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On the embedding of a centerless group in its automorphism group

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ON THE EMBEDDING OF A CENTERLESS GROUP IN ITS AUTOMORPHISM GROUP

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NOTATION

We use $A \subset B$ to denote A is a subgroup of B or, in other context, A is a submodule of B.

$A \overset{\alpha}{\rightarrowtail} B$ denotes that α is a monomorphism, $C \overset{\delta}{\twoheadrightarrow} D$ denotes that δ is an epimorphism and $M \overset{\gamma}{\rightarrowtail\!\!\!\twoheadrightarrow} N$ denotes that γ is both a monomorphism and an epimorphism.

For modules, $A \longrightarrow B \longrightarrow C$ (exact) denotes that the kernel of the homomorphism going out from B equals the image of the map coming into B. Hence $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ (exact) denotes that α is a monomorphism and $C \overset{\beta}{\longrightarrow} D \longrightarrow 0$ (exact) denotes that β is an epimorphism.

A diagram

$$\begin{array}{ccc} A & \overset{\alpha}{\longrightarrow} & B \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\delta} & \\ & 0 & \end{array}$$

is commutative if the composite $\delta\alpha$ equals γ.

$A \lhd B$ denotes that A is a normal subgroup of B.

If $A \overset{\alpha}{\longrightarrow} B$, im($\alpha$) denotes the set of all b in B such that for some a in A, $\alpha(a) = b$.

Z(A) denotes the center of group A. $C_G(A)$ denotes the centralizer of A in G; but if no confusion is likely, the subscript G will be omitted.

If $A \subset C$ and $B \subset C$, $A \cup B$ denotes the smallest subgroup of C that contains A and B. If S is a subset of G, $\langle S \rangle$ is the smallest subgroup of G that contains all of the elements of S.

If G is any group, $\mathcal{A}(G)$ denotes it group of automorphisms and $\mathcal{J}(G)$ denotes its group of inner automorphisms. The usual map of G into $\mathcal{A}(G)$ will in general be denoted by u. Thus for each g in G,

$u(g) = u_g$, where $u_g(x) = g^{-1} x g$ for each x in G.

A group G is called centerless if Z(G) is the identity subgroup of G. This will be denoted by Z(G) = 1. A group G is called complete if it is centerless and has $\mathcal{A}(G) = \mathcal{I}(G)$.

If N = A x B, $proj_A$ is the homomorphism from A x B into A such that $proj_A(a,b) = a$.

For integers t, Z(t) denotes the cyclic group of order t.

$|A|$ denotes the order of group A.

INTRODUCTION

In this paper we investigate some analogies for arbitrary groups to the results of Eckmann and Schopf on injective modules (2).

By definition, an A-module, M, is injective if whenever A-modules P and R satisfy $P \subset R$ and there is an A-homomorphism $\eta : P \longrightarrow M$, then there is an A-homomorphism $\alpha : R \longrightarrow M$ such that $\alpha |_P = \eta$ (i.e., such that the diagram,

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{id} & R \\ & & \eta \downarrow & \swarrow \alpha & \\ & & M & & \end{array}$$

(exact), is commutative).

In that work (2) it was proven that every left A-module M can be embedded in a "smallest" injective left A-module which is called "an injective hull for M". Precisely, a left A-module N is an injective hull of left A-module $M \subset N$ if N is injective and if for any injective left A-module R satisfying $M \subset R \subset N$, we have R = N. Also, by defining a related extension of a left A-module M to be a left A-module N such that $0 \longrightarrow M \xrightarrow{\alpha} N$ (exact) and such that if $K \subset N$ and $K \cap \operatorname{im}(\alpha) = 0$ then K = 0, it was shown that T is an injective hull for M if and only if $M \subset T$ is a related extension of M with the property that for any other related extension of M, say S, if $0 \longrightarrow T \xrightarrow{\delta} S$ (exact) is such that $\delta(m) = m$ for each m in M, then $\delta(T) = S$. It was also shown that an injective hull for a left A-module is "unique up to isomorphism", and, in fact, if we consider any two injective hulls K and W for M such that $0 \longrightarrow M \xrightarrow{id} W$ and $0 \longrightarrow M \xrightarrow{id} K$ are both exact sequences, then there is an isomorphism γ between K and W such that $\gamma |_M = id$.

We may observe that an injective hull for M is a related extension T for M, with $0 \longrightarrow M \xrightarrow{id} T$ (exact), such that if S is any other related extension, say $0 \longrightarrow M \xrightarrow{\gamma} S$ (exact), then there is a homomorphism (which must then be a monomorphism, since S is a related extension) $\eta : S \longrightarrow T$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & 0 & & \\ & & \searrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\gamma} & S \text{ (exact)} \\ & & {\scriptstyle id}\searrow & & \swarrow{\scriptstyle \eta} \\ & & & T & \\ & & & \text{(exact)} & \end{array}$$

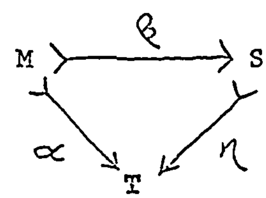
Certainly, if T is an injective hull for M then T is a related extension for M. Also the existence of the map η is assured since T is injective.

To establish the fact that these conditions are sufficient for T to be an injective hull for M, we notice that M has an injective hull, say H, which is a related extension, so there is an isomorphism η : $H \longrightarrow T$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & 0 & & \\ & & \searrow & & \\ 0 & \longrightarrow & M & \xrightarrow{id} & H \\ & & {\scriptstyle id}\searrow & & \swarrow{\scriptstyle \eta} \\ & & & T & \end{array}$$

Now by the result of Eckmann and Schopf mentioned above, H is a related extension of M and if $0 \longrightarrow M \xrightarrow{\delta} N$ (exact) is another related extension of M and $0 \longrightarrow H \xrightarrow{\alpha} N$ (exact) and $\alpha(m) = m$ for each m in M, then $\alpha(H) = N$. Hence, since $0 \longrightarrow M \xrightarrow{id} T$ (exact) is a related extension of M and $0 \longrightarrow H \xrightarrow{\eta} T$ is exact and (by the commutativity of the diagram above) $\eta(m) = m$ for each m in M, $\eta(H) = T$. Thus T is an injective hull for M.

Now suppose we consider arbitrary groups and see if we can characterize those groups M for which there is a group T such that $M \overset{\alpha}{\rightarrowtail} T$ is analogous to a related extension for M, and such that if $M \overset{\beta}{\rightarrowtail} S$ is any other embedding which is analogous to a related extension for M, then there is a monomorphism $\eta: S \rightarrowtail T$ such that the following diagram is commutative:



If such an embedding of M in T exists we could call T a "hull" for M.

Now several possibilities exist for the analogies to a related extension. We could consider an embedding $M \overset{\alpha}{\rightarrowtail} T$ such that no non-trivial subgroup of T is disjoint from $\alpha(M)$. Alternately we might require that the image of α be normal in T and that no non-trival normal subgroup, $K \lhd T$, satisfies $K \cap \alpha(M) = 1$. Or we might require that the image of α be normal in T and that no subgroup of the centralizer of im(α) be disjoint from im(α). The latter two of these alternatives are discussed in this paper. They are equivalent if M is centerless, which is interesting in light of the following characterization which we obtain for finite groups:

> In the category of finite groups, the only "hull" is the embedding of a finite centerless group in its automorphism group.

In obtaining this result we show that if a finite group G has a non-trivial center, then G can be embedded in a group with arbitrarily large order in such a way that the embedding is both an essential embedding and a related embedding. As a corollary of this result, a theorem of Baer (11, p. 94) is extended for finite groups:

> Let G be finite. G is a direct factor of any group in which it appears as a normal subgroup if and only if G is complete.

In the third chapter a "restricted hull" is introduced in an attempt to produce a characterization in terms of the embedding problem when infinite groups are allowed. Not all embeddings of a finite centerless group into its automorphism group satisfy the requirements for being a restricted hull, and a characterization is given for those usual embeddings which satisfy the restricted definition. The restricted definition remains an analogy to the hull problem for modules since only an additional normality condition is added to the original definition of a hull.

