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Large Deviation Results for Random Walks in a Sparse Random Environment

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Large deviation results for random walks in a sparse random environment

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DEDICATION

I would like to dedicate this thesis to my mother Hatun, my wife Neslihan, and to my children Nehir, Emre Fehmi and Melis Defne. Without their love and support, I never would have made this far.

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ABSTRACT

The topic of this thesis is random walks in a sparse random environment (RWSRE) on $\mathbb{Z}$. Basic asymptotic properties of this model were investigated by Matzavinos, Roitershtein and Seol (2016). The purpose of this work is to prove large deviation principles accompanying laws of large numbers for the position of the particle and first hitting times, which have been establish in previous work.

Large deviation principles (LDP) for random walks in i. i. d. environments were first obtained by Greven and den Hollander (1994). Using a different approach, the LDP's were extended to ergodic environments by Comets, Gantert and Zeitouni (1998, 2000). Several refitments of this result due to the same group of authors, Peres, Pisztora, and Povel, have appeared since then. An alternative method of studying the large deviations for random walks in random environments (RWRE) was subsequently suggested by Vardhan and further developed in the work of Yilmaz, Rassoul-Agha, and Rosenbluth.

In this work we obtain quenched and annealed LDP for the RWSRE using a relation between the underlying RWSRE and a random walk in a dual stationary environment, which was introduced by Matzavinos, Roitershtein, and Seol. We first investigate a relation between the sparse environment and its stationary dual, and then obtain LDP's for a random walk in the stationary (and ergodic) dual environment. Next, we transform the quenched LDP in the dual setting to obtain a quenched LDP for the corresponding RWSRE and give a description of the rate function. Finally, we show that the annealed LDP in the dual setting is directly related to an annealed LDP for the RWSRE when the lengths of the cycles are bounded. Our study of the rate functions relies on the approach of Comets, Dembo, Gantert and Zeitouni (2000, 2004).

CHAPTER 1. INTRODUCTION

In this work we are concerned with large deviation principles (LDP) for random walks in a sparse random environment (RWSRE). In the remainder of this chapter we will review the general theory of large deviations (Section 1.1), discuss the general framework of RWRE (Section 1.2), and give a detailed description of RWSRE (Section 1.3). The chapter also contains a brief survey of LDP results for classical RWRE (Section 1.2), a brief summary of basic asymptotic results for RWSRE (Section 1.3), and a short overview of the thesis (Section 1.4).

1.1 Large deviation principles: definition and basic results

In this section we discuss the theory of large deviations and give a brief introduction to basic results. An extensive presentation of the theory can be found, for example, in [29, 8, 16, 23, 33]. Before proceeding with detailed definitions and a description of basic tools in the theory of large deviations, we first focus on the following particular example. The example serves to provide a heuristical motivation to study the subject.

Let $\{X_n\}_{n\geq 1}$ be an i. i. d. sequence of standard Normal random variables. The empirical mean $S_n := \frac{1}{n}\sum_{k=1}^{n} X_k$ has a Normal distribution with zero mean and variance equal to $1/n$. Thus for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|S_n| \geq \epsilon) = 0 \tag{1.1}$$

and, by the Central Limit Theorem, for any interval E,

$$\lim_{n\to\infty} P(\sqrt{n}\; S_n \in E) = \frac{1}{\sqrt{2\pi}} \int_E e^{-x^2/2} dx. \tag{1.2}$$

Thus, one can expect that

$$P(|S_n| \geq \epsilon) \sim 1 - \frac{1}{\sqrt{2\pi}} \int_{-\epsilon\sqrt{n}}^{\epsilon\sqrt{n}} e^{-x^2/2} dx,$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \log P(S_n \geq \epsilon) = -\frac{\epsilon^2}{2}. \tag{1.3}$$

Equation (1.3) can be verified and it provides an example of a large deviation statement in the following sense. A typical value of S_n is of order $1/\sqrt{n}$ by (1.2). However, $|S_n|$ takes relatively large values with small probability of order $e^{-n\epsilon^2/2}$.

In general, large deviation theory is centered around the observation that "nice" functions f of i. i. d. random variables $\{X_n\}_{n\geq 1}$ often have the property that

$$P\big(f(X_1, X_2, \dots, X_n) \in dx\big) \sim e^{nI(x)} \qquad \text{as } n \to \infty$$

that is, the probability that $f(X_1, X_2, \dots, X_n)$ takes values near a point x decays exponentially fast with a rate function I.

More precisely, a *large deviation principle* describes the limiting behavior of a family of probability measures $\{\mu_\epsilon\}$ on a measurable space $(\mathcal{X}, \mathcal{B})$ as $\epsilon \to 0$. Here, $\mathcal{X}$ is a topological space, and the simplest example of the $\sigma-$field $\mathcal{B}$ is the Borel $\sigma-$field $\mathcal{B}_\mathcal{X}$ of open sets in $\mathcal{X}$. Mapping $I : \mathcal{X} \to [0, \infty]$ is a *rate function* if it is not identically infinite and if it is lower semi-continuous, that is, the level sets $\Psi_I(\alpha) := \{x : I(x) \leq \alpha\}$ are closed subsets of $\mathcal{X}$ for all $\alpha \in [0, \infty]$. It is a *good rate function* if all the level sets are compact subsets of $\mathcal{X}$. The *effective domain* of I is the set of points in $\mathcal{X}$ of finite rate and denoted by $\mathcal{D}_I$. We usually refer to $\mathcal{D}_I$ as the domain of I when no confusion arises. Note that if $\mathcal{X}$ is a metric space, I is lower semi-continuous if and only if $\liminf_{n\to\infty} I(x_n) \geq I(x)$ for any sequences $x_n \in \mathcal{X}$ converging to $x \in \mathcal{X}$.

Throughout the thesis, the following notations are used. For any subset $A \subset \mathcal{X}$, $\overline{A}$ denotes the closure of A, A^o the interior of A, and A^c the complement of A. Moreover, for any function f defined on $\mathcal{X}$

$$f(A) := \inf_{x \in A} f(x).$$

We say that the family of probability measures $\{\mu_\epsilon\}$ satisfies the *large deviation principle* (LDP or sometimes full LDP) with a rate function I if, for all $\Gamma \in \mathcal{B}$,

$$-I(\Gamma^o) \le \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \le \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \le -I(\overline{\Gamma}). \tag{1.4}$$

Two equivalent formulations of upper and lower bounds in (1.4) can be given as follows: Suppose that I is a rate function and $\Psi_I(\alpha)$ is its level set. Then,

(a) (Upper bound) For every $\alpha < \infty$ and every measurable set Γ with $\overline{\Gamma} \subset \Psi_I(\alpha)^c$,

$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \le -\alpha. \tag{1.5}$$

(b) (Lower bound) For any $x \in \mathcal{D}_I$ and any measurable set Γ with $x \in \Gamma^o$,

$$\liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \ge -I(x). \tag{1.6}$$

When $\mathcal{B}_{\mathcal{X}} \subset \mathcal{B}$,

(a) (Upper bound) For any closed set $C \subset \mathcal{X}$,

$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(C) \le -I(C). \tag{1.7}$$

(b) (Lower bound) For any open set $O \subset \mathcal{X}$,

$$\liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(O) \ge -I(O). \tag{1.8}$$

Suppose that all the compact subsets of $\mathcal{X}$ belong to $\mathcal{B}$. A family of probability measures $\{\mu_\epsilon\}$ is said to satisfy the *weak LDP* with the rate function I if the upper bound (1.7) holds for all compact sets instead. It is *exponentially tight* if for every $\alpha < \infty$, there exists a compact set $K_\alpha \subset \mathcal{X}$ such that

$$\limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(K_\alpha^c) < -\alpha. \tag{1.9}$$

It can be shown that if an exponentially tight family of probability measures satisfies the weak LDP with a rate function I, then it satisfies the full LDP with the same rate I. Furthermore, the rate function is a good rate function.

The LDP of the empirical mean of a sequence of random variables taking values in finite alphabets is studied via combinatorial techniques. It uses the method of types, and Sanov's Theorem sheds light on the basic LDP results in $\mathbb{R}^d$ such as Cramér Theorem [8]. Specifically, consider the empirical means $S_n := \frac{1}{n}\sum_{i=1}^n X_i$ of $d-$dimensional i.i.d. random vectors $\{X_n\}_{n\geq 1}$ with X_1 distributed according to the probability law μ. Let μ_n denote the law of S_n and define the *logarithmic moment generating function* associated with the law μ by

$$\Lambda(\lambda) := \log E\big[e^{<\lambda, X_1>}\big] \tag{1.10}$$

where $<\lambda, x>:= \sum_{j=1}^d \lambda_j x_j$ is the usual scalar product in $\mathbb{R}^d$, and x_j is the jth coordinate of x. We also define the Fenchel-Legendre transform of $\Lambda(\lambda)$ by

$$\Lambda^*(x) := \sup_{\lambda \in \mathbb{R}^d} \big\{ <\lambda, x> - \Lambda(\lambda) \big\}. \tag{1.11}$$

Cramér's Theorem states that the empirical means of i.i.d. $d-$dimensional random vectors satisfy the LDP with the convex rate function Λ^*. An elegant and standard technique, called the exponential change of measure, is used to prove large deviations lower bound. However, Cramér's Theorem is limited to the i.i.d. case.

An extension to the non-i.i.d. case is given via Gärtner-Ellis Theorem [8], under certain technical assumptions on the rate function and its domain. In particular, consider a sequence of random vectors $Z_n \in \mathbb{R}^d$, where Z_n is distributed according to the law μ_n

and has the logarithmic moment generating function

$$\Lambda_n(\lambda) := \log E\big[e^{<\lambda, Z_n>}\big]. \tag{1.12}$$

Under the assumption of the existence of scaled logarithmic moment generating functions $\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{n}\Lambda_n(n\lambda)$, μ_n satisfies the large deviations upper bound whose proof is essentially similar to the proof of Cramér's Theorem. It satisfies a restricted version of a large deviation lower bound whose proof is based on the large deviation *upper* bound. The proof of the goodness of the rate function Λ is heavily dependent on the convex analysis considerations [25].

LDP for the empirical means of finite state Markov chains can be obtained using the Gärtner - Ellis Theorem. Let $\{Y_k\}$ be a finite state Markov chain taking values in a finite alphabet Σ with an irreducible transition matrix $\mathbf{\Pi} = \{\pi(i,j)\}_{i,j=1}^{|\Sigma|}$. Define the empirical means by

$$Z_n = \frac{1}{n}\sum_{k=1}^{n} X_k \tag{1.13}$$

where $X_k = f(Y_k)$ and $f : \Sigma \to \mathbb{R}^d$ is a given deterministic function. For $\lambda \in \mathbb{R}^d$, define the matrix $\mathbf{\Pi}_\lambda = \{\pi_\lambda(i,j)\}_{i,j=1}^{|\Sigma|}$ such that the entries $\pi_\lambda(i,j)$ are given by

$$\pi_\lambda(i,j) = \pi(i,j)e^{<\lambda, f(j)>}, \qquad i,j \in \Sigma. \tag{1.14}$$

Let $\rho(\mathbf{\Pi}_\lambda)$ denote the Perron-Frobenius eigenvalue of the irreducible $\mathbf{\Pi}_\lambda$ and define

$$I(z) := \sup_{\lambda\in\mathbb{R}^d} \big\{ < \lambda, z > - \log\rho(\mathbf{\Pi}_\lambda)\big\}. \tag{1.15}$$

It can be shown that Z_n satisfies the LDP with the convex, good rate function I [8].

