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Some conditions for the existence of recurrent solutions to systems of ordinary differential equations

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SOME CONDITIONS FOR THE EXISTENCE OF RECURRENT SOLUTIONS TO SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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I. INTRODUCTION

A. History

For many years the theory of dynamical systems has been applied to the solutions of autonomous systems of ordinary differential equations

(1.1) $\quad x' = f(x).$

Recently several authors have applied this theory to non autonomous systems as well. These systems are of the form

(1.2) $\quad x' = f(t,x).$

Deysach and Sell (3) obtain conditions for the existence of almost periodic solutions to a periodic system. Using the periodicity of f they construct a dynamical system on a torus and thus obtain compact motions for all real numbers t.

Miller (8) appears to have been the first to devise a technique whereby a quite arbitrary system of nonautonomous differential equations can define a dynamical system. By introducing a phase space which is the product of the Euclidean space in which the solutions lie and a suitable function space he obtains quite weak conditions for the existence of an a.p. (almost periodic) solution to an a.p. system. By defining a function space based on the properties of a.p. functions Seifert (11) obtains a result similar to that of Miller. Both of these results of Seifert and Miller are stronger in one sense than an existence theorem previously obtained by Seifert (12) through the use of the stability properties of a bounded solution.

In a two part paper Sell (13, 14) studies exhaustively the conditions on the Differential Equation 1.2 and its solutions under which the dynamical systems technique may be applied. In this study the compact open topology seems to play a natural role.

In all of the work noted above the existence of an a.p. solution is established by showing that a recurrent motion exists which is uniformly Lyapunov stable. In the present work we study systems of differential equations and conditions which lead to motions which are recurrent but not necessarily a.p. The functions associated with such motions are known as recurrent and strongly recurrent functions. In the last section we consider the class of almost automorphic functions introduced by Bochner (2) and how it relates to the classes of strongly recurrent and a.p. functions.

B. Preliminaries

In this section we set forth some of the notation to be used throughout this paper together with some of the basic definitions and fundamental theorems from the theory of dynamical systems. A careful development of the theory can be found in the very readable book by Nemytskii and Stepanov (10).

Let A and B be arbitrary sets. Then A x B is the usual Cartesian product of A with B. If n is a positive integer then we define $A^n = A \times A \times \ldots A$, the Cartesian product of A with itself n times.

If A is a subset of a topological space X we denote the closure of A to be A*.

Let (X,d) be a metric space and R the set of real numbers. A dynamical system on X is a mapping

$$\pi: \; R \times X \to X$$

which satisifes the following conditions:

(D1) $\pi(0,x) = x$ for every x in X.

(D2) $\pi(t,\pi(s,x)) = \pi(t + s, x)$.

(D3) π is continuous.

For fixed x in X, the set $\gamma(x) = \{\pi(t,x) \mid t \text{ is in } R\}$ is called the trajectory through x. Also for fixed x the mapping π_x: $R \to X$ is called the motion through x. If we restrict our attention to $t \geq 0$ or $t \leq 0$ we speak of the positive or negative semitrajectories respectively.

For fixed x in X, the omega limit set of the motion through x is

$$\Omega(x) = \{y \text{ in } X \mid \exists \{t_n\} \to \infty \text{ and } d(y, \pi(t_n,x)) \to 0\}.$$

The alpha limit set, A(x), is similarly defined for $\{t_n\} \to (-\infty)$.

A motion is said to be compact (or positively compact or negatively compact) if the closure of its trajectory (or positive or negative semitrajectory) is compact. In (10) Nemytskii and Stepanov call a compact motion Lagrange stable.

A motion $\pi(t,x)$ is said to be recurrent if for every $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that for any t in R and any interval I of length $T(\epsilon)$ there is an s in I such that $d(\pi(t,x), \pi(s,x)) < \epsilon$.

<u>Theorem 1.1</u>: If the motion $\pi(t,x)$ is compact, then it is recurrent if and only if for every $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that for any interval I of length T there is an s in I such that $d(\pi(s,x), x) < \epsilon$. (cf. (10), p. 378.)

A motion is said to be almost periodic if for every $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that for any interval I of length T there is an s in I such that $d(\pi(t,x),\pi(t+s,x)) < \epsilon$ for all t in R.

A motion $\pi(t,x)$ is said to be periodic with period $T > 0$ if $d(\pi(t + T,x), \pi(t,x)) = 0$ for all t in R.

We note that periodic and almost periodic motions are recurrent.

A non-empty subset A of X is invariant if for every x in A, $\pi(t,x)$ is in A for all t in R. We can define positively or negatively invariant in a like manner for $t \geq 0$ or $t \leq 0$.

A subset M of X is a minimal set if it is closed, invariant and has no such proper subset.

<u>Theorem 1.2</u>: Every invariant, closed, compact set contains a minimal set. (cf. (10), p. 374.)

The following two important theorems of Birkhoff (cf. (10, p. 375-377.) relate minimal sets and recurrent motions.

<u>Theorem 1.3</u>: If X is complete and $\pi(t,x)$ is a recurrent motion in X, then the closure of its trajectory is a compact minimal set.

<u>Theorem 1.4</u>: Every trajectory in a compact minimal set is recurrent.

II. DYNAMICAL SYSTEMS

We consider a system of ordinary differential equations

(A) $x' = f(t,x)$.

Let R denote the set of real numbers. The function f is continuous from $R \times R^n$ into R^n. We write: f is in $C[R \times R^n, R^n]$. In what follows f will usually be bounded in t and continuous or uniformly continuous. We write: f is in $BC[R \times R^n, R^n]$ if f is in $C[R \times R^n, R^n]$ and for every compact set K in R^n there is a number B such that $\|f(t,x)\| \leq B$ for (t, x) in $R \times K$. The symbol $\|x\|$ represents for x in R^n any convenient norm. If f is bounded and uniformly continuous on every set of the form $R \times K$ as described above, we write: f is in $BU[R \times R^n, R^n]$. For example the function $x \sin t^2$ is in $BC[R \times R^n, R^n]$ and the function $x^2 \sin t$ is in $BU[R \times R^n, R^n]$.

The construction of a dynamical system associated with System A proceeds in two stages. To begin with we consider an extension of the dynamical system of Bebutov. (cf. (9), p. 8.)

Associated with every function f in $C[R \times R^n, R^n]$ are its translates f_s where $f_s(t,x) = f(t + s,x)$ for all (t,x) in $R \times R^n$ and s is in R. Let F(f) be the collection of all such translates. $F(f) = \{f_s | s \text{ is in } R\}$.