Except for the case of abelian 2-groups with no repeated cyclic factors, we show that a finite abelian group does not have a restricted hull. The situation for those particular abelian 2-groups remains an open question. Using a result which we obtain for groups having a particular type of fixed point free automorphism, we are able to show that some non-abelian groups (for example, those of order p^3 for odd prime p) do not have restricted hulls.

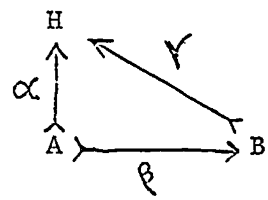
In chapter four we consider a mapping problem which is a dual to the embedding problem. Interchanging monomorphisms and epimorphisms we form a problem of finding "covers" for finite groups. The "covering" of group A by group B is the dual of essentially embedding a group A in a group B, and a "maximal cover" for group A is the dual of a hull for group A.

Whether or not there are any finite groups that have maximal covers remains an open question. We show that in the category of finite groups no abelian group has a maximal cover. We obtain a characterization of a finite cover for A, say $B \xrightarrow{\alpha} A$, in terms of Ker(α) and the Frattini subgroup of B. $B \xrightarrow{\alpha} A$ is a cover $\Longleftrightarrow$ Ker(α) $\subset$ Fr(B). Using this characterization, we show that many classes of finite groups can only be "covered" by groups in that same class. In particular, we show this for cyclic, perfect, nilpotent and solvable groups.

HULLS IN FINITE GROUP THEORY

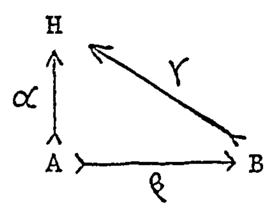
<u>Definition II.1</u> $1 \neq B \overset{\alpha}{\rightarrowtail} G$ is an essential embedding of the group B in the group G if $\mathrm{im}(\alpha) \lhd G$ and if for each $1 \neq N \lhd G$, $N \cap \mathrm{im}(\alpha) \neq 1$.

<u>Definition II.2</u> $1 \neq A \overset{\alpha}{\rightarrowtail} H$ is a hull for A if it is an essential embedding of A in H and if, for each essential embedding $A \overset{\beta}{\rightarrowtail} B$, we have $\gamma : B \rightarrowtail H$ such that the following diagram is commutative:



<u>Definition II.3</u> $1 \neq A \overset{\alpha}{\rightarrowtail} B$ is a related embedding if $\mathrm{im}(\alpha) \lhd B$ and if for each $1 \neq S \subset C(\mathrm{im}(\alpha))$, $S \cap \mathrm{im}(\alpha) \neq 1$.

<u>Definition II.4</u> $1 \neq A \overset{\alpha}{\rightarrowtail} H$ is a related hull for A if it is a related embedding and if, for each related embedding $A \overset{\beta}{\rightarrowtail} B$ there exists $\gamma : B \rightarrowtail H$ such that the following diagram is commutative:



<u>Remark II.5</u> Definitions II.1 and II.3 are equivalent for any non-trivial centerless group A. That is, $A \overset{\alpha}{\rightarrowtail} B$ is a related embedding if and only if $A \overset{\alpha}{\rightarrowtail} B$ is an essential embedding.

Proof: For any two disjoint normal subgroups X and Y, we have $X \subset C(Y)$. Thus, if $A \overset{\alpha}{\rightarrowtail} B$ is a related embedding then $A \overset{\alpha}{\rightarrowtail} B$ is an essential embedding.

The fact that if $A \overset{\alpha}{\rightarrowtail} B$ is an essential embedding then it is also a related embedding is an obvious corollary of the following remark.

Remark II.6 For N centerless, $N \triangleleft G$ is an essential embedding $\Leftrightarrow$ $C(N) = 1$.

($\Leftarrow$) Suppose $H \triangleleft G$ and $H \cap N = 1$. Then for each h in H and for each n in N, $hnh^{-1}n^{-1}$ is in $H \cap N$. Thus $hnh^{-1}n^{-1} = 1$, so $hn = nh$. But then $H \subset C(N)$, so $H = 1$.

($\Rightarrow$) Suppose a is in $C(N) \cap N$. Then $an = na$ for each n in N. Then $a = 1$, since N is centerless.

It is also easily seen that Definition II.1 and Definition II.3 are equivalent in case $A \overset{\alpha}{\rightarrowtail} B$ with A and B abelian.

Remark II.7 Definition II.3 is more restrictive than Definition II.1.

Proof: Consider the dihedral group, D, with 8 elements. We can consider $D = \langle a,b \rangle$ where $a^4=1=b^2$ and $ab = ba^3$. The center of D is $\{a^2,1\}$ and is contained in all of the proper normal subgroups of D. Hence $Z(2) \overset{\alpha}{\rightarrowtail} D$ with $\alpha(\bar{0}) = 1$ and $\alpha(\bar{1}) = a^2$ is an essential embedding of Z(2) in D. But the cyclic group generated by b is contained in $C(\mathrm{im}(\alpha)) = D$, and is disjoint from Z(D).

Some other examples of essential embeddings are: $Z(p) \overset{\alpha}{\rightarrowtail} Z(p^{\infty})$ for arbitrary prime p, with $\alpha(\bar{i}) = i/p$, $i = 0,1,2,\ldots \quad \ldots,p-1$; $Z(2) \overset{\varphi}{\rightarrowtail} Q_8$ with $\mathrm{im}(\varphi)$ being the center of the quaternion group Q_8. $Z(p) \overset{\gamma}{\rightarrowtail} G$, where G is the dihedral group of order 2p, is an essential embedding with im(γ) being the Sylow subgroup (unique) of order p in G. Z(2) can be relatedly embedded in arbitrary quaternion groups, Q_{2^n}, since $Z(Q_{2^n}) \approx Z(2)$ and any proper normal subgroup of a finite p-group cannot be disjoint from its center (11, p. 139).

<u>Remark II.8</u> If $1 \neq A \subset B$ and

i) $\{B_\lambda : \lambda \in \Lambda\}$ is an inclusion chain of subgroups of B, each containing A

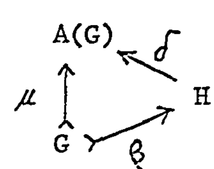
ii) $B = \bigcup_{\lambda \in \Lambda} B_\lambda$

iii) A is relatedly (or essentially) embedded in each B_λ

then A is relatedly (essentially) embedded in B.

Proof: Suppose A is not essentially embedded in B. Then there is $S \subset B$ such that $1 \neq S \triangleleft B$ and $S \cap A = 1$. Notice if g is in B then g is in some B_λ. But $A \overset{inc}{\rightarrowtail} B_\lambda$ is an essential embedding, so $g^{-1}Ag = A$. Thus, $A \triangleleft B$. Now, if $1 \neq s$ is in S, then for some λ in Λ, s is in B_λ. But, $1 \neq (S \cap B_\lambda) \triangleleft B_\lambda$ and $(S \cap B_\lambda) \cap A = 1$ yields a contradiction. Similarly, if $1 \neq S \subset C_B(A) \subset B$, consider $1 \neq s$ in S. Notice that for some λ in Λ, s is in B_λ and hence s is in $C_{B_\lambda}(A)$. But then $\langle s \rangle \subset C_{B_\lambda}(A)$ and $\langle s \rangle \cap A = 1$ yields a contradiction.

<u>Remark II.9</u> The usual embedding of a centerless group in its automorphism group is a hull. If G is a finite centerless group then $G \overset{\mu}{\rightarrowtail} \mathcal{A}(G)$ is a unique hull for G, in the sense that if $G \overset{\beta}{\rightarrowtail} H$ is any hull for G then there is a unique homomorphism $\delta : H \longrightarrow \mathcal{A}(G)$ making the diagram



commutative, and δ is

an isomorphism.

Proof: It is well known that $\mathrm{im}(u) \triangleleft \mathcal{A}(G)$, and that if G has a trivial center, then the centralizer of u(G) in $\mathcal{A}(G)$ must be trivial (10, p. 81). Hence we cannot have $1 \neq N \triangleleft \mathcal{A}(G)$ with $N \cap \mathrm{im}(u) = 1$.

