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Trading cookies with a random walk

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DEDICATION

I would like to dedicate this thesis to my parents Mr. Zhiqiang Sun and Mrs. Hongyu Luo without whose support I would not have been able to complete this work. I would also like to thank my friends and family for their loving guidance and financial assistance during the writing of this work.

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ABSTRACT

The mathematical problem of determining a gambler's risk of ruin involves analyzing decisions of only one agent, namely the "gambler". In this work we consider an extension that introduces two additional players, so called "sellers". These two new agents can boost the probability of success for the gambler by selling to him (using a jargon borrowed from the theory of excited random walks) a "cookie" which is used to increase the probability of moving forward in the next step. The generalized gambler's ruin scenario considers an excited random walk on a finite interval of integer line with two "cookie store" locations and unlimited supply of cookies at each. Each time when the buyer (walker) visits a store location, he has an opportunity to decide whether he is willing to consume the cookie or not. We wish to determine the equilibrium prices and cookie store locations in a formal game associated with this generalized gambler's ruin scenario.

CHAPTER 1. INTRODUCTION

1.1 General background and previous work

Excited random walks (ERW) or *random walks in a cookie environment* on $\mathbb{Z}^d$ is a modification of the nearest neighbor simple random walk such that in several first visits to each site of the integer lattice, the walk's jump kernel gives a preference to a certain direction and assigns equal probabilities to the remaining $(2d-1)$ directions. If the current location of the random walk has been already visited more than a certain number of times, then the walk moves to one of its nearest neighbors with equal probabilities. The model was introduced by Benjamini and Wilson in [3] and extended by Zerner in [13]. In the "cookies" jargon, upon first several visits to every site of the lattice, the walker consumes a cookie providing them a boost toward a distinguished direction in the next step. Many important aspects of the asymptotic behavior of excited random walks on $\mathbb{Z}$ are by now well-understood [11]. An application of the theory of excited random walks to the physics of DNA molecular motors is discussed in [1, 4].

This work continues to investigate a class of models introduced in [10]. In [10] several variations of a two-person (Stackelberg) game between a "buyer" and a "seller", whose major component is a random walk of the buyer on a finite interval of integers were considered. The key element of the game is a gambler's ruin problem [6, 7], where in contrast to the classical version, the walker (buyer) has the option of consuming "cookies", which when used, increase the probability of moving in the desired direction for the next step. The cookies are supplied to the buyer by the second player (seller). The ultimate goal is to determine an equilibrium price policy for the seller and the equilibrium "cookie store" location. The optimization problem which the seller faces is somewhat similar to that of a monopoly whose market is a spatially non-homogeneous Hotelling beach with demand curve varying randomly across the population

[8, 9]. The original Hotelling beach (linear city) model was introduced to illustrate Hotelling's law in economics, namely a general paradigm that in many markets it is rational for producers to make their products as similar as possible.

1.2 Motivation and goals

The game can serve as a simplified model to explore the relationship between economic agents in a risky environment, for instance a firm in an innovative and competitive segment of a hi-tech industry and an experienced consulting company. The firm (buyer) seeks to reduce uncertainty and increase the expected profit by investing in the consulting service at a "bottleneck" point of its production line, while the consultant (seller) wants to optimize the configuration and the price of its service package.

From the probability theory point of view, the models introduced in [10] and in this work attempt

1. To measure the gain of the walker from exploiting a reinforcing mechanism represented by "cookies". It is natural to study this type of problems using a gambler's ruin scenario and within a game-theoretic framework, where exact features of the reinforcing mechanism are determined through the interaction between the walker and sellers. This is in contrast to the usual excited random walk, where the walker, as a price-taker in a large market, has no effect on determining the parameters of the cookie environment.

2. To further contribute to the basic understanding of one-dimensional excited random walks by considering a suitable variation of gambler's ruin problem. It is well-known, see for instance [11], that the asymptotic behavior of a random walk can be inferred from the solution to the corresponding gambler's ruin problem. In particular, the asymptotic behavior of excited random walk is largely governed by a single parameter, its average local drift. Curiously, the main result of this work suggests that the same parameter solely determines the optimal cookie prices for a fixed store location (see Chapter 2 below).

3. The optimization methods employed in this work are, up to a certain point, methods of continuous convex optimization. One could expect that, similarly to the Hotelling linear

city model, the equilibrium configuration will place the cookie stores at the same location. However, it is easy to see that this solution is not available when the cookie's prices can vary and are determined by the sellers. Thus the actual equilibrium design is expected to be affected by a not-so-intrinsic to the problem discrete optimization. A part of our initial motivation was to see whether the discrete design can force the risk-neutral buyer to become effectively risk-adverse. Remarkably, in some particular sense the answer to this question turns out to be affirmative (see the discussion of the main result in Chapter 3). We are planning to consider in the future an extension of this work to a continuous time model based on the "excited stochastic process" considered in [12].

1.3 Overview of the model

This thesis introduces a generalization of the model of [10] to a three-person game with two competing sellers and a buyer. The model is significantly more involved from the technical point of view and computationally extensive because of the required rather detailed analysis of the underlying finite-state Markov chain. More specifically, in this work we introduce the following modification of the classical two-person gambler's ruin scenario, where the buyer has the option to consume cookies supplied by the sellers at two different cookie locations. The cookie serves to instantly increase the probability of moving forward in the next step. Set the starting point of buyer as $n \in \mathbb{N}$ located between 0 and $b \in \mathbb{N}$, $b \geq 2$, and treat the direction from 0 to b as the forward direction. Assume that the buyer performs a nearest-neighbor random walk on the integer line with absorbtion at 0 and b. If the buyer reaches the point b before 0 he is rewarded with R dollars, otherwise he receives a zero payoff. Simultaneously and independently each of other, two sellers set up the "cookie" stores at integer sites n_1 and n_2 within the interval $(0, b)$. The two sellers sell the cookies at fixed prices c_1 and c_2, respectively. At a regular site, the buyer moves one step forward with a fixed probability $p \in (0, 1)$, and backward with a fixed probability $q = 1-p$. If he consumes a cookie at the store locations, then he moves one step forward with a larger probability $p + \epsilon_1 \in (p, 1)$ from n_1 and $p + \epsilon_2 \in (p, 1)$ from n_2. The buyer can choose either accept the cookie for the suggested price or reject it in order to reach the ultimate purpose, which is maximizing the total revenue at the absorbtion time. The goal of this work is to determine an equilibrium price for each cookie and location for the "cookie" store.

