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A SPIN-ONE PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

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TABLE OF CONTENTS

INTRODUCTION

Recently Joos(1), Weinberg(2), and Weaver, Hammer, and Good(3) have developed equivalent descriptions of a free particle with spin s = 0, 1/2, 1, ... These descriptions are of interest because they are analogous to the Dirac theory for a spin-1/2 particle. This means that many of the well known discussions for the spin-1/2 theory can be extended uniformly to apply to particles of arbitrary spin. Joos(1) and Weinberg(2) gave their description in a manifestly covariant form using covariantly defined matrices introduced by Barut, Muzinich, and Williams(4) as the generalization of the Dirac γ_μ matrices. Weaver, Hammer and Good(3) gave their description in Hamiltonian form and found an algorithm for generalizing the Dirac Hamiltonian, $\underline{\alpha}\cdot\underline{p} + \beta m$, to any spin. These two formulations involve essentially the same wave function. These formulations are exactly the same for odd-half-integral spin and are equivalent for integral spin. This wave function forms the basis for the $(s,0) \oplus (0,s)$ irreducible representation of the Lorentz group. It also corresponds to the momentum-space wave function used by Pursey(5) in his treatment of free particles with spin.

In later works most of the detailed properties of the free-particle theory have been worked out. Sankaranarayanan and Good(6) studied the spin-one case in detail, including the polarization operator, and Shay, Song, and Good(7) considered the spin-3/2 case. Sankaranarayanan and Good(8) found the position operators. The density matrices for describing polarization properties were derived by Sankaranarayanan(9) and by Shay, Song, and Good(7). Mathews(10) and Williams, Draayer, and Weber(11) obtained a closed formula for the Hamiltonian for any spin.

This theory has been applied only to the free particle with spin, and the question arises as to how the effect of an electromagnetic field may be included. Since this theory has been successful in describing all aspects of the free particle case, one might hope that electromagnetic effects may also be introduced for any spin. This has not been done before. However, this problem becomes more and more difficult as the spin increases, since a particle of spin s can have anomalous electric and magnetic multipole moments up to the 2^{2s} order.

The purpose of this thesis is to give the theory of a spin-one particle, described by a $(1,0) \oplus (0,1)$ wave function, which interacts with an external electromagnetic field. The effects of both <u>an</u> anomalous magnetic dipole and an anomalous electric quadrupole moment are included.

The spin-one particle in an external field was originally studied by Proca(12) and Kemmer(13) using a ten component wave function. Corben and Schwinger(14) showed how to include an anomalous magnetic dipole term in Proca's theory, and Young and Bludman(15) were able to include the anomalous electric quadrupole. Young and Bludman(15) were also able to get a Hamiltonian of the Sakata-Taketani(16) type which included anomalous magnetic dipole and anomalous electric quadrupole terms. This formulation involves a six component wave function which has complicated Lorentz transformation properties.

The wave equation found here is manifestly covariant, and no auxiliary conditions are required on the wave function. The equation has the usual symmetries with respect to space reflection, time reflection, and charge conjugation. It is derivable by a differentation process from the other formulations as far as the normal and anomalous magnetic moment terms are concerned; the anomalous electric quadrupole terms are of a different type. There are two possible choices for the quadrupole term in this wave equation, and both choices have the proper transformation and inversion properties. They both give identical contributions to the non-relativistic limit to order $1/m^2$. A particle described by this equation also has an intrinsic magnetic dipole moment of $- e\hbar/4mc$, and an intrinsic electric quadrupole moment $e\hbar^2/8m^2c^2$, where e is the particle charge and m the mass. This corresponds to a g factor of 1/2 and a Q factor of - 1/2.

Two new results follow from this reformulation. First, a Lorentz invariant inner product is defined, and covariant equations of motion for operators in the Heisenberg picture follow from this. Second, the Foldy-Wouthuysen(17) transformation is developed by making use of the Lorentz transformation properties of the wave function. This permits the non-relativistic limit of the spin-one wave equation to be rewritten in a Hamiltonian form correct to order $1/m^2$.

PROCA'S FORMULATION

In 1936 Proca(12) gave a covariant formulation which described a spin-one particle of mass m and charge e moving in an external electromagnetic field. In 1940 Corben and Schwinger(14) added an anomalous magnetic dipole term to the equations, and in 1963 Young and Bludman(15) added an electric quadrupole term.

These three formulations describe a spin-one particle by means of a four-vector U_α and an antisymmetric tensor $U_{\alpha\beta}$. The Young-Bludman equations are the most general and are

(1) $$D_\alpha U_{\alpha\nu} - m^2 U_\nu + ie\lambda U_\alpha H_{\alpha\nu} + \frac{ieq}{4m^2} U_{\alpha\beta}\partial_\nu H_{\alpha\beta} = 0 ,$$

and

(2) $$U_{\alpha\beta} = D_\alpha U_\beta - D_\beta U_\alpha + \frac{ieq}{2m^2} U_\nu \partial_\nu H_{\alpha\beta} ,$$

where

$$D_\nu = \partial_\nu - ieA_\nu = \frac{\partial}{\partial x_\nu} - ieA_\nu$$

and

$$H_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha .$$

Summation convention is used for tensor indices with Greek indices running from 1 to 4 and Latin indices 1 to 3. The vector potential and the field tensor associated with the external field are A_α and $H_{\alpha\beta}$. The constants λ and q give the anomalous dipole and quadrupole strengths, respectively.

The Corben-Schwinger equations may be obtained from Equations 1 and 2 by setting $q = 0$. Setting $q = \lambda = 0$ yields the Proca equations.

The extension of the free particle Proca equations to the more general equations is a straight forward process. First, the ∂_ν in the free equations is replaced by D_ν, and then the anomalous terms are added on to the equations. So one should be aware of the fact that Equations 1 and 2 are not a unique set of equations because of the different possible ways of including the anomalous moments.

These Proca type formulations use the vector U_α and the tensor $U_{\alpha\beta}$ to describe a spin-one particle. So, in effect, a ten component wave function is being used to describe a situation which requires only six. Therefore all of the components of U_α and $U_{\alpha\beta}$ cannot be independent. Sakata and Taketani(16) were able to rewrite Proca's equations in terms of six independent components, and Young and Bludman(15) used this same process for their equations with the anomalous dipole and quadrupole moments.

STATEMENT OF THE PROBLEM

Including the external electromagnetic fields in the formulations of Joos, Weinberg, and Weaver, Hammer, and Good is not so straight forward. In order to have the usual gauge invariance, the wave equation must be written in terms of $\pi_\nu = \frac{1}{i}\partial_\nu - eA_\nu(x)$. However, one cannot, as for the Proca equations, just replace $\frac{1}{i}\partial_\nu = p_\nu$ by π_ν and add anomalous moment terms, as will be seen in the following.

The formulation of Joos and Weinberg involves two covariant equations

(3a) $$(p_\alpha p_\beta \gamma_{\alpha\beta} + m^2)\ \psi(x) = 0$$

and

(3b) $$(p_\alpha p_\alpha + m^2)\ \psi(x) = 0\ .$$

The 6x6 covariant matrix $\gamma_{\alpha\beta}$ has the components

$$\gamma_{44} = \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},\ \gamma_{4i} = \gamma_{i4} = -\ i\beta\alpha_i = -\ i\beta \begin{pmatrix} s_i & 0 \\ 0 & -s_i \end{pmatrix},$$

$$\gamma_{ij} = -\beta([\alpha_i, \alpha_j]_+ - \delta_{ij})\ ,$$

where I is the 3x3 unit matrix and the s_i are the three 3x3 spin-one matrices. Equations 3 have six independent, mutual solutions $\psi(x)$ which are six component column matrices. These six solutions are sufficient to describe all of the possible spin-one, free particle-antiparticle states.

