_2 \mu_1}{\rho} \right)^2 \left\{ \frac{(1 + k^2)^2}{\mu} + \frac{(3 + k^2)(1 - k^2)}{\mu_2} \right\},
\]

(2.5.35)
which agrees with that obtained by frequency equation (2.5.18b).

Equation (2.5.34) governs the propagation of the torsional strain. It shows the variation of the wave as expressed in terms of the variation of the displacement gradient where the dominant gradient is the gradient of $H(Z,T)$. The dispersion term arises as a combination of dimensions, densities and elasticities of the two media. The first and second approximation of the phase speed in (2.5.14) and (2.5.16) obtained by frequency equation check with those in (2.5.32g) and (2.5.35) done in this article. The elegance of the technique of asymptotic expansion is greatly appreciated by its easiness in obtaining the solution as illustrated in this article. The discussion of the result is postponed to the next article.

2.6. Discussion of the Results

In the case of linear theory of elasticity, where small deformations are considered, a weak disturbance traveling in an unstrained elastic two-layered plate or rod is affected by dispersion at far-field. The dispersive term (in Jeffrey's equation) depends on the properties and dimensions of the two media. If one considers a single-layer plate or rod, the dispersive term vanishes which is in agreement of the known result (see Art. 2.1). Furthermore, the second approximation of the phase speed, obtained from this dispersive term, is also in agreement with that obtained from the frequency equation.

The resulting equation in this chapter (the analog of Jeffrey's equation in fluid) is new; it brings out the role of interaction of dispersion, material properties, and curvature of the shear wave in the
vicinity of the wave front.

The solutions of the equation (2.5.34) will be shown to be given in terms of Airy functions. These give, to a first approximation, the asymptotic nature of linear dispersive waves of a number of problems. To see this, consider the following example:

$$u_{tt} = u_{xx} + 2\varepsilon \beta_1 u_{txxx}.$$  \hspace{1cm} (2.6.1)

This arises for Rayleigh correction for lateral inertia of longitudinal waves [11]. If we introduce

$$X = x - t, \quad T = \varepsilon t,$$  \hspace{1cm} (2.6.2)

we obtain the following equation, neglecting terms of order $\varepsilon^2$:

$$F_T + \beta_1 F_{xxx} = 0, \quad F = u_x,$$  \hspace{1cm} (2.6.3)

which is the object of the present discussion.

The general solution of (2.6.1), for zero initial conditions and the condition of the vanishing of $u$ as $x$ approaches infinity, can be represented as

$$u_x = \frac{1}{2\pi i} \int_B f(s) \exp(st - s_1 x) ds,$$  \hspace{1cm} (2.6.4)

where $f(s)$ depends on boundary conditions, and $s_1 = s/(1 + 2\varepsilon \beta_1 s^2)^{1/2}$.

It may be noted that the solution considered is for $u_x$, which is $F$. Furthermore, $f(s)$ is unity for a boundary condition $u_x = \delta(t)$ for $x = 0$.

The work by Rausch [17] shows that if
\[ l_k = \frac{1}{2\pi i} \int_{Br} \frac{1}{s^k} \exp \{ (t-x)s + \Delta s^3 \} ds, \quad (2.6.5) \]

where \( \Delta \) is independent of \( s \), then we have

\[ l_1 = \frac{1}{3} + \int_0^\xi \texttt{Ai}(-y)dy, \quad (2.6.6a) \]

\[ l_2 = c_0^x \left\{ d_0 + \frac{1}{3} \xi + \int_0^\xi \int_0^{\xi_1} \texttt{Ai}(-z)dz d\xi_1 \right\}, \quad (2.6.6b) \]

with \( \xi = \frac{t-x}{(3\Delta)^{1/3}} \). The nature of such linear dispersive waves can be seen from Figure 2.2.

If we now assume from well established standard theory that the major large time contribution to (2.6.4) comes from a saddle point, then it is straightforward to verify that the origin is the saddle point for (2.6.4); and the expansion of the argument of the exponent near \( s = 0 \), corresponds exactly to that in (2.6.5).

Now how can we assert that the present asymptotic method really describes the asymptotic nature? A rigorous answer to this question is quite involved, the difficulty of a proof depending on the demand of the rigour. We provide below through exact formulation the plausible argument for such an assertion.

The exact solution for a two-layered plate problem with constant (with respect to \( z \)) delta time type shear stress applied at \( x = 0 \) is given by Laplace and Fourier cosine inverse of
Figure 2.2 Graph of $A_i(\xi)$ showing the nature of dispersion
\[ \overline{v}_1 = \frac{1}{u_1 \alpha_1^2} - c \frac{\cos h \alpha_1 (z-h)}{u_1 \alpha_1 \sinh \alpha_1 h}, \]

\[ \overline{v}_2 = \frac{1}{u_2 \alpha_2^2} + c \frac{\cos h \alpha_2 (z+h)}{u_2 \alpha_2 \sinh \alpha_2 h}, \]

\[ c = \frac{(u_1 \alpha_1^2 - u_2 \alpha_2^2) \tan h \alpha_1 h \tan h \alpha_2 h}{\alpha_1 \alpha_2 D}, \]

\[ D = u_1 \alpha_1 \tan h \alpha_1 h + u_2 \alpha_2 \tan h \alpha_2 h, \]

\[ \alpha_1^2 = \alpha^2 + \frac{s^2}{c_1^2}, \quad \alpha_2^2 = \alpha^2 + \frac{s^2}{c_2^2}, \]

\[ V_1 = \int_{0}^{\infty} v_1 \cos \alpha x \, dx, \]

\[ \overline{V}_1 = \int_{0}^{\infty} V_1 e^{-st} \, dt. \]

It can now be seen that the solution described by the present asymptotic theory is given by one of the zeros of \( D \), with \( s = ic(\alpha) \) and then stationary phase integration with respect to \( \alpha \), with \( \alpha = 0 \) as the point of stationary phase; the contribution from other possible ones must
be negligible.

We stress here that these considerations are purely heuristic. We have explained the general steps involved in the procedures, and outlined the method of attack to point out the plausibility of our arguments.

A rigorous study would require the investigation of all roots of $D = 0$ (besides other singularities) and the proof that the contributions from all others except the one above are damped out.

Such an exhaustive analytical-numerical work on the above lines does not appear to exist in the literature, nor is it undertaken here.

However, some beautiful attempts are worth noting. In discussing the problem of head-on impact of two rods, Skalk [18] obtains, by the use of the method of stationally phase, the asymptotic nature of the wave form. The denominator he has, is the standard frequency equation [19] for longitudinal waves in a rod. The only root he considers corresponds to the long wavelength approximation. The question remains open whether there exists other root, possibly complex, which contribute to the solution.

Another relevant work is a recent study by Karal and Alterman [20], which is a purely numerical study by the use of the computer. Some of their observations are worth quoting, [20]:

For distances as close as two diameters from the end of the rod, the subsequent oscillations differ, while for distances greater than five diameters, they become similar. Our conclusions concerning the far field are in agreement with those held by other research workers.

These and other investigations in the literature support the asymptotic method.
Further, it must be stressed that all "ad hoc" theories of plates and rods follow from this method and since the latter are well verified experimentally, the validity seems to hold, though not proved.
3. SHEAR WAVES IN AN INFINITE VISCOELASTIC MEDIUM - NONLINEAR THEORY

3.1 Introduction

Since about 1957, a large number of studies of nonlinear waves have appeared [21-23]; all of these are based on the singular surface theory. The theory provides a simple technique to describe the evolution of an initial discontinuity into a shock wave. An account of the study of a shear shock wave can be found in [1]. In combination with ray theory, one can study through its use even a nonhomogeneous, anisotropic nonlinear system [23]. This approach has a number of limitations. Firstly the study describes only the wave front; it is applicable only to a fully hyperbolic system and thus the study of a viscous continuum is excluded (except memory-type integral material [24] when real characteristics exist). The most important limitation relevant to the present problem is that the theory predicts no shock in certain cases when shocks are shown to be possible.

To illustrate, consider the final equation, derived by a number of authors for plane nondissipative systems:

\[ \frac{d \xi}{d t} = k_0 \xi^2. \]  (3.1.1)

The solution of (3.1.1) is

\[ \xi = \frac{1}{\xi_0 - k_0 t}. \]

Here \( \xi \) is the value of the discontinuity of some field variable, \( \xi_0 \) is its initial value and \( k_0 \) is a constant depending on the material properties.
Thus, assuming $\xi_0$ and $k_0$ are positive $\xi \rightarrow \infty$ as $t \rightarrow t_c$ with $t_c k_0 = \xi_0$. This is interpreted as a shock. The equation obtained in the case of shear waves is formally identical with (3.1.1) with $k_0 = 0$. This implies the strength of the discontinuities remains constant. If the former is interpreted as a shock formation the latter should imply the absence of shock formation. Analogous arguments given by Morris [25] indicate that it takes an infinite time for a shock to be formed. Some such examples are: (i) shear waves in elasticity [26], (ii) switch-on shock of magnetoelasticity [27], and (iii) electromagnetic wave in a hyperelastic medium [28].

Studies by Whitham [29] first brought out how a small dissipation provides a structure to a wave; this justified neglect of viscosity everywhere except near the wave front. Lighthill [30] further developed the idea to obtain Burgers' equation which governs a weak nonlinear viscous shock layer. This approach relies on the existence of Riemann invariants, which is possible only in a reducible hyperbolic system. Moran and Shen [31] rederived the result by the use of asymptotic expansions; combining these ideas with those of singular surface theory and ray theory as applied to weak waves (strain derivative is discontinuous while strain is continuous), Sedov et al. [32] extended the method to waves of arbitrary configuration in gas dynamics. Since the expansion forms in this last study were based on the study of weak waves by the use of singular theory, this last theory predicts no shock formation for a shear wave; thus the Burgers' equation derived on the use of these ideas turned out to be linear.
Physically such a conclusion is rather intriguing; it is unsatisfactory to conclude that an elastic shear wave, governed by a highly nonlinear basic system, should not be affected by the nonlinearity even in a first approximation. This result brings out the limitations, of the approach and assumptions, involved in such a derivation. The present study reveals that the limitation arises from basing the expansion on the study of weak waves. Based on subsequent studies of shock waves [1, 33-34], which show the existence of weak shear shocks, a different expansion is postulated. Such an amplitude expansion based on the shock relations (discontinuous strain) rather than on weak wave relation (discontinuous strain derivatives) yields a nonlinear Burgers' equation

$$\frac{\partial^2 V}{\partial x \partial t} + \beta_0 \frac{\partial}{\partial x} (V^3) = \delta_0 \frac{\partial^3 V}{\partial x^3},$$

(3.1.2)

where $\beta_0$ is some elastic constant.

Equation (3.1.2) is the equation which governs a viscous shock layer; it is obtained by using a different expansion procedure, supplemented with different similarity hypothesis which physically imply a balance of nonlinearity, unsteadiness and dissipation.

This study leads one to a number of conclusions. Use of singular surface theory to predict shock formation from weak waves has severe limitations; it predicts no shock formation even in cases where shock waves we know to be possible. The amplitude expansions for a viscous shock layer must be based on shock relations. Burgers' equation (3.1.2) governing this wave is also of a novel type, not appearing in the literature thus far. From earlier studies by Morris and Nariboli [28] it is
safe to say that besides the shear wave of isotropic (and certain types of anisotropic cases [23] too) elasticity, the switch-on shock wave of magnetoelasticity and the electromagnetic wave in a hyperelastic dielectric are all governed by a similar linear equation.

