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Reflection group diagrams for a sequence of Gaussian Lorentzian lattices

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by

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DEDICATION

I dedicate this thesis to my mother Alice, for her constant love and encouragement through these past five years. I am also grateful my family for their love and support.

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ABSTRACT

We exhibit a set of three related Gaussian Lorentzian lattices with “Coxeter-like” root diagrams. These root diagrams possess a point of symmetry in complex hyperbolic space, similar to the Weyl vector for positive-definite $\mathbb{Z}$-lattices. For two of the three lattices, this point of symmetry is used to show that the reflections in the diagram roots generate the lattice’s reflection group. It is shown for all three lattices that the lattice’s reflection group has finite index in its automorphism group.

CHAPTER 1. INTRODUCTION

The theory of lattices and their symmetry groups has application in many branches of mathematics: Lie theory (via root lattices and Weyl groups), geometry of numbers (via Minkowski theory), finite group theory (sporadic groups related to interesting lattices), sphere packings, and so on. They also have applications in areas outside pure mathematics, like coding theory and molecular structures in chemistry [CS].

Lie groups and Lie algebras are rich subjects and powerful tools in areas such as geometry, differential equations, and physics. In Lie theory, lattices arose as positive-definite root lattices over $\mathbb{Z}$, associated with the adjoint representation of a simple Lie algebra. Being linear, a Lie algebra is often easier to study than its associated Lie Group. The the idea of a "root" was introduced in the work of Wilhelm Killing and Élie Cartan, aimed at classifying simple Lie algebras over $\mathbb{C}$ [Bou]. In this context, a "root" is a functional on the Cartan subalgebra of a Lie algebra; these roots index the invariant subspaces of the adjoint representation of this Lie algebra [FH]. Understanding the representations of arbitrary Lie algebras is tantamount to understanding those of solvable Lie algebras and semisimple Lie algebras; the representation theory for solvable Lie algebras is relatively straightforward, while the story for semisimple Lie algebras is more complicated. However, it reduces to the case of simple Lie algebras, since every semisimple Lie algebra is a direct sum of simple Lie algebras. In turn, simple Lie algebras are characterized by irreducible root systems, which form Coxeter-Dynkin diagrams and can be used to recover the associated Lie algebra [FH]. These irreducible root systems are of type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, and G_2 [Hum].

In this thesis we investigate a phenomenon in complex Lorentzian lattices which has an analogy with positive-definite root lattices over $\mathbb{Z}$. Namely, any positive-definite integral root lattice embedded in $\mathbb{R}^n$ has a set of roots called *simple roots*, whose reflections (called *simple* *reflections*) generate the *Weyl group* of L. The mirrors of these roots are equidistant (in the spherical metric) from the *Weyl vector*, which is half the sum of the *positive roots*. It also turns out that the mirrors of the simple roots are those closest to the Weyl vector. The simple roots can be placed in a graph called the *Coxeter diagram* where each vertex in the graph represents a simple root. In the case of a simply-laced root system, an edge exists between roots r and r' exactly when the reflections in r and r' braid with each other. Such diagrams originated in Lie theory and are essential to the classification of Lie groups and Lie algebras. In addition to the study of Lie groups and Lie algebras, simply-laced Coxeter-Dynkin diagrams appear in several other classification problems such as: simple surface singularity, finite type quiver, and finite subgroups of $SU(2)$ [HHSV]. Denote by S the positive-definite root lattices over $\mathbb{Z}$. The $L \in S$ with simply laced diagrams have the advantage that distinct L have distinct reflection groups [CS]. Such L fall into five classes of lattices: A_n, D_n, E_6, E_7, and E_8; this explains why the classification problems noted above are called *ADE classification* problems.

At this point we mention a very special positive-definite integral $\mathbb{Z}$-lattice, the *Leech lattice*, which we will call Λ here. We say an integral $\mathbb{Z}$-lattice L is *even* if $|v|^2$ is even for all $v \in L$; otherwise L is said to be *odd*. The Leech lattice has rank 24, and is the unique even *unimodular* lattice that has rank < 32 and no roots [CS]. I.e., its reflection group is trivial, even though its automorphism group is quite large. Another indicator of the significance of Λ is that L. Griess used it to construct the "monster," the largest sporadic finite simple group [CS]. The method used in this dissertation to show $\mathrm{Ref}(L)$ is finitely generated, for the L's we consider, is based on Conway's modification of an algorithm due to E. B. Vinberg. Conway used this algorithm to determine the generators for $\mathrm{Ref}(\Lambda \oplus H)$, where H is a *hyperbolic cell* [CS].

The simple reflections for positive definite root lattices over $\mathbb{Z}$ have the following uniform description: the mirrors of the simple reflections are the walls of a connected component of the complement of the mirrors in the underlying vector space of L. This connected component is the *Weyl chamber*. Because the mirrors in a complex vector space have real codimension 2, for complex lattices we have no direct analog to the *Coxeter chambers* partitioning the ambient space, and a nice uniform geometric description of the generating reflections is missing. Shephard and Todd [ST] were able to fully classify finite complex reflection groups using ad-hoc diagrams similar to Coxeter-Dynkin diagrams. Although the general theory is absent, some of these diagrams seem to have geometric properties [BMR]. These diagrams give deep information about their associated complex reflection groups, such as their invariant degrees and the weak homotopy type of the associated discriminant complement [Be1]. Our work finds and investigates similar diagrams for some infinite complex hyperbolic reflection groups.

Given an integral $\mathbb{Z}$-lattice of even rank L, it is possible to construct a lattice over $\mathcal{G} \subseteq \mathbb{C}$ (the Gaussian integers) with the same underlying set as L whenever the following is satisfied: There is a $\psi \in \mathrm{Aut}(L)$ of order 4 for which ψ, ψ^2, and ψ^3 are fixed-point free. Likewise, we can construct a lattice over $\mathcal{E} \subseteq \mathbb{C}$ (the Eisenstein integers) from L with the same underlying set if there exists an order-3 automorphism $\psi \in \mathrm{Aut}(L)$ which is fixed-point free [CS]. For example [Al1],

$$E_8 = \frac{1}{2}\{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \mid x_j \equiv x_k (\mathrm{mod}\ 2), \sum x_j \in 4\mathbb{Z}\}$$

$$E_8^{\mathcal{G}} = \frac{1}{1+i}\{(x_1, \ldots, x_4) \in \mathcal{G}^4 \mid x_j \equiv x_k (\mathrm{mod}\ 1+i), \sum x_j \in 2\mathcal{G}\}$$

Another example is

$$D_4 = \{(x_1, \ldots, x_4) \in \mathbb{Z}^4 \mid \sum x_j \equiv 0 (\mathrm{mod}\ 2)\}$$

$$D_4^{\mathcal{G}} = \{(x_1, x_2) \in \mathcal{G}^2 \mid x_1 + x_2 \equiv 0 (\mathrm{mod}\ p)\}$$

The literature contains many examples of reflection groups which have nice sets of generators. For instance, almost all finite Euclidean reflection groups: in particular, the aforementioned Weyl groups. Many finite complex reflection groups also furnish examples. This includes all the finite complex reflection groups defined over Gaussian or Eisenstein numbers. Like Weyl groups, they have nice sets of generators that correspond to the vertices of a "complex diagram," and there is a point in the complex vector space which is a sort of point-of-symmetry of the mirror arrangement. The generators are precisely the reflections whose mirrors are closest to this point.

The analogy we explore in this thesis for a complex Lorentzian lattice L can be described as follows. We seek a small set of roots $\mathcal{R}$ (analogous to the simple roots) with "Coxeter-like diagram" D, whose reflections generate the reflection group of L. The projective points of

Table 1.1 Three triplets of lattices

Expository Source	This Thesis	[Ba2]	[Ba1]
Ring R	$\mathcal{G}$	$\mathcal{H}$	$\mathcal{E}$
L_1	$D_4^{\mathcal{G}}$	$E_8^{\mathcal{H}}$	$E_8^{\mathcal{E}}$
Diagram for $L_1 \oplus H$	A_4	A_4	A_6
Diagram for $2L_1 \oplus H$	Octagon	Octagon	12-gon
Diagram for $3L_1 \oplus H$	$\mathbb{P}^2(\mathbb{F}_2)$	$\mathbb{P}^2(\mathbb{F}_2)$	$\mathbb{P}^2(\mathbb{F}_3)$

$V := L \otimes \mathbb{C}$ with negative squared norm can be considered as points in complex hyperbolic $\mathbb{C}H^n$. The reflection group of L naturally acts on $\mathbb{C}H^n$. The mirrors in roots of $\mathcal{R}$ are equidistant from, and the closest mirrors to, a point $\tau \in \mathbb{C}H^n$. This point has a simple geometric characterization relative to these mirrors: they are exactly the mirrors closest to τ. The mirrors in the roots of $\mathcal{R}$ form an undirected version of the diagram D. For this undirected diagram, it is also often true that τ is the unique point in hyperbolic space fixed by a group $G \leq \mathrm{Aut}(L)$, where G acts as the undirected diagram's automorphism group. Such configurations of roots were found in [Ba1] and [Ba2] for two "triplets" of three lattices related to the Leech lattice. One triplet consists of the sequence of lattices $E_8^{\mathcal{E}} \oplus H \subset 2E_8^{\mathcal{E}} \oplus H \subset 3E_8^{\mathcal{E}} \oplus H$. The other triplet is $E_8^{\mathcal{H}} \oplus H \subset 2E_8^{\mathcal{H}} \oplus H \subset 3E_8^{\mathcal{H}} \oplus H$, which are lattices are over $\mathcal{H}$ (the Hurwitz integers). For these lattices the ambient vector space is over the quaternions $\mathbb{H}$. (As of this writing, the author of [Ba2] has not verified the statements about $E_8^{\mathcal{H}} \oplus H$ and $2E_8^{\mathcal{H}} \oplus H$, but strongly believes them to be true.) This thesis deals with a third such triplet of lattices, this time over the Gaussian integers $\mathcal{G}$, namely $D_4^{\mathcal{G}} \oplus H \subset 2D_4^{\mathcal{G}} \oplus H \subset 3D_4^{\mathcal{G}} \oplus H$. (The author of this thesis did not prove that reflections in the diagram roots generate the reflection group of $3D_4^{\mathcal{G}} \oplus H$, although there is much empirical evidence supporting that conclusion.) A comparison of these 3 triplets is summarized in Table 1.

Furthermore, there is a noteworthy connection between one such root configuration in the Eisenstein Lorentzian Leech lattice, and the monster simple group [Ba1], [Al2].

Now we describe in more detail the results in this thesis. Unless otherwise stated L will stand for any one of the lattices $D_4^{\mathcal{G}} \oplus H$, $2D_4^{\mathcal{G}} \oplus H$, or $3D_4^{\mathcal{G}} \oplus H$. Likewise, $B := \mathbb{C}H^k$ (complex hyperbolic space of dimension k), where $k = 3$, 5, or 7 according to L, so that $\mathrm{Aut}(L)$ acts on B. Let $\mathcal{R}$ be a set of roots in L, and let $S := \{r^\perp \mid r \in \mathcal{R}\}$ be the set of corresponding mirrors. We define the *root diagram* of $\mathcal{R}$ to be the graph whose nodes are the roots of $\mathcal{R}$, and which has a directed edge (r_1, r_2) iff $\langle r_1, r_2\rangle = p := 1 + i$. An edge between r_1 and r_2 indicates the i-reflections in these roots braid, i.e. $\phi^i_{r_1}\phi^i_{r_2}\phi^i_{r_1} = \phi^i_{r_2}\phi^i_{r_1}\phi^i_{r_2}$, while absence of an edge indicates they commute. Here, we define the *mirror diagram* of S to be the undirected graph corresponding to the root diagram of $\mathcal{R}$, where the nodes are now the roots' corresponding mirrors.

For each lattice L considered above, we exhibit a small set of roots $\mathcal{R}$ whose root diagram forms a "Dynkin Diagram" with appealing symmetry. There is a $G \leq \mathrm{Aut}(L)$ which acts as the graph automorphism group for the mirror diagram of S. There is a point τ in the complex hyperbolic space that is equidistant from the mirrors in S. The mirrors in the set S play the role of the simple mirrors for Weyl groups while the point τ plays the role of the Weyl vector. These diagrams are A_4 (for $D_4^{\mathcal{G}} \oplus H$), an octagon (for $2D_4^{\mathcal{G}} \oplus H$), and the incidence graph of $\mathbb{P}^2(\mathbb{F}_2)$ (for $3D_4^{\mathcal{G}} \oplus H$). These diagrams also appear in the quaternionic Lorentzian Leech lattice and corresponding sublattices [Ba2].

We have the following results:

Theorem. *The mirrors in S are precisely those closest to τ. For $L := 2D_4^{\mathcal{G}} \oplus H$ or $3D_4^{\mathcal{G}} \oplus H$, the action of G on S is transitive and τ is the unique point in B fixed by G.*

Theorem. *The complex reflections in the mirrors of S generate* $\mathrm{Ref}(L)$. *(This has the status of "expected theorem" for $3D_4^{\mathcal{G}} \oplus H$.)*

Theorem. *The group* $\mathrm{Ref}(L)$ *has finite index in* $\mathrm{Aut}(L)$. *In particular,* $\mathrm{Ref}(L)$ *is arithmetic.*

CHAPTER 2. PRELIMINARIES

2.1 Background on Lattices

Let the ring R be $\mathbb{Z}$, $\mathcal{G} := \mathbb{Z}[i]$ (the Gaussian integers), or $\mathcal{E} := \mathbb{Z}[e^{2\pi i/3}]$ (the Eisenstein integers). An *R-lattice* (or lattice over R) L is a free R-module of finite rank. We define the *underlying vector space* of L to be $V := L \otimes K$, where $K := \mathbb{R}$ or $\mathbb{C}$ depending on R. We can then choose a hermitian form $\langle x, y\rangle : V \times V \to \mathbb{C}$; we will assume throughout this thesis that all hermitian forms are conjugate-linear in the first argument. We define the *norm* of $v \in V$ to be $|v|^2 := \langle v, v\rangle$. When the quadratic form induced by $\langle \cdot, \cdot\rangle$ is positive-(semi)definite, negative-(semi)definite, or indefinite, we give L the same designation. An *automorphism* of L is simply a linear-automorphism of L which preserves $\langle \cdot, \cdot\rangle$. The automorphism group of L is denoted by $\mathrm{Aut}(L)$. If $\langle x, y\rangle \in R$ for all $x, y \in L$ we say L is an *integral R-lattice*. The *dual* lattice of an integral lattice L is $L^\vee := \{v \in V \mid \langle v, x\rangle \in R \text{ for all } x \in L\}$. For an $r \in R$, we say L is *r-modular* if $L^\vee = \frac{1}{r}L$. A lattice is *unimodular* if $L^\vee = L$.

If L is positive-definite, we can define the *covering radius* of L to be $\sup_{v\in V}\left(\inf_{l\in L} |v - l|\right)$. This is the smallest r, such that the collection of closed balls of radius r centered at the points of L covers V. The covering radius therefore gives an idea of the "density" of a lattice.

Suppose L is an R-lattice of rank n, with ordered R-basis $\mathcal{B} := \{x_j\}$ and a hermitian form $\langle \cdot, \cdot\rangle$ that takes values in K. Let $V := L \otimes K$ be the underlying vector space of L. For any $(v_1, \ldots, v_j) \in V^j$, we can form the matrix $\mathsf{gram}(v_1, \ldots, v_j) := (\langle v_k, v_l\rangle)$, called the *Gram matrix* of $(v_1, \ldots, v_j)$. In particular, we can form $M := \mathsf{gram}(\mathcal{B})$; then for all $x, y \in V$ expressed as column vectors with respect to $\mathcal{B}$, we have $\langle x, y\rangle = x^* M y$. Note that if we have a linear-automorphism ψ of L, with matrix S with respect to $\mathcal{B}$, then $\psi \in \mathrm{Aut}(L)$ is equivalent to $S^* M S = M$. If $\langle \cdot, \cdot\rangle$ is degenerate, then $\mathsf{gram}(\mathcal{B})$ is singular for any R-basis $\mathcal{B}$ of L. In this case we call L a *singular* lattice. If $\langle\cdot,\cdot\rangle$ is nondegenerate, we can find a $T \in M_n(K)$ such that $T^*MT = D$, a diagonal matrix with l 1's and m -1's, where $l+m=n$. Sylvester's Law of Inertia assures us that l and m are unique, and we call (l,m) the *signature* of L.

Choose $1 \neq \xi \in R^*$, a unit in R. For $r \in L$ with $|r|^2 \neq 0$, we call the automorphism of V given by

$$\phi_r^\xi(x) := x - (1-\xi)\frac{\langle r,x\rangle}{|r|^2}r$$

the *ξ-reflection in r*. For any $l \in L$, we say l is *primitive* if $l = r\cdot l'$ for some $r \in R$ and $l' \in L \Rightarrow r \in R^*$. I.e., l is primitive if l is not a non-unit multiple of a lattice vector. If ϕ_r^ξ preserves L (i.e., is an automorphism of L) and r has norm 2, we call r a *root* of L. (If the superscript is omitted we assume $\xi = -1$.) If L is generated as an R-module by a set of roots, L is called a *root lattice.*

For a lattice L with sublattices L_1, L_2 such that $L = L_1 + L_2$, we write $L = L_1 \oplus L_2$ if $L_1 \cap L_2 = \{0\}$ and $\langle l_1, l_2\rangle = 0$ for all $l_1 \in L_1$, $l_2 \in L_2$. A lattice is called *even* if all its norms are even integers. One of the simplest examples of lattices, positive-definite even root lattices over $\mathbb{Z}$ have been studied extensively and their structure is well-understood. Such a lattice L has a canonical direct-sum decomposition $L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$, where this decomposition is unique up to isomorphism and order of the summands, and each L_j is isomorphic to a lattice of type A_n, D_n, E_6, E_7, or E_8 [Hum].

A positive-definite integral $\mathbb{Z}$-lattice has a reflection group (or *Weyl group*), denoted $\mathrm{Ref}(L)$. This reflection group has a pleasing characterization, which we now describe. Let Φ be the collection of roots in L, and note that Φ satisfies the following two properties:

For all $r \in \Phi$,

1. $\phi_r(\Phi) = \Phi$.

2. $\mathbb{R}r \cap \Phi = \{+r, -r\}$.

If we choose any vector v_0 in $V := L \otimes \mathbb{R}$ which is not perpendicular to any of the roots, we can define the *positive roots* Φ_+ of L as

$$\Phi_+ := \{r \in \Phi \mid \langle v_0, r\rangle > 0\}.$$

Note that exactly half the roots in Φ are positive. We next define the *Weyl vector* to be

$$w := \frac{1}{2} \sum_{r \in \Phi_+} r.$$

Finally, we define the set Δ of *simple roots* to be those roots $r \in \Phi_+$ for which the "spherical distance" from the mirror $r^\perp$ to w is minimized, i.e. for which $|\langle r, w \rangle|$ is minimized. Then we have the following [Hum]:

Theorem 2.1.1. *The reflection group* $\mathrm{Ref}(L)$ *is generated by reflections in the roots of* Δ*.*

Note that Δ is not uniquely determined for a given L, and ultimately depends on our choice of v_0 above. However, it is also a theorem that any two such Δ's are conjugate under $\mathrm{Ref}(L)$. Suppose $\dim V$ $(= \operatorname{rank} L)$ is n. The mirrors $r^\perp$ in the roots $r \in \Phi$ partition $\mathbb{R}^n - \bigcup_{r \in \Phi} r^\perp$ into connected components called *chambers*. The chamber whose boundary is $\bigcup_{r \in \Delta} r^\perp$ contains the Weyl vector w and is called the *Weyl chamber*. These mirrors in the simple roots (the *simple mirrors*) are precisely the mirrors closest to w in the spherical metric. It is noteworthy that Theorem 2.1.1 can be proved using a "height-reduction argument." The basic idea is to define the "height" of a root $r \in L$ to be $|\langle r, w \rangle|$, essentially the spherical distance from $r^\perp$ to w, and show that the height of (the image of) r can be reduced by reflections in the simple mirrors until it has minimal height. Because the roots of minimal height are precisely (unit multiples of) the simple roots, the conjugation formula

$$\phi_{\phi_u(v)} = \phi_u \phi_v \phi_u$$

shows ϕ_r can be expressed as a product of *simple reflections* (reflections in simple mirrors), as desired.

