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Topics in self-interacting random walks

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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DEDICATION

First, I dedicate this thesis to my family. I would like to thank my mother, Angie, who continues to inspire me with her persistence, patience and compassion. I would also like to thank my sisters, Jamie and Krista, for their advice and support during my time in graduate school. Finally, I'd like to thank my late father, Ronald. While he is not physically here to see me complete this work, I know his spirit will continue to guide, support and empower me.

I also dedicate this thesis to my all of my mathematics instructors over the years. In particular, I would like to thank Jim Brass, my high school math teacher, who helped instill a love of mathematics for me. Also, I would like to thank all of the Concordia College Department of Mathematics faculty, whose passion for mathematics and education influenced me to go to graduate school.

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ABSTRACT

In this thesis, we will show results on two different self-interacting random walk models on $\mathbb{Z}$.

First, we observe the frog model, an infinite system of interacting random walks, on $\mathbb{Z}$ with an asymmetric underlying random walk. For certain initial frog distributions we construct an explicit formula for the moments of the leftmost visited site, as well as their asymptotic scaling limits as the drift of the underlying random walk vanishes. We also provide conditions in which the lower bound can be scaled to converge in probability to the degenerate distribution at 1 as the drift vanishes.

Then, we state and prove a theorem on the bound of the number of favorite (i.e., most visited) sites for the symmetric persistent random walk on $\mathbb{Z}$, a discrete-time process typified by the correlation of its directional history. This is a generalization of a result by Tóth used to partially prove a longstanding conjecture by Erdős and Révész.

We conclude with examples of potential future directions of research in these problems and related topics.

CHAPTER 1. INTRODUCTION

Self-interacting random walks are a broad category of stochastic processes that have attracted a lot of attention among probabilists recently. The non-Markovian nature of these models have inspired researchers to challenge themselves by letting go of the convenient assumptions of independence. In this work, we introduce original results on two different problems involving self-interacting random walk models, with a separate chapter dedicated to each problem. The first chapter concerns the asymptotics of the long-term range of a transient infinite particle system known as the frog model, in which new particles activate from a visit from a pre-existing active particle. This chapter is based on the work in [29]. The second chapter extends a proof of a simple random walk conjecture from Erdős and Révész onto the directionally dependent process called the persistent random walk. The final chapter will cover ideas of future research on these problems.

1.1 The Range of the Transient Frog Model on $\mathbb{Z}$

Consider the following interacting random walks model on $\mathbb{Z}$: initially at each site x there is a fixed number η_x of sleeping particles ("frogs"), and there is a certain number (η_0) of active frogs at the origin. The active frogs perform in discrete time, simultaneously and independently of each other, a biased (say, to the right) nearest-neighbor random walk on $\mathbb{Z}$. When an active frog visits a site x, it activates the η_x sleeping frogs at x, in which each active frog performs the same underlying random walk starting from its initial location, all random walk transitions being independent of each other. The active frogs continue to visit other sleeping frogs and activate them. This model for an infinite number of interacting random walkers is called the frog model on $\mathbb{Z}$ (with drift).

A frog model is called *recurrent* if 0 is visited infinitely often by active frogs with probability (w.p.) 1, and *transient* if 0 is visited only finitely often w.p.1. It is shown in [27] that the zero-one dichotomy actually takes place, namely a one-dimensional frog model is either transient or recurrent. Both necessary and sufficient conditions for recurrence of the frog model on $\mathbb{Z}$ based on the configuration of frogs and the drift of the random walk are provided in [27]. Recurrence for variants of the frog model on more general graphs have been first explored in [59] (for the symmetric random walk on $\mathbb{Z}^d$) and subsequently in [4], [48] and [49]. Shape theorems for the model in $\mathbb{Z}^d$ have been obtained in [2, 3]. For further background on the frog model and its variants, refer to [49]. For an account of the most recent activity in the area see [9, 18, 36, 34]. In particular, [18] generalizes a recurrence criterion of [27] to a model in $\mathbb{Z}^d$, [36] and [34] provide recurrence and transience criteria for the frog model on trees, and [9] studies survival of particles in a one-dimensional variation of the model, also partially extending some of the results of [27].

The frog model can be interpreted as an information spreading network [3, 49]. The underlying idea is that an active frog holds some information and shares it with sleeping frogs when they meet, activating the sleeping frogs who then spread the information along their random walk path. A closely related to our model *particle process on* $\mathbb{Z}$, describing the evolution of a virus in an infinite population (e. g., computer network), has been considered in [41, 42] and [9]. The model is also a discrete-time relative of the one-dimensional *stochastic combustion process* studied in [14, 50].

In Chapter 2, we will explore the behavior of the frog model, in particular its range, when transience is assumed. We specifically introduce a drift component to the random walk and explore how its magnitude affects the range of visited sites in the model. Each active frog will move one integer to the right with probability $p \in (\frac{1}{2}, 1)$, or one integer to the left with probability $1-p$. Thus the underlying random walk is transient to the right. We define the drift constant $\rho := \frac{1-p}{p} \in (0, 1)$. The drift term ρ can be seen as a measure of "transience" of this frog model; small values of ρ indicate more frequent rightward movement by the frogs, whereas values of ρ close to 1 more closely resemble recurrence with a slight rightward drift (see, for instance, formula (2.5) below for a concrete random walk result). Of particular interest is the collective behavior of the frogs as $\rho \uparrow 1$. By Theorem 2.1 in [27], this frog model is transient when we assume an identical distribution of frogs on the nonnegative sites. In particular, w.p.1 there must be only a finite number of visited sites to the left of the origin. That is, in the language of [9], transience implies *local extinction* for our model.

In Chapter 2, while assuming $\eta_x = 0$ for all $x < 0$, we will first explore the single-frog case, i.e., the frog model in which $\eta_x = 1$ for all $x \geq 0$ (cf. [9, 41, 42]). We will provide exact and asymptotic results for the moments of the lower bound of the range, which will be used in convergence theorems. After that, we will move to more general choices for η and show that, under certain conditions, the frog model's lower bound will behave asymptotically similar to that of the single-frog case. Finally, we will provide asymptotic bounds for moments of the frog model range when η is supported on all of $\mathbb{Z}$.

The rest of the chapter is organized as follows. A short Section 2.1 introduces notations and certain technical tools necessary for our proofs. The three subsequent sections constitute the main body of Chapter 2. The single-frog case is considered in Section 2.2. A class of more general initial configurations of frogs η is discussed in Section 2.3. The consideration of configurations supported on $\mathbb{Z}$ is discussed in Section 2.4.

1.2 Favorite Sites of the Persistent Random Walk

Consider a discrete-time random process $\{S_n\}_{n=0}^{\infty}$ on $\mathbb{Z}$, with $S_0 = 0$. After time t, we define the *local time* of a site $x \in \mathbb{Z}$ as the number of values of $n \in \{0, 1, \cdots, t\}$ such that $S_n = x$. Using local time, we define a *favorite site* at time t as a site x whose local time at t is greater than or equal to the local time of all other integer sites at t.

Despite the seemingly intractable behavior of the set of favorite sites, plenty of work has been accomplished on the subject for multiple processes. Erdős and Révész initiated the study of the favorite sites of the symmetric nearest-neighbor random walk on $\mathbb{Z}^d$ in [21, 22, 23] by raising questions about the nature of the set over time. Many of answers to these questions, as well as other results, have been discovered in [8, 15, 16, 43, 62], among other papers. For an overview of the results for favorite sites on the simple random walk, see [52, 56]. Other discrete-time processes with results on favorite sites include random walks on random environment ([35]), biased random walks on trees ([36]) and randomly biased walks on a supercritical Galton-Watson tree ([12]) The continuous-time analogue of favorite sites have also been studied for Brownian motion ([8, 15, 16, 43]), Lévy processes ([45]) and symmetric stable processes ([7, 20]).

One article of special note is [61], in which Tóth partially proved a conjecture from [21] for the simple random walk on $\mathbb{Z}$. He proved that the set of favorite sites stays relatively small for all time; in fact, there are four or more favorite sites only finitely often w.p.1. The method of proof for the main result of [61] involved first representing the event of four simultaneous favorite sites occurring as sojourn times for the simple random walk by way of the Ray-Knight representation as in [39] and [51], and then performing raw calculation and estimates to show that the event occurs only finitely often w.p.1.

In Chapter 3, we will explore a similar result, but for a different discrete-time process on $\mathbb{Z}$ known as the *persistent random walk*, with parameter $\lambda \in (\frac{1}{2}, 1)$. We define the persistent random walk $\{S_n\}_{n=0}^{\infty}$ as follows: initially, $S_0 = 0$, and $S_1 = 1$ or -1 with equal probability. For each time step after that, the particle will move one site over in the direction of its previous time step with probability λ or move one site back with probability $1 - \lambda$. Note that this motion differs from a biased walk on $\mathbb{Z}$, in that the direction that the persistent random walk is biased towards changes whenever the process turns back. When $\lambda = \frac{1}{2}$ is allowed, the persistent random walk is equivalent to the simple random walk. However, for $\lambda \in (\frac{1}{2}, 1)$, the persistent walk is non-Markovian and locally nonsymmetric (although the process is still strongly recurrent).

The earliest works on the persistent random walk come from [26, 58], who each introduced the model as a way of describing certain physical phenomena. The persistent random walk and other related processes have seen applications in other fields of physics and biology, such as random collision models ([55]), ballistic diffusion ([64]), and the movement of animals ([10]), among others. Overviews of applications can be found in [13, 64]. For more information on the general theory of directionally reinforced random walks, see [30, 46].

In Chapter 3, we set out to prove that the persistent random walk on $\mathbb{Z}$, for any choice of $\lambda \in (\frac{1}{2}, 1)$, will have four or more favorite sites only finitely often, as in the case for the simple random walk. This is a somewhat surprising result; if one sets λ arbitrarily close to 1, one can expect large intervals of integers receiving the same number of visits from $\{S_n\}$. However, the size of this interval will almost surely be cut down to a number less then 4 eventually.

Our method of proof will follow closely to that of [61], as the framework of sojourn times provides naturally closed formulations in the extension into the directionally-dependent persistent processes. As such, much of the notation and the key lemmas will appear similar to as they did in [61], albeit under a new random process. However, the extension will not be trivial, as the simple walk case in [61] provided simplifications in the essential formulations that are absent in the persistent case. Our proof for the persistent random walk will utilize some deep results into the studies of probability theory, mathematical statistics, asymptotic analysis and the theory of hypergeometric functions. We hope that the work for this proof will pave the way for the study in the number of favorite sites for other processes outside of the realm of simple random walks.

Chapter 3 will be organized in the following way. After introducing the statement of the main theorem and the key objects of study, we shall provide definitions and notation for the necessary tools used in our proof in Section 3.1. In Section 3.2, we construct a representation of the directional local times of the persistent random walk as a chain of Galton-Watson processes, in the spirit of [39] and [51]. We also introduce the statement of the proposition that was key in our proof of the main theorem. We prove the proposition in Section 3.3 using some technical lemmas on the processes defined in Section 3.2. In Section 3.4, we show results on the transition kernels of the processes in Section 3.2 that serve as intriguing findings on their own. Finally, in section 3.5, we prove the lemmas and relevant sublemmas used in the proof of the proposition.

CHAPTER 2. THE RANGE OF THE TRANSIENT FROG MODEL ON $\mathbb{Z}$

Consider the following interacting random walks model on $\mathbb{Z}$: initially at each site x there is a fixed number η_x of sleeping particles ("frogs"), and there is a certain number (η_0) of active frogs at the origin. The active frogs perform in discrete time, simultaneously and independently of each other, a biased (say, to the right) nearest-neighbor random walk on $\mathbb{Z}$. When an active frog visits a site x, it activates the η_x sleeping frogs at x, in which each active frog performs the same underlying random walk starting from its initial location, all random walk transitions being independent of each other. The active frogs continue to visit other sleeping frogs and activate them. This model for an infinite number of interacting random walkers is called the frog model on $\mathbb{Z}$ (with drift).

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In this chapter, which is based on [29], we will explore the behavior of the frog model, in particular its range, when transience is assumed. We specifically introduce a drift component to the random walk and explore how its magnitude affects the range of visited sites in the model. Each active frog will move one integer to the right with probability $p \in (\frac{1}{2}, 1)$, or one integer to the left with probability $1 - p$. Thus the underlying random walk is transient to the right. We define the drift constant $\rho := \frac{1-p}{p} \in (0, 1)$. The drift term ρ can be seen as a measure of "transience" of this frog model; small values of ρ indicate more frequent rightward movement by the frogs, whereas values of ρ close to 1 more closely resemble recurrence with a slight rightward drift (see, for instance, formula (2.5) below for a concrete random walk result). Of particular interest is the collective behavior of the frogs as $\rho \uparrow 1$. By Theorem 2.1 in [27], this frog model is transient when we assume an identical distribution of frogs on the nonnegative sites. In particular, w.p.1 there must be only a finite number of visited sites to the left of the origin. That is, in the language of [9], transience implies *local extinction* for our model.

