2017

Constructions for cospectral graphs for the normalized Laplacian matrix and distance matrix

Kristin Elizabeth Heysse
Iowa State University

Constructions for cospectral graphs for the normalized Laplacian matrix and distance matrix

by

Kristin Heysse

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Steve Butler, Major Professor
Leslie Hogben
Bernard Lidický
Yiu Tung Poon
Michael Young

Iowa State University

Ames, Iowa

2017

DEDICATION

I would like to dedicate this work to every person who laughed with me, cried with me, and supported me through the last five years. I would not have made it through without your love. At the top of this list are my incredible parents, Mark and Susan, and my wonderful siblings Kevin, Joy, and Billy. Also my amazing friends, too many to list here, but each more helpful than the last. All of you have seen me through this chapter, and I look forward to the next.

"It matters not how strait the gate,
how charged with punishment the scroll,
I am the master of my fate,
I am the captain of my soul!"
W. Henley, *Invictus*

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my thanks to those who helped me with this thesis. First and foremost, Dr. Steve Butler for his unwavering support and guidance throughout my time at Iowa State University. His patience, insight, and commitment were invaluable for my research and overall graduate experience. I would also like to thank Dr. Michael Young for his constant mentorship and reality checks when I needed them most. Both Steve and Michael have provided me with opportunities beyond my dreams, for which I am incredibly thankful. Additionally, I thank my additional committee members, Dr. Leslie Hogben, Dr. Bernard Lidický, and Dr. Yiu Tung Poon, for their contributions to this work. I also extend my thanks to my colleagues in the MSCS department at Macalester College for their support during my final semester as a graduate student in the spring of 2017. Finally, I would like to thank Dr. Jess Lenarz, an undergraduate mentor and friend, for her initial nudge toward graduate school and her consistently good advice.

CHAPTER 1. INTRODUCTION

At the broadest level, graph theory can be summarized as the study of relationships between objects. The objects, called *vertices*, are related by *edges*. One example of a graph can be seen in air travel, where the vertices are the airports and the edges are the direct flights between airports. Another very large graph is the internet, where the vertices are web pages and the (directed) edges are the links between them.

Frequently graphs are quite large, and current computational limitations have required the development of efficient techniques that capture a portion of a graph's structure. *Spectral graph theory* is one such method. By associating a matrix with the graph and considering the eigenvalues (or *spectrum*) of that matrix, we reduce the number of data points from potentially order n^2 to order n. At this point, immediate questions arise concerning what information about the graph is or is not contained in the spectrum. Spectral graph theory explores these questions, and extensive work has already been done (see [3],[10] for an overview).

1.1 Notation and definitions

A graph $G = (V(G), E(G))$ is a pair of sets. When clear context allows, we often write $G = (V, E)$. The definition of an edge varies throughout graph theory, and edges can be weighted, directed, and possibly consist of more than two vertices. A directed edge $u \to v$ is an ordered pair of vertices (u, v), while an undirected edge $\{u, v\}$ is a two element subset of the vertices. Unless specifically stated, we will consider *undirected graphs*, where all edges are undirected. Two vertices are *adjacent*, $u \sim v$, if $\{u, v\} \in E$. A vertex and an edge are *incident* if the vertex is a part of the edge. Further, we can define a *weight function*, w, on the edges, and we say G is a *weighted graph*. Every unweighted graph can be considered as a weighted graph where the weight of every edge is 1. The *degree* of a vertex v, denoted $\deg(v)$, is the sum of the weights of the edges incident to the vertex.

A *path* of length n is a sequence of non-repeating vertices $(v_1, v_2, \ldots v_{n+1})$ such that $v_i \sim v_{i+1}$ for all $i \in \{1, 2, \ldots, n\}$. If the vertices are allowed to repeat, we have a *walk* of length n, and if $v_1 = v_{n+1}$, we have a *closed walk*. Notice that both paths and walks are measured by the number of edges traveled, not the number of vertices in the sequence. The *distance* between two vertices in a graph G, $\text{dist}_G(u, v)$, is the length of the shortest path from u to v. A graph is said to be *connected* if between every pair of vertices there exists a path. If a graph is not connected, we can instead consider its *components*, or connected parts.

Frequently it is useful to consider *subgraphs* of a graph. A subgraph is a graph formed by taking a subset V' of V and E' of $E \cap \binom{V'}{2}$, where $\binom{V'}{2}$ are the two element subsets of V'. In words, we can take any of the edges included in G among the vertices in V'. We will also say a subgraph is *induced* if $E' = E \cap \binom{V'}{2}$, or equivalently we include every edge which is in G among the vertices V'.

Finally, we say a graph is *bipartite* if there exists a partition of the vertices $V = A \cup B$ such that every edge in the graph is formed by exactly one vertex in A and one vertex in B. Equivalently, every edge "goes between" A and B. An equivalent condition to being bipartite is that the graph contains no odd *cycles*. A cycle is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ such that $v_1 = v_n$ and no other vertices are repeated.

For an $n \times n$ square matrix M, the *characteristic polynomial* is the degree n polynomial $p_M(x) = \det(xI - M)$. The roots of this polynomial form a multiset called the *spectrum*. These roots are also known as the *eigenvalues* of M. If for some vector $\mathbf{x}$ we have that $M\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an *eigenvector* and $(\lambda, \mathbf{x})$ is an *eigenpair* of M. We say a square matrix N is a *diagonal matrix* if all nonzero entries occur on the diagonal.

1.2 Common matrices in spectral graph theory

There are a variety of ways to associate a matrix with a graph, and we will consider five: adjacency, Laplacian, signless Laplacian, normalized Laplacian, and distance. We begin here by defining each with respect to G, a graph on n vertices with weighted edges. The following matrices are $n \times n$ and indexed by the vertices of G. We will frequently denote the graph upon which the matrix is built with a superscript, but when the clear context allows, we will drop the superscript.

The (weighted) *adjacency matrix* of G, denoted A^G has entries $A^G_{ij} = w(i,j)$ if i is adjacent to j and 0 otherwise. For an unweighted graph, the powers of the adjacency matrix can be used to count walks. The following proposition can be proven by induction on h.

Proposition 1.2.1 ([3]). *Let h be a nonnegative integer. Then $(A^h)_{xy}$ is the number of walks of length h from x to y.*

The diagonal degree matrix, D^G, is the diagonal matrix where $D^G_{ii} = \deg(i)$. The *Laplacian* of a graph G, denoted L^G, is defined to be $L^G = D^G - A^G$. Similarly, the *signless Laplacian* is $Q^G = D^G + A^G$.

The *normalized Laplacian*, denoted $\mathcal{L}^G$, is defined below for a weighted graph.

$$\mathcal{L}^G_{ij} = \begin{cases} 1 & \text{if } i = j, \text{ and vertex } i \text{ is not isolated;} \\ \frac{-w(i,j)}{\sqrt{\deg(i)\deg(j)}} & \text{if } i \sim j; \\ 0 & \text{otherwise.} \end{cases}$$

If the graph has no isolated vertices, $\mathcal{L}^G = (D^G)^{-1/2} L^G (D^G)^{-1/2}$, where $(D^G)^{-1/2}$ is the diagonal matrix such that $(D^G)^{-1/2}_{ii} = \frac{1}{\sqrt{\deg(i)}}$.

Finally, the *distance matrix*, denoted $\mathcal{D}^G$ of a *connected graph* is the matrix with entries which are the pairwise distances between vertices, or equivalently, $\mathcal{D}^G_{ij} = \mathrm{dist}_G(i,j)$. We require the graph to be connected so the distance between vertices is well defined.

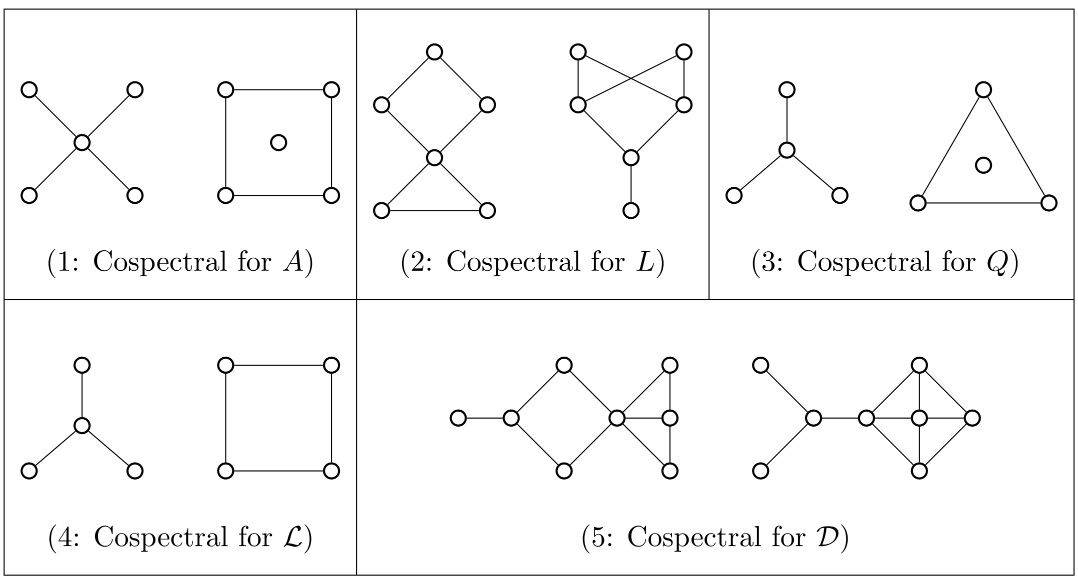
One of the overarching questions in spectral graph theory asks what information about the graph the spectrum contains. An alternative approach to this question considers *cospectral graphs*, or graphs which are nonisomorphic yet have the same spectrum. If cospectral graphs differ in some fundamental way, then the spectrum cannot always capture information about this fundamental difference.

Table 1.1 summarizes known results for four structural properties for the five defined matrices. We immediately note that questions of connectedness do not apply to the distance matrix. A "Yes" in the table corresponds to a theorem which proves the information can be found from the spectrum. A "No" in the table indicates there exists a cospectral pair of graphs which differ in that regard. Further, if a number is listed, an example pair of cospectral graphs is shown in Table 1.2. Table 1.1 can be found in [6] with the row for $\mathcal{D}$ omitted. The question of bipartite distance cospectral graphs is discussed further in Section 4.2.

Table 1.1 What the spectrum knows

	Bipartite	Number of Components	Bipartite Components	Number of Edges
A	Yes	No (1)	No	Yes
L	No (2)	Yes	No	Yes
Q	No	No (3)	Yes	Yes
$\mathcal{L}$	Yes	Yes	Yes	No (4)
$\mathcal{D}$	?	N/A	N/A	No (5)

Table 1.2 Examples of cospectral graphs for various matrices



Having considered a few examples of cospectral graphs, we will now look at results which prove that a structural property about the graph can be determined by the eigenvalues. In other words, we will look at a selection of the "Yes" entries in Table 1.1. We will first consider the column for the number of edges in a graph, because the normalized Laplacian and the distance matrix (the predominant matrices in this thesis) both differ from the other matrices we consider.

Proposition 1.2.2. *For an unweighted graph G, the number of edges can be determined from the spectra of A, L and Q.*

Proof. First, note that the sum of the degrees of a graph is twice the number of edges of a graph, easily proven by double counting. Further, recall that the trace of a matrix is both the sum of the eigenvalues and the sum of the diagonal entries. The result for L and Q follows immediately, as the diagonal entries of both matrices are the degrees of the vertices.

For any matrix M with spectrum $\{\alpha_i\}$ and for any polynomial $p(x)$, the spectrum of the matrix $p(M)$ is the multiset $\{p(\alpha_i)\}$. Consider the diagonal entries of A^2. By Proposition 1.2.1 these are closed walks of length two from a vertex to itself. For every vertex i, there are exactly $\deg(i)$ such walks. Therefore the trace of A^2, which is the squares of the eigenvalues for A, is the sum of the degrees. Thus knowing the spectrum of A, we know the number of edges of the graph. □

The following proposition considers determining if a graph is bipartite from its adjacency spectrum. A proof has been included to illustrate more clearly how the spectrum can be used to prove a structural property (though a shorter proof employing Perron-Frobenius theorem can be seen in [3]).

Proposition 1.2.3. *A graph is bipartite if and only if its adjacency spectrum is symmetric about zero.*

Proof. Let G be a bipartite graph such that $V(G) = U \cup W$. We will order the vertices $U = v_1, \ldots, v_k$ and $W = v_{k+1}, \ldots, v_n$. Because there are no edges between parts, we can write the adjacency matrix of G as the following block matrix:

$$A = \left[\begin{array}{c|c} O & B^T \\ \hline B & O \end{array}\right]$$

where B is some matrix of size $n - k \times k$.

Suppose (λ, x) is an eigenpair of A. We will write $x = [x_1|x_2]^T$, where x_1 is of length k and x_2 is of length $n-k$. Because x is an eigenvector, we know

$$Ax = \left[\begin{array}{c|c} O & B^T \\ \hline B & O \end{array}\right] \left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right] = \left[\begin{array}{c} B^T x_2 \\ \hline B x_1 \end{array}\right] = \lambda \left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right].$$

Let y be the vector $y = [x_1| - x_2]^T$. We claim $(-\lambda, y)$ is an eigenpair for A:

$$Ay = \left[\begin{array}{c|c} O & B^T \\ \hline B & O \end{array}\right] \left[\begin{array}{c} x_1 \\ \hline -x_2 \end{array}\right] = \left[\begin{array}{c} -B^T x_2 \\ \hline B x_1 \end{array}\right] = \left[\begin{array}{c} -\lambda x_1 \\ \hline \lambda x_2 \end{array}\right] = -\lambda \left[\begin{array}{c} x_1 \\ \hline -x_2 \end{array}\right] = -\lambda y.$$

We pause here to note that this transformation of the eigenvectors preserves the dimension of the eigenspace, therefore the geometric multiplicity (and subsequently the algebraic multiplicity, since A is real symmetric) of λ is the same as that of $-\lambda$.

Now suppose G is a graph whose spectrum is symmetric about zero. This means there are an even number of nonzero eigenvalues. We will order the spectrum $\{\lambda_1, \ldots \lambda_{2k}, \lambda_{2k+1} \ldots \lambda_n\}$, where the first $2k$ eigenvalues are the nonzero ones. Further, we say the ordering is such that $\lambda_i = -\lambda_{i+k}$ for $i \in [k]$.

Let $3 \leq \ell = 2j+1$ be an odd integer. We claim there are no closed weighted walks of length ℓ, and prove this by considering the trace of A^ℓ and employing Proposition 1.2.1.