On $C[R \times R^n, R^n]$ we introduce the compact open topology. Since the range of the functions is a metric space Kelley (7),

p. 230, shows that this topology is equivalent to the topology of uniform convergence on compact sets. The space C can be made a metric space by defining a family of pseudo-metrics, d_j. Let f and g be any two functions in $C[R \times R^n, R^n]$. For any positive integer N let I_N be the interval $[-N,N]$ in R and let $J_N = I_N^n$ and $K_N = I_N^{n+1}$ be "cubes" in R^n and R^{n+1} respectively. We then define the pseudo-metric

(2.1) $$d_N(f,g) = \sup_{(t,x) \text{ in } K_N} \| f(t,x)-g(t,x) \| .$$

From the above family of pseudo-metrics we define a metric on $C[R \times R^n, R^n]$ by

(2.2) $$d(f,g) = \sum_{N=1}^{\infty} d_N(f,g)/2^N(1+d_N(f,g)).$$

Kelley (7), p. 231, shows that (C,d) is a complete metric space.

The dynamical system of interest here is defined by the mapping $p: R \times (C,d) \to (C,d)$ where $p(t,f) = f_t$. So p takes a given function f into a translate of f.

That p satisfies the first two axioms of a dynamical system is easily seen. For any f in $C[R \times R^n, R^n]$,

(D1) $p(0,f) = f_o = f$.

For any numbers t, s in R and f in $C[R \times R^n, R^n]$,

(D2) $p(s,p(t,f)) = p(s,f_t) = f_{s+t} = p(s+t,f)$.

The continuity of p is easily verified. Let the sequence $\{t_n\} \to s$ in R and the sequence of functions $\{f^n\} \to g$ in the compact open topology. We must show that $p(t_n, f^n) \to p(s,g) = g_s$ in the compact open topology. Now for each positive integer j we have

$$(2.3) \qquad d_j(f^n_{t_n}, g_s) \leq d_j(f^n_{t_n}, g_{t_n}) + d_j(g_{t_n}, g_s).$$

For any given $\epsilon > 0$ we select N_1 large enough so that $1/2^{N_1} < \epsilon/3$. Now since the t_n's are all in some compact set there exists an $N_2 = N_2(\epsilon)$ such that the first term on the right hand side of (2.3) can be made less than $\epsilon/3$ for $j = 1, 2, \ldots N_1$ if we take $n > N_2$. This is because $f^n \to g$ uniformly on compact sets. The second term on the right in (2.3) can be made less than $\epsilon/3$ for $j = 1, 2, \ldots N_1$ if we take $n > N_3$ where N_3 is determined as follows. Since $\{f^n\}$ is a sequence of continuous functions approaching the function g uniformly on every compact set, g is continuous and hence uniformly continuous on compact sets. So g_s and g_{t_n} are uniformly continuous on compact sets. Hence, for each j there is an M_j such that $d_j(g_{t_n}, g_s) < \epsilon/3$ if $n > M_j$. Let $N_3 = \text{Max}\,\{M_j\}$ $j = 1, 2, \ldots N_1$. From

$$(2.4) \qquad d(f^n_{t_n}, g_s) = \sum_{j=1}^{N_1} d_j(f^n_{t_n}, g_s)/2^j(1 + d_j(f^n_{t_n}, g_s))$$

$$+ \sum_{j=N_1+1}^{\infty} d_j(f^n_{t_n}, g_s)/2^j(1+d_j(f^n_{t_n}, g_s)),$$

and the definition of N_1 it follows that

$$(2.5) \qquad d(f^n_{t_n}, g_s) \le \sum_{j=1}^{N_1} d_j(f^n_{t_n}, g_s)/2^j + \sum_{j=N_1+1}^{\infty} 1/2^j .$$

Applying the inequalities developed above we see that $d(f^n_{t_n}, g_s) \le \epsilon/3 + \epsilon/3 + \epsilon/3$ if $n \ge$ Max $\{N_1, N_2, N_3\}$. So $p(t_n, f^n) = f^n_{t_n}$ approaches $p(s,g) = g_s$ in the compact open topology and p is continuous. Condition D3 is satisfied and the mapping p defines a dynamical system on C.

Now the set of translates F(f) is not in general closed. For example, if $f(t) = \arctan t$ and $\{t_n\}$ is any sequence of real numbers approaching $+\infty$, then f_{t_n} converges to the constant function $\pi/2$ uniformly on compact sets. In order to utilize the classical theorems of dynamical systems we usually study the set F*(f). Then, since (C,d) is a complete metric space and F*(f) is closed, (F*(f),d) is a complete subspace. In terms of the dynamical system defined by the mapping p: $F(f) = \gamma(f)$, the trajectory through f, and $F^*(f) = \gamma^*(f)$, the union of the trajectory with its alpha and omega limit sets.

The dynamical system of Bebutov is defined for a function f in C[R, R]. The mapping p that we introduced above is an extension of that of Bebutov only in as much as it is based on a function f in $C[R \times R^n, R^n]$. Therefore part of the proof of the following lemma differs only in detail from a similar result in (5).

<u>Lemma 2.1</u>: If f is in (C,d), then F*(f) is compact if and only if f is in BU[R x R^n, R^n]. Also, if g is in F*(f), then g is in BU[R x R^n, R^n].

Proof. We apply Ascoli's lemma in the form given by Dieudonné (4), p. 137: If X is a compact metric space and Y a metric space, let (C,ρ) be C[X,Y] with the supremum metric. Then a subset F of C is precompact if and only if F is equicontinuous and uniformly bounded.

Suppose f is in BU[R x R^n, R^n]. For any positive integer j there is a number B_j such that $\|f(t,x)\| \leq B_j$ for (t,x) in R x J_j, so the family F(f) is uniformly bounded on R x J_j and hence on K_j. To see that F(f) is equicontinuous, take $\epsilon > 0$. Then since f is uniformly continuous on R x J_j there is a $\delta > 0$ such that if x and y are in J_j and $\|x - y\| < \delta$, while s and t are in R and $|s - t| < \delta$, then $\|f(t,x) - f(s,y)\| < \epsilon$. Let f_u be any function in F(f). Then $\|f_u(t,x) - f_u(s,y)\| = \|f(u+t\ x) - f(u+s,y)\| < \epsilon$ if only $\|x-y\| < \delta$ and $|s-t| < \delta$. Because δ is independent of u, the family F(f) is equicontinuous. With F(f) satisfying these conditions, it is precompact. So F*(f) is compact (regarding f: $K_j \to R^n$) and every sequence from F(f) has a subsequence which converges uniformly on $K_j (= I_j^{n+1})$.

Let $\{f^n\}$ be the given sequence of functions from F(f). Take j = 1. Then there is a subsequence $\{f_1^n\}$ of $\{f^n\}$ such that $f_1^n \to g^1$ uniformly on K_1. For $j = 2$ we find a subsequence $\{f_2^n\}$ of $\{f_1^n\}$ such that $f_2^n \to g^2$ uniformly on K_2. We note that for (t,x) in K_1, $g^2(t,x) = g^1(t,x)$. Continuing in this manner we extract the "diagonal" subsequence $\{f_n^n\}$ which for each fixed positive integer j converges uniformly to g^j on the compact set K_j. Also, for (t,x) in K_j, $g^{j+1}(t,x) = g^j(t,x)$. Let the function $g(t,x)$ be defined in the following way. If (t,x) is in K_j then $g(t,x) = g^j(t,x)$. Then g is well defined and $\{f_n^n\} \to g$ uniformly on compact sets. Therefore the family $F(f)$ is precompact in the compact open topology and $F^*(f)$ is compact.