1.2 Random walks in a random environment in dimension one

A general definition of random walks in a random environment on infinite graphs with a countable vertex set V can be found in [39]. In this work we focus on a nearest-neighbor random walk on the integer lattice $\mathbb{Z}$. Without loss of generality, all random processes in this work are assumed to be defined in a common probability space $(E, \mathcal{E}, \eta)$.

We now introduce random walks in a random environment as follows: let $\Omega = [0,1]^{\mathbb{Z}}$ and define $\mathcal{F}$ as the Borel $\sigma-$algebra of subsets of Ω. A *random environment* is a random element $\omega = (\omega_i)_{i\in\mathbb{Z}}$ in $(\Omega, \mathcal{F})$ that is supported by $(E, \mathcal{E}, \eta)$. We will also denote the probability measure associated with $\omega \in (\Omega, \mathcal{F})$ by η whenever no confusion occurs. Hence, we use the notation E_η for the corresponding expectation operator throughout this work.

A random walk on $\mathbb{Z}$ in a random environment $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Omega$ is a Markov chain $X = (X_n)_{n\geq 0}$ on $\mathbb{Z}$ governed by the following transition law:

$$P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} \omega_i & \text{if } j = i+1 \\ 1-\omega_i & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

Thus X is an element of $\mathbb{Z}^{\mathbb{Z}_+}$, where $\mathbb{Z}_+$ stands for the set of non-negative integers. We will often refer to $\mathbb{Z}^{\mathbb{Z}_+}$ as the *space of trajectories* of the random walk. The law of the random walk in a fixed environment is usually referred to as a *quenched law* of the random walk. We will use the notation P_ω^x for the quenched probability law when $X_0 = x$ and E_ω^x for the corresponding expectation operator acting in the space of trajectories.

The *annealed* or *averaged* distributions P_η^x of the random walk are obtained by averaging the quenched distributions over all possible environments, that is $P_\eta^x(\cdot) = \int P_\omega^x(\cdot)\eta(d\omega)$ $(P_\eta^x := \eta(d\omega) \otimes P_\omega^x)$. More precisely, let $\mathcal{G}$ be the cylinder σ-algebra of the space of trajectories $\mathbb{Z}^{\mathbb{Z}_+}$. Note that for each $G \in \mathcal{G}$, $P_\omega^x(G) : \Omega \to [0,1]$ is a $\mathcal{F}$-measurable function of ω. The joint probability distribution P_η^x of the random walk and the environment on the product space $\left(\Omega \times \mathbb{Z}^{\mathbb{Z}_+}, \mathcal{F} \otimes \mathcal{G}\right)$ is defined by

$$P_\eta^x(F \times G) = \int_F P_\omega^x(G)\eta(d\omega), \qquad F \in \mathcal{F}, G \in \mathcal{G}. \tag{1.17}$$

The projection of P_η^x on the space of trajectories $\mathbb{Z}^{\mathbb{N}}$ is the annealed law of the random walk. Whenever no confusion occurs, we also denote this marginal as P_η^x, i.e.,

$$P_\eta^x(G) = P_\eta^x(\Omega \times G) = \int_\Omega P_\omega^x(G)\eta(d\omega) \tag{1.18}$$

and corresponding expectation under the law P_η^x by E_η^x. We will usually assume that $X_0 = 0$ and often omit the upper index x in the notations for the underlying probability laws and expectation operators when $x = 0$. For instance, we will write P_ω for P_ω^0 and E_η for E_η^0.

Large deviations for the classical walk in an i.i.d environment was first studied in [15] in both quenched and annealed setting. Using a different approach, the large deviation principles were extended to ergodic environments by Comets, Gantert and Zeitouni [5, 6, 14, 13] who also provided information about rate functions and, in particular, the relation between the quenched and the annealed rate functions. A refitment of the latter result, due to Dembo, Gantert and Zeitouni, can be found in [7, 39]. (See also an efficient adaptation of the technique to a quasi one-dimensional setting by Peterson [19]. Detailed information on the tail for some specific regimes is known due to the work of Dembo, Peres, Pisztora, Povel, and Zeitouni [9, 20, 21]. An alternative method of studying the large deviations for RWRE was subsequently suggested by Vardhan [34] and, in dimension one, further developed in the work of Yilmaz [35, 36, 37], Rassoul-Agha [22], and Rosenbluth [26]. (See also a short introductory survey of Rezakhanlou [24].)

In this work we obtain quenched and annealed LDP for the RWSRE using a relation between the underlying RWSRE and a random walk in a dual stationary environment, which was introduced by Matzavinos, Roitershtein, and Seol. We first investigate a relation between the sparse environment and its stationary dual, and then obtain LDP for a random walk in the stationary (and ergodic) dual environment. Next, we transform the quenched LDP in the dual setting to obtain a quenched LDP for the corresponding RWSRE and give a description of the rate function. Finally, we show that the annealed LDP in the dual setting is directly related to an annealed LDP for the RWSRE when the lengths of the cycles are bounded. Our study of the rate functions relies on the approach of Comets, Dembo, Gantert and Zeitouni [5, 7].

1.3 Sparse environment

The main topic of this thesis is large deviation results for random walks in a sparse random environment. We obtain the LDP for the position of the random walk X_n and hitting times T_n in both quenched and annealed settings and, following [5], provide some information about the relation between the corresponding rate functions.

In dimension one, the RWSRE model is defined as follows. Let $d = (d_n)_{n\in\mathbb{Z}}$ be an i.i.d. sequence of positive integer valued random variables supported by $(E, \mathcal{E}, P)$ and let $\lambda = (\lambda_n)_{n\in\mathbb{Z}}$ be an i.i.d. sequence of random elements in $([0,1], \mathcal{B}([0,1]))$ which is independent of the random sequence d. Throughout, we will assume that there is a compact set $\hat{K}$ such that $\hat{K} = [\epsilon, 1-\epsilon]$ for some fixed $\epsilon \in (0, 1/2)$ and that $\lambda_n \in K$ for all $n \in \mathbb{Z}$. Define the marked points $a = (a_n)_{n\in\mathbb{Z}}$ by

$$a_n = \begin{cases} \sum_{k=1}^{n} d_k & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\sum_{k=n+1}^{0} d_k & \text{if } n < 0. \end{cases}$$

Define a sequence of random elements $\omega = (\omega_n)_{n\in\mathbb{Z}}$ in $(\Omega, \mathcal{F})$ by

$$\omega_n = \begin{cases} \lambda_k & \text{if } n = a_k \text{ for some } k \in \mathbb{Z} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Note that ω can be viewed as a functional of $(a_n, \lambda_n)_{n\in\mathbb{Z}}$. Unless otherwise specified, ω always denotes the sparse environment and we reserve P and $\mathbb{P} := P(d\omega) \otimes P_\omega^0$ for the the sparse environment ω.

An asymptotic analysis of RWSRE was conducted in [27] and [17]. In particular, transience and recurrence criteria, classification of speed regimes, and stable limit laws are presented there. Some of these results are obtained using Palm-type dualities, a technique which we introduce in Chapter 2. Moreover, the asymptotic behavior of the random walk in the (stationary and ergodic) dual environment is related in [27, 17] to its counterpart in the sparse model to discover asymptotic behavior of the RWSRE. We make use of these results and techniques to derive large deviation results for the RWSRE.

1.4 Overview of thesis

The thesis is divided into four chapters, including the first introductory one. We now give a brief review of the content of each chapter.

In Chapter 2 we introduce a stationary dual $\widetilde{\omega}$ of the sparse environment ω. The dual environment is defined in the space of environments using a measure Q which is absolutely continuous with respect to the measure P, and applying a random time shift to the sparse environment that is considered under the law Q. The construction is a particular instance of a general phenomenon: given a stationary sequence of random variables, one can construct a cycle-stationary sequence of random variables via random time shifts. This action is reversible, i.e. one can construct a sequence of stationary random variables from a cycle-stationary sequence by using random time shifts and passing to an equivalent distribution. The stationary dual can be used as an effective tool to obtain asymptotic results for the RWSRE, including large deviation results. We then review asymptotic behavior, transience, and recurrence criteria for both the classical RWSRE and RWSRE. This serves as a preparation phase in our study since we mainly focus on LDP for X_n/n. The dual environment is introduced within this general context.

In Chapter 3 we state and prove quenched LDP for the RWSRE. Since the sparse environment is not stationary, we look at the problem from the dual environment perspective. Our first result is the LDP for the walk in the stationary dual. Because of the equivalence of the measures, main results obtained under the stationarity law Q for the dual environment can be carried over to our underlying setting, with a sparse environment distributed according to the law P. Moreover, applications of the ergodic theorem under the Q measure enable us to justify the existence of two key objects: existence of a *critical exponent* λ_{crit} which implies that both quenched and annealed rate functions are finite in a neighborhood of the asymptotic speed, and existence of the limit of modified logarithmic moment generating function $\Lambda(\cdot)$. Furthermore, we use the transience and recurrence criteria to convert weak large deviation results to full large deviations.

We study annealed large deviations of the sparse model in Chapter 4. We rely on known in literature results for the classical RWRE and exploit the fact that the averaging under the law Q can be shown to have the same effect on the quenched large deviations as the averaging with respect to P. In particular, under the assumption of bounded cycle lengths, we showed that the existence of LDP under $\mathbb{Q}$ measure implies the existence of the LDP under $\mathbb{P}$ measure. More importantly, the rate functions must coincide due to the uniqueness of rate functions.

The main results of the thesis are stated in Section 3.4 (quenched LDP for both the hitting times T_n and X_n) and Section 4.4 (annealed LDP for both T_n and X_n).

CHAPTER 2. DUAL ENVIRONMENT AND ASYMPTOTIC RESULTS FOR RWSRE

2.1 Inroduction

Our primary goal is to prove quenched and annealed large deviation principles for random walks in a sparse random environment (RWSRE). The RWSRE model is introduced in Section 1.3. Large deviations results for random walks in a stationary and ergodic environment are available due to the work of F. Comets, N. Gantert and O. Zeitouni [5]. However, the sparse environment is in general a non-stationary random sequence. We will introduce a dual stationary environment and relate the LDP for the RWSRE to corresponding large deviations results for the RWRE in the dual environment. A similar approach, relying on a dual environment, has been used in [17] to investigate asymptotic results of the RWSRE.

The rest of this chapter is organized as follows. In Section 2.2 we introduce the stationary dual $\widetilde{\omega}$ of the sparse environment. The dual environment is obtained by random time shifts. It has a rich structure that has been well studied in the general ergodic theory and it allows us to create a bridge between RWRE and RWSRE. The transition to a dual environment is facilitated by a general Key Equivalence Theorem that, in particular, generalizes the Key Renewal Theorem for renewal sequences (that is, sequences with i. i. d. increments/life cycles). An adaptation of this general equivalence to our framework is given in Section 2.3. Finally, we revisit some known asymptotic results for RWRE and RWSRE in Section 2.4.

2.2 Dual stationary environment

We first introduce a measurable *shift* θ_k from $(\Omega, \mathcal{F})$ into itself, which is defined for any $k \in \mathbb{Z}$ (possibly random) by

$$(\theta_k \omega)_n = \omega_{n+k}, \qquad n \in \mathbb{Z}.$$

Even though the sparse environment is, in general, a non-stationary sequence of random variables in the sparse model, it is *cycle-stationary* with respect to the probability measure P (Appendix), i.e.,

$$\theta_{d_n} \omega =_D \omega \qquad \text{under } P,\ n \in \mathbb{Z}$$

where $X =_D Y$ means that the random variables X and Y have the same distributions.