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^\ell = \sum_{i=1}^{2k} \lambda_i^\ell &= \sum_{i=1}^{k} \lambda_i^\ell + \sum_{i=k+1}^{2k} \lambda_i^\ell \\
&= \sum_{i=1}^{k} \lambda_i^\ell + \sum_{i=1}^{k} \lambda_{i+k}^\ell = \sum_{i=1}^{k} \lambda_i^\ell + \lambda_{i+k}^\ell \\
&= \sum_{i=1}^{k} \lambda_i^\ell + (-\lambda_i)^\ell = \sum_{i=1}^{k} \lambda_i^{2j+1} + (-\lambda_i)^{2j+1} \\
&= \sum_{i=1}^{k} \lambda_i^{2j} \lambda_i + (-\lambda_i)^{2j}(-\lambda_i) = \sum_{i=1}^{k} \lambda_i^{2j}(\lambda_i - \lambda_i) = 0.
\end{aligned}$$

Since ℓ was arbitrary, there are no odd cycles. This implies the graph is bipartite. □

Similar propositions and proofs exist for the remaining "Yes" entries in Table 1.1. We recommend [3] and [10] for the interested reader.

1.3 Cospectral constructions

In this thesis, we will approach the question of information *not* contained in the spectrum by creating *cospectral constructions*. A cospectral construction is a method for constructing infinitely many pairs of cospectral graphs. One of the most well known is Godsil-McKay switching for the adjacency matrix.

Theorem 1.3.1 ([13]). *Let G be a graph and let $\pi = (C_1, C_2, \ldots, C_k, D)$ be a partition of $V(G)$. Suppose that, whenever $1 \leq i, j \leq k$ and $v \in D$, we have (a) any two vertices in C_i have the same number of vertices in C_j, and (b) v has either 0, $n_i/2$, or n_i neighbors in C_i, where $n_i = |C_i|$. The graph $G^{(\pi)}$ formed by* local switching in G with respect to π *is obtained from G as follows. For each $v \in D$ and $1 \leq i \leq k$, such that v has $n_i/2$ neighbors in C_i, delete these $n_i/2$ edges and join v instead to the other $n_i/2$ vertices in C_i.*

Then G and $G^{(\pi)}$ are cospectral with cospectral complements.

Figure 1.1 shows a pair of adjacency cospectral graphs related by Godsil-McKay switching. In this case, $D = \{v\}$, and $\pi = (V \setminus \{v\}, D)$.

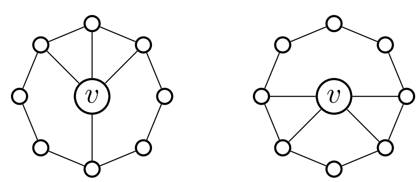


Figure 1.1 A pair of graphs related by Godsil-McKay switching.

The proof of Theorem 1.3.1 proves the adjacency matrices for G and $G^{(\pi)}$ are similar. For summaries of various cospectral constructions for the normalized Laplacian and distance matrix, we refer to Sections 2.1 and 3.1, respectively.

1.4 Thesis organization

The next two chapters of this thesis are papers that are either submitted or published. Chapter 2 contains the paper "A cospectral family of graphs for the normalized Laplacian found by toggling" [8], published jointly with Steve Butler in *Linear Algebra and its Applications.* Mathematics employs alphabetized authorship when publishing papers. As a coauthor, I worked on every aspect of the proof both independently and with Steve, and I did the majority of the writing of the paper. In this paper, we construct three weighted "modules," or small graphs, which can be linked together to form larger graphs. By swapping these modules in a particular way, we create cospectral graphs for the normalized Laplacian. The proof that pairs formed in this way are indeed cospectral relies on the proving that their characteristic polynomials are equal.

Chapter 3 contains the paper "A construction for distance cospectral graphs" [12], submitted. In this paper, I consider creating distance cospectral graphs in two different ways. The first creates distance cospectral graphs of varying edge counts by a graph identification process. The second is a particular localized edge switching. Both constructions rely on a perturbation of the eigenvectors of the graph.

Both Chapter 2 and Chapter 3 have an added unpublished section (Sections 2.8 and 3.5, respectively). These sections will explain any work which occured between submission and the writing of this thesis. We complete this chapter by noting forthcoming notational inconsistencies. In Chapter 2, D denoted the diagonal degree matrix and d_i the degree of vertex i. However, in Chapter 3, D denotes the *distance* matrix and d_{ij} the distance between vertices i and j. This reflects the submissions of two separate papers. Chapter 3 does not utilize the diagonal degree matrix.

CHAPTER 2. A COSPECTRAL FAMILY OF GRAPHS FOR THE NORMALIZED LAPLACIAN FOUND BY TOGGLING

A paper published in *Linear Algebra and its Applications*.

Steve Butler and Kristin Heysse

Abstract

We give a construction of a family of (weighted) graphs that are pairwise cospectral with respect to the normalized Laplacian matrix, or equivalently probability transition matrix. This construction can be used to form pairs of cospectral graphs with different number of edges, including situations where one graph is a subgraph of the other. The method used to demonstrate cospectrality is by showing the characteristic polynomials are equal.

2.1 Introduction

Spectral graph theory studies the relationship between the structure of a graph and the eigenvalues of a particular matrix associated with that graph. There are several matrices that are commonly studied, each with merits and limitations. These limitations exist because graphs can be constructed which have the same spectrum with respect to the matrix and are fundamentally different in some structural aspect. Such graphs are called *cospectral*.

There are many possible matrices to consider, and the matrix we consider in this paper is the normalized Laplacian (see [6; 10]). The rows and columns of this matrix are indexed by the vertices, and for a simple graph the matrix is defined as follows:

$$\mathcal{L}(i,j) = \begin{cases} 1 & \text{if } i = j, \text{ and vertex } i \text{ is not isolated;} \\ \dfrac{-1}{\sqrt{d_i d_j}} & \text{if } i{\sim}j; \\ 0 & \text{otherwise;} \end{cases}$$

where d_i is the degree of vertex i.

In this paper we want to look at the more general setting of edge-weighted graphs, i.e., there is a symmetric, non-negative weight function, $w(i,j)$ on the edges. The degree of a vertex now corresponds to the sum of the weights of the incident edges, i.e., $d_i = \sum_{i\sim j} w(i,j)$. The normalized Laplacian for weighted graphs is defined in the following way:

$$\mathcal{L}(i,j) = \begin{cases} 1 & \text{if } i = j, \text{ and vertex } i \text{ is not isolated;} \\ \dfrac{-w(i,j)}{\sqrt{d_i d_j}} & \text{if } i{\sim}j; \\ 0 & \text{otherwise.} \end{cases}$$

(A simple graph corresponds to the case where $w(i,j) \in \{0,1\}$ for all i,j.) We note that when the graph has no isolated vertices, $\mathcal{L}$ can be written as $\mathcal{L} = D^{-1/2}(D-A)D^{-1/2}$, where $A_{i,j} = w(i,j)$ and D is the diagonal degree matrix. Finally, we point out that this matrix is connected with the probability transition matrix $D^{-1}A$ of a random walk. In particular, two graphs with no isolated vertices are cospectral for $\mathcal{L}$ if and only if they are cospectral for $D^{-1}A$.

There has been some interest in the construction of cospectral graphs for the normalized Laplacian. Cavers [9] showed that a restricted variation of Godsil-McKay switching (see [13]) preserves the spectrum, while Butler and Grout [7] showed that gluing in two different special bipartite graphs into some arbitrary graph resulted in a pair of cospectral graphs. In both cases, the operation preserved the number of edges in the graph.

On the other hand, it is possible for graphs with different number of edges to be cospectral with respect to the normalized Laplacian. The classical example of this is complete bipartite graphs $K_{p,q}$ which have spectrum $\{0, 1^{(p+q-2)}, 2\}$ (here the exponent is indicating multiplicity). For example, the (sparse) star $K_{1,2n-1}$ is cospectral with the (dense) regular graph $K_{n,n}$. Until recently, this was the *only* known construction of cospectral graphs with different number of edges. Butler and Grout [7] gave some examples of small graphs found by exhaustive computation that differ in the number of edges, including some where one graph was a subgraph of the other. Butler [5] expanded on this example to form an infinite family and showed how to construct many pairs of bipartite graphs which were cospectral.

In this paper we introduce a new construction of cospectral graphs for the normalized Laplacian which can differ in the number of edges. The basic idea is to form a ring of linked modules, and then a similar graph where we interchange the role of two of the modules (what we term "toggling"). The resulting pair of graphs are cospectral with respect to the normalized Laplacian. An example of this construction is shown in Figure 2.1. Note that the left graph is a subgraph of the right graph.

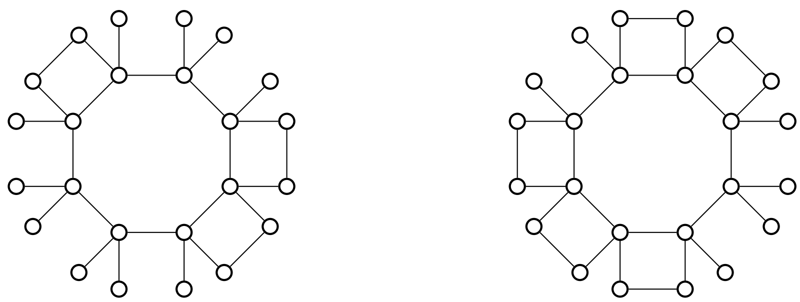


Figure 2.1 A pair of cospectral graphs for $\mathcal{L}$ related by toggling.

In Section 2.2, we give a formal description of this family, of toggling, and state the main result. In Section 2.3, we show how to compute the characteristic polynomial of the normalized Laplacian by using decompositions. We then break the decompositions of a graph in our family into those which contain a "long" cycle (see Section 2.4) and those which do not (see Section 2.5), and in particular conclude the characteristic polynomials are equal so the graphs must be cospectral. In Section 2.6 we show how to go from weighted graphs to simple graphs which are cospectral with respect to the normalized Laplacian.

2.2 Construction

Our family of graphs are formed as a ring composed of three different types of (weighted) modules: the *path* on four vertices, the *cycle* on four vertices, and the *edge* on two vertices, which we label as P, C, and E, respectively. The modules are shown in Figure 2.2 where we have marked the edge weights using a parameter k where $k > 0$ is for now arbitrary. Each module has special vertices marked with "+" and "−" which can be thought of as poles of a magnet to indicate how consecutive modules will connect. In particular, the "+" vertex on one module will connect with the "−" vertex on the next module. We will refer to these two special vertices as the *signed* vertices.

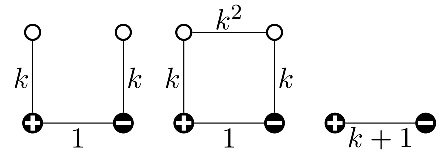


Figure 2.2 The P, C, and E modules, respectively.

A graph in our family is formed by connecting τ modules together in a cycle. In particular, such graph can be associated with a word using the letters P, C and E. As an example, starting with the top module and reading clockwise, the two graphs shown in Figure 2.1 (where $k = 1$) have the words PPCCPPPC and CCPPCCCP. Note that given a graph in our family there are many possible words, i.e., we can choose any module to start and any possible direction. On the other hand, given a word, there is a unique graph.

Definition 1. Given a word $W = \ell_1\ell_2\ldots\ell_\tau$ where $\ell_i \in \{\mathtt{P},\mathtt{C},\mathtt{E}\}$ and $\tau \geq 3$. Then $G(W)$ is the graph obtained by connecting the corresponding τ modules in cyclic order as indicated by the word where consecutive modules connect on the signed vertices, and where the final module will connect to the first module.

We note that the two words we constructed for the graphs in Figure 2.1 are related by interchanging the roles of P and C. This will generalize as follows.

Definition 2. Given a cyclic word W composed of the letters P, C, and E. Then the *toggling* of W is W^T, the word formed by taking W and replacing every P by C and every C by P. The occurrences of E are unchanged.

The motivation for the use of the word "toggling" is to notice that the difference between $G(W)$ and $G(W^T)$ is adding or removing the edge on a module which goes between the non-signed vertices. In essence, we are switching the states of these edges.

We can now state our main result.

Theorem 2.2.1. *For a word W of length at least three using the letters* $\texttt{P}$, $\texttt{C}$, *and* $\texttt{E}$, *$G(W)$ and $G(W^T)$ are cospectral with respect to the normalized Laplacian.*

We note that if W does not contain the same number of occurrences of $\texttt{P}$ and $\texttt{C}$, then the number of edges in $G(W)$ and $G(W^T)$ will differ and are clearly non-isomorphic. Among other things, we can construct cospectral simple graphs which differ by exactly m edges by setting $k = 1$ and using a word in $\texttt{P}$ and $\texttt{C}$ where there are m more instances of $\texttt{C}$ than of $\texttt{P}$. There are also some special words W so that $G(W)$ is a subgraph of $G(W^T)$. One example of this behavior is $W = \texttt{CC}\ldots\texttt{C}$ and $W^T = \texttt{PP}\ldots\texttt{P}$, though others exist (see Figure 2.1).

2.3 Computing the characteristic polynomial

Our approach will involve showing the characteristic polynomials of $G(W)$ and $G(W^T)$ are equal. We start by determining how to compute the characteristic polynomial by the use of generalized cycle decompositions (see [4]). For an $n \times n$ matrix $M = [m_{i,j}]$,

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \underbrace{m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{n,\sigma(n)}}_{:= w_M(\sigma)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) w_M(\sigma).$$

Let G_M denote the digraph which corresponds to M, meaning it has $i \to j$ if and only if $m_{i,j} \neq 0$. We can consider a permutation σ which contributes a nonzero term to $\det(M)$. The factors of $w_M(\sigma)$ correspond to n edges such that each vertex has in-degree and out-degree equal to one, as each vertex will appear as the first and second index somewhere in $w_M(\sigma)$. Such a collection of edges is a *generalized cycle decomposition* of G_M. There are three possible structures in a generalized cycle decomposition: loops (a directed edge that goes into and out of the same vertex), edges (pairs of directed edges $i \to j$ and $j \to i$), and longer directed cycles. More generally, if we think of loops and edges as cycles of length one and two, respectively, then a generalized cycle decomposition is a collection of disjoint cycles so that every vertex is in exactly one cycle.

In the case when the matrix M is symmetric, many of these generalized cycle decompositions will contribute the same factor to the determinant. For example, changing the orientation on a long cycle gives a different decomposition but does not change $\operatorname{sgn}(\sigma)w_M(\sigma)$. With this in mind we consider decompositions.

Definition 3. Let G be an undirected (weighted) graph. Then a *decomposition*, D, is a subgraph consisting of disjoint edges and cycles.