It must be stressed that the result of the present study indicates that the evolution to nonlinear wave can occur for times of order \( \varepsilon^{-2} \) while the earlier work predicts that the wave is linear for time of order \( \varepsilon^{-1} \), where \( \varepsilon \) is the amplitude of the wave.

Viscosity is essentially represented by the ratio of the viscous length to the typical length; so the requirement that this parameter be of order \( \varepsilon^2 \) means that the evolution to the nonlinear wave takes place at a farther distance. The same is implied by the requirement that the time stretching parameter be of order \( \varepsilon^2 \). Therefore, the results are not contradictory but it brings out the inadequacy of the former approach.

A few solutions of equation (3.1.2) are discussed. One is the Taylor's solution describing the steady (in a Galilean frame) profile of a weak viscous shock. The existence of self-similar solution is also shown, on the lines of earlier studies.

A last comment may be pertinent. In the study of weak waves in elasticity, one takes strains as continuous while the strain derivatives are discontinuous. Using compatibility conditions one obtains a nonlinear equation for the discontinuity, which predicts that the discontinuity in the strain derivatives can increase and become infinite in a finite time, predicted by the study; the time is interpreted as formation of shock which is characterised by discontinuous strains. This reasonable inter-
The failure of the theory to predict a shock in some cases even when shock waves are known to exist, indicates some wide gap in the evolutionary interpretation of shock formation.

3.2. Basic Equations and Formulation of the Problem

An infinite, uniform, homogeneous, isotropic and viscoelastic medium, unstrained and at rest, is subjected to a weak shear disturbance. We like to know what is going on in the vicinity of the wave front. We assume the stress is given by two parts, the elastic and viscoelastic. The elastic part is assumed to be that of hyperelastic material and the viscous part is assumed to be that of the rate type viscous material.

We nondimensionalize distances, velocities and time by $L_0$, $c_0$ and $L_0/c_0$ respectively, where $L_0$ is a typical length (see chapter two) and $c_0 = (\mu/\rho_0)^{1/2}$ where $\mu$ is the shear modulus and $\rho_0$ is the density in the undeformed state. Thus, if primes and unprimes denote dimensional and non-dimensional quantities respectively, then

$$u'_i = L_0 u_i, \quad v'_i = c_0 v_i, \quad t' = \frac{L_0}{c_0} t,$$

so that acceleration and strain rate are

$$a'_i = \frac{c_0^2}{L_0} a_i \quad \text{and} \quad d'_{ij} = \frac{c_0}{L_0} d_{ij}.$$

Further elastic stresses, viscous stresses and density are nondimensionalized by

$$T'_{ij} = \mu T_{ij}, \quad \tau'_{ij} = \frac{c_0}{L_0} \tau_{ij} \quad \text{and} \quad \rho' = \rho_0 \rho.$$
In terms of these nondimensional quantities the conservation of mass and linear momentum are given by

\[ \rho = (1 - 2 I + 4 II - 8 III)^{1/2}, \quad (3.2.1) \]

\[ t_{ij,j} = \rho a_i, \quad (3.2.2) \]

where

\[ t_{ij} = T_{ij} + \delta \tau_{ij}, \quad (3.2.3) \]

\[ \tau_{ij} = \lambda^* I^* \delta_{ij} + 2 \mu^* d_{ij}, \quad (3.2.4a) \]

\[ T_{ij} = \rho \frac{\partial U}{\partial e_{kj}} (\delta_{ik} - 2 e_{ik}), \]

or

\[ \frac{T_{ij}}{\rho} = U_1 \{ \delta_{ij} - 2 e_{ij} \} + U_2 \{ I \delta_{ij} - (2 I + 1) e_{ij} + 2 e_{ik} e_{kj} \} \]

\[ + U_3 \{ (II - 2 III) \delta_{ij} - I e_{ij} + e_{ik} e_{kj} \}, \quad (3.2.4b) \]

and

\[ U_1 = \frac{\partial U}{\partial I}, \quad U_2 = \frac{\partial U}{\partial II}, \quad U_3 = \frac{\partial U}{\partial III}, \quad (3.2.5a) \]

\[ U = \frac{\lambda + 2 \mu}{2} I^2 - 2 \mu II + \varepsilon I^3 + \mu I II + n III + p II^2 + \ldots. \quad (3.2.5b) \]

Here \( a_i \) is the acceleration vector. The unstarred and starred of I, II and III are the first, second, and third invariants of the spatial strain tensor, \( e_{ij} \) and the spatial deformation-rate tensor, \( d_{ij} \) respectively. These are defined in terms of the displacement vector \( u_i \) by

\[ 2 e_{ij} = U_{i,j} + u_{j,i} - k_{i,k} u_{k,j}, \quad 2 d_{ij} = v_{i,j} + v_{j,i}, \]

\[ v_i = \frac{\partial u_i}{\partial t} + v_j u_{i,j}, \quad a_i = \frac{\partial v_i}{\partial t} + v_j v_{i,j}, \quad (3.2.6) \]
where \( v \) is the velocity vector. \( \delta = \frac{\mu^*}{\rho_0 c_0 L_0} \) is analogous to the reciprocal of Reynolds' number \([37]\). \( \lambda \) and \( \mu \) are the Lamé constants while \( \lambda^* \) and \( \mu^* \) are the viscous bulk and shear modulii. Equation (3.2.5b) is the polynomial expansion of the internal energy function and written only to the order needed \([1,38]\).

Predebon and Nariboli's \([1]\), and Morris and Nariboli's work \([28]\) show that for some transverse waves the entropy change across a weak shock varies as fourth power in amplitude \( \epsilon \). In the subsequent work we use the terms up to \( \epsilon^3 \). Therefore, we can say that the wave is isentropic up to \( \epsilon^3 \) and the energy equation need not be used. It may be noted that the usual Burgers' equation in gasdynamics also describes an isentropic wave.

Thus, Equations (3.2.1) and (3.2.2) form the full system of equations governing this particular problem and on which the perturbation scheme is to be applied.

It must be noted that shear stress is an odd function of shear strain. This is proved by Coleman and Noll \([39]\) for a certain shear flows and it is also obvious from the studies of Predebon and Nariboli \([1]\) and Morris and Nariboli \([28]\).

Moreover, pure shear does not exist; and in nonlinear wave theory shear is always accompanied by longitudinal strain. Based on these considerations we assume that for small-but-finite waves the amplitude expansions take the following form:

\[
\begin{align*}
u(x,t,\delta,\epsilon) &= \epsilon^2 \bar{u} + \cdots,
\end{align*}
\]
\( v(x,t,\delta,e) = e^{1/2} \psi + e^{3/2} \psi + \ldots \), \( w = 0 \),

where \( e \) is a small parameter characterizing the magnitude of their amplitudes, and \( \delta \) is another small parameter characterizing the viscosity of the material.

### 3.3 Shear Waves in an Infinite Viscoelastic Medium

#### A. Plane shear waves

Geometrically simple but still physically interesting cases of shear waves are of two types: planar and cylindrically symmetric ones. We start with the former which is an easier one.

Let the displacement vector be

\[
\mathbf{u} = [u(x,e,\delta,t), v(x,e,\delta,t), 0];
\]

then the elastic strain tensor, on account of (3.2.6) and (3.2.7), is given by

\[
\varepsilon_{ij} = \begin{bmatrix} e_{xx} & e_{xy} & 0 \\ e_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

with

\[
e_{xx} = e^2 \left( \frac{u^2}{u} - 1/2 \frac{v^2}{v} \right) + O(e^4),
\]

\[
e_{xy} = 1/2 e \frac{1}{v}.
\]

The elastic strain invariants reduce to
\[ I = e_{ii} = e_{xx} + 0(e^4), \]
\[ II = \frac{1}{2} (e_{ij} e_{jj} - e_{ij} e_{ji}) = -e_{xy}^2 \]
\[ = -\frac{1}{4} e_{xy}^2 + 0(e^4), \]
\[ III = \text{det.}(e_{ij}) = 0. \]

Using (3.3.3) on (3.2.1), we have
\[ \rho = (1 - 2 I + 4 II - 8 III)^2 \]
\[ = 1 - e_{xy}^2 u_{x} + 0(e^4). \]

The elastic stress tensor and its components are obtained from (3.2.4b) using (3.3.3). They are:
\[ T_{ij} = \begin{bmatrix}
T_{xx} & T_{xy} & 0 \\
T_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \]
(3.3.5a)

with
\[ T_{xx} = \rho \{ U_1 (1 - 2 e_{xx}) + 2 U_2 e_{xy}^2 \}, \]
\[ T_{xy} = -\rho (2 U_1 + U_2) e_{xy}, \]
(3.3.5b)

Finally, using Equations (3.3.2), (3.3.3) and (3.3.4), the Equations (3.3.5b) reduce to
\[ T_{xx} = \epsilon^2 (1 - \epsilon^2 u_{x}^2) \{ (\lambda + 2 \mu) u_{x}^2 - (\frac{\lambda}{2} + 2 \mu + \frac{m}{4}) v_{x}^2 \} + \ldots, \]
\[ T_{xy} = (1 - \epsilon^2 u_{x}^2) \{ 2 \mu - \epsilon^2 [(m + 2 \lambda + 4 \mu) u_{x}^2 - (m + \lambda + 2 \mu + \frac{P}{2}) v_{x}^2] \}^{1/2} v_{x} + \ldots. \]

If we let
\[ T_{xx} = \epsilon^2 \frac{T_{xx}}{T_{xx}} + \ldots; \quad T_{xy} = \epsilon \frac{T_{xy}}{T_{xy}} + 3 \epsilon \frac{T_{xy}}{T_{xy}} + \ldots; \]
then

\[
\frac{1}{T_{xy}} = \mu v_x ,
\]

(3.3.6a)

\[
\frac{3}{T_{xy}} = -\frac{1}{v_x} [(3\mu + \lambda + \frac{m}{2}) u_x - (\mu + \frac{m+\lambda}{2} + \frac{\mu}{4}) v_x ^2 ],
\]

(3.3.6b)

\[
\frac{2}{T_{xx}} = (\lambda + 2\mu) \frac{2}{v_x} - \left( \frac{\lambda}{2} + 2\mu + \frac{m}{4} \right) v_x ^2 .
\]

(3.3.6c)

The viscous strain tensor, on account of (3.2.6), (3.2.7) and (3.3.2),
is given by

\[
d_{ij} = \begin{bmatrix}
d_{xx} & d_{xy} & 0 \\
d_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(3.3.7a)

with

\[
d_{xx} = \frac{\partial}{\partial t} \left( \frac{\partial u_x}{\partial x} \right) = \varepsilon^2 \left[ \frac{2}{v_x} u_{xt} - \frac{1}{2} v_x ^2 \right] + 0 (\varepsilon^4)
\]

(3.3.7b)

\[
d_{xy} = \frac{1}{2} \varepsilon \frac{1}{v_x} + 0 (\varepsilon^3)
\]

The first viscous strain invariant reduces to \( I^* = d_{kk} = d_{xx} \).