2.1 Preliminaries

For calculations of characteristics of the random variable representing the range of the frog model, we make use of common notation in analytic number theory and combinatorics. For $a \in \mathbb{R}$, $c \in \mathbb{N} \cup \{\infty\}$, $|q| < 1$, the *q-Pochhammer symbol* is defined by $(a; q)_c := \prod_{j=0}^{c-1}(1 - aq^j)$. The following equality can be verified directly from the definition: $(q^{x+1}; q)_\infty = \frac{(q;q)_\infty}{(q;q)_x}$, $x \in \mathbb{N}$.

The q-Pochhammer symbol is one of the key functions in the construction of q-analogs in number theory, and is often used in the theory of basic hypergeometric functions and analytic combinatorics [5, 24, 28]. For example, [28] provides the following identities:

$$(z; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q; q)_n} z^n \qquad \text{and} \qquad \frac{1}{(z; q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n}. \tag{2.1}$$

Thus for a nonzero $q \in (-1, 1)$, $(z; q)_\infty$ and $\frac{1}{(z;q)_\infty}$ are both analytic functions of z on $(0, 1)$. The *q-gamma function* is defined as $\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1 - q)^{1-z}$. The *q-digamma function* ψ_ρ is defined as $\psi_q(z) = \frac{1}{\Gamma_q(z)} \frac{\partial \Gamma_q(z)}{\partial z} = -\ln(1 - q) + \ln q \sum_{n=0}^{\infty} \frac{q^{n+z}}{1-q^{n+z}}$.

To facilitate our calculation of the range moments, we need to develop notation for Bell polynomials. The Bell polynomials are defined (see, for instance, [11, 47]) as the triangular array of polynomials $B_{m,k}$, $m \geq k$, given by

$$B_{m,k}(x_1,\ldots,x_{m-k+1}) = \sum \frac{m!}{k_1!\cdots k_{m-k+1}!}\left(\frac{x_1}{1!}\right)^{k_1}\cdots\left(\frac{x_{m-k+1}}{(m-k+1)!}\right)^{k_{m-k+1}}, \tag{2.2}$$

where the sum is taken over all sequences of nonnegative integers $\{k_1,\ldots,k_{m-k+1}\}$ satisfying

$$k_1+k_2+\cdots+k_{m-k+1}=k \quad \text{and} \quad k_1+2k_2+\cdots+(m-k+1)k_{m-k+1}=m. \tag{2.3}$$

The m^{th} *complete Bell polynomial* is $B_m(x_1,\ldots,x_m) = \sum_{k=1}^{m} B_{m,k}(x_1,\ldots,x_{m-k+1})$. While the Bell polynomials has many intriguing details that can be explored in combinatorial number theory (see, for instance, [11] and references therein), we are mostly concerned with their presence in the celebrated Faà di Bruno's formula [37, 44] for derivatives of composite functions:

$$\left(\frac{d}{dt}\right)^m f\big(g(t)\big) = \sum_{k=1}^{m} f^{(k)}\big(g(t)\big)\; B_{m,k}\big(g'(t),g''(t),\ldots,g^{(m-k+1)}(t)\big). \tag{2.4}$$

Throughout this chapter $f(x) \sim g(x)$ as $x \to c$ stands for $\lim_{x\to c}\frac{f(x)}{g(x)} = 1$.

2.2 The single frog per site case

We will first assume that $\eta_x = 1$ for each nonnegative integer x and $\eta_x = 0$ elsewhere. Let $\rho = \frac{1-p}{p} \in (0,1)$ and consider the corresponding frog model. Let W_ρ represent the random variable for the negative of the minimum of the visited sites in this model. For convenience, we will construct a family of mutually independent random variables $(X_\rho)_{\rho\in(0,1)}$ that all share the same probability space $(\Omega,\mathcal{F},P)$ such that X_ρ and W_ρ share the same distribution.

The distribution function of X_ρ can be easily found by observing that, by the rightward tendency of the initial active frog, all frogs w.p.1 will eventually be woken. Furthermore, if $(S_n)_{n\geq 0}$ has the distribution of the underlying random walk, then

$$P(S_n = 0 \text{ for some } n \geq 1 \,|\, S_0 = 1) = \rho. \tag{2.5}$$

With this observation, we see that for all $x \geq 0$,

$$P(X_\rho \leq x) = P(X_\rho < x+1) = \prod_{j=1}^{\infty}(1-\rho^{x+j}) = (\rho^{x+1};\rho)_\infty. \tag{2.6}$$

It is a simple exercise to see that $P(X_\rho = 0) = \prod_{j=1}^{\infty}(1-\rho^j) = (\rho;\rho)_\infty$. For all $x > 0$, the value of X_ρ's probability density function is

$$\begin{aligned} P(X_\rho = x) = P(X_\rho \le x) - P(X_\rho \le x-1) &= \prod_{j=1}^{\infty}(1-\rho^{x+j}) - \prod_{j=0}^{\infty}(1-\rho^{x+j}) \\ &= (1-(1-\rho^x))\prod_{j=1}^{\infty}(1-\rho^{x+j}) = \rho^x(\rho^{x+1};\rho)_\infty. \end{aligned} \tag{2.7}$$

With the density known, we would now like to study the behavior of X_ρ for values of ρ close to 1, where the frog model more closely resembles the recurrent case. Objects that describe the concentration of the distribution of X_ρ include the central statistics of the random variable, such as the mode and the expectation. For fixed ρ, we define a unique representative of the mode statistic, named M_ρ, by

$$M_\rho := \min\big\{x \ge 0 : P(X_\rho = x) \ge P(X_\rho = n) \text{ for all } n \ge 0\big\}.$$

While the mode statistic is not usually observed compared to other central statistics of a class of random variables, it is a quick calculation that can often provide insight on asymptotic. Also, as we observe later, the concentration of X_ρ around its mode will be influential to its limiting behavior. With that in mind, we present the following result:

Theorem 2.2.1. *For any $\rho \in (0,1)$, $\left\lfloor \frac{\ln(1-\rho)-\ln\rho}{\ln\rho} \right\rfloor \le M_\rho \le \left\lfloor \frac{\ln(1-\rho)-\ln(2-\rho)}{\ln\rho} \right\rfloor$, where $\lfloor a \rfloor$ is the largest integer less than or equal to a. In particular, $M_\rho \sim \frac{\ln(1-\rho)}{\ln\rho}$ as $\rho \to 1$.*

Note that for all $\rho \in (0,1)$, the difference between the two bounds in the theorem's conclusion always belongs to the open interval $(0,2)$. Hence, even for values of ρ that make the bounds extremely large, the theorem narrows down M_ρ to two possibilities.

Proof of Theorem 2.2.1. Considering x to be a continuous variable, we note that

$$\frac{d}{dx}(\rho^{x+1};\rho)_\infty = \frac{d}{dx}\exp\Big(\sum_{j=1}^{\infty}\ln(1-\rho^{x+j})\Big) = \Big(\frac{d}{dx}\sum_{j=1}^{\infty}\ln(1-\rho^{x+j})\Big)\,(\rho^{x+1};\rho)_\infty, \tag{2.8}$$

It is enough to confirm that where $||f||_\infty = \sup\{|f(x)| : x \in [a,b]\}$. Thus the series $\sum_{j=1}^{\infty}||\frac{\rho^{x+j}}{1-\rho^{x+j}}||_\infty$ absolutely converges for any closed interval $[a,b]$ we choose. Hence, the series in (2.8) is differentiable, and furthermore, by [54, Theorem 7.17], the derivative of the series is the series of the derivatives. Thus $\frac{d}{dx}(\rho^{x+1};\rho)_\infty = -\ln(\rho)(\rho^{x+1};\rho)_\infty \sum_{j=1}^\infty \frac{\rho^{x+j}}{1-\rho^{x+j}}$. Using the above derivative, we can find the derivative of the density function for X_ρ:

$$\frac{d}{dx}\Big\{(\rho^{x+1};\rho)_\infty \rho^x\Big\} = \frac{d}{dx}\Big\{\prod_{j=1}^\infty (1-\rho^{x+j})\rho^x\Big\} = \ln(\rho)\rho^x(\rho^{x+1},\rho)_\infty\Big(1-\sum_{j=1}^\infty \frac{\rho^{x+j}}{1-\rho^{x+j}}\Big).$$

Notice that $\ln(\rho)\rho^x(\rho^{x+1},\rho)_\infty$ is nonzero for all $x > 0$. Therefore, by the 1^{st}-derivative test for critical points, M_ρ is an integer within 1 away from the positive value m such that $\sum_{j=1}^\infty \frac{\rho^{m+j}}{1-\rho^{m+j}} = 1$. Since $\rho^{x+j} \leq \frac{\rho^{x+j}}{1-\rho^{x+j}} \leq \frac{\rho^{x+j}}{1-\rho^{x+1}}$ for all $x > 0$ and all $j \in \mathbb{N}$, we obtain $\frac{\rho^{m+1}}{1-\rho} = \sum_{j=1}^\infty \rho^{m+j} \leq \sum_{j=1}^\infty \frac{\rho^{m+j}}{1-\rho^{m+j}} = 1 \leq \sum_{j=1}^\infty \frac{\rho^{m+j}}{1-\rho^{m+1}} = \frac{1}{1-\rho^{m+1}}\frac{\rho^{m+1}}{1-\rho}$. The result follows accounting for the fact that M_ρ is integer-valued. □

We would now like to compare the mode of X_ρ with the moments of the random variable. Calculating the moments directly from definition can be quite a challenge, given the convoluted expression of the density in (2.14). Therefore, we take the more circuitous option of first finding the cumulants of X_ρ. With $M_{X_\rho}(t) := E(e^{tX_\rho})$, the *cumulant generating function* $g_\rho(t)$ of X_ρ is defined as $g_\rho(t) := \log\big(M_{X_\rho}(t)\big)$. We then define the m^{th} *cumulant* $\kappa_\rho^{(m)}$ of X_ρ to be the m^{th} derivative of the cumulant generating function evaluated at 0: $\kappa_\rho^{(m)} := g_\rho^{(m)}(0)$. Cumulants can be used to determine moments through the use of Faà di Bruno's formula (2.4) (see, for instance, [44, 47] and references therein), which is the direction we will take. It turns out that the cumulants of X_ρ, though unable to be written down using fundamental functions, can be expressed as straight-forward series representations.

Lemma 2.2.2. *The cumulant generating function of X_ρ is $g_\rho(t) = \sum_{k=1}^\infty \ln\Big(\frac{1-\rho^k}{1-e^t\rho^k}\Big)$. Furthermore, for each $m \in \mathbb{N}$, the m^{th}-cumulant of X_ρ is $\kappa_\rho^{(m)} = \sum_{k=1}^\infty \frac{k^{m-1}\rho^k}{1-\rho^k}$.*

Proof. Using (2.1), we calculate the moment generating function of X_ρ:

$$M_{X_\rho}(t) := E(e^{tX_\rho}) = (\rho;\rho)_\infty \sum_{x=0}^\infty \frac{e^{tx}\rho^x}{(\rho;\rho)_x} = \frac{(\rho;\rho)_\infty}{(e^t\rho;\rho)_\infty} = \prod_{k=1}^\infty \frac{1-\rho^k}{1-e^t\rho^k}. \tag{2.9}$$

Taking the natural logarithm of (2.9) gives us the desired formula for $g_\rho(t)$. To find $\kappa_\rho^{(m)}$, we first find the m^{th} derivative of $g_\rho(t)$:

$$\begin{aligned} g_\rho^{(m)}(t) &= \sum_{k=1}^{\infty}\Big(\frac{d}{dt}\Big)^m \ln\Big(\frac{1-\rho^k}{1-e^t\rho^k}\Big) = \sum_{k=1}^{\infty}\Big(\frac{d}{dt}\Big)^m\Big(\ln(1-\rho^k)+\sum_{j=1}^{\infty}\frac{(e^t\rho^k)^j}{j}\Big) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} j^{m-1}(e^t\rho^k)^j = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j^{m-1}(e^t\rho^k)^j = \sum_{j=1}^{\infty}\frac{j^{m-1}e^{tj}\rho^j}{1-\rho^j}. \end{aligned}$$

Moving the derivative inside of the summation is justified by Theorem 7.17 in [54], at least for $t \in [-\delta, \delta]$ for small enough $\delta > 0$. Setting $t = 0$ gives us the result for $\kappa_\rho^{(m)}$. □

Finding the moments of a random variable through its cumulants is a well-known technique (see, for instance, [44, 47] and references therein), but we show the details here for completeness. Applying Faà di Bruno's formula (2.4) to $M_{X_\rho}(t) = e^{g_\rho(t)}$ shows that

$$M_{X_\rho}^{(m)}(t) = e^{g_\rho(t)}\sum_{k=1}^{m} B_{m,k}(g_\rho'(t), \ldots, g_\rho^{(m-k+1)}(t)) = M_{X_\rho}(t)B_m(g_\rho'(t), \ldots, g_\rho^{(m)}(t)).$$

Setting $t = 0$, we arrive at the following result.