Conversely, suppose $F^*(f)$ is compact in the compact open topology. Then from each sequence $\{f^n\}$ from $F(f)$ there is a subsequence $\{g^n\}$ which converges uniformly on compact sets. But this implies that for each positive integer j the family $F(f)$ is precompact regarding f restricted to the compact domain K_j. So there exists a number B_j such that for each u in R and (t,x) in K_j, $\| f_u(t,x)\| \le B_j$. So $\| f(s,x)\| \le B_j$ for all (s,x) in $R \times J_j$. The equicontinuity of $F(f)$ implies that for each $\epsilon > 0$ there is a $\delta > 0$ such that if u is in R and (t,x) and (s,y) are in K_j with $|s-t| < \delta$ and $\| x-y\| < \delta$ then $\| f_u(t,x) - f_u(s,y)\| < \epsilon$. So $\| f(t+u,x) - f(s+u,y)\| < \epsilon$ and f is uniformly continuous on $R \times J_j$ since u is any real number. Therefore f is in $BU[R \times R^n, R^n]$.

To show that if g is in $F^*(f)$ then g is in $BU[R \times R^n, R^n]$ we let j be any positive integer. Since $\|f(t,x)\| \leq B_j$ for all (t,x) in $R \times J_j$ and there is a sequence $\{t_n\}$ such that $f_{t_n} \to g$ uniformly on compact sets, we study

$$(2.6) \qquad \|g(t,x)\| \leq \|g(t,x) - f(t+t_n,x)\| + \|f(t+t_n,x)\| .$$

Then if we take any (t,x) in $R \times J_j$ the second term on the right in (2.6) is less than B_j. But the first term on the right can be made arbitrarily small for this (t,x) by choosing n large. So $\|g(t,x)\| \leq B_j$. The uniform continuity of g follows similarly from the uniform continuity of f and an application of the triangle inequality; we omit the details. ###

Remark: Since the family F is determined by t-translates it might seem that $F^*(f)$ would be compact if f satisfied a condition weaker than uniform continuity: Let f(t,x) be in $BC[R \times R^n, R^n]$ and in addition for each $\epsilon > 0$ and each positive integer j let there be a $\delta_j > 0$ such that if $|s-t| < \delta_j$ then $\|f(s,x) - f(t,x)\| < \epsilon$ for all x in J_j. However the function $f(t,x) = \exp(-|tx|)$ satisfies the above conditions, but if $\{t_n\} \to \infty$, f_{t_n} converges pointwise to the discontinuous function which is zero for $x \neq 0$ and is 1 for $x = 0$. So every subsequence must converge pointwise to that same discontinuous function and so no subsequence can converge in the compact open topology, for every such limit must be continuous.

Now for the second stage in the construction of a dynamical system associated with System A we shall find it necessary to make some rather strong assumptions about its solutions. We introduce the following two conditions.

(A1) For each x in R^n the System A has a unique solution $\varphi(t,x,f)$ such that $\varphi(0,x,f) = x$ and $\varphi(t,x,f)$ exists for all t.

(A2) For every function g in $F^*(f)$ Condition A1 holds for the system $x' = g(t,x)$.

Remark: Sell (13) gives an example of a system which satisfies Condition A1 but not A2. He proves however that if f satisfies a Lipschitz condition independent of t, then for every g in $F^*(f)$ and every x in R^n, the system $x' = g(t,x)$ has a unique solution such that $\varphi(0,x,g) = x$.

Let System A satisfy Conditions A1 and A2. Let $\varphi(t,x,f)$ be the unique solution to (A) such that $\varphi(0,x,f) = x$ for x in R^n. Let $X = R^n \times F^*(f)$ be the product of the two metric spaces R^n and $F^*(f)$. If $p = (x,g)$ and $q = (y,h)$ are both in X, then define a metric on X by

$$\rho(p,q) = \|x-y\| + d(g,h). \tag{2.7}$$

We form a dynamical system involving the solutions to (A) by defining the mapping π: $R \times X \to X$.

$$\pi(t;x,g) = (\varphi(t,x,g),g_t); \tag{2.8}$$

t is in R and (x,g) is in X.

To see that π defines a dynamical system we verify Conditions D1, D2, D3. First we set $t = 0$ in (2.8) to get

(2.9) $\quad \pi(0;x,g) = (\varphi(0,x,g),\ g_o) = (x,g).$

So π satisfies Condition D1. To verify Condition D2 we consider

(2.10) $\quad \pi(t,\pi(s;x,g)) = \pi(t;\ \varphi(s,x,g),\ g_s) =$

$$[\varphi(t,\varphi(s,y,g),\ g_s),\ (g_s)_t],$$

while

(2.11) $\quad \pi(t+s;y,g) = (\varphi(t+s,y,g),\ g_{t+s}).$

But $\quad \varphi'(t+s,y,g) = g(t+s,\ \varphi(t+s,y,g)) =$

$$g_s(t,\ \varphi(t+s,y,g)).$$

So $\varphi(t+s,y,g)$ is the solution to $x' = g_s(t,x)$ which for $t = 0$ has the value $\varphi(s,y,g)$. By the uniqueness Hypothesis A2 we have

(2.12) $\quad \varphi(t+s,y,g) = \varphi(t,\varphi(s,y,g),\ g_s).$

Noting that $(g_s)_t = g_{s+t}$ and combining (2.10), (2.11) and (2.12) we have

(2.13) $\quad \pi(t,\pi(s;y,g) = \pi(t+s,y,g).$

So π satisfies Condition D2.

To see that π is continuous and thus satisfies Condition D3 we apply the following lemma of Kamke (6).

Lemma 2.2: (Kamke) (A) Let $\{g^n\}$ be a sequence of functions in $C[R \times R^n,\ R^n]$ and $g^n \to g$ uniformly on compact sets. Let $\varphi(t,\ x_n,\ g^n)$ be the solutions to $x' = g^n(t,x)$ such that

$\varphi(0, x_n, g^n) = x_n$ and $x_n \to x_o$. Then there is a subsequence of $\{g^n\}$ such that the corresponding solutions converge to $\varphi(t,x_o,g)$ uniformly on compact sets in the interval of definition of $\varphi(t,x_o,g)$. (B) Furthermore, if the solutions of $x' = g(t,x)$ are unique, then $\varphi(t,x_n,g^n) \to \varphi(t,x_o,g)$ uniformly on compact sets.