When M is symmetric, we can treat G_M as an undirected graph. Every generalized cycle decomposition now corresponds to a unique decomposition, D, by removing loops and dropping the orientation on the long cycles. Conversely, if we let $s = s(D)$ denote the number of cycles of length at least three in the decomposition D, then each decomposition corresponds to a collection of 2^s different generalized cycle decompositions. Namely, any vertex not in an edge or a cycle has a loop added, edges become cycles of length two, and each of the s cycles of length at least 3 have one of two possible orientations chosen.

If we let $e(D)$ count the number of cycles in the decomposition which have an even number of vertices (including edges), and $F(D)$ be the set of isolated edges in the decomposition D, then we have the following result.

Proposition 2.3.1. *Let G be a weighted graph on n vertices without loops or isolated vertices. Then the characteristic polynomial of the normalized Laplacian matrix is*

$$p(t) = \sum_D (-1)^{e(D)} 2^{s(D)} (t-1)^{n-|V(D)|} \frac{\prod_{\{i,j\}\in E(D)} w(i,j) \prod_{\{i,j\}\in F(D)} w(i,j)}{\prod_{i\in V(D)} d_i}$$

where the sum runs over all decompositions D of the graph G.

Proof. The characteristic polynomial with respect to the normalized Laplacian can be written as

$$\begin{aligned} p(t) &= \det(tI - \mathcal{L}) \\ &= \det\left(tI - D^{-1/2}(D-A)D^{-1/2}\right) \\ &= \det\big(\underbrace{(t-1)I + D^{-1/2}AD^{-1/2}}_{=M}\big). \end{aligned}$$

The graph G_M (ignoring loops) has the same edges and non-edges as G, and so we can use decompositions to compute the determinant.

In particular, every decomposition of G will relate to $2^{s(D)}$ generalized cycle decompositions. For each such generalized cycle decomposition corresponding to a permutation σ, we have $\operatorname{sgn}(\sigma) = (-1)^{e(D)}$. We will have $n - |V(D)|$ loops which each contribute $(t-1)$. The non-loop edges $i \to j$ will contribute $w(i,j)/\sqrt{d_i d_j}$. Now we recall that each vertex in a generalized cycle decomposition has one edge coming in and one edge going out, and therefore for each vertex i in $V(D)$ we will have $\sqrt{d_i}$ occurring twice in the denominator giving us the d_i. Finally, for cycles of length three or greater we only use each edge once in the generalized cycle decomposition, but for cycles of length two we use the same edge for both directions and so we use the edge twice. □

2.4 Decompositions of $G(W)$ with a long cycle

Proposition 2.3.1 shows that we can determine the characteristic polynomial by looking at decompositions of the graph. In this section we will consider the collection of decompositions of a graph $G(W)$ which contain a long cycle, i.e., a cycle which passes through all of the signed vertices in $G(W)$. We denote the set of these decompositions as L.

Lemma 2.4.1. *Let W be a word of length τ with ℓ occurrences of* `P` *and m occurrences of* `C`*. Then for $G(W)$ we have*

$$\sum_{D \in L} (-1)^{e(D)} 2^{s(D)} (t-1)^{n-|V(D)|} \frac{\prod_{\{i,j\} \in E(D)} w(i,j) \prod_{\{i,j\} \in F(D)} w(i,j)}{\prod_{i \in V(D)} d_i} = \frac{(-1)^{\tau-1}(t-1)^{2(m+\ell)}}{2^{\tau-1}(k+1)^{m+\ell}}$$

Proof. Knowing we have a long cycle yields a lot of information about the decomposition D in $G(W)$. In particular, for a module of type `P` or `E`, the decomposition will contain only the edge between the signed vertices. These are shown in Figure 2.3, where edge weights have been removed for clarity.

Figure 2.3 Forced decomposition for `P` and `E`, respectively.

For a module of type `C` the situation is a more interesting as there are three different options for the decomposition. Namely, that the long cycle passes only through the signed vertices; the long cycle passes through the signed vertices and there is an edge between the unsigned vertices; the long cycle passes through all of the vertices. These three possibilities are shown in Figure 2.4.

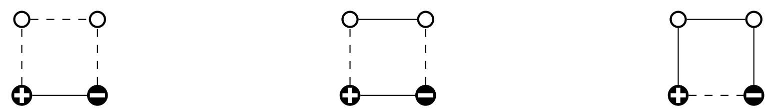


Figure 2.4 Three possible decompositions for `C` in a long cycled decomposition.

Now suppose that among the m modules of type `C` that precisely h of them are the configuration shown on the left in Figure 2.4; i of them are the configuration shown in the center in Figure 2.4; and j of them are the configuration shown on the right in Figure 2.4. The choices of which `C` modules behave in which way is arbitrary. Summing over all the possibilities gives the following.

$$\sum_{D\in L}(-1)^{e(D)}2^{s(D)}(t-1)^{n-|V(D)|}\frac{\prod_{\{i,j\}\in E(D)}w(i,j)\prod_{\{i,j\}\in F(D)}w(i,j)}{\prod_{i\in V(D)}d_i}$$
$$=\sum_{h+i+j=m}\frac{2(-1)^{\tau-1}(k+1)^{\tau-\ell-m}(t-1)^{2\ell}}{\big(2(k+1)\big)^{\tau}}\times$$
$$\binom{m}{h,i,j}\big((t-1)^2\big)^h\left(\frac{(-1)k^4}{\big(k(k+1)\big)^2}\right)^i\left(\frac{k^4}{\big(k(k+1)\big)^2}\right)^j$$

We have $2^{s(D)}=2$ because there is only one cycle of length greater than three, namely the long cycle which contains all the signed vertices. The $e(D)$ will count the number of `C` modules in the middle configuration and possibly the long cycle itself. Regardless of the number of `C` modules in the configuration on the right, the contribution from the long cycle to $(-1)^{e(D)}$ will be $(-1)^{\tau-1}$. Consider first the contributions of isolated vertices and edge weights of the `P` and `E` modules. The $(k+1)^{\tau-\ell-m}$ is the weight of the edges on the long cycle coming from the modules of type `E`. The $(t-1)^{2\ell}$ accounts for the isolated vertices from the modules of type `P`. Further, the $\big(2(k+1)\big)^{\tau}$ is the product of the degrees of the signed vertices (each such vertex has degree $2(k+1)$ as can be seen by noting that in the modules the signed vertices have degree $k+1$ and then we identify two such vertices).

It remains to account for the portions of the decompositions formed on the $\texttt{C}$ modules which are not the signed vertices. The $\binom{m}{h,i,j} = m!/(h!i!j!)$ is the multinomial coefficient for how many ways to choose the different module configurations for $\texttt{C}$, and the final three factors are the contributions from each configuration formed by accounting for isolated vertices, edge weights, and the degrees of vertices in the decomposition (i.e., a pair of isolated vertices or the product of the edge weights over product of degrees). Notice that for the middle configuration, the contribution to $(-1)^{e(D)}$ has been appropriately grouped.

Now we can simplify by pulling out the terms which do not depend on the sum and cancelling. For the terms in the sum we can use the multinomial theorem to simplify. Continuing the above computation, we now have

$$\begin{aligned}
&= \frac{(-1)^{\tau-1}(t-1)^{2\ell}}{2^{\tau-1}(k+1)^{\ell+m}}\left((t-1)^2 - \frac{k^4}{\big(k(k+1)^2\big)^2} + \frac{k^4}{\big(k(k+1)^2\big)^2}\right)^m \\
&= \frac{(-1)^{\tau-1}(t-1)^{2\ell}}{2^{\tau-1}(k+1)^{\ell+m}}(t-1)^{2m} = \frac{(-1)^{\tau-1}(t-1)^{2(m+\ell)}}{2^{\tau-1}(k+1)^{m+\ell}}. \qquad \square
\end{aligned}$$

The important thing to note is that the expression in Lemma 2.4.1 will be the same for W and W^T because $m+\ell$ is invariant under toggling.

2.5 Decompositions of $G(W)$ without a long cycle

Any cycle in a decomposition with edges in consecutive modules would have to go through all of the modules to close up. In particular, if there is not a long cycle in our decomposition D, then the decomposition is composed of only edges and C_4's which lie in individual modules.

We consider what decompositions can happen in a single module and how decompositions in consecutive modules interact. The first task is straightforward to carry out, and in Tables 2.1, 2.2, and 2.3 we show the possible *local* decompositions for each module. To help facilitate the analysis we have grouped the local decompositions by which signed vertices (if any) are used.

The next part is to understand the transitions between modules, i.e., how local decompositions interact. We have already grouped the local decompositions by which of the signed vertices are used. We now note that if signed vertices are used in by a local decomposition in one module, it influences which of the signed vertices are available for use in the next module.

Table 2.1 Decompositions of the P module.

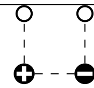
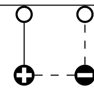
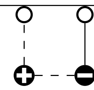
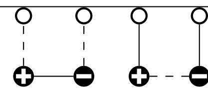
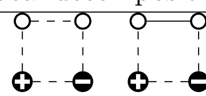
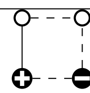
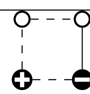
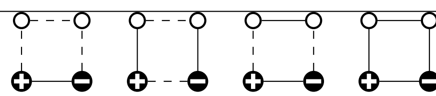
signed vertex	local decomposition	contribution
neither		1
$+$		$\frac{-k}{(t-1)^2(2k+2)}$
$-$		$\frac{-k}{(t-1)^2(2k+2)}$
$+/-$		$\frac{k^2-(t-1)^2}{(t-1)^4(2k+2)^2}$

Table 2.2 Decompositions of the C module.

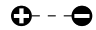
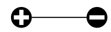
signed vertex	local decomposition	contribution
neither		$1-\frac{k^2}{(t-1)^2(k+1)^2}$
$+$		$\frac{-k}{2(t-1)^2(k+1)^2}$
$-$		$\frac{-k}{2(t-1)^2(k+1)^2}$
$+/-$		$\frac{-1}{4(t-1)^2(k+1)^2}$

This is indicated by the following transition matrix with rows and columns indexed by subsets of the signed vertices:

$$Q = \begin{array}{c} \\ \emptyset \\ + \\ - \\ +/- \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & + & - & +/- \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Using Q we can now count the number of ways that we can have decompositions use the signed vertices in the modules for $G(W)$. This is done using the transfer matrix method (see [11]), and in particular is equal to the number of closed walks in the directed graph corresponding to

Table 2.3 Decompositions of the E module.

used	local decomposition	factor
neither		1
$+$		0
$-$		0
$+/-$		$\dfrac{-1}{4(t-1)^2}$

Q which have the same length as the length of the word. We need to go one step further and for every module add the contribution of the local decomposition.

This final part is done by adding in diagonal weight matrices where the diagonal entries correspond to the contribution of the decomposition for that particular module. These contributions are found by (-1) raised to the number of even cycles (i.e., edges or C_4's) times the product of the edge weights used in the local decomposition (remembering for an edge to use that edge twice), divided by the product of the degrees of any vertex used in the decomposition. The only subtle part is handling the vertices which will not be a part of a decomposition in any module. What we do is assume at the beginning that *every* vertex is isolated and contributes a $(t-1)$ then whenever a vertex becomes a part of the decomposition we divide by $(t-1)$ to correct (the choice of this approach is because signed vertices lie in two modules, hence while it might not be in the decomposition of one module it could be in the decomposition of the other). When there are several possible decompositions in a given case we add them together to form the entry for the weight matrix. The contributions were previously listed in the tables and become the diagonal entries of the weight matrices. We therefore have the following weight matrices.

$$X_{\mathtt{P}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-k}{(t-1)^2(2k+2)} & 0 & 0 \\ 0 & 0 & \frac{-k}{(t-1)^2(2k+2)} & 0 \\ 0 & 0 & 0 & \frac{k^2-(t-1)^2}{(t-1)^4(2k+2)^2} \end{pmatrix}$$

$$X_{\mathtt{C}} = \begin{pmatrix} 1-\frac{k^2}{(t-1)^2(k+1)^2} & 0 & 0 & 0 \\ 0 & \frac{-k}{2(t-1)^2(k+1)^2} & 0 & 0 \\ 0 & 0 & \frac{-k}{2(t-1)^2(k+1)^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{4(t-1)^2(k+1)^2} \end{pmatrix}$$

$$X_{\mathtt{E}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{4(t-1)^2} \end{pmatrix}$$

So for the graph $G(\ell_1\ell_2\cdots\ell_\tau)$, we have the following:

$$\sum_{D\notin L}(-1)^{e(D)}2^{s(D)}(t-1)^{n-|V(D)|}\frac{\prod_{\{i,j\}\in E(D)}w(i,j)\prod_{\{i,j\}\in F(D)}w(i,j)}{\prod_{i\in V(D)}d_i}$$
$$= (t-1)^{|V(G(W))|}\operatorname{trace}(QX_{\ell_1}QX_{\ell_2}\cdots QX_{\ell_\tau}). \quad (2.1)$$

We now focus on rewriting the trace expression in (2.1). To start we note that $Q = RSR^{-1}$ where

$$R = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & -1 \\ \frac{1}{2} & 1 & -2 & 0 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Combining this with $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ we can conclude

$$\begin{aligned}(t-1)^{|V(G(W))|}&\operatorname{trace}(QX_{\ell_1}QX_{\ell_2}\cdots QX_{\ell_\tau}) \\ &= (t-1)^{|V(G(W))|}\operatorname{trace}(RSR^{-1}X_{\ell_1}RSR^{-1}X_{\ell_2}\cdots RSR^{-1}X_{\ell_\tau}) \\ &= (t-1)^{|V(G(W))|}\operatorname{trace}\big((SR^{-1}X_{\ell_1}R)(SR^{-1}X_{\ell_2}R)\cdots(SR^{-1}X_{\ell_\tau}R)\big).\end{aligned}$$

Because S has two rows of 0's this simplifies the matrices that we have to deal with. In particular we have

$$SR^{-1}X_{\mathtt{P}}R = \begin{pmatrix} Y_{\mathtt{P}} & Z_{\mathtt{P}} \\ O & O \end{pmatrix},$$

$$SR^{-1}X_{\mathtt{C}}R = \begin{pmatrix} Y_{\mathtt{C}} & Z_{\mathtt{C}} \\ O & O \end{pmatrix}, \text{ and}$$

$$SR^{-1}X_{\mathtt{E}}R = \begin{pmatrix} Y_{\mathtt{E}} & Z_{\mathtt{E}} \\ O & O \end{pmatrix},$$

where if we let $u := t - 1$ then

$$Y_{\mathtt{P}} = \begin{pmatrix} \frac{16k^2u^4+32ku^4-8k^2u^2+16u^4-8ku^2+k^2-u^2}{12(k+1)^2u^4} & \frac{-8k^2u^4-16ku^4-2k^2u^2-8u^4-2ku^2+k^2-u^2}{6(k+1)^2u^4} \\ \frac{8k^2u^4+16ku^4+2k^2u^2+8u^4+2ku^2-k^2+u^2}{24(k+1)^2u^4} & \frac{16k^2u^4+32ku^4-8k^2u^2+16u^4-8ku^2+k^2-u^2}{12(k+1)^2u^4} \end{pmatrix}$$

$$Y_{\mathtt{C}} = \begin{pmatrix} \frac{16k^2u^2+32ku^2-16k^2+16u^2-8*k-1}{12(k+1)^2u^2} & \frac{-8k^2u^2-16ku^2+8k^2-8u^2-2k-1}{6(k+1)^2u^2} \\ \frac{8k^2u^2+16ku^2-8k^2+8u^2+2k+1}{24(k+1)^2u^2} & \frac{-4k^2u^2-8ku^2+4k^2-4u^2-4k+1}{12(k+1)^2u^2} \end{pmatrix}$$

$$Y_{\mathtt{E}} = \begin{pmatrix} \frac{16u^2-1}{12u^2} & \frac{-8u^2-1}{6u^2} \\ \frac{8u^2+1}{24u^2} & \frac{-4u^2+1}{12u^2} \end{pmatrix}$$

Because we can carry out block matrix multiplication, we note that the resulting upper left block will be the product of the upper left blocks and that the resulting lower right block will be the all zeroes matrix. This allows us to conclude the following:

$$(t-1)^{|V(G(W))|}\,\mathrm{trace}(QX_{\ell_1}QX_{\ell_2}\cdots QX_{\ell_\tau}) = (t-1)^{|V(G(W))|}\,\mathrm{trace}(Y_{\ell_1}Y_{\ell_2}\cdots Y_{\ell_\tau})$$

There is no convenient way to find a simple expression for these decompositions as we did for the long cycles. However, it suffices to show that the toggled words will produce equivalent results, which is what we now show.