Theorem 2.2.3. *Using the notation from Section 2.2 and Lemma 2.2.2, for each $\rho \in (0, 1)$ and for all $m \in \mathbb{N}$,*

$$E(X_\rho^m) = B_m\left(\kappa_\rho^{(1)}, \kappa_\rho^{(2)}, \cdots, \kappa_\rho^{(m)}\right). \tag{2.10}$$

Corollary 2.2.4. $E(X_\rho) = \sum_{x=1}^{\infty}\frac{\rho^x}{1-\rho^x} = \frac{\psi_\rho(1)+\ln(1-\rho)}{\ln\rho}$ *and* $\mathrm{Var}(X_\rho) = \sum_{x=1}^{\infty}\frac{x\rho^x}{1-\rho^x} = \frac{\psi_\rho'(1)}{\ln^2\rho}$, *where ψ_ρ is the q-digamma function defined in Section 2.1.*

The exact calculation of the moments in Theorem 2.2.3 can be quite unwieldy for large values of m. Thus, it's insightful to observe simpler asymptotic formulas instead. In [25], the following were proven for the input series of the Bell polynomials in 2.2.3 as $\rho \uparrow 1$:

$$\sum_{k=1}^{\infty}\frac{\rho^k}{1-\rho^k} \sim \frac{1}{1-\rho}\ln\frac{1}{1-\rho} \sim \frac{\ln(1-\rho)}{\ln\rho} \tag{2.11}$$

$$\sum_{k=1}^{\infty}\frac{k^j\rho^k}{1-\rho^k} \sim \frac{j!\zeta(j+1)}{(1-\rho)^{j+1}} \sim \frac{j!\zeta(j+1)}{-\ln^{j+1}\rho}, \qquad j \geq 1, \tag{2.12}$$

where $\zeta(j) = \sum_{k=1}^{\infty} 1/k^j$ is the Riemann zeta function.

Notice that for each $m \in \mathbb{N}$, the polynomial $B_m(x_1, \ldots, x_m)$ includes a term of the form x_1^m, and this term has coefficient 1. Furthermore, all other terms of $B_m(x_1, \ldots, x_m)$ are a multiple of some variable other than x_1 with a positive coefficient. Indeed, the polynomials $B_{m,k}$ possess this property by virtue of (2.2) and (2.3), because the latter ensures that $k_2 = \cdots = k_{m-k+1} = 0$ and hence $k_1 = k = m$. Clearly, B_m inherits the feature from $B_{m,k}$'s.

Note that from (2.11), $\big(\frac{\ln \rho}{\ln(1-\rho)}\big)^m \big(\kappa_\rho^{(1)}\big)^m \to 1$ as $\rho \uparrow 1$. Also, it follows from (2.12), that for all $j \in \{2, 3, 4, \ldots, m\}$, $\Big(\frac{\ln \rho}{\ln(1-\rho)}\Big)^j \kappa_\rho^{(j)} \sim \frac{(j-1)!\zeta(j)}{-\ln^j(1-\rho)} \to 0$ as $\rho \uparrow 1$. Since every term in (2.10) is a multiple of some cumulant other than $\kappa_\rho^{(1)}$, except for the $(\kappa_\rho^{(1)})^m$ term, multiplying $E(X_\rho^m)$ by $\big(\frac{\ln \rho}{\ln(1-\rho)}\big)^m$ and taking the limit as $\rho \uparrow 1$ eliminates all terms except for the aforementioned term which converges to 1. Therefore, we can obtain the following asymptotic result:

Corollary 2.2.5. *For all $m \in \mathbb{N}$, $E(X_\rho^m) \sim \big(\frac{\ln(1-\rho)}{\ln \rho}\big)^m$ as $\rho \uparrow 1$.*

To further clarify the special role of the number $z_\rho = \frac{\ln(1-\rho)}{\ln \rho}$ in our model consider $U_\rho := \#\{\text{frogs who reached the site } -z_\rho\}$, where, for simplicity and clarity of the subsequent computation, we treat z_ρ as an integer. Then, using (2.5) and the Markov property, observe that $E(U_\rho) = \sum_{x=0}^{\infty} P(S_n = -z_\rho \text{ for some } n \geq 1 \,|\, S_0 = x) = \sum_{x=0}^{\infty} \rho^{x+z_\rho} = \frac{\rho^{z_\rho}}{1-\rho} = 1$. This result can be heuristically interpreted as an illustration of the fact that $-z_\rho$ serves as the most distant place, though barely, is still accessible to the frog population (in average only one frog can reach that far).

We now seek to determine the asymptotic of the distribution of X_ρ when $\rho \uparrow 1$. Theorem 2.2.1 and Corollary 2.2.4 both hint that X_ρ grows at roughly the same rate as $\frac{\ln(1-\rho)}{\ln \rho}$ as ρ rises close to 1. The nature of this growth can be revealed by the asymptotic of the variance, which grows sufficiently slow to guarantee scaling limits for X_ρ.

Theorem 2.2.6. *Let Y_ρ share the same distribution as $\frac{\ln \rho}{\ln(1-\rho)} X_\rho$ for each $\rho \in (0,1)$. Then, as $\rho \uparrow 1$, $Y_\rho \to 1$ in probability. That is, for all $\epsilon > 0$, $\lim_{\rho \to 1^-} P(|Y_\rho - 1| > \epsilon) = 0$.*

Proof. By Corollary 2.2.4 and (2.11), $E(Y_\rho) = \frac{\ln \rho}{\ln(1-\rho)} E(X_\rho) \to 1$. To show the convergence in probability, it is enough to prove that $\mathrm{Var}(Y_\rho) \to 0$ as $\rho \uparrow 1$. By Corollary 2.2.4 and (2.12), $\mathrm{Var}(X_\rho) \sim \frac{1!\zeta(2)}{(1-\rho)^2} \sim \frac{\pi^2}{6} \frac{1}{\ln^2 \rho}$. Thus, $\mathrm{Var}(Y_\rho) = \frac{\ln^2 \rho}{\ln^2(1-\rho)} \mathrm{Var}(X_\rho) \sim \frac{\pi^2}{6} \frac{1}{\ln^2(1-\rho)} \to 0$. Since $E(Y_\rho) \to 1$ and $\mathrm{Var}(Y_\rho) \to 0$, convergence in probability follows. $\square$

Aside from the probabilistic implications of Theorem 2.2.6, we can also use the generating functions of Y_ρ to construct some limit identities involving the q-Pochhammer symbol. Since $Y_\rho \to 1$ in probability (and thus in distribution), the moment generating function of Y_ρ converges pointwise to that of the degenerate variable at 1. Similarly, the characteristic function also converges pointwise to the characteristic function of the same degenerate variable. This observation leads to the following corollary:

Corollary 2.2.7. *For $z > 0$, $\lim_{\rho \uparrow 1} \frac{(\rho;\rho)_\infty}{\left(z^{\frac{\ln(\rho)}{\ln(1-\rho)}}\rho;\rho\right)_\infty} = z$. For $t \in \mathbb{R}$, $\lim_{\rho \uparrow 1} \frac{(\rho;\rho)_\infty}{\left(e^{it\frac{\ln(\rho)}{\ln(1-\rho)}}\rho;\rho\right)_\infty} = e^{it}$.*

We wish to prove a stronger convergence of the sequence of random variables $\{Y_\rho\}_{\rho\in(0,1)}$. Theorem 2.2.6 implies that an appropriate discretization $\{Y_{\rho_n}\}_{n\in\mathbb{N}}$ of $\{Y_\rho\}_{\rho\in(0,1)}$ can be chosen to achieve the almost sure convergence. The following result identifies a class of sequences $\{\rho_n\}_{n\in\mathbb{N}}$ that ensures the almost sure convergence of the discrete sequence Y_{ρ_n}.

Proposition 2.2.8. *Let $\{\rho_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ be a sequence such that $\rho_n \uparrow 1$ and $\frac{1}{\ln(1-\rho_n)} \in \ell^2$. Let $\{Y_{\rho_n}\}_{n\in\mathbb{N}}$ be a sequence of random variables in the probability space $(\Omega, \mathcal{F}, P)$ such that Y_{ρ_n} has the same distribution as $\frac{\ln \rho_n}{\ln(1-\rho_n)} X_{\rho_n}$ for each n. Then, as $n \to \infty$, $Y_{\rho_n} \to 1$ a. s.*

Proof. Let $\epsilon > 0$ be given. By the Borel-Cantelli Lemma, a sufficient condition for a. s. convergence is that $P(|Y_{\rho_n} - 1| > \epsilon) \in \ell^1$. Choose $\rho \in (0,1)$ such that $|E(Y_\rho) - 1| < \frac{\epsilon}{2}$. Then, using Chebyshev's Inequality,

$$
\begin{aligned}
P(|Y_\rho - 1| > \epsilon) &\le P(|Y_\rho - E(Y_\rho)| + |E(Y_\rho) - 1| > \epsilon) \\
&\le P\left(|Y_\rho - E(Y_\rho)| > \frac{\epsilon}{2} \text{ or } |E(Y_\rho) - 1| > \frac{\epsilon}{2}\right) = P\left(|Y_\rho - E(Y_\rho)| > \frac{\epsilon}{2}\right) \le \frac{4}{\epsilon^2}\mathrm{Var}(Y_\rho).
\end{aligned}
$$

As noted above, $\mathrm{Var}(Y_\rho) = \frac{\ln^2 \rho}{\ln^2(1-\rho)}\mathrm{Var}(X_\rho) \sim \frac{\pi^2}{6}\frac{1}{\ln^2(1-\rho)}$. Replacing ρ with the terms of $\{\rho_n\}_{n\in\mathbb{N}}$, we see that $P(|Y_{\rho_n} - 1| > \epsilon) \in \ell^1$, proving our result. ☐

The class of sequences defined in the hypothesis of the proposition above includes those of the form $\rho_n = 1 - e^{-n^c}$, where $c > \frac{1}{2}$ is constant. For further research, we wish to broaden the class of sequences that lead to the a. s. convergence. For instance, instead of assuming that the models corresponding to different values of ρ are independent, one can consider a standard hierarchical coupling of the underlying random walks leading to the setting where

$P(X_{\rho_1} \leq X_{\rho_2}) = 1$ if $\rho_1 < \rho_2$. In that case one can consider for example $\rho_n = 1 - n^{-c}$, $c > \frac{1}{2}$, and imitating Etemadi's proof of the law of large numbers (see, for instance, Section 2.4 in [19]), namely first considering subsequences $k(n) = \lfloor \alpha^n \rfloor$ with an arbitrary $\alpha > 1$ and then using the fact that the ratio of $\frac{\ln \rho_{k(n)}}{\ln(1-\rho_{k(n)})}$ and $\frac{\ln \rho_{k(n+1)}}{\ln(1-\rho_{k(n+1)})}$ converges to α^c when $n \to \infty$, prove the almost sure convergence of Y_{ρ_n} for the sequence $\rho_n = 1 - n^{-c}$ by finally taking α to 1.

2.3 More general frog distributions η

Now let's consider the frog model with drift in which its frog distribution $\eta = \{\eta_x\}_{x=0}^{\infty}$ is a sequence of natural numbers, i.e., $\eta_x \geq 1$ for all $x \geq 0$ and $\eta_x = 0$ elsewhere. Main results of this section are stated in Theorems 2.3.1 and 2.3.2 below.

According to Theorem 2.1 in [27], in order to have transience in the frog model with drift ρ, and hence an almost surely finite minimum of its range, we must assume that $\sum_{x=0}^{\infty} \eta_x \rho^x < \infty$. Since we will be dealing with a continuum of choices for ρ, it will be useful to designate all of the frog distributions that will guarantee transience in the frog model with drift. Hence, we define the following set of integer sequences:

$$\mathbb{H} := \Big\{ \eta \in \mathbb{N}^{\mathbb{N} \cup \{0\}} : \sum_{x=0}^{\infty} \eta_x \rho^x < \infty \text{ for all } \rho \in (0,1) \Big\}.$$

It's worth noting that $\mathbb{H}$ does contain unbounded elements, such as $\eta = \{1, 2, 3, 4, \dots\}$. In fact, any integer sequence η such that $\eta_x = o(\alpha^x)$ as $x \to \infty$ for any $\alpha > 1$ is in $\mathbb{H}$.

Similarly to the single-frog case, we can construct a family of independent random variables $(X_{\rho,\eta})_{\rho \in (0,1), \eta \subset \mathbb{H}}$ on the probability space $(\Omega, \mathcal{F}, P)$ such that for each $\rho \in (0,1)$ and $\eta \in \mathbb{H}$, $X_{\rho,\eta}$ shares the same distribution as the negative of the minimum of the frog model with drift ρ and frog distribution η.