Now to show the continuity of π in R x X suppose $\{(x_n,g^n)\}$ is a sequence in X and $\{t_n\}$ is a sequence in R with $(x_n, g^n) \to (y,g)$ in X and $\{t_n\} \to s$ in R. But $\pi(t_n; x_n, g^n) = (\varphi(t_n, x_n, g^n), g^n_{t_n})$. Since we have already shown that the mapping $p(t,f) = f_t$ is continuous with the compact open topology on $C[R \times R^n, R^n]$ it follows that $g^n_{t_n} \to g_s$. Since $\{g^n\}$ and g satisfy the uniqueness Hypothesis A1 and φ is continuous in t, it follows, using Kamke's Lemma part (B), that $\varphi(t_n, x_n, g^n) \to \varphi(s,y,g)$ uniformly on compact sets. Therefore π satisfies Condition D3 and we can summarize the preceding results as follows.

<u>Theorem 2.1</u>: Let f in $C[R \times R^n, R^n]$ satisfy the uniqueness and global existence properties (A1) and (A2). Let (X,ρ) be the metric space with $X = R^n \times F^*$ and ρ defined as in (1.5). Then the mapping π: $R \times X \to X$ given by (1.6) defines a dynamical system on X.

III. RECURRENT FUNCTIONS

In any study of the behavior of the solutions to the system of differential equations (A) via dynamical systems it will be desirable to have the associated motions compact or positively compact. We therefore apply this terminology to the solutions of the differential equation.

Definition 3.1: The solution $\varphi(t,x,f)$ of System A is compact (or positively compact) if its values are contained in a compact subset of R^n for all t (or $t \geq 0$) in R.

In analogy with recurrent motions in dynamical systems we define recurrent functions.

Definition 3.2: Let K be a compact subset of R^n. A function f in $C[R \times R^n, R^n]$ is said to be recurrent in t uniformly for x in K if for every $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that for any t in R and any interval I in R of length T there is an s in I such that $\| f(t,x) - f(s,x) \| < \varepsilon$ for all x in K. If f is recurrent in t uniformly for x in each compact subset K of R^n, we say simply that f is a recurrent function. This definition coincides with that of Miller (8) for a single compact subset K.

Now if f is a recurrent function it is not true in general that the motion $p(t,f) = f_t$ will be recurrent in the dynamical system defined by p. For example, the function $f(t) = \sin t^2$ is recurrent but not uniformly continuous. However, Theorem 1.3 and Lemma 1.2 together imply that if f_t

is a recurrent motion, f must be uniformly continuous on R x K, K a compact subset of R^n. For this reason we define a stronger type of recurrence as follows.

<u>Definition 3.3</u>: A function f in $C[R \times R^n, R^n]$ is said to be strongly recurrent if for every $\epsilon > 0$ and L a compact subset of R^{n+1} there exists a $T(\epsilon, L) > 0$ such that for any t in R and interval I of length T there is an s in I such that $\| f(t+u,x) - f(s+u,x) \| < \epsilon$ for all (u,x) in L.

<u>Remark</u>: Every strongly recurrent function is recurrent.

We proceed to investigate some of the properties of recurrent and strongly recurrent functions and their connections with recurrent motions in the dynamical systems described in Section II.

<u>Lemma 3.1</u>: If f is in $C[R \times R^n, R^n]$ and is a recurrent function, then every g in F*(f) is a recurrent function.

Proof. Take $\epsilon > 0$ and K compact in R^n. Since f is recurrent there exists a number $T(\epsilon/3, K)$ such that for any t in R and any interval I of length T there is an s in I such that $\| f(t,x) - f(s,x) \| < \epsilon/3$ for all x in K. Now take any t in R and any interval $I = [t_0, t_1]$ of length $T(\epsilon/3, K)$. Let L be a compact subset of R containing t and I. Now let g be any function of F*(f). Then there is a sequence $\{t_n\}$ in R such that $f_{t_n} \to g$ uniformly on compact sets. So there is a t_N such that $\| g(u,x) - f(u+t_N, x) \| < \epsilon/3$ for all (u,x) in L x K. Now there is a number $s+t_N$ in the interval

$I' = [t_o+t_N, t_1+t_N]$ such that $\| f(t+t_N,x) - f(s+t_N,x) \| < \varepsilon/3$ for all x in K. Then

$$(3.1) \qquad \| g(t,x) - g(s,x) \| \leq \| g(t,x) - f(t+t_N,x) \| + \| f(t+t_N,x) - f(s+t_N,x) \| + \| f(s+t_N,x)-g(s,x) \| .$$

But each of the terms on the right in (3.1) is less than $\varepsilon/3$ for all x in K since s and t are both in L. Then g is a recurrent function since s is in I and K is an arbitrary compact subset of R^n.###

Therefore, if f is a recurrent function, every function in F*(f) is recurrent. This same property obtains for strongly recurrent functions but we will verify this fact indirectly via dynamical systems.

The definition of a strongly recurrent function was conceived with the following lemma in mind.

<u>Lemma 3.2</u>: If f is in $C[R \times R^n, R^n]$, then f is a strongly recurrent function if and only if the motion $p(t,f) = f_t$ is recurrent in the extension of the dynamical system of Bebutov defined in Section II.

Proof. Suppose f is a strongly recurrent function. Take $\varepsilon > 0$ and any compact set K in R^{n+1}. Take N large enough so that $1/2^N < \varepsilon/2$ and K is in the cube K_N in R^{n+1} described in Section II. Then, since f is strongly recurrent, there is a $T(\varepsilon/2, K_N)$ such that for any t in R and interval I of length T there is an s in I such that

$$(3.2) \qquad \| f(u+t,x) - f(u+s,x) \| < \varepsilon/2$$

for all (u,x) in K_N. Then $d_j(f_t,f_s) < \varepsilon/2$ for $j = 1, 2, \ldots N$, and

$$d(f_t,f_s) = \sum_{j=1}^{N} \frac{d_j(f_t,f_s)}{2^j(1+d_j(f_t,f_s))} + \sum_{j=N+1}^{\infty} \frac{d_j(f_t,f_s)}{2^j(1+d_j(f_t,f_s))}$$

$$\leq \sum_{j=1}^{N} \frac{\varepsilon/2}{2^j} + \sum_{j=N+1}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{2} + \frac{1}{2^N} < \varepsilon.$$

Therefore, f_t is a recurrent motion.

On the other hand, if f_t is a recurrent motion and we take $0 < \varepsilon < 1$, then for any compact set K in R^{n+1} there is an integer N such that K is in K_N. From the recurrence of the motion f_t we choose $T(\varepsilon/2^{N+1})$. Then, for any interval I of length T there is an s in I such that $d(f_t,f_s) < \varepsilon/2^{N+1}$.

So $\sum_{j=1}^{\infty} \frac{d_j\ (f_t,f_s)}{2^j(1+d_j(f_t,f_s))} < \varepsilon/2^{N+1}$. But each term of the series is less than $\varepsilon/2^{N+1}$ so $d_N(f_t,f_s) < \varepsilon 2^N(1+d_N(f_t,f_s))/2^{N+1}$, from which we get $d_N(f_t,f_s) < \varepsilon/(2-\varepsilon) < \varepsilon$. Using the definition of d_N we have $\| f(t+u,x) - f(s+u,x)\| < \varepsilon$ for all (u,x) in K_N. Since K is in K_N, f is a strongly recurrent function. ###

Birkhoff's theorems (1.3) and (1.4) then yield the lemma for strongly recurrent functions analogous to Lemma 3.1.