(3.3.8)

The viscous stress tensor and its components are obtained from

(3.2.4a) using (3.3.7) and (3.3.8). They are:

\[
\tau_{ij} = \begin{bmatrix}
\tau_{xx} & \tau_{xy} & 0 \\
\tau_{xy} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(3.3.9a)

with

\[
\tau_{xy} = (\lambda^* + 2\mu^*) d_{xx} = \varepsilon^2 \mu^* (\lambda + 2) \left( \frac{2}{v_x} u_{xt} - \frac{1}{v_x} v_x v_{xt} \right) + 0 (\varepsilon^4),
\]

(3.3.9b)

\[
\tau_{xy} = \varepsilon \mu^* \frac{1}{v_x} v_{xt} + 0 (\varepsilon^3)
\]
with
\[ \bar{\lambda} = \lambda^* / \mu^*. \]

Let the velocity vector be
\[ \mathbf{v}_i = [U(x, \varepsilon, \delta, t), V(x, \varepsilon, \delta, t), \omega]; \]
then on account of (3.2.6) and (3.3.1), the vector form of (3.3.10) becomes
\[ \mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + U \frac{\partial \mathbf{u}}{\partial x}, \]
so that
\[ U = \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x}, \]
or
\[ U = \frac{\partial u / \partial t}{1 - \partial u / \partial x} = \frac{\epsilon^2 \partial^2 u / \partial t + 0 (\epsilon^4)}{1 - (\epsilon^2 \partial^2 u / \partial x + 0 (\epsilon^4))} = \epsilon^2 \left( \frac{\partial^2 u / \partial t}{\partial t} + \epsilon \frac{\partial u}{\partial t} \frac{\partial u}{\partial x} \right) + \ldots, \]
and
\[ V = \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = \epsilon \frac{\partial v}{\partial t} + \epsilon^3 \left( \frac{\partial v}{\partial t} + \frac{\partial u}{\partial t} \frac{\partial v}{\partial x} \right) + \ldots. \]

Similarly, from (3.2.6) and (3.3.11), the acceleration vector yields
\[ \mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + U \frac{\partial \mathbf{v}}{\partial x}, \]
hence
\[ a_x = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial t}, \]
or
\[ a_x = \epsilon^2 \frac{\partial U}{\partial t} + 0 (\epsilon^4); \]
and
\[ a_y = \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} \]

\[ = \varepsilon \frac{1}{v_t} + \varepsilon^3 (\frac{3}{v_t} + \frac{2}{u_t} v_x), t + \varepsilon^3 (\frac{1}{u_t v_{tx}}) + O(\varepsilon^5) \]

\[ = \varepsilon \frac{1}{v_t} + \varepsilon^3 (\frac{3}{v_t} + 2 \frac{1}{v_{xt}} + \frac{2}{u_{tt} v_x}) + O(\varepsilon^5). \] \hspace{1cm} (3.3.12c)

From (3.2.2), the component forms of the equation of motion can be written as

\[ \frac{\partial^2 x}{\partial x^2} = \rho \frac{\partial}{\partial x}, \]

\[ \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial}{\partial y}. \] \hspace{1cm} (3.3.13)

Since our interest is on far-field, we introduce the following transformation:

\[ X = x - G_0 t; \quad T = \alpha(\varepsilon) t, \] \hspace{1cm} (3.3.14a)

hence

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial X}; \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial T} + \alpha \frac{\partial}{\partial T}. \] \hspace{1cm} (3.3.14b)

It is found from later work that \( \alpha = \alpha_0 \cdot \varepsilon^2 \) and \( \mu^* = \delta_0 \varepsilon^2 \) result in the most physically meaningful solution.

Using Equations (3.3.6), (3.3.9b), (3.3.4) and (3.3.12), together with this similarity hypothesis, we have a set of partial differential equations of different order of magnitudes to be solved. The terms of order \( \varepsilon \) in (3.3.13) yields

\[ \mu \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2}, \]

or on account of (3.3.14)

\[ \mu = G_0^2, \] \hspace{1cm} (3.3.15)
which is nothing but the linear solution.

The terms of order $\varepsilon^2$ in (3.3.13) lead to

$$
(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - (\lambda + 4\mu + \frac{m}{2}) \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0,
$$
or because of (3.3.14) and (3.3.15)

$$
(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial^2 u}{\partial x^2} = (\lambda + 4\mu + \frac{m}{2}) \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2}. \tag{3.3.16a}
$$

We let $\lambda', m' = \frac{\lambda}{\mu}, \frac{m}{\mu}$ and then drop the prime. Equation (3.3.16a) now reduces to

$$
(\lambda + 1) \frac{\partial^2 u}{\partial x^2} = (\lambda + 4 + \frac{m}{2}) \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2}, \tag{3.3.16b}
$$
or

$$
\frac{\partial^2 u}{\partial x^2} = k_0 \left( \frac{\partial v}{\partial x} \right)^2, \tag{3.3.17}
$$
with

$$
k_0 = \frac{m + 2\lambda + 8}{4 (\lambda + 1)}. \tag{3.3.18}
$$

It is to be noted that (3.3.18) is identical to that for shear shock [1].

Finally, the terms of order $\varepsilon^3$ in (3.3.13), together with (3.3.14) and (3.3.17), result in the following:

$$
2 \alpha_o \frac{\partial^3 V}{\partial x \partial \xi} + \beta_o \frac{\partial^3 V}{\partial x^3} = \delta_o \frac{\partial^3 V}{\partial x^3}. \tag{3.3.19}
$$

Without loss of generality we can let $\alpha_o = \delta_o = 1$; and hence

$$
2 \frac{\partial^3 v}{\partial x \partial \xi} + \beta_o \frac{\partial^3 v}{\partial x^3} = \frac{\partial^3 v}{\partial x^3}, \tag{3.3.19}
$$
with
\[ \beta_0 = 1 + \frac{m + \lambda}{2} + \frac{\rho}{4} - (3 + \lambda + \frac{m}{2}) k_0. \] (3.3.20)

The resulting equation (3.3.19) describes a nonlinear shear layer. It is to be noted that (3.3.19) has a steady solution. A few solutions of this equation will be discussed in Art. 3.4.

B. Cylindrical shear waves

Here we consider the same medium as in Art. 3.3.A, unstrained and at rest, but is subjected to a weak cylindrical shear disturbance. A physical example is a hole, in an infinite medium, subjected to a torsion by some means.

The derivation of the governing equation is quite analogous to that of Art. 3.3.A. The important difference is in the extension of Burgers' equation to more than one dimension. The wave front here is a cylindrical one; thus the coordinate system introduced later does not form a Galilean frame.

Because the propagation has cylindrical symmetry, it is reasonable to study the problem in the cylindrical polar coordinate system.

Let the displacement vector be
\[ u_i = [u(r, \theta, \phi, t), v(r, \theta, \phi, t), 0]. \] (3.3.21)

Then the finite strain tensor in cylindrical polar coordinate system take the following form (see Appendix 8.2):
\[
e_{ij} = \begin{bmatrix}
e_{rr} & e_{r\theta} & 0 \\
e_{\theta r} & e_{\theta\theta} & 0 \\
0 & 0 & 0
\end{bmatrix}, \] (3.3.22a)
with
\[
e_{rr} = u_r - \frac{1}{2} (u_r^2 + v_r^2),
\]
\[
e_{\theta\theta} = \frac{1}{r} \frac{1}{2} u - \frac{1}{2r^2} (u^2 + v^2),
\]
\[
e_{r\theta} = \frac{1}{2} (v_r - \frac{1}{r} v) - \frac{1}{2r} (uv_r - vv_r).
\]

Applying the amplitude expansions (3.2.7), (3.3.22b) becomes
\[
e_{r\theta} = \varepsilon_1 e_{r\theta} + \varepsilon_3^3 e_{r\theta} + ... ,
\]
\[
e_{rr} = \varepsilon_2^2 e_{rr} + ... ,
\]
\[
e_{\theta\theta} = \varepsilon_2^2 e_{\theta\theta} + ... ,
\]

with
\[
\varepsilon_1 = \frac{1}{2} (v_r - \frac{1}{r} v),
\]
\[
\varepsilon_3 = \frac{1}{2} \frac{1}{r} \frac{1}{3} v - \frac{1}{r} (u v_r - v u_r),
\]
\[
\varepsilon_2 = \frac{2}{r} - \frac{1}{2} \frac{1}{2} v_r,
\]
\[
\varepsilon_2 = \frac{1}{r} \frac{2}{r} - \frac{1}{2r^2} \frac{1}{2} v .
\]

The elastic strain invariants reduce to
\[
I = e_{ii} = e_{rr} + e_{\theta\theta},
\]
\[
II = 1/2 (e_{ii} e_{jj} - e_{ij} e_{ji}) = e_{rr} e_{\theta\theta} - e_{r\theta}^2 ,
\]
\[
III = \text{det.}(e_{ij}) = 0 (e^4).
\]

Using (3.3.23) on (3.2.1), we find
\[
\rho = 1 - \varepsilon_2^2 (\frac{2}{r} + \frac{2}{r} - \frac{11}{r} + 0 (e^4)).
\]
The elastic stress tensor and its components are obtained from (3.2.4b) using (3.3.22), (3.3.23) and (3.3.24). They are

\[ T_{ij} = \begin{bmatrix} T_{rr} & T_{r\theta} & 0 \\ T_{r\theta} & T_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix} \]  

(3.3.25a)

with

\[ T_{rr} = \epsilon^2 T_{rr}^0 + O(\epsilon^4), \]

\[ T_{r\theta} = \epsilon T_{r\theta}^0 + \epsilon^2 T_{r\theta}^1 + O(\epsilon^5), \]  

(3.3.25b)

\[ T_{\theta\theta} = \epsilon^2 T_{\theta\theta} + O(\epsilon^4), \]

and

\[ T_{r\theta}^1 = \mu \left( \frac{1}{r} \nabla - \frac{1}{r} \nabla \right), \]

\[ T_{r\theta}^3 = -\left( \frac{1}{r} \nabla - \frac{1}{r} \nabla \right) \left[ (3\mu + \lambda + \frac{m}{2}) \left( \frac{\partial u_r}{r} + \frac{\partial u_r}{r} \right) - (\mu + m + p) \frac{1}{r} \nabla \right] \]

\[ - (\mu + \frac{m + \lambda}{2} + \frac{\mu}{4}) \left( \frac{1}{r} \nabla \right)^2 \left( \sum_{j=1}^{2} \nabla \right) \]  