The simple roots $r \in \Delta$ form a *Coxeter Diagram*, a graph whose vertices are the r's and whose edges are determined and labeled thusly: For any $r, s \in \Delta$, we know $(\phi_r \phi_s)^m = 1$ for some minimal $m \in \mathbb{Z}^+$ because ϕ_r and ϕ_s generate a dihedral group. If $m = 1$ then $r = s$ so no edge is possible. If $m = 2$ then $\phi_r \phi_s = \phi_s \phi_r$ (the reflections commute) and we do not draw an edge between r and s. If $m = 3$ then $\phi_r \phi_s \phi_r = \phi_s \phi_r \phi_s$ (the reflections *braid*) and we draw a single edge between r and s, and either label this edge with a 3 or omit the label. For any other m we draw an edge labeled with m [Hum].

2.2 Complex Hyperbolic Space $\mathbb{C}H^n$

(Unpublished notes of Dr. Tathagata Basak were used as a resource for this section.)

Suppose W is a $\mathbb{C}$-vector space equipped with a Hermitian form $\langle\cdot,\cdot\rangle$. We define $\mathsf{rad}(W) := W^\perp$, and note that W is nonsingular whenever $\mathsf{rad}\, W = 0$. In this case, W has a signature as defined above. If $w \neq 0$ has norm 0 we say w is a *null vector*. We define $W_{\leq} := \{w \in W : |w|^2 \leq 0\}$. Define $W_<$ and $W_>$ similarly.

Let $\mathbb{C}^{n,1}$ be the space whose underlying set is $\mathbb{C}^{n+1}$ with Hermitian form $\langle\cdot,\cdot\rangle$ given by

$$\langle (x_0, x_1, \ldots, x_n), (y_0, y_1, \ldots, y_n)\rangle = \overline{x}_0 y_0 + \cdots + \overline{x}_{n-1} y_{n-1} - \overline{x}_n y_n.$$

Note that $\mathbb{C}^{n,1}$ is a *Lorentzian* space (i.e., it has signature $(n, 1)$); throughout this section we will set $V := \mathbb{C}^{n,1}$.

Define the projective space of V by $\mathbb{P}(V) := \{\mathbb{P}(v) \mid v \in V\}$, where $\mathbb{P}(v) := \{av \mid a \in \mathbb{C} - \{0\}\}$ for $v \in V$. We call the image $\mathbb{P}(v)$ of a null vector v in projective space a *cusp*. If U is the group of isometric automorphisms of V, we can set $\mathbb{P}U := \{\mathbb{P}(g) \mid g \in U\}$, where $\mathbb{P}(g)\mathbb{P}(v) = \mathbb{P}(gv)$.

It will be convenient to define $c : \mathbb{P}(V_<) \times \mathbb{P}(V_<) \to \mathbb{R}$ given by $c(\mathbb{P}(v), \mathbb{P}(w)) = \sqrt{\frac{\langle v,w\rangle\langle w,v\rangle}{\langle v,v\rangle\langle w,w\rangle}}$. We can define a function $V_< \times V_< \to \mathbb{R}$ using the same formula, which we also denote by c. Now we show that $\mathbb{P}(V_<)$ can be made into a metric space with the $\mathbb{P}U$-invariant metric given below; we call this the *n-dimensional complex hyperbolic space* $\mathbb{C}H^n$. Each lattice L, of the three which are the central objects of study in this thesis, has signature $(m, 1)$ for some m. This means L's underlying vector space V is isomorphic to $\mathbb{C}^{m,1}$, so $\mathrm{Aut}(L)$ acts on $\mathbb{C}H^m$.

Theorem 2.2.1. *The function $d : \mathbb{P}(V_<) \times \mathbb{P}(V_<) \to \mathbb{R}_{\geq}$ defined below is a $\mathbb{P}U$-invariant metric on $\mathbb{P}(V_<)$:*

$$d(\mathbb{P}(v), \mathbb{P}(w)) = \cosh^{-1}\left(c(\mathbb{P}(v), \mathbb{P}(w))\right) \tag{2.1}$$

Proof. It is clear that c (and hence d) is well-defined, and that d is $\mathbb{P}U$-invariant. The symmetric and non-negativity properties of a metric are also apparent. Now suppose $d(\mathbb{P}(v), \mathbb{P}(w)) = 0$; then without loss of generality we may assume $|v|^2, |w|^2 = -1$, and we thus have $|\langle v, w\rangle|^2 = 1$.

But then $\mathsf{gram}(v,w)$ is singular, so v and w are linearly dependent and $\mathbb{P}(v)=\mathbb{P}(w)$. It only remains to verify the triangle inequality.

Let $v_1, v_2, v_3 \in V_<$, and set $\alpha := \langle v_1, v_2\rangle, \beta := \langle v_3, v_1\rangle, \gamma := \langle v_2, v_3\rangle$. Without loss of generality assume each $|v_j|^2 = -1$. We next observe that each $\mathsf{span}\{v_j, v_k\}$ either is singular or has signature $(1,1)$, which yields $|\alpha|^2, |\beta|^2, |\gamma|^2 \geq 1$. Now set $M := \mathsf{gram}(v_1, v_2, v_3)$; then M is singular or has signature $(2,1)$, whence

$$0 \leq -\det M = -\begin{vmatrix} -1 & \alpha & \overline{\beta} \\ \overline{\alpha} & -1 & \gamma \\ \beta & \overline{\gamma} & -1 \end{vmatrix} = 1 - |\alpha|^2 - |\beta|^2 - |\gamma|^2 - 2\operatorname{Re}\alpha\beta\gamma$$

$$\leq 1 - |\alpha|^2 - |\beta|^2 - |\gamma|^2 + 2|\alpha\beta\gamma| = (|\alpha|^2-1)(|\beta|^2-1) - (|\gamma| - |\alpha\beta|)^2 \Rightarrow$$

$$|\gamma| \leq \sqrt{|\alpha|^2-1}\sqrt{|\beta|^2-1} + |\alpha||\beta|$$

Let $A = \cosh^{-1}(|\alpha|), B = \cosh^{-1}(|\beta|), C = \cosh^{-1}(|\gamma|)$, so this becomes

$$\cosh(C) \leq \sinh(A)\sinh(B) + \cosh(A)\cosh(B) = \cosh(A+B) \Rightarrow$$

$$\cosh^{-1}(|\gamma|) \leq \cosh^{-1}(|\alpha|) + \cosh^{-1}(|\beta|) \Rightarrow d(v_2, v_3) \leq d(v_2, v_1) + d(v_1, v_3).$$

□

To simplify notation we will sometimes write $d(v,w)$ instead of $d(\mathbb{P}(v), \mathbb{P}(w))$, etc.

2.2.2. Projections and more distance formulas. For $x \in V$ define $h_x : \mathbb{P}(V_<) \to \mathbb{R}_\geq$ by

$$h_x(\mathbb{P}(v)) = \frac{|\langle x, v\rangle|^2}{-|v|^2}.$$

Note that $h_x(\mathbb{P}(v))$ is closely related to $d(\mathbb{P}(x), \mathbb{P}(v))$, and that $h_x(\mathbb{P}(v_1)) < h_x(\mathbb{P}(v_2)) \Leftrightarrow d(\mathbb{P}(x), \mathbb{P}(v_1)) < d(\mathbb{P}(x), \mathbb{P}(v_2))$ whenever $x \in \mathbb{P}(V_<)$. Now suppose $y \in \mathbb{P}(V_<)$ and $A \subseteq \mathbb{P}(V_<)$. We define the *projection of y onto A*, denoted $\mathsf{pr}_A(y)$, to be the unique point $a_0 \in \operatorname{cl} A$ that minimizes the function $a \mapsto d(a,y)$, if such a_0 exists.

Lemma 2.2.3. *Let W be a Lorentzian subspace of V, and suppose $a \in W_<$. Then any $x \in \mathbb{P}(W_<)$ can be represented as $\mathbb{P}(a+w)$, where $w \in \{0\} \cup W_>$ and $\langle a, w\rangle = 0$.*

Proof. Note that $a^\perp \subseteq W$ has signature $(k, 0)$ for some k, so a can be extended to a basis of W given by $a = a_0, a_1, \ldots, a_k$ with $\{a_1, \ldots, a_k\} \subseteq a^\perp \subseteq W_>$. Then any $x \in W_<$ has $x = \sum_{j=0}^k c_j a_j$, for some c's $\in \mathbb{C}$ with $c_0 \neq 0$. Thus $\frac{x}{c_0} = a + \sum_{j=1}^k \frac{c_j}{c_0} a_j$, and $\mathbb{P}(x)$ can be given the desired form. □

Theorem 2.2.4. *Let W be a Lorentzian subspace of V. Suppose $x \in V$ and that either of the following is true:*

(i) $|x|^2 \leq 0$ and $x \notin W$, or

(ii) $|x|^2 > 0$ and $\mathbb{P}(x^\perp_\leq) \cap \mathbb{P}(W_\leq) = \emptyset$

Then we have

(a) $(\mathbb{C}x + W^\perp) \cap W$ is a 1-dimensional space of signature $(0, 1)$

(b) h_x restricted to $\mathbb{P}(W_<)$ has a unique minimum at $\mathbb{P}((\mathbb{C}x + W^\perp) \cap W)$

Proof. (a) From (i) and (ii), $x^\perp$ has no negative norm vector, or $x^\perp \cap W_< = \emptyset$. Thus $W \nsubseteq x^\perp$, so $x \notin W^\perp$, whence $\dim(\mathbb{C}x + W^\perp) = \dim(W^\perp) + 1$. Because $W^\perp$ is definite, $V = W \oplus W^\perp$, so $\dim((\mathbb{C}x + W^\perp) \cap W) = \dim(\mathbb{C}x + W^\perp) + \dim(W) - \dim((\mathbb{C}x + W^\perp) + W) = 1$. Then $(\mathbb{C}x + W^\perp) \cap W$ contains a non-zero vector $a = x + v$ for some $v \in W^\perp$. Since $a \in W$, we have $\langle x, v\rangle = \langle a - v, v\rangle = -|v|^2$. Then we get $|a|^2 = \langle x + v, x + v\rangle = |x|^2 - 2|v|^2 + |v|^2 = |x|^2 - |v|^2$.

If (i) is true, then $|a|^2 \leq -|v|^2$, and we also must have $v \neq 0$ since otherwise $x = a \in W$. Because $v \in W^\perp$ and $W^\perp$ is positive definite, we get $|v|^2 > 0$, so $|a|^2 < 0$.

On the other hand, suppose (ii) is true. Then $(\mathbb{C}x + W^\perp)^\perp = x^\perp \cap W$ is positive definite, so $\mathbb{C}x + W^\perp$ is indefinite. Then there is an $r_1 \in W^\perp$ such that $\mathsf{span}\{x, r_1\}$ has signature $(1, 1)$ and $|r_1|^2 = 1$. Now extend r_1 to an orthogonal basis $\{r_1, \ldots, r_k\}$ of $W^\perp$, where $|r_j|^2 = 1$ for all j. So $v = \sum_{j=1}^k c_j r_j$ for some c_j's $\in \mathbb{C}$. From $a = x + v$ and the fact that $a \in W$, we can compute the c_j's to get $a = x - \sum \langle x, r_j\rangle r_j$. Then

$$|a|^2 = \langle a, x + v\rangle = \langle a, x\rangle = |x|^2 - \sum_{j=1}^k |\langle x, r_j\rangle|^2 = (|r_1|^2|x|^2 - |\langle x, r_1\rangle|^2) - \sum_{j=2}^k |\langle x, r_j\rangle|^2 < 0,$$

where the inequality follows from the fact that $\mathsf{span}\{x, r_1\}$ has signature $(1, 1)$.

(b) We have $a \in W_<$, so by Lemma 2.2.3 any vector in $\mathbb{P}(W_<)$ can be written as $\mathbb{P}(a+w)$, where $w \in W_> \cup \{0\}$ and $\langle a, w\rangle = 0$. We calculate

$$h_x(\mathbb{P}(a+w)) = \frac{|\langle x, a+w\rangle|^2}{-|a+w|^2} = \frac{|\langle x, a\rangle + \langle x, w\rangle|^2}{-|a|^2 - |w|^2} = \frac{|a|^4}{-|a|^2 - |w|^2}.$$

This quantity is clearly minimized when $|w|^2 = 0$, i.e. when $w = 0$. Therefore $\mathbb{P}(a) = \mathbb{P}((\mathbb{C}x + W^\perp) \cap W)$ minimizes h_x on $\mathbb{P}(W_<)$. $\square$

Corollary 2.2.5. *Let $\{r_1, \ldots, r_m\}$ be an orthogonal basis for a positive definite subspace of V. Set $W := \bigcap r_j^\perp$, and let $x \in V_<$. Then the projection of $\mathbb{P}(x)$ onto $\mathbb{P}(W_<)$ is given by $\mathbb{P}(\mathsf{pr}_W(x))$, where*

$$\mathsf{pr}_W(x) = x - \sum_{j=1}^{m} |r_j|^{-2} \langle x, r_j\rangle r_j$$

Proof. If $x \in W$, then we have $\mathsf{pr}_W(x) = x$ and the result holds. Otherwise we can apply Theorem 2.2.4, since it is clear that W is Lorentzian, and by the proof of the Theorem we know $\mathsf{pr}_W(x) \in (\mathbb{C}x + W^\perp) \cap W$. $\square$

Lemma 2.2.6. *Suppose $v, y \in V$ with $|v|^2 \neq 0$ and $|y|^2 = 1$. Let $a = v - \langle v, y\rangle y$. We have (a) $\frac{|\langle v,a\rangle|^2}{|v|^2|a|^2} = \frac{|a|^2}{|v|^2} = 1 - \frac{|\langle v,y\rangle|^2}{|v|^2}$, and (b) If $v \in V_<$ and $y \in V_>$, then $c(v, \mathsf{pr}_{y^\perp}(v))^2 = 1 - c(v,y)^2$.*

Proof. (a) Observe that $\langle y, a\rangle = 0$. Then $\langle v, a\rangle = |a|^2 = \langle v, v - \langle v, y\rangle y\rangle = |v|^2 - |\langle v, y\rangle|^2$, which gives us what we want.

(b) This follows from part (a) and the definition of of $\mathsf{pr}_W(x)$ above, since we may assume $|y|^2 = 1$. $\square$

Theorem 2.2.7. *(a) Let $r \in V_>$ and $x \in V_< \setminus r_<^\perp$. Then*

$$d(\mathbb{P}(x), \mathbb{P}(r_<^\perp)) = \sinh^{-1}(\sqrt{-c(x,r)^2}) \tag{2.2}$$

(b) Let $r, x \in V_>$ such that $\mathsf{span}\{r, w\}$ is Lorentzian (so $r_<^\perp \cap x_<^\perp = \emptyset$). Then

$$d(\mathbb{P}(x_<^\perp), \mathbb{P}(r_<^\perp)) = \cosh^{-1}(c(x,r)) \tag{2.3}$$

Proof. (a) By Corollary 2.2.5 and Lemma 2.2.6, $d(x, r^{\perp}_{<}) = d(x, \mathsf{pr}_{r^\perp}(x)) = \cosh^{-1}\left(\sqrt{1-c(x,r)^2}\right) = \sinh^{-1}\left(\sqrt{-c(x,r)^2}\right)$.

(b) We may assume $|r|^2, |x|^2 = 1$. Let $v \in r^{\perp}_{<}$. Then as in part (a)

$$d(v, x^{\perp}_{<}) = d(v, \mathsf{pr}_{x^\perp}(v)) = \cosh^{-1}\left(c(v, \mathsf{pr}_{x^\perp}(v))\right) = \cosh^{-1}\left(\sqrt{1-c(v,x)^2}\right) = \cosh^{-1}\left(\sqrt{1+h_x(v)}\right).$$

We wish to minimize this expression for $v \in r^{\perp}_{<}$, which amounts to minimizing $h_x(v)$ there. By condition (ii) of Theorem 2.2.4 this occurs at $v = \mathsf{pr}_{r^\perp}(x)$. Therefore by Lemma 2.2.6 we have

$$d(r^{\perp}_{<}, x^{\perp}_{<}) = \cosh^{-1}\left(\sqrt{1-c(\mathsf{pr}_{r^\perp}(x), x)^2}\right) = \cosh^{-1}\left(\sqrt{1-(1-c(x,r)^2)}\right) = \cosh^{-1}\left(c(x,r)\right)$$

$\square$

Theorem 2.2.8. *The following variants of the triangle inequality hold in $\mathbb{C}H^n$.*

(a) Let $z \in V_{>}$ and $x, y \in V_{<}$. Then

$$d(x, z^{\perp}_{<}) \leq d(x,y) + d(y, z^{\perp}_{<}) \tag{2.4}$$

(b) Let $x, z \in V_{>}$ and $y \in V_{<}$. Then

$$d(x^{\perp}_{<}, z^{\perp}_{<}) \leq d(x^{\perp}_{<}, y) + d(y, z^{\perp}_{<}) \tag{2.5}$$

Proof. (a) For any $w \in V_{<}$ we have $d(x,w) \leq d(x,y) + d(y,w)$. Therefore

$$d(x, z^{\perp}_{<}) = \min_{w \in z^{\perp}_{<}} d(x,w) \leq \min_{w \in z^{\perp}_{<}} \left(d(x,y) + d(y,w)\right) = d(x,y) + \min_{w \in z^{\perp}_{<}} d(y,w) = d(x,y) + d(y, z^{\perp}_{<}).$$

(b) For any $v, w \in V_{<}$ we have $d(v,w) \leq d(v,y) + d(y,w)$. Hence

$$d(x^{\perp}_{<}, z^{\perp}_{<}) = \min_{\substack{v \in x^{\perp}_{<} \\ w \in z^{\perp}_{<}}} d(v,w) \leq \min_{\substack{v \in x^{\perp}_{<} \\ w \in z^{\perp}_{<}}} \left(d(v,y) + d(y,w)\right).$$

As $d(v,y)$ does not depend on w, and similarly $d(y,w)$ does not depend on v, the last expression above is equal to

$$\min_{v \in x^{\perp}_{<}} d(v,y) + \min_{w \in z^{\perp}_{<}} d(y,w) = d(x^{\perp}_{<}, y) + d(y, z^{\perp}_{<}).$$

$\square$

2.2.9. Horoballs. Let $\rho \in V$ be a null vector and let $\overline{\rho} := \mathbb{P}(\rho)$ be the corresponding cusp. We define $d_\rho : \mathbb{P}(V_<) \to \mathbb{R}$ as

$$d_\rho(\mathbb{P}(x)) := \log\left(\sqrt{h_\rho(x)}\right) = \log\left(\frac{|\langle \rho, x\rangle|}{\sqrt{-|x|^2}}\right).$$

We define the *open horoball at ρ of radius k* to be

$$B_k(\rho) := \{\mathbb{P}(x) \in \mathbb{P}(V_<) \mid h_\rho(x) < k^2\}.$$

Now suppose $\mathbb{P}(\rho_1) = \mathbb{P}(\rho_2) = \overline{\rho}$ is a cusp, and suppose $x, y \in V_<$. Observe that $d_{\rho_1}(\mathbb{P}(x))$ differs from $d_{\rho_2}(\mathbb{P}(x))$ by a constant independent of x. Then $d_\rho(\mathbb{P}(x)) - d_\rho(\mathbb{P}(y))$ is independent of the lift ρ for $\overline{\rho}$ we choose, so we can think of $\mathbb{P}(x)$ being "closer to" or "farther from" $\overline{\rho}$ than $\mathbb{P}(y)$ is from $\overline{\rho}$, even though $\mathbb{P}(x)$ and $\mathbb{P}(y)$ are at distance infinity from $\overline{\rho}$ in $\mathbb{C}H^n$.

2.2.10. Suppose $\overline{\rho}$ is a cusp with lift $\rho \in V$, $B_k(\rho)$ is a horoball, and $\mathbb{P}(x) \in \mathbb{C}H^n$ is a point not contained in $B_k(\rho)$. The projection $\mathsf{pr}_{B_k(\rho)}(\mathbb{P}(x))$ is contained on the geodesic ray joining $\mathbb{P}(x)$ and $\overline{\rho}$. So if we have two points $\mathbb{P}(x), \mathbb{P}(y) \in \mathbb{C}H^n$ and wish to compare their distances to the cusp $\overline{\rho}$, this is tantamount to comparing their (finite) distances to a horoball $B_k(\rho)$ which doesn't contain them.

