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An a priori error analysis for a Picard-Chebyshev solution to an initial value problem

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TABLE OF CONTENTS

I. INTRODUCTION 1

II. THE PICARD-CHEBYSHEV ITERATION METHOD FOR INITIAL VALUE PROBLEMS 14
   A. Method for First Order Equations 18
   B. Method for Higher Order Equations 20
   C. The van der Pol Equation 22

III. AN INTERVAL DEPENDENT NECESSARY AND SUFFICIENT CONDITION FOR N-PLACE ACCURACY IN A PICARD-CHEBYSHEV SOLUTION TO AN INITIAL VALUE PROBLEM 26
   A. The Accuracy Theorem 28
   B. Proof of the Accuracy Theorem 29

IV. APPLICATIONS OF THE ACCURACY THEOREM 37

V. SUMMARY AND CONCLUSIONS 50

VI. BIBLIOGRAPHY 52

VII. ACKNOWLEDGEMENTS 54

VIII. APPENDICES 55
   A. Picard-Chebyshev Iteration Program 55
   B. Program for Solution to van der Pol's Equation 57
   C. Delta Iteration Program 59
I. INTRODUCTION

The use of Chebyshev polynomials in numerical analysis is based upon least squares approximation theory and the simplification of that theory which is provided by orthogonal polynomials.

Suppose we attempt to approximate a function $F(x)$ with a polynomial $P_R(x)$. For the continuous case a least squares approximation requires that

$$\int_a^b [P_R(x) - F(x)]^2 dx$$

be as small as possible. It should be noted that this does not guarantee that $P_R(x)$ approximates $F(x)$ with the same degree of accuracy throughout $[a,b]$. It only means that $P_R(x)$ is sufficiently close to $F(x)$ so as to guarantee that for any set of points in the domain, the total length of all the intervals measured from $P_R(x)$ to $F(x)$ can be made arbitrarily small. If

$$\lim_{R \to \infty} \int_a^b [P_R(x) - F(x)]^2 dx = 0,$$

we say that the sequence $P_R(x)$ converges in the mean to $F(x)$. The integral in the above limit is referred to as the mean-square deviation of $P_R(x)$ from $F(x)$.

There are two important characteristics of $P_R(x)$ that need to be determined. First, its degree $R$ and, secondly, its
coefficients $a_i$, $i = 0, 1, 2, \ldots, R$. We shall for the moment ignore the problem of determining $R$ and proceed with the theory which establishes the formulae for $a_i$. We may now write

$$
\int_a^b \left[ (a_0 + a_1 x + \ldots + a_R x^R) - F(x) \right]^2 dx.
$$

In order to make our approximation as flexible as possible, we shall introduce a weight function $W(x) > 0$. The purpose of such a weight function is to insure a greater degree of accuracy at certain critical points in $[a, b]$.

Finally, then, the quantity which we wish to minimize will be denoted by $S$ and can be written as follows:

$$
S = \int_a^b W(x) \left[ (a_0 + a_1 x + \ldots + a_R x^R) - F(x) \right]^2 dx
$$

A necessary condition for minimizing $S$ is given by

$$
\frac{\partial S}{\partial a_i} = 0, \quad i = 0, 1, \ldots, R.
$$

Application of this condition yields

$$
\int_a^b W(x)x^i [(a_0 + a_1 x + \ldots + a_R x^R) - F(x)] dx = 0, \quad i = 0, 1, \ldots, R.
$$
This formula provides a linear system $Aa = d$ where

$$a_{ij} = \int_a^b W(x)x^{i+j}dx, \quad d_i = \int_a^b W(x)x^i F(x)dx. \quad \quad (6)$$

The solution of such a system can be simplified considerably with the use of orthogonal polynomials. We shall denote such a set of orthogonal polynomials by $\phi_i(x)$, $i = 0, 1, 2, \ldots, R$. With the appropriate set of coefficients $c_i$, $i = 0, 1, 2, \ldots, R$, we can write

$$p_R(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_R\phi_R(x). \quad \quad (7)$$

Proceeding as before we generate the linear system $Ac = d$ where

$$\int_a^b a_{ij} = 0. \quad \quad (8)$$

The polynomials $\phi_i(x)$ must span the same space as $1, x, \ldots, x^N$ and are chosen so that

$$\int_a^b W(x)\phi_i(x)\phi_j(x)dx = 0, \quad i \neq j. \quad \quad (9)$$

Now the matrix $A$ is diagonal and we can solve for the $c_i$.

$$c_i = \frac{\int_a^b W(x)\phi_i(x)F(x)dx}{\int_a^b W(x)\phi_i^2(x)dx} \quad \quad (10)$$
Sets of orthogonal polynomials have been developed and are usually referred to by various names of famous mathematicians. The literature is extensive on any of the following:

1. Legendre Polynomials
2. Laguerre Polynomials
3. Hermite Polynomials
4. Chebyshev Polynomials

Considerable research has been done on the use of a truncated infinite series of Chebyshev polynomials as a form for the solution to initial value problems.

The motivation for the use of Chebyshev polynomials in this way is provided in part by a consideration of the Chebyshev Minimax Theorem. When we consider $P_R(x)$ as an approximation to $F(x)$, it would be an advantage to select $P_R(x)$ in such a way so as to insure that

$$E_R = \text{MAX}|P_R(x) - F(x)|$$

is a minimum. The Chebyshev Minimax Theorem provides this insurance.

If, as a starting point we take $F(x) = 0$ and restrict ourselves to the interval $[-1,1]$, we can express the minimax property for Chebyshev polynomials by the following theorem:

**MINIMAX THEOREM**

If: $T_R(x) = \cos R\theta$ where $x = \cos \theta$ with leading
coefficient = 2^{R-1},

Then: Among all polynomials \( P_R(x) \) of degree \( R > 0 \) with leading coefficient 1, \( 2^{1-R}T_R(x) \) deviates least from zero in \([-1,1]\). Expressed mathematically as follows:

\[
\max_{-1 \leq x \leq 1} |P_R(x)| > \max_{-1 \leq x \leq 1} |2^{1-R}T_R(x)| = 2^{1-R}.
\]

Proof (20, p. 209): The equality in the above equation follows immediately from

\[
\max_{-1 \leq x \leq 1} |T_R(x)| = \max_{-1 \leq x \leq 1} |\cos \theta| = 1.
\]

To prove the inequality, assume that \( P_R(x) \) is a polynomial of the stated type for which

\[
\max_{-1 \leq x \leq 1} |P_R(x)| < 2^{1-R}.
\]

This hypothesis results in a contradiction.

Notice that the polynomial \( 2^{1-R}T_R(x) = 2^{1-R}\cos \theta \) has the alternating values of \( 2^{1-R}, -2^{1-R}, 2^{1-R}, \ldots, 2^{1-R} \) at the \( N+1 \) points \( x \) that correspond to \( \theta = 0, \pi/R, 2\pi/R, \ldots, R\pi/R = \pi \). By the above assumption,
Q(x) = 2^{1-R}T_R(x) - P_R(x) has the same sign as 2^{1-R}T_R(x) at these points and therefore must have at least R zeros in the interval -1 ≤ x ≤ 1. But Q(x) is a polynomial of degree at most R - 1 and could not, therefore, have more than R - 1 zeros.

Our interest, of course, goes far beyond the trivial function F(x) = 0. A satisfactory account of the general theory for the general function F(x) is provided by the following statement:

For F(x) continuous in -1 ≤ x ≤ 1, it can be shown that for each R (we are using R for the number of terms in the Chebyshev series whereas the article being cited here uses N) there is a polynomial P_R(x) of degree R which is the best approximation to F(x) in the minimax sense. From this fact can be proved the existence of R + 2 real abscissae, in -1 < x < 1, at which e_R(x) = P_R(x) - F(x) has equal and opposite values, and hence that the best approximation of degree R is unique. The simple observation that P_R(x) is formed of a particular linear combination of the basis elements 1, x, x^2, ..., x^R and x^{R+1}, which generate P_{R+1}(x), then shows that E_1 ≥ E_2 ≥ ... ≥ E_R, where E_R = \text{MAX}\{\text{e}_R\}. Finally, the theorem of Weierstrass, that for a continuous function F(x) in a finite range there is a polynomial P_R(x) of sufficiently high degree
such that \(|e_R(x)| \leq \varepsilon\) at all points of the range, shows that our minimax polynomials tend uniformly to \(F(x)\) as \(R \to \infty\) (9, pp. 41-42).

Further motivation for the use of Chebyshev polynomials will be provided when we examine in Chapter II some of the basic properties of the Chebyshev polynomials.