If p_α is replaced by π_α, these two equations are no longer consistent and do not have the required six mutual solutions. If, on the other hand, only Equation 3a is considered, then it will have twelve solutions. This is two times too many.

In the free particle case Equation 3b can be obtained by squaring Equation 3a and choosing one of the two possible choices for the eigenvalue of $p_\alpha p_\alpha$. This occurs because

$$(p_\alpha p_\beta \gamma_{\alpha\beta})^2 = (p_\alpha p_\alpha)^2$$

However, if p_α is replaced by π_α, then $(\pi_\alpha \pi_\beta \gamma_{\alpha\beta})^2$ does not have such a simple form. This is due to the fact that the different components of π_α do not commute. So, there does not appear to be an X such that

$$X \neq \pi_\alpha \pi_\beta \gamma_{\alpha\beta} ,$$

but such that

$$X^2 = (\pi_\alpha \pi_\beta \gamma_{\alpha\beta})^2 .$$

Therefore an equation corresponding to Equation 3b cannot be found from Equation 3a when p_α is replaced by π_α, as can be done in the field free case.

The covariant form of the Weaver-Hammer-Good theory as given by Sankaranarayanan and Good(6) is

(4a) $$\left(p_\alpha p_\beta \gamma_{\alpha\beta} + \frac{i\partial/\partial t}{E}\, m^2\right) \psi(x) = 0$$

and

(4b) $$\left(p_\alpha p_\alpha + m^2\right) \psi(x) = 0 \ ,$$

where

$$E = (p_i p_i + m^2)^{1/2}, \quad i = 1,2,3.$$

This has all the difficulties of Weinberg's formulation, plus the fact that we do not know what energy sign operator $\frac{i\partial/\partial t}{E}$ becomes in the presence of an external field.

FURTHER DIFFICULTIES IN INTRODUCING EXTERNAL FIELDS

Another type of difficulty is encountered when the attempt is made to include external fields in other formulations of the spin-one theory. In the last section it was seen that the replacement of p_ν by π_ν destroyed the mutual consistency of two equations. In the following, the replacement of p_ν by π_ν or ∂_ν by D_ν, causes an additional restriction to be placed on the wave function. The resulting wave function then does not have enough components to describe all the possible polarization states of the particle, and does not have the correct zero field limit.

Consider the spinor equations of Dirac(18) and Fierz and Pauli(19) for a spin-one particle,

(5a) $$\partial_{\dot{\alpha}\beta}\varphi^{\beta\sigma} = m\chi_{\dot{\alpha}}{}^{\sigma}$$

and

(5b) $$\partial^{\alpha\dot{\beta}}\chi_{\dot{\beta}}{}^{\sigma} = m\varphi^{\alpha\sigma},$$

where

$$\varphi^{\alpha\beta} = \varphi^{\beta\alpha}$$

and, in the particle's rest frame,

$$\chi^{R\beta}_{\dot{\alpha}} = \chi^{R\alpha}_{\dot{\beta}} \; .$$

So $\chi_{\dot{\alpha}}{}^{\beta}$ and $\varphi^{\alpha\beta}$ have a total of six independent components. The spinor conventions used here are

$$\partial^{\alpha\dot{\beta}} = -(\sigma_\nu)_{\alpha\beta}\partial_\nu ,$$

$$\varphi^\alpha = i\,(\sigma_2)_{\alpha\beta}\varphi_\beta ,$$

and

$$\varphi_\alpha{}^* = \varphi_{\dot{\alpha}} ,$$

where * denotes complex conjugation. The σ_i are the three 2x2 Pauli matrices and σ_4 is i times the unit matrix I. Summation convention is also used for spinor indices which have the range 1, 2. Simply replacing ∂_ν by D_ν in Equation 5 and using the symmetry of $\varphi^{\alpha\beta}$ gives

$$[H_{4k} - \frac{e}{2}\,\epsilon_{ijk}H_{ij}][(\sigma_k)_{\alpha\tau}\varphi^{\tau\beta} - (\sigma_k)_{\beta\tau}\varphi^{\tau\alpha}] = 0 .$$

This auxiliary condition on the wave function completely eliminates some polarization states, depending on the applied fields, for even arbitrarily small fields. This is not observed in practice and such a theory does not give all of the free particle solutions in the zero field limit. So this process is not a good one. Fierz and Pauli(19) were able to overcome this difficulty only by adding additional components to the wave function.

This shows that caution must be used when inserting external fields into any of the theories based on the Dirac-Pauli-Fierz equations. For example, this same difficulty arises when ∂_ν is simply replaced by D_ν in the Rarita-Schwinger(20) and Bargmann-Wigner(21) theories for spin ≥ 1. The latter theory will be discussed later in more detail for spin-one.

DETERMINATION OF THE WAVE EQUATION

A simple way of including external fields in the equations discussed in the previous sections does not appear to exist. However, the equations with fields related to Proca's are simply related to the free particle equations and are consistent. This must be exploited in some way. This can be done by using the relationship between spinors and tensors given by Laporte and Uhlenbeck(22). By the use of these relations the Young-Bludman equations, for example, can be rewritten in spinor form. This does lead to a consistent spinor theory which describes a spin-one particle-antiparticle in an external field by means of a six component wave function.

A fourth rank spinor $U^{\alpha\dot{\beta}}{}_{\dot{\rho}\epsilon}$ is related to a second rank, antisymmetric tensor $U_{\mu\nu}$ by the equation,

$$U^{\alpha\dot{\beta}}{}_{\dot{\rho}\epsilon} = -(\sigma_\mu)_{\alpha\beta}(\sigma_\nu{}^\dagger)_{\rho\epsilon}U_{\mu\nu} . \tag{6}$$

Further, without loss of generality, this spinor can be expressed in terms of two symmetric spinors $\chi_{\dot{\alpha}\dot{\beta}}$ and $\varphi^{\alpha\beta}$ as follows,

$$U^{\alpha\dot{\beta}}{}_{\dot{\rho}\epsilon} = \delta^{\alpha}_{\epsilon}\chi^{\dot{\beta}}{}_{\dot{\rho}} - \delta^{\dot{\beta}}_{\dot{\rho}}\varphi^{\alpha}{}_{\epsilon} . \tag{7}$$

The Weinberg wave function is written in terms of two symmetric spinors in the following way,

(8)

$$\psi = \begin{pmatrix} \chi_{\dot{1}\dot{1}} \\ \sqrt{2}\ \chi_{\dot{1}\dot{2}} \\ \chi_{\dot{2}\dot{2}} \\ \varphi^{11} \\ \sqrt{2}\ \varphi^{12} \\ \varphi^{22} \end{pmatrix} .$$

Equations 6, 7, and 8 then give the connection between the Proca wave function and the Weinberg wave function. This transformation is one to one, so the inverse also exists. This transformation is not that given by Sankaranarayanan and Good(6), but is related to it by a constant matrix. This difference is due to a difference in representations of the spin-one matrices.

The Weinberg wave function $\psi(x)$ is related directly to the tensor $U_{\mu\upsilon}$. This means that the first step in rewriting the Young-Bludman equations is to eliminate U_μ. The only way that this can be done and still give a theory where the fields appear in a simple, linear way is to let $\lambda = q = 0$ in Equations 1 and 2.