Lemma 2.5.1. *Let $W = \ell_1\ell_2\ldots\ell_\tau$ and $W^T = \gamma_1\gamma_2\ldots\gamma_\tau$. Then*

$$(t-1)^{|V(G(W))|}\,\mathrm{trace}(Y_{\ell_1}Y_{\ell_2}\cdots Y_{\ell_\tau}) = (t-1)^{|V(G(W^T))|}\,\mathrm{trace}(Y_{\gamma_1}Y_{\gamma_2}\cdots Y_{\gamma_\tau}).$$

Proof. Both sides are polynomials, and so it suffices to verify that the relationship holds for $t \neq 0, 1, 2$ (i.e., if two polynomials agree at all but three points, they must agree everywhere). To show that they are equal, we will make use of the following special matrix,

$$U = \begin{pmatrix} 20u^2-2 & -32u^2-4 \\ 8u^2+1 & -20u^2+2 \end{pmatrix}.$$

This matrix has the following special properties, which can be verified by carrying out matrix multiplication:

- $UY_{\mathtt{P}} = Y_{\mathtt{C}}U$.
- $UY_{\mathtt{C}} = Y_{\mathtt{P}}U$.
- $UY_{\mathtt{E}} = Y_{\mathtt{E}}U$.

These properties are key, in that they indicate we can pass U through one of the Y_* matrices but we need to change the matrix in the same way that we do in the toggling operation.

For $t \neq 0, 1, 2$ we have that U is invertible and so by repeated application of the above properties we have

$$\begin{aligned}
(t-1)^{|V(G(W))|}\operatorname{trace}(Y_{\ell_1}Y_{\ell_2}\cdots Y_{\ell_\tau}) &= (t-1)^{|V(G(W))|}\operatorname{trace}(UY_{\ell_1}Y_{\ell_2}\cdots Y_{\ell_\tau}U^{-1}) \\
&= (t-1)^{|V(G(W))|}\operatorname{trace}(Y_{\gamma_1}UY_{\ell_2}\cdots Y_{\ell_\tau}U^{-1}) \\
&= \cdots \\
&= (t-1)^{|V(G(W))|}\operatorname{trace}(Y_{\gamma_1}Y_{\gamma_2}\cdots UY_{\ell_\tau}U^{-1}) \\
&= (t-1)^{|V(G(W))|}\operatorname{trace}(Y_{\gamma_1}Y_{\gamma_2}\cdots Y_{\gamma_\tau}UU^{-1}) \\
&= (t-1)^{|V(G(W))|}\operatorname{trace}(Y_{\gamma_1}Y_{\gamma_2}\cdots Y_{\gamma_\tau}) \\
&= (t-1)^{|V(G(W^T))|}\operatorname{trace}(Y_{\gamma_1}Y_{\gamma_2}\cdots Y_{\gamma_\tau}),
\end{aligned}$$

where in the last we use that toggling does not change the number of vertices in the graph. □

Proof of Theorem 2.2.1. To show that the graphs $G(W)$ and $G(W^T)$ are cospectral we can show that they have the same characteristic polynomial. We use Proposition 2.3.1 and consider all the possible decompositions. Lemma 2.4.1 shows that the sum over all the decompositions which contain a long cycle are equal while Lemma 2.5.1 shows that the sum over all the decompositions which do not contain a long cycle are also equal. Thus the sum over all decompositions is equal, and the theorem is established. □

2.6 Weighted Graphs to Simple Graphs

We have considered graphs with edge weights in terms of a parameter k as shown in Figure 2.2. By letting $k = 1$ and restricting to P and C modules we will produce cospectral simple graphs.

Simple graphs can also be obtained by appropriately "blowing up" our graph. This works by replacing vertices by independent sets. An edge between u and v which has been replaced by r and s vertices respectively then becomes a complete bipartite graphs between the two independent sets with all edge weights $w(u, v)/rs$. (Note that r and s are generally chosen so that this new edge weight is 1, i.e., so the new graph is a simple graph.) Similarly several consecutive E edges with weight $k + 1$ can become $k + 1$ parallel paths. A discussion on how eigenvalues for the normalized Laplacian work for blowups can be found in [5]. In particular, it is known that the eigenvalues of the blowups are determined from the eigenvalues of the original graphs (which we have shown to be cospectral) and the remaining eigenvalues will come from the blowup procedure, which will be the same for both graphs.

As an example in Figure 2.5 we start with the cospectral graphs EEEPCC and EEECPP. This figure also contains the blowups which result by replacing the unsigned vertices in C and P modules with k independent vertices (marked by putting k inside the vertex and making the lines bold to represent complete bipartite graphs), the three consecutive E edges become $k + 1$ parallel paths of length three. In particular, the resulting blowups are simple graphs which are also cospectral.

There are other possibilities. For instance, from the definition of the normalized Laplacian we note that the matrix does not change if we scale all edge weights by a fixed amount. So we can first scale the edge weights and then perform a blowup. A partial example of this is shown in Figure 2.6 where we consider the graph corresponding to ECC. By setting $k = 1$ and then scaling all edge weights by 2 we get a weighted graph which has as a blowup the graph shown on the right in Figure 2.6. By a similar process we could also do the same for EPP to construct a cospectral pair of simple graphs.

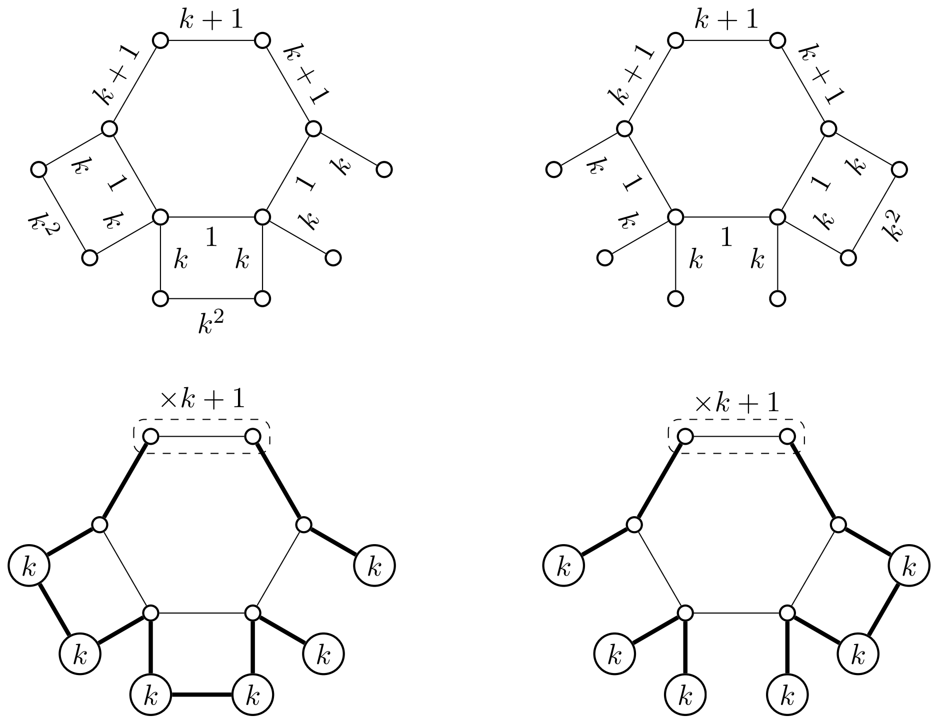


Figure 2.5 The graphs EEEPCC and EEECPP (above) and their respective blowups (below).

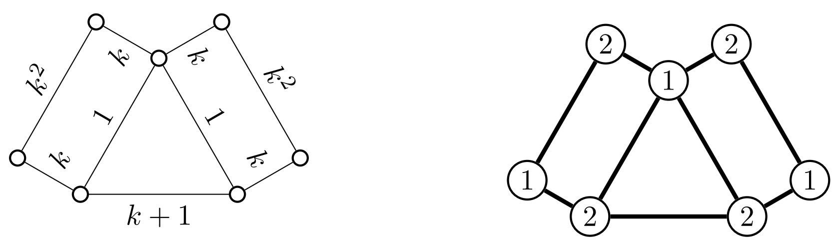


Figure 2.6 The graph ECC and corresponding blowup, where $k = 1$ and all edge weights have been scaled by 2.

2.7 Conclusion

Many, if not most, approaches to establish cospectrality rely on showing that a small perturbation in the graph corresponds to a small, controllable perturbation in the eigenvectors and hence eigenvalues are preserved. This was *not* the case in this construction, which is why we considered the characteristic polynomials. Also, while we show that the characteristic polynomials are equal, we never explicitly computed one. Instead, we showed that the method to determine these polynomials will produce the same answer for a pair of cospectral graphs. It would be interesting to find additional families where this can occur.

We have been able to establish a large family of cospectral graphs for the normalized Laplacian (and hence also probability transition matrix) which have unusual properties, including cospectral graphs with different number of edges and graphs cospectral with subgraphs. There is a vast amount about the spectrum of the normalized Laplacian that is not well understood. We hope to see more of this area explored in future work.

2.8 Post Script

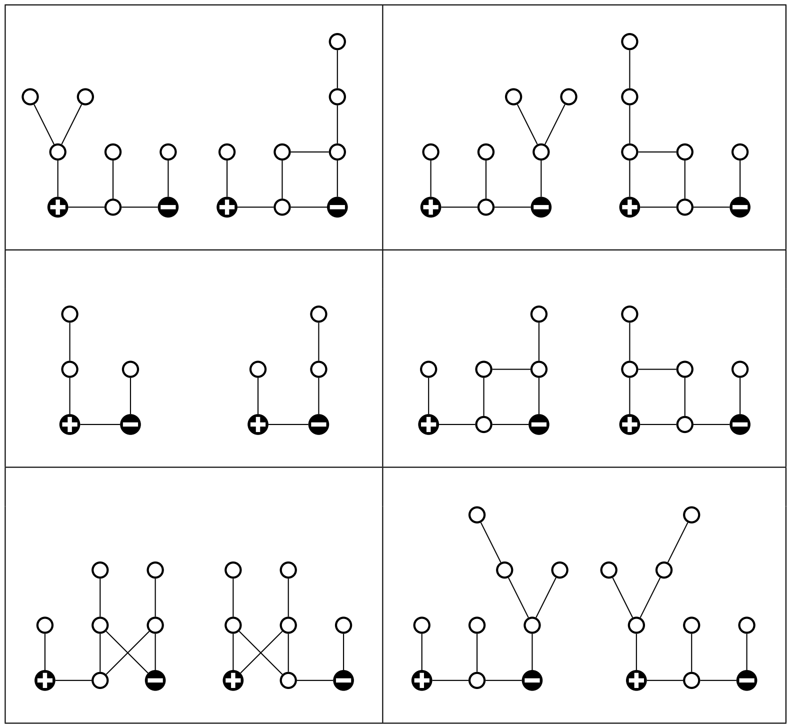
After the submission of this paper, we considered the existence of further modules. The modules given in the paper arose from known examples, but there was no indication from the proof that these were the only graphs which respected this toggling. Indeed, the proof technique could be effectively "turned around" to give further graphs. We looked for graphs G whose weight matrices X_G would follow the proof in Section 2.5.

This was done in SAGE as follows. We immediately knew modules which could be toggled would have to be on the same number of vertices, therefore we fix some number of vertices n. To ease the computation, we consider graphs where all edges are weight one.

For every graph on n vertices, we distinguish two vertices, one positive node and one negative node, to be our connecting vertices. The program then built a weight matrix X_G based on the local decompositions of G. This was accomplished by considering all permutations on n letters (done by the program recursively to cut down on runtime) and determining if the permutation formed a valid decomposition of the graph. If this was the case, the weight of the decomposition was calculated from $\mathcal{L}$ and added to the appropriate diagonal entry of X_G.

At this point, we created the accompanying Y_G matrix and stored the Graph6 string, the pair of signed vertices, and the results of multiplying U on the right and on the left of Y_G. In this master list, it was straightforward to look for the toggling behavior. For every pair of graphs G and H, if $UY_G = Y_HU$ and $UY_H = Y_GU$, then we had found a pair of graphs which were toggling candidates. At this point, these pairs were only candidates because the contribution to the characteristic polynomial of long cycle decompositions needed to be checked. Table 2.4 shows the resulting pairs of graphs.

Table 2.4 Extra toggling modules.



From the table, we see that the only pair of modules which required checking their long cycle decompositions was the first row. This is because the remaining pairs are symmetric, therefore long cycle decompositions for one module are in natural bijection for long cycle decompositions for the other module. Let A be the module without a cycle and let B be the module with a cycle. We show the possible long cycle decompositions in Figures 2.7 and 2.8, respectively, in the same fashion as we did in Figures 2.3 and 2.4. We will do this for the first pair in row one of Table 2.4, noting the other follows by symmetry.