The formulae corresponding to (10) can be obtained for Chebyshev polynomials by considering a special case of the Fourier series representation for \(F(x)\). We have

\[
F(x) = \frac{a_0}{2} + \sum_{K=1}^{R} (a_K \cos Kx + b_K \sin Kx)
\]

where

\[
a_K = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos Kx \, dx \quad \text{and} \quad b_K = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin Kx \, dx.
\]

By letting \(x = \cos \theta\) and considering the range \(-1 \leq x \leq 1\), we define

\[
F(x) = F(\cos \theta) = G(\theta) \quad 0 \leq \theta \leq \pi.
\]

If we expand \(G(\theta)\) in a Fourier series, we obtain

\[
G(\theta) = \frac{a_0}{2} + \sum_{K=1}^{R} a_K \cos K\theta
\]
where

\[ a_K = \frac{2}{\pi} \int_0^{\pi} G(\theta) \cos K\theta \, d\theta. \]

By interpreting (12) in terms of \( x \), we can obtain

\[ \theta = \cos^{-1} x \]

\[ d\theta = -(1 - x^2)^{-1} \, dx \]

\[ \theta = 0 \implies x = +1 \]

\[ \theta = \pi \implies x = -1 \]

\[ \cos K(\cos^{-1} x) = T_K(x) \]

so that

(13) \[ F(x) = \sum_{K=0}^{K=R} a_K T_K(x) \]

where

\[ a_0 = \frac{1}{\pi} \int_{-1}^{1} (1 - x^2)^{-1/2} F(x) \, dx \]

\[ a_K = \frac{2}{\pi} \int_{-1}^{1} T_K(x) (1 - x^2)^{-1/2} F(x) \, dx. \]

It is now appropriate to mention a few remaining points before we move on to a statement of the purpose of this work.
First, the Dirichlet conditions are sufficient to guarantee point-wise convergence of the Fourier series and therefore of the Chebyshev series. This fact is formalized in the following theorem:

If: \( F(x) \) is bounded on the open interval \((-\pi, \pi)\) such that \((-\pi, \pi)\) can be decomposed into a finite number of open subintervals on each of which \( F(x) \) is either nonincreasing or nondecreasing. Or if \( F(x) \) possesses a finite number of infinite discontinuities in \([-\pi, \pi]\), let \( F(x) \) satisfy the preceding conditions except in arbitrarily small neighborhoods of the points of infinite discontinuity and let \( \int_{-\pi}^{\pi} F(x)dx \) be absolutely convergent.

Then: The series

\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx
\]

where

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos kx \, dx \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin kx \, dx
\]

converges at each point \( x = x_0 \) in \((-\pi, \pi)\) to the value

\[
F(x_0 + 0) + F(x_0 - 0)
\]
and at \( x = \pi \) to the value

\[
\frac{F(\pi - 0) + F(-\pi + 0)}{2}
\]

wherever the limits exist (4, pp. 279-280).

Secondly, there are no terminal discontinuities with the Chebyshev series since the entire finite range of the function \( F(x) \) has been transformed to the interval \([-1,1]\).

Thirdly, the rate of convergence of the Fourier series is dependent upon the rate of decrease of its coefficients. For a periodic \( F(x) \) where \( F^{(D-1)}(x) \) is continuous and \( F^{(D)}(x) \) is bounded and \( F^{(i)}(x) \) for \( i = 0,1,2,\ldots,D \) satisfy the Dirichlet conditions, the Fourier coefficients of \( F(x) \) are of the order \( R^{-D-1} \). On the other hand, the Chebyshev series converges like \( 2^{1-R(R^*)^{-1}} \), which for large values of \( R \) is a considerable improvement (9, pp. 24-25; 4, p. 282).

In this work we shall be concerned with the solution to an initial value problem. The differential equation may be of any order and of any degree. As is usually the case, the theoretical considerations will be directed toward the first order equation since higher order equations can be reduced to a system of first order equations. However, the iterative method which we shall use to produce the Chebyshev series solution can be easily modified to handle equations of order greater than 1 without the usual reduction to a system of
first order equations. These methods are discussed and illustrated in Sections A, B, and C of Chapter II.

For the purpose of the Introduction we shall point out the least workable aspect of the basic method, for it is this aspect which forms the basis for this paper. The method requires that the number of terms in the Chebyshev series solution to an initial value problem be known a priori. When expanding a given function in its Chebyshev series, the same problem occurs. However, some extremely interesting techniques have been developed by Elliot which can evaluate and estimate the coefficients in the series (7, pp. 274-284). This information provides a way to determine how long the series should be. The number of terms is, of course, a question of accuracy; but Elliot's techniques refer to a known function whereas the function we are interested in is known only implicitly through the initial value problem for which it is the solution. Elliot's techniques, and others like them, are therefore not applicable to our problem.

A review of the literature indicates that the best solution to the problem is the following suggestion by Clenshaw and Norton:

The minimum value of $R$ (we are using $R$ for the number of terms in the Chebyshev series whereas the article being cited here uses $N$) which is necessary to represent both $y$ and $F(x,y)$ to the desired accuracy will not, in general, be known in advance. Although no harm would result from
a value larger than necessary, this would clearly be uneconomic. Indeed, during the early iterations when the approximation is poor, it may be desirable to use a value of $R$ smaller than ultimately required to achieve full accuracy.

One possibility is to start with a moderate or small value, say $R = 4$, and then introduce further coefficients only when their inclusion appears necessary for further improvement of the solution (2, pp. 88-92). The article goes on to suggest the quantitative ways one could use to determine when it was necessary to increase the value of $R$ and by how much. However, once $R$ has been increased, the entire solution must be rerun and the same criterion for increasing $R$ must be applied again. The article suggests that $R$ be increased by 2 each time. In the following problem,

$$y' = y^2, \quad y(-1) = 0.4,$$

one does not obtain ten places of accuracy until the value of $R$ has reached 26. Starting $R$ at 4 and increasing by 2 each time the criterion requires it would mean that the solution would have to be run in its entirety a total of 12 times. Clearly this is at least as uneconomic as selecting an initial value for $R$ which was larger than necessary.

It is the purpose of this paper to develop a method which will determine the domain of the independent variable $x$ which
for a given value of $R$ is necessary and sufficient for achieving some specified degree of accuracy. This shift in emphasis from the number of terms to the size of the domain is one of the main points in this work.

Rather than the symbol $N$ for the number of terms, which as we have noted is common in much of the literature on this subject, we have been and will continue to use the symbol $R$ and use $N$ instead to refer to the number of leading zeros in the final coefficient of the Chebyshev series. In other words, we shall use $N$ to refer to $N$-place accuracy.

We shall now turn our attention to the iterative method which we shall use to obtain the Chebyshev series solution to an initial value problem. Since it is based on the Picard iterates as well as the Chebyshev series, we have named it the Picard-Chebyshev iteration method for initial value problems.
II. THE PICARD-CHEBYSHEV ITERATION METHOD
FOR INITIAL VALUE PROBLEMS

There exists in the literature several iteration methods whose purpose is to iteratively converge to the first \( R + 1 \) Chebyshev coefficients in the Chebyshev series solution to an ordinary differential equation. We will focus our attention primarily on the technique based on Picard iteration. This technique will illustrate in the most straightforward way the problem which this paper seeks to solve. This problem is common to all of the iterative techniques, and our solution of it will apply to other techniques as well as the Picard-Chebyshev technique.

The formulae for the Picard iteration scheme is based on discrete least squares approximation theory. If \( F(x) \) is to be approximated by \( P_R(x) \) with error \( e_R(x) = P_R(x) - F(x) \), then we seek to find \( P_R(x) \) such that

\[
S = \sum_{K=0}^{K=M} W(x_K) e_R^2(x_K),
\]

where \( M \) is the number of discrete points used, is a minimum. The norm \( E_R = \text{MAX} |e_R(x)| \) is also minimized when \( P_R(x) \) is the minimax polynomial (supra pages 5-7). Following the development for the continuous case, we seek an expansion of the form

\[
P_R(x) = \sum_{r=0}^{r=R} c_r \phi_r(x)
\]
where the functions $\phi_r(x)$ satisfy

\begin{equation}
\sum_{k=0}^{K=M} W(x_k) \phi_r(x_k) \phi_s(x_k) = 0, \ r \neq s.
\end{equation}

The $c_r$ are then given by

\begin{equation}
c_r = \frac{\sum_{k=0}^{K=M} W(x_k) F(x_k) \phi_r(x_k)}{\sum_{k=0}^{K=M} W(x_k) \phi_r^2(x_k)}, \ r = 0, 1, \ldots, R.
\end{equation}

We may now write (14) as follows (9, p. 68):

\begin{equation}
S_{\text{MIN}} = \sum_{k=0}^{K=M} W(x_k) \left\{ F^2(x_k) - \sum_{r=0}^{R} c_r^2 \phi_r^2(x_k) \right\}
\end{equation}

A convenient method for converting the above equations to a set which is relevant to Chebyshev theory is to consider the function $G(\theta) = F(\cos \theta) = F(x_k), 0 \leq \theta \leq \pi$, and begin with the following trigonometric identity:

\begin{equation}
\frac{1}{2} + \cos \theta + \cos 2\theta + \ldots + \cos(R-1)\theta + \frac{1}{2} \cos R\theta
\end{equation}

\begin{equation}
= \frac{1}{2} \sin R\theta \cot \frac{1}{2}\theta
\end{equation}