In this work, we are focus on the a special situation, similar to the one-seller counterpart which is referred to in [10] as a *basic game*, that is we assume $p = q = \frac{1}{2}$. We use the Markov property of the underlying excited random walk and a *subgame perfect Nash equilibrium* to solve the relationship between the price and location. In Section 2 we treat the location of "cookie" stores as a fixed variable to find out the equilibrium price, while the contrary situation will be considered in the Section 3 where the equilibrium store locations are determined. We assume that the buyer is risk-neutral and maximizes its expected game payoff. The proof of the main result is concluded in the last section.

1.4 Game description

In this and next sections, we discuss the basic game. We consider the following scenario with a fixed probability $p = q = \frac{1}{2}$. First fix any $b \in \mathbb{N}$, $b \geq 2$, and set the forward direction as from 0 to b. Then let the buyer starting the random walk at a fixed point $n \in \mathbb{N}$ located between 0 and b. On the other hand, the two sellers, who are seeking for the maximum expected revenue, need to make a decision for the store's location n_1 and n_2 and the price of each cookie c_1 and c_2 independently. In this scenario, there is no product cost for both of the two sellers and the number of cookies η that the sellers provided to the buyer can be infinite many. The buyer can accept the cookie as a instant probability boost strategy, however, he has an option to refuse if he consider it is not worthwhile. Denote $\epsilon_1 \in (0, \frac{1}{6})$ and $\epsilon_2 \in (0, \frac{1}{6})$ as the cookie strategy for the two sellers separately. If the buyer decides to accept the cookie at the first/second cookie store (located at n_1/n_2) he moves forward on $\mathbb{Z}$ according to $\mathbb{P}_{n_1} = p + \epsilon_1/\mathbb{P}_{n_2} = p + \epsilon_2$,otherwise his motion is based on $\mathbb{P} = p$. Notice that, the buyer can only moves step by step. Once the buyer arrives any of the sides, the game is end. If the buyer reaches the point b first he is rewarded with R dollars, in contrast, he gets nothing. Consider the money he paid for cookies as the cost, the buyer seeks to maximize his expected earnings.

The main purpose of this section is to calculate the explicit result for the value of optimal price for each cookie.

Definition 2.1 Game Γ_n

- Γ_n is a three-person game based on the Stackelberg model (the first two players take action independently, the third player observes their action and then decides his own moves). All of the players in the game need to consider a strategy that maximizes their corresponding expected payoff given the chosen strategy pf two other players..

- The first two players are the sellers, and the third player is the buyer. The two sellers move first and inform their action to the buyer separately. Then the buyer determines his game plan and starts a random walk.

- Let $S := [0, \infty) \times \{1, ..., b-1\}$ be the set of strategies of the two sellers. Each pair $(c_1, n_1) \in S$ specifies the cookie's price $c_1 > 0$ and the store's location $n_1 \in \{1, ..., b-1\}$ determined by the first seller. Similarly, each pair $(c_2, n_2) \in S$ specifies the cookie's price $c_2 > 0$ and the store's location $n_2 \in \{1, ..., b-1\}$ determined by the second seller. Notice that, we default $n_1 \leq n_2$ in the whole set.

- Let $B := S^2 \to \{e_{n_1}, e_{n_2}\}$ where $e_k = \{0, 1\}$ be the strategy of the buyer. The buyer can choose to reject the cookie or consume it at the two different stores with the certain price.

Definition 2.2 The buyer's random walk

- Let $X_k \in (0, b)$ denote buyer's location on the integer line at time $k \in \mathbb{Z}_+$.

- Let $M_k \in \mathbb{Z}_+$ be the number of cookies available at the walk's current location at time $k \in \mathbb{Z}_+$.

- Since the buyer can only move one step at a time, the Markov chain transition kernel of buyer's random walk at the "cookie" store n_1 is given by

$$\begin{cases} \mathbb{P}_{n_1}(X_{k+1} = i+1, m_{k+1} = m-1 | X_k = i, M_k = m) = p + \mathbb{1}_{i=n_1, m>0} \cdot \epsilon \\ \mathbb{P}_{n_1}(X_{k+1} = i-1, m_{k+1} = m-1 | X_k = i, M_k = m) = q - \mathbb{1}_{i=n_1, m>0} \cdot \epsilon \end{cases}$$

- Similarly, the Markov chain transition kernel of buyer's random walk at the "cookie" store n_2 is given by

$$\begin{cases} \mathbb{P}_{n_2}(X_{k+1} = i+1, m_{k+1} = m-1 | X_k = i, M_k = m) = p + \mathbb{1}_{i=n_2, m>0} \cdot \epsilon \\ \mathbb{P}_{n_2}(X_{k+1} = i-1, m_{k+1} = m-1 | X_k = i, M_k = m) = q - \mathbb{1}_{i=n_2, m>0} \cdot \epsilon \end{cases}$$

where $\mathbb{1}_A$ is the indicator.

CHAPTER 2. OPTIMAL PRICES c_1^* AND c_2^* FOR A GIVEN STORE LOCATIONS

The goal of this section is to determine optimal prices.

If the price of each cookie is attractive enough, the buyer will choose to consume it. Indeed, if the cookies are free, the buyer will definitely want them, and the claim follows by the continuity. Therefore, the buyer would consume the cookies at both of the two sellers when the price is optimal (does the optimal exist?).

For $n \in \mathcal{I}$, we denote by R_n the expected reward of the buyer who starts their random walk at the location $n \in \mathcal{I}$. By the strong Markov property,

$$R_{n_1} = -c_1 + (q - \varepsilon_1)\alpha R_{n_1} + (p + \varepsilon_1)\beta R_{n_1} + (p + \varepsilon_1)(1 - \beta)R_{n_2},$$

where $\alpha := \mathbb{P}_{n_1-1}(T_{n_1} < T_0)$ is the probability that the buyer returns to n_1 from the backward direction and $\beta := \mathbb{P}_{n_1+1}(T_{n_1} < T_{n_2})$ is the probability that the buyer returns to n_1 from the forward direction.

Similarly,

$$\begin{aligned} R_{n_2} = -c_2 + (q - \varepsilon_2)\gamma R_{n_2} + (q - \varepsilon_2)(1 - \gamma)R_{n_1} \\ + (p + \varepsilon)\delta R_{n_2} + (p + \varepsilon)(1 - \delta)R, \end{aligned}$$

where $\gamma := \mathbb{P}_{n_2-1}(T_{n_2} < T_0)$ is the probability that the buyer returns to n_2 from the backward direction and $\delta := \mathbb{P}_{n_2+1}(T_{n_2} < T_b)$ is the probability that the buyer returns to n_2 from the forward direction.