Now suppose $G \xrightarrow{id} H$ is a hull for G. Let $\delta: H \longrightarrow \mathcal{A}(G)$ such that $\delta(h): G \longrightarrow G$ where for each g in G, $(\delta(h))(g) = h^{-1}gh$. Hence $\delta\big|_G = u$. Obviously δ is a monomorphism, since $\mathrm{Ker}(\delta) \cap G = 1$, and δ preserves products.

Now suppose $\beta: H \longrightarrow \mathcal{A}(G)$ and $\beta \circ id = u$. Now if h is in H and g is in G, then $\beta(hgh^{-1}) = \alpha(hgh^{-1})$. Thus $\beta(h)\, u(g)\, \beta(h)^{-1} = \alpha(h)u(g)\alpha(h)^{-1}$. Hence $\left[\alpha(h)^{-1}\beta(h)\right] u(g) \left[\beta(h)^{-1}\alpha(h)\right] = u(g)$.

Thus $\alpha(h)^{-1}\beta(h)$ is in $C(\mathrm{im}(u)) \subset \mathcal{A}(G)$. But $G \xrightarrow{u} \mathcal{A}(G)$ was shown to satisfy $C(\mathrm{im}(u)) = 1$. Thus $\alpha(h)^{-1}\beta(h) = 1$, for each h in H, and β is unique.

If G is finite, then $\mathcal{A}(G)$ is finite and since H is a hull, there is an isomorphism from $\mathcal{A}(G)$ into H, so δ must be an epimorphism.

Since G was assumed to be centerless, definitions II.3 and II.4 hold also and $G \xrightarrow{u} \mathcal{A}(G)$ is a related hull for G.

This remark is due to Head (5). The ideas were adapted from a paper of Fitting (3) which is discussed in Kurosh (7, pp. 202-209).

<u>Proposition II.10</u> If $A \xrightarrow{\alpha} B$ is an essential embedding and $B \xrightarrow{\beta} C$ is an essential embedding, and $\mathrm{im}(\beta\alpha) \triangleleft C$, then $A \xrightarrow{\beta\alpha} C$ is an essential embedding.

Proof: Assume $1 \neq K \triangleleft C$ and $K \cap \mathrm{im}(\beta\alpha) = 1$. Then since $B \xrightarrow{\beta} C$ is an essential embedding, we have $1 \neq K \cap \text{im}(\beta) = K' \triangleleft \text{im}(\beta)$. But then $1 \neq K' \triangleleft \text{im}(\beta)$ and $\text{im}(\beta\alpha) \cap K' \subset \text{im}(\beta\alpha) \cap K$. Hence $1 = \beta^{-1}[\text{im}(\beta\alpha) \cap K'] = \beta^{-1}[\text{im}(\beta\alpha)] \cap \beta^{-1}(K') = \text{im}(\alpha) \cap \beta^{-1}(K')$. But this is a contradiction, since $1 \neq \beta^{-1}(K') \triangleleft B$.

Proposition II.11 If $A \overset{\alpha}{\rightarrowtail} B$ is an essential embedding then CxA CxB is an essential embedding.

Proof: Suppose $1 \neq K \triangleleft CxB$ and $K \cap (C \times \text{im}(\alpha)) = 1$. Now if k is in K and not in C, then $k = (c,b)$ where b is not in $\text{im}(\alpha)$. Therefore $\text{proj}_B(k)$ is not in $\text{im}(\alpha)$. That is, $\text{proj}_B(K) \cap A = 1$.

But $\text{proj}_B(K) \triangleleft B$, since if $(1,x)$ is in B and $(1,b)$ is in $\text{proj}_B(K)$ then $(1,x)^{-1}(c,b)(1,x) = (c,x^{-1}bx)$ is in K for some c in C. Hence $(1,x^{-1}bx)$ is in $\text{proj}_B(K)$.

Thus $\text{proj}_B(K) = 1$. Thus, $K \subset (C \times 1) \subset C \times \text{im}(\alpha)$. Therefore, either $1 = K$ or else $K \cap (C \times \text{im}(\alpha)) = 1$. In either case we have a contradiction.

Proposition II.12 If $A_1 \overset{\alpha_1}{\rightarrowtail} B_1$ is an essential embedding and $A_2 \overset{\alpha_2}{\rightarrowtail} B_2$ is an essential embedding, then $A_1 \times A_2 \overset{\alpha_1 \times \alpha_2}{\rightarrowtail} B_1 \times B_2$ is an essential embedding.

Proof: $A_1 \times A_2 \overset{id \times \alpha_2}{\rightarrowtail} A_1 \times B_2$ is an essential embedding from proposition 2.2 and for the same reason $A_1 \times B_2 \overset{\alpha_1 \times id}{\rightarrowtail} B_1 \times B_2$ is an essential embedding. The result then follows from proposition II.10.

Proposition II.13 Consider $A = Z(p^k)$ and $B = Z(p^{k+n})$, where p is some prime and k and n are positive integers. Then $A \overset{\alpha}{\rightarrowtail} B$ is an essential embedding of A in B, where $\alpha(1) = \overline{p^n}$.

Proof: Consider $A = \{0,1,2, \ldots \ldots, p^k\}$ and $B = \{\overline{0},\overline{1}, \ldots \ldots, \overline{p^{k+n}}\}$. We first show $\overline{x}$ is an im(α) if and only if $0 \leq x < p^{n+k}$ and $(x,p^{n+k}) = p^t \geq p^n$. Suppose $(x,p^{n+k}) = p^t = p^{n+j}$ with $0 \leq j < k$. Then $x = mp^{n+j}$ where $(m,p) = 1$. Now mp^j is in A and $\alpha(mp^j) = \overline{(mp^{j+n})} = \overline{x}$.

Now if $\overline{x}$ is not in im(α), then $(x,p^{n+k}) = p^t < p^n$. Thus, $p^{n-1-t}\overline{x} = \overline{(p^{n-1-t}x)}$ is in $\langle \overline{x} \rangle$ and $(p^{n-1-t}x, p^{n+k}) = p^{n-1-t}p^t = p^{n-1}$. Hence $\overline{0} \neq p^{n-1-t}\overline{x}$ and $\langle \overline{x} \rangle \cap$ im $(\alpha) \neq 0$. Thus every subgroup in B intersects im(α) non-trivially.

Proposition II.14 Let $A = \sum_{q_j \text{ prime} \neq p}^{m} \sum_{i=1}^{m} Z(q_j^{k_i}) \oplus \sum_{t=1}^{s-1} Z(p^{k_t}) \oplus Z(p^{k_s}) = A_1 \oplus Z(p^{k_s})$, where $k_s > k_t$ if $o < t < s$, and let $B = A_1 + Z(p^{k_s+n})$, where p is a prime and n is a positive integer. Then $A \xrightarrow{id \; + \alpha} B$ is an essential embedding.

Proposition II.15 If A is a finite abelian group, then A has no finite hull.

Remark II.16 Since both A and B in proposition II.14 are abelian, $A \xrightarrow{id \; + \alpha} B$ is also a related embedding. Thus if A is a finite abelian group then A has no finite related hull, since if G is assumed to be a related hull for A, we can construct a related embedding $A \rightarrowtail H$ with H having larger order than that of G, making it impossible for G to contain an isomorphic image of H.

Proposition II.17 If A is a group, y is in Z (A) and n is any positive integer, then there is a group G such that $A \triangleleft G$, $G = \langle A,x \rangle$, $\frac{G}{A} \approx Z(n)$, x is in $C_G(A)$ and $x^n = y$.

Proof: This is Theorem 9.7.2. in (11).

Corollary II.18 If A is a finite p-group and y is in Z(A), then there is a group G such that $A \lhd G$, $G = \langle A,x \rangle$, $\frac{G}{A} \approx Z(p)$, x is in C(A) and $x^p = y$.

Remark II.19 With the above hypotheses, $Z(G) = \langle x,Z(A) \rangle$.

Proof: If g is in G, then $g = ax^t$, thus if c is in Z(A) then $cgc^{-1} = cax^tc^{-1} = x^t(ca)c^{-1} = x^ta = ax^t = g$. Thus c is in Z(G). Hence $\langle x,Z(A)\rangle \subset Z(G)$. Now suppose z is in Z(G) and z is not in $\langle x,Z(A) \rangle$ and let $g = ax^n$ for some a not in Z(A), but in A. Notice $z = a_1x^t$, with a_1 not in Z(A). Now if gz = zg, then $(ax^n)a_1x^t = (a_1x^t)(ax^n)$, so $a_1a = aa_1$. But if we let a run through A-Z(A), we must find an a such that $a_1a \neq aa_1$, since a_1 is not in Z(A). Thus a_1x^t is not in Z(G).