2.3 Analogous and Motivating Examples

The situation for a positive-definite integral $\mathbb{Z}$-lattice, described in Section 2.1, suggested to Allcock and Basak that an analogous situation might hold for lattices L that aren't positive-definite, over a ring $R \not\subseteq \mathbb{R}$. They found several examples where $R = \mathcal{E}$ (the Eisenstein integers), $\mathcal{G}$ (the Gaussian integers), or $\mathcal{H}$ (the Hurwitz integers). In each of these examples the following suggestive features are present.

The lattice L contains a set of roots $\mathcal{R}$ whose reflections generate $\mathrm{Ref}(L)$. The roots in $\mathcal{R}$ form a diagram D, a graph with the following properties: Fix a unit $\xi \in R$. The vertices of D are the roots $r \in \mathcal{R}$, and no edge exists between r_1 and r_2 if $\phi^\xi_{r_1}$ and $\phi^\xi_{r_2}$ commute, while r_1 and r_2 have an edge between them if $\phi^\xi_{r_1}$ and $\phi^\xi_{r_2}$ braid. These roots and the diagram they form are the analogs of the simple roots and their associated Coxeter Diagram. There is a point τ in $V := L \otimes K$ which is equidistant from the $r^\perp$'s for $r \in \mathcal{R}$, where the distance is taken in hyperbolic space. Additionally, these $r^{\perp}$'s are precisely the closest mirrors to τ, and it often has another property. Specifically, τ is often the unique point in hyperbolic space fixed by a group $G \leq \mathrm{Aut}(L)$, where G acts as the graph automorphism group of the undirected graph corresponding to D. This τ is the analog of the Weyl vector.

One example found by Basak is the "complex lorentzian Leech lattice," which has a connection with the bimonster group (the wreath product of the monster sporadic simple group and $\mathbb{Z}/2\mathbb{Z}$) [Ba1]. For this $\mathcal{E}$-lattice L, the set S contains 26 roots (each giving an order 3 reflection) whose diagram D is $\mathrm{Inc}\,\mathbb{P}^2(\mathbb{F}_3)$, the incidence graph of the projective plane over $\mathbb{F}_3$. It turns out that D also describes a presentation for the bimonster, whose generators g are vertices of D and commute or braid accordingly, but where each g has order 2 instead of order 3. The lattice L is isomorphic to $3E_8 \oplus H$ (as $\mathcal{E}$-lattices), and thus contains copies of $E_8 \oplus H$ and $2E_8 \oplus H$. Each of these sublattices also has a "Weyl vector" and corresponding root diagram whose reflections generate $\mathrm{Ref}(L)$.

Basak gives another example in [Bas2], this time with $R = \mathcal{H}$. The lattice L has a root diagram D which is a directed version of the incidence graph of the projective plane over $\mathbb{F}_2$, $\mathrm{Inc}\,\mathbb{P}^2(\mathbb{F}_2)$. There is a "Weyl vector" in L, τ which is fixed by a group $G \leq \mathrm{Aut}(L)$, where G acts as the automorphism group of the undirected version of D. As hoped, the mirrors in roots of D are exactly those mirrors closest to τ in hyperbolic space. The lattice L is isomorphic to $3E_8 \oplus H$; it therefore contains sublattices isomorphic to $E_8 \oplus H$ and $2E_8 \oplus H$ (as $\mathcal{H}$-lattices). It is expected that, as in the prior example, these lattices each have a "Weyl vector" and corresponding root diagram whose reflections generate $\mathrm{Ref}(L)$.

In both of the examples given above, the fact that the mirrors of D generate $\mathrm{Ref}(L)$ can be proved using a "height reduction" argument. However, this height reduction argument differs from that used in the case for positive-definite integral $\mathbb{Z}$-lattices. The difference is that it proceeds in two stages. For the first stage we fix a *cusp* (norm-0 vector in L) called ρ. In hyperbolic space ρ can be thought of as a point at infinity, because it lies on the boundary in V between points of negative norm and points of positive norm. We define a vector's height with respect to ρ, and consider the set of roots with minimal height $\mathcal{R}_0$. By a height reduction argument, we show that an arbitrary root can be reflected down to one in $\mathcal{R}_0$, using reflections in the roots of $\mathcal{R}_0$. There are infinitely many roots in $\mathcal{R}_0$, but the reflections in roots of $\mathcal{R}_0$ are finitely generated. Suppose they are generated by reflections in roots from a finite set $\mathcal{R}_0'$. The second stage of our height reduction argument reflects the roots of $\mathcal{R}_0'$ down to the roots of D by height reduction with respect to the "Weyl vector."

The work presented in this thesis extends the two aforementioned examples. We consider here a nested sequence of three lattices: $3D_4^{\mathcal{G}} \oplus H, 2D_4^{\mathcal{G}} \oplus H,$ and $D_4^{\mathcal{G}} \oplus H$, and prove results for them analogous to those for the examples above.

2.4 Background on graphs

2.4.1. Notation and Conventions about graphs: We maintain the notation of [Ser] regarding graphs. By subgraph we always mean full subgraph. Recall that a combinatorial graph Ψ is specified by giving a set of vertices $v(\Psi)$ and a symmetric relation $e(\Psi) \subseteq v(\Psi) \times v(\Psi)$ called the set of geometric edges of Ψ satisfying $(a, a) \notin e(\Psi)$ for all $a \in v(\Psi)$. If $y = (a, b) \in e(\Psi)$, then $\bar{y} = (b, a)$ denotes the opposite edge and we write $a = o(y) = t(\bar{y})$, $b = t(y) = o(\bar{y})$. An *orientation* on Ψ is given by a subset $e_+(\Psi) \subseteq e(\Psi)$ called the set of (positively) oriented edges satisfying $e(\Psi) - e_+(\Psi) = \{\bar{y} \colon y \in e_+(\Psi)\}$. Equivalently, an orientation on Ψ is given by a function $\omega : e(\Psi) \to \{1, -1\}$ such that $\omega(\bar{y}) = -\omega(y)$ for all $y \in e(\Psi)$. We relate the two ways of defining orientation as follows: given ω, we let $e_+(\Psi) = \{y \in e(\Psi) \colon \omega(y) = 1\}$. A *diagram* (Ψ, ω) is a combinatorial graph Ψ together with an orientation ω. If $(a, b) \in e_+(\Psi)$, then we draw an arrow from a to b in our picture for the diagram. To avoid confusion, we shall write $(a \to b) \in e_+(\Psi)$.

Let $\Psi^{\to} = (\Psi, \omega)$ be a diagram. Let $\mathrm{Aut}(\Psi)$ denote the set of graph automorphisms of the combinatorial graph Ψ and let $\mathrm{Aut}(\Psi, \omega)$ denote the set of oriented graph automorphisms. An automorphism $g \in \mathrm{Aut}(\Psi)$ is determined by a bijection $g : v(\Psi) \to v(\Psi)$ such that $(a, b) \in e(\Psi)$ if and only if $g(a, b) = (g(a), g(b)) \in e(\Psi)$. This automorphism g belongs to $\mathrm{Aut}(\Psi, \omega)$ if and only if $\omega(g(y)) = \omega(y)$ for all $y \in e(\Psi)$.

A *bipartite structure* on a combinatorial graph Ψ is an onto map $\mathsf{type} : v(\Psi) \to \{1, -1\}$ such that $\mathsf{type}(t(y)) = -\mathsf{type}(o(y))$ for each $y \in e(\Psi)$. The vertices v for which $v(\Psi) = 1$ are called *positive vertices* and the rest are called the negative vertices. So each edge of Ψ joins a positive vertex with a negative vertex.

Lemma 2.4.2. *The following are equivalent:*

(a) Each positively oriented edge goes from positive vertex to a negative vertex..

(b) If y is an positively oriented edge, then $o(y)$ is a positive vertex.

(c) One has $\mathsf{type}(o(y)) = \omega(y) = -\mathsf{type}(t(y))$ for each $y \in e(\Psi)$.

We shall say that an orientation ω and an bipartite structure type on Ψ are *compatible* with each other if the equivalent conditions of the above lemma are satisfied.

Lemma 2.4.3. *Let (Ψ, ω) be a connected finite diagram. Let $\mathsf{type} : v(\Psi) \to \{\pm 1\}$ be a bipartite structure compatible with ω.*

(a) Let $g \in \mathrm{Aut}(\Psi)$. If g preserves (resp. reverses) the type of one vertex, then g preserves (resp. reverses) the type of every vertex.

(b) Let $g \in \mathrm{Aut}(\Psi)$. Then g preserves type if and only if g preserves ω.

(c) Assume that there exists $\sigma \in \mathrm{Aut}(\Psi)$ such that $\mathsf{type}(\sigma(v)) = -\mathsf{type}(v)$ for some $v \in v(\Psi)$ (hence for all $v \in v(\Psi)$). Then $\mathrm{Aut}(\Psi, \omega)$ has index 2 in $\mathrm{Aut}(\Psi)$ and σ represents the nontrivial coset.

Proof. (a) Pick $v_0 \in v(\Psi)$. Write $\mathsf{type}(gv_0) = \epsilon\,\mathsf{type}(v_0)$ with $\epsilon = 1$ or $\epsilon = -1$. To show $\mathsf{type}(gv) = \epsilon\,\mathsf{type}(v)$ for every $v \in v(\Psi)$, induct on the distance $d(v_0, v)$. Since Ψ is connected, we can choose $v' \in v(\Psi)$ such that $d(v_0, v') = d(v_0, v) - 1$ and $(v, v') \in e(\Psi)$. So $\mathsf{type}(gv) = -\mathsf{type}(gv') = -\epsilon\,\mathsf{type}(v') = \epsilon\,\mathsf{type}(v)$ where we have the first and the third equality since (v, v') and (gv, gv') are edges of Ψ and the second equality follows from the induction hypothesis.

(b) Suppose g preserves type. Let (a, b) be an oriented edge of Ψ. Then a is of positive type and b is of negative type. Since g is a graph automorphism, either (ga, gb) is an oriented edge, or (gb, ga) is. Since each oriented edge goes from a positive vertex to a negative vertex, it follows that (ga, gb) must be an oriented edge. So g preserves the orientation. Conversely, suppose g preserves orientation. Let (a, b) be a positively oriented edge. Then (ga, gb) is a positively oriented edge. So ga is a positive vertex. Thus g preserves the type of the vertex a, and hence preserves type by part (a).

(c) Part (a) implies that σ reverses type and (b) implies that σ does not preserve ω so $\sigma \in \mathrm{Aut}(\Psi) - \mathrm{Aut}(\Psi, \omega)$. Let $g \in \mathrm{Aut}(\Psi)$. By part (a) either g or $\sigma^{-1}g$ is type preserving, so by part (b), either g or $\sigma^{-1}g$ belongs to $\mathrm{Aut}(\Psi, \omega)$. $\square$

Lemma 2.4.4. *Let $\mathcal{O}$ be one of the three rings : $\mathcal{O} = \mathbb{Z}$ or $\mathcal{E}$ or $\mathcal{G}$. Let K^0 (resp. K) be an $\mathcal{O}$-lattice spanned by $s_1^0, \cdots, s_n^0$ (resp. $s_1, \cdots, s_n$). Assume that K is non-singular, $s_1^0, \cdots, s_n^0$ form a basis of K^0 and $\langle s_i^0, s_j^0 \rangle = \langle s_i, s_j \rangle$. Then $s_i^0 \mapsto s_i$ induces an isometry $K^0 / \mathrm{Rad}(K^0) \simeq K$.*

Proof. Since s_i^0's form a basis of K^0 note that $f : s_i^0 \mapsto s_i$ defines a well defined map from K^0 to K. Since s_i's span K, the function f is onto and we have $\langle f(x), f(y) \rangle = \langle x, y \rangle$ for all $x, y \in K^0$. If $x \in \mathrm{Rad}(K^0)$ then $0 = \langle x, y \rangle = \langle f(x), f(y) \rangle$ for all $y \in K^0$. Since f is onto $\langle f(x), z \rangle = 0$ for all $z \in K$. Since K is nonsingular, $f(x) = 0$, so $x \in \mathrm{Ker}(f)$. On the other hand, if $x \in \ker(f)$, then for all $y \in K^0$, we have $\langle x, y \rangle = \langle f(x), f(y) \rangle = \langle 0, f(y) \rangle = 0$, so $x \in \mathrm{Rad}(K^0)$. So $\mathrm{Rad}(K^0) = \ker(f)$. $\square$

Lemma 2.4.5. *Let $\mathcal{O}$ be as Lemma 2.4.4. Suppose L is an $\mathcal{O}$-lattice, and $\psi \in \mathrm{Aut}(L)$. Define $\widehat{v} := v + \mathrm{Rad}(L)$ for all $v \in L$. Further, define a hermitian form on $L/\mathrm{Rad}(L)$ by $\langle \widehat{v}, \widehat{w} \rangle = \langle v, w \rangle$. Then ψ induces an automorphism $\widehat{\psi} \in \mathrm{Aut}(L/\mathrm{Rad}(L))$ given by $\widehat{\psi}(\widehat{v}) := \widehat{\psi(v)}$.*

Proof. $\langle \cdot, \cdot \rangle$ is well-defined on $L/\mathrm{Rad}(L)$: Suppose $\widehat{v_1} = \widehat{v_2}$ and $\widehat{w_1} = \widehat{w_2}$. Then $v_1 = v_2 + r$ and $w_1 = w_2 + s$ for $r, s \in \mathrm{Rad}(L)$. So $\langle \widehat{v_1}, \widehat{w_1} \rangle = \langle v_1, w_1 \rangle = \langle v_2 + r, w_2 + s \rangle = \langle v_2, w_2 \rangle + \langle r, w_2 \rangle + \langle v_2, s \rangle + \langle r, s \rangle = \langle v_2, w_2 \rangle = \langle \widehat{v_2}, \widehat{w_2} \rangle$.

$\widehat{\psi}$ is well-defined: Suppose $\widehat{v_1} = \widehat{v_2}$, so $v_1 = v_2 + r$ for some $r \in \mathrm{Rad}(L)$. Then $\widehat{\psi}(\widehat{v_1}) = \widehat{\psi(v_1)} = \psi\widehat{(v_2 + r)} = \widehat{\psi(v_2)} + \widehat{\psi(r)} = \widehat{\psi(v_2)} = \widehat{\psi}(\widehat{v_2})$. (We know $\psi(r)$ is in $\mathrm{Rad}(L)$ because ψ preserves $\langle \cdot, \cdot \rangle$.)

$\widehat{\psi}$ is 1-to-1: Suppose $\widehat{\psi}(\widehat{v}) = \widehat{0}$. Then $\psi(v) = r \in \mathrm{Rad}(L)$, so $v = \psi^{-1}(r) \in \mathrm{Rad}(L)$. Thus $\widehat{v} = \widehat{0}$.

$\widehat{\psi}$ is onto: This is immediate since ψ is onto.

$\widehat{\psi}$ preserves $\langle \cdot, \cdot \rangle$: We have $\langle \widehat{\psi}(\widehat{v_1}), \widehat{\psi}(\widehat{v_2}) \rangle = \langle \widehat{\psi(v_1)}, \widehat{\psi(v_2)} \rangle = \langle \psi(v_1), \psi(v_2) \rangle = \langle v_1, v_2 \rangle = \langle \widehat{v_1}, \widehat{v_2} \rangle$. $\square$

2.5 Complex lattices and their diagrams

Definition 2.5.1. If V is an $(n+1)$ dimensional complex vector space with a hermitian form of signature $(n, 1)$, we let $\mathbb{B}(V)$ denote the set of complex lines in $\mathbb{P}(V)$ of negative norm. If K is a subset of V such that $V = \mathrm{span}_{\mathbb{C}}(K)$, then we write $\mathbb{B}(K) = \mathbb{B}(V)$.

Let $\mathcal{G} = \mathbb{Z}[i]$ be the ring of Gaussian integers. Let $p = (1 + i)$. By a *diagram* we mean a combinatorial graph with an orientation. If $\Psi^{\rightarrow}$ is a diagram, then we let Ψ denote the corresponding combinatorial graph. Given a diagram $\Psi^{\rightarrow}$, let $\mathcal{G}(\Psi^{\rightarrow})$ denote the free $\mathcal{G}$-module with basis indexed by the vertices of Ψ and a $\mathcal{G}$-valued hermitian form (linear in second variable) given by $v^2 = 2$ for each vertex $v \in \Psi$ and $\langle w, v \rangle = p = \overline{\langle v, w \rangle}$ if there is an arrow from w to v and $\langle v, w \rangle = 0$ otherwise. Let

$$L^{\mathcal{G}}(\Psi^{\rightarrow}) = \mathcal{G}(\Psi^{\rightarrow}) / \mathsf{Radical}(\mathcal{G}(\Psi^{\rightarrow})).$$

Let $I_{14}^{\rightarrow}$ denote the bipartite diagram $\mathrm{Inc}\,\mathbb{P}^2(\mathbb{F}_2)$ where all the arrows go from the points to the lines. Let $I_8^{\rightarrow}$ denote the diagram with vertex set $\mathbb{Z}/8\mathbb{Z}$ and oriented edges going from the vertices $(2j-1)$ and $(2j+1)$ to the vertex $2j$ for $j = 0, 1, \cdots, 3$. We shall call this diagram an *octagon*. Let $I_4^{\rightarrow}$ be the subdiagram of $I_8^{\rightarrow}$ formed by the vertices $1, 2, 3,$ and 4.

2.5.2. Write $L^0 = L^0(I_8^{\rightarrow})$ with basis vectors $s_0^0, \cdots, s_7^0$. So

$$\langle s_k^0, s_r^0 \rangle = \begin{cases} 2 & \text{if } k = r \\ 1 + (-1)^r i & \text{if } k \text{ and } r \text{ differ by } 1 \\ 0 & \text{otherwise.} \end{cases}$$

where the subscripts are read modulo 8, so $s_8 = s_0$ etc. Let $L = L^{\mathcal{G}}(I_8^{\rightarrow}) = L^0 / \mathrm{Rad}(L^0)$, the projection from L^0 to L sending s_j^0 to s_j. The vectors s_{2j}^0, s_{2j} are called the line roots and the vectors s^0_{2j-1}, s_{2j-1} are called the point roots. For each $k \in \mathbb{Z}/8\mathbb{Z}$, define

$$w^0_k = s^0_{k-1} - (1+(-1)^k i)s^0_k + s^0_{k+1}.$$

Then

$$\langle w^0_k, s^0_j\rangle = \begin{cases} 1+(-1)^k i & \text{if } \ j = k \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular each w^0_k is a null vector and

$$z^0_k = (w^0_k - w^0_{k+4}) \in \operatorname{Rad}(L^0) \ \text{ for all } \ k.$$

The z^0_j's satisfy the relations

$$z_{k-1} + (1-(-1)^k i)z_k + z_{k+1} = 0.$$

Writing the vectors z^0_j in the basis $\{s^0_0, \cdots, s^0_7\}$ we observe that z^0_j and z^0_{j+1} are linearly independent. Verify that that any two of these z^0_j, z^0_{j+1} form a basis of $\operatorname{Rad}(L^0)$.

Now we work in $L(I_8^{\rightarrow}) = L^0(I_8^{\rightarrow})/\operatorname{Rad}(L^0(I_8^{\rightarrow}))$. Let s_j, w_j be the vectors in $L(I_8^{\rightarrow})$ corresponding to s^0_j, w^0_j. The four mirrors $s^{\perp}_{2j}$ intersect in a two dimensional subspace of $L(I_8^{\rightarrow}) \otimes \mathbb{C}$. Note that w_{2k-1} are orthogonal to each s_{2j}. So two linearly independent norm zero vectors in the intersection $\cap_{j=1}^4 s^{\perp}_{2j}$ are given by w_1, w_3.