(3.3.25c)

\[ T_{rr}^2 = (\lambda + 2\mu) \frac{\partial u_r}{r} + \lambda \frac{\partial u_r}{r} - \frac{1}{r} \nabla \left( \frac{\lambda}{2} + 2\mu + \frac{m}{4} \right) - \frac{1}{r^2} \left( \frac{\lambda}{2} + \frac{m}{4} + \mu \right) \]

\[ + \frac{1}{r} \nabla \left( \frac{m}{2} + 2\mu \right), \]

\[ T_{\theta\theta}^2 = \lambda u_r + (\lambda + 2\mu) \frac{\partial u_r}{r} - \frac{1}{r} \nabla \left( \frac{\lambda}{2} + 2\mu + \frac{m}{4} \right) \]

\[ - \frac{1}{r} \nabla \left( \frac{\lambda}{2} + \frac{m}{4} + \mu \right) + \frac{1}{r} \nabla \left( \frac{m}{2} + 2\mu \right). \]
The viscous strain tensor, on account of \((3.2.6), (3.2.7)\) and \((3.3.22)\), is given by

\[
\mathbf{d}_{ij} = \begin{bmatrix}
\frac{d_{rr}}{d_{\vartheta \vartheta}} & \frac{d_{r\vartheta}}{d_{\vartheta r}} & 0 \\
\frac{d_{r\vartheta}}{d_{\vartheta r}} & \frac{d_{\vartheta \vartheta}}{d_{\vartheta r}} & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad (3.3.26a)
\]

with

\[
d_{rr} = e^2 \left( \frac{\frac{1}{r}}{v_{tr}} - \frac{1}{r} \frac{v_{t r}}{v_{r t}} - \frac{1}{r^2} \frac{v_{t t}}{v_{r t}} + \frac{1}{2} \frac{1}{r^2} \right) + 0(e^4),
\]

\[
d_{\vartheta \vartheta} = e^2 \left( \frac{\frac{2}{r}}{v_{tr}} - \frac{1}{r^2} \frac{v_{t t}}{v_{r t}} \right) + 0(e^4), \quad (3.3.26b)
\]

\[
d_{r\vartheta} = \frac{1}{2} e \left( \frac{\frac{1}{v_{tr}}}{r} + 1/2 \frac{e^3}{v_{t t}} \right) + 1/2 e \left[ \frac{\frac{1}{v_{tr}}}{r} (\vartheta - \frac{1}{v_{t r}}) + \frac{1}{v_{r t}} \right] + 0(e^5).
\]

The first viscous strain invariant reduces to

\[
\mathbf{I} = d_{kk} = d_{rr}. \quad (3.3.27)
\]

The viscous stress tensor and its components are obtained from \((3.2.4a)\) using \((3.3.26)\) and \((3.3.27)\). They are:

\[
\mathbf{T}_{ij} = \begin{bmatrix}
T_{rr} & T_{r\vartheta} & 0 \\
T_{r\vartheta} & T_{\vartheta \vartheta} & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad (3.3.28a)
\]

with
\[ \tau_{rr} = (\lambda^* + 2\mu^*) \, d_{rr} + \lambda^* \, d_{\theta\theta} \]
\[ \quad = \epsilon^2 \mu^* \left[ (\lambda + 2) \left( \frac{1}{r} u_t - \frac{1}{r^2} v_t \right) - \frac{1}{r^2} v_t \right] + \left( \frac{1}{r} \right) \frac{1}{r} v_t - 2 v_{tr} \right) \]
\[ + \frac{\lambda}{(u_t - \frac{1}{r^2} v_t - \frac{1}{r^2} v_t^2) + 0 (\epsilon^4) ,} \]
\[ \tau_{\theta\theta} = (\lambda^* + 2\mu) \, d_{\theta\theta} + \lambda^* \, d_{rr} \quad \text{(3.3.28b)} \]
\[ \quad = \epsilon^2 \mu^* \left[ (\lambda + 2) \left( \frac{1}{r} u_t - \frac{1}{r^2} v_t \right) - \frac{1}{r^2} v_t \right] + \left( \frac{1}{r} \right) \frac{1}{r} v_t - 2 v_{tr} \right) \]
\[ - \frac{1}{2} v_{tr}^2 \right] + 0 (\epsilon^4) ,} \]
\[ \tau_{r\theta} = \epsilon^2 \mu^* \left( \frac{1}{r} \frac{1}{r} v_t \right) + \epsilon^2 \mu^* \left[ \frac{1}{r} v_t \left( u_t - v_t \right) + \frac{1}{r} v_t \right] + 0 (\epsilon^5) ,} \]
\[ \text{with} \]
\[ \bar{\lambda} = \frac{\lambda}{\mu} . \]

Since we deal with cylindrical polar coordinate system, mathematics is heavier as we proceed to obtain the velocities and accelerations; but the procedure is the same as that in Art. 3.3A. Therefore, following the same line as is done in Art. 3.3.A, we find the following accelerations (see Appendix 8.2):
\[ a_r = A = \epsilon^2 \left( \frac{2}{A} \right) + 0 (\epsilon^4) , \]
\[ a_\theta = B = \epsilon + \epsilon^3 B + 0 (\epsilon^5) , \quad \text{(3.3.29)} \]
The component forms of the equations of motion in the r and \( \theta \) directions for this problem are given by

\[
\frac{\partial r}{\partial r} + \frac{t r - t \theta}{r} = \rho a_r ,
\]

(3.3.31)

\[
\frac{\partial r}{\partial r} + \frac{2}{r} t \theta = \rho a_\theta .
\]

Since our interest again is on far-field, we introduce the following transformation:

\[
R = r - G_0 t \quad \text{and} \quad T = \alpha (\varepsilon) t ,
\]

(3.3.32a)

hence

\[
\frac{\partial r}{\partial R} = \frac{\partial r}{\partial R} ; \quad \frac{\partial r}{\partial t} = - G_0 \frac{\partial r}{\partial R} + \alpha \frac{\partial r}{\partial R} ; \quad \frac{1}{r} = \frac{\alpha}{c_0 t} + o (\alpha^2) .
\]

(3.3.32b)

Again we use the following similarity hypothesis

\[
\alpha = \alpha_0 \varepsilon^2 ; \quad \mu^* = \delta_0 \varepsilon^2 .
\]

(3.3.33)

Using Equations (3.3.25), (3.3.28b), (3.3.24), (3.3.30) and (3.3.33), we have a set of differential equations. The terms of order \( \varepsilon \) in (3.3.31) yields

\[
\mu \frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 v}{\partial t^2} ,
\]
or on account of (3.3.32b)
\[ \mu = G_o \frac{2}{r}, \quad (3.3.34) \]
which is the linear solution.

The terms of order \( e^2 \) in (3.3.31) lead to
\[ (\lambda + 2\mu) \frac{2}{u_{rr}} - \frac{1}{r} \frac{1}{v_{rr}} (4\mu + \lambda + \frac{m}{2}) = \frac{2}{u_{tt}} - \frac{1}{r} \frac{2}{v}, \quad (3.3.35a) \]
or because of (3.3.32b) and (3.3.34)
\[ (\lambda + \mu) \frac{\partial^2 u}{\partial r^2} = (\lambda + 4\mu + \frac{m}{2}) \frac{\partial v}{\partial r} \frac{\partial^2 v}{\partial r^2}. \quad (3.3.35b) \]

We let \( \lambda', m' = \frac{\lambda}{\mu'}, \frac{m}{\mu} \) and then drop the prime. Equation (3.3.35b) now reduces to
\[ (\lambda + 1) \frac{\partial^2 u}{\partial r^2} = (\lambda + 4 + \frac{m}{2}) \frac{\partial^2 v}{\partial r^2}, \quad (3.3.36) \]
or
\[ \frac{\partial^2 u}{\partial r^2} = k_o \left( \frac{\partial^2 v}{\partial r^2} \right)^2. \quad (3.3.37) \]

It is to be noted that (3.3.37) is identical to that for the planar waves (3.3.18).

Finally, the terms of order \( e^3 \) in (3.3.31), together with (3.3.32) and (3.3.36), result in the following:
\[ \alpha_o \left( 2 \frac{\partial^2 v}{\partial r \partial t} + \frac{1}{r} \frac{\partial v}{\partial r} \right) + \frac{\partial}{\partial r} \left( \frac{\partial^3 v}{\partial r^3} \right)^3 = \delta_o \frac{\partial^3 v}{\partial r^3}. \quad (3.3.38) \]

Without loss of generality we can let \( \alpha_o = \delta_o = 1 \); and hence
with

\[ \beta_0 = 1 + \frac{m+\lambda}{2} + \frac{D}{4} - (3 + \lambda + \frac{m}{2}) k_0, \]  

which is identical to that for planar waves (3.3.20).

The final result given by Equation (3.3.39) describes cylindrical shear waves. It is the generalization of (3.3.19). It is obvious that since \( \alpha_0 \) multiplies the first two terms of (3.3.38), they are both retained or they both drop out. The first describes the unsteadiness while the second gives the effect of curvature. It is clear that (3.3.39) has no steady solution because of the presence of curvature term. Again a few solutions of (3.3.39) will be solved in the next article.

3.4 Discussion of the Results

Considered is the traveling shear wave in an infinite medium. This wave, which is a pure shear wave only in a linear theory, is accompanied by a longitudinal component in a nonlinear theory. The use of correct amplitude expansions, based on shock relations, leads to a nonlinear partial differential equation. The equation reveals the existence of shock-layer for small-but-finite shear waves, which earlier studies fail to predict. The failure of the latter (which is based on singular surface theory) indicates a wide gap in the evolutionary interpretation of shock formation.

The resulting equation, (3.3.39), in this chapter (the analog of
Burgers' equation in gasdynamics) is new; it brings out the role of interaction of dissipation, nonlinearity, and unsteadiness of the shear wave in the vicinity of the wave front.

The nonlinear term, which is a derivative of a square in the case of longitudinal wave [2], is now the derivative of a cube. The requirement of the balance of nonlinearity, viscosity and unsteadiness, leading to such an equation, shows that such a balance is now achieved only after a longer time at a greater distance. If one now considers an infinite plate, dispersion is expected but this turns out to be of smaller order.

A few solutions of Burgers' equation are discussed, on the lines of earlier studies, in what follows.

In the case of planar wave, (3.3.19) becomes Burgers' equation

\[
\frac{\partial e}{\partial t} + \rho_0 \frac{\partial e^3}{\partial x} = \frac{\partial^2 e}{\partial x^2}
\]  

where \( e = \frac{1}{\partial v / \partial x} \) is the fundamental mode of shear waves.

The solution of (3.4.1) is sought in the form

\[
e = f (X - e_0 T) = f (\xi);
\]

then

\[
\frac{df}{d\xi} = f' = \frac{df}{d(X - e_0 T)},
\]

\[
e_x = \frac{\partial e}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial f}{\partial \xi} = f',
\]

\[
e_T = \frac{\partial e}{\partial T} = \frac{\partial f}{\partial T} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial T} = -e_0 f',
\]
hence (3.4.1) becomes
\[-e_0 f' + \beta_0 \frac{\partial}{\partial \xi} f^3 = f'' ,\]
or
\[-e_0 f + \beta_0 f^3 = f' ,\]
assuming zero integration constant.

Let
\[ f = \frac{1}{g} , \]
then
\[- \frac{1}{g} \frac{dg}{d\xi} = \frac{\beta_0}{g^3} - \frac{e_0}{g} , \]
or
\[ \frac{2egdg}{g^2} = 2e_0 d\xi , \]
so that
\[ 2e_0 \xi = \ln \left| g^2 - \frac{\beta_0}{e_0} \right| = \ln \left| \frac{1}{f^2} - \frac{\beta_0}{e_0} \right| , \]
or
\[ \frac{1}{f^2} = \frac{\beta_0}{e_0} + \exp (2e_0 \xi) . \quad (3.4.3) \]

This is the analog of Taylor's solution describing the effect of viscosity on a weak shock. Thus, \[ e \to 0 \quad \text{as} \quad \xi \to \infty \] (far ahead of the wave front) and \[ e \to \sqrt{e_0/\beta_0} \] as \[ \xi \to -\infty \] (far behind the wave front).