Proposition II.20 Let A be a finite group with $1 \neq Z(A) \neq A$. Let prime p divide the order of Z(A). Let y be in Z(A) such that for each c in Z(A), if p^a divides the order of c then p^a divides the order of y. Using proposition II.17, consider the group $G = \langle A,x \rangle$ where $x^p = y$, $\frac{G}{A} \approx Z(p)$ and x is in C(A). Then $A \overset{id}{\rightarrowtail} G = \langle x,A \rangle$ is an essential embedding.

Proof: Recall $Z(G) = \{cx^t : c \text{ is in } Z(A), t \text{ is in } Z^+\}$.

Also recall $A \lhd G$. Suppose $1 \neq N \lhd G$ and $N \cap A = 1$.

(i) Assume $N \cap Z(G) \neq 1$, say $1 \neq n = cx^t$ is in $N \cap Z(G)$. Assume p does not divide t since if p divides t then $N \cap A \neq 1$. Then $n^p = c^px^{pt} = c^py^t$ is in $Z(A) \subset A$. Now $c^py^t \neq 1$, for if $c^p = y^{-t}$, then if p^a divides the order of y we have p^{a+1} divides the order of c and, since c is in Z(A), this contradicts the choice of y. Thus $N \cap A \neq 1$, and we have a contradiction.

(ii) Assume $N \cap Z(G) = 1$. Since $G/A \approx Z(p)$ and $A \cap N = 1$, we have $A \cup N = A \times N = G$. Hence every n in N is in Z(G), contradicting the hypothesis.

<u>Proposition II.21</u> Let A be a finite group with $1 \neq Z(A) \neq A$. Let p be a prime dividing the order of Z(A). Let y be in Z(A) such that for each c in Z(A), if p^a divides the order of c then p^a divides the order of y. Using proposition II.7, consider $G = \langle x,A \rangle$ where $x^p = y$, $G/A \approx Z(p)$ and x is in $C_G(A)$. Then $A \overset{id}{\rightarrowtail} G$ is a related embedding.

Proof: First notice $A \triangleleft G$. Then notice that $C_G(A) = Z(G) = \{cx^t:$ c is in Z(A) and t is a positive integer$\}$. Now suppose $1 \neq N \subset C_G(A)$ and $N \cap A = 1$. Consider any n in N, say $n = gx^r$ with $(r,p) = 1$ and g in Z(A). Then $n^p = g^p x^{rp} = g^p y^r$ is in A. If $n^p = 1$, then $g^p = y^{-r}$ and if p^a divides the order of y we have p^{a+1} divides the order of g, which contradicts the choice of y, since g is in Z(A). Thus $N \cap A \neq 1$.

<u>Proposition II.22</u> Let A be a finite group with $1 \neq Z(A) \neq A$. There is no finite group G which is a hull for A.

Proof: Let p be a prime dividing the order of Z(A). Now there is some integer n such that $p^n|A| > |G|$. We show there is a group A_n such that A is essentially embedded in A_n, with $|A_n| = p^n|A|$. Thus we cannot have an isomorphic copy of A_n in G.

Let p divide the order of Z(A). As in proposition II.20 construct $A \subset A_1 = \langle A, x_1 \rangle \subset A_2 = \langle A_1, x_2 \rangle \subset \ldots \ldots \subset A_{k+1} = \langle A_k, x_{k+1} \rangle \subset \ldots$ where x_{k+1} is in $C(A_k)$ and $x^p_{k+1} = y_k$ where y_k is chosen such that p divides the order of y_k and if p^a divides the order of c for some c in $Z(A_k)$ then p^a divides the order of y_k.

We need only show that for each k, A_{k+1} is essentially embedded in A_{k+1} and that A is normal in A_{k+1}.

Induction shows A is normal in A_{k+1}.

Since $A_k \triangleleft A_{k+1}$ we now suppose $1 \neq N \triangleleft A_{k+1}$ and $A_k \cap N = 1$.

(i) If $N \cap Z(A_{k+1}) \neq 1$, say $1 \neq n = cx^t_{k+1}$ is in $N \cap Z(A_{k+1})$ and c is in $Z(A_k)$. Notice $(p,t) = 1$. Now $n^p = c^p(x^t_{k+1})^p = c^p y^t_k$ is in $Z(A_k)$. But $1 \neq n^p$, for if $n^p = 1$ then $y^t_k = c^{-p}$, which contradicts the choice of y_k.

(ii) Assume $N \cap Z(A_k) = 1$. Since $A_{k+1}/A_k \simeq Z(p)$ and $A_k \cap N = 1$, we have $A_k \cup N = A_k \times N = A_{k+1}$. But $N \simeq Z(p)$, so every element of N is in the center of A_{k+1}.

<u>Proposition II.23</u> Let A be a finite group with $1 \neq Z(A) \neq A$. Let p be a prime dividing the order of Z(A). Then there is a group A_n, for each positive integer n, such that A is relatedely embedded in A_n and $|A_n| = p^n |A|$.

Proof: In proposition II.21, it was shown there is a set $A_1 = \langle A, x \rangle$ with $x^p_1 = y$ in Z(A). Continuing inductively we consider $A_{k+1} = \langle A_k, x_{k+1} \rangle$ with $x^p_{k+1} = y$ where y is some element of $Z(A_k)$ such that if c is in $Z(A_k)$ and p^a divides the order of c then p^a divides the order of y. Now suppose $1 \neq N \subset C_{A_{k+1}}(A_k)$ and $N \cap A = 1$. Consider any n in N. Then $n = gx^r$ with $(r,p) = 1$ and g in A_k. Notice, since n is in $C_{A_{k+1}}(A)$, we have g in $C_{A_k}(A)$. Now $n^p = g^p(x^r)^p = g^p y^r$ is in $C_{A_k}(A)$.

If $n^p \neq 1$ we have a contradiction to $N \cap C_{A_k}(A) = 1$ and since, by inductive assumption, A is relatedly embedded in A_k, we also have a contradiction to $N \cap A = 1$.

If $n^p = 1$, then $y^r = g^{-p}$. But this contradicts the choice of y in $Z(A_k)$.

<u>Corollary II.24</u> Let A be a finite group with $1 \neq Z(A) \neq A$. A has no finite related hull.

<u>Corollary II.25</u> Let C be a finite group. C is a direct summand of every group in which it appears as a normal subgroup $\Longleftrightarrow$ C is complete.

($\Leftarrow$) (9, p.133)

($\Rightarrow$) (i) Let C be centerless but not complete. Then the usual embedding of C in its automorphism group has $C \triangleleft \mathcal{A}(C) \neq C$. But C is essentially embedded in $\mathcal{A}(C)$, so C is not a summand of $\mathcal{A}(C)$. This result is also well known (10, p. 94).

(ii) Let C have a non-trivial center. Then we can form a group $\langle C,x \rangle$, using proposition II.17 in which C is essentially embedded and thus is not a summand.

RESTRICTED HULLS

It appears to be difficult to determine whether finite groups with centers can have hulls among the infinite groups. It is obvious that certain abelian groups cannot have hulls, even among infinite groups, if we require not only that an isomorphic image of each essential embedding appear in a hull, but that the isomorphic image appear as a normal subgroup of the hull. For instance, the cyclic group, Z(3), can be essentially embedded in the symmetric group S_3. This group, S_3, is known to be complete. It is known that any time a complete group appears as a normal subgroup of a group H, it must appear as a direct factor of H. But if H is a hull for Z(3), then it must contain a copy of all other essential embeddings of Z(3) - for instance, Z(9). Thus if $Z(3) \overset{\alpha}{\rightarrowtail} H$ were to be a hull for Z(3) which contains as a normal subgroup a copy of each essential embedding of Z(3), then $H = A \times B$, where $A \neq 1$, $B \approx S_3$ and $\mathrm{im}(\alpha) \subset B$. This contradicts the fact that $Z(3) \overset{\alpha}{\rightarrowtail} H$ must be an essential embedding.