Lemma 2.5.3. *Let L be a Lorentzian integral $\mathcal{G}$-lattice, and let $\phi^i_{r_1}, \phi^i_{r_2} \in \operatorname{Ref}(L)$. Then*
(a) $\phi^i_{r_1}\phi^i_{r_2} = \phi^i_{r_2}\phi^i_{r_1} \Leftrightarrow \langle r_1, r_2\rangle = 0$ (i.e., r_1 and r_2 have no edge between them)
(b) $\phi^i_{r_1}\phi^i_{r_2}\phi^i_{r_1} = \phi^i_{r_2}\phi^i_{r_1}\phi^i_{r_2} \Leftrightarrow |\langle r_1, r_2\rangle|^2 = 2$ (i.e., r_1 and r_2 have an edge between them)

Proof. Set $W := \mathbb{C}r_1 + \mathbb{C}r_2$, $G := \langle\{\phi^i_{r_1}, \phi^i_{r_2}\}\rangle$, and let $\mathcal{B} := \{r_1, r_2\}$ be an ordered basis of W. Then G stabilizes W and fixes $W^{\perp}$ pointwise, so G is isomorphic to its restriction to W. We now identify G with its restriction to W. For $j = 1, 2$ let M_j be the matrix of ϕ_{r_j} with respect to $\mathcal{B}$. Set $\alpha := -(1-i)/2$, $\beta = \langle r_1, r_2\rangle$. We get

$$M_1 = \begin{bmatrix} i & \alpha\beta \\ 0 & 1 \end{bmatrix}, \qquad\qquad M_2 = \begin{bmatrix} 1 & 0 \\ \alpha\overline{\beta} & i \end{bmatrix}$$

$$M_1M_2 = \begin{bmatrix} i+\alpha^2|\beta|^2 & i\alpha\beta \\ \alpha\overline{\beta} & i \end{bmatrix}, \qquad M_2M_1 = \begin{bmatrix} i & \alpha\beta \\ i\alpha\overline{\beta} & i+\alpha^2|\beta|^2 \end{bmatrix}$$

$$M_1M_2M_1 = \begin{bmatrix} -1+i\alpha^2|\beta|^2 & 2i\alpha\beta+\alpha^3|\beta|^2\beta \\ i\alpha\overline{\beta} & i+\alpha^2|\beta|^2 \end{bmatrix}, \quad M_2M_1M_2 = \begin{bmatrix} i+\alpha^2|\beta|^2 & i\alpha\beta \\ 2i\alpha\overline{\beta}+\alpha^3|\beta|^2\overline{\beta} & -1+i\alpha^2|\beta|^2 \end{bmatrix}$$

Now, comparing the $(1,2)$ entries of M_1M_2 and M_2M_1 reveals that $M_1M_2 = M_2M_1 \Leftrightarrow \beta = 0$. This proves (a).

Equating the $(1,1)$ entries of $M_1M_2M_1$ and $M_2M_1M_2$ gives us

$$i+\alpha^2|\beta|^2 = -1+i\alpha^2|\beta|^2 \Leftrightarrow 1+i = (i-1)\alpha^2|\beta|^2 \Leftrightarrow |\beta|^2 = 2$$

Thus $|\beta|^2 = 2$ is a necessary condition to conclude $M_1M_2M_1 = M_2M_1M_2$. It will also be sufficient if we can show $|\beta|^2 = 2$ implies the offdiagonal entries are equal. So assume $|\beta|^2 = 2$, and let us equate the $(1,2)$ entries; equating the $(2,1)$ entries is similar.

$$i\alpha\beta = 2i\alpha\beta + \alpha^3|\beta|^2\beta \Leftrightarrow i = 2i + \alpha^2\cdot 2 \Leftrightarrow \alpha^2 = -\frac{i}{2},$$

which is true by the definition of α. Hence we conclude (b). $\square$

CHAPTER 3. A SEQUENCE OF 3 GAUSSIAN LORENTZIAN LATTICES

3.1 Preface

In this section, we study the reflection groups of three Gaussian Lorentzian lattices and prove mostly similar results about them. First we study the lattice $L = 2D_4^{\mathcal{G}} \oplus H$. We prove that $\mathrm{Ref}(2D_4^{\mathcal{G}} \oplus H)$ is generated by reflections in 8 roots which form an octogonal $I_8^{\rightarrow}$ diagram. The mirrors of these roots have a central point τ in $\mathbb{C}H^5$ which is equidistant from the mirrors, and these mirrors are the closest ones to τ. The point τ is also the unique point fixed by the induced action of the dihedral D_8 group acting on the 8 mirrors. We also prove $\mathrm{Ref}(2D_4^{\mathcal{G}} \oplus H)$ has finite index in $\mathrm{Aut}(2D_4^{\mathcal{G}} \oplus H)$.

We prove analogous results for $D_4^{\mathcal{G}} \oplus H$ and $3D_4^{\mathcal{G}} \oplus H$, but with 2 differences. For $L := D_4^{\mathcal{G}} \oplus H$, τ is no longer the unique fixed point of the graph automorphism group. For $L := 3D_4^{\mathcal{G}} \oplus H$, we do not prove that $\mathrm{Ref}(L)$ is generated by the mirrors closest to τ, although we present strong empirical evidence that this is true.

Definition 3.1.1. Let $p := 1 + i$. We define $D_4^{\mathcal{G}} := \{(x, y) \in \mathcal{G}^2 \mid x + y \equiv 0 \mod p\}$ as a positive-definite $\mathcal{G}$-lattice with hermitian form given by $\langle (x_1, x_2), (y_1, y_2)\rangle = \overline{x}_1 y_1 + \overline{x}_2 y_2$. Define the $\mathcal{G}$-lattice H, called the *hyperbolic cell*, by $H := \mathcal{G}^2$ with hermitian form $\langle (x_1, x_2), (y_1, y_2)\rangle =$
$\begin{bmatrix} \overline{x}_1 & \overline{x}_2 \end{bmatrix} \begin{bmatrix} 0 & \overline{p} \\ p & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

Throughout this chapter we will abuse notation and identify a point $x \in \mathbb{C}^{n,1}$, $|x|^2 < 0$ with its image $\mathbb{P}(x)$ in $\mathbb{C}H^n$.

Let K be a $\mathcal{G}$-lattice of rank m. We can forget the $\mathcal{G}$ structure on K to create a $\mathbb{Z}$-lattice of rank $2m$, denoted by K_{real} [CS], which we call the *underlying Z-lattice of K*. The points of K_{real} are given by $(\operatorname{Re} z_1, \frac{1}{i}\operatorname{Im} z_1, \ldots, \operatorname{Re} z_m, \frac{1}{i}\operatorname{Im} z_m)$ whenever $(z_1, \ldots, z_m)$ is in K. The bilinear form on K_{real} is simply the real part of the Hermitian form on K.

We recall now the definition of D_4:

$$D_4 := \{(x_1, \ldots, x_4) \in \mathbb{Z}^4 \mid \sum x_j \equiv 0 \mod 2\}$$

with bilinear form $((x_1, \ldots, x_4), (y_1, \ldots, y_4)) = \sum x_j y_j$.

Lemma 3.1.2. *The underlying $\mathbb{Z}$-lattice of $D_4^{\mathcal{G}}$ is D_4.*

Proof. Suppose $(z_1, z_2) \in D_4^{\mathcal{G}}$, where $z_j := x_j + iy_j$ with $x_j, y_j \in \mathbb{Z}$. Then $p|(z_1 + z_2) \Rightarrow 2|(\overline{p}z_1 + \overline{p}z_2) \Rightarrow 2|\operatorname{Re}(\overline{p}z_1 + \overline{p}z_2) \Rightarrow 2|(x_1 + y_1 + x_2 + y_2) \Rightarrow (x_1, y_1, x_2, y_2) \in D_4$. Conversely, suppose $(x_1, y_1, x_2, y_2) \in D_4$, and set $z_j := x_j + iy_j$. We have $(x_1, y_1, x_2, y_2) \in D_4 \Rightarrow 2|(x_1 + y_1 + x_2 + y_2), 2|(-x_1 + y_1 - x_2 + y_2) \Rightarrow 2|\overline{p}(z_1 + z_2) \Rightarrow p|(z_1 + z_2) \Rightarrow (z_1, z_2) \in D_4^{\mathcal{G}}$. Finally, it is immediately checked that the bilinear form on D_4 is the real part of the hermitian form on $D_4^{\mathcal{G}}$. □

Lemma 3.1.3. *Both H and $D_4^{\mathcal{G}}$ are p-modular lattices.*

Proof. $H^\vee \subseteq \frac{1}{p}H$: Let $v = (a, b) \in H^\vee$. Then $\langle (0, 1), v\rangle = r_1 \in \mathcal{G}$, so $pa = r_1 \Rightarrow a = \frac{r_1}{p}$. Similarly we get $b = \frac{r_2}{p}$ for some $r_2 \in \mathcal{G}$, and thus $v \in \frac{1}{p}H$.

$\frac{1}{p}H \subseteq H^\vee$: Let $v = (\frac{r_1}{p}, \frac{r_2}{p}) \in \frac{1}{p}H$. Choose any $w = (a, b) \in H$. Then $\langle v, w\rangle = p\frac{\overline{r}_2}{\overline{p}}a + \overline{p}\frac{\overline{r}_1}{\overline{p}}b = i\overline{r}_2 a + \overline{r}_1 b \in \mathcal{G}$, so $v \in H^\vee$.

$(D_4^{\mathcal{G}})^\vee \subseteq \frac{1}{p}D_4^{\mathcal{G}}$: Let $v := (a, b) \in (D_4^{\mathcal{G}})^\vee$. Then $\langle (\overline{p}, 0), v\rangle = r_1 \in \mathcal{G} \Rightarrow pa = r_1 \Rightarrow a = \frac{r_1}{p}$. Similarly $b = \frac{r_2}{p}$. Furthermore, we have $\langle (1, 1), v\rangle = \frac{r_1}{p} + \frac{r_2}{p} \in \mathcal{G}$, so $p|(r_1 + r_2)$. Therefore $(r_1, r_2) \in D_4^{\mathcal{G}}$ so $v \in \frac{1}{p}D_4^{\mathcal{G}}$.

$\frac{1}{p}D_4^{\mathcal{G}} \subseteq (D_4^{\mathcal{G}})^\vee$: Let $v = (\frac{r_1}{p}, \frac{r_2}{p}) \in \frac{1}{p}D_4^{\mathcal{G}}$. Choose any $w = (a, b) \in D_4^{\mathcal{G}}$. Then $\langle v, w\rangle = \frac{1}{\overline{p}}(\overline{r}_1 a + \overline{r}_2 b) = \frac{1}{\overline{p}}[a(\overline{r}_1 + \overline{r}_2) - \overline{r}_2(a - b)] = \frac{1}{\overline{p}}[a(\overline{r}_1 + \overline{r}_2) - \overline{r}_2(a + b - 2b)] = a(\frac{\overline{r}_1 + \overline{r}_2}{\overline{p}}) - \overline{r}_2(\frac{a+b}{\overline{p}} - pb) \in \mathcal{G}$. So $v \in (D_4^{\mathcal{G}})^\vee$. □

Corollary 3.1.4. *Both H and D are even lattices.*

Proof. Let $L = H$ or D, and take any $v \in L$. Then by Lemma 3.1.3, $p \mid |v|^2$, but then $|v|^2 \in \mathbb{Z} \Rightarrow 2 \mid |v|^2$. □

Observe that the direct sum of p-modular (even) lattices is p-modular (even). The covering radius of $D_4^{\mathcal{G}}$ is 1 [CS]; we can use the following Lemma 3.1.5 to get the covering radii of $2D_4^{\mathcal{G}}$ and $3D_4^{\mathcal{G}}$.

Lemma 3.1.5. *Suppose L and L' are positive-definite lattices with respective covering radii r and r'. Then the covering radius of $L \oplus L'$ is $\sqrt{r^2 + (r')^2}$.*

Proof. Let the underlying vector spaces of L and L' be V and V', respectively. Denote the covering radius of $L \oplus L'$ by r''.

$(r'')^2 \leq r^2 + (r')^2$: Let $(v, v') \in V \oplus V'$. There are $l \in L$, $l' \in L'$ with $|v - l|^2 \leq r^2$ and $|v' - l'|^2 \leq (r')^2$. So we have $|(v, v') - (l, l')|^2 \leq r^2 + (r')^2$, so $(r'')^2 \leq r^2 + (r')^2$ as desired.

$(r'')^2 \geq r^2 + (r')^2$: Take any $\epsilon > 0$. There are $v \in V$ and $v' \in V'$ with $\min_{l \in L} |v - l|^2 > r^2 - \epsilon/2$ and $\min_{l' \in L'} |v' - l'|^2 > (r')^2 - \epsilon/2$. Then

$$\min_{(l,l') \in L \oplus L'} |(v, v') - (l, l')|^2 = \min_{(l,l') \in L \oplus L'} (|v - l|^2 + |v' - l'|^2) =$$

$$\min_{l \in L} |v - l|^2 + \min_{l' \in L'} |v' - l'|^2 > r^2 + (r')^2 - \epsilon.$$

The conclusion follows. □

3.2 The Lattice $L = 2D_4^{\mathcal{G}} \oplus H$

3.2.1. The Root Diagram of L. Let roots $s_0, \ldots, s_7 \in L$ be as follows:

$$s_0 = (0, 0, 0, p, -1, 0)$$

$$s_1 = (0, 0, 0, 0, -1, -1)$$

$$s_2 = (0, p, 0, 0, -1, 0)$$

$$s_3 = (-1, 1, 0, 0, 0, 0)$$

$$s_4 = (-p, 0, 0, 0, 0, 0)$$

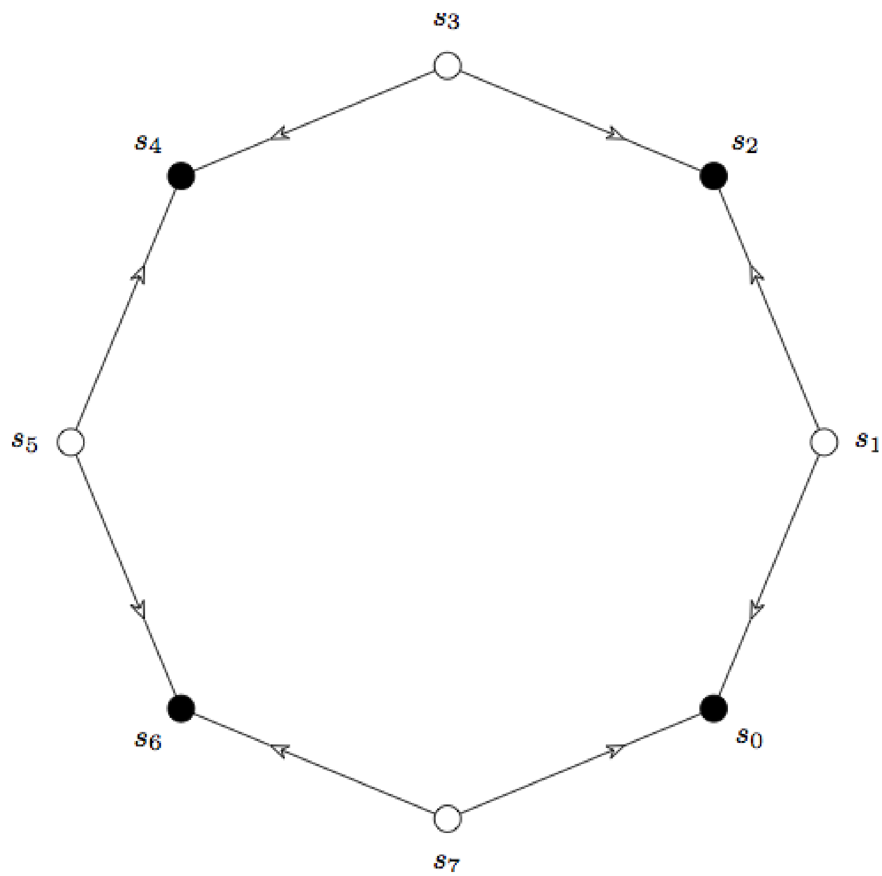
$$s_5 = (-1, -1, -1, -1, -i, -1)$$

$s_6 = (0, 0, -p, 0, 0, 0)$

$s_7 = (0, 0, -1, 1, 0, 0)$

These s_j's give the octogonal $I_8^{\rightarrow}$ diagram shown below, where an arrow goes from s_k to s_l whenever $\langle s_k, s_l \rangle = p$, and no edge means $\langle s_k, s_l \rangle = 0$.

Figure 3.1 The root diagram $I_8^{\rightarrow}$



Consider the singular $\mathcal{G}$-lattice L^0, with basis $s_0', \ldots, s_7'$ whose products are given by $I_8^{\rightarrow}$. Because $\mathrm{Aut}(I_8^{\rightarrow})$ permutes these basis elements, it induces a subgroup of $\mathrm{Aut}(L_0)$. Then Lemmas 2.4.4 and 2.4.5 tell us $\mathrm{Aut}(I_8^{\rightarrow})$ induces a subgroup of $\mathrm{Aut}(L)$. There is a $\sigma \in \mathrm{Aut}(L)$ defined by $s_{2k} \mapsto i s_{1-2k}$, $s_{1-2k} \mapsto s_{2k}$ (where subscripts are taken $\bmod 8$). This σ does not act on the roots $s_0, \ldots, s_7$ in L. However, $\mathbb{P}\sigma$ does act on their images in $\mathbb{P}(L \otimes \mathbb{C})$, and therefore acts on their mirrors in $\mathbb{C}H^5$. Note that $\sigma^2(x) = ix$, so $(\mathbb{P}\sigma)^2 = 1$. Thus $Q := \langle \mathrm{Aut}(I_8^{\rightarrow}), \sigma \rangle$ has a D_8 action on the set of mirrors $\{s_j^{\perp}\}$. (The group D_8 here is the octogonal dihedral group.)

3.2.2. The Central Point of $I_8^{\rightarrow}$ in $\mathbb{C}H^5$. Set $\tau := \sum_{j \text{ odd}} s_j + (-e^{-\pi i/4}) \sum_{j \text{ even}} s_j$, and note $|\tau|^2 < 0$. We see that $\mathrm{Aut}(I_8^{\rightarrow})$ fixes τ in $L \otimes \mathbb{C}$, and $\sigma(\tau) = -e^{\pi i/4}\tau$. This tells us that the action of Q on $\mathbb{C}H^5$ fixes τ's image there. If we denote the hyperbolic distance metric by $d(\cdot,\cdot)$, the $I_8^{\rightarrow}$ diagram shows that $c := d(\tau, s_j^{\perp})$ is constant for all j. This is because Q fixes τ and acts transitively on the mirrors $\{s_j^{\perp}\}$.

Theorem 3.2.3. *The $s_j^{\perp}$'s are the closest mirrors in $\mathbb{C}H^5$ to τ.*

Proof. We describe the algorithm which was implemented as computer code to show the desired result. Suppose $r := (r_1, \ldots, r_6)$ is a root of L with $d(\tau, r^{\perp}) \leq c$. By (2.5) we have for all j

$$d(r^{\perp}, s_j^{\perp}) \leq 2c. \tag{3.1}$$

If $r^{\perp}$ and $s_j^{\perp}$ don't intersect in $\mathbb{C}H^5$, then this and (2.3) give us $|\langle r, s_j\rangle| \leq 2\cosh 2c < \sqrt{5}$. If $r^{\perp}$ and $s_j^{\perp}$ do intersect, then $\mathrm{span}\{r, s_j\}$ must be positive-definite. This implies $|\langle r, s_j\rangle| \leq 2$, so $|\langle r, s_j\rangle| < \sqrt{5}$ holds in this case too.