The right-hand side vanishes if $\theta = K\pi/R$ for any integral value of $K$. Since

\begin{equation}
2 \cos r\theta \cos s\theta = \cos(r + s)\theta + \cos(r - s)\theta,
\end{equation}
it follows that the independent functions \( \phi_r(\theta) = \cos r\theta \) satisfy the orthogonality conditions

\[
\sum_{K=0}^{K=R} \phi_r(\theta_K)\phi_s(\theta_K) = 0 \quad \theta_K = \frac{K\pi}{R} \quad r \neq s
\]

where the first and last terms of the sum are taken with factor 1/2, as indicated by the double prime ("'). Further, we find from (19) and (20) that the normalization factors for these orthogonal functions are

\[
\sum_{K=0}^{K=R} \phi_r^2(\theta_K) = \begin{cases} 
\frac{1}{2}R, & r \neq 0, R \\
R, & r = 0, R.
\end{cases}
\]

A trigonometric least squares approximation for the function \( G(\theta) \), over the \( R + 1 \) equally spaced points \( \theta_K = K\pi/R, \quad K = 0, 1, 2, \ldots, R \), with weights 1/2 at first and last points and unity elsewhere, is then given by

\[
P_N(\theta) = \sum_{r=0}^{r=N} c_r \cos r\theta, \quad \text{where}
\]

\[
c_r = \frac{2}{R} \sum_{K=0}^{K=R} G(\theta_K) \cos r\theta_K, \quad \theta_K = \frac{K\pi}{R},
\]

where \( N \) is an arbitrary truncation of the approximating polynomial and where the single prime (') indicates that only the first term of the sum is taken with factor 1/2. From the last part of (22) it can be deduced that if \( N = R \), we will
produce the exact fit with the "interpolation" formula

\[ P_R(\theta) = \sum_{r=0}^{R} c_r \cos r\theta, \text{ where} \]

\[ c_r = \frac{2}{R} \sum_{K=0}^{R} G(\theta_K) \cos r\theta_K, \theta_K = \frac{K\pi}{R}. \]

Substituting \( x = \cos \theta \) into (23) so that \( G(\theta) = F(x) \) in 
\(-1 \leq x \leq 1\) yields the corresponding Chebyshev least squares
approximation, and we have

\[ P_N(x) = \sum_{r=0}^{N} c_r T_r(x), \text{ where} \]

\[ c_r = \frac{2}{R} \sum_{K=0}^{R} F(x_K) T_r(x_K), \text{ where} \]

\[ x_K = \cos \left( \frac{K\pi}{R} \right) \]

with a double prime in the first summation if \( N = R \) for the
exact fit. Here the data points are not at equal intervals
in \( x \), but they are equally spaced in \( \theta = \cos^{-1}x \).

In addition to these basic formulae there exists a set
of formulae for integrating a Chebyshev series. For simple
integration we have

\[ \int T_r(x) \, dx = -\int \cos r\theta \sin r\theta \, d\theta \]
\[ = -\frac{1}{2} \left\{ \sin(r+1)\theta - \sin(r-1)\theta \right\} \]
\[ = \frac{1}{2} \left\{ \frac{1}{r+1} T_{r+1}(x) - \frac{1}{r-1} T_{r-1}(x) \right\} \]
For special cases $r = 0$ and $r = 1$ we find

\[
\int T_0(x)\,dx = T_1(x), \quad \int T_1(x)\,dx = \frac{1}{4} \left\{ T_0(x) + T_2(x) \right\}.
\]

If $P_R'(x) = \sum_{r=0}^{r=R-1} c_r T_r(x)$, then we may use (26) to obtain

\[ P_R(x) = \sum_{r=0}^{r=R-1} a_r T_r(x) = \frac{1}{4} c_0 T_0(x) + \frac{1}{2} c_0 T_1(x) + \frac{1}{4} c_1 T_2(x) \]

\[ + \frac{1}{2} \sum_{r=2}^{r=N-1} c_r \left\{ \frac{T_{r+1}(x)}{r+1} - \frac{T_{r-1}(x)}{r-1} \right\}, \]

and by equating coefficients of $T_r(x)$ on each side we find

\[ a_0 = \frac{1}{4} c_1 \]

\[ a_r = \frac{1}{2r} (c_{r-1} + c_{r+1}), \quad r = 1, 2, \ldots, R-2 \]

\[ a_{R-1} = \frac{1}{2(R-1)} c_{R-2}, \quad a_R = \frac{1}{2R} c_{R-1}. \]

We are now ready to turn our attention to the method itself.

A. Method for First Order Equations

The existence and uniqueness theorem for the initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]
is usually stated in the following way:

If: \( F(x,y) \) is bounded in \( E^2 \)

\( F(x,y) \) is continuous in \( E^2 \)

\( F(x,y) \) satisfies a Lipschitz condition in \( E^2 \)

Where \( E^2 \) is some region of the \( xy \)-plane,

Then: For \( |x - x_0| < H \) there exists a unique function \( y(x) \) such that \( y' = F(x,y) \) and \( y(x_0) = y_0 \).

The proof is somewhat varied from author to author but can generally be thought of as consisting of the following four steps:

Step 1: Determine \( H \).

Step 2: Prove that the Picard iterates are:

(a) Continuous.

(b) \(< MH \) where \( |F(x,y)| \leq M \).

Step 3: Prove that the Picard iterates converge uniformly to \( y(x) \).

Step 4: Prove that \( y(x) \) is unique.

The method which we are about to describe can be applied to any initial value problem for which the above theorem is valid and is basically a matter of constructing the Picard iterates in terms of their Chebyshev series. The method is outlined in the following six steps:

Step 1: After determining the domain for a given value of \( R \) and a given value of \( N \) by the method of Chapter III of this paper, express the initial condition \( y_0 \) in its Chebyshev series. When this
is done, all the coefficients except the first one, \( a_0 \), will be zero. \( a_0 \) will be \( 2y_0 \).

**Step 2:** Evaluate the current Chebyshev series for \( y(x) \) at \( x(i) \) where \( x(i) = \cos(\pi i/R) \), \( i = 0,1,\ldots,R \).

**Step 3:** Evaluate \( F(x(i),y(i)) \), \( i = 0,1,2,\ldots,R \). Note: It is at this point in the method where we avoid the usual difficulties associated with the nonlinear forms of \( F(x,y) \).

**Step 4:** Compute the Chebyshev series coefficients using the discrete least squares formula given as equation (25).

**Step 5:** The Chebyshev series for \( F(x,y) \) can now be integrated using equations (27) and (28). This will produce all but \( a_0 \) in the series for \( y(x) \) and \( a_0 \) can be determined from the initial values.

**Step 6:** If the accuracy criteria is satisfied, the solution is complete. If not, return to Step 2.

The computer program, written in a subset of Fortran IV, which carries out the above steps for a 28-term Chebyshev series solution to an initial value problem satisfying the conditions of the Picard-dependent existence and uniqueness theorem is provided in Appendix A.

**B. Method for Higher Order Equations**

As was mentioned in the Introduction, this method is easily modified to handle equations of order \( \geq 2 \). It should
be pointed out, however, that for boundary problems such as the one we will use as an example of the method for higher order equations, the convergence of the method is no longer guaranteed.

The method is based on three relatively simple modifications of the method for first order equations, specifically at Steps 1, 2, and 5. For an \( m \)th order problem we have values at \( x_0 \) for the first \( m - 1 \) derivatives as well as the equation

\[
y^{(m)} = F(x, y, y'(1), \ldots, y^{(m-1)}).
\]

At Step 1 we begin with the Chebyshev series for \( y^{(m-1)} \) where the leading coefficient will be \( 2^s y^{(m-1)}(x_0) \), and all remaining coefficients will be zero. Application of (28) will produce the Chebyshev series for \( y^{(m-2)} \) where the value of the leading coefficient is unknown and the value of the second coefficient has been determined by (28). Since \( y^{(m-2)}(x_0) \) is known, we can use it to determine the leading coefficient. We now proceed to integrate this series for \( y^{(m-3)} \) where once again the leading coefficient is determined by the value of \( y^{(m-3)}(x_0) \). This process is continued until we have Chebyshev series for \( y \) and its first \( m - 1 \) derivatives.

At Step 2 we proceed as before except that we will be evaluating each of the Chebyshev series for \( y \) and its first \( m - 1 \) derivatives at \( x(i) \).
Steps 3 and 4 are the same. At Step 5 we will perform the integrations as in Step 1 where each time the leading coefficient of the Chebyshev series for \( y^{(L-1)} \), \( L = 1, 2, \ldots, m \) is determined by the value of \( y^{(L-1)}(x_0) \).

Step 6 is unaltered.