Lemma 3.3: If f is in $C[R \times R^n, R^n]$ and is strongly recurrent, then every g in $F^*(f)$ is a strongly recurrent function.

Proof. By the preceding lemma f_t is a recurrent motion. Since (C,d) is a complete metric space, Theorem 1.3 yields the result that $\gamma^*(f) = F^*(f)$ is a compact minimal set.

Theorem 1.4 states that every motion in a compact minimal set is recurrent. So, if g is in F*(f), $g_{\cdot}$ is a recurrent motion and, again by Lemma 3.2, g is a strongly recurrent function. ###

Before going on to the next lemma, which is quite important in what follows, we shall discuss the concepts of recurrent motions and their relationship to almost periodic functions that have been studied by Auslander and Hahn (1), Nemytskii (9), and Sell (13). We start with a definition.

<u>Definition 3.4</u>: Let π be a dynamical system on the metric space (X,d). Then for x in X the motion $\pi(t,x)$ is pseudo-recurrent if for every $\varepsilon > 0$ there is a $T(\varepsilon) > 0$ such that in every interval I of length T there is a number s such that $d(\pi(0,x), \pi(s,x)) < \varepsilon$.

Clearly, every recurrent motion is pseudo-recurrent. Nemytskii and Stepanov (10), p. 378, show that if a pseudo-recurrent motion is compact, then it is recurrent.

Let us define for f and g in $BC[R \times R^n, R^n]$

$$(3.3) \qquad \delta_j(f,g) = \sup_{(t,x) \text{ in } R\times J_j} \| f(t,x) - g(t,x) \| .$$

$$\delta(f,g) = \sum_{j=1}^{\infty} \delta_j(f,g)/2^j(1+\delta_j(f,g)).$$

Then δ is a metric on C and it is well known (cf. (1), (13)) that in the dynamical system p: $R \times (C,\delta) \to (C,\delta)$ the concepts of pseudo-recurrent motions, recurrent motions, and almost periodic motions are identical. In fact, f in (C,δ) is an almost periodic function if and only if the motion f_t is recurrent.

The situation is radically altered, however, when the metric space is (C,d). We then have the following implications. Let K be any compact subset of R^n. If f(t,x) is almost periodic in t uniformly for x in K, then the motion f_t is recurrent. Also, as pointed out above, a recurrent motion is pseudo-recurrent. But for neither of these last two statements is the converse true.

Auslander and Hahn (1) give interesting examples of functions whose motions are pseudo-recurrent, but not recurrent. In these examples F*(f) is not compact, while for a.p. functions and functions with recurrent motions F*(f) is always compact.

Nemytskii and Stepanov (10), p. 391, give an example of a motion on a torus which is recurrent, but not a.p. The following theorem (cf. Nemytskii (9)) can then be used to show the existence of a strongly recurrent function which is not a.p.

<u>Theorem 3.1</u>: (Bebutov) Every dynamical system situated in a compact metric space with no more than one equilibrium point can be mapped continuously into a dynamical system of Bebutov.. The mapping preserves trajectories.

The example cited above is in a compact space T, a torus in R^3, and has no equilibrium points. The function from T into (C,d) is in fact shown to be one-to-one and continuous both ways. Since T is compact, the continuity is uniform and therefore a.p. motions go into a.p. motions and recurrent motions go into recurrent motions. So the image f in (C,d) of the recurrent, but not a.p. motion on the torus will be a function which is strongly recurrent, but not a.p.

Another point of interest is the fact that if f is a.p. and g is in F*(f), then f is in F*(g) with either the metric d or δ on $C[R \times R^n, R^n]$. However, this property does not hold in general for recurrent functions (cf. Definition 3.2) with the metric d on $C[R \times R^n, R^n]$ as the following example shows.

<u>Example 3.1</u>: Let $f(t) = \sin (t+\exp(-t^2))$. Now f is a recurrent function and g(t) = sin t is in F*(f) using the metric d. But f is clearly not in F*(g) since every function in F*(g) is periodic.

For the case of strongly recurrent functions the property does hold as Professor Sell has pointed out.

<u>Lemma 3.4</u>: (Sell) If f is in $BU[R \times R^n, R^n]$, then g is in F*(f) implies f is in F*(g) if and only if the motion f_t is recurrent.

Proof: If f_t is a recurrent motion and g is in F*(f) then g_t is a recurrent motion and F*(f) is a compact minimal set. But F(g) is in F*(f) since F*(f) is invariant, and F*(g) = F*(f) since both are minimal sets. So, since f is in F*(f), f is in F*(g).

Conversely, let g in F*(f) imply that f is in F*(g). Now by Lemma 2.1 F*(f) is compact. Since it is invariant, it contains a recurrent motion, say g_t. By hypothesis, f is in F*(g) and so by Lemma 3.3 the motion f_t is recurrent. ###

IV. EXISTENCE OF RECURRENT SOLUTIONS

In this section we shall see what information the classical theory of dynamical systems as applied in the previous sections to nonautonomous systems of ordinary differential equations yields about the existence of recurrent solutions to those systems.

As before, we consider the system

(A) $\quad x' = f(t,x).$

In addition we consider the associated system

(B) $\quad x' = g(t,x).$

Here the function f is in $C[R \times R^n, R^n]$ and g is in $F^*(f)$ with the compact open topology used throughout on $C[R \times R^n, R^n]$.

We will also set forth the following set of conditions:

(A1) For each x in R^n the System A has a unique solution $\varphi(t,x,f)$ such that $\varphi(0,x,f) = x$ and $\varphi(t,x,f)$ exists for all t.

(A2) For every function g in $F^*(f)$ Condition A1 holds for the System B.

(A3) The function f is in $BU[R \times R^n, R^n]$.

(A4) There exists a solution $\varphi(t,x,f)$ to System A and a compact subset K of R^n such that $\varphi(t,x,f)$ is in K for all $t \geq 0$. That is, System A has a positively compact solution.

Now a simple example will show that Conditions A1 through A4 are not sufficient to guarantee the existence of recurrent solutions to System A.

Example 4.1: Let $f(t,x) = 1/(1+t^2)$. It is easy to verify that Conditions A1 through A4 are satisfied. But the solutions of $x' = f(t)$, $\varphi(t,x,f) = x + \arctan t$, are clearly not recurrent.

Although in Example 4.1 System A has no recurrent solutions we note that the zero function is in $F^*(f)$. This function is recurrent and every solution to $x' = 0$ is constant and hence recurrent. The following theorem shows that we can always expect recurrent solutions to at least some of the systems $x' = g(t,x)$ where g is in $F^*(f)$ and f satisfies Conditions A1 through A4.