The solution to the classical gambler's ruin problem [6] yields:

$$\alpha = \frac{n_1 - 1}{n_1}, \qquad \beta = \frac{n_2 - n_1 - 1}{n_2 - n_1}$$
$$\gamma = \frac{n_2 - n_1 - 1}{n_2 - n_1}, \qquad \delta = \frac{b - n_2 - 1}{b - n_2}$$

Therefore,

$$R_{n_1} = \frac{(p+\epsilon_1)(1-\beta)R_{n_2} - c_1}{1-(q-\epsilon_1)\alpha-(p+\epsilon_1)\beta} \tag{2.1}$$

$$\begin{aligned} R_{n_2} = & \frac{[(q-\epsilon_1)(p+\epsilon_2)(n_2-n_1)+(p+\epsilon_1)(p+\epsilon_2)n_1)]}{(q-\epsilon_1)(q-\epsilon_2)(b-n_2)+(q-\epsilon_1)(p+\epsilon_2)(n_2-n_1)+(p+\epsilon_1)(p+\epsilon_2)n_1}R \\ & -\frac{(q-\epsilon_2)(b-n_2)n_1}{(q-\epsilon_1)(q-\epsilon_2)(b-n_2)+(q-\epsilon_1)(p+\epsilon_2)(n_2-n_1)+(p+\epsilon_1)(p+\epsilon_2)n_1}c_1 \\ & -\frac{[(q-\epsilon_1)(n_2-n_1)(b-n_2)+(p+\epsilon_1)(b-n_2)n_1]}{(q-\epsilon_1)(q-\epsilon_2)(b-n_2)+(q-\epsilon_1)(p+\epsilon_2)(n_2-n_1)+(p+\epsilon_1)(p+\epsilon_2)n_1}c_2 \end{aligned}$$

Definition 2.3 Subgame perfect Nash equilibrium[8]

- A subgame perfect Nash equilibrium means that the strategy serves best for each player and it satisfied that every player is playing in a Nash equilibrium in every subgame.

Since both of the two sellers seek for the maximum profit simultaneously, the optimal c_1^* and c_2^* have to satisfied the following conditions. Without loss of generality, when the first seller fix the price of each cookie at c_1^*, the second seller would not change the price c_2^* for a bigger benefit.

Therefore,if we fixed n_1 and n_2, then the revenue R_{n_1} and R_{n_2} should be larger if the buyer consumes at both of the two cookie stores. This condition is equivalent to imposing the following set of four inequalities.

$$R_{n_1}(\epsilon_1, c_1, \epsilon_2, c_2) \geqslant R_{n_1}(0, 0, \epsilon_2, c_2) \tag{2.2}$$

$$R_{n_1}(\epsilon_1, c_1, \epsilon_2, c_2) \geqslant R_{n_1}(\epsilon_1, c_1, 0, 0) \tag{2.3}$$

$$R_{n_2}(\epsilon_1, c_1, \epsilon_2, c_2) \geqslant R_{n_2}(0, 0, \epsilon_2, c_2) \tag{2.4}$$

$$R_{n_2}(\epsilon_1, c_1, \epsilon_2, c_2) \geqslant R_{n_2}(\epsilon_1, c_1, 0, 0) \tag{2.5}$$

For convenient, we first rewrite $R_{n_2}(\epsilon_1, c_1, \epsilon_2, c_2)$ and $R_{n_2}(0, 0, \epsilon_2, c_2)$ in (2.4) as

$$R_{n_2}(\epsilon_1, c_1, \epsilon_2, c_2) = \frac{X_1 R - X_2 c_1 - X_3 c_2}{K_1}$$

$$R_{n_2}(0, 0, \epsilon_2, c_2) = \frac{X_1^1 R - X_3^1 c_2}{K_1^1}$$

with

$$X_1 = (q-\epsilon_1)(p+\epsilon_2)(n_2-n_1)+(p+\epsilon_1)(p+\epsilon_2)n_1 \quad (2.6)$$

$$X_2 = (q-\epsilon_2)(b-n_2)n_1 \quad (2.7)$$

$$X_3 = (q-\epsilon_1)(n_2-n_1)(b-n_2)+(p+\epsilon_1)(b-n_2)n_1 \quad (2.8)$$

$$X_1^1 = q(p+\epsilon_2)(n_2-n_1)+p(p+\epsilon_2)n_1 \quad (2.9)$$

$$X_3^1 = q(n_2-n_1)(b-n_2)+p(b-n_2)n_1 \quad (2.10)$$

$$K_1 = (q-\epsilon_1)(q-\epsilon_2)(b-n_2)+(q-\epsilon_1)(p+\epsilon_2)(n_2-n_1) \quad (2.11)$$

$$+ (p+\epsilon_1)(p+\epsilon_2)n_1 \quad (2.12)$$

$$> 0 \quad (2.13)$$

$$K_1^1 = q(q-\epsilon_2)(b-n_2)+q(p+\epsilon_2)(n_2-n_1)+p(p+\epsilon_2)n_1 \quad (2.14)$$

$$> 0 \quad (2.15)$$

Then (2.4) can be write as

$$(X_1R-X_2c_1-X_3c_2)K_1^1-(X_1^1R-X_3^1c_2)K_1 \geqslant 0$$

$$\Rightarrow \quad (X_1K_1^1-X_1^1K_1)R-X_2K_1^1c_1-(X_3K_1^1-X_3^1K_1)c_2 \geqslant 0$$

where

$$X_1K_1^1-X_1^1K_1=(q-\epsilon_2)(b-n_2)n_1\cdot(p+\epsilon_2)\epsilon_1$$

$$X_2K_1^1=(q-\epsilon_2)(b-n_2)n_1\cdot[q(q-\epsilon_2)(b-n_2)+q(p+\epsilon_2)(n_2-n_1)+p(p+\epsilon_2)n_1]$$

$$X_3K_1^1-X_3^1K_1=(q-\epsilon_2)(b-n_2)n_1\cdot(b-n_2)\epsilon_1$$

By algebraic simplification, we get following result:

$$(p+\epsilon_2)\epsilon_1R-[q(q-\epsilon_2)(b-n_2)+q(p+\epsilon_2)(n_2-n_1)+p(p+\epsilon_2)n_1]c_1 \geqslant (b-n_2)\epsilon_1c_2$$

We then use the method to deal with (2.5), let

$$R_{n_2}(\epsilon_1,c_1,\epsilon_2,c_2)=\frac{X_1R-X_2c_1-X_3c_2}{K_1}$$

$$R_{n_2}(\epsilon_1,c_1,0,0)=\frac{X_1^2R-X_2^2c_1}{K_1^2}$$

with

$$X_1^2 = p(q-\epsilon_1)(n_2-n_1)+p(p+\epsilon_1)n_1 \quad (2.16)$$

$$X_2^2 = q(b-n_2)n_1 \quad (2.17)$$

$$K_1^2 = q(q-\epsilon_1)(b-n_2)+p(q-\epsilon_1)(n_2-n_1)+p(p+\epsilon_1)n_1 \quad (2.18)$$

$$> 0 \quad (2.19)$$

and X_1, X_2, X_3, K_1 has been defined in (2.6),(2.7),(2.8)and(2.11).