Usually \( m \) is quite small and most of the examples in the literature are for \( m = 2 \). The most common example is the van der Pol equation, and so it is provided here as an illustration of a second order equation. This example does have an additional value since it is actually a boundary value problem. Basically this means that instead of evaluating the leading coefficient of the Chebyshev series for \( y^{(1)} \), we will simply continue with the integration of \( y^{(1)} \) and thus produce the series for \( y \) with the first two coefficients unknown. These can then be evaluated simultaneously from the known boundary values.

C. The van der Pol Equation

The equation can be expressed as follows:

\[
\frac{d^2 y}{dx^2} = (1 - y^2) \frac{dy}{dx} - y
\]

\[
y\left(\frac{-1}{4}\right) = 0, \quad y\left(\frac{1}{4}\right) = 2
\]
By introducing the following change in variable:

\[ t = \frac{1}{x} \]

we obtain

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 4 \frac{dy}{dx}
\]

\[
\frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dx} \cdot \frac{dx}{dt} \right) = \frac{d}{dt} \left( 4 \frac{dy}{dx} \right)
\]

\[
= 4 \frac{d}{dx} \left( \frac{dy}{dx} \right) \frac{dx}{dt}
\]

\[
= 16 \frac{d^2y}{dx^2}.
\]

Substitution yields

\[
16 \frac{d^2y}{dx^2} + 4(y^2 - 1) \frac{dy}{dx} + y = 0
\]

\[
\frac{d^2y}{dx^2} = \frac{1}{4} (1 - y^2) \frac{dy}{dx} - \frac{1}{16} y
\]
Initially \( \frac{dy}{dx} \) can be written as

\[
\frac{dy}{dx} = \frac{b_0}{2}
\]

where \( b_0 \) is unknown.

Integration yields

\[
y = \frac{a_0}{2} + a_1 T_1(x)
\]

where \( a_1 = \frac{b_0}{2} \) and \( a_0 \) is unknown.

We have then the following set of simultaneous equations:

\[
y(-1) = \frac{a_0}{2} - \frac{b_0}{2} = 0
\]

\[
y(1) = \frac{a_0}{2} + \frac{b_0}{2} = 2
\]

Solving we obtain

\[
a_0 = 2
\]

\[
b_0 = 2
\]

This completes Step 1 with the initial Chebyshev series for \( \frac{dy}{dx} \) being

\[
\frac{dy}{dx} = 1
\]
The remainder of the solution is carried out by the program which appears in Appendix B. In this particular solution the value of $R$ was arbitrarily set at 8, and the values of the coefficients in the Chebyshev series after 10 iterations were as follows:

\[
\begin{align*}
    a_0 &= 2.0680487796 \\
    a_1 &= 1.0239810217 \\
    a_2 &= -0.0327932076 \\
    a_3 &= -0.0248556255 \\
    a_4 &= -0.0013669784 \\
    a_5 &= 0.0009011440 \\
    a_6 &= 0.0001374549 \\
    a_7 &= -0.0000265403 \\
    a_8 &= -0.000016588
\end{align*}
\]
III. AN INTERVAL DEPENDENT NECESSARY AND SUFFICIENT CONDITION FOR N-PLACE ACCURACY IN A PICARD-CHEBYSHEV SOLUTION TO AN INITIAL VALUE PROBLEM

In this chapter we will attempt to formulate and prove a theorem which will relate the following:

1. N, where N is the number of places of decimal accuracy in the last coefficient of a finite Chebyshev series,

2. R, where R is the index of the last coefficient in a finite Chebyshev series,

3. δ, where δ is the length of the interval measured from x₀ on which the solution is constructed, and,

4. n, where x₀ < n < x₀ + δ.

Before turning to the problem of accuracy in the resultant finite Chebyshev series solution, let us first point out that the question of accuracy can be dealt with in terms of the magnitude of the coefficient which appears last in the series. This is due to the fact that each term in the Chebyshev series consists of a coefficient multiplied by a Chebyshev polynomial. The Chebyshev polynomials have a maximum absolute value equal to one. Thus, the largest contribution any term can make is equal to the absolute value of its coefficient. Furthermore, the value of the coefficients decreases in proportion to the reciprocal of \( R!2^{R-1} \) for sufficiently large R. Therefore,
the last term will provide an accurate indication of the accuracy afforded by the finite series in question. For example, if \( a_R \) has \( N \) leading zeros, then the series which terminates with \( a_R T_R(x) \) will provide \( N \)-place accuracy.

Experience with actual solutions provides ample illustrations of the above, although there are some exceptions. For example, the initial value problem

\[
y' = 1 - \sqrt{y} + \cos(\pi x), \quad y(-1) = 0.962556070550,
\]

when solved by the Picard-Chebyshev iteration method yields the following:

\[
a_{14} = -0.000000034904
\]

If one assumed that the series terminating with \( a_{14} T_{14}(x) \) would yield 7-place accuracy, he would error because

\[
a_{15} = -0.000000237369.
\]

This value implies by the above considerations only 6-place accuracy. It should be pointed out, however, that none of the test case problems solved by this author have ever yielded a decrease in accuracy by more than one decimal place when considering the next coefficient, and most of the time the coefficients are monotone decreasing after the first few terms.
Finally, we should point out that in an initial value problem's solution we do not solve for the infinite series and then truncate it to reflect the desired accuracy. We truncate it before the solution is started. As a result, truncation error could conceivably contribute enough to the last term in the finite series so as to make our accuracy considerations, which depend on that last term, invalid. It turns out, fortunately, that this is not a serious problem. We will consider it in sufficient detail in Chapter IV.

A. The Accuracy Theorem

\[ a_R = 10^{-N} \]

If and only if:

1. There exists a number \( S \) such that the finite Chebyshev series containing \( R + S + 1 \) terms produces the same number of leading zeros in \( a_R \) as the corresponding infinite Chebyshev series, and,

2. The Picard-Chebyshev iteration method is carried out in the domain \([x_0, x_0 + \delta]\) where \( \delta \) satisfies

\[
\delta = \left( \frac{R!2^{2R-1}}{|P(R)(\eta)| \cdot 10^N} \right)^{1/R}
\]

where:

(a) \( R + 1 \) is the number of terms in the finite Chebyshev series for \( y \), and,
(b) \( n \) is some number such that \( x_0 < n < x_0 + \delta \), and,

(c) \( N \) is the number of leading zeros desired in \( a_R \).

The number \( S \) will be discussed in Chapter IV and need not concern us at this point. Our first concern in the proof will be to derive an expression for \( a_R \).

B. Proof of the Accuracy Theorem

The Chebyshev series for the function \( F(x) \), \( a \leq x \leq b \), is produced by first transforming the domain \([a,b]\) to the domain \([-1,1]\). The formula for the \( R \)th coefficient of the Chebyshev series is then written as follows:

\[
(30) \quad a_R = \frac{2}{\pi} \int_{-1}^{1} (1-\xi^2)^{-1/2} F(a\xi + \beta) T_R(\xi) d\xi
\]

where

\[
\alpha = \frac{b - a}{2} \quad \text{and} \quad \beta = \frac{b + a}{2}.
\]

Since our concern is with derivatives of \( F(x) \), it is convenient to revise (30) by expanding \( F(a\xi + \beta) \) in its Taylor series about the point \( \xi = 0 \). We obtain

\[
(31) \quad F(a\xi + \beta) = \sum_{K=0}^{K=R-1} \frac{(a\xi)^K}{K!} \frac{d^K F(\beta)}{du^K} + \frac{(a\xi)^R}{R!} \frac{d^R F(\xi)}{du^R}
\]
where

\[ u = \alpha \xi + \beta \]

and

\[ a < \eta < b. \]

Since all powers of \( \xi \) can be expressed as a finite summation of Chebyshev polynomials each of which is of the same degree or less as the power of \( \xi \) and since the Chebyshev polynomials are orthogonal on \([-1,1]\), we may substitute (31) into (30) and obtain

\[
a_{R} = \frac{2}{\pi} \int_{-1}^{1} (1-\xi^2)^{-1/2} \frac{d^{(R)}(\eta)}{R!} \frac{d^{(R)}T_{R}(\xi)}{d\xi} d\xi
\]

\[
= \frac{a}{R!} \frac{2}{\pi} \frac{d^{(R)}P(n)}{d\xi} \int_{-1}^{1} (1-\xi^2)^{-1/2} R_{R}(\xi) d\xi.
\]

Following Elliot (7, pp. 274-275), we now convert the above integral into the complex plane. Cauchy's integral formula provides the following:

\[
G(\xi) = \frac{1}{2\pi i} \int_{C} \frac{G(z)}{z-\xi} dz
\]

where \( G(\xi) \) is analytic in and on the contour \( C \), where \( C \) is any contour containing the interval \(-1 \leq \xi \leq 1\). By letting \( G(\xi) = \xi^{R} \), we can rewrite the expression in (32) as follows:
(34) \[ a_R = \frac{2}{R!} \frac{d^{(R)}(x)}{d u_R} \left( \int_C \frac{z^R dz}{z-\xi} \right) (1-\xi^2)^{-1/2} T_R(\xi) \, d\xi \]

Since \( C \) has been chosen so as to contain \([-1,1]\) and since \( z^R \) is analytic everywhere, we may reverse the order of integration to obtain

(35) \[ a_R = \frac{\alpha_R}{\pi^2} \left( -i \right) \frac{d^{(R)}(x)}{d u_R} \int_C 1 (1-\xi^2)^{-1/2} (z-\xi)^{-1} T_R(\xi) \, d\xi \, dz. \]

The bracketed integral can be evaluated as follows. First, for each nonzero complex number \( z \) we let \( z^{1/2} \) denote the principle square root of \( z \), that is, if \( \theta \) is the argument of \( z \) where \(-\pi < \theta \leq \pi\), then \( z^{1/2} = |z|^{1/2} e^{i\theta/2} \). We begin with the bracketed integral in (35).