Note that $s_3, s_4, s_6, s_7 \in 2D_4^{\mathcal{G}}$ as their last 2 coordinates are 0, so if we denote by $\hat{v}$ the vector consisting of the first 4 coordinates (the "$2D_4^{\mathcal{G}} \otimes \mathbb{C}$ part") of a vector $v \in L \otimes \mathbb{C}$, we get

$$|\langle \hat{r}, \hat{s_j}\rangle| < \sqrt{5} \tag{3.2}$$

for $j = 3, 4, 6, 7$. This can be written in matrix form as

$$\hat{r}^* = R^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

where R is the invertible 4x4 matrix whose rows are $\hat{s}_3, \hat{s}_4, \hat{s}_6, \hat{s}_7$ in that order, and the $a_i \in \mathcal{G}$ satisfy $|a_i| < \sqrt{5}$. There are only finitely many possibilities for such a_i's, so the first 4 coordinates of r belong to a finite set, which can be found by iterating through the possible a_i's. To determine the last 2 coordinates r_5 and r_6 of r, we proceed as follows. Define $w \in \mathbb{C}^{5,1}$ as $w := (0^{(4)}; 1, -1)$, where $\langle w, w\rangle < 0$ so $w \in \mathbb{C}H^5$. Either $r^{\perp}$ and $s_1^{\perp}$ intersect so $|\langle r, s_1\rangle| \leq 2$, or from $j = 1$ in (3.1) we get $|pr_5 + \bar{p}r_6| \leq 2\cosh 2c$. In either case we have

$$|r_5 - ir_6| \leq \sqrt{2}\cosh 2c. \tag{3.3}$$

From (2.4) we get $d(r^\perp, w) \leq d(r^\perp, \tau) + d(\tau, w) \leq c + e$, where $e := d(\tau, w)$. If $w \in r^\perp$ then $\langle w, r \rangle = 0$. Otherwise (2.2) yields $|\langle w, r \rangle| = |-pr_5 + \bar{p}r_6| \leq 2\sinh(c + e)$, so in both cases we have

$$|-r_5 - ir_6| \leq \sqrt{2}\sinh(c + e) \tag{3.4}$$

From (3.3) and (3.4) we thus obtain

$$2|r_5| = |2r_5| = |(r_5 - ir_6) - (-r_5 - ir_6)| \leq \sqrt{2}(\cosh 2c + \sinh(c + e))$$

hence

$$|r_5| \leq \frac{\cosh 2c + \sinh(c + e)}{\sqrt{2}} < \sqrt{5}$$

and similarly we see $|r_6|$ satisfies the same inequality. So we have that r_5 and r_6 belong to a finite set as well, and altogether there are a finite number of choices for the coordinates of a root r whose mirror is at least as close to τ as the mirrors of the s_j's, and these choices can be checked iteratively. In this way we can verify that $s_0^\perp, \ldots, s_7^\perp$ are precisely the closest mirrors to τ. $\square$

Theorem 3.2.4. *τ is the unique point in $\mathbb{C}H^5$ fixed by the group $Q := \langle \mathrm{Aut}(I_8^\to), \sigma \rangle$.*

Proof. Consider $\psi \in \mathrm{Aut}(I_8^\to)$ which acts by reflecting the diagram about the line through s_1 and s_5. In cyclic notation, it permutes the vertices in $I_8^\to$ according to

$$\psi := (s_0\ s_2)(s_3\ s_7)(s_4\ s_6).$$

Take the ordered basis $\mathcal{B} := (s_0, s_1, s_2, s_3, s_6, s_7)$ of $L \otimes \mathbb{C}$. The matrices M_σ, M_ψ of σ, ψ with respect to $\mathcal{B}$ are given by

$$M_\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{bmatrix} \qquad M_\psi = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1+i & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1-i & 0 \end{bmatrix}$$

A computer calculation reveals that M_σ and M_ψ have exactly two common eigenvectors (up to scalar multiple), one of which is τ. Of the two, only τ has negative norm and thus is in $\mathbb{C}H^5$. Therefore τ is the only point in $\mathbb{C}H^5$ fixed by the group Q. $\square$

3.2.5. We define $\rho := (\underline{0}; 0, 1) \in L$. Then for $v = (\lambda; a, b) \in L \otimes \mathbb{C}$ set $\mathrm{ht}(v) := |\langle v, \rho\rangle|/\sqrt{2} = |a|$. We call this the *height* of v. Note that this is related to the "distance" from $\mathbb{P}(v)$ to the cusp $\mathbb{P}(\rho)$, which is tantamount to the distance from $\mathbb{P}(v)$ to a horoball $B_\rho(k)$. (See Section 2.2.10 for more information.) Reducing the height of v, as we have defined it here, is essentially the same as reducing the distance from $\mathbb{P}(v)$ to $B_\rho(k)$.

Lemma 3.2.6. *Let $y := (l; 1, \overline{p}^{-1}(\alpha - l^2/2))$ be a multiple of a root in $L \otimes \mathbb{C}$ with height > 1. Then there is a root r in L of height 1 such that we can reduce the height of y to a smaller height, by reflecting it in an i-reflection of r.*

Proof. We have $|y|^2 \in (0, 2)$, so $\operatorname{Re}\alpha \in (0, 1)$. Set $r := (\lambda; 1, \overline{p}^{-1}(1 - \lambda^2/2 + \beta + n))$ where $\beta \in \operatorname{Im}\mathbb{C}$ is chosen so that $\overline{p}^{-1}(1 - \lambda^2/2 + \beta) \in \mathcal{G}$. We will choose $\lambda \in 2D_4^{\mathcal{G}}$ and $n \in \langle p\rangle \cap \operatorname{Im}\mathcal{G} = 2\operatorname{Im}\mathcal{G}$ later. We have

$$\begin{aligned} \langle r, y\rangle &= \overline{pp}^{-1}(\alpha - l^2/2) + pp^{-1}(1 - \lambda^2/2 + \overline{\beta} + \overline{n}) + \langle \lambda, l\rangle \\ &= \alpha - l^2/2 + 1 - \lambda^2/2 - \beta - n + \langle \lambda, l\rangle \\ &= 2a + 2b \end{aligned}$$

where $2a := 1 + \operatorname{Re}\alpha - \frac{1}{2}|l-\lambda|^2 \in \mathbb{R}$ and $2b := \operatorname{Im}\alpha + \operatorname{Im}\langle\lambda, l\rangle - \beta - n \in \operatorname{Im}\mathbb{C}$. So $ht(\phi_r^i(y)) = 1-(1-i)(a+b)$. We want $|1-(1-i)(a+b)|^2 < 1$, i.e.

$$\left|\frac{1}{2} - a\right|^2 + \left|\frac{1}{2}i - b\right|^2 < \frac{1}{2} \tag{3.5}$$

As noted above, $\operatorname{Re}\alpha \in (0,1)$. Thus we may choose λ so that $a \in (\frac{1}{2} - \frac{1}{4}c^2, 1) = (0,1)$ where $c = \sqrt{2}$ is the covering radius of $2D_4^{\mathcal{G}}$. This implies $\left|\frac{1}{2} - a\right|^2 < \frac{1}{4}$. Next, observe $i - 2b \in \operatorname{Im}\mathbb{C}$ and we may choose $n \in 2\operatorname{Im}\mathcal{G}$ so that $|i-2b| \leq 1$, since the covering radius of $2\operatorname{Im}\mathcal{G}$ is 1. Therefore we may choose n so that $\left|\frac{1}{2}i - b\right|^2 \leq \frac{1}{4}$, and (3.5) follows. □

Theorem 3.2.7. *The reflections in roots of height 0 and 1 in L generate Ref(L).*

Proof. Let r be a root of L. By Lemma 3.2.6, if r has height > 1, it can be reflected to a root with strictly lower height; since the set of possible heights for lattice vectors is $\mathbb{Z}_{\geq 0}$, by induction r can be reflected to a root of height 0 or 1. This shows that every reflection is conjugate, via reflections in roots of height 1, to a reflection in a root of height 0 or 1, which proves the theorem. □

3.2.8. The Heisenberg group of Translations on L. Define the *group of translations* $\mathbb{T}$ *of L* to be the subgroup of $\operatorname{Aut}(L)$ which fixes ρ and acts trivially on $\rho^\perp/\langle\rho\rangle$.

Theorem 3.2.9. *We have*

$$\mathbb{T} = \{T_{\lambda,z} \mid \lambda \in 2D_4^{\mathcal{G}}, z = (|\lambda|^2/2 + 2k)i \text{ for some } k \in \mathbb{Z}\}$$

(note this is equivalent to $\mathbb{T} = \{T_{\lambda,z} \mid \lambda \in 2D_4^{\mathcal{G}}, z \in i\cdot(2\mathbb{Z} + \frac{|\lambda|^2}{2} (\operatorname{mod} 2))\}$)

where $T_{\lambda,z} : L \longrightarrow L$ acts via

$$\begin{aligned} T_{\lambda,z}(\rho) &= \rho \\ T_{\lambda,z}(\underline{0}; 1, 0) &= (\lambda; 1, \frac{1}{\bar{p}}(-|\lambda|^2/2 + z)) \\ T_{\lambda,z}(x; 0, 0) &= (x; 0, -\frac{1}{\bar{p}}\langle\lambda, x\rangle) \end{aligned}$$

Proof. Let $\psi \in \mathbb{T}$, and choose an arbitrary $x \in 2D_4^{\mathcal{G}}$. Because ψ preserves $\langle \cdot, \cdot \rangle$, in particular it preserves the products and norms of ρ, $(\underline{0}; 1, 0)$, and $(x; 0, 0)$. By the definition of $\mathbb{T}$ we have $\psi : \rho \mapsto \rho$. We can see that $\rho^{\perp} = \{(x; 0, b) \mid x \in 2D_4^{\mathcal{G}}, b \in \mathcal{G}\}$. Then because ψ acts trivially on $\rho^{\perp}/\langle \rho \rangle$ we know $\psi : (x; 0, 0) \mapsto (x; 0, b_1)$ for some $b_1 \in \mathcal{G}$. Finally, let us write $\psi : (\underline{0}; 1, 0) \mapsto (\lambda; a, b_2)$ for some $\lambda \in 2D_4^{\mathcal{G}}$ and $a, b_2 \in \mathcal{G}$.

Then we have

$$\overline{p} = \langle (\underline{0}; 1, 0), \rho \rangle = \langle (\lambda; a, b), \rho \rangle = \overline{pa} \Rightarrow a = 1$$

Also,

$$0 = \langle (\underline{0}; 1, 0), (x; 0, 0) \rangle = \langle (\lambda, a, b_2), (x; 0, b_1) \rangle = \langle \lambda, x \rangle + \overline{pa} b_1 \Rightarrow b_1 = -\frac{1}{\overline{p}} \langle \lambda, x \rangle$$

Note that $b_1 \in \mathcal{G}$ because $2D_4^{\mathcal{G}}$ is p-modular. Furthermore,

$$0 = |(\underline{0}; 1, 0)|^2 = |(\lambda; a, b_2)|^2 = |\lambda|^2 + 2 \operatorname{Re} pa\overline{b_2} = |\lambda|^2 + 2 \operatorname{Re} p\overline{b_2} \Rightarrow$$

$$\operatorname{Re} \overline{p} b_2 = -|\lambda|^2/2 \Rightarrow \overline{p} b_2 = -|\lambda|^2/2 + \beta i \text{ for some } \beta \in \mathbb{Z}.$$

So $b_2 = \frac{1}{\overline{p}}(-|\lambda|^2/2 + \beta i)$. For b_2 to be in $\mathcal{G}$ it is necessary and sufficient that $\beta = -|\lambda|^2/2 + 2l$ for some $l \in \mathbb{Z}$. Because $|\lambda|^2$ is even, this is equivalent to $\beta = |\lambda|^2/2 + 2k$ for some $k \in \mathbb{Z}$. Thus ψ is of the form $T_{\lambda,z}$ for a $\lambda \in 2D_4^{\mathcal{G}}$ and with $z = (|\lambda|^2/2 + 2k)i$ for some $k \in \mathbb{Z}$.

Conversely, take any such $T_{\lambda,z}$. That $T_{\lambda,z}$ fixes ρ and acts trivially on $\rho^{\perp}/\langle \rho \rangle$ is immediate. Because $2D_4^{\mathcal{G}}$ is even and p-modular we see $T_{\lambda,z}$ preserves L. Now choose any $x \in 2D_4^{\mathcal{G}}$. To see $T_{\lambda,z}$ preserves $\langle \cdot, \cdot \rangle$, it suffices to check it preserves products between and norms of ρ, $(\underline{0}; 1, 0)$, and $(x; 0, 0)$. This is a straightforward calculation. □

Lemma 3.2.10. *The following identities hold in* $\mathbb{T}$*:*

$$T_{\lambda_1,z_1} T_{\lambda_2,z_2} = T_{\lambda_1+\lambda_2, z_1+z_2+\operatorname{Im}\langle \lambda_2, \lambda_1 \rangle} \tag{3.6}$$

$$T_{\lambda,z} T_{0,0} = T_{0,0} T_{\lambda,z} = T_{\lambda,z} \tag{3.7}$$

$$T_{\lambda,z}^{-1} = T_{-\lambda,-z} \tag{3.8}$$

$$T_{\lambda_1,z_1} T_{\lambda_2,z_2} T_{\lambda_1,z_1}^{-1} T_{\lambda_2,z_2}^{-1} = T_{0, 2\operatorname{Im}\langle \lambda_2, \lambda_1 \rangle} \tag{3.9}$$

Proof. We prove identity (3.6), from which the others follow. To do this, we show that the LHS and RHS of (3.6) have the same action on ρ, $(\underline{0};1,0)$, and $(x;0,0)$, where $x \in 2D_4^{\mathcal{G}}$ is arbitrary. First we calculate:

$$\begin{aligned} T_{\lambda_1+\lambda_2, z_1+z_2+\mathrm{Im}\,\langle\lambda_2,\lambda_1\rangle}(\rho) &= \rho \\ T_{\lambda_1+\lambda_2, z_1+z_2+\mathrm{Im}\,\langle\lambda_2,\lambda_1\rangle}(\underline{0};1,0) &= (\lambda_1+\lambda_2;1,\frac{1}{p}(-|\lambda_1+\lambda_2|^2/2+z_1+z_2+\mathrm{Im}\,\langle\lambda_2,\lambda_1\rangle)) \\ T_{\lambda_1+\lambda_2, z_1+z_2+\mathrm{Im}\,\langle\lambda_2,\lambda_1\rangle}(x;0,0) &= (x;0,-\frac{1}{p}\langle\lambda_1+\lambda_2,x\rangle) \end{aligned}$$

We also have

$$\begin{aligned} T_{\lambda_2,z_2}(\rho) &= \rho \\ T_{\lambda_2,z_2}(\underline{0};1,0) &= (\lambda_2,1;\frac{1}{p}(-|\lambda_2|^2/2+z_2)) \\ T_{\lambda_2,z_2}(x;0,0) &= (x;0,-\frac{1}{p}\langle\lambda_2,x\rangle) \end{aligned}$$

and therefore

$$\begin{aligned} T_{\lambda_1,z_1}(T_{\lambda_2,z_2}(\rho)) &= \rho \\ T_{\lambda_1,z_1}(T_{\lambda_2,z_2}(\underline{0};1,0)) &= (\lambda_2,0;-\frac{1}{p}\langle\lambda_1,\lambda_2\rangle)+(\lambda_1;1,\frac{1}{p}(-|\lambda_1|^2/2+z_1))+(0;0,\frac{1}{p}(-|\lambda_2|^2/2+z_2)) \\ &= (\lambda_1+\lambda_2;1,\frac{1}{p}(-|\lambda_1|^2/2-|\lambda_2|^2/2-\langle\lambda_1,\lambda_2\rangle+z_1+z_2)). \end{aligned}$$

This last expression is equal to

$$\begin{aligned} &(\lambda_1+\lambda_2;1,\frac{1}{p}(-\frac{1}{2}(|\lambda_1|^2+|\lambda_2|^2+2\,\mathrm{Re}\,\langle\lambda_1,\lambda_2\rangle)-\mathrm{Im}\,\langle\lambda_1,\lambda_2\rangle+z_1+z_2)) \\ &= (\lambda_1+\lambda_2;1,\frac{1}{p}(-|\lambda_1+\lambda_2|^2/2+z_1+z_2+\mathrm{Im}\,\langle\lambda_2,\lambda_1\rangle)). \end{aligned}$$

Finally,

$$\begin{aligned} T_{\lambda_1,z_1}(T_{\lambda_2,z_2}(x;0,0)) &= (x;0,-\frac{1}{p}\langle\lambda_1,x\rangle)+(0;0,-\frac{1}{p}\langle\lambda_2,x\rangle) \\ &= (x;0,-\frac{1}{p}\langle\lambda_1+\lambda_2,x\rangle). \end{aligned}$$

Hence we see the actions are the same, as desired. □

Note: In what follows up through Theorem 3.2.13, where appropriate we will implicitly be considering roots to be defined "up to units." E.g., Lemma 3.2.11 below is really a statement *about T's action on the equivalence classes of height-1 roots defined up to a unit multiple. This doesn't pose a problem, because the purpose is to prove Theorem 3.2.13, in which we are considering reflections in a set of roots, and thus unit multiples don't change the reflections so determined.*

Lemma 3.2.11. *The group $\mathbb{T}$ acts freely and transitively on the set of height-1 roots in L.*

Proof. Let r_1 be a height-1 root, and set $\delta := (0; 1, 1)$, which is another height-1 root. We know $r_1 = (x; 1, \frac{1}{\overline{p}}(1 - |x|^2/2 + \beta i))$ for some $x \in 2D_4^{\mathcal{G}}$ and $\beta \in \mathbb{Z}$. We also have $\beta = 1 - \frac{|x|^2}{2} + 2k$ some $k \in \mathbb{Z}$. Now set $z := (\beta + 1)i = (-\frac{|x|^2}{2} + 2(k+1))i$. It is immediate to check that $T_{x,z}$ is in $\mathbb{T}$ and maps $\delta \mapsto r_1$. If r_2 is another height-1 root, we have $T_{x_2,z_2}T_{x_1,z_1}^{-1}(r_1) = r_2$, for x_j, z_j defined as above. Hence $\mathbb{T}$ acts transitively on the roots of height 1 in L.

To see $\mathbb{T}$ acts freely, calculate/observe that there is at most one $T_{\lambda,z} \in \mathbb{T}$ which maps a given $l_1 \in L$ to another given $l_2 \in L$. □

Lemma 3.2.12. *Let G be a group acting transitively and freely on a set X. Suppose $H \leq G$, and let $\{g_\alpha\}_{\alpha \in A}$ be a complete set of right coset representatives for $H \backslash G$. Then for any $x \in X$, the union $\cup_{\alpha \in A}\{g_\alpha \cdot x\}$ is disjoint and is a complete set of orbit representatives for the action of H on X.*

Proof. Let $x \in X$ be arbitrary. Because G acts transitively on X, and since $\{Hg_\alpha \mid \alpha \in A\}$ partitions G, we see that $\{g_\alpha \cdot x\}_{\alpha \in A}$ contains representatives for each orbit of H's action on X. To show $g_\alpha \cdot x$ and $g_\beta \cdot x$ are in different H-orbits for $\alpha \neq \beta$, choose $\alpha, \beta \in A$ with $g_\alpha \cdot x$ and $g_\beta \cdot x$ in the same orbit of H. Then $g_\alpha \cdot x = hg_\beta \cdot x$ for some $h \in H$. This gives $h^{-1}g_\alpha g_\beta^{-1} \cdot x = x$, so $h^{-1}g_\alpha g_\beta^{-1} \in \mathrm{Stab}_G\, x$; as G acts freely, this says that $g_\alpha = hg_\beta$. So g_α and g_β are in the same coset of $H \backslash G$ and thus $\alpha = \beta$. □

Theorem 3.2.13. $\mathrm{Ref}(L)$ *is finitely generated.*

Proof. Set $m := \dim_{\mathcal{G}} 2D_4^{\mathcal{G}} = 4$. We proceed similarly to [Ba1]. Let $r_1 = (\underline{0}; 1, 1)$, $r_2 = (\underline{0}; 1, i)$. A matrix calculation reveals that

$$T_{\lambda,z}\phi_{r_1}^i\phi_{r_2}^i T_{\lambda,z}^{-1}(\phi_{r_1}^i\phi_{r_2}^i)^{-1} = T_{\overline{p}\lambda, i|\lambda|^2} \tag{3.10}$$

for all $T_{\lambda,z} \in \mathbb{T}$. If we choose $\lambda_1, \ldots, \lambda_m$ to be a $\mathcal{G}$-basis for $2D_4^{\mathcal{G}}$, we get $\lambda_1, \ldots, \lambda_m, i\lambda_1, \ldots, i\lambda_m$ as a $\mathbb{Z}$-basis. We may also choose z_j for each j such that T_{λ_j,z_j} and $T_{i\lambda_j,z_j}$ are in $\mathbb{T}$. It then follows from (3.10) that reflections in $T(r_1)$ and $T(r_2)$ for these T's, together with $\phi^i_{r_1}$ and $\phi^i_{r_2}$, generate a group that contains the group of translations

$$S := \langle T_{\overline{p}\lambda_1, i|\lambda_1|^2}, \ldots, T_{\overline{p}\lambda_m, i|\lambda_m|^2}, T_{\overline{p}i\lambda_1, i|\lambda_1|^2}, \ldots, T_{\overline{p}i\lambda_m, i|\lambda_m|^2}\rangle.$$