We now construct a stationary dual $\widetilde{\omega}$ of the sparse environment [31, 32]. One can assume without loss of generality that the underlying probability space supports a uniformly distributed random variable U on the interval $[0, 1)$ that is independent of the sparse environment ω. Let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$, i.e., $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$. Define $(\widetilde{\omega}, \widetilde{a}) = (\widetilde{\omega}_n, \widetilde{a}_n)_{n \in \mathbb{Z}}$ by

$$\widetilde{a}_n = a_n + \lfloor U d_0 \rfloor \qquad \text{and} \qquad \widetilde{\omega}_n = \begin{cases} \lambda_k & \text{if } n = \widetilde{a}_k \text{ for some } k \in \mathbb{Z} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

The following theorems constitute an adaptation to our model of the Palm's dualities [31, Chapter 8] between the distribution of ω under P and that of $\widetilde{\omega}$ under the probability measure Q, a probability measure equivalent to P.

Theorem 2.2.1 (see Theorem 2 in [32]). *Assume that $(\lambda_n, d_n)_{n \in \mathbb{N}}$ is a stationary and ergodic sequence under P and $E_P(d_0) < \infty$. Define a new probability measure Q on $(\Omega, \mathcal{F})$ by*

$$dQ = \frac{d_0}{E_P(d_0)} dP, \qquad \text{(stationary } Q). \tag{2.1}$$

Then:

(a) The dual environment $\widetilde{\omega} = \theta_{-\lfloor U d_0 \rfloor}\omega$ *is stationary and ergodic under the probability measure* Q*.*

(b) $E_Q(1/d_0) = 1/E_P(d_0) > 0$*.*

(c) $Q(\widetilde{a}_0 = k) = P(d_0 > k)/E_P(d_0)$*.*

(d) $Q(\omega \in \cdot | \widetilde{a}_0 = k) = P(\omega \in \cdot | d_0 > k), \ k \geq 0$*.*

The following corollary is an immediate consequence of Theorem 2.2.1.

Corollary 2.2.2 (see Corollary 2.2 in [17])**.** *Assume the conditions of Theorem 2.2.1. Then*

(a) $E_P(d_0^2) = E_P(d_0)E_Q(d_0)$*.*

(b) $E_Q(d_0) = 2E_Q(\widetilde{a}_0) + 1$*.*

The next Theorem is a reverse result which in particular implies that there is a unique cycle-stationary dual to any stationary environment. The Theorem exhibits an explicit reverse relation between the pairs Q and P and ω and $\widetilde{\omega}$.

Theorem 2.2.3 (see Theorem 1 in [32])**.** *Assume that the environment* $\widetilde{\omega}$ *is stationary and ergodic under the probability measure* Q*. Let* $\widetilde{a} = (\widetilde{a}_n)_{n\in\mathbb{Z}}$ *be the increasing two-sided sequence of points with* $\widetilde{a}_{-1} < 0 \leq \widetilde{a}_0$*. Define the probability measure* P *on* $(\Omega, \mathcal{F})$ *by*

$$dP = \frac{1}{d_0 E_Q(1/d_0)} dQ, \qquad \text{(length-debiasing } P\text{)}. \tag{2.2}$$

Then:

(a) The environment defined by $\omega = \theta_{\lfloor U d_0 \rfloor}\widetilde{\omega}$ *is cycle-stationary.*

(b) Conditionally on d_0 *the distribution of* $\widetilde{a}_0$ *is uniform on* $\{0, 1, \ldots, d_0 - 1\}$*.*

(c) $E_P(d_0) = 1/E_Q(1/d_0) < \infty$*.*

(d) $Q(\widetilde{a}_0 = k) = P(d_0 > k)/E_P(d_0)$.

(e) $Q(\omega \in \cdot | \widetilde{a}_0 = k) = P(\omega \in \cdot | d_0 > k),\ k \geq 0$.

From now on, we will denote $\mathbb{Q} := Q(d\widetilde{\omega}) \otimes P_{\widetilde{\omega}}$, i.e., for any measurable $G \in \mathcal{G}$

$$\mathbb{Q}(G) := \int_{\Omega} P_{\widetilde{\omega}}(G) Q(d\widetilde{\omega}) := E_Q\big(P_{\widetilde{\omega}}(G)\big).$$

2.3 Key equivalence theorem

Proof of Theorems 2.2.1 and 2.2.3 makes use of the Key Equivalence Theorem [32]. In particular, the equivalence of (a) and (e) in Theorem 2.3.1 is used to prove conditional uniformity and cycle-stationarity in Theorem 2.2.3. We next introduce an adaptation of the Key Equivalence Theorem to our setting. The result will be directly used in the upcoming chapters.

For non-negative integers k, define $N_k = \min\{n \geq 0 : \widetilde{a}_n \geq k\}$.

Theorem 2.3.1 (see Theorem 6 in [32])**.** *Under the assumptions of Theorem 2.2.1 and Theorem 2.2.3, the following statements are equivalent:*

(a) $\widetilde{\omega}$ *is stationary and ergodic under* Q.

(b) For all non-negative measurable functions f *and non-negative integers* n,

$$E_Q\Big[\sum_k 1_{\{\widetilde{a}_0 < k \leq \widetilde{a}_{N_n}\}} f(\theta_k \widetilde{\omega})/d_{N_k}\Big] = n E_Q\big[f(\widetilde{\omega})/d_0\big]. \tag{2.3}$$

(c) For all non-negative measurable functions f *and all non-negative integers* n *and* m,

$$E_Q\Big[\sum_{1 \leq i \leq N_n} f(\theta_{\widetilde{a}_i} \widetilde{\omega}) 1_{\{d_i > m\}}/d_i\Big] = n E_Q[1_{\{\widetilde{a}_0 = m\}} f(\omega)/d_0]. \tag{2.4}$$

(d) For all non-negative measurable functions f *and non-negative integers* n,

$$E_Q\Big[\sum_{1 \leq i \leq N_n} f(\theta_{\widetilde{a}_i} \widetilde{\omega})\Big] = n E_Q[f(\omega)/d_0], \tag{2.5}$$

and conditionally on ω the variable $\widetilde{a}_0$ is uniform on $\{0, 1, \ldots, d_0 - 1\}$, i.e.,

$$E_Q\big[1_{\{\widetilde{a}_0 = m\}} f(\omega)\big] = E_Q\big[1_{\{d_0 > m\}} f(\omega)/d_0\big], \qquad m \geq 0. \tag{2.6}$$

(e) For all non-negative measurable functions f and integers i,

$$E_Q\big[f(\theta_{\widetilde{a}_i}\widetilde{\omega})/d_0\big] = E_Q\big[f(\omega)/d_0\big] \tag{2.7}$$

and conditionally on ω the variable $\widetilde{a}_0$ is uniform on $\{0, 1, \ldots, d_0 - 1\}$.

Proof of this theorem is based on a "splitting of points" technique, which is standard in the context of Palm dualities (see, for instance, [31] and the proof of Kac's recurrence lemma in [11]).

2.4 Recurrence and transience criteria and asymptotic speed regimes for classical RWRE and RWSRE

Let ω be a fixed environment with $|\log \rho_k| < \infty$ for each $k \in \mathbb{Z}$. Define

$$\phi_{\omega,k}(m_-, m_+) := P_\omega^k(T_{-m_-} < T_{m_+})$$

for any $m_-,\ m_+ \in \mathbb{N}$ and $k \in [-m_-, m_+] \cap \mathbb{Z}$, i.e., $\phi_{\omega,k}(m_-, m_+)$ is the probability of the random walk hitting $-m_+$ before hitting m_+ when it starts from the location k. As a function of k, $\phi_{\omega,k}(m_-, m_+)$ is harmonic for the random walk by the Markov Property. That is, it satisfies the equation

$$\phi_{\omega,k}(m_-, m_+) = (1 - \omega_k)\phi_{\omega,k-1}(m_-, m_+) + \omega_k \phi_{\omega,k+1}(m_-, m_+) \tag{2.8}$$

with the boundary conditions $\phi_{\omega,-m_-}(m_-, m_+) = 1$ and $\phi_{\omega,m_+}(m_-, m_+) = 0$. Then an explicit formula can be given for the hitting probabilities [39, Section, 2.1]:

$$\phi_{\omega,k}(m_-, m_+) = \frac{\sum_{i=k+1}^{m_+} \prod_{j=k+1}^{i-1} \rho_j}{\sum_{i=k+1}^{m_+} \prod_{j=k+1}^{i-1} \rho_j + \sum_{i=-m_-+1}^{k} \prod_{j=i}^{k} \rho_j^{-1}}. \tag{2.9}$$

Let

$$S(\omega) = \sum_{k=1}^{\infty} \rho_1 \rho_2, \ldots, \rho_k \qquad \text{and} \qquad F(\omega) = \sum_{k=0}^{\infty} \rho_1^{-1} \rho_2^{-1}, \ldots, \rho_k^{-1}. \tag{2.10}$$

Let $T_0 = 0$. Define $T_k = \min\{n : X_n = k\}$ for $k \in \mathbb{Z}$ and

$$\tau_k = \begin{cases} T_k - T_{k-1} & \text{if } k > 0 \\ T_k - T_{k+1} & \text{if } k > 0 \end{cases} \tag{2.11}$$

with the convention that $\infty - \infty = \infty$. The next two propositions play key roles in the recurrence and transience criteria for the RWRE [39], and are deduced from asymptotic behavior of exit probabilities 2.9 as $|m_\pm| \to \infty$.

Proposition 2.4.1. *Let ω be a stationary and ergodic environment and let $X = (X_n)_{n \geq 0}$ be the random walk in this environment. Assume that $E_P(\log \rho_0) \in \overline{\mathbb{R}}$ is well-defined (exists as an extended real number). Then*

(a) $P(S(\omega) < \infty) = 1$ then, $X_n \to +\infty$ as $n \to \infty$, $\mathbb{P}-a.s.$,

(b) $P(F(\omega) < \infty) = 1$ then, $X_n \to -\infty$ as $n \to \infty$, $\mathbb{P}-a.s.$,

(c) $P(S(\omega) = F(\omega) = \infty) = 1$ then, $\limsup_{n\to\infty} X_n = +\infty$ and $\liminf_{n\to\infty} X_n = -\infty$, $\mathbb{P}-a.s.$

Proposition 2.4.2. *Let ω be a stationary and ergodic environment and let $X = (X_n)_{n \geq 0}$ be the random walk in this environment. Assume that $E_P(\log \rho_0) \in \overline{\mathbb{R}}$ is well-defined (exists as an extended real number). Then*

(a) $P(S(\omega) < \infty) = 1$ if and only if $E_P(\log \rho_0) < 0$,

(b) $P(F(\omega) < \infty) = 1$ if and only if $E_P(\log \rho_0) > 0$,

(c) $P(S(\omega) = F(\omega) = \infty) = 1$ if and only if $E_P(\log \rho_0) = 0$.

As a result, we have the following recurrence and transience criteria from [28] and [2].