For cylindrical shear waves, (3.3.39) becomes the generalized Burgers' equation.
To solve this equation, we let
\[ e = T^{-\frac{(m+n)}{2}} f_m(\xi), \]  
where \( \xi \) is a similarity variable as shown below
\[ \xi = \frac{R}{2T^{1/2}}, \]

hence
\[
\frac{\partial}{\partial R} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial R} = \frac{1}{2T^{1/2}} \frac{\partial}{\partial \xi},
\]
\[
\frac{\partial}{\partial T} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial T} = -\frac{\xi}{2T} \frac{\partial}{\partial \xi}.
\]

With this transformation, (3.4.4) yields

\[
\frac{d^2f_m}{d\xi^2} = -2mf_m - 2\xi \frac{df_m}{d\xi} + 2f_m^2 \frac{df_m}{d\xi}, \quad m = 1/2 - n. \]  

It does not appear to be possible to obtain solutions of this last equation.

Lastly, for the linearized Burgers' equation, (3.4.7) take the form

\[
\frac{d^2g}{d\xi^2} = -2(\xi \frac{dg}{d\xi} + mg), \]  

where
\[
g = \frac{d^m f_0}{d\xi^m}. \]

The solution of (3.4.8) is
\[ f_0(\xi) = \text{Berfc}(\xi), \quad (3.4.9) \]

where \( B = -\frac{\sqrt{\pi}}{2} C \), \( C \) is an integration constant.

The nature of such linear dissipative waves can be seen from Fig. 3.1. Clearly \( m = 0 \) corresponds to Heaviside type initial conditions and \( m > 0 \) to more singular distributions. These generalize Lighthill's law [40] of decay for dissipative cases.

Finally, as stretched time \( T = \alpha(\varepsilon) t \) is introduced in the derivation, this \( \alpha(\varepsilon) \) multiplies both the first two terms of (3.4.4); the first describes unsteadiness and the second one represents curvature; the assumption \( \alpha(\varepsilon) = O(\varepsilon^2) \), as is done here, retains both these, while if \( \alpha(\varepsilon) = o(\varepsilon^2) \), both drop out, in both cases, no Taylor-type solution exists (except for planar wave).

The limitation of this approach is that it does not apply to memory type (integral) viscoelastic material when real characteristics exist.
Figure 3.1 Graph of $\text{erfc}(\xi)$ indicating the nature of dissipation
4. FORCED TRANSVERSE OSCILLATIONS - NONLINEAR THEORY

4.1. Introduction

Nonlinear oscillatory motion governed by partial differential equations have received less attention until 1966 although Carrier [41] studied the string and Stoker [42] studied a special nonlinear wave equation. Perturbation method, well known for ordinary differential equation, has been first applied to partial differential equations for water waves by Tadjbakhsh and Keller [43]. The existence of periodic solutions of certain nonlinear partial differential equations has been proved by Cesari [44] and Rabinowitz [45]. Then in 1966 Keller and Ting gave a perturbation method for calculating finite amplitude free oscillations of nonlinear wave equations and applied it to a variety of problems including the transverse oscillations of an elastic string and circular membrane. Since then a few studies of nonlinear wave equations have appeared [46 - 48]. All of these are based on perturbation method.

In 1970 Collins [3] showed that, for certain one-dimensional forced oscillation - problems for nonlinear wave equations, solutions appear to exist at all frequencies when the amplitudes of the oscillations are small. Support for his result comes from the work of Chester [49] on forced oscillations in a closed gas-filled tube produced by the oscillations of a piston at one end, a problem for which the method of Keller and Ting breaks down. The perturbation procedure he used is essentially an extension of that of Keller and Ting, which has been extended to other
problems by Millman and Keller [46].

If stress can be expressed in terms of strain as follows

\[ t(v_x) = av_x + bv_x^2 + cv_x^3 + \ldots, \]

where \( a, b, c \) are constant, Collins showed that shock waves exist if \( b \neq 0 \) (longitudinal waves) whereas shock waves do not exist if \( b = 0 \), \( c \neq 0 \) (shear waves); of course, \( a \neq 0 \) in both cases is assumed to be a linear one. Furthermore, these considerations hold at the resonant frequency of the linear theory.

The essential feature of the present method is to solve three nonlinear wave equations instead of one nonlinear wave equation as in the case of one-dimensional problem. The procedure, in line with earlier work with a few modifications, involves expanding the unknown frequency in powers of amplitude. Once this is done the usual perturbation method leads to a set of boundary value problems. The first of these yields a homogeneous equation; but the boundary conditions are nonhomogeneous. The rest are nonhomogeneous equations subjected to homogeneous boundary conditions. The solvability conditions (orthogonality conditions) for the last of these leads to the solution which brings in a very important feature, i.e., the effect of the longitudinal strain of higher power in amplitude \( \epsilon \) in the solution which is overlooked by earlier studies.

Equations governing this wave are also of a novel type, not appearing in the literature thus far.

A limitation of the result is that it is only valid for problems with its forcing frequencies away from the natural frequency of the system.
4.2. Basic Equations and Formulation of the Problem

We here consider the physical problem of the oscillations of an isotropic elastic layer of infinite extent in the y and z directions and extending between \( x = 0 \) and \( x = L_0 \pi \) in the x direction. Such a layer is subjected to a forced oscillation in the y direction at \( x = L_0 \pi \) and fixed at \( x = 0 \).

It is well known that pure shear waves are not possible; it is always accompanied by a displacement in the x direction.

All lengths are measured in units of the standard length \( L_0 \). Elastic moduli in units of \( \mu \), the shear modulus of the linear theory; and density in terms of the initial density \( \rho_0 \). From these we get a standard velocity \( c_0 \) \( (\sqrt{\mu/\rho}) \). Time is measured in unit of \( L_0/c_0 \). Then the basic equations are

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},
\]

\[
\rho \frac{\partial^2 \nu}{\partial t^2} = \frac{\partial^2 \nu}{\partial y^2},
\]

with symbols with obvious meanings.

However, it should be stressed these are all nondimensional.

The expression for stress, acceleration and density in terms of the displacement vector

\[
u = [u(x,t), \nu(x,t), 0]
\]

will be presently noted only to the needed order. These equations are to be solved subjected to the conditions

\[
u(0,t) = 0, \nu(\pi,t) = \epsilon A_1 \cos \omega t;
\]

\[
u(x,t + \frac{2\pi}{\omega}) = \nu(x,t),
\]

(4.2.1)
where $\epsilon$ is a small parameter characterizing the amplitude, $A_j$ is a constant, and $\omega$ is the frequency of a forcing function.

It is important in the approach developed by Keller to regard $\omega$ also as subjected to perturbation. To this end, time is again transformed as $t' = t\omega$ and drop the prime for convenience. Then Equations (4.2.1) yield

$$
\omega^2 \frac{\partial^2 u}{\partial t^2} - \omega^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial T_{xx}}{\partial x} + \cdots,
$$

$$
\omega^2 \frac{\partial^2 v}{\partial t^2} - \omega^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial T_{xy}}{\partial x} + Ay,
$$

where

$$
\eta^2 = \frac{\lambda + 2\mu}{\mu};
$$

$$
T_{xx} = B \frac{\partial^2 v}{\partial x^2}, \quad B = -(\lambda + 4 + m); \quad (4.2.6)
$$

$$
T_{xy} = \frac{\partial}{\partial x} (\alpha v_x u_x - \beta (v_x)^2), \quad \alpha = 2\mu + \lambda + \frac{m}{2}, \quad \beta = \mu + \frac{m + \lambda}{2} + p;
$$

$$
Ay = v_x u_{tt} + 2u_{t}v_x t_{tt} - u_x v_{tt}.
$$

4.3. Forced Transverse Oscillations

We assume amplitude and frequency expansions of the form

$$
v = \epsilon v + \epsilon^3 v + \cdots,
$$

$$
u = \epsilon^2 u + \cdots, \quad (4.3.1)
$$

$$
\omega^2 = \omega_0^2 + \epsilon^2 \omega_1^2 + \cdots,
$$

where $\omega_0$ is the frequency away from the natural frequency of the linear problem. The form of these are based on earlier study [1]. Another support comes from Collins' work (see Art. 4.1).
Then we can write the system to be

\[ \frac{d^2 v}{dt^2} = \frac{1}{\omega^2} \frac{d^2 v}{dx^2}, \quad 0 < x < \pi, \quad 0 < t < \infty, \]
\[ v(0,t) = 0, \quad v(\pi,t) = A_1 \cos t, \quad (4.3.2) \]
\[ v(x, t + 2\pi) = v(x,t); \]

and

\[ \frac{d^2 u}{dt^2} - \frac{d^2 u}{dx^2} = B(v_x)_x, \quad 0 < x < \pi, \quad 0 < t < \infty, \]
\[ u(0,t) = 2u(\pi,t) = 0, \quad (4.3.3) \]
\[ u(x, t + 2\pi) = u(x,t); \]

and

\[ \frac{d^3 v}{dt^3} - \frac{d^3 v}{dx^3} = F, \quad 0 < x < \pi, \quad 0 < t < \infty, \]
\[ v(0,t) = 3v(\pi,t) = 0, \quad (4.3.4a) \]
\[ v(x, t + 2\pi) = v(x,t), \quad (4.3.4c) \]

where

\[ F = \omega^2 \frac{d^2 v}{dt^2} + \alpha (v_x)_x - \beta (v_x)_x + \omega^2 \frac{d^2 v}{dx^2} + 2 \omega \frac{d^2 v}{dx dt} - \omega^2 \frac{d^2 v}{dx dt}. \]

The general solution of (4.3.2) is

\[ v = \frac{A_1}{\sin \omega \pi} \sin \omega \pi \cos t \]
\[ = A \sin \omega \pi \cos t. \quad (4.3.5) \]

Then

\[ (v_x)_x = - \frac{\omega^3}{2} \sin 2 \omega \pi (1 + \cos 2 t). \quad (4.3.6) \]

Using (4.3.6) in (4.3.3) and solving for \( u \), we have

\[ u = - \frac{A^2 B \omega}{8} \frac{\sin 2 \omega \pi}{\eta^2} - \frac{1}{1 - \eta^2} \sin 2 \omega \pi \cos 2 t. \quad (4.3.7) \]
It must be remembered here that the attention is in the shear oscillation, thus only that part of the solution for \( u \) which vanishes with \( v \) is of significance. So for solution of \( u \) we have rejected the complementary function.