Theorem 4.1: If $f(t,x)$ and its associated systems of ordinary differential equations satisfy Conditions A1 through A4 above, then there exists in $\Omega(f)$ a set of functions M such that if g is in M, then g is strongly recurrent and there is a solution $\varphi(t,y,g)$ to (B) which is a recurrent function.

Proof. With f satisfying the above conditions, we know from the results of the previous sections that the mapping $\pi(t;x,f) = (\varphi(t,x,f), f_t)$ defines a dynamical system mapping $R \times X$ into X, where $X = R^n \times F^*(f)$.

Let $\varphi(t,x,f)$, the positively compact solution to (A), be contained in the compact set K. $F^*(f)$ is compact by (A3) and Lemma 2.1. Let $Y = \Omega(x,f)$, the omega limit set of the motion $\pi(t;x,f)$. Then Y is a closed, invariant subset of the compact set $K \times F^*(f)$. So Y is compact and by Theorem

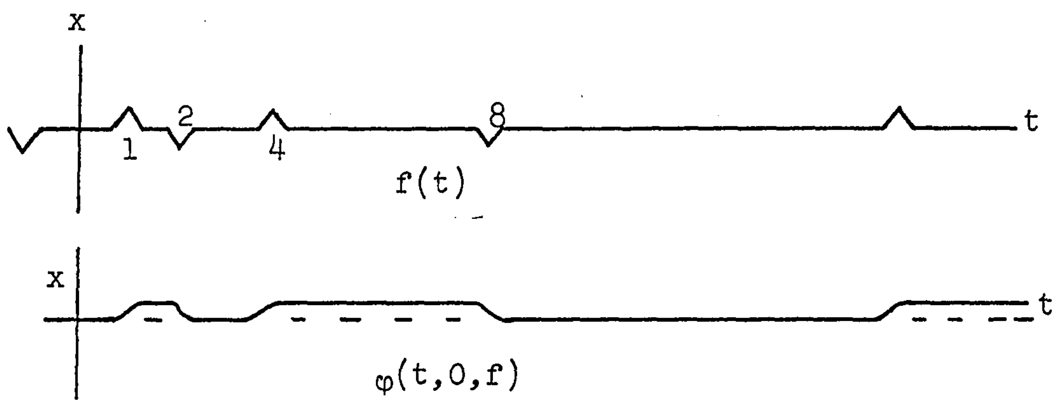
1.2, Y contains a minimal set Z. Since Z is closed, it is compact and by Theorem 1.4 every trajectory in Z is recurrent.

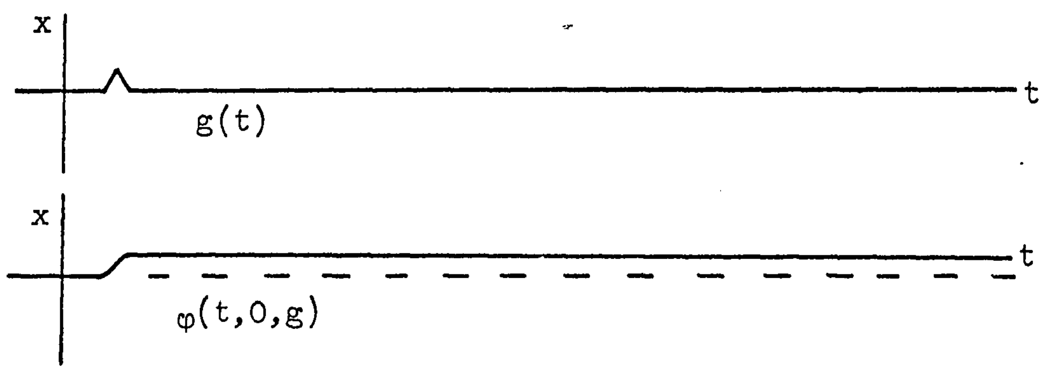
Let (y,g) be in Z. Then g is in $\Omega(f)$ and $\pi(t;y,g)$ is a recurrent motion. But a direct application of the definitions of recurrence and the metric ρ yields the result that $\varphi(t,y,g)$ is a recurrent function and $p(t,g) = g_t$ is a recurrent motion. By Lemma 3.2 the function g(t,x) is strongly recurrent. ###

Remarks: (1) We note that in the proof above, the recurrence of the function $\varphi(t,y,g)$ and the motion g_t are similar in the following sense. For every $\epsilon > 0$ there is a $T(\epsilon)$ such that for each t in R and interval I of length $T(\epsilon)$ there is in I a number s such that $\|\varphi(t,y,g) - \varphi(s,y,g)\| < \epsilon$ and $d(g_t,g_s) < \epsilon$. (2) Since $Y = \Omega(x,f)$ is an invariant set, $\varphi(t,y,g)$ is in K for all t.

Now in Example 4.1, using the notation of Theorem 4.1, Y = Z. That is, the whole omega limit set $\Omega(x,f)$ is recurrent. The following example shows that this need not always be the case.

Example 4.2: Let f(t) be an odd function in C[R,R] which is everywhere zero except for isosceles triangular pulses of height one and base one erected with vertices at points $t = 2^j$, j = 0, 1, 2, ... The pulses are alternatively positive and negative. Then f and the differential equation $x' = f(t)$ satisfy Conditions A1 through A4 as may easily be verified. For $t_j = 3.2^j$ the sequence of translates $f(t+t_j)$ converges in the compact open topology to the zero function. All solutions to $\dot{x} = 0$ are recurrent and for this sequence of translates $\pi(t_j;0,f)$ converges to a point in Z, the minimal set of the recurrent motions described in Theorem 4.1. However, if $t_j = 2^{2j} - 1$, $f(t+t_j)$ converges to a function g in $\Omega(f)$ which has exactly one positive pulse at t = 1. Clearly, this g(t) is not strongly recurrent, nor are the solutions to $x' = g(t)$ recurrent functions; see graphs below.





We continue to examine conditions for the existence of recurrent solutions to various systems. The next result was proved in weaker form by Miller (8). There he assumes that f is a.p. and shows the existence of a recurrent solution.

Theorem 4.2: If $f(t,x)$ is a strongly recurrent function and the system $x' = f(t,x)$ satisfies Conditions A1 through A4 then there exists a recurrent solution to System A.

Proof. By Theorem 4.1 there is a minimal set of recurrent motions in the omega limit set of the motion $\pi(t;x,f)$. Let $\varphi(t,x,f)$ be the positively compact solution from Condition A4. Let (y,g) be in the above minimal set and $\{t_n\} \to \infty$, $\varphi(t_n,x,f) \to y$, and $f_{t_n} \to g$ in the compact open topology. Since f_t is a recurrent motion and g is in $\Omega(f)$, then by Lemma 3.4 f is in $\Omega(g) = F^*(g)$. So there is a sequence $\{s_n'\} \to \infty$ such that $g_{s_n'} \to f$. Since $\Omega(y,g)$ is a compact minimal set, there is a subsequence $\{s_n\}$ of $\{s_n'\}$ such that $\pi(s_n;y,g) = (\varphi(s_n,y,g),g_{s_n}) \to (w,f)$ where w is in K. But by Theorem 1.4, every motion in the compact minimal set $\Omega(y,g)$ is recurrent. So $\varphi(t,w,f)$ is a recurrent function.###

We can now state the following corollary.