Theorem 2.4.3. *[2, 28] Let ω be a stationary and ergodic environment and let $X = (X_n)_{n\geq 0}$ be the random walk in this environment. Assume that $E_P(\log\rho_0) \in \overline{\mathbb{R}}$ is well-defined. Then*

(a) If $E_P(\log\rho_0) < 0$ then, $X_n \to +\infty$ as $n \to \infty$, $\mathbb{P}-a.s.$,

(b) If $E_P(\log\rho_0) > 0$ then, $X_n \to -\infty$ as $n \to \infty$, $\mathbb{P}-a.s.$,

(c) If $E_P(\log\rho_0) = 0$ then $\limsup_{n\to\infty} X_n = +\infty$ and $\liminf_{n\to\infty} X_n = -\infty$, $\mathbb{P}-a.s.$

Next, we discuss the law of large numbers for the classical RWRE. Define

$$\overline{S} := \sum_{i=1}^{\infty} \frac{1}{\omega_{-i}} \prod_{j=0}^{i-1} \rho_{-j} + \frac{1}{\omega_0} = 1 + 2\sum_{i=0}^{\infty}\prod_{j=0}^{i} \rho_{-j} \tag{2.12}$$

and

$$\overline{F} := \sum_{i=1}^{\infty} \frac{1}{1-\omega_i} \prod_{j=0}^{i-1} \rho_j^{-1} + \frac{1}{1-\omega_0}. \tag{2.13}$$

The criteria for the asymptotic speed of the walk is given in the following theorem.

Theorem 2.4.4. *[28, 2] Let ω be a stationary and ergodic environment and let $X = (X_n)_{n\geq 0}$ be the random walk in this environment. Assume that $E_P(\log\rho_0) \in \overline{\mathbb{R}}$ is well-defined. Then*

(a) If $E_P(\overline{S}) < \infty$, then $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{E_P(\overline{S})}$, $\mathbb{P} - \text{a. s.}$

(b) If $E_P(\overline{F}) < \infty$, then $\lim_{n\to\infty} \frac{X_n}{n} = -\frac{1}{E_P(\overline{F})}$, $\mathbb{P} - \text{a. s.}$

(c) If $E_P(\overline{S}) = \infty$ and $E_P(\overline{S}) = \infty$, then $\lim_{n\to\infty} \frac{X_n}{n} = 0$, $\mathbb{P} - \text{a. s.}$

Furthermore, the three cases listed above are exhausting all the possibilities.

In the case when the environment is i.i.d, conditions (a)–(c) of the Theorem 2.4.4 can be replaced by $E_p(\rho_0) < 1$, $E_P(\rho_0^{-1}) < 1$ and $E_P(\rho_0)^{-1} \leq 1 \leq E_P(\rho_0^{-1})$, respectively. We then have

Corollary 2.4.5. *Let ω be a stationary and ergodic environment and let $X = (X_n)_{n\geq 0}$ be the random walk in this environment. Assume that $E_P(\log \rho_0) \in \overline{\mathbb{R}}$ is well-defined. Then*

(a) If $E_P(\rho_0) < 1$, then $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-E_P(\rho_0)}{1+E_P(\rho_0)}$, $\mathbb{P} - \text{a.s.}$

(b) If $E_P(\rho_0^{-1}) < 1$, then $\lim_{n\to\infty} \frac{X_n}{n} = -\frac{1-E_P(\rho_0^{-1})}{1+E_P(\rho_0^{-1})}$, $\mathbb{P} - \text{a.s.}$

(c) If $E_P(\rho_0)^{-1} \leq 1 \leq E_P(\rho_0^{-1})$, then $\lim_{n\to\infty} \frac{X_n}{n} = 0$, $\mathbb{P} - \text{a.s.}$

The proof of the Theorem 2.4.4 is based on the hitting time decompositions, c.f., (3.18). In particular, the next three lemmas (details in [39]) are used to relate the asymtotics of hitting times T_n to the asymptotics of the walk X_n.

Lemma 2.4.6. *Let ω be a stationary and ergodic environment and let $X = (X_n)_{n\geq 0}$ be the random walk in this environment. Assume that $E_P(\log \rho_0) \in \overline{\mathbb{R}}$ is well-defined. Then*

(a) $\mathbb{E}(\tau_1) = E_P(\overline{S})$.

(b) $\mathbb{E}(\tau_{-1}) = E_P(\overline{F})$.

Lemma 2.4.7. *Let ω be a stationary and ergodic environment and let $X = (X_n)_{n\geq 0}$ be the random walk in this environment. Assume that $E_P(\log \rho_0) \in \overline{\mathbb{R}}$ is well-defined. Suppose that $\limsup_{n\to\infty} X_n = +\infty$, $\mathbb{P}-\text{a.s.}$ Then the sequence $\{\tau_n\}_{n\geq 1}$ defined in (2.11) is stationary and ergodic. In particular,*

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \tau_i = \mathbb{E}(\tau_1) = E_P(\overline{S}), \qquad \mathbb{P} - \text{a.s.}$$

Lemma 2.4.8. *Let ω be a stationary and ergodic environment and let $X = (X_n)_{n\geq 0}$ be the random walk in this environment. Assume that $E_P(\log \rho_0) \in \overline{\mathbb{R}}$ is well-defined and let $\alpha \in (0, +\infty]$. Then*

(a) If $E_P(\log \rho_0) < 0$ and $\lim_{n\to\infty} \frac{T_n}{n} = \alpha$, *then* $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\alpha}$, $\mathbb{P} - \text{a.s.}$

(b) If $E_P(\log\rho_0) > 0$ *and* $\lim_{n\to\infty}\frac{T_{-n}}{n} = \alpha$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = -\frac{1}{\alpha}$, $\mathbb{P}-\mathrm{a.\,s.}$

(c) If $E_P(\log\rho_0) = 0$ *and* $\lim_{n\to\infty}\frac{T_n}{n} = \lim_{n\to\infty}\frac{T_{-n}}{n} = \infty$, *then* $\lim_{n\to\infty}\frac{X_n}{n} = 0$, $\mathbb{P}-\mathrm{a.\,s.}$

The next two theorems give the transience and recurrence criteria for the RWSRE. An auxiliary Markov chain is defined via the times of successive visits to the random point set and recurrence and transience criteria for the walk X_n deduced from the recurrence and transience criteria of this Markov chain.

Theorem 2.4.9. *[17, 27] Suppose that the following three conditions are satisfied:*

1. *The sequence of pairs* $(d_n, \lambda_n)_{n\in\mathbb{Z}}$ *is stationary and ergodic*
2. $E_P(\log\xi_0)$ *exists (possibly infinite)*
3. $E_P\big(\log d_0\big) < +\infty$.

Then:

(a) $E_P(\log\xi_0) < 0$ *implies* $\lim_{n\to\infty} X_n = +\infty$, $\mathbb{P}-\mathrm{a.\,s.}$

(b) $E_P(\log\xi_0) > 0$, *implies* $\lim_{n\to\infty} X_n = -\infty$, $\mathbb{P}-\mathrm{a.\,s.}$

(c) $E_P(\log\xi_0) = 0$ *implies* $\liminf_{n\to\infty} X_n = -\infty$ *and* $\limsup_{n\to\infty} X_n = +\infty$, $\mathbb{P}-\mathrm{a.\,s.}$

Theorem 2.4.10. *[17, 27] Suppose that the following conditions hold:*

1. *The sequence of pairs* $(d_n, \lambda_n)_{n\in\mathbb{Z}}$ *is stationary and ergodic*
2. *The random variables* d_n *are i.i.d.*
3. $E_P\big(|\log\xi_0|\big) < +\infty$ *while* $E_P(\log d_0) = +\infty$.

Then, $\liminf_{n\to\infty} X_n = -\infty$ *and* $\limsup_{n\to\infty} X_n = +\infty$, $\mathbb{P}-\mathrm{a.\,s.}$

We use these criteria to specify the types of quenched large deviations for the RWSRE in Chapter 3.

CHAPTER 3. QUENCHED LDP RESULTS

3.1 Introduction

In order to prove a LDP for a sequence of probability measures, we usually identify a candidate rate function by using basic probability inequalities (such as Chebyshev's exponential inequality) and then verify large deviations upper and lower bounds. A common technique to prove a LDP result for a RWRE is first to obtain a LDP for the hitting times T_n and then to transform it into a LDP for the position of the random walk X_n. We use a similar approach for the dual environment, and obtain a quenched LDP for the position of RWSRE through a corresponding result for the hitting times $T_.$

The rest of the chapter is organized as follows. In Section 3.2 we review the LDP for RWRE and give a brief description of methods used to obtain these results. We observe in Section 3.3 that a quenched LDP for the random walk X_n in the stationary dual environment is a direct consequence of the results in [5]. The main results of the chapter are contained in Section 3.4, where the large deviation results included in Section 3.3 are transformed into corresponding P_ω-quenched statements under the underlying law P of the environment ω. In Section 3.5 we study basic properties of the quenched rate function, in particular its relation to the logarithmic moment generating function computed under the underlying law P. Some of the results obtained in Section 3.5 are used in Chapter 4 to facilitate the proof of annealed LDP for RWSRE.

3.2 Quenched LDP for RWRE

Recall that a function $I : \mathbb{R} \to [0, \infty]$ is a rate function if it is not identically infinite and is lower semi-continuous, that is its level sets are closed. It is a good rate function if its level sets are compact. We say that a sequence of real-valued random variables $\{X_n\}$ satisfies a quenched large deviation principle with speed n and deterministic rate function I if, for any Borel set A,

$$-I(A^\circ) \le \liminf_{n\to\infty} \frac{1}{n} \log P_\omega(X_n \in A) \le \limsup_{n\to\infty} \frac{1}{n} \log P_\omega(X_n \in A) \le -I(\overline{A}), \qquad \eta - \text{a.s.} \tag{3.1}$$

A sequence of real-valued random variables $\{X_n\}$ satisfies the annealed LDP with speed n and rate function I if, for any Borel set A,

$$-I(A^\circ) \le \liminf_{n\to\infty} \frac{1}{n} \log P_\eta(X_n \in A) \le \limsup_{n\to\infty} \frac{1}{n} \log P_\eta(X_n \in A) \le -I(\overline{A}) \tag{3.2}$$

A LDP is called weak if the upper bound in 3.1 or 3.2 holds only with $\overline{A}$ compact. Let $\{A_n\}$ be a sequence of events, subsets of $\Omega \times \mathbb{Z}^{\mathbb{N}}$. Then,

$$c := \limsup_{n\to\infty} \frac{1}{n} \log P_\eta(A_n) \ge \limsup_{n\to\infty} \frac{1}{n} \log P_\omega(A_n), \qquad \eta - \text{a.s.} \tag{3.3}$$

Further,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\eta(A_n) \ge \liminf_{n\to\infty} \log P_\omega(A_n), \qquad \eta - \text{a.s.} \tag{3.4}$$

In particular, if a sequence of $\mathbb{R}$-valued random variables $\{X_n\}$ satisfies both annealed and quenched LDP's with rate functions $I_a(\cdot)$ and $I_q(\cdot)$, respectively, then

$$I_a(x) \le I_q(x), \qquad \forall x \in \mathbb{R}.$$

Intuitively speaking, annealed deviation probabilities allow for atypical fluctuations of the environment, and hence are not smaller than corresponding quenched deviation probabilities [39]. It is also well known that if the LDP hold for a sequence of probability measures then the rate function is uniquely defined. (See, for instance, [8, Lemma 4.1.4].)