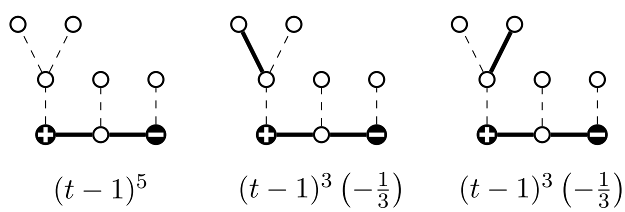


Figure 2.7 Possible long cycle decompositions for module A.

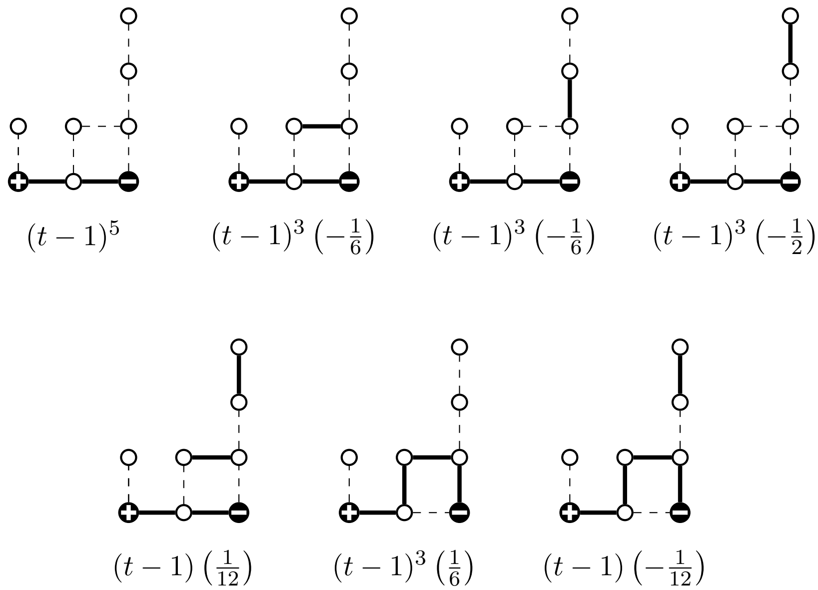


Figure 2.8 Possible long cycle decompositions for module B.

Further, for each forced decomposition, we have indicated the factor which each decomposition contributes to $w(D)$ for the unsigned vertices. This is similar to the consideration given to the forced decompositions of the C modules in the proof of Lemma 2.4.1. A similar computation using multinomial coefficients proves that the long cycle decompositions for a graph and its toggling contributes the same term to the characteristic polynomial. The result is that all of the pairs in Table 2.4 are valid toggling modules.

CHAPTER 3. A CONSTRUCTION FOR DISTANCE COSPECTRAL GRAPHS

A paper submitted to *Linear Algebra and its Applications*.

Kristin Heysse

Abstract

The distance matrix of a connected graph is the symmetric matrix with columns and rows indexed by the vertices and entries that are the pairwise distances between the corresponding vertices. We give a construction for graphs which differ in their edge counts yet are cospectral with respect to the distance matrix. Further, we identify a subgraph switching behavior which constructs additional distance cospectral graphs. The proofs for both constructions rely on a perturbation of (most of) the distance eigenvectors of one graph to yield the distance eigenvectors of the other.

3.1 Introduction

Spectral graph theory explores the relationship between a graph and the eigenvalues (i.e., spectrum) of a matrix associated with that graph. There are a handful of common ways to associate a matrix to a graph, and the spectrum of each matrix holds a variety of information about the graph (see [3]). However, each matrix also has limitations in what information its spectrum can contain. This is seen in the existence of *cospectral graphs*, or graphs that are fundamentally different yet yield the same spectrum for a particular matrix.

By exploring cospectral graphs, we further our understanding of the limitations of each type of matrix. One of the most well-known constructions of cospectral graphs for the adjacency matrix is Godsil-McKay switching. This is done by defining specific subsets of the vertices of a particular graph and constructing a cospectral mate by exchanging edges and non-edges between these subsets. Godsil and McKay [13] prove the adjacency matrices of two graphs related by this edge switching are similar, and therefore the graphs are cospectral.

In this paper, we consider cospectral graphs for the *distance matrix*. The distance matrix $D^{(G)} = \left[d_{ij}^{(G)}\right]$ of a connected graph $G = (V(G), E(G))$ is a symmetric matrix such that $d_{ij}^{(G)}$ is the distance, or length of the shortest path, between vertices i and j. Its multiset of eigenvalues is the *distance spectrum* of G and two graphs are considered to be distance cospectral if their distance spectra are the same. There has been extensive work done on the distance spectra of graphs (see [2] for a survey of recent results).

However, relatively little is known in regard to distance cospectral pairs. McKay [14] gives a construction for distance cospectral trees by considering any rooted tree and identifying the root with the root of one of two particular trees. Further, he proves the complement graphs of trees constructed in this fashion are also distance cospectral. Both proofs rely on manipulation of the distance characteristic polynomial. This is the only known distance cospectral graph construction in the literature, and we note that pairs constructed in this manner must contain the same number of edges. In particular, prior to this paper it was not known whether a family could be constructed where distance cospectral pairs could have differing numbers of edges.

In this paper, we give a construction for distance cospectral graphs with differing numbers of edges in Section 3.2, and in Section 3.3 we describe a local edge switching behavior which produces more distance cospectral graphs. While these distance switching pairs do not differ in number of edges, they do account for all distance cospectral pairs on seven vertices (see Figure 3.3). Finally, in Section 3.4, we consider further questions of interest for the distance matrix. We complete the introduction with an elementary discussion of graph identification, a process which will be used in subsequent sections.

3.1.1 Graph Identification

Throughout our constructions, we will frequently make use of *graph identification*, therefore we define it here and state some observations about distances between vertices in graphs formed in this way. Let G, K be graphs and let $u \in V(G)$, $v \in V(K)$. We construct the graph $GK(u, v)$ by identifying the vertices u and v into a new vertex uv in the graph $G \cup K$. When clear context allows, we will denote this graph GK.

Consider calculating the distance between two vertices x, y of GK. We can easily do this by considering if x and y are in the G portion of GK or the K portion of GK.

- If x, y are both in the G portion, $d_{xy}^{(GK)} = d_{xy}^{(G)}$.
- If x, y are both in the K portion, $d_{xy}^{(GK)} = d_{xy}^{(K)}$.
- If x is in the G portion and y is in the K portion, $d_{xy}^{(GK)} = d_{xu}^{(G)} + d_{vy}^{(K)}$.

These claims can be verified by noticing that a shortest path between vertices in the same portion will be fully contained in that portion. Further, if two vertices are not in the same portion, any path between them must include the vertex uv.

3.2 Distance cospectral graphs with differing numbers of edges

Consider the two graphs G and H shown in Figure 3.1, each with vertices labeled zero through nine.

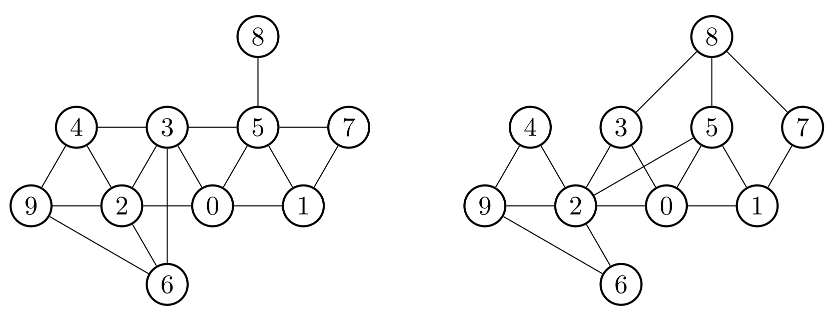


Figure 3.1 Graphs G (left) and H (right).

We immediately note G has 17 edges and H has 16 edges. For future reference, we give the distance matrices of both graphs below.

$$D^{(G)} = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 3 & 1 & 3 & 1 & 2 & 3 \\ 1 & 2 & 0 & 1 & 1 & 2 & 1 & 3 & 3 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 3 & 1 & 1 & 0 & 2 & 2 & 3 & 3 & 1 \\ 1 & 1 & 2 & 1 & 2 & 0 & 2 & 1 & 1 & 3 \\ 2 & 3 & 1 & 1 & 2 & 2 & 0 & 3 & 3 & 1 \\ 2 & 1 & 3 & 2 & 3 & 1 & 3 & 0 & 2 & 4 \\ 2 & 2 & 3 & 2 & 3 & 1 & 3 & 2 & 0 & 4 \\ 2 & 3 & 1 & 2 & 1 & 3 & 1 & 4 & 4 & 0 \end{pmatrix} \quad D^{(H)} = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 3 & 1 & 3 & 1 & 2 & 3 \\ 1 & 2 & 0 & 1 & 1 & 1 & 1 & 3 & 2 & 1 \\ 1 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 2 \\ 2 & 3 & 1 & 2 & 0 & 2 & 2 & 4 & 3 & 1 \\ 1 & 1 & 1 & 2 & 2 & 0 & 2 & 2 & 1 & 2 \\ 2 & 3 & 1 & 2 & 2 & 2 & 0 & 4 & 3 & 1 \\ 2 & 1 & 3 & 2 & 4 & 2 & 4 & 0 & 1 & 4 \\ 2 & 2 & 2 & 1 & 3 & 1 & 3 & 1 & 0 & 3 \\ 2 & 3 & 1 & 2 & 1 & 2 & 1 & 4 & 3 & 0 \end{pmatrix}$$

Theorem 3.2.1. *For any graph K and any vertex $v \in V(K)$, and for $u \in \{0,1\}$, the graphs $GK(u,v)$ and $HK(u,v)$ are distance cospectral.*

Proof. When identifying the graph K onto G, we will enforce that the vertices will be labeled as follows. The vertex v will have the same label as u, and the remaining vertices will be labeled with the set $\{10, 11, 12, \ldots, n\}$. We similarly label HK. Let $D^{(GK)}$ be the distance matrix of GK and similarly $D^{(HK)}$ for HK. The proof given will handle the case where $u = 0$. The case where $u = 1$ is done similarly.

Let (λ, x) be an eigenpair for $D^{(GK)}$ where $\lambda \neq -\frac{1}{2}$. We claim the vector $y := x + \Delta$ is an eigenvector of $D^{(HK)}$ for eigenvalue λ, where

$$\Delta_i = \begin{cases} 0 & i \in \{0, 1, 10, 11, 12, \ldots, n\} \\ \alpha & i \in \{2, 9\} \\ -\alpha & i \in \{4, 6\} \\ \beta & i \in \{3, 7\} \\ -\alpha - \beta & i = 5 \\ \alpha - \beta & i = 8 \end{cases}$$

and α and β are defined to be

$$\alpha = \frac{-x_3 - x_5 - x_7 - x_8}{2\lambda + 1} \text{ and } \beta = \frac{\lambda+1}{2\lambda+1}\left(x_5 + x_8\right) - \frac{\lambda}{2\lambda+1}\left(x_3 + x_7\right).$$

To prove this, we will consider $(D^{(HK)}y)_i$ for all i. By inspection of the two matrices, $d_{ij}^{(H)} = d_{ij}^{(G)}$ for all $j \in \{0, 1, 2, \ldots 9\}$ for $i \in \{0, 1\}$. A straightforward algebraic substitution and simplification proves $(D^{(HK)}y)_i = \lambda y_i$ for $i \in \{0, 1\}$. Consider $i \in \{10, 11, \ldots, n\}$. We will fully elaborate the steps taken in the following work, as similar processes will be repeated frequently.

$$(D^{(HK)}y)_i = \sum_{j=0}^{n} d_{ij}^{(HK)} y_j = \sum_{j=0}^{9} d_{ij}^{(HK)}(x_j + \Delta_j) + \sum_{j=10}^{n} d_{ij}^{(HK)}(x_j + \Delta_j)$$

We immediately break the summation into the first ten vertices and the rest, as we will need to treat each group separately. We also substitute the definition of y. Next, we use the observations from Section 3.1.1 to break the distances in HK to distances in H and K, recalling that v is the vertex in the graph K we identify with 0 in H to create HK.

$$= \sum_{j=0}^{9}(d_{iv}^{(K)} + d_{0j}^{(H)})(x_j + \Delta_j) + \sum_{j=10}^{n} d_{ij}^{(K)}(x_j + \Delta_j)$$
$$= \sum_{j=0}^{9}(d_{iv}^{(K)} + d_{0j}^{(G)})(x_j + \Delta_j) + \sum_{j=10}^{n} d_{ij}^{(K)}(x_j + \Delta_j)$$

We can substitute $d_{0j}^{(G)}$ for $d_{0j}^{(H)}$ by inspection of the first rows of the matrices $D^{(G)}$ and $D^{(H)}$. We continue by regrouping terms and recombining sums of distances in G and K to be distances in GK, again by the observations from Section 3.1.1.

$$\begin{aligned}
&= \sum_{j=0}^{9}(d_{iv}^{(K)} + d_{0j}^{(G)})x_j + \sum_{j=10}^{n} d_{ij}^{(K)} x_j + \sum_{j=0}^{9}(d_{iv}^{(K)} + d_{0j}^{(G)})\Delta_j \\
&= \sum_{j=0}^{n} d_{ij}^{(GK)} x_j + \sum_{j=0}^{9}(d_{iv}^{(K)} + d_{0j}^{(G)})\Delta_j \\
&= \lambda x_i + \alpha(3d_{iv}^{(K)} - 3d_{iv}^{(K)} + d_{02}^{(G)} - d_{04}^{(G)} - d_{05}^{(G)} - d_{06}^{(G)} + d_{08}^{(G)} + d_{09}^{(G)}) \\
&\quad + \beta(2d_{iv}^{(K)} - 2d_{iv}^{(K)} + d_{03}^{(G)} - d_{05}^{(G)} + d_{07}^{(G)} - d_{08}^{(G)}) \\
&= \lambda x_i = \lambda y_i.
\end{aligned}$$

The last few steps result from the fact that (λ, x) is an eigenpair for $D^{(GK)}$ and by direct computation and substitution. This proves $(D^{(HK)}y)_i = \lambda y_i$ for $i \in \{10, 11, \ldots, n\}$.