Corollary 4.1: Let $f(t,x)$ and its associated systems of ordinary differential equations satisfy Conditions A1 through A4 and in addition let every g in $\Omega(f)$ be a strongly recurrent function. Then for every function g in $\Omega(f)$ the system $x' = g(t,x)$ has a recurrent solution.

Proof. We need only verify that the systems $x' = g(t,x)$ satisfy Conditions A1 through A4. Let $\varphi(t,x,f)$ be the positively compact solution to $x' = f(t,x)$. Let $\pi(t_n,x,f) \to (y,g)$ in $\Omega(x,f)$. Since g is in $F^*(f)$, Condition A1 holds for g. To see that Condition A1 holds for every function h in $F^*(g)$ we note that if g is in $F^*(f)$, then $F(g)$ is in $F^*(f)$. Since $F^*(f)$ is closed, $F^*(g)$ is in $F^*(f)$. Thus, h is in $F^*(f)$ and Condition A2 is satisfied with f replaced by g. Lemma 2.1 states that g is in $BU[R \times R^n, R^n]$ so Condition A3 is satisfied for g. Now $\Omega(y,g)$ is compact, (as is shown in the proof of Theorem 4.1) and since it is invariant under the mapping π, the motion through (y,g) is compact and therefore the solution $\varphi(t,y,g)$ is compact. Thus, Condition A4 is satisfied for the system $x' = g(t,x)$. Since g is a strongly recurrent function, Theorem 3.2 implies the existence of a recurrent solution to the system $x' = g(t,x)$. But g is an arbitrary function of $\Omega(f)$, so the proof is complete. ###

We now consider a case in which the omega limit set $\Omega(f)$ is entirely composed of recurrent motions, while f itself is not strongly recurrent. This occurs when f is of the form $g(t,x) + r(t,x)$ where $g(t,x)$ is a strongly recurrent function and $r(t,x) \to 0$ as $t \to \infty$. We establish a preliminary result.

Lemma 4.1: Let f and g both be in $BU[R \times R^n, R^n]$. Let g be strongly recurrent and $f(t,x) = g(t,x) + r(t,x)$ where

$r(t,x) \to 0$ as $t \to \infty$. Then $\Omega(f) = \Omega(g)$ and $\Omega(f)$ is a minimal set of recurrent motions.

Proof. Let $h(t,x)$ be any function in $\Omega(f)$. Then there is a monotone sequence of real numbers $\{t_n\} \to \infty$ such that $d(f_{t_n},h) \to 0$. Let j be any positive integer. Take any ε such that $0 < \varepsilon < 1$. Then there is a positive integer N_j, such that if $n > N_j$, $d(f_{t_n},h) < \varepsilon/2^{j+1}$. As in the proof of Lemma 3.2, this implies that $d_j(f_{t_n},h) < \varepsilon/(2-\varepsilon) < \varepsilon$. So $\| f(t+t_n,x) - h(t,x)\| < \varepsilon$ for all (t,x) in K_j provided $n > N_j$.

On the other hand, since $r(t,x) \to 0$ and r is uniformly continuous in $R \times J_j$; then there is a positive integer M_j such that if $t > t_{M_j} - j$ and x is in J_j then $\| f(t,x)-g(t,x)\| < \varepsilon$. So if n is greater than N_j and M_j,

$$(4.1) \qquad d_j(g_{t_n},h) = \sup_{(t,x) \text{ in } K_j} \| g(t+t_n,x) - h(t,x)\| < 2\varepsilon.$$

Statement 4.1 is true because

$$(4.2) \qquad \| g(t+t_n,x)-h(t,x)\| \leq \| g(t+t_n,x)-f(t+t_n,x)\| + \| f(t+t_n,x) - h(t,x)\| .$$

From the discussion above we see that each of the terms on the right in (4.2) is less than ε for (t,x) in K_j. This is true because if $|t| < j$ and $n > M_j$, then $t > -j$ and $t_{M_j} - j < t_n + t$. So for each positive integer j there is a corresponding positive integer $P_j = \text{Max}\{N_j, M_j\}$ such that if $n > P_j$, $d_j(g_{t_n},h) < 2\varepsilon$.

Finally, take $\bar{P}$ a positive integer large enough so that $1/2^{\bar{P}} < \varepsilon$. Then if $P = \text{Max}\,\{\bar{P}, P_1, P_2, \ldots P_{\bar{P}}\}$, and $n > P$ we have

$$(4.3) \qquad d(g_{t_n},h) = \sum_{j=1}^{\infty} d_j(g_{t_n},h)/2^j\ (1+d_j(g_{t_n},h))$$

$$\leq \sum_{j=1}^{\bar{P}} d_j(g_{t_n},h)/2^j + \sum_{j=\bar{P}+1}^{\infty} 1/2^j$$

$$< 2\varepsilon + 1/2^{\bar{P}} < 3\varepsilon.$$

So h is in $\Omega(g)$ and since g is strongly recurrent, h is also. ###

Remark: The author feels that the converse of Lemma 4.1 is not true, but he has so far been unable to construct a counter example.

The preceding lemma enables us to prove the following existence theorem.

Theorem 4.3: Let $f = g+r$ be as in Lemma 4.1. Let $x' = f(t,x)$ be a system of ordinary differential equations satisfying Conditions A1 through A4. Then for every function h in $\Omega(f)$ there exists a recurrent solution to the differential equation $x' = h(t,x)$. In particular, there is a recurrent solution to the equation $x' = g(t,x)$.

Proof. Lemma 4.1 and Corollary 4.1 together imply that if h is in $\Omega(f)$ then there exists a recurrent solution to the system $x' = h(t,x)$. The second conclusion follows directly from the fact that $\Omega(f) = \Omega(g)$. Now since g is strongly recurrent, $\Omega(g) = F^*(g)$. But g is in $F^*(g)$ so g is in $\Omega(f)$. ###

Consider the following result for dynamical systems. (cf. (10), p. 327.)

Theorem 4.4: If π: R x X $\to$ X defines a dynamical system on the metric space (X,d) then for every $\epsilon > 0$ and $T > 0$ there is a $\delta > 0$ such that if $d(p,q) < \delta$, then $d(\pi(t,p), \pi(t,q)) < \epsilon$ for all t such that $|t| < T$.

Theorem 4.4 implies that under the conditions of Corollary 4.1 and Theorem 4.3, the positively compact solution from Condition A4 approaches a recurrent function uniformly on compact sets. That is, for every sequence $\{t_n\} \to \infty$ there is a subsequence $\{s_n\}$ and a recurrent function $\psi(t)$ such that for every $\epsilon > 0$ and $T > 0$ there is an N such that if $n > N$ then $\|\varphi(t+s_n,x,f) - \psi(t)\| < \epsilon$ for all t such that $|t| < T$. However, we cannot claim that there is a T' such that the above inequality holds for all $t > T'$.