For this special case, Equations 1 and 2 give the spinor equations,

(9a) $$\pi_{\dot{\alpha}\rho}\theta_{\dot{\beta}}{}^{\rho} + \pi_{\dot{\beta}\rho}\theta_{\dot{\alpha}}{}^{\rho} = 2m\chi_{\dot{\alpha}\dot{\beta}} ,$$

(9b) $$\pi^{\alpha\dot{\rho}}\theta_{\dot{\rho}}{}^{\beta} + \pi^{\beta\dot{\rho}}\theta_{\dot{\rho}}{}^{\alpha} = 2m\varphi^{\alpha\beta} ,$$

and

(9c) $$\pi^{\alpha\dot{\rho}}\chi_{\dot{\rho}\dot{\beta}} + \pi_{\dot{\beta}\rho}\varphi^{\rho\alpha} = 2m\theta_{\dot{\beta}}{}^{\alpha} ,$$

where

$$\theta_{\dot{\alpha}\beta} = - im(\sigma_{\mu}{}^{+})_{\alpha\beta}U_{\mu} .$$

Eliminating $\theta_{\dot{\alpha}}{}^{\beta}$ from Equations 9 and using the matrix ordering given in Equation 8 gives

(10) $$(\pi_{\alpha}\pi_{\beta}\gamma_{\alpha\beta} + m^2)\ \psi(x) =$$

$$= -(\pi_{\alpha}\pi_{\alpha} + m^2 + \frac{ie}{12} H_{\alpha\beta}[\gamma_{\alpha\tau},\gamma_{\beta\tau}]_{-})\ \psi(x) ,$$

where the covariant matrix $\gamma_{\alpha\beta}$ has already been given.

A magnetic dipole term already appears in Equation 10, so an anomalous term can be included just by inserting a numerical factor λ.

The quadrupole term is a bit more difficult. This term must contain $\partial_j E_i\ [\Sigma_i,\Sigma_j]_+$, where $\Sigma_i = \begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix}$ and the s_i are the 3x3 spin-one matrices. The covariant extension of $\partial_j E_i$ is $\partial_\nu H_{\alpha\beta}$. However, this cannot be contracted with a covariant matrix to give a scalar operator since, as was shown in Reference 6, all the spin-one covariant matrices have an even number of tensor indices. This suggests that the quadrupole term be constructed with $(\partial_\nu H_{\alpha\beta})\pi_\mu$.

This gives three possible quadrupole terms;

1) $\frac{Qe}{m^2}\,(\partial_\nu H_{\alpha\beta})\,\pi_\beta\gamma_{\alpha\nu}$,

2) $\frac{Qe}{72m^2}\,(\partial_\nu H_{\alpha\beta})\,\pi_\mu[[\gamma_{\alpha\rho},\gamma_{\beta\rho}]_-,[\gamma_{\nu\tau},\gamma_{\mu\tau}]_-]_+$,

3) $\frac{Qe}{6m^2}\,(\partial_\nu H_{\alpha\beta})\,\pi_\mu\gamma_{6,\alpha\beta,\nu\mu}$,

where $\gamma_{6,\alpha\beta,\nu\mu}$ is

$$\gamma_{6,\alpha\beta,\nu\mu} = [\gamma_{\alpha\nu},\gamma_{\beta\mu}]_+ - [\gamma_{\alpha\mu},\gamma_{\beta\nu}]_+ + $$
$$+ 2\delta_{\alpha\nu}\delta_{\beta\mu} - 2\delta_{\alpha\mu}\delta_{\beta\nu} ,$$

as was given in Reference 6. The first choice contains the term $(\partial_j E_i)\,\pi_4\,[\Sigma_i,\Sigma_j]_+\beta$. Although this term contains an extra factor β there is no basis for ignoring it.

Expressions 2 and 3 both contain $(\partial_j E_i)\,\pi_4\,[\Sigma_i,\Sigma_j]_+$, and, in fact, the two expressions are identical if $\partial_\beta H_{\alpha\beta} = 0$. As a result, the covariant equation describing a spin-one particle, with anomalous magnetic and electric quadrupole moments, moving in an arbitrary external electromagnetic field is

(11) $$(\pi_\alpha\pi_\beta\gamma_{\alpha\beta} + \pi_\alpha\pi_\alpha + 2m^2 + \frac{ie\lambda}{12}\,H_{\alpha\beta}\,[\gamma_{\alpha\tau},\gamma_{\beta\tau}]_- + $$
$$+ Q)\,\psi(x) = 0 ,$$

where,

$$Q = \frac{\alpha e}{6m^2} (\partial_\nu H_{\alpha\beta}) \pi_\mu \gamma_{6,\alpha\beta,\nu\mu} =$$

$$= \frac{\alpha e}{72m^2} (\partial_\nu H_{\alpha\beta}) \pi_\mu [[\gamma_{\alpha\rho}, \gamma_{\beta\rho}]_-, [\gamma_{\nu\tau}, \gamma_{\mu\tau}]_-]_+ ,$$

when

$$\partial_\beta H_{\alpha\beta} = 0 ;$$

or, as another possibility,

$$Q = \frac{\alpha e}{m^2} (\partial_\nu H_{\alpha\beta}) \pi_\beta \gamma_{\alpha\nu} ,$$

and λ and q are the multipole moment strengths.

In the field free case, Equation 11 reduces to

$$(12) \qquad (p_\alpha p_\beta \gamma_{\alpha\beta} + m^2) \psi(x) = - (p_\alpha p_\alpha + m^2) \psi(x) .$$

Multiplying Equation 12 by $p_\alpha p_\beta \gamma_{\alpha\beta}$ and using the identity,

$$(p_\alpha p_\beta \gamma_{\alpha\beta})^2 = (p_\alpha p_\alpha)^2 ,$$

yields the equation,

$$m^2 (p_\alpha p_\alpha + m^2) \psi(x) = 0 .$$

For non-zero m, then, the solutions of the single Equation 12 are precisely the six solutions of the Weinberg equations.

So with Weinberg's formulation rewritten with only one equation, the theory can be generalized to the case of external fields simply by replacing p_α by π_α and adding anomalous multipole terms. This resulting equation is covariant with respect to proper Lorentz transformations, space and time inversion, and charge conjugation, for either choice of Q. This covariance is discussed in the Appendix.

In Weinberg's formulation there is no Hamiltonian for integral spin. It is also true for Equation 11 that an exact Hamiltonian does not exist.

The transformation from $U_{\alpha\beta}$ to $\psi(x)$ is reversible. So, using the connection between $U_{\alpha\beta}$ and $\psi(x)$ in Equation 11 gives

(13) $$D_\alpha D_\nu U_{\nu\beta} - D_\beta D_\nu U_{\nu\alpha} - m^2 U_{\alpha\beta} + \frac{ie}{2}(1-\lambda)(H_{\alpha\nu}U_{\nu\beta} - H_{\beta\nu}U_{\nu\alpha}) + Q_{\alpha\beta} = 0 ,$$

where

$$Q_{\alpha\beta} = \frac{a ie}{3m^2}[3(\partial_\nu H_{\alpha\beta})D_\mu + (\partial_\alpha H_{\nu\mu})D_\beta - (\partial_\beta H_{\nu\mu})D_\alpha + (\partial_\nu H_{\alpha\mu})D_\beta - (\partial_\nu H_{\beta\mu})D_\alpha]\, U_{\nu\mu} + \frac{a ie}{2m^2}\{[(\partial_\beta H_{\nu\mu}) + (\partial_\mu H_{\nu\beta})]D_\nu U_{\alpha\mu} - [(\partial_\alpha H_{\nu\mu}) + (\partial_\mu H_{\nu\alpha})]D_\nu U_{\beta\mu}\}$$

for

$$Q = \frac{ae}{6m^2}(\partial_\nu H_{\alpha\beta})\pi_\mu \gamma_{6,\alpha\beta,\nu\mu} ;$$

and

$$Q = -\frac{a ie}{2m^2}\{[(\partial_\beta H_{\nu\mu}) + (\partial_\mu H_{\nu\beta})]D_\nu U_{\alpha\mu} -$$

$$- [(\partial_\alpha H_{\nu\mu}) + (\partial_\mu H_{\nu\alpha})] D_\nu U_{\beta\mu}\}$$

for

$$Q = \frac{\alpha e}{m^2} (\partial_\nu H_{\alpha\beta})\pi_\beta \gamma_{\alpha\nu} .$$

The vector U_α is not determined here and so it could be arbitrarily defined in terms of $U_{\alpha\beta}$.