By using similar ideas as in the single-frog case, we can find the distribution of $X_{\rho,\eta}$:

$$P(X_{\rho,\eta} \leq x) = \prod_{k=0}^{\infty} \big(1 - \rho^{x+k+1}\big)^{\eta_k}. \tag{2.13}$$

For simplicity, we will define the integer sequence $\{\Delta_k\}_{k=0}^{\infty}$ by $\Delta_0 = \eta_0$ and for all $k \geq 1$, $\Delta_k = \eta_k - \eta_{k-1}$. The density of $X_{\rho,\eta}$ is then

$$\begin{aligned} P(X_{\rho,\eta} = x) &= \prod_{k=0}^{\infty} \left(1 - \rho^{x+k+1}\right)^{\eta_k} - \prod_{k=0}^{\infty} \left(1 - \rho^{x+k}\right)^{\eta_k} \\ &= \prod_{k=0}^{\infty} \left(1 - \rho^{x+k+1}\right)^{\eta_k} \Big(1 - \prod_{k=0}^{\infty} (1 - \rho^{x+k})^{\Delta_k}\Big). \end{aligned} \tag{2.14}$$

One special case to consider is $\eta_x = n \in \mathbb{N}$ for all $x \geq 0$. Then, (2.13) and (2.14) become

$$P(X_{\rho,\eta} \leq x) = \prod_{k=0}^{\infty} (1 - \rho^{x+k+1})^n = (\rho^{x+1}; \rho)_{\infty}^n,$$

$$P(X_{\rho,\eta} = x) = \big(1 - (1 - \rho^x)^n\big)(\rho^{x+1}; \rho)_{\infty}^n.$$

With frog distributions η that differ from the single-frog case, we could assume that the moments of $X_{\rho,\eta}$ grow at different rates than $\frac{\ln(1-\rho)}{\ln \rho}$ found in Corollary 2.2.5. However, by the theorem below, if η grows at a "slow enough" rate, the moments of $X_{\rho,\eta}$ will behave asymptotically similar to those of the single-frog case.

Theorem 2.3.1. *For each $\rho \in (0,1)$, let $z_\rho = \frac{\ln(1-\rho)}{\ln \rho}$. Suppose that $\{\eta_k\}_{k=0}^{\infty} \in \mathbb{H}$ is such that $\lim_{\rho \uparrow 1} (1-\rho)^{1+\delta} \sum_{k=0}^{\infty} \eta_k \rho^k = 0$ for all $\delta > 0$. Then, for all $m \in \mathbb{N}$, $E(X_{\rho,\eta}^m) \sim z_\rho^m$ as $\rho \uparrow 1$.*

The proof of the theorem is given below in this section, after a short discussion of the result. Note that according to the theorem, any even frog distribution $\eta = \{n, n, \dots\}$, where $n \in \mathbb{N}$, will produce the same asymptotic rate for the moments. Not only that, but there exist unbounded choices for η that produce the same rate as well. One simple example of such a choice is $\eta = \{\lceil \log(n+1) \rceil\}_{n=1}^{\infty}$.

A consequence of Theorem 2.3.1 is an analogue to Theorem 2.2.6 in the single-frog case revealing that $\frac{\ln(1-\rho)}{\ln \rho}$ is also an appropriate scaling for $X_{\rho,\eta}$'s convergence in probability as $\rho \uparrow 1$.

Theorem 2.3.2. *Let $\eta \in \mathbb{H}$ satisfy the conditions of Theorem 2.3.1. Let $Y_{\rho,\eta}$ share the same distribution as $\frac{\ln \rho}{\ln(1-\rho)} X_{\rho,\eta}$ for each $\rho \in (0,1)$. Then, as $\rho \uparrow 1$, $Y_{\rho,\eta} \to 1$ in probability.*

To clarify the intuition behind this result it is instructive to consider the case of an even frog configuration $\eta_x = m$ for all $x \geq 0$, where $m \in \mathbb{N}$ is a fixed integer, and observe that the corresponding model can be thought of as a composition of m independent models with $\eta_x = 1$.

In this case, Theorem 2.3.2 is a direct implication of the result in Theorem 2.2.6 following by a simple observation that since the random variable Y_ρ is asymptotic to a constant, the same is true for its analogue $Y_{\rho,\eta}$ in Theorem 2.3.2 which is the minimum of m independent copies of Y_ρ. From this perspective, Theorems 2.3.1 and Theorem 2.3.2 can be viewed as an indirect extension of this argument to sequences η_x growing sufficiently slowly, so that they can be well enough approximated by initial configurations with an even distribution of frogs (notice that the further is the initial placement of a frog from the origin the less relevant it is for the asymptotic of $Y_{\rho,\eta}$).

Remark 2.3.3. *The proof of Proposition 2.2.8 can be carried over, and hence its conclusion remains valid, for sequences $\eta \in \mathbb{H}$ that satisfy the conditions of Theorem 2.3.1.*

Before we begin the proof of Theorem 2.3.1, we must first introduce a lemma.

Lemma 2.3.4. *For each $\rho \in (0,1)$, let $z_\rho = \frac{\ln(1-\rho)}{\ln \rho}$. Then, for all $\delta > 0$ and $m \in \mathbb{N}$,*

$$\sum_{x=1}^{\infty} \big(z_\rho(1+\delta)+x\big)^m \rho^x \sim \frac{z_\rho^m (1+\delta)^m}{1-\rho} \qquad as \qquad \rho \uparrow 1. \tag{2.15}$$

Proof of Lemma 2.3.4. For the quotient of the left- and right-hand sides in (2.15) we have

$$\begin{aligned}
(1-\rho)\sum_{x=1}^{\infty}\Big(1+\frac{x}{z_\rho(1+\delta)}\Big)^m \rho^x &= (1-\rho)\sum_{x=1}^{\infty}\rho^x \sum_{j=0}^{m}\binom{m}{j}\frac{x^j}{z_\rho^j(1+\delta)^j} \\
&= (1-\rho)\sum_{j=0}^{m}\frac{1}{z_\rho^j(1+\delta)^j}\binom{m}{j}\Big(\sum_{x=1}^{\infty}x^j\rho^x\Big) \le (1-\rho)\sum_{j=0}^{m}\frac{1}{z_\rho^j(1+\delta)^j}\binom{m}{j}\frac{j!}{(1-\rho)^{j+1}} \\
&= \sum_{j=0}^{m}\frac{m!}{(m-j)!}\frac{1}{[z_\rho(1+\delta)(1-\rho)]^j}.
\end{aligned}$$

Now, as $\rho \uparrow 1$, $\frac{\ln(\rho)}{1-\rho} \to -1$ and $\ln(1-\rho) \to -\infty$. Thus, $z_\rho(1-\rho) \to \infty$. Hence, for any $j > 0$, the j^{th} term of the above sum goes to 0 as $\rho \uparrow 1$. So when taking the limit, the only term in the sum that survives would be the 0^{th}, which is equal to 1 for all ρ. $\square$

We now proceed with the proof of the above theorem.

Proof of Theorem 2.3.1. Since the frog model corresponding to $X_{\rho,\eta}$ contains more frogs than the single-frog case, by Corollary 2.2.5, $\liminf_{\rho\uparrow 1} \frac{E(X^m_{\rho,\eta})}{z_\rho^m} \ge 1$. For the other inequality, note that for any $\eta \in \mathbb{H}$, $E(X^m_{\rho,\eta}) \le E(X^m_{\rho,\theta})$, where $\theta_k = \max\{\eta_j : j = 1,2,\ldots,k\}$. Thus, we can assume without loss of generality that η is a nondecreasing sequence, and hence $\Delta_k \geq 0$ for all k.

Choosing $\delta > 0$, we consider the sum for $E(X^m_{\rho,\eta})$ and split it at the point $\lfloor z_\rho(1+\delta)\rfloor$. For the tail sum, we find that

$$\sum_{x=\lfloor z_\rho(1+\delta)\rfloor+1}^{\infty} x^m P(X_{\rho,\eta}=x) \leq \sum_{x=\lfloor z_\rho(1+\delta)+1\rfloor}^{\infty} x^m\Big(1-\prod_{k=0}^{\infty}(1-\rho^{x+k})^{\Delta_k}\Big)$$
$$\sim \sum_{x=1}^{\infty}\big(z_\rho(1+\delta)+x\big)^m\Big(1-\prod_{k=0}^{\infty}\big(1-\rho^{z_\rho(1+\delta)}\rho^{x+k}\big)^{\Delta_k}\Big).$$

Since $\rho^{z_\rho}=(1-\rho)$, expanding the infinite product up to the first-order terms, we obtain:

$$\sum_{x=\lfloor z_\rho(1+\delta)\rfloor+1}^{\infty} x^m P(X_{\rho,\eta}=x) \leq \sum_{x=1}^{\infty}\big(z_\rho(1+\delta)+x\big)^m\Big(1-\prod_{k=0}^{\infty}\big(1-(1-\rho)^{1+\delta}\rho^{x+k}\big)^{\Delta_k}\Big)$$
$$\leq \sum_{x=1}^{\infty}\big(z_\rho(1+\delta)+x\big)^m\Big(\sum_{k=0}^{\infty}\Delta_k(1-\rho)^{1+\delta}\rho^{x+k}\Big)$$
$$=(1-\rho)^{1+\delta}\Big(\sum_{k=0}^{\infty}\Delta_k\rho^k\Big)\sum_{x=1}^{\infty}\big(z_\rho(1+\delta)+x\big)^m\rho^x \sim (1-\rho)^{1+\delta}\Big(\sum_{k=0}^{\infty}\eta_k\rho^k\Big)z_\rho^m(1+\delta)^m.$$

The equivalence result in last line comes from Lemma 2.3.4, combined with the fact that $\sum_{k=0}^{\infty}\Delta_k\rho^k=\eta_0+(1-\rho)\sum_{k=1}^{\infty}\eta_k\rho^k$. We also derive the following upper bound for the finite sum:

$$\sum_{x=0}^{\lfloor z_\rho(1+\delta)\rfloor} x^m\, P(X_{\rho,\eta}=x) \leq \lfloor z_\rho\rfloor^m(1+\delta)^m P\big(X_{\rho,\eta}\leq z_\rho(1+\delta)\big) \leq z_\rho^m(1+\delta)^m.$$

Thus, we can derive an upper bound for the following limit:

$$\limsup_{\rho\uparrow 1}\frac{E(X^m_{\rho,\eta})}{z_\rho^m} \leq \lim_{\rho\uparrow 1}(1+\delta)^m\Big(1+(1-\rho)^{1+\delta}\cdot\sum_{k=0}^{\infty}\eta_k\rho^k\Big)=(1+\delta)^m.$$

Since $\delta>0$ is arbitrary, $\limsup_{\rho\uparrow 1}\frac{E(X^m_{\rho,\eta})}{z_\rho^m}\leq 1$, and this completes the proof. □

2.4 Initial configuration η supported on the whole $\mathbb{Z}$

In this section we will provide asymptotic bounds for the minimum of the frog model's range under the assumption that η is supported on all of $\mathbb{Z}$. The main result of this section is stated in Theorem 2.4.1 below.

Up until now, we have assumed that there were no sleeping frogs on any of the negative sites. With this assumption, all of the frogs on $\mathbb{Z}_+$ would eventually wake w.p.1, and we only needed to observe the collective minima of those frogs. However, when we consider the transient frog model with configuration η supported on all of $\mathbb{Z}$, we now have a random number of active frogs originating from the negative sites that have the potential of expanding the range. We begin to explore the moments of the minimum of this case by the groundwork laid in the previous sections for the η supported only on nonnegative sites.

Fix any $n \in \mathbb{N}$. We will assume throughout this section that $\eta_x = n$ for any $x \in \mathbb{Z}$, that is exactly n frogs are initially placed at each site of $\mathbb{Z}$. Our proofs in this section rely on the following description of the "avalanche structure" of the model. We refer to the frogs initially located in the nonnegative sites of $\mathbb{Z}$ as the "first wave". If we just observe the nonnegative frogs, we can locate the leftmost site visited by the frogs from the first wave. We consider the frogs on the negative sites down to the minimum ever visited by the first wave to be the "second wave" of frogs being activated. Tracking the leftmost site visited by the frogs from the second wave, we designate a "third wave" of frogs activated between the subsequent minimums. We will continue to label these activated frogs in terms of waves. Since we assume a transient model, there will eventually be a final wave of frogs w.p.1 that never venture any more to the left than their initial locations. In this section, let the negative of the leftmost site visited by any of the active frogs be $\widetilde{X}_{\rho,n}$.

The following theorem provides upper and lower bounds for the m^{th} moment of $\widetilde{X}_{\rho,n}$ for any given $m \in \mathbb{N}$. While the bounds contain the familiar $z_\rho = \frac{\ln(1-\rho)}{\ln(\rho)}$ term from Sections 2.2 and 2.3, they are not immediately obvious from the previous results.