We illustrate this behavior with the following example in which we exhibit a system which satisfies the conditions of Theorem 4.3 but no positively compact solution approaches a recurrent function asymptotically for all t greater than some T' as described above.

Example 4.3: Let r and θ be the usual polar coordinates in the plane. Take

$$(4.4) \qquad r' = \begin{cases} 0 & \text{if } r \leq 1, \\ (1-r)/2(1+|t|) & \text{if } r > 1, \end{cases}$$

and

$$\theta' = \begin{cases} 1 & \text{if } r \leq 1, \\ 1 + (r-1)/\sqrt{1+|t|} & \text{if } r > 1. \end{cases}$$

For $t = 0$, let $r = r_o$, $\theta = \theta_o$.

The right hand members in (4.4) are bounded and uniformly continuous in every compact set $r < R$. In fact, its rectangular form is

$$(4.5) \qquad x' = \begin{cases} -y & \text{if } r \leq 1, \\ [(1-r)(x+2yr\sqrt{1+|t|})-2yr(1+|t|)]/2r(1+|t|) & \text{if } r > 1. \end{cases}$$

$$y' = \begin{cases} x & \text{if } r \leq 1, \\ [(1-r)(y-2xr\sqrt{1+|t|})+2xr(1+|t|)]/2r(1+|t|) & \text{if } r > 1. \end{cases}$$

Here $r = \sqrt{x^2+y^2}$.

The system is asymptotically autonomous and if $r_o > 1$, the periodic solution $r = 1$, $\theta = t + \theta_o$ is asymptotically, orbitally stable (from one side) but without asymptotic phase. We can show this if we integrate the first equation of (4.5) and substitute in the second.

$$(4.6) \qquad (1-r_o)/(1-r) = \sqrt{1+|t|}\ .$$

$$(4.7) \qquad \theta' = 1+ (r_o-1)/(1+|t|).$$

Integrating (4.7) yields

$$(4.8) \qquad \theta(t) = \theta_o + t + (r_o-1)\ln\,(1+|t|).$$

Since the last term in (4.8) is unbounded, no point of $\varphi(t, r_o, \theta_o, f)$ will stay near a point of the periodic solution $r = 1$, $\theta = t + \theta_o$ for all t greater than some T'.

V. ALMOST AUTOMORPHIC FUNCTIONS

In a recent paper Bochner (2) introduced the class of almost automorphic functions.

Definition 5.1: A continuous, complex valued function is almost automorphic if for every sequence $\{\beta_n\}$ there is a subsequence $\{\alpha_n\}$ such that

(5.1) $$\lim_{m\to\infty} \lim_{n\to\infty} f(t+\alpha_n-\alpha_m) = f(t).$$

There is no assumption of uniformity in the above limits.

Bochner shows that if f is a.p. (almost periodic) then f is a.a. (almost automorphic) and leaves open the question of whether the class of a.a. functions is really larger than the class of a.p. functions.

That question was partially answered by one of Professor Bochner's students, W. A. Veech, in a paper (15) where he exhibits a function which is a.a., but not a.p. However, this example is defined on the integers.

In what follows we apply some results from the preceding sections to show that if f is defined on the real line and is uniformly continuous, then an a.a. function is strongly recurrent. Let us now define almost automorphic functions as we shall use them in this paper.

Definition 5.2: A function $f(t,x)$ in $C[R \times R^n, R^n]$ is a.a. (almost automorphic) if for every sequence $\{\beta_n\}$ in R there is a subsequence $\{\alpha_n\}$ such that

$$(5.2)\qquad \lim_{n\to\infty} f(t + \alpha_n, x) = g(t,x), \text{ and}$$

$$\lim_{n\to\infty} g(t - \alpha_n, x) = f(t,x).$$

The above limits are assumed to be uniform for x's in compact sets.

Definition 5.3: If f is a.a. and the limits (5.2) are uniform for (t,x) in compact subsets of R^{n+1}, then we write: f is a.a.c.

Remarks: (1) If f is a.a.c. then the function g in (5.2) is in F*(f) and F*(f) is compact. (2) If f is a.a. then f is in $BC[R \times R^n, R^n]$. For if f were not bounded on compact sets in R^n, there would exist a sequence with no subsequence for which the first limit in (5.2) exists.

We state a preliminary result.

Lemma 5.1: If f is a.a. and f is in $BU[R \times R^n, R^n]$, then f is a.a.c.

Proof. Since f is in $BU[R \times R^n, R^n]$, F*(f) is compact in the compact open topology by Lemma 2.1. So if $f(t+\alpha_n,x) \to g(t,x)$ pointwise, there is a subsequence of $\{\alpha_n\}$ for which the limit exists uniformly on compact sets. This limit function will be g(t,x), so g is in F*(f). But by Lemma 2.1 g is in $BU[R \times R^n, R^n]$ and from the above subsequence of $\{\alpha_n\}$ we can obtain a second subsequence $\{\beta_n\}$ so that $\{g(t-\beta_n,x)\}$ converges uniformly on compact sets. Since f is a.a. $g(t-\beta_n,x) \to f(t,x)$ and f is a.a.c. ###

The following result is immediate.

Theorem 5.1: If f is a.a.c. then f is strongly recurrent.

Proof. If g is in F*(f) then there is a sequence $\{\beta_n\}$ such that $f(t+\beta_n,x) \to g(t,x)$ uniformly on compact sets. Since f is a.a.c. there is a subsequence $\{\alpha_n\}$ of $\{\beta_n\}$ such that $g(t-\alpha_n,x) \to f(t,x)$ uniformly on compact sets, so f is in F*(g). Recalling the remark following Definition 5.3 we see that F*(f) is compact, so f is in $BU[R \times R^n, R^n]$ by Lemma 2.1. Lemma 3.4 then implies that f is strongly recurrent. ###

So, for functions f in $BU[R \times R^n, R^n]$ the following implications hold: If f is a.p., it is a.a., and if it is a.a. it is strongly recurrent. Nemytskii (9) suggests the problem of characterizing those functions which are strongly recurrent, but not a.p. If it could be shown that the set of a.a. functions is the same as the set of a.p. functions, then a strongly recurrent function which is not a.p. (cf. discussion preceding Example 3.1) could be characterized by the existence of a sequence $\{\alpha_n\}$ containing no subsequence for which (5.2) holds. Put another way, for such a function f there would exist, since F*(f) is compact, a sequence $\{\alpha_n\}$ such that $f(t+\alpha_n,x) \to g(t,x)$ and $g(t-\alpha_n,x) \to h(t,x)$. Both of the limits above are uniform on compact sets in R^{n+1} and $h \neq f$.

VI. LITERATURE CITED

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