Hence,(2.5) is

$$(X_1R - X_2c_1 - X_3c_2)K_1^2 - (X_1^2R - X_2^2c_1)K_1 \geqslant 0$$

$$\Rightarrow \quad (X_1K_1^2 - X_1^2K_1)R - (X_2K_1^2 - X_1^2K_1)c_1 - X_3K_1^2c_2 \geqslant 0$$

where

$$X_1K_1^2 - X_1^2K_1 = [(q-\epsilon_1)(n_2-n_1+(p+\epsilon_1)n_1](b-n_2)\cdot(q-\epsilon_1)\epsilon_2$$

$$X_2K_1^2 - X_1^2K_1 = -[(q-\epsilon_1)(n_2-n_1)+(p+\epsilon_1)n_1](b-n_2)\cdot n_1\epsilon_2$$

$$X_3K_1^2 = [(q-\epsilon_1)(n_2-n_1)+(p+\epsilon_1)n_1](b-n_2)$$
$$\cdot[q(q-\epsilon_1)(b-n_2)+p(q-\epsilon_1)(n_2-n_1)+p(p+\epsilon_1)n_1]$$

Therefore,

$$(q-\epsilon_1)\epsilon_2R + n_1\epsilon_2c_1 \geqslant [q(q-\epsilon_1)(b-n_2)+p(q-\epsilon_1)(n_2-n_1)+p(p+\epsilon_1)n_1]c_2$$

After solving the last two inequalities (2.4) and (2.5), we need figure out their the relationship with the first two (2.2) and (2.3).

Plug the value of R_{n_1} defined in (2.1) into (2.2), we have

$$\frac{(p+\epsilon_1)(1-\beta)R_{n_2}(\epsilon_1,c_1,\epsilon_2,c_2)-c_1}{1-(q-\epsilon_1)\alpha-(p+\epsilon_1)\beta} \geqslant \frac{p(1-\beta)R_{n_2}(0,0,\epsilon_2,c_2)}{1-q\alpha-p\beta}$$
$$\Rightarrow \quad \frac{(p+\epsilon_1)n_1R_{n_2}(\epsilon_1,c_1,\epsilon_2,c_2)-(n_2-n_1)n_1c_1}{(q-\epsilon_1)(n_2-n_1)+(p+\epsilon_1)n_1} \geqslant \frac{pn_1}{q(n_2-n_1)+pn_1}R_{n_2}(0,0,\epsilon_2,c_2)$$
$$\Rightarrow \quad R_{n_2}(\epsilon_1,c_1,\epsilon_2,c_2) \geqslant R_{n_2}(0,0,\epsilon_2,c_2)$$

Similarly, for (2.3),

$$\frac{(p+\epsilon_1)(1-\beta)R_{n_2}(\epsilon_1,c_1,\epsilon_2,c_2)-c_1}{1-(q-\epsilon_1)\alpha-(p+\epsilon_1)\beta} \geqslant \frac{(p+\epsilon_1)(1-\beta)R_{n_2}(\epsilon_1,c_1,0,0)-c_1}{1-(q-\epsilon_1)\alpha-(p+\epsilon_1)\beta}$$
$$\Rightarrow \quad R_{n_2}(\epsilon_1,c_1,\epsilon_2,c_2) \geqslant R_{n_2}(\epsilon_1,c_1,0,0)$$

Therefore, the inspection of (2.1) shows that in fact the first two inequalities imply the last two in the above system. By using the closed form expressions for R_1 and R_2 the system is reduced to the following set of four inequalities.

$$\begin{cases} l_1 : c_1 \geqslant 0 \\ l_2 : c_2 \geqslant 0 \\ l_3 : A_1 \cdot R + B_1 \cdot c_1 \geqslant D_1 \cdot c_2 \\ l_4 : A_2 \cdot R + B_2 \cdot c_1 \geqslant D_2 \cdot c_2 \end{cases}$$

with

$$A_1 = (p + \epsilon_2)\epsilon_1 \tag{2.20}$$

$$A_2 = (q - \epsilon_1)\epsilon_2 \tag{2.21}$$

$$B_1 = -[q(q - \epsilon_2)(b - n_2) + q(p + \epsilon_2)(n_2 - n_1) + p(p + \epsilon_2)n_1] \tag{2.22}$$

$$B_2 = n_1\epsilon_2 \tag{2.23}$$

$$D_1 = (b - n_2)\epsilon_1 \tag{2.24}$$

$$D_2 = q(q - \epsilon_1)(b - n_2) + p(q - \epsilon_1)(n_2 - n_1) + p(p + \epsilon_1)n_1 \tag{2.25}$$

The set of solutions in the (c_1, c_2)-plane is non-empty and bounded by two (in general, oblique) straight straight and two axes. The intersection point of two slanting lines is $N(c_{N1}, c_{N2})$, where

$$c_{N1} = \frac{A_2D_1 - A_1D_2}{B_1D_2 - B_2D_1}R \qquad \text{and} \qquad c_{N2} = \frac{A_2B_1 - A_1B_2}{B_1D_2 - B_2D_1}R$$

Since we are focusing on the $p = q = \frac{1}{2}$, then we can compute out the value of c_{N1} and c_{N2}.

$$c_{N1} = \frac{\frac{1}{2}(\frac{1}{2} - \epsilon_2)(\frac{1}{2} - \epsilon_1)\epsilon_1 b + (\frac{1}{2} - \epsilon_1)\epsilon_1\epsilon_2 n_2 + (\frac{1}{2} + \epsilon_2)\epsilon_1^2 n_1}{(\frac{1}{2}(\frac{1}{2} - \epsilon_2)b + \epsilon_2 n_2)(\frac{1}{2}(\frac{1}{2} - \epsilon_1)b + \epsilon_1 n_1) + \epsilon_1\epsilon_2(b - n_2)n_1}R \tag{2.26}$$

$$= \frac{2\epsilon_1 R}{b} \tag{2.27}$$

$$c_{N2} = \frac{\frac{1}{2}(\frac{1}{2} - \epsilon_1)(\frac{1}{2} - \epsilon_2)\epsilon_2 b + (\frac{1}{2} - \epsilon_1)\epsilon_2^2 n_2 + (\frac{1}{2} + \epsilon_2)\epsilon_1\epsilon_2 n_1}{(\frac{1}{2}(\frac{1}{2} - \epsilon_2)b + \epsilon_2 n_2)(\frac{1}{2}(\frac{1}{2} - \epsilon_1)b + \epsilon_1 n_1) + \epsilon_1\epsilon_2(b - n_2)n_1}R \tag{2.28}$$

$$= \frac{2\epsilon_2 R}{b} \tag{2.29}$$

For a graphical illustration we refer to Fig. 2 below. Note that for any fixed $c_2 < c_{N2}$ there is a constant $c_1 < c_{l1}$ such that the value of R_{n_1} at c_1 is bigger then at c_{l1} (see Fig. 2). Similarly, for an arbitrary c_1 one can find $c_2' < c_{N2}$ such that the value of R_{n_2} at c_2' is bigger then at c_2. Hence, neither of two sellers would benefit form changing the price unilaterally if and only if $c_1^* = c_{N1}$ and $c_2^* = c_{N2}$.