The Young-Bludman equations cannot be put in the same form as Equation 13 since U_α cannot be eliminated from Equations 1 and 2. However, $U_{\alpha\beta}$ may be eliminated from Equations 1 and 2 to yield a second order equation for U_α. So Equation 13 describes a spin-one particle in terms of a second rank tensor $U_{\alpha\beta}$ which satisfies a second order wave equation. The Young-Bludman theory essentially uses a four vector U_α which obeys a second order wave equation.

THE NON-RELATIVISTIC APPROXIMATION

The non-relativistic limit of Equation 11 can be found by using a method analogous to that used by Foldy and Wouthuysen(17) for the Dirac equation with fields. For the spin-one case the process involves first applying the Foldy-Wouthuysen transformation, which is related to the pure Lorentz lab to rest transformation for a free spin-one particle, reducing the order of the time derivatives in the resulting equation, and then picking out the "large" components of the wave function. The resulting equation is in the Hamiltonian form and involves only these "large" components.

By writing out the tensor components of $\gamma_{\alpha\beta}$, Equation 11 may be rewritten in the form

$$(14) \qquad [\pi_\alpha\pi_\alpha(I + \beta) + 2m^2 + 2i\underline{\pi}\cdot\underline{\alpha}\pi_4\beta - 2(\underline{\pi}\cdot\underline{\alpha})^2\beta - e(i\underline{E}\cdot\underline{\alpha} + \underline{B}\cdot\underline{\Sigma})\beta + e\lambda(i\underline{E}\cdot\underline{\alpha} - \underline{B}\cdot\underline{\Sigma}) + Q]\psi(x) = 0,$$

where

$$Q = \frac{qe}{72m^2}(\partial_\nu H_{\alpha\beta})\pi_\mu[[\gamma_{\alpha\rho},\gamma_{\beta\rho}]_-,[\gamma_{\nu\tau},\gamma_{\mu\tau}]_-]_+$$

or

$$Q = \frac{qe}{m^2}(\partial_\nu H_{\alpha\beta})\pi_\beta\gamma_{\alpha\nu} ;$$

The vectors $\underline{E}$ and $\underline{B}$ are the electric and magnetic fields, respectively.

Now, the pure Lorentz transformation for a free spin-one particle to its rest frame is represented by the matrix operator

$$T = \exp\,(-\hat{\underline{p}}\cdot\underline{\alpha}\,\sinh^{-1}\frac{p}{m})\ .$$

If the $\sinh^{-1}\frac{p}{m}$ term is expanded in a power series and then $\underline{p}$ is replaced by $\underline{\pi}$, the first term of the new operator is

$$T_1 = \exp\,(-\underline{\pi}\cdot\underline{\alpha}/m).$$

A new wave function $\psi_1(x)$ can be defined by

(15) $$\psi_1(x) = \exp\,(-\underline{\pi}\cdot\underline{\alpha}/m)\psi(x)\ .$$

Then Equation 15 can be substituted into Equation 14. By neglecting all terms of order $1/m^2$ or higher, with the understanding that $-i\pi_4$ is of order m, this substitution yields the following equation for $\psi_1(x)$,

(16) $$\{[I+\beta - \frac{2}{m}\,\underline{\pi}\cdot\underline{\alpha}\beta + \frac{2}{m^2}(\underline{\pi}\cdot\underline{\alpha})^2\beta - \frac{4}{3m^3}\,(\underline{\pi}\cdot\underline{\alpha})^3\beta]\pi_4{}^2 +$$
$$+ [2i\underline{\pi}\cdot\underline{\alpha}\beta - \frac{2e}{m}\underline{E}\cdot\underline{\alpha}(I-\beta) - \frac{4i}{m}\,(\underline{\pi}\cdot\underline{\alpha})^2\beta -$$
$$- \frac{4e}{m^2}\,(\underline{\pi}\cdot\underline{\alpha})(\underline{E}\cdot\underline{\alpha})\beta + \frac{e}{m^2}\,[\underline{\pi}\cdot\underline{\alpha},\underline{E}\cdot\underline{\alpha}]_-(I+\beta) +$$
$$+ \frac{4i}{m^2}\,(\underline{\pi}\cdot\underline{\alpha})^3\beta]\pi_4 + 2m^2 + \underline{\pi}\cdot\underline{\pi}\,(I+\beta - \frac{2}{m}\,\underline{\pi}\cdot\underline{\alpha}\beta) -$$
$$- 2(\underline{\pi}\cdot\underline{\alpha})^2\beta + \frac{4}{m}\,(\underline{\pi}\cdot\underline{\alpha})^3\beta - ie\,\underline{E}\cdot\underline{\alpha}\beta - e\underline{B}\cdot\underline{\Sigma}\beta +$$

$$+ ie\lambda \underline{E}\cdot\underline{\alpha} - e\lambda \underline{B}\cdot\underline{\Sigma} + \frac{2ie}{m}(\underline{\pi}\cdot\underline{\alpha})(\underline{E}\cdot\underline{\alpha})\beta +$$

$$+ \frac{ie}{m}[\underline{\pi}\cdot\underline{\alpha},\underline{E}\cdot\underline{\alpha}]_{+}\beta - \frac{ie\lambda}{m}[\underline{\pi}\cdot\underline{\alpha},\underline{E}\cdot\underline{\alpha}]_{-} +$$

$$+ \frac{e}{m}\frac{\partial \underline{E}\cdot\underline{\alpha}}{\partial t}(I-\beta) + \frac{ie}{m}[\underline{B}\times\underline{\pi} - \underline{\pi}\times\underline{B}]\cdot\underline{\alpha}(I+\beta) +$$

$$+ \frac{e}{m}[\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_{+}\beta + \frac{e\lambda}{m}[\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_{-} + Q\}\psi_1(x) = 0.$$

The first step in reducing the order of Equation 16 in the time derivative is now taken. This is done by defining $\psi_2(x)$ as

(17) $$\psi_2(x) = \exp(i\,mt)\,\psi_1(x)$$

and assuming

$$- i\pi_4\ \psi_2(x) << m\ \psi_2(x)\ .$$

This is equivalent to saying that the bulk of the time dependence of $\psi_1(x)$ is contained in a factor exp (-imt). This is consistent with the fact that the kinetic energy of a non-relativistic particle is much less than the particle's rest mass energy. In fact, it is consistent to say

$$- i\pi_4\ \psi_2(x) = \mathcal{O}\left(\frac{\pi^2}{m}\right)\ \psi_2(x),$$

that is, $i\pi_4\ \psi_2(x)$ is of order $\frac{1}{m}\ \psi_2(x)$. In choosing this time dependence, the wave function is now limited to describing particle states.