(36) \[ \int_{-1}^{1} (1-\xi^2)^{-1/2} (z-\xi)^{-1} T_R(\xi) \, d\xi \]

We assume that \( z \) in (36) is not a real number on the closed interval \([-1,1]\). A change in variable from \( \xi \) to \( \theta \) using

(37) \[ T_R(\xi) = \cos R\theta, \quad \xi = \cos \theta \]

yields

(38) \[ \int_{-1}^{1} (1-\xi^2)^{-1/2} (z-\xi)^{-1} T_R(\xi) \, d\xi = \frac{1}{2} \int_{0}^{2\pi} (z-\cos \theta)^{-1} \cos R\theta \, d\theta. \]
Let $\Gamma$ be the positively oriented unit circle in the complex $\zeta$-plane with center at the origin. We now proceed to interpret the integral on the right side of (38) to obtain

$$\int_{\Gamma} \frac{\zeta^{2R+1}}{\zeta^R [\zeta-((z+(z^2-1)^{1/2})][\zeta-((z-(z^2-1)^{1/2})]} \, d\zeta$$

$$= i \left\{ \int_{\Gamma} \frac{\zeta^R}{[\zeta-((z+(z^2-1)^{1/2})][\zeta-((z-(z^2-1)^{1/2})]} \, d\zeta$$

$$+ \int_{\Gamma} \frac{1}{\zeta^R [\zeta-((z+(z^2-1)^{1/2})][\zeta-((z-(z^2-1)^{1/2})]} \, d\zeta \right\}.$$ 

Since $(z+(z^2-1)^{1/2})(z-(z^2-1)^{1/2}) = 1$, we can consider the case where

$$|z+(z^2-1)^{1/2}| < 1 \text{ and } |z-(z^2-1)^{1/2}| > 1.$$ 

The case where the inequalities are reversed can be treated in a similar fashion. We let

(39) \hspace{1cm} a = z+(z^2-1)^{1/2}.
The residue of

\[ \frac{\zeta^R}{(\zeta-a)(\zeta-1/a)} \]

at a is

\[ \frac{a^{R+1}}{a^2-1}. \]

The residue of

\[ \frac{1}{\zeta^R(\zeta-a)(\zeta-1/a)} \]

at a is

\[ \frac{1}{a^{R-1}(a^2-1)} \]

and at 0 is

\[ \frac{a}{a^2-1} \cdot \left( a^R - \frac{1}{a^R} \right). \]

Hence

\[ \int_{-1}^{1} (1-\xi^2)^{-1/2}(z-\xi)^{-1}T_{R}(\xi)\,d\xi = -2\pi \frac{1}{a-1/a} \cdot a^R \]

\[ = \frac{-\pi}{(z^2-1)^{1/2}(z-(z^2-1)^{1/2})^R}. \]
Thus for the case where

\[ |z+(z^2-1)^{1/2}| > 1 \]
\[ |z-(z^2-1)^{1/2}| < 1 \]

we obtain

\[
\int_{-1}^{1} (1-\xi^2)^{-1/2} (z-\xi)^{-1} \tau_R(\xi) d\xi = \frac{\pi}{(z^2-1)^{1/2}(z+(z^2-1)^{1/2})^R }.
\]

Substitution of (41) in (35) gives

\[
a_R = \frac{a_R^{(-1)}}{\pi} \frac{F(R)(\eta)}{C} \int (z^2-1)^{-1/2} (z+(z^2+1)^{1/2})^{-R} z^P dz.
\]

The following transformation is then applied to (42):

\[
z = 1/2(a+1/a)
\]

so that

\[
(z^2-1)^{1/2} = 1/2(a-1/a)
\]

and

\[
dz = 1/2(1-1/a^2) da.
\]
Substitution of (43), (44), and (45) into (42) yields

\[ a_R = \frac{a^R(i)F(R)(\eta)}{R!\pi} \int_C a^{-R-1}[1/2(a-1/a)]^R da. \]

The binomial expansion of \(1/2(a+1/a)^R\) is also the Laurent expansion and therefore the contour integral becomes

\[ \int_C a^{-R-1}[1/2(a+1/a)]^R da = 2\pi i (1/2)^R. \]

Substitution of (47) into (46) now gives the final version of the formula for \(a_R\).

\[ a_R = \frac{a^R}{R!2^{R-1}} F(R)(\eta) \]

Let us now assume that the interval beginning at \(x_0\) is \([x_0,x_0+\delta]\) and transform \([x_0,x_0+\delta]\) to \([-1,1]\). The equations for doing so are given in (30) of this chapter. We have

\[ a = \frac{(x_0+\delta)-x_0}{2} = \frac{\delta}{2} \]

\[ \beta = \frac{(x_0+\delta)+x_0}{2} = \frac{\delta+2x_0}{2}. \]
Substituting (49) in (48) we obtain

\[ a_R = \frac{\delta/2}{R!2^{R-1}} f(R) (n) = \frac{\delta f(R) (n)}{R!2^{R-1}} \quad x_0 < n < x_0 + \delta \]

It is now a simple matter to show that if we would like the series for \( y \) to be

\[ y = \sum_{r=0}^{\infty} a_r T_r(x) \]

with \( N \)-place accuracy in \( a_R \); that is,

\[ |a_R| = 10^{-N} \]

then by substitution of (50) into (52) we obtain

\[ \delta = \left( \frac{R!2^{2R-1}}{|f(R) (n)| \cdot 10^N} \right)^{1/R} \]

This and equation (63) on page 41, which derives the value for \( S \) (supra page 28), complete the proof.

Applying the theorem we have just proved does present some difficulties, and we will discuss each of them in the next chapter.
IV. APPLICATIONS OF THE ACCURACY THEOREM

An obvious question which might be asked at this juncture has to do with using (50) just as it stands. To do so would, of course, yield only an estimate for $a_R$ because the values of $\delta$ and $n$ would have to be estimated. And, in the case of the initial value problem, $F(x)$ is actually $y(x)$; and, therefore, its derivatives could not be calculated exactly except at $x_0$. Even if $n$ was equal to $x_0$, it would still be extremely cumbersome to calculate the derivatives at $x_0$ for even the simplest of problems. Therefore, (50) is unworkable especially for large values of $R$.

There is a way, however, to make significant use of (50) with regard to the solution of an initial value problem. This is accomplished by abandoning the following question:

(1) How many terms must we use in the Chebyshev series solution to obtain $N$-place accuracy?

in favor of the question:

(2) How large a domain, beginning at $x_0$, can we use in order to obtain $N$-place accuracy in a Chebyshev series solution to $y' = F(x,y)$, $y(x_0) = y_0$ which contains $R + 1$ terms?

This shift in emphasis from number of terms to size of domain can be facilitated by (53) and avoids the unworkable aspects of (50).