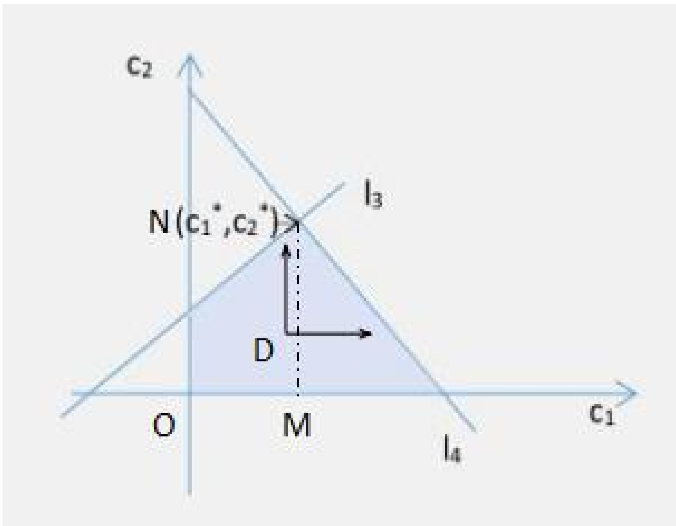

Figure 2.1 The relationship between c_1 and c_2.

As illustrated in the figure, the plot can be classified into four areas by the straight lines l_3 and l_4. Furthermore, the shadows D in the plot represents the price that the buyer would accept. Otherwise, the buyer would reject the cookies at least one of the two stores. Let point M be the projection of point N on $c_1 - axis$. If the pair of price (c_1, c_2) lies on l_3, then the first seller can always increase c_1 and push the price point into area D. And once (c_1, c_2) appears in D, the second seller will definitely choose to rise his price, which leads two results. If the price point is on the right of NM, then the action would force the point to l_4 line. Otherwise, the two sellers would reach the agreement of price on point N. On the other hand, if the price point is on l_4, increasing the price is reasonable only for the second seller. Finally, turns out that the only price point satisfied Nash equilibrium for the two sellers is point $N = (c_1^*, c_2^*)$. Since R_{n_1} can be treated as a function of R_{n_2}, then the only thing left in this system is that we need to check if

$$R_{n_2}(\epsilon_1, c_1^*, \epsilon_2, c_2^*) \geqslant R_{n_2}(0, 0, 0, 0)$$

Proof Since c_1^* and c_2^* satisfied the inequality (2.5), then we know that

$$R_{n_2}(\epsilon_1, c_1^*, \epsilon_2, c_2^*) \geqslant R_{n_2}(\epsilon_1, c_1^*, 0, 0)$$

The only thing left that need to be proved is

$$R_{n_2}(\epsilon_1, c_1^*, 0, 0) \geqslant R_{n_2}(0, 0, 0, 0)$$

Plug the value of c_1^* into (2.1), we get

$$\begin{aligned} R_{n_2}(\epsilon_1, c_1^*, 0, 0) &= \frac{(\frac{1}{2} - \epsilon_1)b + 2\epsilon_1 n_1}{(\frac{1}{2} - \epsilon_1)b - 2\epsilon_1 n_1} \frac{n_2 R}{b} \\ &= (1 + \frac{4\epsilon_1 n_1}{(\frac{1}{2} - \epsilon_1)b - 2\epsilon_1 n_1}) \frac{n_2 R}{b} \\ &\geqslant (1 + \frac{4\epsilon_1 n_1}{(\frac{1}{2} - 3\epsilon_1)b}) \frac{n_2 R}{b} \\ &\geqslant \frac{n_2 R}{b} \\ &= R_{n_2}(0, 0, 0, 0) \end{aligned}$$

Therefore, $R_{n_2}(\epsilon_1, c_1^*, \epsilon_2, c_2^*) \geqslant R_{n_2}(0, 0, 0, 0)$.

CHAPTER 3. OPTIMAL STORE LOCATIONS n_1^* AND n_2^*

In this section we continue to investigate the game-theoretic framework introduced somewhere before. The main purpose of this section is to explicitly identify the optimal location for each cookie store.

In the last section we have figured out the optimal prices c_1^* and c_2^* as a function of the two store locations. This will be used here to determine the optimal store location n_1^* and n_2^* for the two sellers according to the optimal price. We denote by W_{n_1} the revenue of the seller (located at n_1) if the buyer starts at n_1 (that is, $n = n_1$), and by W_{n_2} the revenue of the seller (located at n_2) if the buyer starts at n_2 (that is, $n = n_2$). Then we can write,

$$W_{n_1} = \eta_{n_1}^{n_1} \cdot c_1 \qquad \textit{and} \qquad W_{n_2} = \eta_{n_2}^{n_2} \cdot c_2,$$

where η_y^x stands for the number of visits of the buyer to the store located at x when he starts at y. If the buyer starts at n_1, then the strong Markov property implies

$$\begin{aligned}\eta_{n_1}^{n_1} &= 1 + (q-\epsilon_1)\frac{n_1-1}{n_1}\eta_{n_1}^{n_1} + (p+\epsilon_1)\frac{n_2-n_1-1}{n_2-n_1}\eta_{n_1}^{n_1} + (p+\epsilon_1)\frac{1}{n_2-n_1}\eta_{n_1}^{n_2} \\ \eta_{n_1}^{n_2} &= (q-\epsilon_2)\frac{1}{n_2-n_1}\eta_{n_1}^{n_1} + (p+\epsilon_2)\frac{b-n_2-1}{b-n_2}\eta_{n_1}^{n_2} + (q-\epsilon_2)\frac{n_2-n_1-1}{n_2-n_1}\eta_{n_1}^{n_2}\end{aligned}$$

Thus we get

$$\eta_{n_1}^{n_2} = \frac{(1-(q-\epsilon_1)\frac{n_1-1}{n_1} - (p+\epsilon_1)\frac{n_2-n_1-1}{n_2-n_1})\cdot \eta_{n_1}^{n_1} - 1}{\frac{p+\epsilon_1}{n_2-n_1}}$$