Substituting Equation 17 into Equation 16 and again neglecting all terms of order $1/m^2$ or higher give the equation

$$\{[(I+\beta) - \frac{2}{m}\,\underline{\pi}\cdot\underline{\alpha}\beta]\,\pi_4^{\,2} + 2im\pi_4(I+\beta) - 2i\underline{\pi}\cdot\underline{\alpha}\beta\pi_4 + m^2(I-\beta) + \underline{\pi}\cdot\underline{\pi}(I+\beta) - \frac{2}{m}\,\underline{\pi}\cdot\underline{\pi}\;\underline{\pi}\cdot\underline{\alpha}\beta + \frac{4}{3m}(\underline{\pi}\cdot\underline{\alpha})^3\beta + ie\underline{E}\cdot\underline{\alpha}\beta + ie(\lambda-2)\underline{E}\cdot\underline{\alpha} - e\underline{B}\cdot\underline{\Sigma}\beta - e\lambda\underline{B}\cdot\underline{\Sigma} + \frac{ie}{m}(1-\lambda)[\underline{\pi}\cdot\underline{\alpha},\underline{E}\cdot\underline{\alpha}]_- + \frac{e}{m}\frac{\partial \underline{E}\cdot\underline{\alpha}}{\partial t}(I-\beta) + \frac{ie}{m}[\underline{B}\times\underline{\pi} - \underline{\pi}\times\underline{B}]\cdot\underline{\alpha}(I+\beta) + \frac{e}{m}[\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_+\beta + \frac{e\lambda}{m}[\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_- + imQ_4\}\,\psi_2(x) = 0\,, \tag{18}$$

where

$$Q_4 = \frac{qe}{72m^2}(\partial_\nu H_{\alpha\beta})\,[[\gamma_{\alpha\rho},\gamma_{\beta\rho}]_-,\ [\gamma_{\nu\tau},\gamma_{4\tau}]_-]_+$$

or

$$Q_4 = \frac{qe}{m^2}(\partial_\beta H_{\alpha 4})\,\gamma_{\alpha\beta}\ .$$

Now Equation 18 is multiplied by $\frac{1}{2}(I+\beta)$ and $\frac{1}{2}(I-\beta)$ to give

$$\{2\pi_4^{\,2} + i\pi_4 + \frac{\pi\cdot\pi}{2m} - \frac{e}{4m}(1+\lambda)\underline{B}\cdot\underline{\Sigma} + \frac{ie}{4m^2}(1-\lambda)\,[\underline{\pi}\cdot\underline{\alpha},\underline{E}\cdot\underline{\alpha}]_- + \frac{i}{8}(I+\beta)\,Q_4\}\,\psi_L(x) + \tag{19a}$$

$$+ \{\frac{2}{m}\ \underline{\pi}\cdot\underline{\alpha}\pi_4{}^2 + \frac{i}{2m}\ \underline{\pi}\cdot\underline{\alpha}\pi_4 + \frac{1}{2m^2}\ \underline{\pi}\cdot\underline{\pi}\ \underline{\pi}\cdot\underline{\alpha} -$$

$$- \frac{1}{3m^2}\ (\underline{\pi}\cdot\underline{\alpha})^3 + \frac{ie}{4m}\ (\lambda-3)\underline{E}\cdot\underline{\alpha} + \frac{e}{2m^2}\ \frac{\partial\underline{E}\cdot\underline{\alpha}}{\partial t} -$$

$$- \frac{e}{4m^2}\ [\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_+ + \frac{e\lambda}{4m^2}\ [\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_- +$$

$$+ \frac{i}{8}\ (I+\beta)\ Q_4\}\ \psi_S(x) = 0$$

and

(19b)
$$\{- \frac{2}{m}\ \underline{\pi}\cdot\underline{\alpha}\pi_4{}^2 - i\underline{\pi}\cdot\underline{\alpha}\pi_4 - \frac{1}{m}\ \underline{\pi}\cdot\underline{\pi}\ \underline{\pi}\cdot\underline{\alpha} + \frac{2}{3m}(\underline{\pi}\cdot\underline{\alpha})^3 +$$

$$+ \frac{ie}{2}\ (\lambda-1)\underline{E}\cdot\underline{\alpha} + \frac{ie}{m}\ [\underline{B}\times\underline{\pi}-\underline{\pi}\times\underline{B}]\cdot\underline{\alpha} +$$

$$+ \frac{e}{2m}\ [\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_+ + \frac{e\lambda}{2m}\ [\underline{\pi}\cdot\underline{\alpha},\underline{B}\cdot\underline{\Sigma}]_- +$$

$$+ \frac{im}{4}\ (I-\beta)\ Q_4\}\ \psi_L(x) +$$

$$+\{m^2 + \frac{e}{2}\ (1-\lambda)\underline{B}\cdot\underline{\Sigma} + \frac{ie}{2m}\ (1-\lambda)\ [\underline{\pi}\cdot\underline{\alpha},\underline{E}\cdot\underline{\alpha}]_- +$$

$$+ \frac{im}{4}\ (I-\beta)\ Q_4\}\ \psi_S(x) = 0\ ,$$

respectively, where

$$\psi_L(x) = \frac{1}{2}\ (I+\beta)\ \psi_2(x)$$

and

$$\psi_S(x) = \frac{1}{2}\ (I-\beta)\ \psi_2(x)\ .$$

Equation 19b says that

$$\psi_S(x) = \mathcal{O}\,(\frac{\pi^2}{m^2})\ \psi_L(x)\ .$$

This means that to order $(1/m^2)$ $\psi_L(x)$, Equation 19a may be rewritten as

$$(20) \qquad -i\pi_4 \; \psi_L(x) = H\psi_L(x) + \frac{\pi_4^{\,2}}{2m} \psi_L(x) \; ,$$

where

$$H = \frac{\underline{\pi}\cdot\underline{\pi}}{2m} - \frac{e}{4m}(1+\lambda)\; \underline{B}\cdot\underline{\Sigma} + $$
$$+ \frac{e}{4m^2}\left[1-q-\tfrac{1}{2}(1+\lambda)\right](\partial_j E_i)\,[\Sigma_i,\Sigma_j]_+ +$$
$$+ \frac{e}{4m^2}\left[1-\tfrac{1}{2}(1+\lambda)\right]\left[\underline{E}\times\underline{\pi}-\underline{\pi}\times\underline{E}\right]\cdot\underline{\Sigma} \; .$$

Equation 20 may be rewritten in the form

$$-i\pi_4 \; \psi_L(x) = H\psi_L(x) + \frac{i\pi_4}{2m}\left(H + \frac{\pi_4^{\,2}}{2m}\right)\psi_L(x) \; .$$

To order $1/m^2$ this can be rewritten in the form

$$(21) \qquad i\frac{\partial}{\partial t}\Psi(x) = (H+e\varphi)\;\Psi(x) \; ,$$

where

$$\Psi(x) = \exp\,(H/2m)\;\psi_L(x)$$

and φ is the electric potential.

This last transformation, exp (H/2m), is not analogous to that used by Foldy and Wouthuysen(17). It should not be regarded as a further step in taking the non-relativistic limit of the wave function, but rather as a device for reducing the order in the time derivative of the wave equation.

Equation 21 and this Hamiltonian are true to order $(1/m^2)\Psi(x)$ and follow for either choice of Q_4. It should be noted here that the external fields were taken to satisfy the equation

$$\partial_\beta H_{\alpha\beta} = 0 \ .$$

Therefore no Darwin term, $\underline{\partial}\cdot\underline{E}$, appears in the Hamiltonian. If field sources were admitted into the problem, the quadrupole terms 2 and 3 would no longer be the same. Quadrupole terms 1 and 3 would give rise to a Darwin term and term 2 would not. All three terms, however, do give the same quadrupole term in the non-relativistic Hamiltonian.