Let $M_1(\Omega)$, $M_1^s(\Omega)$, $M_1^e(\Omega)$ be the spaces of probability measures, stationary probability measures, and ergodic probability measures on Ω respectively. These spaces are equipped with the topology induced by the weak convergence of measures. We denote

$$M_1^e(\Omega)^+ := \{\eta \in M_1^e(\Omega) : \int \log \rho_0(\omega)\eta(d\omega) \leq 0\}.$$

Let $K \subset (0,1)$ be some fixed compact subset of $(0,1)$. For any subset $A \subset M_1(\Omega)$, we define

$$M^K = M \cap \{\eta : \text{supp}(\eta_0) \subset K \subset (0,1)\},$$

where η_0 denotes the marginal of the law η. Let $\omega = (\omega_k)_{k\in\mathbb{Z}} \in \Omega$ be a sequence of random variables which is used as an environment for the random walk, and define

$$\rho_k = \rho_k(\omega) = \frac{1-\omega_k}{\omega_k}.$$

Define the maps $F : \Omega \to \Omega$ by $(F\omega)_k = 1 - \omega_k$, and $\text{Inv} : \Omega \to \Omega$ by $(\text{Inv}\ \omega)_k = (F\omega)_{-k} = 1 - \omega_{-k}$.

Introduce, for any $t \in \mathbb{R}$,

$$\begin{aligned} \varphi(t,\omega) &= E_\omega\big(e^{t\tau_1} I(\tau_1 < \infty)\big), \qquad f(t,\omega) = \log \varphi(t,\omega) \\ \overline{\varphi}(t,\omega) &= E_\omega\big(e^{t\tau_{-1}} I(\tau_{-1} < \infty)\big), \qquad \overline{f}(t,\omega) = \log \overline{\varphi}(t,\omega) \\ G(t,\eta,u) &= tu - E_\eta\Big(f(t,\omega)\Big). \end{aligned}$$

Throughout, we use the notation f' to denote the derivative of f with respect to the first variable. Define a quenched rate function

$$I_\eta^{\tau,q}(u) = \sup_{t\in\mathbb{R}} G(t,\eta,u).$$

Note that the environment in the classical RWRE model is always assumed to be stationary. Under the measure Q, the environment ω of the RWSRE becomes a Markov chain. The random shift $\omega \to \widetilde{\omega}$ introduced in Chapter 2 can be thought as an explicit transformation making the Markov chain's initial distribution be its stationary one. Therefore, when using the above notation for the law Q one should consider its stationary "variant", namely $G(t, Q, u)$ should be defined in terms of $\widetilde{\omega}$ as $E_Q\big(f(t, \widetilde{\omega})\big)$. For future references, we explicitly define

$$I_Q^{\tau,q}(u) = \sup_{t \in \mathbb{R}} \Big(tu - -E_\eta \Big[\log E_{\widetilde{\omega}} \big(e^{t\tau_1} I(\tau_1 < \infty) \big) \Big] \Big). \tag{3.5}$$

We next state the main results of [5].

Theorem 3.2.1. *[5] Assume that $P \in M_1^e(\Omega)^K$. Then the distributions of T_n/n under P_ω satisfy a weak LDP for $P-$a.e. environment ω with the convex rate function $I_P^{\tau,q}$ (independent of the environment ω).*

The statement of Theorem 3.2.1 can be strengthened to a full LDP if $P \in M_1^e(\Omega)^{+,K}$, namely if the underlying random walk is transient to the right (see Theorem 2.4.3 in Chapter 2).

Proposition 3.2.2. *[5] Assume that $P \in M_1^e(\Omega)^K$. Then*

$$\int \overline{f}(t, \omega) \eta(d\omega) = \int f(t, \omega) P(d\omega) + \int \log \rho_0(\omega) P(d\omega). \tag{3.6}$$

Moreover, if $P \in M_1^e(\Omega)^{+,K}$, then the distributions of T_{-n}/n under P_ω satisfy a weak LDP for $P-$a.e. environment ω with the rate function

$$I_P^{-\tau,q}(u) := I_P^{\tau,q}(u) - \int \log \rho_0(\omega)) P(d\omega), \qquad 1 \le u < \infty. \tag{3.7}$$

To state a quenched LDP for the distributions of X_n/n, using the quenched rate functions $I_P^{\tau,q}$ and $I_P^{-\tau,q}$, we define the rate function

$$I_\eta^q(v) = \begin{cases} v I_\eta^{\tau,q}\big(\frac{1}{v}\big) & 0 \le v \le 1 \\ |v| I_\eta^{-\tau,q}\big(\frac{1}{|v|}\big) & -1 \le v \le 0. \end{cases} \tag{3.8}$$

Theorem 3.2.3. *[5] Assume that $P \in M_1^e(\Omega)^K$.*

(a) If $E_P(\log \rho_0) \le 0$, then, for $\eta-$a.e. environment ω, the distributions of X_n/n under P_ω satisfy a LDP with convex rate function I_P^q and

$$I_P^q(0) = \lim_{v \downarrow 0} v I_P^{\tau,q}\Big(\frac{1}{v}\Big).$$

(b) *If* $E_P(\log \rho_0) > 0$, *define* $P^{Inv} := P \circ Inv^{-1}$. *Then* $E_{P^{Inv}}(\log \rho_0) < 0$, *and the LDP for* X_n/n *holds with good convex rate function*

$$\hat{I}^q_P(v) := I^q_{P^{Inv}}(-v).$$

Proof of these theorems make use of the following lemma that sheds light on the key features of the rate functions in quenched and annealed settings.

Lemma 3.2.4. *[5] Assume* $P \in M^e_1(\Omega)^{+,K}$. *Then*

(a) *The convex function* $I^{\tau,q}_P : \mathbb{R} \to [0, \infty]$ *is non-increasing on* $[1, E_P(\overline{S})]$, *and non-decreasing on* $[E_P(\overline{S}), \infty)$. *Furthermore, if* $E_P\big(\overline{S}(\omega)\big) < \infty$ *then* $I^{\tau,q}_P\big(E_P(\overline{S})\big) = 0$.

(b) *For any* $1 < u < E_P(\overline{S})$, *there exists a unique* $t_0 = t_0(u, P)$ *such that* $t_0 < 0$ *and*

$$u = \int f'(t_0, \omega)P(d\omega). \tag{3.9}$$

Further,

$$\inf_{P \in M^e_1(\Omega)^{+,K}} t_0(u, P) > -\infty. \tag{3.10}$$

(c) *There is a deterministic* $t_{\text{crit}} \in [0, \infty)$, *depending only on* P, *such that for* $t < t_{\text{crit}}$, $E_\omega\big[e^{t\tau_1}\big] < \infty$ *for* $P-$*a.e. and for* $t_{\text{crit}} < t$, $E_\omega\big[e^{t\tau_1}\big] = \infty$ *for* $P-$*a.e.*

(d) *With*

$$u_{crit} = \begin{cases} \infty & E_P\Big[\dfrac{E_\omega(\tau_1 e^{t_{\text{crit}}\tau_1})}{E_\omega(e^{t_{\text{crit}}\tau_1})}\Big] = \infty \\ \int f'(t_{\text{crit}}, \omega)P(d\omega) & E_P\Big[\dfrac{E_\omega(\tau_1 e^{t_{\text{crit}}\tau_1})}{E_\omega(e^{t_{\text{crit}}\tau_1})}\Big] < \infty \end{cases}$$

and $E_P(\overline{S}) \le u < u_{crit}$, *there exists a unique* $t_0 := t_0(u, P)$ *such that* $t_0 \ge 0$ *and* *3.16* *holds.*

3.3 $P_{\widetilde{w}}$-quenched LDP under the stationary law Q for $\widetilde{w}$

Throughout the rest of this chapter we assume that the conditions of Theorem 2.4.9 are satisfied. That is, we assume the following:

Assumption 3.3.1. *The following three conditions are satisfied:*

1. *The sequence of pairs* $(d_n, \lambda_n)_{n\in\mathbb{Z}}$ *is stationary and ergodic*

2. $E_P(\log \xi_0)$ *exists (possibly infinite)*

3. $E_P\big(\log d_0\big) < +\infty$.

Recall the rate function $I_Q^{\tau,q}$ from (3.5). By virtue of Theorem 2.2.1, the following results are an immediate consequence of, respectively, Theorem 3.2.1 , Proposition 3.2.2, and Theorem 3.2.3.

Theorem 3.3.2. *Let Assumption 3.3.1 hold. Then the distributions of* T_n/n *under* $P_{\widetilde{\omega}}$ *satisfy a weak LDP, for* $Q-$*a.e. environment* $\widetilde{w}$*, with deterministic convex rate function* $I_Q^{\tau,q}$*. Further,* $I_Q^{\tau,q}(\cdot)$ *is decreasing on* $[1, E_Q\big(\overline{S}(\widetilde{w})\big)]$ *and increasing on* $[E_Q\big(\overline{S}(\widetilde{w})\big), \infty)$*. If* v_P *is finite then* $I_P^{\tau,q}\big(1/\mathrm{v}_P\big) = 0$.

Proposition 3.3.3. *Let Assumption 3.3.1 hold. Then the distribution of* T_{-n}/n *under* $P_{\widetilde{\omega}}$ *satisfy for* $Q-$*a.e. environment* $\widetilde{w}$ *a weak LDP with deterministic rate function*

$$I_Q^{-\tau,q}(u) := I_Q^{\tau,q}(u) - E_Q\big(\log \rho_0(\widetilde{\omega})\big), \qquad 1 \le u < \infty.$$

Moreover,

$$E_Q\big[\overline{f}(t,\widetilde{\omega})\big] = E_Q\big[f(t,\widetilde{\omega})\big] + E_Q\big(\log \rho_0(\widetilde{\omega})\big). \tag{3.11}$$

Theorem 3.3.4. *Let Assumption 3.3.1 hold. Then, for* $Q-$*a.e. environment* $\widetilde{w}$*, the distributions of* X_n/n *under the law* $P_{\widetilde{\omega}}$ *satisfy a large deviation principle with convex rate function* I_Q^q.

Note that since $\omega_k \in \hat{K}$, we have that $Q \in M_1^e(\Omega)^{\hat{K}}$. Moreover, if we assume that $E_P\big(\log \xi_0(\widetilde{\omega})\big) \le 0$, the weak LDP for T_n/n becomes full LDP, i.e., $Q \in M_1^e(\Omega)^{+,\hat{K}}$ [17] (see also Theorem 2.4.9 in Chapter 2 of this thesis).

3.4 P_ω-quenched LDP under the law P

This section includes main results we obtained in Chapter 2 of this thesis.

In view of Theorem 2.2.1, the original law P of the environment is absolutely continuous with respect to the measure Q. Therefore, the results in the previous section hold for P-a.e. environment $\widetilde{\omega}$. Recall from Section 2.2 that

$$\widetilde{a}_n = a_n + \lfloor U d_0 \rfloor \qquad \text{and} \qquad \widetilde{\omega}_n = \begin{cases} \lambda_k & \text{if } n = \widetilde{a}_k \text{ for some } k \in \mathbb{Z} \\ \frac{1}{2} & \text{otherwise.} \end{cases},$$

where U is a random variable uniformly distributed on the interval $[0,1)$ which is independent of the sparse environment ω, and $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$, that is $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \le x\}$. Furthermore, $Q\big(\omega = (\widetilde{\omega})\big) = 1/E_P(d_0) > 0$. Thus, the results stated in the previous section remain to be valid for P-a.e. environment ω if the quenched measure $P_{\widetilde{\omega}}$ is replaced by the underlying quenched law P_ω. Recall from [17] that $E_Q\big(\overline{S}(\widetilde{w})\big) = 1/\mathrm{v}_P$, where $\mathrm{v}_P = \lim_{n\to\infty} X_n/n$, $P-$a.s., denotes the asymptotic speed. Observe that

$$E_Q\big(\log \rho_0(\widetilde{\omega})\big) \;=\; \frac{1}{E_P(d_0)} E_P\big(d_0 \cdot \log \xi_0 \cdot P_\omega(U=0)\big) = \frac{1}{E_P(d_0)} E_P(\log \xi_0).$$

Summarizing the above discussion, we have proved the following:

Theorem 3.4.1. *Let Assumption 3.3.1 hold. Then:*

(i) The distributions of T_n/n under P_ω satisfy a weak LDP, for $P-$a.e. environment w, with deterministic convex rate function $I_Q^{\tau,q}$.