We then substitute $\eta_{n_1}^{n_2}$ into this equation:

$$(1-(q-\epsilon_2)\frac{n_2-n_1-1}{n_2-n_1} - (p+\epsilon_2)\frac{b-n_2-1}{b-n_2})\cdot \eta_{n_1}^{n_2} = \frac{q-\epsilon_2}{n_2-n_1}\cdot \eta_{n_1}^{n_1}$$

And thus,

$$\eta_{n_1}^{n_1} = \frac{(p+\epsilon_2)(n_2-n_1)n_1 + (q-\epsilon_2)(b-n_2)n_1}{(q-\epsilon_1)(p+\epsilon_2)(n_2-n_1) + (q-\epsilon_1)(q-\epsilon_2)(b-n_2) + (p+\epsilon_1)(p+\epsilon_2)n_1}$$

Similarly, if the buyer starts at n_2,

$$\eta_{n_2}^{n_2} = 1 + (q-\epsilon_2)\frac{n_2-n_1-1}{n_2-n_1}\eta_{n_2}^{n_2} + (p+\epsilon_2)\frac{b-n_2-1}{b-n_2}\eta_{n_2}^{n_2} + (q-\epsilon_2)\frac{1}{n_2-n_1}\eta_{n_2}^{n_1}$$
$$\eta_{n_2}^{n_1} = (q-\epsilon_1)\frac{n_1-1}{n_1}\eta_{n_2}^{n_1} + (p+\epsilon_1)\frac{n_2-n_1-1}{n_2-n_1}\eta_{n_2}^{n_1} + (p+\epsilon_1)\frac{1}{n_2-n_1}\eta_{n_2}^{n_2}$$

Then, we have

$$\eta_{n_2}^{n_1} = \frac{(1-(q-\epsilon_2)\frac{n_2-n_1-1}{n_2-n_1} - (p+\epsilon_2)\frac{b-n_2-1}{b-n_2})\cdot\eta_{n_2}^{n_2} - 1}{\frac{q-\epsilon_2}{n_2-n_1}}$$

Substituted into the equation:

$$(1-(q-\epsilon_1)\frac{n_1-1}{n_1} - (p+\epsilon_1)\frac{n_2-n_1-1}{n_2-n_1})\cdot\eta_{n_2}^{n_1} = \frac{p+\epsilon_1}{n_2-n_1}\cdot\eta_{n_2}^{n_2}$$

Thus,

$$\eta_{n_2}^{n_2} = \frac{(p+\epsilon_1)(b-n_2)n_1 + (q-\epsilon_1)(n_2-n_1)(b-n_2)}{(q-\epsilon_1)(p+\epsilon_2)(n_2-n_1) + (q-\epsilon_1)(q-\epsilon_2)(b-n_2) + (p+\epsilon_1)(p+\epsilon_2)n_1}$$

Considering this problem in the reality situation, we have the following cases:

Case 1: When the buyer starts at n, where $n_1 = n < n_2$

When the buyer starts at $n = n_1$, we first define $W_{n_1}^1$ as the actual benefit of the first seller(located at n_1) and $W_{n_2}^1$ as the actual benefit of the second seller(located at n_2). Then we can write:

$$W_{n_1}^1 = W_{n_1} = \eta_{n_1}^{n_1}c_1^*$$
$$W_{n_2}^1 = \eta_{n_1}^{n_2}c_2^*$$

Hence,

$$W_{n_1}^1 = \frac{(\frac{1}{2}+\epsilon_2)(n_2-n_1)n_1 + (\frac{1}{2}-\epsilon_2)(b-n_2)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times\frac{2\epsilon_1 R}{b}$$
$$W_{n_2}^1 = \frac{(\frac{1}{2}-\epsilon_2)(b-n_2)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times\frac{2\epsilon_2 R}{b}$$

Case 2: When the buyer starts at n, where $n_1 < n_2 = n$

When the buyer starts at $n = n_2$, we first define $W^2_{n_1}$ as the actual benefit of the first seller(located at n_1) and $W^2_{n_2}$ as the actual benefit of the second seller(located at n_2). Therefore we can write:

$$W^2_{n_1} = W_{n_1} = \eta^{n_1}_{n_2} c^*_1$$
$$W^2_{n_2} = \eta^{n_2}_{n_2} c^*_2$$

We can get the value of $\eta^{n_1}_{n_2}$ and $\eta^{n_2}_{n_2}$ directly from the above analysis. Therefore,

$$\begin{aligned} W^2_{n_1} &= \frac{(\frac{1}{2}+\epsilon_1)(b-n_2)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1)+(\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2)+(\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \\ &\quad \times \frac{2\epsilon_1 R}{b} \\ W^2_{n_2} &= \frac{(\frac{1}{2}+\epsilon_1)(b-n_2)n_1+(\frac{1}{2}-\epsilon_1)(b-n_2)(n_2-n_1)}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1)+(\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2)+(\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \\ &\quad \times \frac{2\epsilon_2 R}{b} \end{aligned}$$

Case 3: When the buyer starts at n, where $n_1 < n < n_2$

When the buyer starts at n, we first define $W^3_{n_1}$ as the actual benefit of the first seller(located at n_1) and $W^3_{n_2}$ as the actual benefit of the second seller(located at n_2). Therefore we can write:

$$W^3_{n_1} = \alpha_n W_{n_1} + (1-\alpha_n)\beta W_{n_1}$$

where α_n is the probability that the buyer starts at n and reaches to n_1 before n_2, and β is the probability that that the buyer starts from n_2 and gets to n_1 before b.

$$W^3_{n_2} = (1-\alpha_n)W_{n_2} + \alpha_n(1-\gamma)W_{n_2}$$

where γ is the probability that that the buyer starts from n_1 and gets to 0 before n_2. Based on the property of Markov chain, we know that:

$$\alpha_n = \frac{n_2 - n}{n_2 - n_1}$$
$$\beta = (p+\epsilon_2)\frac{b-n_2-1}{b-n_2}\beta + (q-\epsilon_2)\frac{1}{n_2-n_1} + (q-\epsilon_2)\frac{n_2-n_1-1}{n_2-n_1}\beta$$
$$\Rightarrow \beta = \frac{(q-\epsilon_2)(b-n_2)}{(q-\epsilon_2)(b-n_2)+(p+\epsilon_2)(n_2-n_1)}$$
$$\gamma = (q-\epsilon_1)\frac{1}{n_1} + (q-\epsilon_1)\frac{n_1-1}{n_1}\gamma + (p+\epsilon_1)\frac{n_2-n_1-1}{n_2-n_1}\gamma$$
$$1-\gamma = \frac{(p+\epsilon_1)n_1}{(q-\epsilon_1)(n_2-n_1)+(p+\epsilon_1)n_1}$$