By (3.6) we know that S has a translation $T_{\overline{p}\lambda,z}$ for every $\lambda \in 2D_4^{\mathcal{G}}$. Now take any $T_{\overline{p}\lambda,z}$ and $T_{\overline{p}\lambda',z'}$ in $\mathbb{T}$. Let z'' and z''' be such that $T_{\overline{p}(\lambda+\lambda'),z''} = T_{\overline{p}\lambda,z}T_{\overline{p}\lambda',z'}$ and $T_{\overline{p}(\lambda+\lambda'),z'''} = T_{\overline{p}\lambda',z'}T_{\overline{p}\lambda,z}$. Then (3.6) says that $z'' \equiv z'''(\text{mod } 4)$. This tells us that for any fixed $\lambda \in 2D_4^{\mathcal{G}}$, all $T_{\overline{p}\lambda,z} \in S$ have z of the form $(e_\lambda + 4k)i$, where $k \in \mathbb{Z}$ and $e_\lambda \in \{0, 2\}$ depends only on λ. Furthermore, because we may choose λ_1 and λ_2 so that $\langle\lambda_1, \lambda_2\rangle = p$, we get from (3.9) that $\{T_{0,4ki} \mid k \in \mathbb{Z}\}$ is contained in S. Altogether, we get that S contains

$$\mathbb{T}' := \{T_{\overline{p}\lambda, i(e_\lambda+4k)} \mid \lambda \in 2D_4^{\mathcal{G}}, k \in \mathbb{Z}\} \leq \mathbb{T}.$$

We now enumerate a (finite) complete set of coset representatives for $\mathbb{T}/\mathbb{T}'$, where there may be some repetition of the represented cosets. Take any $T_{\lambda,z} \in \mathbb{T}$. Observe that $(2, 0, 0, 0)$ and $(2i, 0, 0, 0)$ (and all permutations of coordinates) are in $\overline{p} \cdot 2D_4^{\mathcal{G}}$. Therefore we can find a $T_{\overline{p}x_0,z_0} \in \mathbb{T}'$ such that $T_{\lambda',z'} := T_{\lambda,z} \cdot T_{\overline{p}x_0,z_0} = T_{\lambda+\overline{p}x_0,*}$ has all coordinates of $\lambda' := (a_1', b_1', a_2', b_2')$ in $\{0, 1, i, p\}$. Because $\lambda' \in 2D_4^{\mathcal{G}}$ we have $p|a_j' + b_j'$. Thus the possible combinations for (a_j', b_j') are $(0, 0), (0, p), (p, 0), (p, p), (1, 1), (1, i), (i, 1)$, and (i, i). We next note that $(u_1p, u_2p) \in \overline{p} \cdot D_4^{\mathcal{G}}$ for any units $u_1, u_2 \in \mathcal{G}^*$. Hence we can find a $T_{\overline{p}x_1,z_1} \in \mathbb{T}'$ such that $T_{\lambda'',z''} := T_{\lambda',z'} \cdot T_{\overline{p}x_1,z_1}$ has $\lambda'' := (a_1'', b_1'', a_2'', b_2'')$ with the following properties: for each j independently, either $a_j'' = 0$ and $b_j'' = 0$ or p, or $a_j'' = 1$ and $b_j'' = 1$ or i. In summary, we can find an $x \in 2D_4^{\mathcal{G}}$ such that $T_{\lambda'',z''} = T_{\lambda,z} \cdot T_{\overline{p}x,i(e_x+4k)}$, where we may choose $k \in \mathbb{Z}$ so that $z'' \in \{0, i, 2i, 3i\}$. Then the following elements of $\mathbb{T}$ form a (possibly redundant) complete set of coset representatives for $\mathbb{T}/\mathbb{T}'$:

$$\{T_{\lambda'',z''} \mid \lambda'' = (a_1'', b_1'', a_2'', b_2''), \text{ where the } a_j'' \text{ and } b_j'' \text{ are as above,}$$
$$\text{and } z'' \in \{0, i, 2i, 3i\},\ z'' = (|\lambda''|^2/2 + 2j)i \text{ for some } j \in \mathbb{Z}\}.$$

Let $T_1, \ldots, T_n$ be an enumeration of this set. Note that $n = 2^{m+1}$, so $[\mathbb{T} : \mathbb{T}'] \leq n < \infty$.

Set $\lambda_j' := T_j(\underline{0}; 1, 1)$ for each j. Then Lemma 3.2.12 and the fact that $\mathbb{T}$ acts freely and transitively on the height-1 roots in L shows that $\cup_{j=1}^n \mathbb{T}'\lambda_j' = \{\text{height-1 roots in } L\}$. Because

$$A\phi_r A^{-1} = \phi_{Ar} \tag{3.11}$$

for any $A \in \mathrm{Aut}(L)$, we deduce that $\mathbb{T}' \cup \{\phi^i_{\lambda_j'}\}_{j=1}^n$ generates all reflections in height-1 roots of L. All such reflections are therefore generated by i-reflections in the following finite set of roots: $r_1, r_2, \{T_{\lambda_j,z_j}(r_1)\}_{j=1}^m, \{T_{\lambda_j,z_j}(r_2)\}_{j=1}^m, \{T_{i\lambda_j,z_j}(r_1)\}_{j=1}^m, \{T_{i\lambda_j,z_j}(r_2)\}_{j=1}^m, \{\lambda_j'\}_{j=1}^n$.

We now turn our attention to the roots of height 0, which have the form $(x; 0, b)$, where x is a root in $2D_4^{\mathcal{G}}$ and $b \in \mathcal{G}$ is arbitrary. We first show that for each such x, the subset of the height-0 roots $R_x := \{(x; 0, b) \mid b \in \mathcal{G}\}$ has exactly two orbits under $\mathbb{T}'$.

Take any $T_{\overline{p}\lambda,z} \in \mathbb{T}'$ and any $(x; 0, b) \in R_x$. Because x is primitive (as it is a root) and $2D_4^{\mathcal{G}}$ is p-modular, there is a $\lambda \in 2D_4^{\mathcal{G}}$ with $\langle \lambda, x\rangle = \overline{p}$. For this choice of λ we get $T_{\overline{p}\lambda,z}(x; 0, b) = (x; 0, b - \frac{1}{\overline{p}}\langle \overline{p}\lambda, x\rangle) = (x; 0, b - p)$. Because $[\mathcal{G} : p\mathcal{G}] = 2$, we can set λ' to be an appropriate $\mathcal{G}$-multiple of λ to get $T_{\overline{p}\lambda',z}(x; 0, b) = (x; 0, 0)$ or $(x; 0, 1)$, depending on b's congruence class mod $p\mathcal{G}$. These are representatives of the two orbits.

Note that $2D_4^{\mathcal{G}}$ has only finitely many roots. Then, as in the height-1 case above, we can generate all reflections in height-0 roots of L with i-reflections in the following finite set of roots: $r_1, r_2, \{T_{\lambda_j,z_j}(r_1)\}_{j=1}^m, \{T_{\lambda_j,z_j}(r_2)\}_{j=1}^m, \{T_{i\lambda_j,z_j}(r_1)\}_{j=1}^m, \{T_{i\lambda_j,z_j}(r_2)\}_{j=1}^m, \{(x; 0, 0) \mid x \in 2D_4^{\mathcal{G}} \text{ is a root}\}, \{(x; 0, 1) \mid x \in 2D_4^{\mathcal{G}} \text{ is a root}\}$. $\square$

Theorem 3.2.14. $\mathrm{Ref}(L)$ *is generated by reflections in the* s_i*'s.*

Proof. This result can be verified by computer program. The proof of theorem 3.2.13 provides a finite set of roots $\{q_1, q_2, \ldots, q_N\}$ whose reflections generate $\mathrm{Ref}(L)$. We demonstrate that the q_j's can all be reflected, by reflections in the s_j's, into $\{s_j\}$. The operative technique is another height reduction, where this time height is measured with respect to the central point τ. The basic algorithm can be described as follows:

Start by setting the root q equal to one of the finite generating roots q_k. Try reflecting q in each of the $\phi_{s_j}^{\xi_m}$'s in turn, until one of these gives $d(\phi_{s_j}^{\xi_m}(q^\perp), \tau) < d(q^\perp, \tau)$. If such an s_j exists, we set $q \leftarrow \phi_{s_j}^{\xi_m}(q)$ and iterate. If no such s_j exists, we check whether $q^\perp = s_l^\perp$ for some l. If so, we report success. Otherwise we report failure (this does not occur in practice). □

We conclude the section for $L = 2D_4^{\mathcal{G}}$ by showing that $[\mathrm{Aut}(L) : \mathrm{Ref}(L)]$ is finite.

Lemma 3.2.15. *Let* $\mathrm{Ref}_1(L)$ *be the group in* $\mathrm{Aut}(L)$ *generated by reflections in roots of height 1. Let* $\mathrm{Cus}(L)$ *be the set of primitive norm-0 vectors in* L*. Then* $\mathrm{Ref}_1(L)$ *acts transitively (up-to-units) on* $\mathrm{Cus}(L)$*.*

Proof. Suppose $y := (l; 1, \overline{p}^{-1}(\alpha - l^2/2)) \in L \otimes \mathbb{C}$ is a multiple of a lattice vector $v \in \mathrm{Cus}(L)$ with non-0 height. Because $|y|^2 = 0$ we must have $\mathrm{Re}\,\alpha = 0$. Set $r := (\lambda; 1, \overline{p}^{-1}(1 - \lambda^2/2 + \beta + n))$ where $\beta \in \mathrm{Im}\,\mathbb{C}$ is chosen so that $\overline{p}^{-1}(1 - \lambda^2/2 + \beta) \in \mathcal{G}$. The lattice vector $\lambda \in 2D_4^{\mathcal{G}}$ and $n \in \langle p \rangle \cap \mathrm{Im}\,\mathcal{G} = 2\,\mathrm{Im}\,\mathcal{G}$ will be chosen later. We have $\langle r, y \rangle = 2a + 2b$, where $2a := 1 - \frac{1}{2}|l - \lambda|^2 \in \mathbb{R}$ and $2b := \mathrm{Im}\,\alpha + \mathrm{Im}\,\langle \lambda, l \rangle - \beta - n \in \mathrm{Im}\,\mathbb{C}$, whence $ht(\phi_r^i(y)) = 1 - (1 - i)(a + b)$.

Using the covering radius $\sqrt{2}$ of $2D_4^{\mathcal{G}}$ we may choose λ so that $a \in [0, \frac{1}{2}]$. Then $\left|\frac{1}{2} - a\right|^2 \leq \frac{1}{4}$. Next, observe $i - 2b \in \mathrm{Im}\,\mathbb{C}$ and we may choose $n \in 2\,\mathrm{Im}\,\mathcal{G}$ so that $|i - 2b| \leq 1$, since the covering radius of $2\,\mathrm{Im}\,\mathcal{G}$ is 1, so $\left|\frac{1}{2}i - b\right|^2 \leq \frac{1}{4}$, and we therefore can strictly decrease the height of y and thus of v. It follows that after a finite number of reflections in $\mathrm{Ref}_1(L)$, v can be brought to a unit multiple of $(\underline{0}; 0, 1)$, and we see $\mathrm{Ref}_1(L)$ acts transitively on $\mathrm{Cus}(L)$ up-to-units. □

Theorem 3.2.16. *The reflection group* $\mathrm{Ref}(L)$ *has finite index in* $\mathrm{Aut}(L)$*.*

Proof. (A similar result is contained in [Al1].) Define $\rho := (0; 0, 1), \rho' := (0; 1, 0)$. Choose any $\psi \in \mathrm{Aut}(L)$; by Lemma 3.2.15 there is an $r \in \mathrm{Ref}(L)$ such that $r\psi\rho = u \cdot \rho$ for some $u \in R^*$. Then we have $\langle \rho', \rho \rangle = \langle r\psi\rho', u \cdot \rho \rangle$, so $r\psi\rho' = (*; u, *)$. It is clear that we may choose $t \in \mathbb{T}$ with $tr\psi\rho' = u \cdot \rho'$, and since $\mathbb{T}$ fixes ρ, we also have $tr\psi\rho = t(u \cdot \rho) = u \cdot \rho$. Therefore k acts on H via multiplication by u, and thus each such k is in a group K isomorphic to $\mathrm{Aut}(2D_4^{\mathcal{G}}) \times \mathbb{Z}_4$, which is finite.

From the proof of Theorem 3.2.13 we know $\mathrm{Ref}(L)$ contains a subgroup $\mathbb{T}'$ which has finite index in $\mathbb{T}$. Choose a finite set $\mathcal{T}$ of coset representatives of $\mathbb{T}'$ in $\mathbb{T}$, and let $t = t_0 t'$ where $t' \in \mathbb{T}'$ and $t_0 \in \mathcal{T}$. Then $\psi = r^{-1}t'^{-1}t_0^{-1}k$ with $r^{-1}t'^{-1} \in \mathrm{Ref}(L)$. As $\psi \in \mathrm{Aut}(L)$ was arbitrary, this says $\mathrm{Aut}(L) = \bigcup_{k\in K, t_0 \in \mathcal{T}} \mathrm{Ref}(L)t_0^{-1}k$. I.e., $\mathrm{Ref}(L)$ has finite index in $\mathrm{Aut}(L)$. □

3.3 The Lattice $L = D_4^{\mathcal{G}} \oplus H$

3.3.1. The Root Diagram of L. Define the roots $s_1, \ldots, s_4$ to be:

$$s_1 = (0, 0, -1, -1)$$

$$s_2 = (0, p, -1, 0)$$

$$s_3 = (-1, 1, 0, 0)$$

$$s_4 = (-p, 0, 0, 0)$$

Note that the s_j's here are projections of corresponding "s_j's" from the $2D_4^{\mathcal{G}} \oplus H$ case above. In particular, define the projection

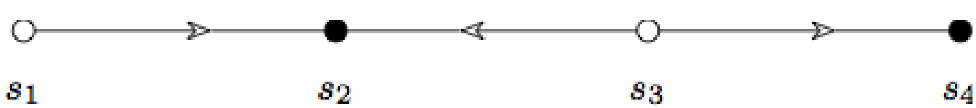
$$f : 2D_4^{\mathcal{G}} \oplus H \to D_4^{\mathcal{G}} \oplus H$$

$$f : (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1, x_2, x_5, x_6)$$

If we relabel each "s_j" from the $2D_4^{\mathcal{G}} \oplus H$ case as s_j', we have $s_j = f(s_j')$ for $j = 1, 2, 3, 4$.

These s_j's form the root diagram $I_4^{\rightarrow}$ which is a chain of four vertices with alternating arrow directions.

Figure 3.2 The root diagram $I_4^{\rightarrow}$



3.3.2. The Central Point of $I_4^{\rightarrow}$ in $\mathbb{C}H^3$. Similarly to the $2D_4^{\mathcal{G}} \oplus H$ case, define $\sigma \in \mathrm{Aut}(L)$ via $s_{2k} \mapsto i s_{1-2k}$, $s_{1-2k} \mapsto s_{2k}$, where subscripts are taken $\mod 4$. The graph automorphism group $\mathrm{Aut}(I_4^{\rightarrow})$ is trivial, and we can extend it by σ to get $Q := \langle \sigma \rangle$. Here Q has a $\mathbb{Z}_2$ action on the mirrors $s_j^{\perp}$, which is not transitive. If, by analogy with the $2D_4^{\mathcal{G}} \oplus H$ case, we define $\tau_0 := (s_1 + s_3) + (-e^{-\pi i/4})(s_2 + s_4)$, we find $|\tau_0|^2 < 0$ and τ_0's image in $\mathbb{C}H^3$ is fixed by Q. However, τ_0 is not equidistant from the $s_j^{\perp}$'s this time. We will instead define our τ for the $D_4^{\mathcal{G}} \oplus H$ case using a projection.

Let τ' be the "τ" from the $2D_4^{\mathcal{G}} \oplus H$ case, and set $\tau = f(\tau')$. Because the 3rd and 4th coordinates of s_j' are 0, for $j = 1, 2, 3, 4$, we see that $|\langle s_j, \tau\rangle| = |\langle s_j', \tau'\rangle|$ is constant. Therefore the mirrors in $s_1, s_2, s_3,$ and s_4 are equidistant from τ. It can be verified using the method given in the proof of Theorem 3.2.3 that these mirrors are precisely those closest to τ. More specifically, because the 3rd and 4th coordinates of s_3 and s_4 are 0, we are able to obtain an invertible matrix R from the first 2 coordinates of these roots. The rest of the calculations and proof are entirely analogous. We also verify by computer that τ is fixed in $\mathbb{C}H^3$ by Q.

Theorem 3.3.3. $\mathrm{Ref}(L)$ *is generated by reflections in the* s_j*'s.*

Proof. The proof is similar to that for $2D_4^{\mathcal{G}} \oplus H$. The covering radius of $D_4^{\mathcal{G}}$ is smaller than that of $2D_4^{\mathcal{G}}$, which enables the height reduction arguments to go through essentially unmodified. □

We also have a finite-index theorem, whose proof again goes through for the $D_4^{\mathcal{G}}$ case due to covering-radius considerations.

Theorem 3.3.4. *The reflection group* $\mathrm{Ref}(L)$ *has finite index in* $\mathrm{Aut}(L)$*.*

3.4 The Lattice $L = 3D_4^{\mathcal{G}} \oplus H$

3.4.1. The Root Diagram of L. The following roots in L:

$$s_1 = (0, 0, 0, 0, 0, 0, -1, -1)$$

$$s_2 = (-1, 1, 0, 0, 0, 0, 0, 0)$$

$$s_3 = (0, 0, -1, 1, 0, 0, 0, 0)$$

$$s_4 = (0, 0, 0, 0, -1, 1, 0, 0)$$

$$s_5 = (0, 0, -1, -1, -1, -1, -i, -1)$$

$$s_6 = (-1, -1, 0, 0, -1, -1, -i, -1)$$

$$s_7 = (-1, -1, -1, -1, 0, 0, -i, -1)$$

$$s_8 = (0, p, 0, p, 0, p, ip, p)$$

$s_9 = (0, p, 0, 0, 0, 0, -1, 0)$

$s_{10} = (0, 0, 0, p, 0, 0, -1, 0)$

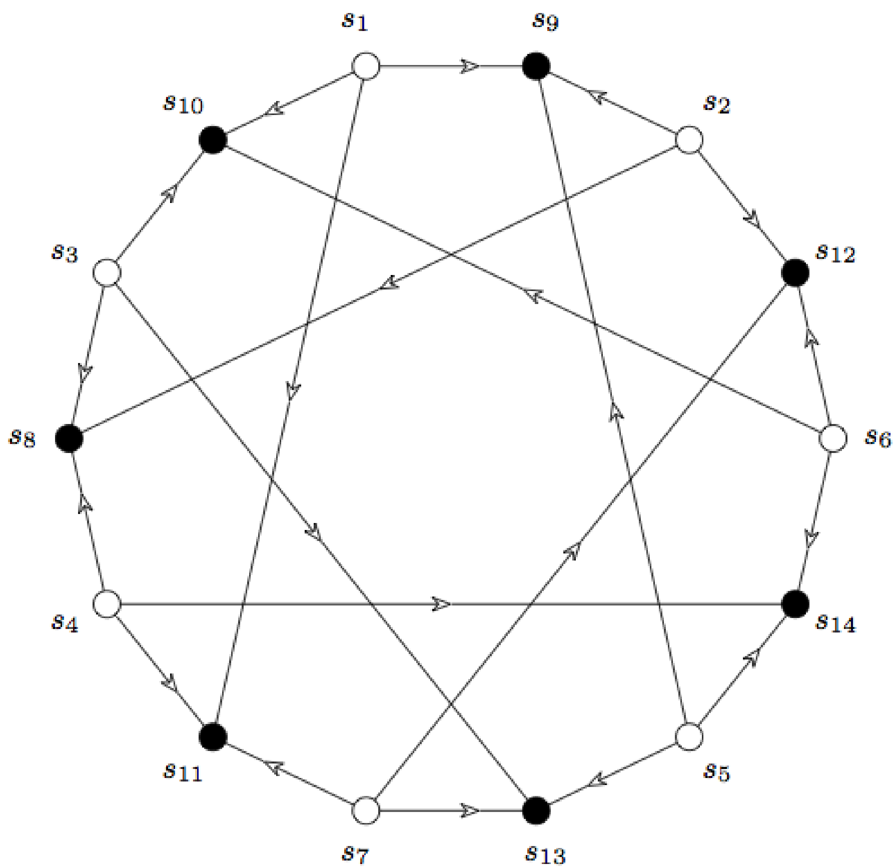
$s_{11} = (0, 0, 0, 0, 0, p, -1, 0)$

$s_{12} = (-p, 0, 0, 0, 0, 0, 0, 0)$

$s_{13} = (0, 0, -p, 0, 0, 0, 0, 0)$

$s_{14} = (0, 0, 0, 0, -p, 0, 0, 0)$

Figure 3.3 The root diagram $I_{14}^{\rightarrow}$



form the root diagram $I_{14}^{\rightarrow} = \mathrm{Inc}\,(\mathbb{P}^2(\mathbb{F}_2))$, where an arrow goes from r_i to r_j whenever $\langle r_i, r_j \rangle = p$, and no edge indicates $\langle r_i, r_j \rangle = 0$. $\mathrm{Inc}\,(\mathbb{P}^2(\mathbb{F}_2))$ is bipartite with parts corresponding to the set of points and set of lines in $\mathbb{P}^2(\mathbb{F}_2)$, which we will call the “points and lines of $I_{14}^{\rightarrow}$,” defined such that $\langle x, l\rangle = p$ whenever x is a point incident with line l. We denote the set of points by $\mathcal{P}$ and the set of lines by $\mathcal{L}$. Define $\Sigma_{\mathcal{P}}$ and $\Sigma_{\mathcal{L}}$ to be the sum of the points and sum of the lines, respectively. As in the case for $2D_4^{\mathcal{G}} \oplus H$, Lemmas 2.4.4 and 2.4.5 imply $\mathrm{Aut}(I_{14}^{\rightarrow}) \cong PGL_3(\mathbb{F}_2)$ induces a subgroup of $\mathrm{Aut}(L)$. Note that $PGL_3(\mathbb{F}_2)$ acts by permuting $\mathcal{P}$ and by permuting $\mathcal{L}$.