The linear differential operator on the left side of (4.3.4) is the same differential operator which occurs in (4.3.2); hence (4.3.4) is an inhomogeneous form of (4.3.2) and will have a solution only if an appropriate solvability condition is satisfied. The necessity of this condition is shown by multiplying both sides of (4.3.4) by \( v(x,t) \) and integrate with respect to \( x \) from \( 0 \) to \( \pi \) and with respect to \( t \) from \( 0 \) to \( 2\pi \), so that

\[
\int_0^{2\pi} dt \int_0^\pi dx \left( \frac{1}{\omega_0} \dddot{v} - \frac{3}{\nu} \dddot{f} \right) = \int_0^{2\pi} dt \int_0^\pi dx vF. \tag{4.3.8}
\]

Integrating by parts once with respect to \( t \) and \( x \) respectively, we find the integral of the left side of (4.3.8) gives

\[
\omega_0^2 \left\{ \int_0^\pi \frac{3}{\nu} \frac{2}{\nu} \frac{1}{\nu} \int_0^\pi dx - \int_0^\pi dx \int_0^\pi \frac{3}{\nu} \frac{1}{\nu} \frac{1}{\nu} \int_0^\pi dt \right\}
\]

\[
- \int_0^{2\pi} \left[ \frac{3}{\nu} \frac{1}{\nu} \frac{1}{\nu} \int_0^\pi dx \right] dt + \int_0^{2\pi} dt \int_0^\pi \frac{3}{\nu} \frac{1}{\nu} \frac{1}{\nu} dx.
\]

Integrating by parts once more, and rearranging the terms, we obtain

\[
\int_0^{2\pi} \int_0^\pi \left( \omega_0^2 \frac{2}{\nu} \frac{1}{\nu} - \frac{1}{\nu} \frac{1}{\nu} \frac{1}{\nu} \right) dx dt + \int_0^{2\pi} \omega_0^2 \left[ \frac{3}{\nu} \frac{1}{\nu} - \frac{3}{\nu} \frac{1}{\nu} \right] \int_0^\pi dx
\]

\[
+ \int_0^{2\pi} \left[ \frac{3}{\nu} \frac{1}{\nu} - \frac{3}{\nu} \frac{1}{\nu} \right] dt.
\]
In the above expression, the first term vanishes due to (4.3.2); the second and third terms vanish because of (4.3.4b) and (4.3.4c). Therefore, the left side of (4.3.8) vanishes.

By making use of (4.3.5), the right side of (4.3.8) yields

\[
\int_0^{2\pi} \int_0^{\pi} \frac{1}{r} f(x,t) \, dx \, dt - \int_0^{2\pi} \int_0^{\pi} A^2 \omega_x^2 \sin^2 \omega \cos^2 t \, dx \, dt = \int_0^{2\pi} \int_0^{\pi} A \sin \omega \cos t f(x,t,\omega) \, dx \rightarrow \leftarrow 0.2222^2 \quad \frac{A^2 \omega_x^2 \pi^2}{2},
\]

where \( f = F - \frac{1}{\omega_x^2} v_{tt} \)

\[
= \alpha \left( \frac{1}{2} v_x \right)_x - \beta \left( \frac{1}{2} v_x \right)_x^2 + \omega \left( \frac{1}{2} v_{xt} + \frac{1}{2} v_{tt} - \frac{1}{2} v_{xt} \right). \quad (4.3.10)
\]

Thus, the condition of \( \omega_2 \) such that \( \dot{v} \) has a solution is found to be

\[
\frac{A^2 \pi^2}{2} \omega_2^2 = \int_0^{2\pi} \int_0^{\pi} \sin \omega \cos t f(x,t,\omega) \, dx. \quad (4.3.11)
\]

From which and (4.3.1), we find (see Appendix 8.3.)

\[
\omega^2(\varepsilon) = \omega_0^2 + \varepsilon^2 \frac{A^3 \omega_0^3}{32\pi} \left\{ \frac{8(2 - 3n^2)}{\pi^2 (1 - n^2)} \left[ \omega \pi (\alpha - 1) + (\alpha + 1) \sin 2\omega \pi 
\right.ight.
\]
\[
\left. - \frac{1}{4} (3\alpha + 1) \sin 4\omega \pi \right] \left. - \frac{9\beta}{2} (\sin 2\omega \pi - \frac{1}{2} \sin 4\omega \pi) \right\}
\]
\[
+ O(\varepsilon^4), \quad (4.3.12)
\]

with \( A, \eta, B, \alpha \) and \( \beta \) defined in (4.2.3), (4.3.5) and (4.2.6), respectively.

Next we insert (4.3.12) into the right side of (4.3.4) and solve for \( v \).
Finally, collecting the results and reintroducing the original nondimensional variable $t$, we have (see Appendix 8.3.)

$$v = \varepsilon A \sin \omega_o \pi \sin \omega_o \times \cos \omega t + \varepsilon^3 (\gamma_1 \sin \omega_o \times \cos \omega t$$

$$+ \gamma_2 \sin \omega_o \times \cos 3 \omega t + \gamma_3 \sin 3 \omega_o \times \cos \omega t + \gamma_4 \sin 3 \omega_o$$

$$\times \cos 3 \omega t) + O(\varepsilon^5), \quad (4.3.13)$$

where

$$\gamma_1 = - \frac{A \omega_2^2}{8} \frac{A^3 \omega_o^4 B(\alpha + 1)}{1 + \frac{1}{\eta^2} + \frac{1}{2(1 - \eta^2)}},$$

$$\gamma_2 = - \frac{A^3 \omega_o^4 B (5 + \alpha)}{128 (1 - \eta^2)},$$

$$\gamma_3 = \frac{A^3 \omega_o^4 B}{64} (3\alpha - 1) \left[ \frac{1}{\eta^2} - \frac{1}{2(1 - \eta^2)} \right] - \frac{98}{2B},$$

$$\gamma_4 = \frac{A^3 \omega_o^4 B}{16} \left( \frac{3\alpha - 5}{1 - \eta^2} \right) \left( \frac{3\beta}{B} \right),$$

with $A$, $\eta$, $B$, $\alpha$ and $\beta$ again defined in (4.2.3), (4.3.5), and (4.2.6), respectively.

Equations (4.3.12) and (4.3.13) are the solutions of the problem.

4.4. Discussion of the Results

Considered is forced oscillatory shear waves for a single elastic layer.

The resulting Equation (4.3.12) is new. It shows that frequency of these oscillations depends on the amplitude ($\varepsilon A_1$).
In the solution (4.3.13), also not appearing in the literature, the effect of longitudinal amplitude, $u$, on transverse amplitude, $v$, is well illustrated by the presence of $\eta$, square root of the ratio of dilatational wave speed to shear wave speed.

In the course of calculation, higher order terms can be retained but then energy equation comes into play.

A limitation of the result is that the solution is only valid for problems with its forcing frequencies away from the natural frequency of the system.

The layer considered is assumed to be finite in extent in one direction and infinite in the other two directions. Therefore, specifically, these are shear oscillations in an infinite plate, the shear motion being parallel to the plane faces of the plate.

However, if as indicated by a detailed study (see Art. 3.4.), we assume dispersion is negligible to a high order, it seems reasonable to assume that this describes the torsional oscillations of a finite circular cylinder and transverse oscillations of a plate strip sheared parallel to the parallel edges.
5. REVIEW AND PROSPECTS

All studies investigated deal with shear waves. We started with some problems of two-layered isotropic bounded media for which dispersion vanishes in the single-layered case. An asymptotic expansion leads to the Jeffery wave equation which brings out the dispersive nature of these waves if wave speeds of the two media are not equal. Self-similar solution which has the nature of Airy's function is obtained.

Next, shear wave in an infinite viscoelastic medium is studied. Using an amplitude expansion based on shock wave studies, a nonlinear Burgers' equation is derived. This indicates the existence of shear shock layer. Furthermore, this illustrates that the nonlinear effects come into operation at a much larger time and at a greater distance from the source than that for longitudinal waves. For planar waves, a steady solution exists. Self-similar solution for linearized Burgers' equation exhibits the nature of the waves in the vicinity of the wave front.

Finally, an interesting property of forced nonlinear oscillations for a single elastic layer is studied. The result indicates the frequency of these oscillations depends on the amplitude. The solution is also obtained.

All solutions in the first two cases are far-field ones. The complete solution, of course, is obtained by matching the interior solutions to the boundary layer solution. This is not done here. Similarly, in the last case one strictly has an initial value problem. Solution of such a problem is not discussed here. Such a study should be most interesting
especially in the case of resonant oscillations.

The importance of the present study is two-fold: (1) it reveals the power and straight-forwardness of the method of asymptotic expansions even in some linear problems (see Art. 2.5.); (2) it provides the method of analysis for wave propagation in an isotropic medium.

To conclude, some related problems of interest are: (1) the matching problem as stated above; (2) a problem to bring out the dispersive nature of a composite material of equal to or more than three layers; (3) a problem to derive the generalized B.K.D.V. equation for an infinite viscous medium by taking higher order terms in asymptotic expansions into consideration; (4) an application of the method to more general continua; (5) resonant shear oscillations.


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8. APPENDIX

8.1. Derivation of Equation (2.5.13)

The power series representation of a Bessel function of the first kind of order \( n \), and the modified Bessel functions of the first and second kind of order \( n \) are [16].

\[
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+n}}{k! (n+k)!},
\]

\[
I_n(x) = \left( \frac{1}{2} x \right)^n \sum_{k=0}^{\infty} \frac{\left( \frac{1}{4} x^2 \right)^k}{k! (n+k)!},
\]

\[
K_n(z) = \frac{1}{2} \left( \frac{1}{2} z \right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left( -\frac{1}{4} z^2 \right)^k + (-1)^{n+1} \ln \left( \frac{1}{2} z \right) I_n(z)
\]

\[+ (-1)^n \frac{1}{2} \left( \frac{1}{2} z \right)^n \sum_{k=0}^{\infty} \left\{ \psi(k+1) + \psi(n+k+1) \right\} \frac{\left( \frac{1}{4} z^2 \right)^k}{k! (n+k)!},\]

where

\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \text{ is a Psi function,}
\]

\[
\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1},
\]

with \( \Gamma \) as a gamma function and

\[
\gamma = \lim_{m \to \infty} \left[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} - \ln m \right] = 0.57721\ldots.
\]

Thus

\[
J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots,
\]
\[ J_1(x) = \frac{x}{2} \left( 1 - \frac{x^2}{8} + \frac{x^4}{192} - \cdots \right) = \frac{x}{2} \left[ 1 - \left( \frac{x^2}{8} - \frac{x^4}{192} - \cdots \right) \right], \]

\[ I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \cdots, \]

\[ I_1(x) = \frac{x}{2} \left( 1 + \frac{x^2}{8} + \frac{x^4}{192} + \cdots \right). \]

By using binomial expansions

\[ \frac{1}{1 - x} = 1 + x + x^2 + \cdots; \quad \frac{1}{1 + x} = 1 - x + x^2 \cdots, \]

the above expressions yield

\[ \frac{1}{J_1(x)} = \frac{2}{x} \left( 1 + \frac{x^2}{8} + \frac{x^4}{96} \right), \quad \frac{1}{I_1(x)} = \frac{2}{x} \left( 1 - \frac{x^2}{8} + \frac{x^4}{96} \right), \]

so that (2.5.8a) through (2.5.8c) result in the following:

\[ A_1 = -\frac{x^2}{4} \left( 1 + \frac{x^2}{24} + \cdots \right), \]

\[ A_2 = \frac{x^2}{4} \left( 1 - \frac{x^2}{24} + \cdots \right), \]

\[ A_3 = \frac{y^2}{4x} \left( 1 + \frac{y^2}{12} - \frac{x^2}{8} + \cdots \right). \]

Note here that, each of \( A_1, A_2 \) and \( A_3 \) has the form \( \alpha_0 k^2 + \alpha_1 k^4 \) as \( k \to 0 \) and so \( A_1 A_3 \) has the form \( k^4(1 + 0(k^2)) \).