Therefore,

$$W_{n_1}^3 = \frac{(\frac{1}{2}-\epsilon_2)(b-n_2)n_1 + (\frac{1}{2}+\epsilon_2)(n_2-n)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times \frac{2\epsilon_1 R}{b}$$
$$W_{n_2}^3 = \frac{(\frac{1}{2}-\epsilon_1)(n-n_1)(b-n_2) + (\frac{1}{2}+\epsilon_1)(b-n_2)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times \frac{2\epsilon_2 R}{b}$$

In order to figure out the relationship between $W_{n_1}^3$ and n_1, we assume n_2 is fixed , then

$$W_{n_1}^3 = \frac{(\frac{1}{2}-\epsilon_2)(b-n_2)n_1 + (\frac{1}{2}+\epsilon_2)(n_2-n)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times \frac{2\epsilon_1 R}{b}$$
$$= \frac{(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_2)(n_2-n)}{\frac{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)n_2 + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2)}{n_1} + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)} \frac{2\epsilon_1 R}{b}$$

Therefore, $W_{n_1}^3$ is monotonic increasing as $n_1 < n$ increased. The trend of $W_{n_1}^3$ is logically force the n_1 goes to n as close as possible, which force the **case 3** into the **case 2**.

Similarly, if we fixed n_1 and focus on $W_{n_2}^3$ and n_2, then

$$W_{n_2}^3 = \frac{(\frac{1}{2}-\epsilon_1)(n-n_1)(b-n_2) + (\frac{1}{2}+\epsilon_1)(b-n_2)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times \frac{2\epsilon_2 R}{b}$$
$$= \frac{(\frac{1}{2}-\epsilon_1)(n-n_1) + (\frac{1}{2}+\epsilon_1)n_1}{\frac{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1}{b-n_2} + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)} \frac{2\epsilon_2 R}{b}$$

Then for the part

$$Z = \frac{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1)+(\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1}{b-n_2}$$

we can rewrite it as

$$\begin{aligned} Z &= \frac{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)n_2+2\epsilon_1(\frac{1}{2}+\epsilon_2)n_1}{b-n_2} \\ &= -(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)+\frac{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)b+2\epsilon_1(\frac{1}{2}+\epsilon_2)n_1}{-n_2+b} \end{aligned}$$

It it obviously that Z is decreasing as n_2 decreased, so that $W^3_{n_2}$ is monotone increasing as n_2 decreased. Then to make $W^3_{n_2}$ reaches a higher benefit, we want to push n_2 to n as close as possible, which actually turns **case 3** into the **case 1**.

Case 4: When the buyer starts at n, where $n_1 < n_2 < n$

When the buyer starts at n, we first define $W^4_{n_1}$ as the actual benefit of the first seller(located at n_1) and $W^4_{n_2}$ as the actual benefit of the second seller(located at n_2). Therefore we can write:

$$W^4_{n_1} = \alpha_n \beta W_{n_1}$$

where α_n is the probability that the buyer starts at n and reaches to n_2 before b, and β is the probability that that the buyer starts from n_2 and gets to n_1 before b.

$$W^4_{n_2} = \alpha_n W_{n_2}$$

Here, we have

$$\alpha_n = \frac{b-n}{b-n_2} \qquad \textit{and} \qquad \beta = \frac{(q-\epsilon_2)(b-n_2)}{(q-\epsilon_2)(b-n_2)+(p+\epsilon_2)(n_2-n_1)}$$

Thus,

$$\begin{aligned} W^4_{n_1} &= \frac{(\frac{1}{2}-\epsilon_2)(b-n)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1)+(\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2)+(\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \\ &\times \frac{2\epsilon_1 R}{b} \\ W^4_{n_2} &= \frac{(\frac{1}{2}-\epsilon_1)(n_2-n_1)(b-n)+(\frac{1}{2}+\epsilon_1)(b-n)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1)+(\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2)+(\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \\ &\times \frac{2\epsilon_2 R}{b} \end{aligned}$$

Since $n_1 < n_2 < n$, we can focus on the relationship between $W^4_{n_2}$ and n_2. Fixed n_1, we have,

$$\begin{aligned} W^4_{n_2} &= \frac{(\frac{1}{2}-\epsilon_1)(n_2-n_1)(b-n)+(\frac{1}{2}+\epsilon_1)(b-n)n_1}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1)+(\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2)+(\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \\ &\times\frac{2\epsilon_2 R}{b} \\ &= \frac{(\frac{1}{2}-\epsilon_1)n_2+2\epsilon_1 n_1}{2(\frac{1}{2}-\epsilon_1)\epsilon_2 n_2+(\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)b+2\epsilon_1(\frac{1}{2}+\epsilon_2)n_1}\frac{2\epsilon_2(b-n)R}{b} \\ &= \Big(\frac{1}{2\epsilon_2}+\frac{Z}{2(\frac{1}{2}-\epsilon_1)\epsilon_2 n_2+(\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)b+2\epsilon_1(\frac{1}{2}+\epsilon_2)n_1}\Big)\frac{2\epsilon_2(b-n)R}{b} \end{aligned}$$

where

$$Z = -\frac{1}{2\epsilon_2}\Big((\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)b+2\epsilon_1(\frac{1}{2}+\epsilon_2)n_1\Big)+2\epsilon_1 n_1 < 0$$

This indicates that $W^4_{n_2}$ is monotone increasing as n_2 increased. In order to maximize the earning of the second seller, we will require n_2 goes to n, which turns **case 4** into **case 2**

Case 5: When the buyer starts at n, where $n < n_1 < n_2$

When the buyer starts at n, we first define $W^5_{n_1}$ as the actual benefit of the first seller(located at n_1) and $W^5_{n_2}$ as the actual benefit of the second seller(located at n_2). Therefore we can write:

$$W^5_{n_1} = \alpha_n W_{n_1}$$

where α_n is the probability that the buyer starts at n and reaches to n_1 before 0.

$$W^5_{n_2} = \alpha_n(1-\gamma)W_{n_2}$$

where γ is the probability that that the buyer starts from n_1 and gets to 0 before n_2. As what we discussed before,

$$\alpha_n = \frac{n}{n_1} \qquad \textit{and} \qquad 1-\gamma = \frac{(p+\epsilon_1)n_1}{(q-\epsilon_1)(n_2-n_1)+(p+\epsilon_1)n_1}$$

Thus,

$$W^5_{n_1} = \frac{(\frac{1}{2}-\epsilon_2)(b-n_2)n + (\frac{1}{2}+\epsilon_2)(n_2-n_1)n}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times \frac{2\epsilon_1 R}{b}$$

$$W^5_{n_2} = \frac{(\frac{1}{2}+\epsilon_1)(b-n_2)n}{(\frac{1}{2}-\epsilon_1)(\frac{1}{2}+\epsilon_2)(n_2-n_1) + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)(b-n_2) + (\frac{1}{2}+\epsilon_1)(\frac{1}{2}+\epsilon_2)n_1} \times \frac{2\epsilon_2 R}{b}$$