(ii) $I_Q^{\tau,q}(\cdot)$ is decreasing on $[1, \mathrm{v}_P]$ and increasing on $[1/\mathrm{v}_P, \infty)$.

(iii) If v_P is finite, then $I_P^{\tau,q}\big(1/\mathrm{v}_P\big) = 0$.

Proposition 3.4.2. *Let Assumption 3.3.1 hold. Then the distribution of T_{-n}/n under P_ω satisfy, for $P-$a.e. environment w, a weak LDP with deterministic rate function*

$$I_Q^{-\tau,q}(u) := I_Q^{\tau,q}(u) - \frac{1}{E_P(d_0)} E_P(\log \xi_0), \qquad 1 \le u < \infty.$$

Moreover,

$$E_Q\big[\overline{f}(t,\widetilde{\omega})\big] = E_Q\big[f(t,\widetilde{\omega})\big] + \frac{1}{E_P(d_0)} E_P(\log \xi_0). \tag{3.12}$$

Furthermore, if $E_P(\log \xi_0) < 0$, the weak LDP for T_n/n becomes a full LDP.

We next transform the LDP for the hitting times into a LDP for the position of the random walk. The argument is similar to the one for RWRE [5]. For the sake of completeness we outline below the argument giving the asymptotic behavior of $P_\omega\Big(\frac{X_n}{n} \geq v\Big)$ for $v > \mathrm{v}_P$.

Theorem 3.4.3. *Let Assumption 3.3.1 hold. Then, for $Q-$a.e. environment $\widetilde{w}$, the distributions of X_n/n under the law $P_{\widetilde{\omega}}$ satisfy a large deviation principle with convex rate function I_Q^q. Moreover, if $E_P\big(\log \xi_0\big) \leq 0$, then the weak LDP for T_n/n becomes a full LDP.*

We will next verify the theorem for the upper tails $P_\omega\Big(\frac{X_n}{n} \geq v\Big)$ and $v > \mathrm{v}_P$. We have

$$P_\omega\Big(\frac{X_n}{n} \geq v\Big) \leq P_\omega\Big(T_{\lfloor nv \rfloor} \leq n\Big) = P_\omega\Big(\frac{T_{\lfloor nv \rfloor}}{\lfloor nv \rfloor} \leq \frac{n}{\lfloor nv \rfloor}\Big).$$

The LDP for the hitting times and the monotonicity of $I_Q^{\tau,q}$ now imply

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{X_n}{n} \geq v\Big) \leq -v I_Q^{\tau,q}\Big(\frac{1}{v}\Big),$$

which yields large deviation upper bound for upper tail by the monotonicity of I_Q^q. The same argument can be run for large deviation upper bound for lower tail for $v < 0$, by considering the hitting times $T_{\lceil nv \rceil}$.

Similarly, for any $0 < \beta < \delta/2$,

$$P_\omega\Big(v - \delta \leq \frac{X_n}{n} \leq v + \delta\Big) \geq P_\omega\big((1-\beta)n \leq T_{\lceil nv \rceil} \leq n\big).$$

It then follows from the LPD for the hitting times that for $v \geq 0$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{X_n}{n} \in (v-\delta, v+\delta)\Big) \geq -v I_Q^{\tau,q}\Big(\frac{1-\beta}{v}\Big), \qquad P-\text{a.e.},$$

and the lower bound is obtained by letting $\beta \to 0$. We conclude this section with the remark that the same argument also yields the lower bounds for $v < 0$, using this time the function $I_Q^{-\tau,q}$.

3.5 Properties of the quenched rate function $I_Q^{\tau,q}$

Many features of the quenched rate functions for RWRE, and hence of $I_Q^{\tau,q}$, are known due to the work of [9, 13, 20, 21] and others. It is an interesting direction for future work to verify counterparts of those properties for the RWSRE. In this section we focus only on a few very basic properties of the rate function.

To translate properties of the quenched rate function $I_Q^{\tau,q}$ into terms directly related to the law P, we will again exploit the fact that the measures P and Q are equivalent along with the observation $Q\big(\omega = (\widetilde{\omega})\big) = 1/E_P(d_0) > 0$. These two observations combined together imply that any claim that holds true for Q-a.e. environment $\widetilde{\omega}$ also holds true for P-a.e. environment ω. This enables us to carry over to the (ω, P)-setting several auxiliary lemmas of [5] that, by virtue of their arguments, hold true in the $(\widetilde{\omega}, Q)$- (stationary and ergodic) setting.

We turn now to study properties of the quenched rate function $I_Q^{\tau,q}$, which is introduced in (3.5). For $t \in \mathbb{R}$, define

$$\Lambda(t) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(t, \theta_i\omega) \tag{3.13}$$

and let $I_P^{\tau,q}$ be the Fenchel-Legendre transform of Λ, i.e.,

$$I_P^{\tau,q}(u) := \Lambda^*(u) := \sup_{t\in\mathbb{R}} \big[tu - \Lambda(t)\big]. \tag{3.14}$$

Lemma 3.5.1. *Let Assumption 3.3.1 hold. Then $\Lambda(t)$ exists for each $t \in \mathbb{R}$ $P-$a.s.*

Proof. It suffices to show that the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(t, \theta_i\omega) = E_Q\big(f(t, \omega)\big)$$

exists $Q-$a.e. for each $t \leq 0$.

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(t,\theta_i\omega) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\Big(t,\theta_i\big(\theta_{\lfloor Ud_0\rfloor}\widetilde{\omega}\big)\Big)\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\Big(t,\theta_{\lfloor Ud_0\rfloor}\big(\theta_i\widetilde{\omega}\big)\Big)\\ &= E_Q\Big(f\big(t,\theta_{\lfloor Ud_0\rfloor}\widetilde{\omega}\big)\Big)\\ &= E_Q\big(f(t,\omega)\big) \qquad Q-\text{a.s.}\end{aligned}$$

where the first equality is simply due to reversing the definition of $\widetilde{\omega}$. We changed the order of the left shift operators in the third inequality. Since $f\big(t,\theta_{\lfloor Ud_0\rfloor}(\cdot)\big)$ is a non-negative Borel measurable function of the dual environment, and $E_Q\big|f\big(t,\omega\big)\big| < \infty$, the limit exists by the ergodic theorem. However, this convergence is a $P-$a.s. convergence by the definition of the law Q for each $t \geq 0$. The proof of the case when $t \leq 0$ is similar. Finally, if we apply the change of measure in the limit function, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(t,\theta_i\omega) = \frac{1}{E_P(d_0)}E_P\big(d_0 f(t,\omega)\big) =: \Lambda(t) \qquad P-\text{a.s.} \tag{3.15}$$

for each $t \in \mathbb{R}$. □

Next, using the argument outlined in the beginning of this section, we have:

Lemma 3.5.2. *Let Assumption 3.3.1 hold. Then there is a deterministic* $t_{\text{crit}} := t_{\text{crit}}(P) \in [0,\infty]$ *such that*

$$\varphi(t,\omega)\begin{cases} <\infty & t < t_{\text{crit}}, \qquad P-a.s.\\ =\infty & t > t_{\text{crit}}, \qquad P-a.s.\end{cases}$$

Lemma 3.5.3. *Let Assumption 3.3.1 hold. Then*

(a) For any $1 < u < 1/\mathrm{v}_P$, *there exists a unique* $t_0 = t_0(u,P)$ *such that* $t_0 < 0$ *and*

$$u = \int f'(t_0,\omega)Q(d\omega). \tag{3.16}$$

(b) With

$$u_{crit} = \begin{cases} \infty & E_Q\Big[\frac{E_{\widetilde{\omega}}(\tau_1 e^{t_{\text{crit}}\tau_1})}{E_{\widetilde{\omega}}(e^{t_{\text{crit}}\tau_1})}\Big] = \infty \\ \int f'(t_{\text{crit}}, \omega) Q(d\widetilde{\omega}) & E_Q\Big[\frac{E_{\widetilde{\omega}}(\tau_1 e^{t_{\text{crit}}\tau_1})}{E_{\widetilde{\omega}}(e^{t_{\text{crit}}\tau_1})}\Big] < \infty \end{cases} \tag{3.17}$$

and $1/\mathrm{v}_P \leq u < u_{crit}$*, there exists a unique* $t_0 := t_0(u, P)$ *such that* $t_0 \geq 0$ *and* (3.16) *holds.*

The last lemma is a crucial ingredient for the proof of the LDP under the stationary law Q. In particular, it implies that that the logarithmic generating function $\Lambda(t)$ is smooth enough to guarantee that its Fenchel-Legendre transform can be defined as a maximum and unique critical point of a smooth function.

We conclude this section with a discussion of the continued fraction representation of $\varphi(t, \omega)$ under the law P. Let $\tau_1' + 1$ be the first hitting time of 0 after time 1 and let $1 + \tau_1' + \tau_1''$ be the first hitting time of $+1$ after time $1 + \tau_1'$. We can write a pathwise decomposition of τ_1 as

$$\tau_1 = I(X_1 = 1) + I(X_1 = -1)(1 + \tau_1' + \tau_1''). \tag{3.18}$$

Then,

$$\begin{aligned} \varphi(t, \omega) &= E_\omega\big(e^{t\tau_1} I(\tau_1 < \infty)\big) \\ &= P_\omega\big(X_1 = 1\big) E_\omega\big(e^{t\tau_1} I(\tau_1 < \infty) | X_1 = 1\big) \\ &\qquad + P_\omega\big(X_1 = -1\big) E_\omega\big(e^{t\tau_1} I(\tau_1 < \infty) | X_1 = -1\big) \\ &= \omega_0 e^t + (1 - \omega_0) E_\omega\big(e^{t(\tau_1 \circ \theta_{-1})} I(\tau_1 \circ \theta_{-1} < \infty)\big) E_\omega\big(e^{t\tau_1} I(\tau_1 < \infty)\big) e^t \\ &= \omega_0 e^t + (1 - \omega_0) e^t \varphi(t, \theta_{-1}\omega) \varphi(t, \omega). \end{aligned}$$

The existence of t_{crit} allows us to write, for any $t < t_{\text{crit}}$,

$$\varphi(t, \omega) = \frac{\omega_0 e^t}{1 - (1 - \omega_0) e^t \varphi(t, \theta_{-1}\omega)}.$$

Dividing both the numerator and denominator by $e^t\omega_0$ yields

$$\varphi(t,\omega) = \frac{1}{(1+\rho_0)e^{-t} - \rho_0\varphi(t,\theta_{-1}\omega)}.$$

Iterating, we obtain the following P-a. s. continued fraction representation of $\varphi(t,\omega)$:

$$\varphi(t,\omega) = \frac{1}{(1+\rho_0)e^{-t} - \frac{\rho_0}{(1+\rho_{-1})e^{-t} - \frac{\rho_{-1}}{(1+\rho_{-2})e^{-t} - \frac{\rho_{-2}}{\cdots}}}}.$$

CHAPTER 4. ANNEALED LDP

4.1 Introduction

The goal of this chapter is to derive an annealed LDP for RWSRE. The rate function for the sparse model is constructed using the dual environment. In contrast to the quenched case, we make ellipticity (bounded away from zero and infinity) assumptions for both d_0 and ρ_0 in order to meet the assumptions for stationary and ergodic (dual) environment posed in [5]. Under the conditions on P assumed in this chapter, the dual environment becomes a functional of an auxiliary Markov process having uniformly bounded transitions. The LDP for this Markov process implies process level LDP under the dual environment, and this fact constitutes a key ingredient of our proof of the annealed LDP.