Next,

\[ K_1(x) = \frac{1}{x} + \frac{1}{2} \times \ln x - \frac{x}{2} \left( \ln 2 - \gamma + \frac{1}{2} \right) + O(x^3 \ln x), \]

\[ K_0(x) = -\ln x + (\ln 2 - \gamma) - \frac{1}{4} \times \ln x + 0(x^2 \ln x), \]
so that (2.5.8d) becomes

\[ B_1(x) = xK_0 + 2K_1 \]

\[ = \frac{2}{x} - \frac{1}{2}x + 0 \left( x^3 \ln x \right) \]

\[ = \frac{2}{x} \left( 1 - \frac{1}{4}x^2 + 0 \left( x^4 \ln x \right) \right). \]

Then

\[ A_1B_1(y_2) = -\frac{x_1^2}{4}(1 + \frac{x_1^2}{24} + \ldots) \frac{2}{y_2} \left( 1 - \frac{1}{4}y_2^2 + \ldots \right) \]

\[ = -\frac{x_1^2}{2y_2} \left( 1 + \frac{x_1^2}{2y_2} - \frac{y_2^2}{4} + \ldots \right), \text{ to } k^3; \]

\[ A_1A_2K_1(x_2) = \left( -\frac{x_1^2}{4} \right) \left( \frac{y_2^3}{4x_2} \right) \left( \frac{1}{x_2} \right) = -\frac{x_1^2}{16x_2^2}, \text{ to } k^3; \]

\[ A_3B_1(x_2) = \frac{2}{x_2^2} \left( 1 - \frac{1}{4}x_2^2 \right) \frac{y_2^3}{4x_2} \left( 1 + \frac{y_2^2}{12} - \frac{x_2^2}{8} \right) \]

\[ = \frac{y_2^3}{2x_2^2} \left( 1 + \frac{y_2^2}{12} - \frac{3}{8}x_2^2 + \ldots \right), \text{ to } k^3; \]

and

\[ A_2B_1(y_2) = \frac{x_2^2}{4} \left( 1 - \frac{x_1^2}{24} \right) \frac{2}{y_2} \left( 1 - \frac{1}{4}y_2^2 + \ldots \right) \]

\[ = \frac{x_2^2}{2y_2} \left( 1 - \frac{x_1^2}{24} - \frac{y_2^2}{4} + \ldots \right), \text{ to } k^3. \]

Therefore, to order \( k^3 \), (2.5.10) becomes

\[ \mu_1 \left\{ \frac{x_1^2}{2y_2} \left( 1 + \frac{x_1^2}{24} - \frac{y_2^2}{4} + \ldots \right) + \frac{x_1^2}{24} \frac{y_2^3}{4x_2} \right\} + \mu_2 \left\{ \frac{x_1^2}{2y_2} \left( 1 - \frac{x_1^2}{24} - \frac{y_2^2}{4} \right) \right\} \]

\[ - \frac{y_2^3}{2x_2^2} \left( 1 + \frac{y_2^2}{12} - \frac{3}{8}x_2^2 \right) \} = 0. \]
Multiply the last equation by \(2y_2\), rearranging, and cancelling \(k^2\), we have (2.5.13).

8.2. Derivation of Finite Strain and Acceleration In Cylindrical Coordinates

The following is the derivation of the physical components of finite strain in rotationally symmetrical cylindrical coordinates. This can be done in two ways. By calculating the Christoffel symbols we can first obtain tensor components and then transform them to physical components. Alternatively, we can perform calculations directly with the physical components using base vectors.

In cylindrical coordinates, the displacement vector \(u_i = (u, v, w)\) has the form [50]

\[
u_i = u p_i + v q_i + w r_i, \tag{8.2.1}\]

where \(p_i, q_i, r_i\) are unit base vectors in the \(r, \theta, z\) directions.

These are

\[
p_i = (\cos \theta, \sin \theta, 0), \quad q_i = (-\sin \theta, \cos \theta, 0), \quad r_i = (0, 0, 1). \tag{8.2.2}\]

Further, we have [51]

\[
\frac{\partial}{\partial r} (p_i, q_i, r_i) = \frac{\partial}{\partial z} (p_i, q_i, r_i) = 0,
\]

\[
\frac{\partial}{\partial \theta} (p_i, q_i, r_i) = (q_i, -p_i, 0),
\]

\[
p_i p_i = q_i q_i = r_i r_i = 1. \tag{8.2.3}\]

For rotationally symmetric case \(u, v, w = u, v, w (r, z)\);
hence,
\[
\begin{align*}
  u_{i,j} &= (p_j \frac{\partial}{\partial r} + \frac{1}{r} q_j \frac{\partial}{\partial \theta} + r_j \frac{\partial}{\partial z}) (u_{p_i} + v_{q_i} + w_{r_i}) \\
  &= p_j (p_i u_r + q_i v_r + r_i w_r) + \frac{1}{r} q_i (u_{q_i} - v_{p_i}) \\
  &+ r_i (p_i u_z + q_i v_z + r_i w_z).
\end{align*}
\]

(8.2.4)

Now the finite strain tensor is given by [52]
\[
2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j},
\]

(8.2.5)

and the physical components are
\[
\begin{align*}
  e_{rr} &= e_{ij}p_ip_j = u_{ij,pjpj} - \frac{1}{2}(u_{k,i}p_j) (u_{k,j}p_j) \\
  &= \frac{\partial u}{\partial r} - \frac{1}{2}(u_r^2 + v_r^2 + w_r^2),
\end{align*}
\]
\[
\begin{align*}
  e_{\theta\theta} &= e_{ij}q_iq_j = u_{ij,qiqj} - \frac{1}{2}(u_{k,i}q_j) (u_{k,j}q_j) \\
  &= \frac{u}{r} - \frac{1}{2r^2} (u^2 + v^2),
\end{align*}
\]
\[
\begin{align*}
  e_{zz} &= e_{ij}r_ir_j = u_{ij,rjrj} - \frac{1}{2}(u_{k,i}r_j) (u_{k,j}r_j) \\
  &= w_z - \frac{1}{2} (u_z^2 + v_z^2 + w_z^2),
\end{align*}
\]

(8.2.6)
\[
\begin{align*}
  e_{r\theta} &= e_{ij}p_ip_j = \frac{1}{2} u_{ij,jqj}p_i + (\frac{1}{2} u_{ij,ipj})q_j - \frac{1}{2}(u_{k,i}p_j) (u_{k,j}q_j) \\
  &= \frac{1}{2} (-v_r + v_r) - \frac{1}{2r}(uv_r - vu_r),
\end{align*}
\]
\[
\begin{align*}
  e_{\theta z} &= e_{ij}q_iq_j = \frac{1}{2} u_{ij,rjrj}q_i + (\frac{1}{2} u_{ij,qjrj})r_j - \frac{1}{2}(u_{k,i}q_j) (u_{k,j}r_j) \\
  &= \frac{1}{2} v_z - \frac{1}{2r}(uv_z - vu_z),
\end{align*}
\]
\[ e_{zr} = e_{ij} r_j p_j \]
\[ = \left( \frac{1}{2} u_{i,j}^r r_i + \left( \frac{1}{2} u_{j,i}^r r_j \right) p_j - \frac{1}{2} (u_k, i r_i) (u_k, j p_j) \right) \]
\[ = \frac{1}{2} (w_r + u_z) = - \frac{1}{2} (u_z u_r + v_z v_r + w_z w_r). \]

For the problem in chapter 3
\[ u = u(r,t), \quad v = v(r,t), \quad w = 0; \]
hence Equations (8.2.6) reduce to
\[ e_{rr} = u_r - \frac{1}{2} (u_r^2 + v_r^2), \]
\[ e_{\theta\theta} = \frac{u}{r} - \frac{1}{2r^2} (u^2 + v^2), \]
\[ e_{\theta r} = \frac{1}{2} (v_r - v) - \frac{1}{2r} (uv_r - vu_r), \]
\[ e_{zr} = e_{zz} = e_{\theta z} = 0. \quad (8.2.7) \]
\[ = (3.3.22b) \]

Next we derive the acceleration. Let the velocity vector be
\[ \mathbf{V}_i = (U, V, W), \quad (8.2.8) \]
or
\[ \mathbf{V}_i = p_i U + q_i V + r_i W. \quad (8.2.9) \]

Since
\[ \mathbf{V}_i = \frac{\partial u_i}{\partial t} + v_j u_{i,j}, \quad (8.2.10) \]
where
\[ u_i \] is defined by (8.2.1) and \( u_{i,j} \) is defined by (8.2.4), thus
\[ U = p_i V_i = p_i \frac{\partial u_i}{\partial t} + v_j p_i u_{i,j}, \]
Using (8.2.1) and (8.2.4), we have

\[ p_i \frac{\partial u_i}{\partial t} = \frac{\partial u}{\partial t}, \quad q_i \frac{\partial u_i}{\partial t} = \frac{\partial v}{\partial t}, \quad r_i \frac{\partial u_i}{\partial t} = \frac{\partial w}{\partial t}, \]

\[ V_j (p_i u_{i,j}) = V_j (p_j u_r - \frac{1}{r} q_j v + r_j w) \]

\[ = U v_r - \frac{V}{r} v + W u_z, \]

\[ V_j (q_i u_{i,j}) = V_j (p_j v_r + \frac{1}{r} q_j u + r_j v_z) \]

\[ = U v_r + \frac{V}{r} u + W v_z, \]

\[ V_j (r_i u_{i,j}) = V_j (p_j w_r + r_j w_z) \]

\[ = U w_r + W w_z. \]

Therefore, Equations (8.2.11) reduce to

\[ U = \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial r} - \frac{V v}{r} + W \frac{\partial u}{\partial z}, \]

\[ V = \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial r} + \frac{V u}{r} + W \frac{\partial v}{\partial z}, \]

\[ W = \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial r} + W \frac{\partial w}{\partial z}. \]

(8.2.12)

Now we apply amplitude expansions used in chapter 3

\[ u = \epsilon u + \cdots, \]
\[ v = e^v + e^3v + \cdots, \]
\[ w = 0. \] \hfill (8.2.13)

Then Equations (8.2.12) reduce to
\[ u = \frac{u_t - Vv/r}{1 - u/r} = (u_t - Vv/r) (1 + u/r + \cdots) \]
\[ = e^2 \left( \frac{1}{u_t - Vv/r} \right) + O(e^4), \]
\hfill (8.2.14)

\[ V = (v_t + u_t v_r)/\left[1 - (u/r - v_r v/r)\right] \]
\[ = 1 + e^3 \left[ v_t + \frac{1}{3} \left( \frac{u_t - Vv_r}{r} \right) + \frac{2}{3} \right] + O(e^5), \]
\hfill (8.2.15)

\[ W = 0. \]

Substituting (8.2.15) into (8.2.14), we have
\[ u = e^2 \left( \frac{1}{u_t - Vv/r} \right) + O(e^4). \]
\hfill (8.2.16)

Similarly, we let the acceleration vector be
\[ a_i = (A, B, C) \]
\hfill (8.2.17)
or
\[ a_i = p_i A + q_i B + r_i C. \]
\hfill (8.2.18)

Since
\[ a_i = \frac{\partial v_i}{\partial t} + V_j v_{i,j}, \]
\hfill (8.2.19)
by same procedure as is done in obtaining the velocities, we find

\[ A = \frac{\partial u}{\partial t} + U \frac{\partial u}{\partial r} - \frac{v^2}{r} + W \frac{\partial u}{\partial z}, \]

\[ B = \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial r} + \frac{W u}{r} + W \frac{\partial v}{\partial z}, \]

\[ C = \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial r} + W \frac{\partial w}{\partial z}. \]

Using (8.2.15) and (8.2.16), Equations (8.2.20) yield

\[ A = \epsilon^2 \left[ u_{tt} - \frac{\partial}{\partial t} \left( 1 \frac{\partial v}{\partial t} / r \right) - \frac{1}{r} (v_t)^2 \right] + O(\epsilon^4), \]

\[ B = \epsilon \cdot \frac{1}{v_{tt}} \]

\[ + \epsilon^3 \left[ v_{tt} + \frac{1}{v_t} \left( u - v v_r / r \right) + \frac{2}{v_t} v_r \right] _{t} + \frac{1}{v_{rt}} \left( u_t - v v_t / r \right) \]

\[ + O(\epsilon^5), \]

\[ C = 0. \]

Equations (8.2.21) yields Equations (3.3.30).