Since $n < n_1 < n_2$, the only thing left is the relationship between $W^5_{n_1}$ and n_1. We then fixed n_2,

$$\begin{aligned} W^5_{n_1} &= \frac{-(\frac{1}{2}+\epsilon_2)n_1 + (\frac{1}{2}-\epsilon_2)b + 2\epsilon_2 n_2}{2(\frac{1}{2}-\epsilon_1)\epsilon_2 n_2 + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)b + 2\epsilon_1(\frac{1}{2}+\epsilon_2)n_1} \frac{2\epsilon_1 nR}{b} \\ &= \Big(-\frac{1}{2\epsilon_1} + \frac{Z}{2(\frac{1}{2}-\epsilon_1)\epsilon_2 n_2 + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)b + 2\epsilon_1(\frac{1}{2}+\epsilon_2)n_1}\Big)\frac{2\epsilon_1 nR}{b} \end{aligned}$$

where

$$Z = -(-\frac{1}{2\epsilon_1})(2(\frac{1}{2}-\epsilon_1)\epsilon_2 n_2 + (\frac{1}{2}-\epsilon_1)(\frac{1}{2}-\epsilon_2)b) + (\frac{1}{2}-\epsilon_2)b + 2\epsilon_2 n_2 > 0$$

Therefore, $W^5_{n_1}$ is monotone increasing as n_1 decreased. In order to maximize the earning of the first seller, we will require n_1 goes to n., which turns **case 5** into **case 1**

In conclude, **case 1** and **case 2** are the best choice for the above 5 different cases. And we still need to consider a following special case.

Special Case: When the buyer starts at n, where $n_1 = n_2 = n$

Under this situation, both of the two stores coincide with the starting point,$n_1 = n_2 = n$. Therefore the number of visits to the two stores should be same, write as η^n_n. Meanwhile, define $W^6_{n_1}$ as the actual benefit of the first seller(located at n_1) and $W^6_{n_2}$ as the actual benefit of the second seller(located at n_2).

$$\begin{aligned} \eta^n_n &= 1 + (q-\epsilon^*)\frac{n-1}{n}\eta^n_n + (p+\epsilon^*)\frac{b-n-1}{b-n}\eta^n_n \\ W^6_{n_1} &= \frac{1}{2}\eta^n_n c^*_1 \\ W^6_{n_2} &= \frac{1}{2}\eta^n_n c^*_2 \end{aligned}$$

where

$$\epsilon^* = \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

then, we get

$$\eta_n^n = \frac{2(b-n)n}{b+(\epsilon_1+\epsilon_2)(2n-b)}$$

Therefore,

$$W_{n_1}^6 = \frac{(b-n)n}{b+(\epsilon_1+\epsilon_2)(2n-b)}\frac{2\epsilon_1 R}{b}$$
$$W_{n_2}^6 = \frac{(b-n)n}{b+(\epsilon_1+\epsilon_2)(2n-b)}\frac{2\epsilon_2 R}{b}$$

In order to find out the best location for the two stores based on the Nash equilibrium, we need to compare their payoff under **case 1** $(n_1 = n < n_2)$, **case 2** $(n_1 < n_2 = n)$ and **special case** $(n_1 = n = n_2)$. As we have mentioned before, a Nash equilibrium problem is to make both of the two seller want to stay at their location.

Under **case 1**,we want to know if the second seller is willing to stay. From the above analysis, we know that $W_{n_2}^1$ is increasing as n_2 decreased, which indicates that n_2 should get as close as possible to n. Since the location are integers, then the maximum $W_{n_2}^1$ happens when $n_2 = n+1$. Hence we want to compare the actual payoff of the second seller when $n_1 = n, n_2 = n+1$ and $n_1 = n_2 = n$. By substitution and calculation, one can easily show that $W_{n_2}^6 > W_{n_2}^1(n_2 = n+1)$, which indicates that the second seller would like to move from $n_2 = n+1$ to $n_2 = n$. This result force the **case 1** to be the **special case**. However, this special case is actually not a Nash equilibrium result. Because if the two seller share the same store location, then any one of them can slightly reduce their price to gain all the trading opportunity with the buyer.

Under **case 2**, we want to know if the first seller is still want to keep the original location. As we have discussed before, $W_{n_1}^2$ increased as n_1 increased, which means that the maximum $W_{n_1}^2$ happens when $n_1 = n-1$. Then compare the actual payoff of the first seller when $n_1 = n-1, n_2 = n$ and $n_1 = n_2 = n$. Through calculation, one can show that $W_{n_1}^2(n_1 = n-1) > W_{n_1}^6$, which indicates that the first seller would like to stay at $n_1 = n-1$. As for the second seller, if he move forward to $n_2 = n+1$, then the profit of the first seller would decrease, and so that the first guy would move forward to $n_1 = n$, which is actually stay in the same case. If the second seller want to move backward to $n_2 = n-1$, then this situation changes into special case, which is not a Nash problem. Hence, the second seller would also stay at the original setting.

From what we discussed above, we can conclude that the best store location for both of the two sellers are actually $n_1 = n - 1$ and $n_2 = n$, where n is the starting point of the buyer.

CHAPTER 4. CONCLUSION

We discussed a 2-dimensional modification gambler's ruin scenario, which has some common performances as the excited regular nearest-neighbor random walk on $\mathbb{Z}$,except being localized to the two certain point. In the whole Markov chain, the states 0 and b are recurrent states (with the transition probability equal to 1) and other states are transient states(with the transition probability smaller than 1). General speaking, the deformation of transition kernel at two different point can be described as two sellers that providing an instantaneously increased probability(smaller than $\frac{1}{6}$) in the forward direction when the buyer visits the stores. In this game, the goal of the buyer is to maximize his expected earning which can be expressed in terms of a difference between the revenue and the cost. The cost of the buyer is determined by the price of a cookie, which has been negotiated between the buyer and the sellers. Through this paper, we discussed a special situation, a fair moving probability for the buyer when he face the parts without modification. Based on the analysis, the equilibrium price and the stores' location are two independent variables. Since the starting point of the buyer can be anywhere between 0 and b, the sellers need to choose the stores' location to maximize their expected benefit. For conclusion, we include all the reasonable relationship between the starting point and the stores' location, which turns out that the nash equilibrium store location for the two sellers are $n_1 = n - 1$ and $n_2 = n$, where n is the starting point of the buyer. However, since the result is just established on the certain case that $p = q = \frac{1}{2}$ and $\epsilon_1, \epsilon_2 < \frac{1}{6}$, this assumption may not be true for all the situations. In fact, we do analysis the different situations, and some results indicates that the equilibrium price is somehow depending on the store's location. The further results need an even deeper analysis.

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