<u>Definition III. 1</u> An embedding $A \overset{\alpha}{\rightarrowtail} B$ is a hull restricted by normality condition for essential embeddings - or, simply, a restricted hull - if

i) $A \overset{\alpha}{\rightarrowtail} B$ is an essential embedding

ii) if $A \overset{\gamma}{\rightarrowtail} C$ is an essential embedding, then there is a monomorphism β such that the diagram $\begin{array}{ccc} A & \overset{\alpha}{\rightarrowtail} & B \\ & {\scriptstyle\gamma}\searrow \; C \; \nearrow{\scriptstyle\beta} & \end{array}$ is commutative and $\mathrm{im}(\beta) \lhd B$.

Notice a restricted hull for A. is a hull for A.

<u>Remark III.2</u> Let G be a finite centerless group. The usual embedding

of G in its automorphism group is a restricted hull for G if and only if $\mathcal{A}(G)$ has the property that for every subgroup K such that $\text{im}(u) \subset K \subset \mathcal{A}(G)$, there exists $M \lhd \mathcal{A}(G)$ such that $\text{im}(u) \subset M$ and there exists $\delta : K \rightarrowtail\!\!\twoheadrightarrow M$ such that the following diagram is commutative.

$$\begin{array}{ccc} \text{im}(u) & \xrightarrow{id} & K \\ & \searrow^{id} & \downarrow \delta \\ & & M \end{array}$$

Proof: ($\Rightarrow$) Notice that if $\text{im}(u) \subset K \subset \mathcal{A}(G)$, then $G \xrightarrow{\mu} K$ is an essential embedding. For, suppose not, then $C_K(\text{im}(u)) \neq 1$. But $C_K(\text{im}(u)) \subset C_{\mathcal{A}(G)}(\text{im}(u))$. Hence this latter centralizer is nontrival. But the centralizer of a normal subgroup is normal and im(u) is centerless, so we obtain a contradiction to $G \xrightarrow{\mu} \mathcal{A}(G)$ being an essential embedding. But if $G \xrightarrow{\mu} \mathcal{A}(G)$ is a restricted hull and $\text{im}(u) \subset K \subset \mathcal{A}(G)$, there is a map η_K such that $\text{im}(\eta_K) \lhd \mathcal{A}(G)$ and the following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{\mu} & \mathcal{A}(G) \\ & {}_{\mu}\searrow \quad \nearrow_{\eta_K} & \\ & K & \end{array}$$

($\Leftarrow$) If $\mathcal{A}(G)$ does not have the property mentioned, then $\mathcal{A}(G)$ cannot be a restricted hull for G. But no other group could possibly be a restricted hull for G, since a restricted hull is also a hull and G has a unique hull.

<u>Remark III. 3</u> From this last proposition it is obvious that all finite complete groups have unique restricted hulls. This includes all

finite symmetric groups S_n for $n \geq 3$, except S_6. But it is known that $\mathcal{A}(S_6) / S_6 \approx Z(2)$ (6, p. 92). Hence S_6 has a restricted hull. It is also known that for n = 3 or 4 or 5 or $n > 6$, the alternating group A_n has S_n as its automorphism group (8, pp. 166-168). Hence A_n has a unique restricted hull for $n \geq 4$ if $n \neq 6$.

There exist finite centerless groups that do not have a restricted hull. Let A = Z(3) x Z(3). Let D be the associated generalized dihedral group. $\mathcal{A}(D)$ is isomorphic to the holomorph of A, Hol(A). It is known that Hol(A) is the split extension of A by $\mathcal{A}(A)$. That is, Hol(A) = $A \cup \mathcal{A}(A)$, with $A \cap \mathcal{A}(A) = 1$ and $A \triangleleft \mathrm{Hol}(A)$ -(8, p. 169) and (11, p. 214). It is also known that $\mathcal{A}(A) \approx GL(2,3)$ (11, p. 125), and that GL(2,3) is a split extension of Q_8 by S_3 (10, p. 119). Also $GL(2, 3) / Z(GL(2, 3)) \approx S_4$. But GL(2, 3) consists of the 2 by 2 matrices which have entries in the field of integers (mod 3) and have inverses. Z(GL (2, 3)) contains only two matrices, namely 2I and I. GL(2, 3) has only this one normal subgroup of order precisely two. Thus $\mathcal{A}(D) / D \approx \frac{A \cup GL(2, 3)}{A \cup Z(GL(2, 3))} \approx \left[\frac{A \cup GL(2, 3)}{A}\right] \Big/ \left[\frac{A \cup Z(GL(2, 3))}{A}\right] \approx S_4$. Now consider K such that the index of D in K is three and $D \subset K \subset \mathcal{A}(D)$. Such a subgroup is known to exist, due to Sylow's theorem. But it is known that S_4 has no subgroup which is normal and has order 3. Thus D has no restricted hull.

Definition III. 4 The dihedral group D_n is a group of order 2n generated by two elements s and t which satisfy the relations $s^n = 1 = t^2$ and $tst = s^{-1}$.

Definition III. 5 If A is any abelian group, there is an associated dihedral group $D = \langle A, x \rangle$ where $x^2 = 1$ and $xax = a^{-1}$, for each a in A.

<u>Remark III. 6</u> (8, p. 169) If D is the dihedral group associated with the abelian group A of odd order then D is centerless.

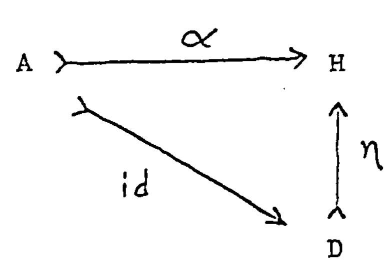
Proof: First notice $A \triangleleft D$ and each element in D can be expressed as ax^w where a is in A and w is zero or one. If $a \neq 1$, then $a \neq a^{-1}$, since the order of a must divide the order of A. $1x^1$ is not in Z(D), neither is ax^0 if $a \neq 1$. Also, if $a \neq 1$, ax is not in A(D), since $(ax)x = a$ and $x(ax) = a^{-1}$.

<u>Remark III. 7</u> $A \overset{id}{\rightarrowtail} D$ is an essential embedding, if A is an abelian group of odd order and D is the associated generalized dihedral group.

Proof: The index of A in D is two, so if there is a normal subgroup N which is disjoint from A, then $D = A \times N$ with $N \approx Z(2)$. But then $N \subset Z(D)$, which contradicts the preceding remark.

<u>Proposition III. 8</u> If A is any abelian group of odd order, then A has no restricted hull.

Proof: Suppose $A \overset{\alpha}{\rightarrowtail} H$ were a hull for A. Recall $A \overset{id}{\rightarrowtail} D$, where D is the associated dihedral group for A, is an essential embedding. Thus the following diagram is commutative and the centerless group, $\mathrm{im}(\eta)$ is normal in H.



Now suppose $1 \neq N \triangleleft H$ and $N \cap \mathrm{im}(\eta) = 1$. Then $N \cap \mathrm{im}(\alpha) = 1$, which is impossible. Thus $\mathrm{im}(\eta)$ is essentially embedded in H. But $\mathrm{im}(\eta)$ is finite and centerless and hence cannot be essentially embedded in an infinite group, from remark II. 9. But recalling that a restricted hull is also a hull and no finite abelian group has a finite hull, we see that H must be infinite and we obtain a contradiction.

Remark III. 9 If $A \subset G$ and $B \triangleleft G$ and $B \cap A = 1$, then there is a group N in G that is maximal with respect to i) $A \cap N = 1$, ii) $B \subset N$, and $N \triangleleft G$. This is proven by a straight forward application of Zorn's lemma.

Proposition III. 10 If H is a restricted hull for A x B and B is normal in H and N is a subgroup that is maximal with respect to being normal in H, containing B and being disjoint from A, then H/N is a restricted hull for A.

Proof: Assume $A \times B \xrightarrow{\alpha} H$ is a restricted hull for A x B and $B \triangleleft H$.
Then consider $A \xrightarrow{inj} A \times B \xrightarrow{\alpha} H \xrightarrow{nat} H/N$.