3.4.2. The Central Point of $I_{14}^{\rightarrow}$ in $\mathbb{C}H^7$. Choose a line l of $I_{14}^{\rightarrow}$. Using the defining properties of $I_{14}^{\rightarrow}$, we see that

$$w_{\mathcal{P}} := -\overline{p}l + \sum_{x\in l} x \tag{3.12}$$

is perpendicular to the points, has norm -2, and satisfies $\langle w_{\mathcal{P}}, \sum_{l\in\mathcal{L}} l\rangle = 7p$. Because there is only one such vector, it follows that $w_{\mathcal{P}}$ is independent of the choice of l. We also see that $\mathcal{P} \cup \{w_{\mathcal{P}}\}$ is a basis for $V := L \otimes \mathbb{C}$.

Now we determine which points of V are common eigenvectors for $PGL_3(\mathbb{F}_2)$. Let $\mathcal{P} = \{x_1, \ldots, x_7\}$, where increasing indices arrange the points in counterclockwise order around $I_{14}^{\rightarrow}$. Let $v \in V$ be such a common eigenvector. Then for each $g \in PGL_3(\mathbb{F}_2)$ we have $g(v) = k_g v$ for some $k_g \in \mathbb{C}$. Write $v = \sum_{j=1}^{7} c_j x_j + c_{\mathcal{P}} w_{\mathcal{P}}$, where the c's are in $\mathbb{C}$. Since $w_{\mathcal{P}}$ is fixed by $PGL_3(\mathbb{F}_2)$, we have for any g that $g(v) = k_g v \Rightarrow$

$$\sum_{j=1}^{7} c_j g(x_j) + c_{\mathcal{P}} w_{\mathcal{P}} = \sum_{j=1}^{7} k_g c_j x_j + k_g c_{\mathcal{P}} w_{\mathcal{P}}.$$

If $c_{\mathcal{P}} \neq 0$ this implies $k_g = 1$, while if $c_{\mathcal{P}} = 0$ the terms involving $w_{\mathcal{P}}$ vanish. In either case, we get

$$\sum_{j=1}^{7} c_j g(x_j) = \sum_{j=1}^{7} k_g c_j x_j. \tag{3.13}$$

The action of $PGL_3(\mathbb{F}_2)$ is 2-transitive on the points, so for any $k \neq l$ there is a $g_0 \in PGL_3(\mathbb{F}_2)$ such that $g_0(x_k) = x_l$ and $g_0(x_l) = x_k$. Using (3.13) with $g = g_0$ and applying g_0 to both sides yields $k_{g_0}^2 = 1 \Rightarrow k_{g_0} = \pm 1$. This then tells us that for any $k \neq l$ we have $c_k = \pm c_l$. Now choose $g_1 \in PGL_3(\mathbb{F}_2)$ to be the automorphism of $I_{14}^{\rightarrow}$ that rotates the points by $2\pi/7$ clockwise, i.e., g_1 maps $x_j \mapsto x_{j+1}$ (where indices are taken mod 7). Then $g_1^7(v) = v \Rightarrow k_{g_1}^7 = 1$, so k_{g_1} is a 7th root of unity. Then for $k \neq l$ (3.13) implies $c_k = \xi_7 c_l$ where ξ_7 is a 7th root of unity. But $c_k = \pm c_l$, so we must have $c_k = c_l$. Thus the common eigenvectors of $PGL_3(\mathbb{F}_2)$ are given by

$W := \text{span}\{\Sigma_{\mathcal{P}}, w_{\mathcal{P}}\}$, and these vectors are fixed by $PGL_3(\mathbb{F}_2)$. (Verify that $\Sigma_{\mathcal{P}}$ and $w_{\mathcal{P}}$ are linearly independent.)

Now consider the $\sigma \in \text{Aut}(L)$ which acts as a "reflection" about the central vertical line of $I_{14}^{\rightarrow}$. For a point x and line l identified by this "reflection," σ maps $l \mapsto x$ and $x \mapsto -il$. Then $Q := \langle \sigma, PGL_3(\mathbb{F}_2)\rangle = 8 \cdot PGL_3(\mathbb{F}_2)$ acts as the graph automorphism group of the undirected graph corresponding to $I_{14}^{\rightarrow}$. The group Q acts by automorphisms on L and therefore on $L \otimes \mathbb{C}$. In particular Q acts on $\mathbb{C}H^7$.

If we apply σ to $w_{\mathcal{P}}$, we have

$$w_{\mathcal{L}} := -px + \sum_{x \in l} l \tag{3.14}$$

doesn't depend on our choice of x. This says $w_{\mathcal{L}}$ is fixed by $PGL_3(\mathbb{F}_2)$. We can then check that $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ are linearly independent, so $W = \text{span}\{w_{\mathcal{P}}, w_{\mathcal{L}}\}$ is the 2-dimensional subspace fixed by $PGL_3(\mathbb{F}_2)$. We have the signature of W is $(1, 1)$. Also, $\sigma(w_{\mathcal{P}}) = -iw_{\mathcal{L}}$ and $\sigma(w_{\mathcal{L}}) = w_{\mathcal{P}}$, whence Q stabilizes W and has a unique fixed point in $\mathbb{C}H^7$. We can see that $PGL_3(\mathbb{F}_2)$ also fixes $\text{span}\{\Sigma_{\mathcal{P}}, \Sigma_{\mathcal{L}}\}$ while similarly $\sigma(\Sigma_{\mathcal{P}}) = -i\Sigma_{\mathcal{L}}$ and $\sigma(\Sigma_{\mathcal{L}}) = \Sigma_{\mathcal{P}}$. Thus we can calculate the unique fixed-point of the action of Q on $\mathbb{C}H^7$, which is given by the image of $\Sigma_{\mathcal{P}} - \xi\Sigma_{\mathcal{L}}$ or $w_{\mathcal{P}} - \xi w_{\mathcal{L}}$, where $\xi := e^{-\frac{\pi}{4}i}$. We call this fixed vector the *Weyl vector*:

$$\tau = (\Sigma_{\mathcal{P}} - \xi\Sigma_{\mathcal{L}})/14 \tag{3.15}$$

We note some of the products between the special vectors that we need later: Let $(\tau_1, \cdots, \tau_{14}) := (x_1, \cdots, x_7, -\xi l_1, \cdots, -\xi l_7)$, where the x_j's are the points and the l_j's are the lines of $I_{14}^{\rightarrow}$. We have $\langle \tau_s, \tau_t\rangle$ is equal to $-\sqrt{2}$ or 0 according to whether the two nodes are joined or not joined in the diagram $I_{14}^{\rightarrow}$. We have, for $s = 1, \cdots, 14$,

$$\langle \tau, \tau_s\rangle = |\tau|^2 = -1/(2 + 3\sqrt{2}) \tag{3.16}$$

From (3.12) and (3.14) we get

$$|w_{\mathcal{P}}|^2 = |w_{\mathcal{L}}|^2 = -2 \text{ and } \langle \tau, w_{\mathcal{P}}\rangle = \langle \tau, -\xi w_{\mathcal{L}}\rangle = -\sqrt{2}/2 \tag{3.17}$$

We use the Weyl vector τ to define the *height of a root* r as

$$\mathrm{ht}(r) := |\langle \tau, r\rangle|/|\tau|^2 \tag{3.18}$$

Our definition of "height" here is tantamount to the distance of $r^\perp$ from τ in $\mathbb{C}H^7$.

Proposition 3.4.3. *The 14 roots of the diagram $I_{14}^{\rightarrow}$ are the only roots (up to units) having the minimum height 1.*

Proof. Let r be a root of the lattice L with $\mathrm{ht}(L) = |\langle \tau, r\rangle|/|\tau|^2 \leq 1$. We want to prove that r is a unit multiple of one of the 14 roots of $I_{14}^{\rightarrow}$.

Let x be a point in $I_{14}^{\rightarrow}$. Either $|\langle x, r\rangle| \leq 2$, or using the triangle inequality $d(r^\perp, x^\perp) \leq d(r^\perp, \tau) + d(x^\perp, \tau)$ along with the distance formulae (2.2) and (2.3) above we get

$$|\langle x, r\rangle| \leq 2\cosh\left(2\sinh^{-1}\left(|\tau|/\sqrt{2}\right)\right) \approx 2.32$$

So we must have $|\langle x, r\rangle|^2$ equal to 0, 2, or 4.

Similarly from $d(r^\perp, w_{\mathcal{P}}) \leq d(r^\perp, \tau) + d(w_{\mathcal{P}}, \tau)$, (2.1) and (2.2) we get

$$|\langle w_{\mathcal{P}}, r\rangle| \leq 2\sinh\left(\sinh^{-1}\left(|\tau|/\sqrt{2}\right) + \cosh^{-1}\left(1/(2|\tau|)\right)\right) \approx 2.26$$

It follows that $|\langle w_{\mathcal{P}}, r\rangle|^2$ is equal to 0, 2, or 4.

We can write

$$r = \sum_{x \in \mathcal{P}} x\langle x, r\rangle/2 - w_{\mathcal{P}}\langle w_{\mathcal{P}}, r\rangle/2 \tag{3.19}$$

and taking the product with r on both sides of (3.19) we get

$$2 = \sum_{x \in \mathcal{P}} |\langle x, r\rangle|^2/2 - |\langle w_{\mathcal{P}}, r\rangle|^2/2 \tag{3.20}$$

There are only a few cases to consider. Multiplying r by a unit, we may assume that $\langle w_{\mathcal{P}}, r\rangle$ is either 0, p, or 2. In the following let u_1, u_2, etc. denote units in $\mathcal{G}^*$ and x_1, x_2, etc. denote points of $I_{14}^{\rightarrow}$.

If $\langle w_{\mathcal{P}}, r\rangle = 0$, from (3.20) we get $\sum |\langle x_s, r\rangle|^2 = 4$. Then the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, 0^6)$ or $(u_1 p, u_2 p, 0^5)$. So either r is a unit multiple of x_s (in which case it has height equal to one) or $r = (u_1 p x_1 + u_2 p x_2)/2$. Using diagram automorphisms (which are

2-transitive on points of $I_{14}^{\rightarrow}$) we can assume that $x_1 = s_1$ and $x_2 = s_2$ and then check that there is no such root r.

If $\langle w_{\mathcal{P}}, r\rangle = p$, then $\sum |\langle x_s, r\rangle|^2 = 6$; the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1, u_2 p, 0^5)$ or $(u_1 p, u_2 p, u_3 p, 0^4)$. In the first case we get $r = 2u_1/2x_1 + u_2 p/2x_2 - p/2w_{\mathcal{P}}$. Taking product with τ and using $\langle \tau, w_{\mathcal{P}}\rangle/|\tau|^2 = 3+\sqrt{2}$ we get $\langle \tau, r\rangle/|\tau|^2 = u_1 + u_2/\bar{p} - (3+\sqrt{2})/\bar{p}$ which clearly has norm greater than one.

In the second case we get $r = \sum_{s=1}^{3} u_s p/2x_s - p/2w_{\mathcal{P}}$ which implies $\langle \tau, r\rangle/|\tau|^2 = (u_1 + u_2 + u_3 - 3 - \sqrt{2})/\bar{p}$. Again this quantity has norm at least one. We now show that the only way it can be equal to one is if r is one of $l_1, \cdots, l_7$.

The only way one can have $\mathrm{ht}(r) = 1$ in the above paragraph is if r has product p with three of the points x_1, x_2, x_3 and is orthogonal to the others. If x_1, x_2, x_3 do not all lie on a line then there is a line l that avoids all these three points. Taking product with r in equation (3.12) gives $p = -\bar{p}\langle l, r\rangle$, contradicting $L = pL^{\vee}$. So x_1, x_2, x_3 are points on a line l_1. It follows that r and l_1 have the same product with each element of $\mathcal{P}$ and with $w_{\mathcal{P}}$. So r equals l_1.

If $\langle w_{\mathcal{P}}, r\rangle = 2$, and $\sum |\langle x_s, r\rangle|^2 = 8$, the unordered tuple $(\langle x_1, r\rangle, \cdots, \langle x_7, r\rangle)$ is equal to $(2u_1 p, 0^6)$, $(2u_1, 2u_2, 0^5)$, $(2u_1, u_2 p, u_3 p, 0^4)$, or $(u_1 p, \cdots, u_4 p, 0^3)$. Using a similar calculation as above, we get $\langle \tau, r\rangle/|\tau|^2$ is equal to $(u_1 p - 3 - \sqrt{2})$, $(u_1 + u_2 - 3 - \sqrt{2})$, $(u_1 + u_2/\bar{p} + u_3/\bar{p} - 3 - \sqrt{2})$, or $((u_1 + \cdots + u_4)/\bar{p} - 3 - \sqrt{2})$ respectively. Again each of these quantities are clearly seen to have norm strictly bigger than one. □

Theorem 3.4.4. *The reflection group* $\mathrm{Ref}(L)$ *has finite index in* $\mathrm{Aut}(L)$*.*

Proof. Although $3D_4^{\mathcal{G}}$ has a larger covering radius than $2D_4^{\mathcal{G}}$, it is still small enough for the arguments to work as in the $2D_4^{\mathcal{G}}$ case. □

Theorem 3.4.5. (Conjecture.) $\mathrm{Ref}(L)$ *is generated by reflections in the* s_i*'s.*

Here, the author was unable to show that reflections in height-0 and height-1 roots generate $\mathrm{Ref}(L)$. The covering radius of $3D_4^{\mathcal{G}}$ is too big for the arguments which worked before to get the needed height reduction. The author attempted to show $\mathrm{Ref}(L)$ is generated by reflections in roots of height $\sqrt{2}$, in addition to reflections in height-0 and height-1 roots, but without success. However, there is considerable experimental evidence that the conjecture is true. Over 100,000 roots with large coordinates (on the order of 1000 for the real and imaginary parts) were generated, and all could be reflected into $\{s_j\}$ using the reflections in the s_j's.

CHAPTER 4. FUTURE DIRECTIONS

4.1 Finishing the $3D_4^{\mathcal{G}} \oplus H$ Case

Proving the 14 diagram mirrors generate the reflection group of $3D_4^{\mathcal{G}} \oplus H$ is an obvious direction for further research. As noted earlier, the experimental evidence for this conjecture is quite strong. The major sticking point seems to be the height reduction argument relative to a cusp. Here the argument used in the other examples fails because the covering radius of $3D_4^{\mathcal{G}}$ is too big.

But there is hope that a modified version of the height-reduction argument may work using the mirrors of height 0, 1, and $\sqrt{2}$ around the cusp. This could involve covering the vector space underlying $3D_4^{\mathcal{G}}$ using two kinds of balls centered around two sets of points related to the lattice. A similar idea has been used to prove Theorem 6.2 of [Al1].

4.2 Fundamental Group of the Discriminant Complement

The discriminant complement of a Lorentzian lattice L is its hyperbolic space $\mathbb{C}H^n$ minus the mirrors of L's roots, quotiented by L's reflection group. It would be interesting to study the fundamental group π_1 of the discriminant complement for the three main lattices considered in this thesis, similarly to [AB1] [AB2]. In particular, to investigate whether the loops corresponding to the mirrors in S generate π_1, and whether these loops braid and commute according to the diagram for S. This holds in the case of real Weyl groups, where the fundamental group of the discriminant complement is known to be an Artin group. A similar result was obtained by D. Bessis for finite complex reflection groups [Be2]. Another example is furnished by a fourteen dimensional Lorentzian lattice over the Eisenstein integers [Ba3] [AB1] [AB2]. It is conjectured in [Al3] that a quotient of the fundamental group of this lattice's discriminant complement is isomorphic to the bimonster group, the wreath product of the monster sporadic group with $\mathbb{Z}_2$.

4.3 Reflection Groups of Other Lattices

Finding more examples of similar phenomena would be significant. Specifically, the author studied the reflection group of the lattice $L := E_8^{\mathcal{G}} \oplus H$. This example appears to be challenging, and the root diagram of the generating roots may be significantly more complicated than the prior considered cases. The author searched without success for such a root diagram on 6 vertices. Once it appears feasible, the entire sequence of lattices $E_8^{\mathcal{G}} \oplus H$, $2E_8^{\mathcal{G}} \oplus H$, and $3E_8^{\mathcal{G}} \oplus H$ could be studied. The last lattice here is particularly interesting, as it is isomorphic to $\Lambda^{\mathcal{G}} \oplus H$, where $\Lambda^{\mathcal{G}}$ is the Gaussian Leech lattice. There are a couple of other promising examples where the root diagram happens to be the Petersen graph and the 1-complex of the cube.

Also, it would be good to verify that the believed root diagrams do exist for the lattices $E_8^{\mathcal{H}} \oplus H$ and $2E_8^{\mathcal{H}} \oplus H$. Furthermore, Dr. Tathagata Basak has found a hexagonal root diagram for the $\mathcal{E}$-lattice $2D_4^{\mathcal{E}} \oplus H$. This picture could be completed by studying $D_4^{\mathcal{E}} \oplus H$ and $3D_4^{\mathcal{E}} \oplus H$.