The rest of the chapter is organized as follows. We first state the conditions we use in this chapter and briefly review basic results known in the literature for RWRE (random walks in stationary ergodic environments) in Section 4.2. We then state the main results of the chapter in Section 4.3. Next, in Section 4.4 we explore a relation between two environments, the sparse model and its dual, that is crucial for the transformation of the LDP under the stationary law Q to the underlying law P. In Section 4.5 we show that the law Q under the dual environment satisfies the assumptions that are needed to prove annealed LDP for the walk in stationary dual environment.

4.2 Annealed LDP for classical RWRE

The approach of [5] to LDP in the annealed setting also makes use of the hitting times T_n and T_{-n}. Define the empirical process

$$R_n(\omega) := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{\theta_i\omega}.$$

Note that R_n takes values in the space $M_1(\Omega)$. Introduce the specific relative entropy $h(\cdot|\alpha) : M_1(\Omega) \to [0,\infty]$ by

$$h(\eta|\alpha) := \begin{cases} \lim_{N\to\infty}\frac{1}{N}H(\eta_N|\alpha_N) & , \ \eta \text{ stationary} \\ \infty & , \text{ otherwise} \end{cases}$$

where η_N, α_N denote the restriction of η, and Q to the first N coordinates $\{\omega_i\}_{i=0}^{N-1}$ respectively and $H(\cdot|\cdot)$ denotes the relative entropy:

$$H(\mu|\nu) = \begin{cases} \int \log\Big(\frac{d\mu}{d\nu}(x)\Big)\mu(dx), & \mu \ll \nu \\ \infty, & \text{otherwise.} \end{cases}$$

We say that α is *locally equivalent to the product of its marginals* if its restriction to $M_1([0,1]^n)$ is equivalent to $\prod_{i=1}^n \alpha_i$ for each $n \in \mathbb{N}$, that is, if for any measurable $\Gamma \in [0,1]^n$, $\alpha(\Gamma) = 0$ if and only if $\prod_{i=1}^n \alpha_i(\Gamma)$. We say that the sequence of random variables $(R_n)_{n\in\mathbb{N}}$ satisfies under α the process level LDP if the distributions of the random variables R_n satisfy the LDP in $M_1(\Omega)$ with the rate function $h(\cdot|\alpha)$. Let $\mathcal{F}_n := \sigma(\{\omega_0, \omega_1, \ldots, \omega_n\})$. We also define $\zeta := \alpha(d\omega)\otimes P_\omega$ to avoid possible confusion with $\mathbb{P} = P(d\omega)\otimes P_\omega$ which we reserved for the sparse model. The following assumption on α [5, Assumption (A)] is made in [5] to derive an annealed LDP for classical RWRE.

Assumption 4.2.1. *(C1) α is stationary and ergodic*

(C2) There exists $\varepsilon \in (0, 1/2)$ such that $\alpha\big(\omega \in (1-\varepsilon, \varepsilon)\big) = 1$.

(C3) $(R_n)_{n\in\mathbb{N}}$ satisfies under α the process level LDP with the good rate function $h(\cdot|\alpha)$. α is locally equivalent to the product of its marginals and, for each $\eta \in M_1^s(\Omega)^K$, there

is a sequence $\{\eta^n\}$ *of ergodic measures with* $\eta^n \to \eta$ *weakly and* $h(\eta^n|\alpha) \to h(\eta|\alpha)$ *as* $n \to \infty$.

(C4) α *is locally equivalent to the product of its margins (any finite dimensional distributions are) and, for any stationary distribution* $\eta \in M_1(\Omega)$ *there exists a sequence of stationary and ergodic distributions* η^n *such that, as* $n\text{ø}\infty$, η^n *converges to* η *weakly in* $M_1(\Omega)$ *and* $h(\eta^n|P)$ *converges to* $h(\eta|P)$ *in* $\mathbb{R}$.

(C5) α *is "extremal", a strong ellipticity condition which is automatically satisfied for both* P *and* Q *in our framework by assuming that* P *is locally equivalent to its margins and having* $P(\omega_0 = 1/2) > 0$ *and* $Q(\omega_0 = 1/2 > 0)$.

Following [5], we remark that conditions (C3)-(C5) are satisfied for i. i. d. sequences and Markov chains with a uniformly bounded (away from zero and one) transition kernel.

For $u \geq 1$, let

$$I_\alpha^{\tau,a}(u) = \inf_{\eta \in M_1^e(\Omega)} \big[I_\eta^{\tau,q}(u) + h(\eta|\alpha)\big]. \tag{4.1}$$

Theorem 4.2.2. *[5] Suppose that Assumption 4.2.1 holds and let* $\alpha \in M_1^e(\Omega)$. *Then the distributions of* T_n/n *under* ζ *satisfy a weak LDP with convex rate function* $I_\alpha^{\tau,a}$.

Next, we define the rate function in the form that is used in [5] to obtain an annealed LDP for the X_n process. Let

$$I_\alpha^a(v) = \begin{cases} vI_\alpha^{\tau,a}\big(\frac{1}{v}\big) & v \in [0,1] \\ |v|I_{\alpha^{\text{Inv}}}^{\tau,a}\big(\frac{1}{|v|}\big) & v \in [-1,0] \end{cases} \tag{4.2}$$

and note that the relation between the quenched rate function I_η^q and the annealed rate function I_α^a is given by the variational formula

$$I_\alpha^a(v) = \inf_{\eta \in M_1^e(\Omega)} \big[I_\eta^q(v) + |v|h(\eta|\alpha)\big] \tag{4.3}$$

where $vh(\eta|\alpha) = \infty$ if $h(\eta|\alpha) = \infty$. We now have

Theorem 4.2.3. *[5] Suppose that Assumption 4.2.1 holds. Then, the distributions of* X_n/n *under* ζ *satisfy an annealed LDP with the convex rate function* I_Q^a.

4.3 Annealed LDP in a stationary environment under the dual law Q

In this chapter we make the following assumption on RWSRE:

Assumption 4.3.1. *Let Assumption 3.3.1 hold. Furthermore, assume that the ellipticity condition C_2 in Assumption 4.2.1 is satisfied and that $P(d_0 \leq M) = 1$ for some (deterministic) integer $M \in \mathbb{N}$.*

Since the dual environment $\widetilde{\omega}$ is stationary and ergodic, and hence $Q \in M_1^e(\Omega)^{\hat{K}}$, we have annealed LDP for the hitting time T_n/n and for the walk X_n/n by the following lemma.

Lemma 4.3.2. *Let Assumption 4.3.1 hold. Then Q satisfies the level process LDP. Moreover, the Assumption 4.2.1 holds for Q, that is, the law Q is locally equivalent to the product of its marginals and, for any stationary measure $\eta \in M_1^s(\Omega)^{\hat{K}}$, there is a sequence $\{\eta^n\}$ of ergodic measures with $\eta^n \to \eta$ weakly and $h(\eta^n|\alpha) \to h(\eta|\alpha)$ as $n \to \infty$.*

The proof of the Lemma 4.3.2 is deferred to Section 4.5. We now give annealed large deviation results obtained under the $\mathbb{Q}$ measure.

Theorem 4.3.3. *Let Assumption 4.3.1 hold. Then,*

(a) The distributions of T_n/n under $\mathbb{Q}$ satisfy a weak LDP with speed n and rate function $I_Q^{\tau,a}(\cdot)$.

(b) The distributions of X_n/n under $\mathbb{Q}$ satisfy a LDP with convex rate function I_Q^a.

Proof. It is immediate by the Lemma 4.3.2, Theorem 4.2.2 and Theorem 4.2.3. □

4.4 Transition from the stationary dual environment to the sparse environment

In this Section 4.3, we obtained the annealed LDP for both hitting time T_n/n and X_n/n under the stationary law $\mathbb{Q}$. Annealed large deviation results under $\mathbb{P}$ measure are derived by using annealed large deviations under $\mathbb{Q}$ measure. We now state the main result of this chapter.

Theorem 4.4.1. *Let Assumption 4.3.1 hold. Then,*

(a) The distributions of T_n/n under $\mathbb{P}$ satisfy a weak LDP with speed n and rate function $I_Q^{\tau,a}(\cdot)$.

(b) The distributions of X_n/n under $\mathbb{P}$ satisfy a LDP with convex rate function I_Q^a.

The proof of this theorem is based on the following observation.

Lemma 4.4.2. *Suppose that each d_i is a bounded random variable with probability one. Then the averaged LDP hold under $\mathbb{P}$ if and only if the averaged LDP hold under $\mathbb{Q}$. In this case, corresponding rate functions coincide.*

Proof. Suppose d_0 assumes values from $\{1, 2, \dots, M\}$. The proof is by a direct comparison of the distributions $\mathbb{P}$ and $\mathbb{Q}$. Let A be a Borel measurable set. Then for the upper bound we have:

$$\begin{aligned}
\mathbb{P}(X_n \in A) &= E_P\Big[P_\omega(X_n \in A)\Big] \\
&= E_Q\Big[\frac{E_P(d_0)}{d_0} P_\omega(X_n \in A)\Big] = E_P(d_0) E_Q\Big[\frac{P_{\theta_{\lfloor Ud_0 \rfloor}\widetilde{\omega}}(X_n \in A)}{d_0}\Big] \\
&= E_P(d_0) E_Q\Big[\sum_{k=1}^{M}\sum_{j=0}^{k-1} I(d_0 = k)\frac{P_{\theta_j\widetilde{\omega}}(X_n \in A)}{k}\Big] \\
&= E_P(d_0) E_Q\Big[\sum_{j=0}^{M-1}\sum_{k=j+1}^{M} I(d_0 = k)\frac{P_{\theta_j\widetilde{\omega}}(X_n \in A)}{k}\Big] \\
&\le E_P(d_0) E_Q\Big[\sum_{j=0}^{M-1}\sum_{k=j+1}^{M} I(d_0 = k) P_{\theta_j\widetilde{\omega}}(X_n \in A)\Big] \\
&\le E_P(d_0) E_Q\Big[\sum_{j=0}^{M-1} I(d_0 > j) P_{\theta_j\widetilde{\omega}}(X_n \in A)\Big] \\
&\le E_P(d_0) E_Q\Big[\sum_{j=0}^{M-1} P_{\theta_j\widetilde{\omega}}(X_n \in A)\Big] \\
&\le C E_Q\Big[P_{\widetilde{\omega}}(X_n \in A)\Big] = C\mathbb{Q}(X_n \in A),
\end{aligned}$$

where the constant C is equal to $ME_P(d_0)$. Note that the first equality is by the definition of annealed law while the second equality is due to the change of measure. We represent ω by inverting the definition of $\widetilde{\omega}$ since $\widetilde{\omega} = \theta_{-\lfloor Ud_0 \rfloor}\omega$ in the third equality. Since d_0 assumes finitely many positive integers from the set $\{1, 2, \ldots, M\}$ and for each $k \in \{1, 2, \ldots, M\}$ $\lfloor Ud_0 \rfloor$ ranges between 0 to $k-1$, we partition this average to obtain the fourth equality. After changing the order of summation, and using the fact that $\frac{1}{k} \le 1$ for any $k \in \{1, 2, \ldots, M\}$, we arrive at the first inequality. Moreover, the collection of events $\{d_0 = k\}_{k=j+1}^{M}$ is a partition of the event $\{d_0 > j\}$, and hence the second inequality is established. The third inequality is simply due to the fact that the indicator function is always less than or equal to 1. Finally, we obtain the last inequality by using the fact that the dual environment $\widetilde{\omega}$ is stationary under the Q measure. Note also that the constant C at the end of the last inequality is simply equal to $ME_P(d_0)$ because there we use the stationarity of the dual environment M times.