The function $\Psi(x)$ is taken to be the non-relativistic wave function. It is related to the exact wave function by the following singular transformation

$$\Psi(x) = \frac{1}{2}\,(I+\beta)\,\exp\,(H/2m)\,\exp\,(-\underline{\pi}\cdot\underline{\alpha}/m)\times \times\exp\,(imt)\psi(x) \ . \tag{22}$$

An inverse of this transformation does not exist because of the presence of the matrix $\frac{1}{2}\,(I+\beta)$.

In Reference 15, Young and Bludman give the non-relativistic approximation of their generalized Sakata-Taketani Hamiltonian

$$H = e\varphi + m + \frac{\pi\cdot\pi}{2m} + \frac{e}{4m^2}(g-1+q)(\partial_j E_i)[\Sigma_i, \Sigma_j]_+$$

$$- \frac{eg}{2m}\underline{B}\cdot\underline{\Sigma} + (1-g)\frac{e}{4m^2}(\underline{E}\times\underline{\pi}-\underline{\pi}\times\underline{E})\cdot\underline{\Sigma}$$

to order $1/m^2$. If g were replaced by $\frac{1}{2}(1+\lambda)$, the form of this H would differ from our H only by the additive term m and the sign of the quadrupole term.

Examination of Equation 21 and the Young-Bludman Hamiltonian reveals the following. Equation 21 describes a spin-one particle of mass m, charge e, intrinsic magnetic moment e/4m (or a g factor of 1/2), and an intrinsic electric quadrupole moment of $-e/8m^2$ (or a Q factor of -1/2). On the other hand, the Young-Bludman Hamiltonian describes a spin-one particle of mass m, charge e, intrinsic magnetic moment e/2m (or a g factor of 1), and an intrinsic electric quadrupole moment of $e/4m^2$ (or a Q factor of 1). The intrinsic moments are defined here as those which arise in the Hamiltonian when $H_{\alpha\beta}$ does not appear explicitly in the covariant wave equation.

However, despite the similarity of the two Hamiltonians, the two formulations are not simply related. They are both obtained from two inequivalent generalizations of the Proca equations in different ways. These generalizations differ in the way that multipole terms are included and, correspondingly, the Hamiltonians differ in the multipole terms.

However, there is a very fundamental difference between the two formulations. The Young-Bludman generalized Sakata-Taketani formulation is not manifestly covariant but has an exact Hamiltonian. Our generalization of the Weinberg formulation is manifestly covariant but has no exact Hamiltonian. The generalization of the Weinberg equations is written as a second order, covariant wave equation which may, in the approximation given, be reduced to a first order Hamiltonian equation.

THE BARGMANN-WIGNER AND KEMMER FORMULATIONS

In the Bargmann-Wigner formulation a spin-one particle is described by the ten component wave function, $\psi^{BW}_{\zeta_1\zeta_2}$ where $\zeta_1, \zeta_2 = 1,2,3,4$ and $\psi^{BW}_{\zeta_1\zeta_2} = \psi^{BW}_{\zeta_2\zeta_1}$. While Bargmann and Wigner(21) wrote their equations in terms of momentum space wave functions, the configuration space wave functions will be used in the following. The relationship between the Bargmann-Wigner quantities and spinors is summarized in the following

$$\psi^{BW} = \begin{pmatrix} \psi^{BW}_1 \\ \psi^{BW}_2 \\ \psi^{BW}_3 \\ \psi^{BW}_4 \end{pmatrix} = \begin{pmatrix} \chi_{\dot{1}} \\ \chi_{\dot{2}} \\ \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} \chi \\ \varphi \end{pmatrix} = \psi(x) .$$

By means of this correspondence, the spinor equations, Equations 9, may be rewritten in the form

$$(23) \qquad \tfrac{1}{2}\, \pi_\alpha [(\gamma_\alpha)_{\zeta_1\rho_1} \delta_{\zeta_2\rho_2} + \delta_{\zeta_1\rho_1} (\gamma_\alpha)_{\zeta_2\rho_2}]\, \psi^{BW}_{\rho_1\rho_2} - im\, \psi^{BW}_{\zeta_1\zeta_2} = 0 ,$$

where

$$\gamma_\alpha = \begin{pmatrix} 0 & (-i\sigma_\alpha)^\dagger \\ -i\sigma_\alpha & 0 \end{pmatrix} .$$

This may also be rewritten

(24) $$\frac{1}{2}\,\pi_\alpha\,[\gamma_\alpha^{(1)} I^{(2)} + I^{(1)}\gamma_\alpha^{(2)}]\,\psi^{BW} - im\,\psi^{BW} = 0\ ,$$

where $\gamma_\alpha^{(i)}$ operates only in the i th index and $I^{(i)}$ is the unit matrix acting only on the i th index.

The wave function $\psi^{BW}_{\zeta_1\zeta_2}$ has ten components while $\chi_{\dot\alpha\dot\beta}$, $\varphi^{\alpha\beta}$, and $\theta_{\dot\alpha}{}^{\beta}$ also total up to ten components. In fact Equation 23 or 24 is equivalent to Equation 9 and so is equivalent to the Proca equation with fields.

Another point to be made is that if β_α is defined as

$$\beta_\alpha = \frac{1}{2}\,(\gamma_\alpha^{(1)} I^{(2)} + I^{(1)}\gamma_\alpha^{(2)})\ ,$$

then

(25) $$\beta_\alpha\beta_\nu\beta_\epsilon + \beta_\epsilon\beta_\nu\beta_\alpha = \delta_{\alpha\nu}\beta_\epsilon + \delta_{\nu\epsilon}\beta_\alpha\ ;$$

and Equation 24 may be rewritten in the form

(26) $$(\pi_\alpha\beta_\alpha - im)\,\psi^{BW} = 0\ .$$

Equation 26 is precisely the form of the Kemmer equation,

$$(\pi_\alpha\beta'_\alpha - im)\,\psi^{K} = 0\ ,$$

where the 10x10 matrices β'_α satisfy Equation 25. The Kemmer wave function has ten components, and there exists a numerical matrix T such that

$$\psi^{BW} = T\psi^{K}.$$

The specific form of T depends on the manner in which the components $\psi^{BW}_{\zeta_1\zeta_2}$ are ordered into a column matrix.

In the field free case, Equation 24 becomes

$$(27) \qquad \frac{1}{2}\, p_\alpha \, [\gamma^{(1)}_\alpha I^{(2)} + I^{(1)}\gamma^{(2)}_\alpha]\, \psi^{BW} - im\, \psi^{BW} = 0 \; .$$

This does not, at first glance, appear to be the same as the Bargmann-Wigner equations,

$$(28) \qquad p_\alpha \gamma^{(i)}_\alpha \psi^{BW} - im\, \psi^{BW} = 0 \; ,$$

for i = 1,2. However, it can easily be shown that Equation 27 implies

$$(p_\alpha p_\alpha + m^2)\psi^{BW} = 0$$

for $m \neq 0$, and this then implies

$$p_\alpha \gamma^{(1)}_\alpha \psi^{BW} = p_\alpha \gamma^{(2)}_\alpha \psi^{BW} \; .$$

So, Equation 24 corresponds to a generalization of the Bargmann-Wigner equations to include external fields, and this generalization is essentially the same as the Kemmer equation.

The replacement of p_α by π_α in Equation 28 will not lead to a good equation. This replacement is exactly equivalent to replacing ∂_α by D_α in the Dirac-Pauli-Fierz equations and leads to an additional condition on the wave function involving the fields which eliminates some states. In particular, it leads to the condition

$$H_{\alpha\beta}\gamma_{\alpha}^{(1)}\gamma_{\beta}^{(2)} \psi^{BW} = 0 .$$

However, by combining the Bargmann-Wigner equations in the form of Equation 27, which is entirely equivalent to Equation 28, the simple replacement of p_{α} by π_{α} does lead to a consistent theory, the Kemmer theory.