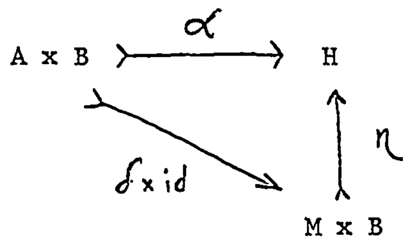
The composite $nat \cdot \alpha \cdot inj$ is a monomorphism, since $N \cap \mathrm{im}(\alpha \cdot inj) = 1$.

Using the correspondence theorem for homomorphisms, $\mathrm{im}(nat \cdot \alpha \cdot inj) \triangleleft H/N$.

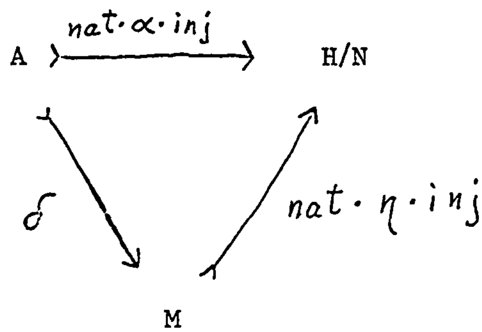
Now suppose $1 \neq K/N \triangleleft H/N$ and $K/N \cap nat(A) = 1$. Then $nat^{-1}(K/N) = K$ must be disjoint from A. But this contradicts the maximality of N. Hence $A \xrightarrow{nat \cdot \alpha \cdot inj} H/N$ is an essential embedding.

Now suppose $A \xrightarrow{\delta} M$ is any essential embedding of A. Then $A \times B \xrightarrow{\delta \times id} M \times B$ is an essential embedding. Thus M x B appears as a normal subgroup of H containing A x B. Now suppose $M \cap N = T \neq 1$. Then $T \triangleleft M$, so $T \cap A \neq 1$. Hence $M \cap N = 1$. Thus $\frac{M \cup N}{N} \approx M \subset H/N$. But $M \cup N = M \cup (B \cup N) = (M \cup B) \cup N \triangleleft H$, so $\frac{M \cup N}{N} \triangleleft H/N$.

Since $A \times B \xrightarrow{\alpha} H$ is a hull for A, the following diagram is commutative.



Now consider the following diagram.



This diagram is commutative, since if a is in A then $nat \cdot \eta \cdot inj\ \delta(a) = nat \cdot \eta(\delta(a), 1)$; but $\eta(\delta(a), 1) = \alpha(a, 1)$ from the preceding diagram, so $nat \cdot \eta(\delta(a), 1) = nat \cdot \alpha(a, 1)$.

<u>Corollary III. 11</u> If A is a finite abelian group and some odd prime divides its order, then A has no restricted hull.

Proof: $A = C \times D$ where C is the Sylow 2 - subgroup of A and, hence, D is an abelian group of odd order. If $C = 1$, proposition III. 8 covers this case. If $C \neq 1$ then, since C is characteristic in A, C is normal in H and proposition III. 10 applies to yield a restricted hull for D. This contradicts the fact that D cannot have a restricted hull.

Remark III. 12 (8, p. 110) If $G \simeq \prod_{i=1}^{r} Z(2^n)$ with n a positive integer and r a positive integer greater than one, then G has no characteristic subgroup of order 2.

Remark III. 13 (4, p. 86) If G is abelian and $G \overset{id}{\rightarrowtail} Hol(G)$ then G is its own centralizer in Hol(G).

Remark III. 14 Let G be a finite abelian group which is a direct product of at least two isomorphic cyclic groups of order some fixed power of two, as in remark III. 12. Then Hol(G) is centerless.

Proof: $Z(Hol(G)) \subset C_{Hol(G)}(G) = G$. But if $Z(Hol(G)) \neq 1$, then there is an element x in Z(Hol(G)) with order two. Hence $\langle x \rangle$ is a group of order two that is normal in Hol(G). But it is known (8, p. 214) that if K is a subgroup of G and K is normal in Hol(G), then K is characteristic in G. This contradicts remark III. 12.

Remark III. 15 If G is a finite abelian group, $G \overset{id}{\rightarrowtail} Hol(G)$ is an essential embedding. This follows from remark III. 13.

Remark III. 16 If $G \simeq \prod_{i=1}^{r} Z(2^n)$ for some fixed positive integer n and some integer r greater than one then G has no hull. This follows from remarks III. 14 and III. 15 using an argument as in proposition III. 8.

Remark III. 17 If G is an abelian group such that $G = Z(2^a) \times Z(2^a) \times B$ where B has order a power of 2 and a is a positive integer, then

G has no hull.

Proof: Assume $G \overset{\alpha}{\rightarrowtail} H$ is a hull for G. Since $A = Z(2^a) \times Z(2^a)$ is contained in Hol(A) as an essential embedding, a copy of Hol(A) x B appears as a normal subgroup of H containing G. Now B is the center of Hol(A) x B and hence B is characteristic in Hol(A) x B and thus B is normal in H. Now, using proposition III. 10, we obtain a hull for $Z(2^a) \times Z(2^a)$, which contradicts the preceding remark.

Remark III. 18 Let G be a finite group with $Z(G) \neq 1$ such that some α in $\mathcal{A}(G)$ has prime order, say p, and fixes no element in $Z(G)$ other than 1. Let $x^p = 1$ and let $G^* = \langle G, x \rangle$ such that $x^{-1}gx = \alpha(g)$ for each g in G. Then G^* is centerless and $G \overset{id}{\rightarrowtail} G^*$ is an essential embedding.

Proof: Notice $G \triangleleft G^*$ and hence each w in G^* can be expressed as $w = x^r g$ for some g in G and some non-negative integer r less than p. No element of the form $1.g \neq 1$ is in $Z(G^*)$, since the only elements we need to consider are those in $Z(G)$, but no elements in $Z(G)$ commute with x.

Now suppose $x^r g$ is in $Z(G^*)$ with $0 < r < P$. Then $x^r = (x^r g)g^{-1} = g^{-1}(x^r g)$. Now let $z \neq 1$ be in $Z(G)$ and notice $x^r z = (x^r g)g^{-1}z = z\, g^{-1}(x^r g) = z\, x^r$. Thus $\langle x^r \rangle = \langle x \rangle$ is contained in the centralizer of z. This contradicts the hypothesis, so G^* is centerless.

Now suppose N is normal in G^* and $N \cap G = 1$. Then, since $[G^* : G] = p$, we have $G^* = G \times N \cong G \times Z(p)$. But this contradicts G^* being centerless.

Remark III. 19 Let G be a finite group with $Z(G) \neq 1$. such that some α in $\mathcal{A}(G)$ has prime order, say p, and fixes no element in $Z(G)$ other than 1. Then G has no restricted hull.

The proof follows from the proof of proposition III. 8 where it was shown that if G can be essentially embedded in a finite centerless group then G can have no restricted hull.

Remark III.19 generalizes proposition III. 8, since for any odd ordered abelian group the mapping of each element of G into its inverse is an automorphism of order 2 which fixes no elements in G.

It is known that there are precisely two non-abelian groups of order p^3 for any odd prime p. These are listed on page 52 in (5) as follows:

i) $\langle a,b \rangle$: $a^{p^2} = 1 = b^p$, $b^{-1}ab = a^{1+p}$

ii) $\langle a,b,c \rangle$: $b^{-1}ab = a = c^{-1}ac$, $b^{-1}cb = ac$.

Consider case i). Each element can be written uniquely in the form $b^r a^s$ for some positive integers r and s with r less than p and s less than p^2. By induction we can show $a^m b^n = b^n a^{m(1+p)^n}$. Define $\alpha: G \longrightarrow G$ by $\alpha(b^r a^s) = b^r a^{-s}$. Now suppose $x = b^r a^s$ and $y = b^u a^v$. Then $\alpha(x)\alpha(y) = (b^r a^{-s})(b^u a^{-v}) = b^r b^u (a^{-s})^{(1+p)^u} a^{-v}$. But $\alpha(xy) = \alpha(b^r a^s b^u a^v) = \alpha(b^{r+u} a^{s(1+p)^u} a^v) = b^{r+u} a^{-s(1+p)^u - v}$, so α is an automorphism.

Now consider case ii). It can be shown that each element can be expressed uniquely in the form $a^r b^s c^t$. Also, by induction, we can show that $c^r b^n = a^{rn} b^n c^r$. Define $\alpha: G \longrightarrow G$ by $\alpha(a^r b^s c^t) = a^{-r} c^s b^t$.