Theorem 2.4.1. *The following holds for any $m \in \mathbb{N}$:*

(a) The m^{th} moment of $\widetilde{X}_{\rho,n}$ is bounded above by a function $\phi : (0,1) \to [0,\infty)$ such that

$$\phi(\rho) \sim z_\rho^m \Big(\frac{1-\rho}{2\pi}\Big)^{\frac{n}{2}} \exp\Big\{\frac{\pi^2}{6}\frac{mn}{1-\rho}\Big\} \quad \text{as} \quad \rho \uparrow 1. \tag{2.16}$$

(b) For any function $\delta : (0,1) \to (0,1)$ such that $\lim_{\rho\uparrow 1}\delta(\rho) = 0$ and $\lim_{\rho\uparrow 1}(1-\rho)^{\delta(\rho)} = 0$, the m^{th} moment of $\widetilde{X}_{\rho,n}$ is bounded below by a function $\psi_\delta : (0,1) \to [0,\infty)$ such that

$$\psi_\delta(\rho) \sim m!\, z_\rho^m \exp\Big\{\frac{mn}{(1-\rho)^{\delta(\rho)}}\Big\} \quad \text{as} \quad \rho \uparrow 1. \tag{2.17}$$

Remark 2.4.2. *An example of a function* δ *that satisfies the conditions of Theorem 2.4.1 (b) is* $\delta(\rho) = \big\{\ln\big(\frac{1}{1-\rho}\big)\big\}^{-\alpha}$ *for a constant* $\alpha \in (0,1)$ *and* $\rho > 1-e^{-1}$ *(as long as we are only interested in the asymptotic as* $\rho \uparrow 1$*, the values of* $\delta(\rho)$ *can be assigned arbitrarily for* $\rho \le 1-e^{-1}$*). In this case,* $\psi_\delta(\rho) \sim m!\, z_\rho^m \exp\big\{mn \exp\big(|\ln(1-\rho)|^{1-\alpha}\big)\big\}$ *as* $\rho \uparrow 1$*. For the sake of comparison with the upper bound, notice that the latter can be written as* $\phi(\rho) \sim z_\rho^m \big(\frac{1-\rho}{2\pi}\big)^{\frac{n}{2}} \exp\big\{mn \exp\big(|\ln(1-\rho)|\big)\big\}$ *as* $\rho \uparrow 1$*, and that the parameter* $\alpha \in (0,1)$ *can be chosen arbitrarily close to zero.*

To motivate and clarify the intuition behind the coupling construction employed in the proof of Theorem 2.4.1 given below, we precede the proof by the following observation.

Remark 2.4.3. *Consider the model described in Section 2.2, namely* $\eta_x = 1$ *for* $x \ge 0$ *and* $\eta_x = 0$ *for* $x < 0$*. Fix any* $\delta > 0$ *and consider the probability* $P_{\rho,\delta}$ *that no one of the frogs initially placed to the right of* $z_\rho(1+\delta)$ *will ever reach zero. For simplicity and without loss of generality we will treat* $z_\rho(1+\delta)$ *as an integer. Then, since* $\rho^{z_\rho(1+\delta)} = (1-\rho)^{1+\delta}$*, we have* $P_{\rho,\delta} = \prod_{j=1}^\infty \big(1-\rho^{z_\rho(1+\delta)+j}\big) = \prod_{j=1}^\infty \big(1-(1-\rho)^{1+\delta}\rho^j\big)$*. Since* $1-x > e^{-2x}$ *for all* $x > 0$ *small enough, we obtain that* $P_{\rho,\delta} \ge \exp\Big\{-2(1-\rho)^{1+\delta}\sum_{j=1}^\infty \rho^j\Big\} = \exp\big\{-2\rho(1-\rho)^\delta\big\}$ *for all values of* ρ *sufficiently close to* 1*, and hence* $\lim_{\rho\uparrow 1} P_{\rho,\delta} = 1$*.*

Now, recall that the results of Section 2.2 indicate tight concentration of the distribution of X_ρ *around* z_ρ *as* $\rho \uparrow 1$*. On the other hand, heuristically,* $\lim_{\rho\uparrow 1} Q_{\rho,\delta} = 0$ *indicates that for large values of* ρ *only the first* z_ρ *frogs are relevant to the dynamics of the model. To further support this claim, consider the probability* $Q_{\rho,\delta}$ *that no one of the* $z_\rho(1-\delta)$ *frogs initially placed at the first* $z_\rho(1-\delta)$ *nonnegative integers will ever reach* $-z_\rho(1-\delta)$*. Similarly as before, we treat* $z_\rho(1-\delta)$ *as an integer. Then*

$$
\begin{aligned}
Q_{\rho,\delta} &= \prod_{j=0}^{z_\rho(1-\delta)-1} \big(1-\rho^{z_\rho(1-\delta)+j}\big) \le \exp\Big\{-(1-\rho)^{1-\delta} \sum_{j=0}^{z_\rho(1-\delta)-1} \rho^j\Big\} \\
&= \exp\Big\{-(1-\rho)^{-\delta}\big(1-\rho^{z_\rho(1-\delta)}\big)\Big\} = \exp\Big\{-(1-\rho)^{-\delta}\big(1-(1-\rho)^{1-\delta}\big)\Big\} \\
&\le \exp\Big\{-(1-\rho)^{-\delta}\big(1-e^{-\rho(1-\delta)}\big)\Big\}.
\end{aligned}
$$

Thus $\lim_{\rho\uparrow 1} Q_{\rho,\delta} = 0$*.*

Heuristically, $\lim_{\rho\uparrow 1} P_{\rho,\delta} = 1$ *along with* $\lim_{\rho\uparrow 1} Q_{\rho,\delta} = 0$ *tell us that the dynamics of the model considered in Section 2.2 is for large values of* ρ *similar to the dynamics of a modification* *where* η_x *equals* 1 *only if* $0 \le x < z_\rho$ *and is* 0 *otherwise. The proof of Theorem 2.4.1 given below is using an interpretation of the model considered in this section as an "avalanche" of the models described in Section 2.3 and is using the above heuristic observation to produce upper and lower bounds of Theorem 2.4.1 for the model range.*

Proof of Theorem 2.4.1.

(a) We first provide an upper bound for the moments of $\widetilde{X}_{\rho,n}$ by coupling the frog model with the following variant. First observe the minimum location reached by the frogs that are initially placed on the nonnegative sites and obtain a second wave of active frogs. Modify the original second wave in the following way. Put n more frogs on each site to the right of the minimum reached by the first wave and suppose that only the second wave can activate them. Note that without the consideration of activation times, this set of frogs is a translation of the configuration of nonnegative frogs considered in Sections 2.2 and 2.3. Find the minimum bound for this modified configuration and, for the resulting next wave, add frogs to all of the right-side sites in a similar fashion. Since there is a positive probability for the nonnegative frogs never reach -1, and each wave of frogs is a translation of this case, the waves will terminate w.p.1. Let $W_{\rho,n}$ be the negative of the minimum bound for this model.

A formal definition of the distribution of $W_{\rho,n}$ can be given in the following manner. Define the sequence $\widetilde{\eta} = \{\widetilde{\eta}_x\}_{x\in\mathbb{Z}} \in \mathbb{Z}_+^{\mathbb{Z}}$ as follows: $\eta_x = n$ if $x \ge 0$ and $\eta_x = 0$ if $x < 0$. Such a configuration has been considered in Section 2.3. Let $X_{\rho,n}$ be the negative of the range of the corresponding model and let $X_{\rho,n}^{(k)}$, $k \in \mathbb{Z}$, be independent copies of this random variable. Let $T_{\rho,n} = \inf\{k \in \mathbb{N} : X_{\rho,n}^{(k)} = 0\}$ and $W_{\rho,n} = \sum_{k=1}^{T_{\rho,n}-1} X_{\rho,n}^{(k)}$, where, as usual, the empty sum (when $T_{\rho,n} = 1$) is interpreted as zero.

To facilitate our computations we will actually use the following equivalent modification of this definition. Let $\varepsilon_{\rho,n} = P(X_{\rho,n} = 0) = \big((\rho;\rho)_\infty\big)^n$, where $X_{\rho,n}$ is the minimum of the range of the case with n frogs on each nonnegative site, introduced in Section 2.3. Thus $\varepsilon_\rho = P\big(X_{\rho,n}^{(k)} = 0\big)$ for any $k \in \mathbb{N}$. Let $\widetilde{T}_{\rho,n}$ be a geometric random variable with parameter ε_ρ. Namely, $P\big(\widetilde{T}_{\rho,n} = k\big) = (1-\varepsilon_\rho)^k \varepsilon_\rho$, $k = 0, 1, \ldots$. Notice that, $\lim_{\rho\to 1} \varepsilon_{\rho,n} = 0$, and hence $\widetilde{T}_{\rho,n}$ converges to infinity in probability as $\rho \uparrow 1$. Clearly, $\widetilde{T}_{\rho,n}$ has the same distribution as $T_{\rho,n} - 1$. Let $\mathcal{Y}_{\rho,n} = \{Y_{\rho,n}^{(k)}\}_{k\in\mathbb{Z}}$ be an i.i.d. sequence independent of $\widetilde{T}_{\rho,n}$ and such that for any $j \in \mathbb{N}$,

$P(Y_{\rho,n}=j)=P(X_{\rho,n}=j|X_{\rho,n}>0)=\frac{P(X_{\rho,n}=j)}{1-P(X_{\rho,n}=0)}$. Finally, let $\widetilde{W}_{\rho,n}=\sum_{k=1}^{\widetilde{T}_{\rho,n}}Y_{\rho,n}^{(k)}$. Clearly, $\widetilde{W}_{\rho,n}$ has the same distribution as $W_{\rho,n}$.

For $m\in\mathbb{N}$, we look at the m^{th} moment of $\widetilde{W}_{\rho,n}$ conditioned on $\widetilde{T}_{\rho,n}$:

$$E\big(\widetilde{W}_{\rho,n}^m\big|\widetilde{T}_{\rho,n}\big)=E\Big[\Big(\sum_{k=1}^{\widetilde{T}_{\rho,n}}Y_{\rho,n}^{(k)}\Big)^m\Big|\widetilde{T}_{\rho,n}\Big]\le\Big(\sum_{k=1}^{\widetilde{T}_{\rho,n}}E\big[\big(Y_{\rho,n}^{(k)}\big)^m\big]^{\frac{1}{m}}\Big)^m=\big(\widetilde{T}_{\rho,n}\big)^mE\big[\big(Y_{\rho,n}^{(1)}\big)^m\big],$$

where we use Minkowski inequality and the fact that $Y_{\rho,n}^{(k)}$ are independent of $\widetilde{T}_{\rho,n}$. From this conditioned expectation, we approximate the m^{th} moments of $W_{\rho,n}$ as $\rho\uparrow 1$:

$$E\big(W_{\rho,n}^m\big)=E\big(E\big(\widetilde{W}_{\rho,n}^m\big|\widetilde{T}_{\rho,n}\big)\big)\le E\big[\big(\widetilde{T}_{\rho,n}\big)^m\big]\cdot E\big[\big(Y_{\rho,n}^{(1)}\big)^m\big]\sim\varepsilon_{\rho,n}^{-m}z_\rho^m, \tag{2.18}$$

where $z_\rho=\frac{\ln(1-\rho)}{\ln\rho}$ as in Section 2.3. To evaluate the moments of $\widetilde{T}_{\rho,n}$ we used the following known result whose short proof is supplied for reader's convenience.

Lemma 2.4.4. *For $\varepsilon\in(0,1)$, let T_ε be a geometric random variable with parameter ε_ρ. Namely, $P\big(T_\varepsilon=k\big)=(1-\varepsilon)^k\varepsilon$, $k=0,1,\ldots$. Then, for any $m\in\mathbb{N}$, $E(T_\varepsilon^m)\sim\varepsilon^{-m}$ as $\varepsilon\uparrow 1$.*

Proof of Lemma 2.4.4. We have as $\varepsilon\uparrow 1$,

$$\begin{aligned}E(T_\varepsilon^m)&=\int_0^\infty P(T_\varepsilon^m>x)\,dx=\int_0^\infty P(T_\varepsilon>y)\cdot my^{m-1}\,dy\sim\int_0^\infty(1-\varepsilon)^{y+1}\cdot my^{m-1}\,dy\\&=m(1-\varepsilon)\int_0^\infty e^{\ln(1-\varepsilon)y}y^{m-1}\,dy=m(1-\varepsilon)\frac{\Gamma(m)}{|\ln(1-\varepsilon)|^m}\sim\frac{m!}{\varepsilon^m}.\end{aligned}$$

The proof of the lemma is complete. □

Using (2.18), we finally arrive at the asymptotic bound in *(a)* through the asymptotic of the q-Pochhammer symbol derived from [32]: for $\rho=e^{-t}$, as $t\downarrow 0$, $(\rho;\rho)_\infty\sim\sqrt{\frac{2\pi}{t}}\exp\Big(\frac{-\pi^2}{6t}\Big)$. Note that $t=-\ln\rho\sim 1-\rho$, and we have *(a)*. □

(b) Now, consider another variant of the frog model. Define a function $\delta(\rho):(0,1)\to(0,1)$. To simplify notation, we will occasionally use δ_ρ for $\delta(\rho)$. For a given ρ, consider the configuration $\hat{\eta}$ with $\hat{\eta}_k=n$ if $k\in\{0,1,2,\ldots,z_\rho(1-\delta_\rho)\}$ and $\hat{\eta}_k=0$ elsewhere. For simplicity and without loss of generality, we will assume that $z_\rho(1-\delta_\rho)$ is integer-valued. Start the model, and see if the frogs eventually reach the site $-z_\rho(1-\delta_\rho)$. If they do, activate n frogs on each of the $z_\rho(1-\delta_\rho)$ sites to the left of the origin. Now observe if the newly activated frogs reach the site $-2z_\rho(1-\delta_\rho)$. If they do, activate n frogs on all the $z_\rho(1-\delta_\rho)$ sites to the left of the sites

previously activated. Continue this procedure indefinitely. Note that at any observed time step, this model will always have fewer active frogs than the frog model with initial configuration η with $\eta_k = n$ for all $k \in \mathbb{Z}$. By Theorem 2.1 in [27], the frog model with this configuration is transient. Hence, the variant model is transient, and the process will eventually stop producing new sets of frogs from the left. Let $V_{\rho,\delta}$ be the negative of the minimum of the range of the variant, and let $\tau_{\rho,\delta}$ be the number of activated blocks of the length $z_\rho(1-\delta_\rho)$, not including the initial one located within $\mathbb{Z}_+$.