CURRENT CONSERVATION, THE INVARIANT INTEGRAL, AND THE HEISENBERG PICTURE

A conserved four-vector current, i.e., one whose four-divergence vanishes, may be found from Equation 11 by multiplying it on the left by $\overline{\psi}(x) = \psi^{\dagger}(x)\beta$ and subtracting the hermitian conjugate of the expression. The result is that

$$\partial_{\nu} J_{\nu}(x) = 0 ,$$

where

$$J_{\alpha}(x) = \overline{\psi}(x)\pi_{\beta}\gamma_{\alpha\beta}\psi(x) - [\overline{\pi}_{\beta}\overline{\psi}(x)]\gamma_{\alpha\beta}\psi(x) + \tag{29}$$

$$+ \overline{\psi}(x)\pi_{\alpha}\psi(x) - [\overline{\pi}_{\alpha}\overline{\psi}(x)]\,\psi(x) +$$

$$+ \overline{\psi}(x)\, Q_{\alpha}\psi(x) .$$

In the above equation

$$Q_{\alpha} = \frac{qe}{6m^2}\,(\partial_{\beta}H_{\nu\epsilon})\;\gamma_{6,\alpha\beta,\epsilon\nu}$$

or

$$Q_{\alpha} = \frac{qe}{m^2}\,(\partial_{\beta}H_{\nu\alpha})\;\gamma_{\beta\nu}$$

and

$$\overline{\pi}_{\alpha} = \frac{1}{i}\,\partial_{\alpha} + eA_{\alpha} .$$

Some additional terms may be added to the current vector, for example, terms like

$$J^{D}_{\alpha}(x) = \partial_{\nu}\,[\overline{\psi}(x)[\gamma_{\alpha\beta},\gamma_{\nu\beta}]_{-}\,\psi(x)]\ ,$$

$$J^{Q}_{\alpha}(x) = \partial_{\nu}\partial_{\beta}\,[\overline{\psi}(x)\pi_{\epsilon}\gamma_{6,\alpha\beta,\epsilon\nu}\psi(x)]\ ,$$

and

$$J^{Q'}_{\alpha}(x) = \partial_{\nu}\,[\overline{\psi}(x)\partial_{\beta}H_{\epsilon\nu}\gamma_{\beta\epsilon}\psi(x)]\ .$$

These vectors are conserved in themselves,

$$\partial_{\nu}J^{D}_{\nu}(x) = \partial_{\nu}J^{Q}_{\nu}(x) = \partial_{\nu}J^{Q'}_{\nu}(x) = 0\ ;$$

and, while they contribute to the current density, they contribute nothing to the total current, $\int d^3x\ J_{\alpha}(x)$, integrated over all space.

The integral over all configuration space of the fourth component of a conserved four-vector is a Lorentz invariant quantity. So, for this theory, the invariant integral is given by

$$(30)\qquad (\psi_r,\psi_s) = \frac{1}{4im}\int d^3x\ \{\overline{\psi}_r(x)M_4\psi_s(x) - \overline{[M_4\psi_r(x)]}\ \psi_s(x)\}\ ,$$

where

$$M_4 = \pi_{\alpha}\gamma_{4\alpha} + \pi_4 + \frac{1}{2}\,Q_4$$

and

$$Q_4 = \frac{qe}{6m^2}\,(\partial_i H_{\alpha\beta})\ \gamma_{6,\alpha\beta,i4}$$

or

$$Q_4 = \frac{qe}{m^2} (\partial_\nu H_{i4}) \gamma_{\nu i} \; .$$

The matrix element of a general operator T may be defined in the following manner

(31) $$T_{rs} = (\psi_r, T\psi_s) = \frac{1}{4im} \int d^3x \; \{\overline{\psi}_r(x) M_4 T\psi_s(x) - \overline{[M_4 \psi_r(x)]} \; T\psi_s(x)\}.$$

The time rate of change of T_{rs} is

(32) $$\frac{dT_{rs}}{dt} = \frac{1}{4im} \int d^3x \; \overline{\psi}_r(x) \; [T,W]_- \; \psi_s(x) \; ,$$

where

$$W = \pi_\alpha \pi_\beta \gamma_{\alpha\beta} + \pi_\alpha \pi_\alpha + m^2 + \frac{ie\lambda}{12} H_{\alpha\beta} [\gamma_{\alpha\tau}, \gamma_{\beta\tau}]_- + Q \; .$$

This is true whether or not T is time dependent. So $W\psi(x) = 0$ is the wave equation, Equation 11, and the above holds for both forms of Q. In this formulation the equation of motion of an operator in the Heisenberg picture has a simple form. This occurs even though an exact Hamiltonian does not exist. However, since no Hamiltonian exists, the operator M_4 is non-hermitian and, in the zero field limit the invariant integral is not positive definite.

The non-relativistic limit of the invariant integral of Equation 21 can be found. By neglecting terms of order

$1/m^3$ or higher, this integral becomes, with the help of Equation 22,

$$(\psi_r, \psi_s) = \int d^3x \ \Psi_r^+(x) \ \Psi_s(x) \ .$$

This has a nice positive definite form; however, it must be remembered that the non-relativistic limit taken here implies a choice of particle states only.

DISCUSSION AND CONCLUSIONS

The covariant equations with external fields which are generalizations of the Proca, Weinberg, and Bargmann-Wigner free particle equations differ only in the way that the anomalous dipole and quadrupole terms appear. This occurs because the free particle wave functions which satisfy these equations are all connected by a constant transformation "matrix" which is unaffected by the addition of external fields. The resulting equations are inequivalent only because of the different ways in which the anomalous moments are added. So the problem of consistently inserting fields into all spin-one equations has not been solved in general. A solution has been shown for a group of simply related equations.

For the formulations of Joos(1) and Weinberg(2), in particular, the covariant Weinberg equations, the difficulty of inserting external fields into the wave equations arises because the free particle wave function is a simultaneous solution of two differential equations. Merely replacing p_α by π_α in these equations destroys their consistency. If an attempt were made to avoid this problem by considering only the wave equation, Equation 3a, then the resulting equation would have twelve solutions. An additional condition on the wave function would then be needed to separate the physical from the non-physical solutions. Equation 3b plays this role in the free particle case, but a corresponding equation with fields is unknown. So one is right back to the problem of finding two mutually consistent equations with fields.

For the spin-one Weinberg equations it is possible to combine Equations 3a and 3b in such a manner that the resulting equation has only the six physical solutions. It is possible to insert the external fields into this single equation in a straight forward manner without mutual consistency problems. This same type of procedure works for the spin-one, Bargmann-Wigner(21) equations yielding an equation with fields which is essentially the Kemmer(13) equation. It is also possible to insert fields in the Dirac-Pauli-Fierz(18,19) spinor equations by writing them in a more symmetric form, Equations 9, than is customary. These combined and symmetrized equations are related directly to the Proca equations by the constant transformations between the wave functions.

The anomalous magnetic dipole and electric quadrupole terms can be included in the generalization, Equation 11, of the Weinberg equations. The dipole term presents no problem; but there is a choice of two quadrupole terms, one involving $(\partial_\nu H_{\alpha\beta})\pi_\beta \gamma_{\nu\alpha}$ and the other involving $(\partial_\nu H_{\alpha\beta})$

$\pi_\mu \gamma_{6,\alpha\beta,\nu\mu}$. On the basis of inversion and charge conjugation properties and the contribution to the non-relativistic, approximate Hamiltonian, there is no reason for choosing one above the other. However, the first choice is peculiar to spin-one because of the $\gamma_{\alpha\beta}$, while the latter choice may be generalized to any spin, since a matrix with the symmetry and transformation properties of $\gamma_{6,\alpha\beta,\nu\mu}$ appears in the set of covariant matrices for any spin $\geq$ one. On the basis of its more general usefulness, the second choice is much more desirable than the first.