Let $x = a^r b^s c^t$ and $y = a^u b^v c^w$. Then $\alpha(x)\ \alpha(y) = (a^{-r} c^s b^t)(a^{-u} c^v b^w) = a^{-r-u} c^s (b^t c^v) b^w = a^{-r-u} c^s (a^{-tv} c^v b^t) b^w = a^{-r-u-tv} c^{s+v} b^{t+w}$. But $\alpha(xy) = \alpha(a^r b^s c^t a^u b^v c^w) = \alpha(a^{r+u} b^s a^{tv} b^v c^t c^w) = \alpha(a^{r+u+tv} b^{s+v} c^{t+w}) = \alpha(x)\ \alpha(y)$, so α is in $\mathcal{A}(G)$.

A DUAL PROBLEM

<u>Definition IV.1</u>   Call $B \xrightarrow{\alpha} \!\!\!\!\rightarrow A$ a cover for $A \neq 1$, if whenever $X \xrightarrow{\delta} B \xrightarrow{\alpha} \!\!\!\!\rightarrow A$ is such that $\alpha\delta$ is an epimorphism, then $X \xrightarrow{\delta} B$ is an epimorphism.

<u>Definition IV.2</u>   Call $C \xrightarrow{\pi} A$ a maximal cover for A, if it is a cover for A and if for each cover for A, say $D \xrightarrow{\eta} \!\!\!\!\rightarrow A$, there is a homomorphism $\gamma : C \longrightarrow D$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & D \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\eta} & \\ & A & \end{array}$$

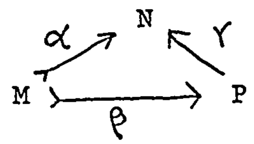
is commutative. (Notice then $C \xrightarrow{\gamma} \!\!\!\!\rightarrow D$).

This is a dual of essential embeddings and the hull problem, since $M \overset{\alpha}{\rightarrowtail} N$ (with im $(\alpha) \triangleleft$ N) is an essential embedding of M in N if whenever $M \overset{\alpha}{\rightarrowtail} N \xrightarrow{\gamma} P$ is such that $M \xrightarrow{\gamma\alpha} P$ is a monomorphism, then $N \xrightarrow{\gamma} P$ is a monomorphism.

Proof: Suppose $\gamma(n) = 1$, then n is in Ker(γ). But Ker(γ) $\triangleleft$ N and since there is no normal subgroup in N disjoint from im(α) - assuming that $M \overset{\alpha}{\rightarrowtail} N$ is an essential embedding - we see that n is in im(α), say $n = \alpha(h)$. But if $1 = \gamma(n) = \gamma(\alpha(h))$ then $h = 1$, so $n = 1$.

Now suppose $M \overset{\alpha}{\rightarrowtail} N$ is not an essential embedding. Let $\eta : N \twoheadrightarrow \frac{N}{K}$ be the natural mapping with $1 \neq K \triangleleft N$ and $K \cap \text{im}(\alpha) = 1$. Then $\eta\alpha$ is a monomorphism but η is not a monomorphism.

Now recall that a hull for M is a group N and a monomorphism $M \overset{\alpha}{\rightarrowtail} N$ such that $M \overset{\alpha}{\rightarrowtail} N$ is an essential embedding such that if $M \overset{\beta}{\rightarrowtail} P$ is any essential embedding, we have a homomorphism $\gamma : P \longrightarrow N$ such that the diagram

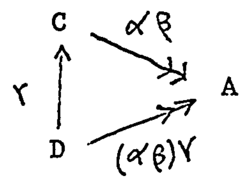


is communtative and γ is a monomorphism.

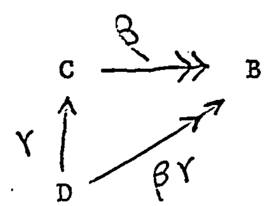
Thus by changing the direction of the arrows and interchanging "one to one" and "onto" we obtain a dual problem.

<u>Remark IV.3</u> If $B \overset{\alpha}{\twoheadrightarrow} A$ is a cover for A and $C \overset{\beta}{\twoheadrightarrow} B$ is a cover for B, then $C \overset{\alpha\beta}{\twoheadrightarrow} A$ is a cover for A.

Proof: Assume



. Then $D \overset{\beta\gamma}{\longrightarrow} B \overset{\alpha}{\twoheadrightarrow} A$, so $D \overset{\beta\gamma}{\twoheadrightarrow} B$. Hence, since



, we have $C \overset{\gamma}{\twoheadrightarrow} D$.

<u>Remark IV.4</u> Let B be finite. Let $B \overset{\alpha}{\twoheadrightarrow} C$ be an epimorphism with ker(α) = N. $B \overset{\alpha}{\twoheadrightarrow} C$ is a cover for C $\Longleftrightarrow$ if $K \subset B$ and $\frac{K}{K \cap N} \approx C$ then K = B.

($\Rightarrow$) Suppose $K \subset B$ and $\frac{K}{K \cap N} \approx C \approx \frac{B}{N}$, but $K \neq B$. Then consider $K \overset{id}{\longrightarrow} B \overset{\alpha}{\twoheadrightarrow} C$. Notice im($\alpha \cdot id$) $\approx$ C, so $\alpha \cdot id$ is an epimorphism.

Hence, $B \xrightarrow{\alpha} C$ is not a cover for C, since id is not an epimorphism.

($\Leftarrow$) Suppose $B \xrightarrow{\alpha} C$ is not a cover for C. Then there is a map $A \xrightarrow{\eta} B$ with $A \xrightarrow{\alpha\eta} C$, but such that η is not an epimorphism. Now consider $\mathrm{im}(\eta) \subset B$. Now $\mathrm{im}(\alpha\eta) \approx \frac{\mathrm{im}(\eta)}{\mathrm{im}(\eta) \cap \mathrm{Ker}(\alpha)} \approx C$, but $\mathrm{im}(\eta) \neq B$.

Proposition IV.5 Let B be a finite group. $B \xrightarrow{\alpha} C$ is a cover for C. $\Longleftrightarrow$ if $K \subset B$ and $K \cup \mathrm{Ker}(\alpha) = B$, then $K = B$.

Proof: Since B is a finite, this follows from $\frac{K \cup \mathrm{Ker}(\alpha)}{\mathrm{Ker}(\alpha)} \approx \frac{K}{K \cap \mathrm{Ker}(\alpha)}$.

Proposition IV.6 If B and C are finite groups, and $B \xrightarrow{\alpha} A$ is a cover for A, then $B\mathrm{x}C \xrightarrow{\alpha \times id} A\mathrm{x}C$ is a cover for AxC.

Proof: Let $K \subset B\mathrm{x}C$ such that $\frac{K \cup (\mathrm{Ker}(\alpha) \times 1)}{\mathrm{Ker}(\alpha) \times 1} \approx A\mathrm{x}C$. We need only show that $K = B\mathrm{x}C$. Let (b,1) be in B. Then (b,1) is in $K \cup (\mathrm{Ker}(\alpha) \times 1)$, say $(b,1) = (k_1,k_2)(m,1) = (k_1m,k_2)$. Thus, $(k_1,1)$ is in K and $(k_1,1)(m,1) = (b,1)$. Thus, since $\{(k_1,1) \text{ in } K\} \cup \mathrm{Ker}(\alpha) = B$, we have $\{(k_1,1) \text{ in } K\} = B$, using the fact that $B \xrightarrow{\alpha} A$ is a cover for A and using proposition IV.5. Now let (1,c) be in C. Then (1,c) is in $K \cup (\mathrm{Ker}(\alpha) \times 1)$, say $(1,c) = (k_3,k_4)(w,1) = (k_3w, k_4)$, so (k_3,c) is in K. But $(k_3^{-1}, 1)$ is in K, so (1,c) is in K. Thus, $K = B\mathrm{x}C$, since $B \subset K$ and $C \subset K$.

Proposition IV.7 For any prime p, and for any positive integer n, $Z(p^{n+1})$ is a cover for $Z(p^n)$.

Proof: Let $Z(p^{n+1}) = \langle x \rangle$ and $Z(p^n) = \langle y \rangle$.