Finally an adequate lower bound can be established by considering only the case when $j = 0$, that is:

$$\begin{aligned}\mathbb{P}(X_n \in A) &= E_P(d_0)E_Q\Big[\sum_{j=0}^{M-1}\sum_{k=j+1}^{M} I(d_0 = k)\frac{P_{\theta_j\tilde{\omega}}(X_n \in A)}{k}\Big]\\ &\geq E_P(d_0)E_Q\Big[\sum_{k=1}^{M} I(d_0 = k)\frac{P_{\theta_j\tilde{\omega}}(X_n \in A)}{k}\Big]\\ &\geq cE_Q\Big[\sum_{k=1}^{M} I(d_0 = k)P_{\theta_j\tilde{\omega}}(X_n \in A)\Big] = c\mathbb{Q}(X_n \in A),\end{aligned}$$

where the constant c is equal to M^{-1}. We used the fact that each term is a non-negative quantity, and only took the case $j = 0$ into consideration in the first inequality. Since $\dfrac{1}{k} \geq \dfrac{1}{M}$ for any $k \in \{1, 2, \ldots, M\}$, the second inequality follows. Note that the number $c = \dfrac{E_P(d_0)}{M}$. As a result, we have shown that

$$c\mathbb{Q}(X_n \in A) \leq \mathbb{P}(X_n \in A) \leq C\mathbb{Q}(X_n \in A). \tag{4.4}$$

By the uniqueness of rate functions, we have

$$I_Q^{\tau,a} = I_P^{\tau,a} \qquad \text{and} \qquad I_Q^a = I_P^a.$$

□

We are now in a position to prove the main result of this chapter.

Proof of the Theorem 4.4.1. The result in Theorem 4.4.1 is an immediate consequence of Theorem 4.3.3 combined together with Lemma 4.4.2. □

4.5 Proof of Lemma 4.3.2

This section is devoted to the deferred proof of Lemma 4.3.2. Note that product measures and Markov processes with bounded transition kernels satisfy the process level LDP, c.f. [10], as well as Assumption 4.2.1, c.f. [12], and Lemma 4.8. We refer to a transition kernel $P(x, A)$ on a state space $\big(\mathcal{S}, \mathcal{B}(\mathcal{S})\big)$ as bounded if there is a probability measure π and a constant $c > 0$ such that

$$c^{-1}\pi(A) \leq P(x, A) \leq cP(x, A), \qquad \forall\ x \in \mathcal{S},\ A \in \mathcal{B}(\mathcal{S}).$$

Similarly to [17], we introduce the following auxiliary Markov process. The marked points $(a_n)_{n\in\mathbb{N}}$ is a renewal sequence if the cycle lengths $(d_n)_{n\in\mathbb{Z}}$ are independent. Let N_t denote the number of renewals in the time interval $[0, t]$ for $t \geq 0$. More precisely,

$$N_t = \inf\{n \geq 0 : a_n > t\}. \tag{4.5}$$

Define the age at t and the residual life at t by $A_t = t - a_{N_t - 1}$ and $B_t = a_{N_t} - t$, respectively. Further, define the Markov process $\xi_n = (\omega_n, \sigma_n)$, where σ_n is the distance from the last marked site up to n. That is,

$$\sigma_n = n - a_{\eta_n} = n - \sup\{k \in \mathbb{Z} : k \leq n \text{ and } k \in \mathcal{A}\}, \qquad n \in \mathbb{Z}.$$

Note that $\sigma_{a_k} = 0,\ k \in \mathbb{N}$, and

$$\sigma_{n+1} - \sigma_n = 1 \text{ if } a_{\eta_n} \leq n < a_{\eta_n+1}.$$

If $\sigma_n = 0$, then $\omega_n = \lambda_{\eta_n} \neq 1/2$ is on the marked site and if $\sigma_n > 0$ then $\omega_n = 1/2$. Note that if the conditions of Theorem 4.4.2 hold for the dual environment, then $\xi = (\xi_n)_{n\in\mathbb{Z}}$ under the law Q is a positive recurrent Markov chain on $\Sigma = (0, 1) \times \mathbb{Z}_+$ whose transition probability function is given by

$$Q(\xi_{n+1} \in (B, j) | \xi_n \in (A, i)) = \begin{cases} P(\lambda_0 \in B)P(d_0 > i + 1 | d_0 > i) & \text{if } j = i + 1 \\ P(\frac{1}{2} \in B)P(d_0 = i + 1 | d_0 > i) & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Assuming, without loss of generality that $P(d_0 = M) > 0$, this transition kernel is clearly bounded under Assumption 4.3.1. As we mentioned in the beginning of this section, the result in the lemma is a direct implication of this observation.

CHAPTER 5. SUMMARY AND DISCUSSION

The main results of the thesis are stated in Section 3.4 (quenched LDP for both the hitting times T_n and X_n) and Section 4.4 (annealed LDP for both T_n and X_n). The following appear to be natural directions for future related work:

1. Is it possible to remove the assumption that d_0 is a bounded random variable? It seems plausible that while this assumption is an artifact of our proof (cf. [4]), some restrictions are necessary in order to get a usual annealed LDP. Note that on the scale of large deviations it is not obvious, even on a heuristic level, that annealed LDP under $\mathbb{Q}$ implies a regular annealed LDP under $\mathbb{P}$. Indeed, it is well known that the statement that any LDP for a stationary and ergodic sequence implies its counterpart for the cycle-stationary dual (e. g., regenerative sequences) is incorrect (see, for instance, [1]).

2. A closely related question: Is it possible to prove some form of the annealed LDP (perhaps for deviations of the magnitude vn for large $v > 0$ only)?

3. It is well known that the rate function $I_Q^{\tau,q}$ might be zero in the interval $(0, \lambda_c^{-1})$ for some $\lambda_c > 0$. For this regime, so called sub-exponential tail estimates (a refinement of the LDP) are known for the classical RWRE [9, 13, 20, 21] in both annealed and quenched settings. Is it possible to carry these refined estimates to the RWSRE framework?

4. To use the dual environment we had to assume that $E_P(d_0) < \infty$. Can this assumption be relaxed to $E_P(\log d_0) < \infty$?

5. Can the LDPs be obtained for mixing environments, say for strongly mixing environments with a super-exponential mixing rate (which would guarantee the process level LDP for Q, cf. [3])?

APPENDIX . ADDITIONAL MATERIAL

Cycle stationarity of the sparse environment

Lemma .0.1. *The sparse environment ω is cycle-stationary.*

Proof. Define the cycles as the vectors of random variables with random lengths

$$C_n = (\omega_{a_{n-1}}, \omega_{a_{n-1}+1}, \ldots, \omega_{a_n-1}; d_n).$$

Then the sequence of random vectors $C = (C_n)_{n\in\mathbb{Z}}$ is stationary, i.e., for any k and $n \in \mathbb{N}$

$$(C_{i_1}, C_{i_2}, \ldots, C_{i_k}) =_D (C_{i_1+n}, C_{i_2+n}, \ldots, C_{i_k+n},)$$

Let $A_j \in \mathcal{B}([0,1]^{h_j} \times \mathbb{N})$ for $j \in \{1, 2, \ldots, k\}$ be measureable sets of the form

$$(-\infty, x_{j1}] \times \ldots (-\infty, x_{jh_j}] \times \{h_j\}$$

then

$$\begin{aligned}
P\left((C_{i_j})_{j=1}^k \in \prod_{j=1}^k A_j\right) &= P(C_{i_1} \in A_1; C_{i_2} \in A_2; \ldots; C_{i_k} \in A_k) \\
&= P\big\{(\omega_{a_{i_1-1}} \le x_{11}, \omega_{a_{i_1-1}+1} \le x_{12}, \ldots, \omega_{a_{i_1}-1} \le x_{1h_1}; d_{i_1} = h_1); \\
&\quad (\omega_{a_{i_2-1}} \le x_{21}, \omega_{a_{i_2-1}+1} \le x_{22}, \ldots, \omega_{a_{i_2}-1} \le x_{2h_2}; d_{i_2} = h_2); \ldots; \\
&\quad (\omega_{a_{i_k-1}} \le x_{k1}, \omega_{a_{i_k-1}+1} \le x_{k2}, \ldots, \omega_{a_{i_k}-1} \le x_{kh_k}; d_{i_k} = h_k)\big\}
\end{aligned}$$

$$
\begin{aligned}
&= P\big\{(\lambda_{i_1-1} \le x_{11}, \frac{1}{2} \le x_{12}, \dots, \frac{1}{2} \le x_{1h_1}; d_{i_1} = h_1);\\
&\quad (\lambda_{i_2-1} \le x_{21}, \frac{1}{2} \le x_{22}, \dots, \frac{1}{2} \le x_{2h_2}; d_{i_2} = h_2); \dots;\\
&\quad (\lambda_{i_k-1} \le x_{k1}, \frac{1}{2} \le x_{k2}, \dots, \frac{1}{2} \le x_{kh_k}; d_{i_k} = h_k)\big\}\\
&= P\big\{(\lambda_{i_1+n-1} \le x_{11}, \frac{1}{2} \le x_{12}, \dots, \frac{1}{2} \le x_{1h_1}; d_{i_1+n} = h_1);\\
&\quad (\lambda_{i_2+n-1} \le x_{21}, \frac{1}{2} \le x_{22}, \dots, \frac{1}{2} \le x_{2h_2}; d_{i_2+n} = h_2); \dots;\\
&\quad (\lambda_{i_k+n-1} \le x_{k1}, \frac{1}{2} \le x_{k2}, \dots, \frac{1}{2} \le x_{kh_k}; d_{i_k+n} = h_k)\big\}\\
&= P\big\{(\omega_{a_{i_1+n-1}} \le x_{11}, \omega_{a_{i_1+n-1}+1} \le x_{12}, \dots, \omega_{a_{i_1+n}-1} \le x_{1h_1}; d_{i_1+n} = h_1);\\
&\quad (\omega_{a_{i_2+n-1}} \le x_{21}, \omega_{a_{i_2+n-1}+1} \le x_{22}, \dots, \omega_{a_{i_2+n}-1} \le x_{2h_2}; d_{i_2+n} = h_2); \dots;\\
&\quad (\omega_{a_{i_k+n-1}} \le x_{k1}, \omega_{a_{i_k+n-1}+1} \le x_{k2}, \dots, \omega_{a_{i_k+n}-1} \le x_{kh_k}; d_{i_k+n} = h_k)\big\}\\
&= P(C_{i_1+n} \in A_1; C_{i_2+n} \in A_2; \dots; C_{i_k+n} \in A_k)\\
&= P\big((C_{i_j+n})_{j=1}^k \in \prod_{j=1}^k A_j\big).
\end{aligned}
$$

$\square$

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