APPENDIX COMPUTER PROGRAMS

We collect here the computer programs used to verify Theorems 3.2.3 and 3.2.14, and their analogs for the $D_4^{\mathcal{G}} \oplus H$ case. They are written in the **PARI/GP** programming language, which is freely available from https://pari.math.u-bordeaux.fr/. Documentary comments are included within the listings.

```
\\**************************************************************
\\ Name    : DiagramMirrorsClosest2D4+H.gp
\\ Purpose : This program applies to the case L := 2D_4+H. It
\\           shows that the closest mirrors to the point tau in
\\           hyperbolic space CH^5 are precisely the 8 diagram
\\           mirrors.

p = 1 + I;

s = vector(8);

\\ Our diagram roots.
s[1] = [    0,    0,    0,    0,   -1,   -1];
s[2] = [    0,    p,    0,    0,   -1,    0];
s[3] = [   -1,    1,    0,    0,    0,    0];
s[4] = [   -p,    0,    0,    0,    0,    0];
s[5] = [   -1,   -1,   -1,   -1,   -I,   -1];
s[6] = [    0,    0,   -p,    0,    0,    0];
s[7] = [    0,    0,   -1,    1,    0,    0];
s[8] = [    0,    0,    0,    p,   -1,    0];

\\ Determines whether two inputs are equal up to
\\ a unit multiple.
equal_upto_units(v1, v2)=
{
  my(units = [[1, -1], [I, -I]]);

  for( i = 1, 2,
    for ( j = 1, 2,
      if( v1 == units[i][j]*v2,
        return(true) )));

  return(false);
}

\\ Is the input complex number a Gaussian integer
is_gaussian_integer(z)=
{
  if( ( round(real(z)) == real(z) ) &&
      ( round(imag(z)) == imag(z) ),
      return(true), return(false));
}

\\ Is our vector in the lattice D_4
is_in_D_4(v)=
{
```

```
  for( i = 1, 2,
    if( (is_gaussian_integer(v[i]) == false),
      return(false) ));

  if( is_gaussian_integer((v[1]+v[2])/p) == false,
      return(false) );

  return(true);
}

\\ The product of two vectors in 2D_4+H
ip(x,y)=
{
  return( sum(i = 1, 4, conj(x[i])*y[i])
    + p*y[5]*conj(x[6]) + conj(p)*conj(x[5])*y[6] );
}

\\ Determine whether a give root is equal to one of the
\\ diagram roots, up to units
is_among_8_roots(r_1)=
{
  for(i = 1, 8,
    if(equal_upto_units(r_1, s[i]),
      return( true );
    );
  );

  return( false );
}

\\ Try the various a_j possibilities described in Theorem 3.2.3.
\\ If this gives a root r whose mirror is at least as close to
\\ tau as one of the diagram mirrors, report r as new.
main()=
{
root_list = listcreate(100);

tau = [1.00000000000000, 2.41421356237309, 1.00000000000000,
2.41421356237309, -0.707106781186548 + 4.12132034355964*I,
3.41421356237309];

R = [-1,    1,    0,    0;
     -p,    0,    0,    0;
      0,    0,   -p,    0;
      0,    0,   -1,    1];

R_inv = R^(-1);
```

```
gi = List( [
0,
1,
-1,
I,
-I,
1 + I,
1 - I,
-1 + I,
-1 - I,
2,
-2,
2*I,
-2*I
] );

const = abs(ip(s[1], tau));
roots_found = 0;
new_roots_found = 0;

for( i = 1, 13,
  for( j = 1, 13,
    for( k = 1, 13,
      for( l = 1, 13,
        a_vec = [gi[i], gi[j], gi[k], gi[l]];
        r_hat_conj = (R_inv*(a_vec~))~;
        if( (is_in_D_4(r_hat_conj) == true),
          r_hat = conj(r_hat_conj);
          for( m = 1, 13,
            for( n = 1, 13,
              r = [r_hat[1], r_hat[2], r_hat[3], r_hat[4],
                   gi[m], gi[n]];
              if( (abs(ip(r, r) - 2) < 0.0001) &&
                  (abs(ip(r, tau)) - const < 0.0001),
                print("=+=+=+=+=+=+=+=+=+=+=+=+=+=+");
                roots_found = roots_found + 1;
                print(r);
                print(abs(ip(r, tau)));
                if(is_among_8_roots(r) == true,
                  print("r is already among 8 diagram roots.");
                );
                if(is_among_8_roots(r) == false,
                  print("r is NOT in diagram!");
                  new_roots_found = new_roots_found+1;
                );
              );
```

```
            );
          );
        );
      );
    );
  );
);

print("Done!");
print("Number of roots found = ", roots_found);
print("Number of roots we found NOT in diagram = ",
new_roots_found);
}
```

```
\\*************************************************************
\\ Name    : DiagramMirrorsClosestD4+H.gp
\\ Purpose : This program applies to the case L := D_4+H. It
\\           shows that the closest mirrors to the point tau in
\\           hyperbolic space CH^3 are precisely the 4 diagram
\\           mirrors.

p = 1 + I;

s = vector(4);

\\ Our diagram roots.
s[1] = [    0,    0,   -1,   -1];
s[2] = [    0,    p,   -1,    0];
s[3] = [   -1,    1,    0,    0];
s[4] = [   -p,    0,    0,    0];

\\ Determines whether two inputs are equal up to
\\ a unit multiple.
equal_upto_units(v1, v2)=
{
  my(units = [[1, -1], [I, -I]]);

  for( i = 1, 2,
    for ( j = 1, 2,
      if( v1 == units[i][j]*v2,
        return(true) )));

  return(false);
}

\\ Is the input complex number a Gaussian integer
is_gaussian_integer(z)=
{
  if( ( round(real(z)) == real(z) ) &&
      ( round(imag(z)) == imag(z) ),
      return(true), return(false));
}

\\ Is our vector in the lattice D_4
is_in_D_4(v)=
{
  for( i = 1, 2,
    if( (is_gaussian_integer(v[i]) == false),
      return(false) ));

  if( is_gaussian_integer((v[1]+v[2])/p) == false,
```

```
      return(false) );

  return(true);
}

\\ The product of two vectors in D_4+H
ip(x,y)=
{
  return( sum(i = 1, 2, conj(x[i])*y[i])
    + p*y[3]*conj(x[4]) + conj(p)*conj(x[3])*y[4] );
}

\\ Determine whether a given root is equal to one of the
\\ diagram roots, up to units
is_among_4_roots(r_1)=
{
  for(i = 1, 4,
    if(equal_upto_units(r_1, s[i]),
      return( true );
    );
  );

  return( false );
}

\\ Try the various a_j possibilities described in Theorem 3.2.3.
\\ If this gives a root r whose mirror is at least as close to
\\ tau as one of the diagram mirrors, report r as new.
main()=
{
root_list = listcreate(100);

tau = [1.00000000000000, 2.41421356237309,
-0.707106781186548 + 4.12132034355964*I, 3.41421356237309];

R = [-1,    1;
     -p,    0];

R_inv = R^(-1);

gi = List( [
0,
1,
-1,
I,
-I,
1 + I,
```

```
1 - I,
-1 + I,
-1 - I,
2,
-2,
2*I,
-2*I
] );

const = abs(ip(s[1], tau));
roots_found = 0;
new_roots_found = 0;

for( k = 1, 13,
  for( l = 1, 13,
    a_vec = [gi[k], gi[l]];
    r_hat_conj = (R_inv*(a_vec~))~;
    if( (is_in_D_4(r_hat_conj) == true),
      r_hat = conj(r_hat_conj);
      for( m = 1, 9,
        for( n = 1, 9,
          r = [r_hat[1], r_hat[2], gi[m], gi[n]];
          if( (abs(ip(r, r) - 2) < 0.0001) &&
              (abs(ip(r, tau)) - const < 0.0001),
            print("=+=+=+=+=+=+=+=+=+=+=+=+=+=+");
            roots_found = roots_found + 1;
            print(r);
            print(abs(ip(r, tau)));
            if(is_among_4_roots(r) == true,
              print("r is already among 4 diagram roots.");
            );
            if(is_among_4_roots(r) == false,
              print("r is NOT in diagram!");
              new_roots_found = new_roots_found+1;
            );
          );
        );
      );
    );
  );
);

print("Done!");
print("Number of roots found = ", roots_found);
print("Number of roots we found NOT in diagram = ",
new_roots_found);
}
```

```
\\***********************************************************
\\ Name    : DiagramMirrorsGenerate2D4+H.gp
\\ Purpose : This program applies to the case L := 2D4+H. It
\\           shows that the finite set of roots in the proof of
\\           Theorem 3.2.13 can all be reflected down to one of
\\           the 8 diagram roots, using reflections in the
\\           diagram roots.

p = 1 + I;

s = vector(8);

\\ Our diagram roots.
s[1] = [    0,    0,    0,    0,   -1,   -1];
s[2] = [    0,    p,    0,    0,   -1,    0];
s[3] = [   -1,    1,    0,    0,    0,    0];
s[4] = [   -p,    0,    0,    0,    0,    0];
s[5] = [   -1,   -1,   -1,   -1,   -I,   -1];
s[6] = [    0,    0,   -p,    0,    0,    0];
s[7] = [    0,    0,   -1,    1,    0,    0];
s[8] = [    0,    0,    0,    p,   -1,    0];

refl_units = vector(3);

\\ The units we can reflect in.
refl_units[1] = -1;
refl_units[2] = I;
refl_units[3] = -I;

\\ Determines whether two inputs are equal up to
\\ a unit multiple.
equal_upto_units(v1, v2)=
{
  my(units = [[1, -1], [I, -I]]);

  for( i = 1, 2,
    for ( j = 1, 2,
      if( v1 == units[i][j]*v2,
        return(true) )));

  return(false);
}

\\ The product of 2 vectors 2D_4+H tensor C.
ip_L(x,y)=
{
  return( sum(i = 1, 4, conj(x[i])*y[i])
```

```
    + p*y[5]*conj(x[6]) + conj(p)*conj(x[5])*y[6] );
}

\\ The product of 2 vectors 2D_4 tensor C.
ip_Lambda(x,y)=
{
  return( sum(i = 1, 4, conj(x[i])*y[i]) );
}

\\ Heisenberg group of translations T
T(lambda,z,v)=
{
  my(x = [0, 0, 0, 0]);
  x[1] = v[1]; x[2] = v[2]; x[3] = v[3]; x[4] = v[4];
  my(r_v1 = [0, 0, 0, 0, 0, v[6]]);
  my(r_v2 = v[5]*[lambda[1], lambda[2], lambda[3], lambda[4], 1,
            1/conj(p)*(-ip_Lambda(lambda, lambda)/2 + z)]);
  my(r_v3 = [x[1], x[2], x[3], x[4], 0,
            -1/conj(p)*ip_Lambda(lambda, x)]);

  return( r_v1 + r_v2 + r_v3 );
}

\\ Perform unit-reflection on y in mirror of x
reflect(x,unit,y)=
{
  return( y - (1-unit)*(ip_L(x,y)/ip_L(x,x))*x );
}

\\ Determine whether a given root is equal to one of the
\\ diagram roots, up to units
is_among_8_roots(r_1)=
{
  for(i = 1, 8,
    if(equal_upto_units(r_1, s[i]),
      return( true );
    );
  );

  return( false );
}

\\ The main loop, that tries to reflect each of the finite set
\\ of roots from Theorem 3.2.13 down to one of the diagram
\\ roots, using reflections in the diagram roots.
main()=
{
```

```
\\ Our central point
tau = [1.00000000000000, 2.41421356237309, 1.00000000000000,
2.41421356237309, -0.707106781186548 + 4.12132034355964*I,
3.41421356237309];

\\ Before the main loop, we build our list of roots that we
\\ want to reflect down to the diagram roots.

gen = listcreate(1000);

r = listcreate(2);
listput(r, [0, 0, 0, 0, 1, 1]);
listput(r, [0, 0, 0, 0, 1, I]);

x_0 = [0, 0, 0, 0, 1, 1];

\\ G-basis for 2D4
basis = listcreate(4);
listput(basis, [1, I, 0, 0]);
listput(basis, [p, 0, 0, 0]);
listput(basis, [0, 0, 1, I]);
listput(basis, [0, 0, p, 0]);

\\ Coset representatives for D_4 parts of coordinates
a = listcreate(2);
listput(a, 0);
listput(a, 1);

b_0 = listcreate(2);
listput(b_0, 0);
listput(b_0, 1 + I);

b_1 = listcreate(2);
listput(b_1, 1);
listput(b_1, I);

b = listcreate(2);
listput(b, b_0);
listput(b, b_1);

\\ Roots of D_4
D4rootsuptounit = listcreate(6);
listput(D4rootsuptounit, [p, 0]);
listput(D4rootsuptounit, [0, p]);
listput(D4rootsuptounit, [1, 1]);
listput(D4rootsuptounit, [1, -1]);
listput(D4rootsuptounit, [1, I]);
```

```
listput(D4rootsuptounit, [1, -I]);

D4roots = listcreate(24);
for(i = 0, 3,
  for( j = 1, 6,
    listput(D4roots, (I^i)*D4rootsuptounit[j]);
  );
);

\\ Add r_1, r_2 to list
for(i = 1, 2,
  listput(gen, r[i]);
);

\\ Add T_{lambda, z}(r_j) to list
for(i = 0, 1,
for(j = 1, 2,
for(k = 1, 4,
  lambda = basis[k] * (I^i);
  z = ip_Lambda(lambda, lambda)/2*I;
  listput(gen, T(lambda, z, r[j]));
);
);
);

\\ Add the lambda' roots to the list
for( i = 1, 2,
for( j = 1, 2,
for( k = 1, 2,
for( l = 1, 2,
for( m = 0, 1,
  lambda = [0, 0, 0, 0];
  lambda[1] = a[i];
  lambda[2] = b[i][j];
  lambda[3] = a[k];
  lambda[4] = b[k][l];
  z = ((ip_Lambda(lambda, lambda)/2 + 2*m)%4)*I;
  listput(gen, T(lambda, z, x_0));
);
);
);
);
);

\\ Add the (x; 0, 0) and (x; 0, 1) roots to list
for( i = 0, 1,
for( j = 1, 24,
```

```
for( k = 0, 1,
  ht0root = [0, 0, 0, 0, 0, 0];

  ht0root[2*i + 1] = D4roots[j][1];
  ht0root[2*i + 2] = D4roots[j][2];
  ht0root[6] = k;

  listput(gen, ht0root);
);
);
);

\\ Absolute value of the product that a diagram mirror
\\ has with tau.
abs_mirror_ip = abs(ip_L(s[1], tau));

num_got_there = 0;

gen_count = length(gen);

\\ Loop, trying to bring the mirrors in "gen" closer to
\\ the diagram mirrors, until we get there or time out.
for( i = 1, gen_count,
  got_there = false;

  r_1 = gen[i];
  r_start = r_1;

  for( j = 1, 1000000,
    found_closer = false;

    for( k = 1, 8,
      for( l = 1, 3,
        r_2 = reflect(s[k],refl_units[l],r_1);
        if( abs(ip_L(r_2, tau)) <
            abs(ip_L(r_1, tau)),
          r_1 = r_2;
          found_closer = true;
        );
      )
    );

    if( found_closer == false,
      if( abs(abs(ip_L(r_1, tau)) - abs_mirror_ip) < 0.0001,
        if(is_among_8_roots(r_1),
          got_there = true;
          num_got_there = num_got_there + 1;
```

```
        );
      );

      if( got_there == false,
        print("COULDN'T get to one of 8 diagram roots with:");
        print(r_1);
      );

      break;
    );
  );

  if( (got_there == false) && (found_closer == true),
    print("Timed out trying to get to one of the 8 mirrors.");
    print("Starting with:");
    print(r_start);
  );
);

print("");
print("+_+_+_+_+_+_+_+_+_+_");
print("");
print("Number of roots in generator set = ", gen_count);
print("Number got there = ", num_got_there);
}
```

```
\\***************************************************************
\\ Name    : DiagramMirrorsGenerateD4+H.gp
\\ Purpose : This program applies to the case L := D4+H. It
\\           shows that a finite set of roots analogous to those
\\           in the proof of Theorem 3.2.13 can all be reflected
\\           down to one of the 4 diagram roots, using
\\           reflections in the diagram roots.

p = 1 + I;

s = vector(4);

\\ Our diagram roots.
s[1] = [    0,    0,   -1,   -1];
s[2] = [    0,    p,   -1,    0];
s[3] = [   -1,    1,    0,    0];
s[4] = [   -p,    0,    0,    0];

refl_units = vector(3);

\\ The units we can reflect in.
refl_units[1] = -1;
refl_units[2] = I;
refl_units[3] = -I;

\\ Determines whether two inputs are equal up to
\\ a unit multiple.
equal_upto_units(v1, v2)=
{
  my(units = [[1, -1], [I, -I]]);

  for( i = 1, 2,
    for ( j = 1, 2,
      if( v1 == units[i][j]*v2,
        return(true) )));

  return(false);
}

\\ The product of 2 vectors D_4+H tensor C.
ip_L(x,y)=
{
  return( sum(i = 1, 2, conj(x[i])*y[i])
    + p*y[3]*conj(x[4]) + conj(p)*conj(x[3])*y[4] );
}

\\ The product of 2 vectors D_4 tensor C.
```

```
ip_Lambda(x,y)=
{
  return( sum(i = 1, 2, conj(x[i])*y[i]) );
}

\\ Heisenberg group of translations T
T(lambda,z,v)=
{
  my(x = [0, 0]);
  x[1] = v[1]; x[2] = v[2];
  my(r_v1 = [0, 0, 0, v[4]]);
  my(r_v2 = v[3]*[lambda[1], lambda[2], 1,
           1/conj(p)*(-ip_Lambda(lambda, lambda)/2 + z)]);
  my(r_v3 = [x[1], x[2], 0,
           -1/conj(p)*ip_Lambda(lambda, x)]);

  return( r_v1 + r_v2 + r_v3 );
}

\\ Perform unit-reflection on y in mirror of x
reflect(x,unit,y)=
{
  return( y - (1-unit)*(ip_L(x,y)/ip_L(x,x))*x );
}

\\ Determine whether a give root is equal to one of the
\\ diagram roots, up to units
is_among_4_roots(r_1)=
{
  for(i = 1, 4,
    if(equal_upto_units(r_1, s[i]),
      return( true );
    );
  );

  return( false );
}

\\ The main loop, that tries to reflect each of the finite set
\\ of roots from an analog of Theorem 3.2.13 down to one of the
\\ diagram roots, using reflections in the diagram roots.
main()=
{
\\ Our central point
tau = [1.00000000000000, 2.41421356237309,
-0.707106781186548 + 4.12132034355964*I, 3.41421356237309];
```

```
\\ Before the main loop, we build our list of roots that we
\\ want to reflect down to the diagram roots.

gen = listcreate(1000);

r = listcreate(2);
listput(r, [0, 0, 1, 1]);
listput(r, [0, 0, 1, I]);

x_0 = [0, 0, 1, 1];

\\ G-basis for D4
basis = listcreate(2);
listput(basis, [1, I]);
listput(basis, [p, 0]);

\\ Coset representatives for D_4 parts of coordinates
a = listcreate(2);
listput(a, 0);
listput(a, 1);

b_0 = listcreate(2);
listput(b_0, 0);
listput(b_0, 1 + I);

b_1 = listcreate(2);
listput(b_1, 1);
listput(b_1, I);

b = listcreate(2);
listput(b, b_0);
listput(b, b_1);

\\ Roots of D_4
D4rootsuptounit = listcreate(6);
listput(D4rootsuptounit, [p, 0]);
listput(D4rootsuptounit, [0, p]);
listput(D4rootsuptounit, [1, 1]);
listput(D4rootsuptounit, [1, -1]);
listput(D4rootsuptounit, [1, I]);
listput(D4rootsuptounit, [1, -I]);

D4roots = listcreate(24);
for(i = 0, 3,
  for( j = 1, 6,
    listput(D4roots, (I^i)*D4rootsuptounit[j]);
  );
```

```
);

\\ Add r_1, r_2 to list
for(i = 1, 2,
  listput(gen, r[i]);
);

\\ Add T_{lambda, z}(r_j) to list
for(i = 0, 1,
for(j = 1, 2,
for(k = 1, 2,
  lambda = basis[k] * (I^i);
  z = ip_Lambda(lambda, lambda)/2*I;
  listput(gen, T(lambda, z, r[j]));
);
);
);

\\ Add the lambda' roots to the list
for( i = 1, 2,
for( j = 1, 2,
for( k = 0, 1,
  lambda = [0, 0];
  lambda[1] = a[i];
  lambda[2] = b[i][j];
  z = ((ip_Lambda(lambda, lambda)/2 + 2*k)%4)*I;
  listput(gen, T(lambda, z, x_0));
);
);
);

\\ Add the (x; 0, 0) and (x; 0, 1) roots to list
for( i = 1, 24,
for( j = 0, 1,
  ht0root = [0, 0, 0, 0];

  ht0root[1] = D4roots[i][1];
  ht0root[2] = D4roots[i][2];
  ht0root[4] = j;

  listput(gen, ht0root);
);
);

\\ Absolute value of the product that a diagram mirror
\\ has with tau.
abs_mirror_ip = abs(ip_L(s[1], tau));
```

```
num_got_there = 0;

gen_count = length(gen);

\\ Loop, trying to bring the mirrors in "gen" closer to
\\ the diagram mirrors, until we get there or time out.
for( i = 1, gen_count,
  got_there = false;

  r_1 = gen[i];
  r_start = r_1;

  for( j = 1, 1000000,
    found_closer = false;

    for( k = 1, 4,
      for( l = 1, 3,
        r_2 = reflect(s[k],refl_units[l],r_1);
        if( abs(ip_L(r_2, tau)) <
            abs(ip_L(r_1, tau)),
          r_1 = r_2;
          found_closer = true;
        );
      );
    );

    if( found_closer == false,
      if( abs(abs(ip_L(r_1, tau)) - abs_mirror_ip) < 0.0001,
        if(is_among_4_roots(r_1),
          got_there = true;
          num_got_there = num_got_there + 1;
        );
      );

      if( got_there == false,
        print("COULDN'T get to one of 4 diagram roots with:");
        print(r_1);
      );

      break;
    );
  );

  if( (got_there == false) && (found_closer == true),
    print("Timed out trying to get to one of the 4 mirrors.");
    print("Starting with:");
```

```
    print(r_start);
  );
);

print("");
print("+_+_+_+_+_+_+_+_+_+_");
print("");
print("Number of roots in generator set = ", gen_count);
print("Number got there = ", num_got_there);
}
```

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