Let $\theta_{\rho,\delta} = \prod_{j=0}^{z_\rho(1-\delta_\rho)-1}\big(1-\rho^{z_\rho(1-\delta_\rho)}\rho^j\big)^n = \big((\rho^{z_\rho(1-\delta_\rho)};\rho)_{z_\rho(1-\delta_\rho)}\big)^n$. Thus $\theta_{\rho,\delta}$ is the probability that none of the frogs located at the first $z_\rho(1-\delta_\rho)$ nonnegative sites will ever reach the half-line on the left of $-z_\rho(1-\delta_\rho)$. Viewing the event of a block of active frogs reaching $z_\rho(1-\delta_\rho)$ units to the left as a "failure", we see that $\tau_{\rho,\delta}$ is geometrically distributed with parameter $\theta_{\rho,\delta}$. Namely, $P(\tau_{\rho,\delta}=k) = (1-\theta_{\rho,\delta})^k\theta_{\rho,\delta}$, $k = 0,1,\ldots$. Since $\rho^{z_\rho} = 1-\rho$,

$$\begin{aligned}\theta_{\rho,\delta} &= \prod_{j=0}^{z_\rho(1-\delta_\rho)-1}\big(1-(1-\rho)^{1-\delta_\rho}\rho^j\big)^n \le \exp\Big(-n(1-\rho)^{1-\delta_\rho}\sum_{j=0}^{z_\rho(1-\delta_\rho)-1}\rho^j\Big)\\ &= \exp\Big(-n\frac{1-\rho^{z_\rho(1-\delta_\rho)}}{(1-\rho)^{\delta_\rho}}\Big) = \exp\Big(-n\frac{1-(1-\rho)^{1-\delta_\rho}}{(1-\rho)^{\delta_\rho}}\Big).\end{aligned}$$

With the constraints specified in the theorem's hypotheses, $\theta_{\rho,\delta} \to 0$ as $\rho \uparrow 1$. Hence, by Lemma 2.4.4, $E(\tau_{\rho,\delta}^m) \sim m!\,\theta_{\rho,\delta}^{-m}$ as $\rho \uparrow 1$. Clearly, $\widetilde{X}_{\rho,n}$ is stochastically dominated from below by $V_{\rho,\delta} = \tau_{\rho,\delta}\cdot z_\rho(1-\delta_\rho)$. The lower bound for the moments of $\widetilde{X}_{\rho,n}$ is therefore

$$E(\tau_{\rho,\delta}^m)\cdot z_\rho^m(1-\delta_\rho)^m \sim m!\,z_\rho^m\exp\Big\{mn\frac{1-(1-\rho)^{1-\delta_\rho}}{(1-\rho)^{\delta_\rho}}\Big\} \sim m!\,z_\rho^m\exp\Big\{\frac{mn}{(1-\rho)^{\delta_\rho}}\Big\},$$

as $\rho \uparrow 1$. The proof of the theorem is complete. □

CHAPTER 3. FAVORITE SITES OF THE PERSISTENT RANDOM WALK

Let $\lambda \in \left[\frac{1}{2}, 1\right)$. Let $\{X_s\}_{s=1}^{\infty}$ be a discrete-time Markov chain on the state space $\{-1, 1\}$. X_1 is either 1 or -1 with equal probability, and for each $s > 1$, the Markov chain has the transition probabilities for values $c \in \{-1, 1\}$

$$\begin{aligned} P(X_s = c | X_{s-1} = c) &= \lambda, \\ P(X_s = -c | X_{s-1} = c) &= 1 - \lambda. \end{aligned} \tag{3.1}$$

Define the symmetric nearest-neighbor persistent random walk $\{S_t\}_{t=0}^{\infty}$ by

$$S_t := \sum_{s=1}^{t} X_s,$$

with the convention $S_0 = 0$ w.p.1. Intuitively, S_t is similar to a simple symmetric random walk on $\mathbb{Z}$, except the direction of the motion of S_t has a bias towards the same direction its previous step. As a matter of fact, if $\lambda = \frac{1}{2}$ is permitted, then the persistent random walk can be seen as a generalization of the simple random walk.

We begin this study on the persistent random walk with defining relevant notation. We define the *local time* of a site $x \in \mathbb{Z}$ at time $t > 0$ as the number of visits x receives from the walk up to time t:

$$L(x, t) := \#\{0 < s \leq t : S_s = x\}.$$

For every time t, we also define the set of *favorite sites*, that is, the sites of $\mathbb{Z}$ that have been visited by the random walk the most by time t:

$$\mathcal{K}(t) := \{y \in \mathbb{Z} : L(y, t) = \max_{z \in \mathbb{Z}} L(z, t)\}.$$

Since the range of S_t is finite at any given point in time, $\#\mathcal{K}(t) < \infty$ w.p.1 for any t. Note that there are only two ways in which $\mathcal{K}(t)$ could change from $\mathcal{K}(t-1)$: either the local time of a site outside of $\mathcal{K}(t-1)$ becomes a maximum local time at time t, in which $\#\mathcal{K}(t) = \#\mathcal{K}(t-1)+1$, or one of the sites in $\mathcal{K}(t-1)$ receives one more visit at time t, in which $\#\mathcal{K}(t) = 1$.

Finally, we define the random variable $f(r)$ to be the number of times $\#\mathcal{K}(t)$ becomes $r \in \mathbb{N}$:

$$f(r) := \#\{t \geq 1 : S_t \in \mathcal{K}(t), \#\mathcal{K}(t) = r\}.$$

In the simple walk case ($\lambda = \frac{1}{2}$), it was shown that $f(1) = f(2) = \infty$ w.p.1 in [21] and [8]. In [21], [22] and [23], Erdős and Révész conjectured that $f(r)$ was finite w.p.1 for $r \geq 3$. The conjecture was partially proven in [61], in which it was shown that $f(4)$ was finite w.p.1, hence $f(r)$ for $r \geq 5$ as well. Our main result in this chapter reveals that the set of favorite sites for the persistent random walk behaves similarly, regardless of the amount of local directional bias.

Theorem 3.0.1 (Main Theorem)**.** *For any choice of* $\lambda \in (\frac{1}{2}, 1)$*,*

$$E(f(4)) < \infty.$$

In particular, $f(4) < \infty$ *w.p.1.*

This theorem extends the result found for the simple random walk in [61]. It's somewhat surprising of a result for the persistent case; for λ close to 1 the persistent walk will cover the same large intervals of integers with the same number of visits. One would presume that the intervals of favorite sites will stay large, but the theorem shows that over time, the number of favorite sites will eventually be bounded above by 3.

3.1 Definitions

Before we begin to prove the theorem, we first need to establish the preliminary definitions and observations. First, we define the upcrossings and downcrossings, respectively, of a site x:

$$
\begin{aligned}
U(x,t) &:= \#\{0 < s \le t : S_s = x, S_{s-1} = x-1\}, \\
D(x,t) &:= \#\{0 < s \le t : S_s = x, S_{s-1} = x+1\}.
\end{aligned}
$$

A couple of things to note here: $U(x,t)$ and $D(x,t)$ can be seen as a partition of the total local time $L(x,t)$, in that $L(x,t) = U(x,t) + D(x,t)$. Also, $U(x,t)$ and $D(x,t)$ are related to each other given the relative position of S_t in the following way:

$$
U(x,t) - D(x-1,t) = \mathbb{1}_{\{0<x\le S_t\}} - \mathbb{1}_{\{S_t<x\le 0\}}, \tag{3.2}
$$

$$
D(x,t) - U(x+1,t) = -\mathbb{1}_{\{0<x\le S_t\}} + \mathbb{1}_{\{S_t<x\le 0\}}. \tag{3.3}
$$

Using (3.2) and (3.3), we can rewrite the local time all in terms of either upcrossings or downcrossings:

$$
\begin{aligned}
L(x,t) &= D(x,t) + D(x-1,t) + \mathbb{1}_{\{0<x\le S_t\}} - \mathbb{1}_{\{S_t<x\le 0\}} && (3.4) \\
&= U(x,t) + U(x+1,t) - \mathbb{1}_{\{0<x\le S_t\}} + \mathbb{1}_{\{S_t<x\le 0\}}. && (3.5)
\end{aligned}
$$

Next, we define the following stopping times for the upcrossings and downcrossings above: for any $x \in \mathbb{Z}$ and $k \ge 0$,

$$
\begin{aligned}
T^U_{x,k} &:= \inf\{t \ge 1 : U(x,t) = k\}, \\
T^D_{x,k} &:= \inf\{t \ge 1 : D(x,t) = k\}.
\end{aligned}
$$

We can use these stopping times to help partition $f(4)$ into infinite random variables based on the location and visiting direction of the new favorite sites in the following way:

$$
\begin{aligned}
u_x(4) &:= \sum_{t=1}^{\infty} \mathbb{1}_{\{\Delta_t=1, x\in\mathcal{K}_t, \#\mathcal{K}(t)=4\}} \\
&= \sum_{k=1}^{\infty} \mathbb{1}_{\{x\in\mathcal{K}(T^U_{x,k}), \#\mathcal{K}(T^U_{x,k})=4\}}.
\end{aligned}
$$

$$\begin{aligned} d_x(4) &:= \sum_{t=1}^{\infty} \mathbb{1}_{\{\Delta_t=-1, x\in\mathcal{K}_t, \#\mathcal{K}(t)=4\}}, \\ &= \sum_{k=1}^{\infty} \mathbb{1}_{\{x\in\mathcal{K}(T^D_{x,k}), \#\mathcal{K}(T^D_{x,k})=4\}}. \end{aligned}$$

From here, we can see that

$$f(4) = \sum_{x\in\mathbb{Z}} (u_x(4) + d_x(4))$$

Note that, due to the symmetry of our persistent random walk model (in the sense that for any $x \in \mathbb{Z}$ and $t \geq 0$, $P(S_t = x|S_0 = 0) = P(S_t = -x|S_0 = 0)$), $u_x(4)$ is equal in distribution to $d_{-x}(4)$ for any $x \in \mathbb{Z}$. Hence, for the expectation of $f(4)$, we only need to concern ourselves with the nonnegative sites:

$$E(f(4)) = 2\sum_{x=1}^{\infty} E(u_x(4)) + 2\sum_{x=0}^{\infty} E(d_x(4)). \tag{3.6}$$

We can prove $E(f(4))$ is finite by showing the series on the right-hand side of (3.6) are both finite. For the rest of this work, we will set out to prove the following:

$$\sum_{x=1}^{\infty} E(u_x(4)) = \sum_{x=1}^{\infty}\sum_{k=1}^{\infty} P\left(x \in \mathcal{K}(T^U_{x,k}), \#\mathcal{K}(T^U_{x,k}) = 4\right) < \infty. \tag{3.7}$$

The proof that $\sum_{x=0}^{\infty} E(d_x(4)) < \infty$ is a similar exercise left to the reader.

3.2 Ray-Knight Representation

Now we introduce the offspring distribution for a sequence of critical branching processes which will be vital in the theorem's proof. For every $t \geq -1$ and $i \geq 1$, consider the random variable $\zeta_{t,i}$ with distribution

$$P(\zeta_{t,i} = j) = \begin{cases} 1-\lambda & \text{if } j = 0 \\ \lambda^2(1-\lambda)^{j-1} & \text{if } j \geq 1. \end{cases} \tag{3.8}$$

A note about this distribution is that its expectation is 1 and its variance is $2\frac{1-\lambda}{\lambda}$, as the computation of a couple of geometric series reveals.

For any given positive site $x \geq 1$, this random variable will represent the number of times a persistent particle will move from $x+1$ to x until eventually returning to $x-1$. When the particle first moves rightward onto x, it has a $1-\lambda$ probability of going against its rightward bias and moving leftward to $x-1$. If the particle goes right instead, the particle will take an excursion before returning to x again, which includes a downcrossing from $x+1$. This time, the particle has a $1-\lambda$ probability of moving right and starting another excursion, or a λ probability of moving left and ending the "trials".

Whenever a particle visits x from the left again after arriving at $x-1$, the memory of the Markov chain that dictate the particle's transition probabilities does not include any of its previous excursions to the right of x. Thus, every trial of $(x+1)$-to-x downcrossings for each x-to-$(x-1)$ downcrossing will be independent and identically distributed with each other. So the number of downcrossings between two adjacent sites will be a Markov chain dependent on the number of downcrossings between the next lower pair of sites.

For each $t \geq 0$ and $i \geq 1$, we will make i.i.d. copies of $\zeta_{t,i}$, call them $\zeta^*_{t,i}$ and $\zeta'_{t,i}$. The motivation for these new random variables are slightly different from that of $\zeta_{t,i}$, but they will be used for similar representations. Fix $k \geq 0$ and $x \geq 1$.