We now consider the vertices $\{2, 3, \ldots, 8\}$ by showing the case where $i = 2$ and considering how the work generalizes. For this case, notice that $d_{2j}^{(H)} = d_{2j}^{(G)}$ for $j \notin \{5, 8\}$ and $d_{2j}^{(H)} = d_{2j}^{(G)} - 1$ for $j \in \{5, 8\}$.

$$\begin{aligned}
(D^{(HK)}y)_2 &= \sum_{j=0}^{n} d_{2j}^{(HK)} y_j \\
&= \sum_{j=0}^{9} d_{2j}^{(HK)} (x_j + \Delta_j) + \sum_{j=10}^{n} d_{2j}^{(HK)} (x_j + \Delta_j) \\
&= \sum_{j=0}^{9} d_{2j}^{(H)} (x_j + \Delta_j) + \sum_{j=10}^{n} (d_{20}^{(H)} + d_{vj}^{(K)})(x_j + \Delta_j) \\
&= \sum_{\substack{j=0 \\ j \neq 5,8}}^{9} d_{2j}^{(G)} (x_j + \Delta_j) + (d_{25}^{(G)} - 1)(x_5 + \Delta_5) \\
&\quad + (d_{28}^{(G)} - 1)(x_8 + \Delta_8) + \sum_{j=10}^{n} (d_{20}^{(G)} + d_{vj}^{(K)})(x_j + \Delta_j) \\
&= \sum_{j=0}^{9} d_{2j}^{(G)} x_j + \sum_{j=10}^{n} (d_{20}^{(G)} + d_{vj}^{(K)}) x_j + \sum_{j=0}^{9} d_{2j}^{(G)} \Delta_j - x_5 - x_8 - \Delta_5 - \Delta_8 \\
&= \sum_{j=0}^{n} d_{2j}^{(GK)} x_j + \sum_{j=0}^{9} d_{2j}^{(G)} \Delta_j - x_5 - x_8 - \Delta_5 - \Delta_8 \\
&= \lambda x_2 + \alpha(d_{22}^{(G)} - d_{24}^{(G)} - d_{25}^{(G)} - d_{26}^{(G)} + d_{28}^{(G)} + d_{29}^{(G)}) \\
&\quad + \beta(d_{23}^{(G)} - d_{25}^{(G)} + d_{27}^{(G)} - d_{28}^{(G)}) - x_5 - x_8 - \Delta_5 - \Delta_8 \\
&= \lambda x_2 - \beta - x_5 - x_8 - (-\alpha - \beta) - (\alpha - \beta) \\
&= \lambda(x_2 + \alpha) - \lambda\alpha + \beta - x_5 - x_8 \\
&= \lambda y_2 - \lambda\alpha + \beta - x_5 - x_8
\end{aligned}$$

Let c_2 be the "remainder" terms, specifically $c_2 := -\lambda\alpha + \beta - x_5 - x_8$. To finish the claim that $(D^{(HK)}y)_2 = \lambda y_2$, it would suffice to show $c_2 = 0$:

$$\begin{aligned}
-\lambda\alpha + \beta - x_5 - x_8 &= -\lambda \left(\frac{-x_3 - x_5 - x_7 - x_8}{2\lambda + 1} \right) - x_5 - x_8 \\
&\quad + \left(\frac{\lambda + 1}{2\lambda + 1} (x_5 + x_8) - \frac{\lambda}{2\lambda + 1} (x_3 + x_7) \right) \\
&= (x_3 + x_7) \left(\frac{\lambda}{2\lambda + 1} - \frac{\lambda}{2\lambda + 1} \right) + (x_5 + x_8) \left(\frac{\lambda}{2\lambda + 1} - 1 + \frac{\lambda + 1}{2\lambda + 1} \right) = 0.
\end{aligned}$$

Therefore, by the definition of α and β, the claim holds for $i = 2$. Repeating this process, we calculate the remainder terms c_i for $i \in \{3, 4, \ldots, 8\}$ (meaning $(D^{(HK)}y)_i = \lambda y_i + c_i$ for all i) in a similar fashion. These are listed below.

$$c_2 = -\lambda\alpha + \beta - x_5 - x_8$$

$$c_3 = -\lambda\beta - 2\alpha - \beta + x_4 + x_5 + x_6 - x_8$$

$$c_4 = \lambda\alpha + \alpha + \beta + x_3 + x_7$$

$$c_5 = \lambda\alpha + \lambda\beta + 3\beta - x_2 + x_3 + x_7 - x_9$$

$$c_6 = \lambda\alpha + \alpha + \beta + x_3 + x_7$$

$$c_7 = -\lambda\beta - 2\alpha - \beta + x_4 + x_5 + x_6 - x_8$$

$$c_8 = -\lambda\alpha + \lambda\beta - 2\alpha + \beta - x_2 - x_3 - x_7 - x_9$$

$$c_9 = -\lambda\alpha + \beta - x_5 - x_8$$

Similarly to the case where $i = 2$, our goal is to show that all remaining c_i are equal to zero. Substitution of α and β suffices for c_4. To prove c_3 and c_5, we consider combinations of particular rows of $D^{(GK)}$. We claim the following three equations hold:

$$2x_3 + x_4 + 2x_5 + x_6 + 2x_7 = \lambda\left(x_2 - x_4 - x_5 - x_6 + x_8 + x_9\right), \tag{3.1}$$

$$x_2 - x_3 - 3x_5 - x_7 - 3x_8 + x_9 = \lambda\left(-x_2 - x_3 + x_4 + 2x_5 + x_6 - x_7 - x_9\right), \tag{3.2}$$

and

$$x_2 + x_3 + x_4 - x_5 + x_6 + x_7 - 3x_8 + x_9 = \lambda\left(-x_3 - x_7 + x_5 + x_8\right). \tag{3.3}$$

We will only prove (3.1), as this proof can be generalized into proofs for (3.2) and (3.3). Let $D_i^{(GK)}$ denote the ith row of the matrix $D^{(GK)}$. Further, let e_i be the ith standard row vector. Consider the following sum and difference of rows of $D^{(GK)}$

$$m := D_2^{(GK)} - D_4^{(GK)} - D_5^{(GK)} - D_6^{(GK)} + D_8^{(GK)} + D_9^{(GK)}.$$

By definition, the ith entry of m is

$$m_i = d_{i2}^{(GK)} - d_{i4}^{(GK)} - d_{i5}^{(GK)} - d_{i6}^{(GK)} + d_{i8}^{(GK)} + d_{i9}^{(GK)}.$$

If $i \in \{0, 1, \dots, 9\}$, $d_{ij}^{(GK)} = d_{ij}^{(G)}$ for $j \in \{0, 1, \dots, 9\}$, therefore the first 10 entries of m can be computed directly from $D^{(G)}$. Consider $i \in \{10, 11, \dots, n\}$. Recall $d_{ij}^{(GK)} = d_{iv}^{(K)} + d_{0j}^{(G)}$ for $j \in \{0, 1, \dots, 9\}$. In this case, the ith entry of m is

$$\begin{aligned} m_i &= 3d_{iv}^{(K)} - 3d_{iv}^{(K)} + d_{02}^{(G)} - d_{04}^{(G)} - d_{05}^{(G)} - d_{06}^{(G)} + d_{08}^{(G)} + d_{09}^{(G)} \\ &= d_{02}^{(G)} - d_{04}^{(G)} - d_{05}^{(G)} - d_{06}^{(G)} + d_{08}^{(G)} + d_{09}^{(G)} = 0. \end{aligned}$$

therefore we can write the following equation

$$D_2^{(GK)} - D_4^{(GK)} - D_5^{(GK)} - D_6^{(GK)} + D_8^{(GK)} + D_9^{(GK)} = 2e_3 + e_4 + 2e_5 + e_6 + 2e_7.$$

Because x is an eigenvector of $D^{(GK)}$, $D_i^{(GK)}x = (D^{(GK)}x)_i = \lambda x_i$ for all i. By multiplying by x on both sides of the equation above on the right, we see

$$\begin{aligned} \left(D_2^{(GK)} - D_4^{(GK)} - D_5^{(GK)} - D_6^{(GK)} + D_8^{(GK)} + D_9^{(GK)}\right) x &= (2e_3 + e_4 + 2e_5 + e_6 + 2e_7)\, x \\ \lambda(x_2 - x_4 - x_5 - x_6 + x_8 + x_9) &= 2x_3 + 2x_5 + 2x_7 + x_4 + x_6 \end{aligned}$$

which is (3.1). Equations (3.2) and (3.3) follow similarly by considering appropriate row combinations.

With these three equations, we can prove c_3 and c_5 are zero. We begin the work for c_3 by substituting the definitions of α and β:

$$\begin{aligned} c_3 &= -\lambda\beta - 2\alpha - \beta + x_4 + x_5 + x_6 - x_8 \\ &= (-\lambda - 1)\left(\frac{\lambda+1}{2\lambda+1}(x_5 + x_8) - \frac{\lambda}{2\lambda+1}(x_3 + x_7)\right) \\ &\quad - 2\left(\frac{-x_3 - x_5 - x_7 - x_8}{2\lambda+1}\right) + x_4 + x_5 + x_6 - x_8 \\ &= \frac{\lambda^2(x_3 - x_5 + x_7 - x_8) + \lambda(x_3 + 2x_4 + 2x_6 + x_7 - 4x_8)}{2\lambda+1} \\ &\quad + \frac{2x_3 + x_4 + 2x_5 + x_6 + 2x_7}{2\lambda+1} \end{aligned}$$

Consider the last term above. The numerator is the left hand side of (3.1), and we can substitute the right hand side.

$$= \frac{\lambda^2(x_3 - x_5 + x_7 - x_8) + \lambda(x_3 + 2x_4 + 2x_6 + x_7 - 4x_8)}{2\lambda+1}$$

$$+\frac{\lambda\left(x_2-x_4-x_5-x_6+x_8+x_9\right)}{2\lambda+1}$$
$$=\frac{\lambda^2(x_3-x_5+x_7-x_8)+\lambda(x_2+x_3+x_4-x_5+x_6+x_7-3x_8+x_9)}{2\lambda+1}$$

Here we see the linear combination of terms that is multiplied by λ is the left hand side of (3.3). Similarly to before, we will substitute the right hand side and cancel.

$$=\frac{\lambda^2(x_3-x_5+x_7-x_8)+\lambda\left(\lambda\left(-x_3-x_7+x_5+x_8\right)\right)}{2\lambda+1}=0$$

This proves $c_3=0$. A similar substitution of (3.2) and (3.3) yield $c_5=0$. Finally, notice $c_8=c_5-2c_6$, and thus c_8 is also zero. This validates the claim that y is an eigenvector of $D^{(HK)}$.

We note the mapping of eigenpairs of $D^{(GK)}$ where $\lambda\neq-\frac{1}{2}$ to those of $D^{(HK)}$ where $\lambda\neq-\frac{1}{2}$ is injective. Suppose $(\lambda,x),(\lambda,x')$ are eigenpairs of $D^{(GK)}$ such that $y=y'$, or equivalently $x+\Delta=x'+\Delta'$. We will show $\Delta=\Delta'$ by showing $\alpha=\alpha'$ and $\beta=\beta'$.

$$y_3+y_5+y_7+y_8=y_3'+y_5'+y_7'+y_8'$$
$$x_3+x_5+x_7+x_8+2\alpha-2\alpha+2\beta-2\beta=x_3'+x_5'+x_7'+x_8'+2\alpha'-2\alpha'+2\beta'-2\beta'$$
$$x_3+x_5+x_7+x_8=x_3'+x_5'+x_7'+x_8'$$
$$\frac{x_3+x_5+x_7+x_8}{2\lambda+1}=\frac{x_3'+x_5'+x_7'+x_8'}{2\lambda+1}$$
$$-\alpha=-\alpha'$$
$$\alpha=\alpha'$$

To prove $\beta=\beta'$, we recall that

$$-c_2=\lambda\alpha-\beta+x_5+x_8=0$$

therefore, since $\lambda\alpha=\lambda\alpha'$,

$$\lambda\alpha-\beta+x_5+x_8=\lambda\alpha'-\beta'+x_5'+x_8'$$
$$-\beta+x_5+x_8=-\beta'+x_5'+x_8'$$
$$-\beta+x_5-\alpha-\beta+x_8+\alpha-\beta+2\beta=-\beta'+x_5'-\alpha'-\beta'+x_8'+\alpha'-\beta'+2\beta'$$
$$y_5+y_8+\beta=y_5'+y_8'+\beta'$$
$$\beta=\beta'.$$

Therefore the mapping is injective as claimed. Further, the mapping is also surjective, as we could have started with the graph HK and performed the perturbation in reverse to get eigenpairs of $D^{(GK)}$.

What remains to be considered are eigenpairs where $\lambda = -\frac{1}{2}$, if any exist. However, since the map is bijective where defined, the dimensions of the eigenspaces for all eigenvalues not equal to $-\frac{1}{2}$ must be the same for both $D^{(GK)}$ and $D^{(HK)}$. Because the sum of the dimensions of all eigenspaces must be n, the multiplicity of $-\frac{1}{2}$ as an eigenvalue must be the same for both $D^{(GK)}$ and $D^{(HK)}$. Therefore the dimensions of all eigenspaces are the same, and the graphs GK and HK are distance cospectral as claimed. □

We note that the theorem yields a construction for large distance cospectral families with a variety of edge counts. Consider identifying k copies of G at a single vertex, namely vertex 0 of each copy. By repeated applications of the theorem, we can exchange out copies of G with copies of H one at a time. Doing this, we construct $k + 1$ graphs which are mutually distance cospectral and with edge counts $\{16k, 16k + 1, \ldots, 17k\}$.

3.3 Distance switching

The proof in Section 3.2 relied on a perturbation of the distance eigenvectors of one graph to yield the distance eigenvectors of another. In this section, we explore a similar technique when considering pairs of distance cospectral graphs related by restricted edge switching. Suppose a graph G has the following two properties. First, G has one of the graphs in Figure 3.2 as an induced subgraph.

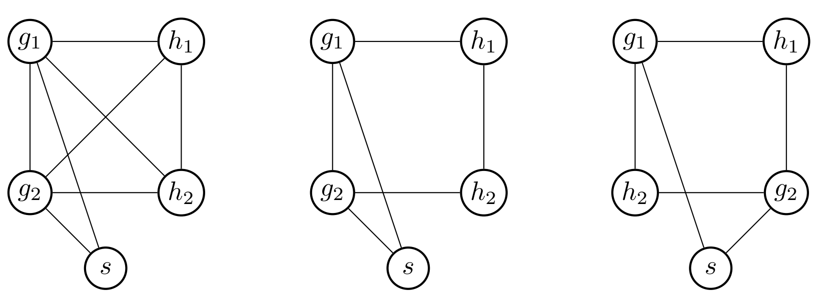


Figure 3.2 Subgraph switching candidates.