Let $f: \langle x \rangle \longrightarrow \langle y \rangle$ such that $f(x^k) = y^r$, where $k = qp^s + r$ using division algorithm. Notice $\mathrm{Ker}(f) = \{ g \text{ in } \langle x \rangle : g^p = 1\} \approx Z(p)$. Hence Ker(f) is a subgroup of any non-trivial subgroup of $\langle x \rangle$. Thus, if $M \subset \langle x \rangle$ and $M \cup \mathrm{Ker}(f) = \langle x \rangle$ then $M = \langle x \rangle$. We also notice f is an epimorphism, since $|\mathrm{im}(f)| = p^n$.

<u>Proposition IV.8</u> If A is an abelian group, then there is no finite group B which is a maximal cover for A.

Proof: Suppose $G \xrightarrow{\alpha} A$ is a maximal cover for A. Then if H is any other cover for A, there must be a homomorphism $\eta: G \twoheadrightarrow H$, so $|G| \geq |H|$. Let A_1 be a cyclic factor of A, say $A_1 \approx Z(p^n)$ for some prime p and some positive integer n. Using proposition IV.7 and the remark on the transitivity property of covers, we can obtain a cover, say B, for A having arbitrarily large (finite) order of the form $p^{n+k} > |G|$. Now $A = A_1 \times A_2$ for some group A_2. We now use proposition IV.6 and $B \times A_2$ is a cover for A with $|B \times A_2| > |G|$.

<u>Remark IV.9</u> Let G be a finite group and $1 \neq \frac{G}{K}$ perfect, and assume $G \xrightarrow{nat} \frac{G}{K}$ is a cover for G/K. Then G is perfect (i.e. equals its commutator subgroup, G^1).

Proof: Deny. Let G be the finite group of smallest order such that G is not perfect, but G is a cover for G/K with $1 \neq K \neq G$.

Now $G^1 \triangleleft G$, $\frac{G}{G^1}$ is abelian and $1 \neq G^1 \neq G$. Notice $\frac{G}{K \cap G^1}$ is not perfect, since $\left[\frac{G}{K \cap G^1}\right] \Big/ \left[\frac{G^1}{K \quad G^1}\right] \approx \frac{G}{G^1}$.

(i) Suppose $K \cap G^1 \neq 1$. Then $\left|\frac{G}{K \cap G^1}\right| < |G|$ and we have a contradiction, provided the natural map with kernel $\frac{K}{K \cap G^1}$, from $\frac{G}{K \cap G^1}$ onto $\frac{G}{K}$ is a cover for G/K. Now assume

$$\frac{H}{K \cap G^1} \subset \frac{G}{K \cap G^1} \quad \text{and} \quad \frac{H}{K \cap G^1} \cup \frac{K}{K \cap G^1} = \frac{G}{K \cap G^1}.$$

We will show that $H \cup K = G$, thus $H = G$ and we have $\frac{H}{K \cap G^1} = \frac{G}{K \cap G^1}$. Let g be in G, then $g(K \cap G^1) = h(K \cap G^1)\, k(K \cap G^1) = (hk)(K \cap G^1)$ for some h in H and k in K. Thus $hkg^{-1} = t$ in $K \cap G^1$, so $t^{-1}hk = g$ and, since $K \lhd G$, $g = h\, t_1\, k$ in $H \cup K$.

(ii) Now suppose $K \cap G^1 = 1$. If $K \cup G^1 = G$, then we have a contradiction to the hypothesis that G is a cover for G/K. So assume $K \cup G^1 \neq G$. Notice $(K \cup G^1) \lhd G$, so $\frac{K \cup G^1}{K} \lhd G/K$. But then

$$\frac{G}{K \cup G^1} \approx \frac{G/K}{\left[\frac{K \cup G^1}{K}\right]}$$

yields the contradiction that $1 \neq \frac{G}{K \cup G^1}$ is abelian whereas $\frac{G}{K}$ is perfect and, thus, can not have a non-trivial abelian homomorphic image.

<u>Definition IV.10</u> The Frattini subgroup, Fr(G), of a group G is the intersection of all maximal proper subgroups of G [and Fr(G) = G if there are no maximal proper subgroups of G].

<u>Definition IV.11</u> An element x of G is a non-generator if whenever S is a subset of G such that $\langle S, x \rangle = G$ then $\langle S \rangle = G$.

<u>Remark IV.12</u> If G is a group, then Fr(G) is the set of all non-generators of G. For a proof see remark 7.3.2, (11, p. 159).

<u>Remark IV.13</u> Fr(G) is characteristic in G (11, p. 159). This remark follows from the fact that if M is maximal in G and Υ is an automorphism of G, then $\Upsilon(M)$ is maximal in G. Hence the intersection

of all of the maximal subgroups of G remains fixed under any automorphism of G.

We can now give another characterization of a covering of a group A by a finite group G, involving Fr(G).

<u>Lemma IV.14</u> If G is a group, $H \subset G$, Fr(G) is finitely generated, and $G = Fr(G) \cup H$, then $H = G$. This follows directly from the fact that Fr(G) is the set of non-generators of G, and is proven as remark 7.3.8, (11, p. 160).

<u>Proposition IV.15</u> Given A and G finite. $G \overset{\alpha}{\twoheadrightarrow} A$ is a cover for A if and only if $Ker(\alpha) \subset Fr(G)$.

($\Rightarrow$) Suppose $Ker(\alpha)$ is not contained in Fr(G). Let x be in $Ker(\alpha)$ and not in Fr(G). Now there is a set S such that $\langle S, x \rangle = G$ and $\langle S \rangle \neq G$. Call $\langle S \rangle = M \subset G$. Then $M \cup Ker(\alpha) = G$, but $M \neq S$. Hence by the remark preceding proposition IV.5, $G \overset{\alpha}{\twoheadrightarrow} A$ is not a cover for A.

($\Leftarrow$) Assume $Ker(\alpha) \subset Fr(G)$. Now suppose for some $M \subset G$ we have $M \cup Ker(\alpha) = G$. Then $M \cup Fr(G) \supset M \cup Ker(\alpha) = G$. But then, using the lemma above we have $M = G$.

<u>Remark IV.16</u> If G is finite, then Fr(G) is nilpotent. This is proven as remark 7.3.13(11).

<u>Remark IV.17</u> If G is finite, $A \triangleleft G$, $B \triangleleft G$, $B \subset Fr(G)$ and A/B is nilpotent, then A is nilpotent (11, p. 168).

<u>Remark IV.18</u> If B is finite and A nilpotent and $B \overset{\alpha}{\twoheadrightarrow} A$ is a cover, then B is nilpotent.

Proof: $Ker(\alpha) \subset Fr(B)$ and $\frac{B}{Ker(\alpha)}$ is nilpotent. Thus, by the preceding remark, B is nilpotent.

Remark IV.19 If B is finite, A solvable and $B \xrightarrow{\alpha} A$ is a cover, then B is solvable.

Proof: It is known that nilpotent implies solvable, that any subgroup of a solvable group is solvable, and any extension of a solvable group by a solvable group is solvable (remarks 6.4.7, 2.6.2 and 2.6.3, (11)). Hence, since $\mathrm{Ker}(\alpha)$ is a subgroup of the solvable group Fr(G), the result follows.

Remark IV.20 (11, p. 165). If G is finite and $\frac{G}{Fr(G)}$ is cyclic, then G is cyclic.

Proof: $\frac{G}{Fr(G)} = \langle x\, Fr(G) \rangle$ for some x in G. Let $S = \{x\}$. Let $M = \langle S \rangle$. Then for each g in G we have $g\, Fr(G) = x^r\, Fr(G)$ for some integer r. Hence $g = x^r y$ for some y in Fr(G). Thus $G = M \cup Fr(G)$. Hence, by a previous remark, we have $G = M = \langle x \rangle$.

Remark IV.21 If B is finite and $B \xrightarrow{\alpha} A$ is a cover for the cyclic group A, then B is cyclic. (The proof of this remark is essentially the same as that of the preceding remark). It is known that every homomorphic image of a nilpotent (abelian) (perfect) group is nilpotent (abelian) (perfect), hence it is obvious that the only cover for a non-nilpotent (non-abelian) (non-perfect) group is a non-nilpotent (non-abelian) (non-perfect) group.

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