First, we will define a Galton-Watson process Y_t with $\{\zeta_{t,i}\}_{t=-1,i=1}^{\infty}$ as the offspring it produces each generation. Define the initial state $Y_{-1} = k$ and, for each $-1 \leq t < \infty$, let $Y_{t+1} := \sum_{i=1}^{Y_t} \zeta_{t+1,i}$. Y_t is then a Markov chain with transition probabilities $\pi(i,j)$, given by

$$\begin{aligned}\pi(i,j) &:= P\left(Y_{t+1} = j | Y_t = i\right) \\ &= \begin{cases} \delta_{0,j}, & i = 0 \\ \left(\frac{\lambda^2}{1-\lambda}\right)^i (1-\lambda)^j \sum_{k=(i-j)_+}^{i-1} \binom{i}{k}\binom{j-1}{i-k-1}\left(\frac{1-\lambda}{\lambda}\right)^{2k}, & i \geq 1 \end{cases}\end{aligned} \tag{3.9}$$

where $(a)_+ := \max\{a, 0\}$. Note that the right-hand side of (3.9) is the calculated i-fold convolution of (3.8). As a note of interest, setting $\lambda = \frac{1}{2}$ in (3.9) will reduce the right-hand side to the equivalent transition probabilities seen in [51] and [39], due to the Chu-Vandermonde identity (see [53]).

Next, define a Galton-Watson process Z_t with $\{\zeta^*_{t,i}\}_{t=-1,i=1}^{\infty}$ offspring and one intruder particle entering each generation. Let $Z_0 = k$ be the initial state and, for each $0 \leq t \leq x-1$, let $Z_{t+1} := \sum_{i=1}^{Z_t+1} \zeta^*_{t+1,i}$. Then Z_t is also a Markov chain with transition probabilities $\rho(i,j)$ given by

$$\begin{aligned}\rho(i,j) &:= P(Z_{t+1} = j | Z_t = i) \\ &= \left(\frac{\lambda^2}{1-\lambda}\right)^{i+1} (1-\lambda)^j \sum_{k=(i+1-j)_+}^{i} \binom{i+1}{k}\binom{j-1}{i-k}\left(\frac{1-\lambda}{\lambda}\right)^{2k}.\end{aligned} \tag{3.10}$$

Before defining the final process in this set, we first need to define a new random variable η with distribution

$$P(\eta = j) = \begin{cases} \frac{1}{2} & \text{if } j = 0 \\ \frac{1}{2}\lambda(1-\lambda)^{j-1} & \text{if } j \geq 1 \end{cases}$$

This variable describes the number of downcrossings from 0 to -1 until a first visit to 1. This will be used to define the third Galton-Watson process Y'_t, with initial state $Y'_0 = Z_{x-1}$, $Y'_1 := \eta \cdot \delta_{\{S_1=-1\}} + \sum_{i=1}^{Y'_0} \zeta'_{1,i}$, and $Y'_{t+1} := \sum_{i=1}^{Y'_t} \zeta'_{t+1,i}$ for each $1 \leq t < \infty$. We exclude the calculation of the transition probabilities of Y'_t, as they are not needed for this proof.

With these three processes defined, we are now ready to build our Ray-Knight type representation of the local times of S_t. For each $y \in \mathbb{Z}$, define $\Delta_{x,k}(y)$ by

$$\Delta_{x,k}(y) := \begin{cases} Y_{y-x} & \text{if} \quad x-1 \leq y < \infty \\ Z_{x-y-1} & \text{if} \quad 0 \leq y \leq x-1 \\ Y'_{-y} & \text{if} \quad -\infty < y \leq 0 \end{cases}.$$

By this construction, we arrive at the Ray-Knight type representation for the downcrossings of the persistent walk:

$$(\Delta_{x,k}(y), y \in \mathbb{Z}) \stackrel{\mathcal{D}}{=} \left(D(T^U_{x,k+1}, y), y \in \mathbb{Z}\right), \tag{3.11}$$

in which $\stackrel{\mathcal{D}}{=}$ means equal in distribution. Plainly speaking, $\Delta_{x,k}(y)$ represents the random number of downcrossings into site y before the $(k+1)^{\text{th}}$ upcrossing to x for any $y \in \mathbb{Z}$.

Now define the following random variable for each $y \in \mathbb{Z}$:

$$\Lambda_{x,k}(y) := \Delta_{x,k}(y) + \Delta_{x,k}(y-1) + \mathbb{1}_{\{0<y\leq x\}}. \tag{3.12}$$

$\Lambda_{x,k}(y)$ serves as the local time of y stopped at $T^U_{x,k+1}$, based on (3.4). Hence, using (3.4) and (3.12), we get the following Ray-Knight representation:

$$(\Lambda_{x,k}(y), y \in \mathbb{Z}) \stackrel{\mathcal{D}}{=} \big(L(T^U_{x,k+1}, y), y \in \mathbb{Z}\big)$$

The following is a list of random variables and events that we will use for the more technical aspects of the main theorem's proof:

$$\widetilde{Y}_t := Y_t + Y_{t-1}, \qquad \widetilde{Z}_t := Z_t + Z_{t-1} + 1, \qquad \widetilde{Y}'_t := Y'_t + Y'_{t-1}.$$

$$\begin{aligned}
\sigma_h &:= \inf\{t \geq 0 : Y_t \geq h\} \\
\omega &:= \inf\{t \geq 0 : Y_t = 0\} \\
\sigma'_h &:= \inf\{t \geq 0 : Y'_t \geq h\} \\
\omega' &:= \inf\{t \geq 0 : Y'_t = 0\} \\
\tau_h &:= \inf\{t \geq 0 : Z_t \geq h\}
\end{aligned}$$

$$\begin{aligned}
\widetilde{\sigma}_{h,0} &:= 0, & \widetilde{\sigma}_{h,i+1} &:= \inf\{t > \widetilde{\sigma}_{h,i} : \widetilde{Y}_t \geq h\}, \\
\widetilde{\sigma}_h &:= \widetilde{\sigma}_{h,1} & & \\
\widetilde{\sigma}'_{h,0} &:= 0, & \widetilde{\sigma}'_{h,i+1} &:= \inf\{t > \widetilde{\sigma}'_{h,i} : \widetilde{Y}'_t \geq h\}, \\
\widetilde{\sigma}'_h &:= \widetilde{\sigma}'_{h,1} & & \\
\widetilde{\tau}_{h,0} &:= 0, & \widetilde{\tau}_{h,i+1} &:= \inf\{t > \widetilde{\tau}_{h,i} : \widetilde{Z}_t \geq h\}, \\
\widetilde{\tau}_h &:= \widetilde{\tau}_{h,1} & &
\end{aligned}$$

$$\begin{aligned} A_{h,p} &:= \left\{\max_{1\le t<\infty} \widetilde{Y}_t \le h, \#\{1 \le t < \infty : \widetilde{Y}_t = h\} = p\right\} \\ &:= \{\widetilde{\sigma}_{h,p} < \infty = \widetilde{\sigma}_{h,p+1},\ \widetilde{Y}_{\widetilde{\sigma}_{h,i}} = h \text{ for } i = 1, \cdots, p\} \\ A'_{h,p} &:= \left\{\max_{1\le t<\infty} \widetilde{Y}'_t \le h, \#\{1 \le t < \infty : \widetilde{Y}'_t = h\} = p\right\} \\ &:= \{\widetilde{\sigma}'_{h,p} < \infty = \widetilde{\sigma}'_{h,p+1},\ \widetilde{Y}'_{\widetilde{\sigma}'_{h,i}} = h \text{ for } i = 1, \cdots, p\} \\ B_{x,h,p} &:= \left\{\max_{1\le t<x} \widetilde{Z}_t \le h, \#\{1 \le t < x : \widetilde{Z}_t = h\} = p\right\} \\ &:= \{\widetilde{\tau}_{h,p} < x \le \widetilde{\tau}_{h,p+1},\ \widetilde{Z}_{\widetilde{\tau}_{h,i}} = h \text{ for } i = 1, \cdots, p\} \end{aligned}$$

Plainly speaking, $\widetilde{\sigma}_{h,i}$, $\widetilde{\sigma}'_{h,i}$ and $\widetilde{\tau}_{h,i}$ are the i^{th} hitting times of the interval $[h, \infty)$ by their respective processes, and ω and ω' are the extinction times of their respective processes. Furthermore, $A_{h,p}$, $A'_{h,p}$ and $B_{x,h,p}$ are the events that the respective processes hit exactly p times its maximum level h either before extinction or, in $B_{x,h,p}$'s case, before time x.

With these events defined and the Ray-Knight representation established, we arrive at the following expression for $E(u_x(4))$ for any x:

$$\begin{aligned} E(u_x(4)) = \sum_{p+q+r=3} \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} & P(A_{h,p} | Y_0 = h - k - 1) \\ & \times \pi(k, h - k - 1) \\ & \times P(B_{x,h,q}, Z_{x-1} = \ell | Z_0 = k) \\ & \times P(A'_{h,r} | Y'_0 = \ell), \end{aligned}$$

which then leads to an upper bound for the left-hand side of (3.7):

$$\begin{aligned} \sum_{x=1}^{\infty} E(u_x(4)) \le \sum_{p+q+r=3} \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} & P(A_{h,p} | Y_0 = h - k - 1) \\ & \times \pi(k, h - k - 1) \\ & \times \left(\sum_{x=1}^{\infty} P(B_{x,h,q} | Z_0 = k) \right) \\ & \times \left(\sup_{\ell \ge 0} P(A'_{h,r} | Y'_0 = \ell) \right) \end{aligned} \tag{3.13}$$

We now introduce bounds to the values in the right-hand side of (3.13). The first set of bounds comes in the form of a proposition, which shall be proven in the next section:

Proposition 3.2.1. *For any $\epsilon > 0$ there exists a finite constant $C < \infty$ such that for any $h \geq 1$ and $k \geq 0$:*

1. *For any $p \geq 0$,*
$$\sum_{x=1}^{\infty} P(B_{x,h,p}|Z_0 = k) \leq Ch \tag{3.14}$$

2. *If either $k \in [(h - h^{1/2+\epsilon})/2, h + h^{1/2+\epsilon})/2]$ or $p \geq 1$ holds, then*

$$P(A_{h,p}|Y_0 = k) \leq (Ch^{-1/2+\epsilon})^{p+1} \tag{3.15}$$
$$\sum_{x=1}^{\infty} P(B_{x,h,p}|Z_0 = k) \leq (Ch^{-1/2+\epsilon})^{p+1}h \tag{3.16}$$

Next is a lemma on the sum of some extreme values of $\pi(i,j)$. The proof of this lemma will be postponed until Section 3.5. For organizational purposes, we will begin an alphabetical ordering of the lemmas which will be proven in Section 3.5, starting with the following lemma.

Lemma A. *For any $\epsilon > 0$, there exist constants $C, \gamma > 0$ such that, for any $h \geq 1$,*

$$\sum_{k:|h-2k|>h^{1/2+\epsilon}} \pi(k, h-1-k) < C\exp(-\gamma h^{2\epsilon}).$$

Using Proposition 3.2.1 and Lemma A, we can bound the terms of the right-hand side of (3.13) for each fixed h. To show this, first fix h, p, q, r and $\epsilon \in (0, \frac{1}{10})$, then decompose the right-hand side of (3.13) into two series, one for values of k such that $|h - 2k| \leq h^{1/2+\epsilon}$ and the other for $|h - 2k| > h^{1/2+\epsilon}$. The bounds of each of these sums will be represented in the following lines as a left term of a sum and a right term, respectively.

For the case in which $r = 0$, we have through Proposition 3.2.1 and Lemma A

$$\begin{aligned}\sum_{x=1}^{\infty} E(u_x(4)) &\leq \sum_{h=1}^{\infty} (Ch^{-1/2+\epsilon})^{p+q+2}h + (Ch)(C\exp(-\gamma h^{2\epsilon})) \\ &\leq \sum_{h=1}^{\infty} C'h^{-3/2+5\epsilon} < \infty\end{aligned}$$

for a large enough $C' < \infty$. If $r > 0$, we have

$$\sum_{x=1}^{\infty} E(u_x(4)) \le \sum_{h=1}^{\infty} (Ch^{-1/2+\epsilon})^{p+q+r+3} h + (Ch)(C\exp(-\gamma h^{2\epsilon})) \le \sum_{h=1}^{\infty} C' h^{-2+6\epsilon} < \infty$$

for another large $C' < \infty$. This shows (3.7), which then completes the proof of Theorem 3.0.1.

3.3 Proof of Proposition 3.2.1

To prove Proposition 3.2.1, we rely primarily on four main lemmas, whose proofs will be reserved for Section 3.5 along with the proof of Lemma A. We shall continue the alphabetical labeling of the lemmas. For all of the lemmas, fix $\epsilon > 0$.

The first lemma shows that the jumps of the Markov chains Y_t and Z_t is unlikely to be greater than $h^{1/2+\epsilon}$ until the Markov chains reach h.

Lemma B. *Define the maximal jumps of Y_t and Z_t:*

$$M_h := \sup\{|Y_t - Y_{t-1}| : 1 \le t \le \sigma_h \wedge \omega\},$$
$$N_h := \sup\{|Z_t - Z_{t-1}| : 1 \le t \le \tau_h\}.$$

There exist two constants, $C < \infty$ and $\gamma > 0$, such that for any $h \ge 1$ and $k \ge 0$, we have

$$P(M_h > h^{1/2+\epsilon} | Y_0 = k) < C\exp(-\gamma h^{2\epsilon}),$$
$$P(N_h > h^{1/2+\epsilon} | Z_0 = k) < C\exp(-\gamma h^{2\epsilon}).$$

The next lemma are bounds on the probabilities that $\widetilde{Y}_t$ and $\widetilde{Z}_t$ enter the interval $[h, \infty)$ at exactly h.