There are several difficulties which arise in the applications of the accuracy theorem which we will now discuss.
prior to the actual applications themselves. First, equation (50) is true only for the appropriate value of \( n \), and we do not have access to this value. We can, however, by considering the weighting function for Chebyshev polynomials, namely \((1-\xi^2)^{-1/2}\), which is designed to make the error small at -1 and +1, argue that the best value for \( n \) is the one which minimizes the weight function and therefore maximizes the error. This value occurs at \( \xi = 0 \). \( \xi = 0 \) maps to the midpoint of \([x_0, x_0+\delta]\). Therefore we take \( n \) to be

\[
(54) \quad n = \frac{\delta + 2x_0}{2}
\]

Secondly, equation (53) must now be solved iteratively for \( \delta \) since (54) introduces \( \delta \) on the right of (53). Theoretically, we are assured of the existence of a value for \( n \) which would satisfy the following:

\[
(55) \quad |a_R| = \frac{\delta^R |F(R)(n)|}{R!2^{2R-1}} = 10^{-N}
\]

Our interpretation of \( n \) would alter (55) as follows:

\[
(56) \quad \frac{\delta^R |F(R)\left(\frac{\delta + 2x_0}{2}\right)|}{R!2^{2R-1}} \approx 10^{-N}
\]
We can solve this approximate equality for $\delta$ and obtain the following iteration for $\delta$:

$$(57) \quad \delta_{i+1} = \left( \frac{R! \cdot 2^{2R-1}}{F(R) \left( \frac{\delta + 2x_0}{2} \right)^N} \right)^{1/R}$$

Appendix C contains a program which performs the above iteration and describes the technique for computing

$$\left| F(R) \left( \frac{\delta + 2x_0}{2} \right)^N \right| .$$

Once we have a value for $\delta$ we can proceed with the Picard-Chebyshev solution with the assurance that

$$|a_R| \approx 10^{-N}.$$ 

Finally, we need to mention the problem of truncation error referred to on page 28. It can be shown that the relationship between the coefficients of the infinite series which we shall denote as $B_R$ and the coefficients of the finite series $A_R$ is such that $S = 2$ (supra page 28) is for all practical purposes sufficient to insure that truncation error will not contribute significantly to the last coefficient in the finite series (9, p. 68). We proceed as follows:
For any function, \( F(x) \), the infinite Chebyshev series for the continuous case is denoted by

\[
F(x) = \sum_{r=0}^{\infty} B_r T_r(x)
\]

where

\[
B_r = \frac{2}{\pi} \int_{-1}^{1} (1-x^2)^{-1/2} F(x) T_r(x) \, dx = \frac{2}{\pi} \int_0^{\pi} F(\cos \theta) \cos r \theta \, d\theta.
\]

Similarly, the finite Chebyshev series is denoted by

\[
P_R(x) = \sum_{r=0}^{R} A_r T_r(x)
\]

where

\[
A_r = \frac{2}{R} \sum_{K=0}^{K=R} F(x_K) T_r(x_K), \quad x_K = \cos \frac{K\pi}{R}.
\]

Substitution of (58) into (59) yields

\[
A_r = \frac{2}{R} \sum_{K=0}^{K=R} \left( T_r(x_K) \sum_{r=0}^{r=\infty} B_r T_r(x_K) \right)
\]

The orthogonality of the Chebyshev polynomials with respect to summation as well as the following:
\[(60) \quad T_r(x_K) = \cos \frac{x_K \pi}{R} = \cos \left[ \left(2PR^2r\right) \frac{K \pi}{R} \right] = T_{2PR^2r}(x_K), \quad p = 1, 2, \ldots \]

yields the following expression:

\[(61) \quad A_r = B_r + (B_{2r-r} + B_{2r+r}) + (B_{4r-r} + B_{4r+r}) + \ldots \]

In the event that \(B_{R+2}, B_{R+3}, \ldots\) are negligible, which in our case is very likely since \(|B_R| < 10^{-N}\) and the Chebyshev coefficients decrease in proportion to the reciprocal of \(R! \times 2^{R-1}\), we can use (61) to show that \(A_R\) agrees with \(B_R\) within our accuracy specifications except when \(r = R - 1\). In this case

\[(62) \quad A_{R-1} = B_{R-1} + B_{R+1} \]

By selecting \(S = 2\) (supra pages 28 and 36)

\[(63) \quad A_{R+1} = B_{R+1} + B_{R+3} \quad \text{and,} \quad A_R = B_R + B_{R+2} \]

where \(B_{R+2}\) is negligible and \(A_R\) would therefore be unaffected by the truncation. This is the approach we have taken in the
solutions presented in this chapter, and for all the solutions it was more than sufficient.

There are several ways one could illustrate the applications of the accuracy theorem. Our approach shall be based on the following six problems:

(1) \( y' = -y \) \[ y(-1) = 2.718281828459 \] (2, p. 90)

(2) \( y' = y^2 \) \[ y(-1) = 0.4 \] (12, pp. 79-81)

(3) \( y' = \exp(-y) \) \[ y(-1) = 0.0 \] (24, p. 360)

(4) \( y' = \sin y \) \[ y(-1) = 0.705026843560 \] (12)

(5) \( y' = x - y^2 \) \[ y(-1) = -0.018971824750 \] (12)

(6) \( y' = 1 - \sqrt{y} + \cos \pi x \) \[ y(-1) = 0.962556070550 \] (12)

These were selected as being representative and are taken from the literature with only slight modifications.

The first three problems have simple analytic solutions and thus offer an opportunity to compare the results of the Picard-Chebyshev method to the true solution. The fourth and fifth problems also have analytic solutions which are provided in Norton's article (12), but these analytic solutions are much more difficult and complicated. The sixth problem has no finite analytical solution.

For each of these problems we shall do the following:

(1) Solve the problem using 28 terms in the Picard-Chebyshev method with \( \delta = 2 \).
(2) Compute a δ which will produce N-place accuracy in $a_R$ where N is different from the accuracy of $a_R$ that was obtained in the 28-term solution.

(3) Solve the problem using $R + 2$ terms and check $a_R$ for N-place accuracy.

Selected delta values are shown in tabular and graphic form for each problem. The coefficients of the series produced by doing the above are arranged in two tables, the first of which shows the solution for $\delta = 2$ and $R$ set arbitrarily at 27. The second shows the solution for a selected value of $R$ and one of the computed values of $\delta$. The accuracy of $a_R$ should be greater than or equal to the value of $N$ which corresponds to the delta which was used in the solution. The values which need to be compared are enclosed in the table by two horizontal lines. Due to spacial consideration we have only displayed the first 11 coefficients of each solution.
Problem 1

\[ y' = -y \]
\[ y(-1) = 2.718281828459 \]

Results of Delta Iteration for \( R = 8 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.595772782445</td>
</tr>
<tr>
<td>6</td>
<td>2.533073503352</td>
</tr>
<tr>
<td>7</td>
<td>1.818237611597</td>
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<tr>
<td>8</td>
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</tr>
<tr>
<td>9</td>
<td>0.992974132835</td>
</tr>
<tr>
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<td>0.266249922131</td>
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</tbody>
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Results of Picard-Chebyshev Iteration

\[ \delta = 2 \quad \text{\( R = 27 \)} \]

<table>
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<tbody>
<tr>
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<tr>
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<tr>
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\[ \delta = 1.818\ldots \quad \text{\( R = 10 \)} \]

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</table>
Problem 2

\[ y' = y^2 \quad y(-1) = 0.4 \]

Results of Delta Iteration for \( R = 8 \)

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<tr>
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<tbody>
<tr>
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<td>4</td>
<td>1.891638484579</td>
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<td>1.574932007775</td>
</tr>
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<td>6</td>
<td>1.299565543998</td>
</tr>
<tr>
<td>7</td>
<td>1.064295194214</td>
</tr>
</tbody>
</table>

\[ \delta = F(N) \]

Results of Picard-Chebyshev Iteration

For \( \delta = 2 \) and \( R = 27 \):

<table>
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<tbody>
<tr>
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<tr>
<td>2</td>
<td>0.260990337042</td>
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<tr>
<td>3</td>
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</tr>
<tr>
<td>7</td>
<td>0.002122012930</td>
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<td>8</td>
<td>0.000810536165</td>
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<td>9</td>
<td>0.0003099597514</td>
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<tr>
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<td>0.000118255727</td>
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For \( \delta = 1.574 \ldots \) and \( R = 10 \):

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<td>2</td>
<td>0.066505723388</td>
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<td>3</td>
<td>0.010581390045</td>
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<tr>
<td>4</td>
<td>0.001486276452</td>
</tr>
<tr>
<td>5</td>
<td>0.000189735157</td>
</tr>
<tr>
<td>6</td>
<td>0.000022423459</td>
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<td>7</td>
<td>0.000002486439</td>
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</tr>
<tr>
<td>10</td>
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</tbody>
</table>
Problem 3

\[ y' = \exp(-y) \quad y(-1) = 0.0 \]

Results of Delta Iteration for \( R = 8 \)

<table>
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<tbody>
<tr>
<td>4</td>
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<tr>
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<td>6</td>
<td>1.391791146061</td>
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<tr>
<td>7</td>
<td>0.916878431466</td>
</tr>
<tr>
<td>8</td>
<td>0.349698333258</td>
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<tr>
<td>9</td>
<td>0.289871721778</td>
</tr>
</tbody>
</table>

Results of Picard-Chebyshev Iteration

\[ \delta = 2 \quad R = 27 \]

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<thead>
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<tbody>
<tr>
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<tr>
<td>3</td>
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</tr>
<tr>
<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>0.000552487242</td>
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<td>6</td>
<td>-0.000123365425</td>
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<tr>
<td>7</td>
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<td>8</td>
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<td>9</td>
<td>0.000001582193</td>
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\[ \delta = 1.391... \quad R = 8 \]

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<td>3</td>
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<td>5</td>
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<td>7</td>
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<td>0.00000092473</td>
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Problem 4

\[ y' = \sin y \quad y(-1) = 0.705026843560 \]