It is possible to get a non-relativistic limit of the covariant wave equation with fields and anomalous moments by using a transformation analogous to that used by Foldy and Wouthuysen(17) for the spin-1/2 case. Further, it is possible to reduce the order of the resulting equation to get a Hamiltonian equation correct to order $1/m^2$, Equation 21. This Hamiltonian explicitly displays the anomalous and intrinsic multipole moments, and it bears a striking resemblance to the non-relativistic, generalized, Sakata-Taketani Hamiltonian given by Young and Bludman(15) also correct to order $1/m^2$.

In spite of this similarity, the generalized Weinberg and the Young-Bludman formulations are not equivalent. They both are related to the Proca(12) equations, but in quite a different way. Essentially the difference arises from the fact that the Proca equations may be rewritten entirely in terms of the four-vector U_α or the antisymmetric tensor $U_{\alpha\beta}$. Either of these quantities alone is sifficient to describe all the states of a spin-one particle. The generalized Weinberg equation is related to the generalization of the Proca equation involving $U_{\alpha\beta}$. The Young-Bludman equations represent a generalization of the other Proca equation which involves U_α. Although the Young-Bludman equations are written in terms of both U_α and $U_{\alpha\beta}$, the tensor is merely a subsidiary quantity which is completely determined by the U_α.

A conserved four-vector current, Equation 29, follows from the generalized Weinberg equation, Equation 11, and, from this, it is possible to derive an invariant integral, the matrix element of an operator, and the time derivative of this matrix element, Equations 30, 31, and 32, respectively. This invariant integral is not positive definite. This occurs because a Hamiltonian does not exist. This same situation arises in the usual Joos and Weinberg formulation for spin-one.

Weaver, Hammer, and Good(3) were able to get a Hamiltonian formulation and Sankaranarayanan and Good(6) a covariant formulation which are both equivalent to that of Joos and Weinberg. The covariant equation was obtained by inserting an energy sign operator $\frac{i\partial/\partial t}{E}$, $E = (-\nabla^2 + m^2)^{1/2}$, into the Weinberg wave equation (see Equation 4a). In this case a Hamiltonian exists and the invariant integral is positive definite. Fields have not been put into this formulation yet. The difficulty arises from the presence of E in the denominator of the Hamiltonian and of the energy sign operator in the wave equation. It has not yet been decided what E becomes when fields are inserted. Its presence in the denominator of the equations complicates the problem.

Now for the question of whether the process of inserting fields into the spin-one equations can be generalized to higher spins. Unfortunately, as such, the process discussed here is unsatisfactory for higher spins. In the higher spin case, one has the same problem of the mutual consistency of two (or more in the Bargmann-Wigner formulation) differential equations that occurs for spin-one. If these equations are combined into one equation, just as was done here, the resulting equation has extraneous solutions. This holds true for the Weinberg and the Bargmann-Wigner equations. It seems to be a characteristic of the spin-one case, only, that this combination introduces no extra solutions.

The problem of what to do is a difficult one. The problem of keeping two or more equations consistent when inserting fields is unsolved. It may be that there is a more general way of combining the equations into one, which will not introduce extra solutions for spins $\geq 3/2$. Only a special case of this process, that for spin-one, is known so far; and it is the simplest, non-trivial case.

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APPENDIX

The covariance of Equation 11 under the proper Lorentz transformations follows immediately from the property

$$\Lambda^{-1}\gamma_{\alpha\beta}\Lambda = a_{\alpha\rho}a_{\beta\nu}\gamma_{\rho\nu} ,$$

where

$$\psi'(x') = \Lambda\psi(x)$$

and

$$x'_{\alpha} = a_{\alpha\beta}x_{\beta} .$$

It remains, however, to verify the covariance of Equation 11 under space and time inversion and charge conjugation.

According to Weinberg(2), the spin-one wave function has the following properties for space inversion

$$x'_{i} = -x_{i}, \quad t' = t ,$$

and

$$\psi'(x') = \beta\psi(x) ;$$

for charge conjugation

$$x'_{\alpha} = x_{\alpha}$$

and

$$\psi'(x') = [C\psi(x)]^{*} ,$$

where

$$C = \begin{pmatrix} 0 & C_s \\ C_s & 0 \end{pmatrix} ,$$

$$C_s s_i = - s_i^* C_s ,$$

and * denotes complex conjugate; for time inversion

$$x_i' = x_i, \quad t' = -t ,$$

and

$$\psi'(x') = \beta[C\psi(x)]^*$$

Multiplying Equation 11 by β and using

$$\beta\gamma_{44}\beta = \gamma_{44} , \quad \beta\gamma_{4i}\beta = -\gamma_{4i} ,$$

and

$$\beta\gamma_{ij}\beta = \gamma_{ij}$$

give

(A.1) $$\{\pi_\alpha'\pi_\beta'\gamma_{\alpha\beta} + \pi_\alpha'\pi_\alpha' + 2m^2 + \frac{ie\lambda}{12} H_{\alpha\beta}' [\gamma_{\alpha\tau},\gamma_{\beta\tau}]_- + \frac{ae}{6m^2} (\partial_\nu' H_{\alpha\beta}') \pi_\tau' \gamma_{6,\alpha\beta,\nu\tau}\} \psi'(x') = 0 ,$$

where

$$\pi_i' = -\pi_i , \quad \pi_4' = \pi_4 ,$$

$$x_i' = -x_i , \quad x_4' = x_4 ,$$

$$H_{4i}' = -H_{4i} , \quad H_{ij}' = H_{ij} ,$$

and

$$\psi'(x') = \beta\psi(x) .$$

This equation shows the proper covariance under a spatial inversion.

Now, multiplying Equation 11 by C, taking the complex conjugate of the result, and using

$$C^* = C^{-1}$$

and

$$C\gamma_{44}C^{-1} = \gamma_{44}^* , \quad C\gamma_{4i}C^{-1} = -\gamma_{4i}^* ,$$

$$C\gamma_{ij}C^{-1} = \gamma_{ij}^* ,$$

give an equation of the form of Equation A.1; but where

$$x'_\alpha = x_\alpha ,$$

$$H'_{\alpha\beta} = H_{\alpha\beta} ,$$

$$\pi'_\alpha = p_\alpha + eA_\alpha ,$$

and

$$\psi'(x') = [C\psi(x)]^* .$$

So the charge conjugation of Equation 11 merely sends e into -e. Equation 11 is therefore covariant under charge conjugation.

Finally, the multiplication of Equation 11 by βC and taking the complex conjugate also give Equation A.1. In this case one has

$$\pi'_\nu = p'_\nu + eA'_\nu \ , \ p'_\nu = \frac{1}{i} \ \frac{\partial}{\partial x'_\nu}$$

$$x'_i = x_i \ , \quad t' = -t \ ,$$

$$A'_i = -A_i \ , \quad A'_4 = A_4 \ ,$$

$$H'_{ij} = -H_{ij} \ , \quad H'_{4i} = H_{4i} \ ,$$

and

$$\psi'(x') = \beta[C\psi(x)]^* \ .$$

Equation 11 is also covariant under a combined time reversal. So, as a result, Equation 11 is covariant with respect to both proper and improper Lorentz transformations and charge conjugation. While this was shown here for one choice of Q, it is also true for the other choice.