Second, we can partition the vertices in $V(G) \setminus \{g_1, g_2, h_1, h_2\}$ into two sets, A and B, such that for all $v \in A$

$$d_{vg_1}^{(G)} + d_{vg_2}^{(G)} - d_{vh_1}^{(G)} - d_{vh_2}^{(G)} = -2,$$

and for all vertices $v \in B$

$$d_{vg_1}^{(G)} + d_{vg_2}^{(G)} - d_{vh_1}^{(G)} - d_{vh_2}^{(G)} = 0.$$

We construct a new graph H as follows. Let $V(H) = V(G)$, and

$$E(H) = E(G) \setminus \{(s, g_1), (s, g_2)\} \cup \{(s, h_1), (s, h_2)\}.$$

We note this switching is somewhat similar to Godsil-McKay switching. Godsil and McKay's construction for local switching requires a switching set D and a partition of the remaining vertices into sets $\{C_i\}$ where for every vertex in $v \in D$ and every set C_i, v is either adjacent to all, none, or exactly half of the vertices in C_i. The switching is done by exchanging edges for non-edges between D and the sets C_i where the vertices in D are adjacent to half of the vertices in C_i. See Section 2.1 of [13] for a full explanation of the construction, including further requirements on the sets C_i not stated here. If we consider the switching set D to be the singleton s and one of the C_i of the partition to be $\{g_1, g_2, h_1, h_2\}$, the construction of H can be likened to Godsil and McKay's construction.

Because $V(H) = V(G)$ and because we will be referencing distances between vertices in both G and H, we will frequently reference the vertex set as simply V.

Theorem 3.3.1. *If for all $v \in B$, $d_{vu}^{(H)} = d_{vu}^{(G)}$ for all $u \in V$ and if for all $w \in A$, $d_{wu}^{(H)} = d_{wu}^{(G)}$ for all $u \in V \setminus \{g_1, g_2, h_1, h_2\}$ and*

$$d_{wg_i}^{(H)} = d_{wg_i}^{(G)} + 1 \quad \textit{and} \quad d_{wh_i}^{(H)} = d_{wh_i}^{(G)} - 1$$

for $i \in \{1, 2\}$, then G and H are distance cospectral.

Proof. We first define a function c on the vertices to be

$$c(v) = d_{vg_1}^{(G)} + d_{vg_2}^{(G)} - d_{vh_1}^{(G)} - d_{vh_2}^{(G)}.$$

By our assumptions on G and direct computation, we can establish that

$$c(v) = \begin{cases} -2 & v \in A \\ 0 & v \in B \\ -k & v \in \{g_1, g_2\} \\ k & v \in \{h_1, h_2\} \end{cases}$$

where $k = 1$ for the subgraph on the left in Figure 3.2, $k = 2$ for the subgraph in the middle of Figure 3.2, and $k = 0$ for the subgraph on the right of Figure 3.2.

Suppose (λ, x) is an eigenpair for the matrix $D^{(G)}$ for $\lambda \neq -k$. We claim $y := x + \Delta$ is an eigenvector of $D^{(H)}$ for eigenvalue λ, where

$$\Delta_i = \begin{cases} 0 & i \notin \{g_1, g_2, h_1, h_2\} \\ \frac{\sum_{j \in A} x_j}{\lambda + k} & i \in \{g_1, g_2\} \\ -\frac{\sum_{j \in A} x_j}{\lambda + k} & i \in \{h_1, h_2\}. \end{cases}$$

To prove y is indeed an eigenvector of $D^{(H)}$, we will show $(D^{(H)}y)_i = \lambda y_i$ for each vertex i. First suppose $i \in B$. We immediately note that $d_{iv}^{(G)} = d_{iv}^{(H)}$ for all $v \in V$ by the hypotheses of the theorem. Further, recall $c(i) = 0$ for all $i \in B$. We therefore have

$$\begin{aligned} (D^{(H)}y)_i &= \sum_{j \in V} d_{ij}^{(H)} y_j \\ &= \sum_{j \in V} d_{ij}^{(G)} (x_j + \Delta_j) \\ &= \sum_{j \in V} d_{ij}^{(G)} x_j + \frac{\sum_{j \in A} x_j}{\lambda + k} \left(d_{ig_1}^{(G)} + d_{ig_2}^{(G)} - d_{ih_1}^{(G)} - d_{ih_2}^{(G)} \right) \\ &= \sum_{j \in V} d_{ij}^{(G)} x_j + \frac{\sum_{j \in A} x_j}{\lambda + k} c(i) \\ &= \sum_{j \in V} d_{ij}^{(G)} x_j \\ &= \lambda x_i = \lambda y_i. \end{aligned}$$

Now suppose $i \in \{g_1, g_2\}$. We know that for all vertices $v \in B$, $d_{iv}^{(H)} = d_{iv}^{(G)}$. Further, for all vertices $u \in A$, $d_{ui}^{(H)} = d_{ui}^{(G)} + 1$ and $c(i) = -k$ for $i \in \{g_1, g_2\}$. Combining these facts, we see that

$$
\begin{aligned}
(D^{(H)}y)_i &= \sum_{j\in V} d_{ij}^{(H)} y_j \\
&= \sum_{j\in A}(d_{ij}^{(G)}+1)(x_j+\Delta_j) + \sum_{j\in B} d_{ij}^{(G)}(x_j+\Delta_j) + \sum_{j\in\{g_1,g_2,h_1,h_2\}} d_{ij}^{(G)}(x_j+\Delta_j) \\
&= \sum_{j\in V} d_{ij}^{(G)} x_j + \sum_{j\in A} x_j + \sum_{j\in\{g_1,g_2,h_1,h_2\}} d_{ij}^{(G)}\Delta_j \\
&= \sum_{j\in V} d_{ij}^{(G)} x_j + \sum_{j\in A} x_j + \frac{\sum_{j\in A} x_j}{\lambda+k}\left(d_{ig_1}^{(G)} + d_{ig_2}^{(G)} - d_{ih_1}^{(G)} - d_{ih_2}^{(G)}\right) \\
&= \sum_{j\in V} d_{ij}^{(G)} x_j + \sum_{j\in A} x_j + \frac{\sum_{j\in A} x_j}{\lambda+k} c(i) \\
&= \lambda x_i + \sum_{j\in A} x_j - k\frac{\sum_{j\in A} x_j}{\lambda+k} \\
&= \lambda x_i + \frac{\lambda\sum_{j\in A} x_j}{\lambda+k} \\
&= \lambda\left(x_i + \frac{\sum_{j\in A} x_j}{\lambda+k}\right) = \lambda y_i.
\end{aligned}
$$

A similar algebraic computation suffices for the case $i \in \{h_1, h_2\}$. What remains to be checked are the vertices $i \in A$. We know $d_{iu}^{(H)} = d_{iu}^{(G)}$ for all $u \in V \setminus \{g_1, g_2, h_1, h_2\}$. Further, $d_{ig_\ell}^{(H)} = d_{ig_\ell}^{(G)} + 1$ and $d_{ih_\ell}^{(H)} = d_{ih_\ell}^{(G)} - 1$ for $\ell \in \{1,2\}$. Finally, recall that $c(i) = -2$ for all $i \in A$. Therefore

$$
\begin{aligned}
(D^{(H)}y)_i &= \sum_{j\in V} d_{ij}^{(H)} y_j \\
&= \sum_{\substack{j\in V \\ j\notin\{g_1,g_2,h_1,h_2\}}} d_{ij}^{(G)}(x_j+\Delta_j) + \sum_{j\in\{g_1,g_2\}} (d_{ij}^{(G)}+1)(x_j+\Delta_j) + \sum_{j\in\{h_1,h_2\}} (d_{ij}^{(G)}-1)(x_j+\Delta_j) \\
&= \sum_{j\in V} d_{ij}^{(G)} x_j + x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2} \\
&\qquad + \frac{\sum_{j\in A} x_j}{\lambda+k}\left(d_{ig_1}^{(G)} + 1 + d_{ig_2}^{(G)} + 1 - (d_{ih_1}^{(G)} - 1) - (d_{ih_2}^{(G)} - 1)\right) \\
&= \sum_{j\in V} d_{ij}^{(G)} x_j + x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2} + \frac{\sum_{j\in A} x_j}{\lambda+k}\left(d_{ig_1}^{(G)} + d_{ig_2}^{(G)} - d_{ih_1}^{(G)} - d_{ih_2}^{(G)} + 4\right)
\end{aligned}
$$

$$= \sum_{j \in V} d_{ij}^{(G)} x_j + x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2} + \frac{\sum_{j \in A} x_j}{\lambda + k} (c(i) + 4)$$

$$= \sum_{j \in V} d_{ij}^{(G)} x_j + x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2} + \frac{2 \sum_{j \in A} x_j}{\lambda + k}.$$

We pause here to prove the following equality:

$$x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2} + \frac{2 \sum_{j \in A} x_j}{\lambda + k} = 0.$$

To do this, let $D_i^{(G)}$ denote the ith row of the matrix $D^{(G)}$ and e_i denote the ith standard row vector. We claim

$$D_{g_1}^{(G)} + D_{g_2}^{(G)} - D_{h_1}^{(G)} - D_{h_2}^{(G)} = -2 \sum_{j \in A} e_j - k e_{g_1} - k e_{g_2} + k e_{h_1} + k e_{h_2}.$$

Suppose $m := D_{g_1}^{(G)} + D_{g_2}^{(G)} - D_{h_1}^{(G)} - D_{h_2}^{(G)}$, and consider the jth entry of m:

$$m_j = d_{jg_1}^{(G)} + d_{jg_2}^{(G)} - d_{jh_1}^{(G)} + d_{jh_2}^{(G)}.$$

This by definition is $c(j)$, and the claim follows.

With this in mind, we multiply both sides by x on the right. Because x is an eigenvector, we know $D_j^{(G)} x = (D^{(G)} x)_j = \lambda x_j$ for all j. Therefore we have

$$\left(D_{g_1}^{(G)} + D_{g_2}^{(G)} - D_{h_1}^{(G)} - D_{h_2}^{(G)} \right) x = \left(-2 \sum_{j \in A} x_j - k e_{g_1} - k e_{g_2} + k e_{h_1} + k e_{h_2} \right) x$$

$$\lambda x_{g_1} + \lambda x_{g_2} - \lambda x_{h_1} - \lambda x_{h_2} = -2 \sum_{j \in A} x_j - k x_{g_1} - k x_{g_2} + k x_{h_1} + k x_{h_2}$$

$$(\lambda + k)(x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2}) = -2 \sum_{j \in A} x_j$$

$$x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2} = \frac{-2 \sum_{j \in A} x_j}{\lambda + k}$$

which proves the equality. Returning to our case,

$$(D^{(H)} y)_i = \sum_{j \in V} d_{sj}^{(G)} x_j + x_{g_1} + x_{g_2} - x_{h_1} - x_{h_2} + \frac{2 \sum_{j \in A} x_j}{\lambda + k}$$

$$= \lambda x_i = \lambda y_i$$

which finishes the case for $i \in A$. Thus the vector y is an eigenvector as claimed.

We note that this mapping of eigenpairs of $D^{(G)}$ with $\lambda \neq -k$ to eigenpairs of $D^{(H)}$ for $\lambda \neq -k$ is bijective. Suppose there are two distinct eigenvectors x and x' for $D^{(G)}$ with the same eigenvalue $\lambda \neq -k$ that map to the same eigenvector y for $D^{(H)}$. Then $y_i = x_i = x'_i$ for all $i \notin \{g_1, g_2, h_1, h_2\}$. If $i \in \{g_1, g_2\}$, then for all $j \in A$,

$$x_i + \frac{\sum_{j \in A} x_j}{\lambda + k} = y_i = x'_i + \frac{\sum_{j \in A} x'_j}{\lambda + k}$$
$$x_i + \frac{\sum_{j \in A} x_j}{\lambda + k} = x'_i + \frac{\sum_{j \in A} x_j}{\lambda + k}$$
$$x_i = x'_i$$

and similarly if $i \in \{h_1, h_2\}$. This implies $x = x'$, and the mapping is injective. Certainly the map is also surjective because we could have perturbed instead the eigenvectors of H. Because of this, we notice that the dimensions of all eigenspaces are the same for $D^{(G)}$ and $D^{(H)}$ for $\lambda \neq -k$.

What remains to be considered are eigenpairs (λ, x) for $\lambda = -k$, if any exist. However, since the mapping is bijective where defined, the dimensions of the eigenspaces for all eigenvalues not equal to $-k$ are the same for both $D^{(G)}$ and $D^{(H)}$. Because the sum of all eigenspaces must be the order of G, the multiplicity of $-k$ must be the same for both graphs. Thus the graphs G and H are distance cospectral as claimed. □

We note that while this edge switching behavior may seem restrictive, it does explain all pairs of distance cospectral graphs on seven vertices, checked by exhaustive search. Figure 3.3 shows these graphs, arranged in the table in order of the induced subgraph contained from Figure 3.2. We point out that the vertex s is shown at the bottom of every embedding.

Further, once we have found a pair G, H that follows this switching behavior, we claim that we can construct infinitely many more pairs using graph identification.

Corollary 3.3.1. *Let G, H be a distance cospectral pair of graphs given by Theorem 3.3.1, and let $u \in V(G) \setminus \{g_1, g_2, h_1, h_2\}$. For any graph K and any vertex $v \in V(K)$, the graphs $GK(u, v), HK(u, v)$ are distance cospectral.*

Proof. We first require some notation. Let A_G and B_G be the partition of $V(G) \setminus \{g_1, g_2, h_1, h_2\}$ given by the construction preceding Theorem 3.3.1.