Results of Delta Iteration for \( R = 7 \)

<table>
<thead>
<tr>
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<tbody>
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<td>3</td>
<td>2.967528139860</td>
</tr>
<tr>
<td>4</td>
<td>1.847133735376</td>
</tr>
<tr>
<td>5</td>
<td>1.450160842701</td>
</tr>
<tr>
<td>6</td>
<td>1.184471706612</td>
</tr>
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<td>7</td>
<td>0.747525901063</td>
</tr>
<tr>
<td>8</td>
<td>0.509896862474</td>
</tr>
</tbody>
</table>

\[ \delta = F(N) \]

Results of Picard-Chebyshev Iteration

\( \delta = 2 \quad R = 27 \)

<table>
<thead>
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<th>( a_r )</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>1</td>
<td>0.895867258385</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>-0.031670934242</td>
</tr>
<tr>
<td>4</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>5</td>
<td>0.0016686839892</td>
</tr>
<tr>
<td>6</td>
<td>-0.000000000000</td>
</tr>
<tr>
<td>7</td>
<td>-0.0000101626744</td>
</tr>
<tr>
<td>8</td>
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</tr>
<tr>
<td>9</td>
<td>0.000000000000</td>
</tr>
<tr>
<td>10</td>
<td>-0.000000000000</td>
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</table>

\( \delta = 1.847... \quad R = 9 \)

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<td>2</td>
<td>0.013022139346</td>
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<td>3</td>
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<tr>
<td>4</td>
<td>-0.001487294116</td>
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<td>0.001091712193</td>
</tr>
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<td>6</td>
<td>0.000113872373</td>
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<td>0.000001488254</td>
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<tr>
<td>10</td>
<td>0.000000000000</td>
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</table>
Problem 5

\[ y' = x - y^2 \]

\[ y(-1) = -0.018971824750 \]

Results of Delta Iteration for \( R = 6 \)

<table>
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<tbody>
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<tr>
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<td>1.060429589678</td>
</tr>
<tr>
<td>6</td>
<td>0.631841353289</td>
</tr>
<tr>
<td>7</td>
<td>0.365642349412</td>
</tr>
</tbody>
</table>

\[ \delta = F(N) \]

Results of Picard-Chebyshev Iteration

\( \delta = 2 \) \( R = 27 \)

<table>
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<tbody>
<tr>
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<tr>
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<tr>
<td>2</td>
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\( \delta = 0.631... \) \( R = 8 \)

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<tr>
<td>6</td>
<td>0.00000183131</td>
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<td>7</td>
<td>-0.00000038337</td>
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<tr>
<td>8</td>
<td>0.000000000056</td>
</tr>
<tr>
<td>9</td>
<td>* ************</td>
</tr>
<tr>
<td>10</td>
<td>* ************</td>
</tr>
</tbody>
</table>
Problem 6

\[ y' = 1 - \sqrt{y} + \cos \pi x \quad y(-1) = 0.962556070550 \]

Results of Delta Iteration for \( R = 10 \)

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<tr>
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<td>3.076876327097</td>
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<tr>
<td>4</td>
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<tr>
<td>6</td>
<td>1.087264432934</td>
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<tr>
<td>7</td>
<td>0.885596343309</td>
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</tbody>
</table>

Results of Picard-Chebyshev Iteration

\( \delta = 2 \) \( R = 27 \)

<table>
<thead>
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<th>( a_r )</th>
</tr>
</thead>
<tbody>
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<tr>
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<td>-0.000015381966</td>
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</table>

\( \delta = 1.488... \) \( R = 12 \)

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<td>9</td>
<td>0.000009750956</td>
</tr>
<tr>
<td>10</td>
<td>-0.000002586350</td>
</tr>
</tbody>
</table>
V. SUMMARY AND CONCLUSIONS

The most general statement that can be made about what we have said is that the question of accuracy, that is, the question of error analysis, is not to be thought of in the usual way when constructing the solution to an initial value problem in the form of a finite Chebyshev series. The usual way has been to fix the length of the interval and concentrate on the number of terms one should use in the series. Instead we have suggested fixing the length of the series and concentrate on the length of the interval in which the solution fits the accuracy requirements. This allows the use of a relatively short Chebyshev series while at the same time meeting the demands of accuracy. In the event that the interval is smaller than desired, the series can be evaluated at the end of that interval and a new series constructed from the "initial value" thus obtained for the next segment of the desired interval.

For example, in Problem #2, 16 terms were required for six-place accuracy over the interval [-1,1] whereas only 8 terms were required for a solution over [-1.0, 0.574932007775]. If the remainder of the interval from 0.574932007775 to 1.0 could be described with six-place accuracy by a Chebyshev series containing 8 or less terms, then we would be better off with a two-series solution over [-1,1] than with the one-series solution over [-1,1]. The possibility of fewer terms in an N-series solution than in a one-series solution seems quite
high to this author, although further research would be necessary to establish this from an analytical point of view. It is suggested strongly in actual practice.

Several general conclusions are possible. First, working with the Picard-Chebyshev method has convinced this author that it is superior to most of the methods commonly used. It avoids the problem of matrix inversion altogether, and the nonlinear case is handled exactly as the linear, whereas in other methods the nonlinear case requires special treatment. Secondly, the form of the solution is superior to methods which only produce tables of data as a solution. The form, a finite Chebyshev series, is easily differentiated, integrated, and evaluated. No interpolation is ever necessary. And, finally, with reference to the basic contribution of this paper, it is possible to construct a solution and know that all accuracy requirements will be met without having to either form an extremely lengthy Chebyshev series with the hope that one has made it "long enough," or rerun the solution with several additional terms each time until the accuracy requirements are satisfied. The method is limited as far as initial value problems are concerned to problems satisfying the hypotheses of the existence theorem (supra page 18).
VI. BIBLIOGRAPHY


VII. ACKNOWLEDGEMENTS

The author wishes to acknowledge his wife, Linda, and children, Heather, Chad, and Shannan, for their patience and understanding during the development and writing of this thesis.

I would also like to express my appreciation to Graceland College and in particular Dr. Bruce M. Graybill, Dr. Harold L. Condit, and Dr. William T. Higdon, all of whom not only encouraged me to pursue my graduate studies at Iowa State University, but also provided a two-year leave of absence by which it could be accomplished. Also, a special thanks to Michelle Jones, Secretary of the Division of Science and Mathematics, for her assistance in typing the manuscript.

A word of appreciation is also due to my professors at Iowa State University and especially Dr. Fred M. Wright and Dr. Roy F. Keller who made special arrangements for me at a particularly difficult time.

Thanks are due to my committee members for their time and efforts in my behalf.

But most of all I am indebted to Dr. Clair G. Maple whose constant encouragement and assistance proved to be a necessary and sufficient condition for the completion of my studies. I would like to acknowledge this fact by formally dedicating this thesis to him.
VIII. APPENDICES

A. Picard-Chebyshev iteration program

C**************************************************************
C PROGRAM TO SOLVE ORDINARY DIFFERENTIAL EQUATIONS  
C IN TERMS OF CHEBYSHEV SERIES. EQUATION MUST BE  
C FIRST ORDER, LINEAR, OR NONLINEAR. THE NUMBER  
C OF TERMS IS SET AT R+1 IN THIS LISTING. R.LE.27.  
C**************************************************************

INTEGER R
DIMENSION X(28), A(28), Y(28), F(28), B(28)

C THE COEFFICIENTS OF THE INITIAL CHEBYSHEV SERIES  
C ARE THOSE WHICH REFLECT THE INITIAL CONDITION.  
C THEREFORE, THEY WILL ALL EQUAL ZERO EXCEPT A(1).  
C THIS PROGRAM REQUIRES THE INITIAL CONDITION AT X=-1.  

DATA A/1.925112141100,27*0.0/
R=12
RR=R

DELTA PRODUCES A SOLUTION ON (-1,-1+DELTA)  
THIS DELTA GUARANTEES 5 LEADING ZEROS IN A(10).  

DELTA=1.488840646665
PI=3.141592653589
ITER=12

C EVALUATION OF THE CURRENT CHEBYSHEV SERIES FOR  
C Y AT X(I).  