8.3. Derivation of Equations (4.3.12) and (4.3.13)

Rewriting Equation (4.3.11), we have

\[ \omega_2^2 = \frac{2}{A_\pi} \int_0^\pi dx \int_0^{2\pi} \sin \omega_0 \cos \theta f(x,t,\omega_0) \, dt, \]

\[ \text{(8.3.1)} \]
where \( f \) is defined in (4.3.10). This together with (4.3.5) and (4.3.7) gives

\[
f = \alpha \left( \frac{1}{2} \frac{\partial}{\partial x} (v_x u_x) \right)_x - \beta \left( \frac{1}{2} \frac{\partial}{\partial t} (v_x)^3 \right)_x + \omega^2 \left( \frac{1}{2} \frac{\partial}{\partial t} (v_x u_t) + 2u_t \frac{\partial}{\partial t} (v_x) - \frac{1}{2} \frac{\partial}{\partial t} (v_x u_t) \right),
\]

(8.3.2)

\[
\begin{align*}
\frac{1}{2} v &= A \sin \omega x \cos t, \\
\frac{3}{2} u &= A^2 B \frac{\sin 2\omega x}{8} - \frac{1}{2} \sin 2\omega x \cos 2t.
\end{align*}
\]

(8.3.3)

(8.3.4)

We define

\[
I = \int_0^\pi \int_0^{2\pi} \sin \omega x \cos t f(x,t,\omega) \, dt.
\]

(8.3.5)

Since \( f(x,t,\omega) \) has five terms as shown in (8.3.2) we perform the integration \( I \) in five steps. First we consider the term \( \alpha \frac{\partial}{\partial x} (v_x u_x) \) in (8.3.2). Making use of (8.3.3) and (8.3.4) it can be shown that

\[
\frac{\partial}{\partial x} (v_x u_x) = -\frac{A^2 B}{8} \left( \sin \omega x + \sin 3\omega x \right) \left[ \frac{\cos t}{2} + \frac{1}{1-\eta^2} \cos t + \cos 3t \right].
\]

since

\[
\int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1-\cos 2t}{2} \, dt = \pi,
\]

(8.3.6)

and

\[
\int_0^{2\pi} \cos t \cos 3t \, dt = \int_0^{2\pi} \frac{\cos 2t - \cos 4t}{2} \, dt = 0,
\]

we have
\[ I_1 = \alpha \int_0^\pi dx \int_0^{2\pi} \sin \omega_x \cos t \frac{\partial}{\partial x} \left( \frac{1}{2} \dot{v}_x \right) dt \]

\[ = \frac{\alpha A^3 \omega_o^4 B \pi (2 - 3\eta^2)}{16 \eta^2 (1 - \eta^2)} \int_0^\pi \sin \omega_x (\sin \omega_x + 3 \sin 3\omega_x) dx \]

\[ = \frac{\alpha A^3 \omega_o^4 B \pi (2 - 3\eta^2)}{16 \eta^2 (1 - \eta^2)} \left( \frac{\pi}{2} + \frac{1}{2 \omega_o} \sin 2\omega_o \pi - \frac{3}{8 \omega_o} \sin 4\omega_o \pi \right). \quad (8.3.7) \]

Next consider the term \( \beta (v_x^1)^3 \) in (8.3.2). By virtue of (8.3.3), we find

\[ \frac{\partial}{\partial x} (v_x^1)^3 = \frac{3}{16} A^3 \omega_o^4 \sin 3\omega_x (3 \cos t + \cos 3t); \]

hence

\[ I_2 = -\beta \int_0^\pi dx \int_0^{2\pi} \sin \omega_x \cos t \frac{\partial}{\partial x} (v_x^1)^3 dt \]

\[ = -\frac{9}{64} \beta A^3 \omega_o^3 \frac{3}{\pi} \sin \omega_x (\sin 2 \omega_o \pi - 1/2 \sin 4\omega_o \pi), \quad (8.3.8) \]

where use is made of (8.3.6).

Thirdly, due to (8.3.3) and (8.3.4), we obtain

\[ \frac{1}{2} v_x^{tt} = - \frac{A^3 B \omega_o^2}{8 (1 - \eta^2)} (\sin \omega_x + \sin 3\omega_x) \cos t + \cos 3t), \]

\[ I_3 = \omega_o^2 \int_0^\pi dx \int_0^{2\pi} \sin \omega_x \cos t \frac{1}{2} v_x^{tt} dt \]

\[ = - \frac{A^3 B \omega_o^4}{8 (1 - \eta^2)} \pi \int_0^\pi \sin \omega_x \sin (\sin \omega_x + \sin 3\omega_x) dx \]
\[
= - \frac{A^3 B \omega_0^4 \pi}{8(1 - \eta^2)} \left( \frac{\pi}{2} - \frac{3}{8\omega_0} \sin 4\omega_0 \pi + \frac{1}{2\omega_0} \sin 2\omega_0 \pi \right), \quad (8.3.9)
\]

using (8.3.6).

Equations (8.3.3) and (8.3.4) yield
\[
2 \mathbf{1} \mathbf{v}_{xt} = \frac{A^3 B \omega_0^2}{16(1 - \eta^2)} \left( \sin \omega_0 x + \sin 3\omega_0 x \right) \left( \cos t - \cos 3t \right);
\]

thus,
\[
I_4 = 2\omega_0^2 \int_0^\pi dx \int_0^{2\pi} \sin \omega_0 x \cos t \left( \frac{1}{2} \mathbf{1} \mathbf{v}_{xt} \right) dt
\]
\[
= \frac{A^3 B \omega_0^- 4 \pi}{8(1 - \eta^2)} \left( \frac{\pi}{2} - \frac{3}{8\omega_0} \sin 4\omega_0 \pi + \frac{1}{2\omega_0} \sin 2\omega_0 \pi \right) = -I_3, \quad (8.3.10)
\]

where (8.3.6) is used.

Lastly (8.3.3) and (8.3.4) give
\[
2 \mathbf{1} \mathbf{x}_{tt} = \frac{A^3 B \omega_0^2}{8} \left( \sin 3\omega_0 x - \sin \omega_0 x \right) \left[ \frac{\cos t}{\eta^2} + \frac{1}{1 - \eta^2} \frac{\cos t + \cos 3t}{2} \right];
\]

thus,
\[
I_5 = -\omega_0^2 \int_0^\pi dx \int_0^{2\pi} \sin \omega_0 x \cos t \left( \frac{1}{2} \mathbf{x}_{tt} \right) dt
\]
\[
= -\frac{A^3 B \omega_0^- 4}{8} \frac{\pi(2 - 3\eta^2)}{2\eta^2(1 - \eta^2)} \int_0^\pi \sin \omega_0 x \left( \sin 3\omega_0 x - \sin \omega_0 x \right) dx
\]
\[
= \frac{A^3 B \omega_0^4 (2 - 3\eta^2)}{16\pi^2(1 - \eta^2)} \left( \frac{\pi}{2} - \frac{1}{2\omega_0} \sin 2\omega_0 \pi + \frac{1}{8\omega_0} \sin 4\omega_0 \pi \right), \quad (8.3.11)
\]
through the use of (8.3.6).

Combining the above five integrations (8.3.7) through (8.3.11) and rearranging, and making use of (8.3.1), (8.3.5) and (4.3.1), we have

\[ \omega^2(\varepsilon) = \omega_o^2 + \varepsilon^2 \omega_2^2 + \ldots, \]

\[ = \omega_o^2 + \varepsilon^2 \frac{A^2 \omega_o^3}{32\pi} \left\{ \frac{B(2 - 3\eta^2)}{\eta^2(1 - \eta^2)} \left[ \frac{\omega_o \pi}{32} (\alpha - 1) + (\alpha + 1) \sin 2\omega_o \pi \right] \right\} \]

\[ - \frac{1}{4} (3\alpha + 1) \sin 4\omega_o \pi \right\} \]

\[ + O(\varepsilon^4), \quad (8.3.12) \]

with \( A, \eta, B, \alpha \) and \( \beta \) defined in (4.2.3), (4.3.5) and (4.2.6), respectively.

We now insert (8.3.12) into the right side of (4.3.4a) and find

\[ \omega_o^2 \frac{v_{tt}^3}{v_{xx}^3} = -A\omega_o^2 \sin \omega_o x \cos t \]

\[ + \frac{\alpha A^3 \omega_o^4}{8} (\sin \omega_o x + 3 \sin 3\omega_o x) \left[ \frac{1}{n^2} \cos t + \frac{1}{1-n^2} \cos 3t \right] \]

\[ - \frac{3 \beta A^3 \omega_o^4}{16} \sin 3\omega_o x (3 \cos t + \cos 3t) \]
\[
- \frac{A^3B \omega_o^4}{4(1 - \eta^2)} (\sin \omega_o x + \sin 3\omega_o x) \cos 3t 
\]

with \( \omega_2^2 \) as shown in (8.3.12).

The solution of (8.3.13), (4.3.4b), and (4.3.4c) is

\[
v = \gamma_1 \sin \omega_o x \cos t + \gamma_2 \sin \omega_o x \cos 3t + \gamma_3 \sin 3\omega_o x \cos t \\
+ \gamma_4 \sin 3\omega_o x \cos 3t,
\]

where

\[
\gamma_1 = -A \omega_2^2 + \frac{A^3 \omega_o^4 B (\alpha + 1)}{8} \left[ \frac{1}{\eta^2} + \frac{1}{2(1 - \eta^2)} \right], \\
\gamma_2 = -\frac{A^3 \omega_o^4 B (\alpha + 5)}{128 (1 - \eta^2)}, \\
\gamma_3 = \frac{A^3 \omega_o^4 B}{64} \left\{ (3\alpha - 1) \left[ \frac{1}{\eta^2} - \frac{1}{2(1 - \eta^2)} \right] - \frac{98}{28} \right\}, \\
\gamma_4 = \frac{A^3 \omega_o^4 B}{16} \left( \frac{3\alpha - 5}{1 - \eta^2} - \frac{38}{8} \right),
\]
with $A$, $C$, $B$, $\alpha$ and $\beta$ defined in (4.2.3), (4.3.5) and (4.2.6), respectively.

Now we collect the results and reintroduce the original non-dimensional variable $t$, so that

$$v = e A \sin \omega \pi \sin \omega \pi \cos \omega t + e^3 (\gamma_1 \sin \omega \pi \cos \omega t + \gamma_2 \sin \omega \pi \cos 3 \omega t + \gamma_3 \sin 3 \omega \pi \cos \omega t + \gamma_4 \sin 3 \omega \pi \cos 3 \omega t) + O(e^5), \quad (8.3.15)$$

with $\gamma_i$, $i = 1, 2, 3, 4$ defined in (8.3.14).