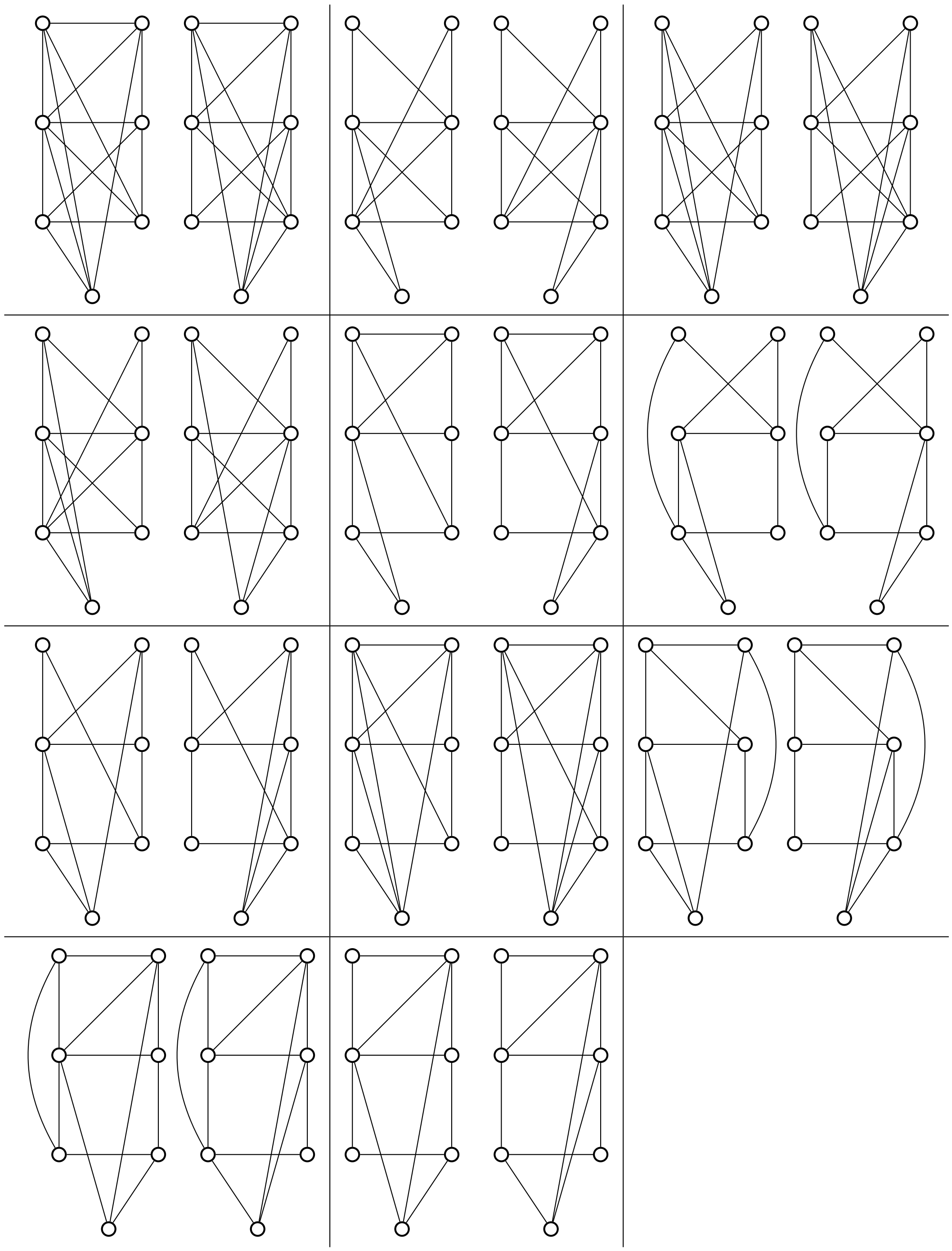


Figure 3.3 All distance cospectral pairs on seven vertices.

We need only to show that this new pair of graphs satisfies the original switching construction and the hypotheses of Theorem 3.3.1. Certainly GK contains one of the induced subgraphs in Figure 3.2 because G does.

To construct the partition A, B of GK, we will extend the partition A_G, B_G in a predicable way. Notice first that for any vertex x in the G portion of GK and for any vertex $w \in \{g_1, g_2, h_1, h_2\}$,

$$d_{xw}^{(GK)} = d_{xw}^{(G)}$$

by the construction of the graph. Therefore, if $x \in A_G$, it follows that $x \in A$ for GK, and similarly for $x \in B_G$.

We now aim to partition the vertices in the K portion of GK. Suppose w is such a vertex. For any vertex x in the G portion of GK, by Section 3.1.1, we know

$$d_{wx}^{(GK)} = d_{wv}^{(K)} + d_{ux}^{(G)}.$$

This implies

$$\begin{aligned} d_{wg_1}^{(GK)} + d_{wg_2}^{(GK)} - d_{wh_1}^{(GK)} - d_{wh_2}^{(GK)} &= d_{wv}^{(K)} + d_{ug_1}^{(G)} + d_{wv}^{(K)} + d_{ug_2}^{(G)} - d_{wv}^{(K)} - d_{uh_1}^{(G)} - d_{wv}^{(K)} - d_{uh_2}^{(G)} \\ &= d_{ug_1}^{(G)} + d_{ug_2}^{(G)} - d_{uh_1}^{(G)} - d_{uh_2}^{(G)} \end{aligned}$$

which is either -2 or 0, depending on if u is in A_G or in B_G. Thus the necessary partition holds for all vertices in GK. Specifically, if $u \in A_G$, all vertices in the K portion of GK are in A. Similarly, if $u \in B_G$, all vertices in the K portion of GK are in B.

Certainly the graph HK is the graph which is formed by the switching construction on GK. We now must prove HK meets the hypotheses of the theorem. We start by showing that for any two vertices x, y such that neither is in $\{g_1, g_2, h_1, h_2\}$, $d_{xy}^{(GK)} = d_{xy}^{(HK)}$. Suppose x, y are two such vertices. First we consider if both are in the K portion of HK. Then, by the construction of the graph

$$d_{xy}^{(GK)} = d_{xy}^{(K)} = d_{xy}^{(HK)}.$$

If both are in the H portion of HK, then because H met the hypotheses of the theorem applied to the pair G, H, we know $d_{xy}^{(GK)} = d_{xy}^{(HK)}$. Now consider if $x \in H$ and $y \in K$. We again use the fact that any path between vertices in H and K must pass through the identified vertex, and we can write

$$d_{xy}^{(HK)} = d_{xu}^{(H)} + d_{vy}^{(K)}$$

and we have two instances of the previous cases, where both vertices are in H and K.

We now need to consider the distances between $\{g_1, g_2, h_1, h_2\}$ and the remaining vertices in the graph. Suppose $w \in A$. If w is in the H portion of HK, then because H meets the conditions of Theorem 3.3.1,

$$d_{wg_i}^{(HK)} = d_{wg_i}^{(GK)} + 1 \text{ and } d_{wh_i}^{(HK)} = d_{wh_i}^{(GK)} - 1$$

for $i \in \{1, 2\}$.

If w is in the K portion of HK, then we notice that by the extension of A_G and B_G into A and B, we know $u \in A_G$. This means

$$d_{ug_i}^{(H)} = d_{ug_i}^{(G)} + 1 \text{ and } d_{uh_i}^{(H)} = d_{uh_i}^{(G)} - 1$$

for $i \in \{1, 2\}$.

We can therefore write

$$d_{wg_i}^{(HK)} = d_{wv}^{(K)} + d_{ug_i}^{(H)} = d_{wv}^{(K)} + d_{ug_i}^{(G)} + 1 = d_{wg_i}^{(GK)} + 1$$

and

$$d_{wh_i}^{(HK)} = d_{wv}^{(K)} + d_{uh_i}^{(H)} = d_{wv}^{(K)} + d_{ug_i}^{(G)} - 1 = d_{wg_i}^{(GK)} - 1$$

for $i \in \{1, 2\}$.

If $w \in B$, we follow a parallel argument and use the fact that u must be in B_G. Therefore GK, HK meet the conditions of Theorem 3.3.1, and GK and HK are distance cospectral. □

3.4 Conclusion

We have established two constructions for distance cospectral pairs (and indeed, large distance cospectral families), including one where graphs have differing numbers of edges. It is interesting to note that distance cospectral graphs with differing numbers of edges are rare.

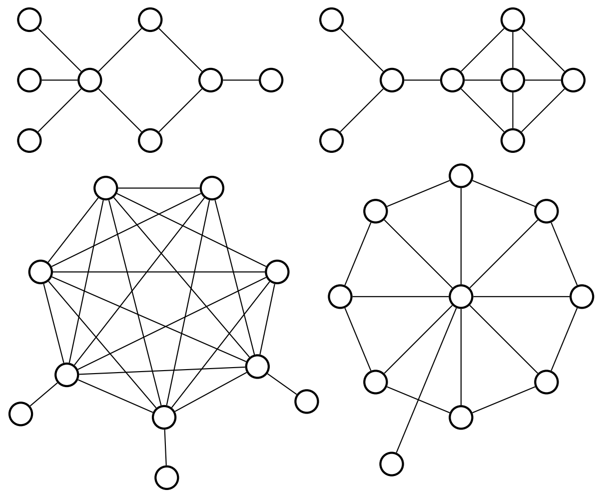


Figure 3.4 Distance cospectral graph pairs with differing numbers of edges.

Other than the graphs show in Section 3.2, there are only two distance cospectral pairs on ten vertices or fewer. These are shown in Figure 3.4.

This emphasis on the edge count fits in a larger question of what the spectrum of any matrix can tell about the graph's structure. For well studied matrices, the questions of whether cospectral pairs exist with differing number of components or whether pairs exist where one graph is bipartite and one is not have been answered. Only one of these questions is relevant for the distance matrix, since the distance matrix is not defined for disconnected graphs. It would be interesting to know if distance cospectral pairs exist where one graph is bipartite and the other is not; no such pair exists on ten vertices or fewer. We hope to see exploration of this problem and more work for distance cospectral constructions in the future.

3.5 Post Script

Since the submission of this paper, another cospectral construction for the distance matrix was created by Abiad et al. considering distance cospectral graphs with different diameters and Weiner indices [1]. To answer the question of the existence of infinitely many distance cospectral graphs with differing diameters and Weiner indices, Abiad et al. prove another construction for distance cospectral graphs. The q-coclique extension of a graph G, denoted G_q, is the graph with vertex set $V \times \{1, ..., q\}$, where (x, i) is adjacent to (y, j) if and only if x is adjacent to y in G. The q-clique extension of a graph G, denoted G_q^+, is the graph with vertex set $V(G) \times \{1, ..., q\}$ with (x, i) adjacent to (y, j) if and only if x is adjacent to y in G or $x = y$ and $i \neq j$. With these two constructions comes the following theorem.

Theorem 3.5.1 ([1])**.** *Let G and H be distance cospectral graphs. Then G_q and H_q are distance cospectral, and G_q^+ and H_q^+ are distance cospectral.*

The proof for both constructions arises from considering the distance matrices of q-clique and q-coclique expansions as sums, differences, and Kronecker products of the identity matrix, the all ones matrix, and the distance matrices of G and H.

CHAPTER 4. CONCLUSIONS

4.1 General conclusions

The overarching goal of spectral graph theory is to discern structural properties of a graph from information contained in the spectrum. A handful of results and counterexamples were discussed in Section 1.3. In this thesis, we considered examples of cospectral constructions to further understand what information is not contained in the spectrum of a graph. In particular, we focused on the inability of the normalized Laplacian and the distance matrix to always capture information about the number of edges in a graph. This not only differs from the majority of the matrices considered in spectral graph theory, it is also somewhat surprising. The number of edges is a fundamental graph property, yet it is not contained in either the normalized Laplacian or distance spectrum.

In Chapter 2, we gave a construction for an infinite family of weighted graphs which are cospectral with respect to the normalized Laplacian. Further, we discussed the expansion of these weighted graphs into simple graphs, creating even more cospectral pairs. In Chapter 3, we introduced the idea of distance switching and answered in the affirmative whether distance cospectral graphs can have differing numbers of edges. Indeed, for any positive integer k, we can construct distance cospectral graphs which have edge sets of sizes that differ by k.

It is worth noting that the proof techniques for Chapters 2 and 3 vary considerably. In Chapter 2, we considered only the characteristic polynomial, while in Chapter 3 we worked with eigenvectors. The proof techniques reflect the matrix in question. The idea of generalized cycle decompositions does not translate well to the distance matrix because every off diagonal entry is nonzero. Similarly, the normalized Laplacian eigenvectors of the graphs related by toggling do not behave predictably, therefore a perturbation approach was not feasible. In developing a variety of proof techniques, we create a set of tools which may help us answer further questions about cospectral graphs.

4.2 Further questions

In Table 1.1, there is a question mark in the entry for determining if a graph is bipartite by using the distance matrix. The question whether distance cospectral graphs exist where one is bipartite and one is not is still open. However, SAGE calculations reveal that such a pair does not exist on ten or fewer vertices. In addition, it seems that bipartite distance cospectral graphs are rare. Only six of the 11962 families of distance cospectral graphs on nine vertices are bipartite and 51 of the 648943 distance cospectral families on ten vertices are bipartite. This information gives rise to the following conjecture.

Conjecture. *The distance spectrum of a graph determines if the graph is bipartite.*

Another direction to consider for future research is further exploration of distance switching. How many pairs of distance cospectral graphs on eight vertices can be explained by switching? Certainly not all, as the graph in Table 1.2 is a pair on eight vertices. Are there any other subgraph candidates? Are our three pairs only a limited view of a larger behavior? Answering these questions will hopefully add more to our knowledge of the distance matrix.

BIBLIOGRAPHY

[1] A. Abiad, B. Brimkov, A. Erey, L. Leshock, X. Martínez-Rivera, S. O, S.Y. Song, J. Williford. "On the Wiener index, distance cospectrality and transmission regular graphs", preprint.

[2] M. Aouchiche and P. Hansen. "Distance spectra of graphs: A survey", *Linear Algebra Appl.* 458 (2014). 301–386.

[3] A. E. Brouwer and W. H. Haemers. *Spectra of Graphs.* New York: Springer-Verlag. (2012).

[4] R.A. Brualdi, and H.J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, Cambridge, 1991.

[5] S. Butler. "Using twins and scaling to construct cospectral graphs for the normalized Laplacian", *Electronic Journal of Linear Algebra* **28** (2015), 54–68.

[6] S. Butler, and F. Chung. "Spectral graph theory", in *Handbook of Linear Algebra*, 2nd edition, ed. L. Hogben, CRC Press, 2014.

[7] S. Butler, and J. Grout. "A construction of cospectral graphs for the normalized Laplacian", *Electronic Journal of Combinatorics* **18** (2011), #P231, 20pp.

[8] S. Butler and K. Heysse. "A cospectral family of graphs for the normalized Laplacian found by toggling", *Linear Algebra and its Applications.* **507** (2016), pg 499–512.

[9] M. Cavers, *The normalized Laplacian matrix and general Randic index of graphs*, Ph.D. Thesis, University of Regina, 2010.

[10] F. Chung, *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics, 92. American Mathematical Society, Provedince, RI, 1997.

[11] I. M. Gessel and R. P. Stanley. “Algebraic Enumeration”, in *Handbook of Combinatorics*, Vol. 2, ed. R. L. Graham et al., Elsevier, 1995, 1021–1061.

[12] K. Heysse. “A construction for distance cospectral graphs”, submitted.

[13] C. D. Godsil, and B. McKay. “Constructing cospectral graphs”, *Aequationes Mathematicae* **25** (1982), 257–268.

[14] B. D. McKay. “On the spectral characterization of trees”, *Ars Combinatoria* **3** (1977). 219–232.