1 DO 3 I=1,R+1
   X(I)=DELTA/2.0*(COS((I-1)*PI/RR)+1.0)-1.0
   AT=ATAN(SQRT(1.0-X(I)*X(I))/X(I))
   IF(X(I))11,12,12
   11 AT=AT+PI
   12 Y(I)=A(I)/2.0
   DO 2 J=2,R+1
   2 Y(I)=Y(I)+A(J)*COS((J-1)*AT)

C EVALUATION OF F(X(I),Y(X(I)))  

3 F(I)=1.0-SQRT(Y(I))+COS(PI*X(I))
C*****************************************************************************
C*****************************************************************************
DO 5 I=1,R+1
   B(I)=F(I)/2.0
DO 4 J=2,R
   4 B(I)=B(I)+F(J)*COS(PI*(I-1)*(J-1)/RR)
   5 B(I) = (B(I) + (-1)**(I-1)*F(R+1)/2.0)*2.0/RR
C*****************************************************************************
C EVALUATION OF THE COEFFICIENTS IN THE CHEBYSHEV SERIES FOR Y. FOX AND PARKER, PAGE 61.
C*****************************************************************************
DO 6 I=2,R
   6 A(I)=(B(I-1)-B(I+1))/(2.0**(I-1))
   A(R+1)=B(R)/(2.0*RR)
   T=A(1)
   A(1)=1.925112141100
   DO 7 I=2,R+1
      7 A(I)=A(1)+2.0*A(I)*(-1.0)**I
      ITER=ITER+1
      IF(ITER-12)9,8
   8 WRITE(6,20)
      ITER=0
   9 WRITE(6,30)(A(I),I=1,25,4),(A(I),I=2,26,4),(A(I),I=3,27,4),(A(I),I=4,28,4)
C*****************************************************************************
C THE ITERATION WILL TERMINATE WHEN THE INITIAL COEFFICIENT IS ACCURATE TO TEN DECIMAL PLACES.
C*****************************************************************************
D=ABS(T-A(1))
   IF(D<0.00000000001)10,10,1
10 STOP
20 FORMAT(1H1,4X,'A(0,1,2,3)',8X,'A(4,5,6,7)',8X,'A(8,9,10,11)',6X,'A(12,13,14,15)',4X,'A(16,17,18,19)',4X,'A(20,21,22,23)',4X,'A(24,25,26,27)')
30 FORMAT(/(IX,7F18.12))
END
B. Program for solution to van der Pol's equation

C******************************************************************************************
C SOLUTION TO VAN DER POL'S EQUATION. *
C******************************************************************************************
DIMENSION D(9), A(9), Y(9), F(9), B(9), YP(9)
PI=3.141592653589
C******************************************************************************************
C STEP 1 AS DESCRIBED ON PAGES 19 AND 21. *
C******************************************************************************************
A(1)=2.0
A(2)=1.0
D(1)=2.0
1 WRITE (6,30) (A(I),I=1,9)
C******************************************************************************************
C STEP 2 AS DESCRIBED ON PAGES 19 AND 21. *
C******************************************************************************************
DO 3 I=1,9
Y(I)=A(I)/2.0
YP(I)=D(I)/2.0
DO 2 J=2,9
YP(I)=YP(I)+D(J)*COS(PI*(I-1)*(J-1)/8.0)
2 Y(I)=Y(I)+A(J)*COS(PI*(I-1)*(J-1)/8.0)
C******************************************************************************************
C STEP 3 AS DESCRIBED ON PAGE 20. *
C******************************************************************************************
3 F(I)=0.25*(1.0-Y(I)*Y(I))*YP(I)-0.062500*Y(I)
DO 5 I=1,9
B(I)=F(I)/2.0
C******************************************************************************************
C STEP 4 AS DESCRIBED ON PAGE 20. *
C******************************************************************************************
DO 4 J=2,8
4 B(I)=B(I)+F(J)*COS(PI*(I-1)*(J-1)/8.0)
5 B(I)=(B(I)+(-1)***(I-1)*F(9)/2.0)*0.25
C******************************************************************************************
C STEP 5 AS DESCRIBED ON PAGES 20 AND 22. *
C******************************************************************************************
DO 6 I=2,8
D(I)=(B(I-1)-B(I+1))/(2.0*(I-1))
D(9)=B(8)/16.0
DO 7 I=3,8
7 A(I)=(D(I-1)-D(I+1))/(2.0*(I-1))
\[ A(9) = A(8) / 16.0 \]
\[ T = A(1) \]
\[ A(1) = 2.0 - 2.0 \times (A(3) + A(5) + A(7) + A(9)) \]
\[ A(2) = 1.0 - (A(4) + A(6) + A(8)) \]
\[ D(1) = 2.0 \times A(2) + D(3) \]

C**************************************************************
C     STEP 6 AS DESCRIBED ON PAGE 20.
C     THE ITERATION WILL TERMINATE WHEN THE INITIAL
C     COEFFICIENT IS ACCURATE TO TEN DECIMAL PLACES.
C**************************************************************
E = \text{ABS}(T - A(1))
\text{IF}(E - 0.00000000001)10,10,1
10 STOP
30 FORMAT (1H0, 9F14.10)
END
C. Delta iteration program

**C PROGRAM FOR ITERATION TO OBTAIN DELTA AS DEFINED IN EQUATION 57 ON PAGE 39. R AND N MUST BE PROVIDED BY THE USER AS DISCUSSED ON PAGE 36. X(1) AND Y(1) ARE INITIAL VALUES. H=STEP SIZE. NS=NUMBER OF POINTS IN (-1,+1). R=INDEX ON FINAL CHEBYSHEV COEFFICIENT. N=NUMBER OF PLACES OF ACCURACY IN RTH COEFFICIENT.**

```fortran
INTEGER E,R
DIMENSION X(41),Y(41),D(10,40)

C THE ITERATION BEGINS BY SOLVING THE INITIAL VALUE PROBLEM USING THE RUNGE KUTTA METHOD TO OBTAIN THE FIRST THREE VALUES AND THEN PROCEEDS USING THE ADAMS-BASHFORTH, ADAMS-MOULTON PREDICTOR-CORRECTOR.

READ(5,1)X(1),Y(1),H,NS,R,N
1 FORMAT(3F20.12,3I3)
DO 2 I=1,3
   RKCl=H*F(X(I),Y(I))
   RKc2=H*F(X(I)+H/2.0,Y(I)+RKCl/2.0)
   RKc3=H*F(X(I)+H/2.0,Y(I)+RKc2/2.0)
   RKc4=H*F(X(I)+H,Y(I)+RKc3)
   Y(I+1)=Y(I)+(RKCl+2.0*RKc2+2.0*RKc3+RKc4)/6.0
2   X(I+1)=X(I)+H
DO 3 I=4,NS-1
   Y(I+1)=Y(I)+H/24.0*(55.0*F(X(I),Y(I))-59.0*F(X(I-1),
   CY(I-1))+37.0*F(X(I-2),Y(I-2))-9.0*F(X(I-3),Y(I-3)))+
   X(I+1)=X(I)+H
3   Y(I+1)=Y(I)+H/24.0*(9.0*F(X(I+1),Y(I+1))+19.0*F(X(I),
   CY(I)))+5.0*F(X(I-1),Y(I-1))+F(X(I-2),Y(I-2))
DO 4 I=1,NS-1
4   D(I,I)=(Y(I+1)-Y(I))/H
```

C THE UPPER LIMIT OF DO-LOOP 4 IS THE NUMBER OF STEPS REQUIRED TO TRAVERSE THE INTERVAL (-1,1) BY H.
C***********************************************************
C THE UPPER LIMIT OF DO-LOOP 5 DETERMINES THE HIGHEST *
C ORDER DERIVATIVE APPROXIMATION. THE APPROXIMATION *
C IS ACCOMPLISHED USING DIVIDED DIFFERENCES. R.LE.NS. *
C***********************************************************
DO 5 J=2,R
DO 5 I=1,NS-J
5 D(J,I)=(D(J-1,I+1)-D(J-1,I))/H
C***********************************************************
C CALCULATION OF THE CONSTANT PORTION OF EQUATION 57 *
C***********************************************************
CON=1.0
DO 6 I=1,R
6 CON=CON*2.0*I/10.0
CON=CON*2.0**(R-I)/(10.0**(N-R))
C***********************************************************
C BEGINNING OF DELTA ITERATION *
C***********************************************************
WRITE(6,66)
66 FORMAT('l',8X/DELTA',15X,'X(1)',16X,'Y(1)',
C,16X,'H NS R N')
DELTA=1.0
WRITE(6,67)DELTA,X(1),Y(1),H,NS,R,N
67 FORMAT(3F20.12,F12.4,3I4)
7 T=(DELTA+2.0*X(1))/2.0
C***********************************************************
C LOCATION OF (DELTA+2*X(1))/2) IN X VECTOR *
C***********************************************************
DO 8 I=1,NS
IF(T-X(I))9,8,8
8 CONTINUE
9 J=I-R/2-1
FR=ABS(D(R,J))
DELTAN=(CON/FR)**(1.0/R)
WRITE(6,10)DELTAN
10 FORMAT(F20.12)
C***********************************************************
C TEST FOR CONVERGENCE *
C***********************************************************
DIFF=ABS(DELTA-DELTAN)
IF(DIFF-0.000001)12,11,11
11 DELTA=DELTAN
GO TO 7
12 STOP
END
FUNCTION F(X,Y)
PI=3.141592653589
F=1.0-SQRT(Y)+COS(PI*X)
RETURN
END

C******************************
C INITIALIZATION DATA SAMPLE *
C*****************************************************************************
-1.0 0.962556070550 0.05 41 10 7