Lemma C. *There exists a constant $C < \infty$ such that for any $h \ge 1$ and $k \ge 0$, we have*

$$P(\widetilde{\sigma}_h < \infty, \widetilde{Y}_{\widetilde{\sigma}_h} = h | Y_0 = k) < Ch^{-1/2+\epsilon}$$
$$P(\widetilde{Z}_{\widetilde{\tau}_h} = h | Z_0 = k) < Ch^{-1/2+\epsilon}$$

Next is a bound on the probability that $\widetilde{Y}_t$ does not enter $[h, \infty)$ before extinction, given that Y_0 is close to $h/2$.

Lemma D. *There exists a constant $C < \infty$ such that for any $h \geq 1$ and $k \in \left[\frac{h-h^{1/2+\epsilon}}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$,*

$$P(\widetilde{\sigma}_h = \infty | Y_0 = k) < Ch^{-1/2+\epsilon}.$$

Finally, we give upper bounds to the expectation of the hitting times $\widetilde{\tau}_h$.

Lemma E. *There exists a constant $C < \infty$ such that for any $h \geq 1$ the following holds:*

1. *For any k,*

$$E(\widetilde{\tau}_h | Z_0 = k) < Ch.$$

2. *For $k \in \left[\frac{h-h^{1/2+\epsilon}}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$,*

$$E(\widetilde{\tau}_h | Z_0 = k) < Ch^{1/2+\epsilon}.$$

Proof of Proposition 3.2.1. Using the strong Markov property of Y_t and Z_t, we arrive at the following recurrence relations for $p \geq 1$:

$$\begin{aligned} P(A_{h,p}|Y_0 = k) &= \sum_{\ell=0}^{\infty} P(\widetilde{\sigma}_h < \infty, Y_{\widetilde{\sigma}_h - 1} = h - \ell, Y_{\widetilde{\sigma}_h} = \ell | Y_0 = k) \times P(A_{h,p-1}|Y_0 = \ell), \\ \sum_{x=1}^{\infty} P(B_{x,h,p}|Z_0 = k) &= \sum_{\ell=0}^{\infty} P(Z_{\widetilde{\tau}_h - 1} = h - \ell, Z_{\widetilde{\tau}_h} = \ell | Z_0 = k) \times \left(\sum_{x=1}^{\infty} P(B_{x,h,p-1}|Z_0 = \ell)\right). \end{aligned} \tag{3.17}$$

Note that

$$\sum_{x=1}^{\infty} P(B_{x,h,0}|Z_0 = k) = \sum_{x=1}^{\infty} P(\widetilde{\tau}_h \geq x | Z_0 = k) = E(\widetilde{\tau}_h | Z_0 = k),$$

so we have the first part of the proposition when $p = 0$ by Lemma E.

Now consider the case $p = 1$. We divide the right-hand sides of (3.17) into two disjoint sums: one such that $\ell \in \left[\frac{h-h^{1/2+\epsilon}}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$ and for all other values of ℓ. From Lemmas C and D, we have for the first sum of the first series

$$\sum_{\ell: |h-2\ell| \leq h^{1/2+\epsilon}}^{\infty} P(\widetilde{\sigma}_h < \infty, Y_{\widetilde{\sigma}_h - 1} = h - \ell, Y_{\widetilde{\sigma}_h} = \ell | Y_0 = k) \times P(A_{h,0}|Y_0 = \ell) \leq \left(Ch^{-1/2+\epsilon}\right)\left(Ch^{-1/2+\epsilon}\right).$$

Also, from Lemma B, we have for the second sum

$$\sum_{\ell: |h-2\ell| > h^{1/2+\epsilon}}^{\infty} P(\widetilde{\sigma}_h < \infty, Y_{\widetilde{\sigma}_h - 1} = h - \ell, Y_{\widetilde{\sigma}_h} = \ell | Y_0 = k) \times P(A_{h,0}|Y_0 = \ell)$$

$$\leq P(M_h > h^{1/2+\epsilon} | Y_0 = k) < C \exp(-\gamma h^{2\epsilon})$$

As for the other relation, we obtain similar results using Lemmas B, C and E:

$$\sum_{\ell:|h-2\ell|\le h^{1/2+\epsilon}}^{\infty} P(Z_{\widetilde{\tau}_h-1}=h-\ell, Z_{\widetilde{\tau}_h}=\ell|Z_0=k)\times\left(\sum_{x=1}^{\infty}P(B_{x,h,0}|Z_0=\ell)\right)\le\left(Ch^{-1/2+\epsilon}\right)\left(Ch^{-1/2+\epsilon}\right),$$

$$\begin{aligned}&\sum_{\ell:|h-2\ell|>h^{1/2+\epsilon}}^{\infty} P(Z_{\widetilde{\tau}_h-1}=h-\ell, Z_{\widetilde{\tau}_h}=\ell|Z_0=k)\times\left(\sum_{x=1}^{\infty}P(B_{x,h,0}|Z_0=\ell)\right)\\&\le P(N_h>h^{1/2+\epsilon}|Z_0=k)\left(\sup_{\ell\ge 0}\sum_{x=1}^{\infty}P(B_{x,h,0}|Z_0=\ell)\right)\\&<(C\exp(-\gamma h^{2\epsilon}))(Ch).\end{aligned}$$

These inequalities yield the second part of the proposition for $p=1$. The cases of $p=2,3$ follow directly from the $p=1$ case and from the recurrence relations in (3.17).

□

3.4 Preliminary Results on $\pi(i,j)$ and $\rho(i,j)$

Before we prove the lemmas introduced in Sections 3.2 and 3.3, we first need to establish some preliminary facts about the transition kernels $\pi(i,j)$ and $\rho(i,j)$ introduced in (3.9) and (3.10) respectively. For the simple random walk ($\lambda=\frac{1}{2}$), [39] and [51] showed that the variables Y_t and Z_t followed a negative binomial distribution, which was used to great effect in [61]. While the distributions of Y_t and Z_t in the persistent case are related to negative binomial distributions, there are enough differences to warrant a more meticulous kind of analysis.

The majority of the effort shown in this section will focus more on $\pi(i,j)$, as a proof of a result for $\pi(i,j)$ will closely resemble that for $\rho(i,j)$ with minor differences. However, analogous results of both kernels will be seen.

3.4.1 Expectation and variance

Observation 3.4.1. *For each $i\ge 0$ and $t\ge 1$,*

$$E(Y_t|Y_{t-1}=i)=i,\qquad \mathrm{Var}(Y_t|Y_{t-1}=i)=2\frac{1-\lambda}{\lambda}i \tag{3.18}$$

$$E(Z_t|Z_{t-1}=i)=i+1,\qquad \mathrm{Var}(Z_t|Z_{t-1}=i)=2\frac{1-\lambda}{\lambda}(i+1) \tag{3.19}$$

This is a trivial result, since both random variables are sums of i i.i.d. copies of the same variable $\zeta_{t,i}$ with distribution given in (3.8). Still, we will make use of these calculations in the next section.

3.4.2 Log-concavity

Here we explain the concept of logarithmic-concavity for sequences, which will be used to justify the unimodality of the distributions $\pi(i,\cdot)$ and $\rho(i,\cdot)$.

Definition 3.4.2. *A nonnegative sequence $\{a_k\}_{k=0}^{\infty}$ is* log-concave *if, for every $k \geq 1$,*

$$a_k^2 \geq a_{k-1}a_{k+1}.$$

If $\{a_k\}_{k=0}^{\infty}$ is a positive sequence, then this is equivalent to the sequence of ratios $\{\frac{a_{k+1}}{a_k}\}_{k=0}^{\infty}$ being nonincreasing.

For more information on the concept of log-concave sequences, we refer to [6], [57] and [63]. For now, we present this fact:

Theorem 3.4.3 (Corollary 3.3 in [63])**.** *The convolution of two log-concave sequences is also log-concave.*

Given the distribution in (3.8), it is a straightforward exercise to show that $\{\pi(1,j)\}_{j=0}^{\infty}$ and $\{\rho(1,j)\}_{j=0}^{\infty}$ are both log-concave sequences. Thus, since $\{\pi(i,j)\}$ is a convolution of $\{\pi(i-1,j)\}$ and $\{\pi(1,j)\}$ for each $i > 1$, $\{\pi(i,j)\}$ is log-concave for any $i \geq 1$ by mathematical induction. Similarly, $\{\rho(i,j)\}$ is log-concave as well. The log-concavity feature of these transition kernels ensures unimodality in the distribution, which is the next topic of discussion.

3.4.3 Mode of $\pi(i,j)$

Since $\{\pi(i,j)\}$ is unimodal, there exists $k \geq 0$ such that for all $j \geq 0$,

$$\pi(i,j) \leq \pi(i,k). \tag{3.20}$$

It is in our interests to find exactly which values this k could take for each fixed i and λ. While the exact values could depend on λ, we have found that they do not stray very far from the expectation of the random variable found in (3.18).

Theorem 3.4.4. *Fix $\lambda \in \left(\frac{1}{2}, 1\right)$ and $i \geq 1$. Suppose there is an integer k such that (3.20) holds. Then $k \in \{i-1, i\}$.*

Note that, by the log-concavity of $\{\pi(i,j)\}$ and the resulting monotonicity of $\left\{\frac{\pi(i,j+1)}{\pi(i,j)}\right\}$, we have the following corollary:

Corollary 3.4.5.

$$\pi(i, j-1) \leq \pi(i,j) \text{ if } j \leq i-1,$$
$$\pi(i, j+1) \leq \pi(i,j) \text{ if } j \geq i.$$

Our proof of Theorem 3.4.4 will require a reinterpretation of $\pi(i,j)$. First, allow us to define the Gauss hypergeometric function as in [1]:

Definition 3.4.6. *For each a, b and c, the* Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ *is the function mapping $\{z : |z| < 1\}$ to $\mathbb{C}$ of the form*

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(m)_n = \prod_{k=0}^{n-1}(m+k)$ is the Pochhammer symbol.

Define $z := \left(\frac{1-\lambda}{\lambda}\right)^2$, and observe that $z \in (0,1)$. Then the following representation can be formulated for $i \geq 1$ and $j \geq 0$:

$$\pi(i,j) = \begin{cases} \sqrt{z}^{\,j-i}(1-\sqrt{z})^{j+i} \binom{j-1}{i-1} \; {}_2F_1(j+1, j; 1+j-i; z), & i \leq j \\ \frac{\sqrt{z}^{\,i-j}}{(1+\sqrt{z})^{i+j}} \binom{i}{i-j} \; {}_2F_1(1-j, -j; 1+i-j; z), & i > j. \end{cases} \tag{3.21}$$

There are multiple ways of representing $\pi(i,j)$ with hypergeometric functions, particularly using the Euler transformation

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b} \; {}_2F_1(c-a, c-b; c; z).$$

For the purposes of this proof, we shall use the representation in (3.21). In particular, we wish to observe the following instances of $\pi(i,j)$:

$$\pi(i,i+1) = i(1-\sqrt{z})^{2i}(\sqrt{z}-z)\ {}_2F_1\left(i+2,i+1;2;z\right) \tag{3.22}$$

$$\pi(i,i) = (1-\sqrt{z})^{2i}\ {}_2F_1\left(i+1,i;1;z\right) \tag{3.23}$$

$$\pi(i,i-1) = i\frac{\sqrt{z}}{(1+\sqrt{z})^{2i-1}}\ {}_2F_1\left(2-i,1-i;2;z\right) \tag{3.24}$$

$$\pi(i,i-2) = \frac{i(i-1)}{2}\frac{z(1+\sqrt{z})}{(1+\sqrt{z})^{2i-1}}\ {}_2F_1\left(3-i,2-i;3;z\right) \tag{3.25}$$

We recognize other transformations of the hypergeometric functions, in particular the following (incomplete) list of Gauss' contiguous relations. A note on notation: $F = {}_2F_1(a,b;c;z)$, $F(a\pm) = {}_2F_1(a\pm 1,b;c;z)$, and all other parameter changes use similar notation.

$$\begin{aligned} z\frac{ab}{c}F(a+,b+,c+) &= a(F(a+)-F) \\ &= b(F(b+)-F) \\ &= \frac{(c-b)F(b-)+(b-c+az)F}{1-z} \\ &= \frac{z}{c(1-z)}\left((c-a)(c-b)F(c+)+c(a+b-c)F\right) \end{aligned}$$

Using these contiguous relations, one can find the following equalities for every $i \geq 1$:

$${}_2F_1(i+2,i+1;2;z) = \frac{1-z}{z(2i+1)}\ {}_2F_1(i+2,i+1,1,z) - \frac{1}{z(2i+1)}\ {}_2F_1(i+1,i,1,z) \tag{3.26}$$

$$\begin{aligned} {}_2F_1(3-i,2-i;3;z) = \frac{2}{i(1+z)-2}\ {}_2F_1(2-i,1-i;2;z) \\ - \frac{(i-1)(1-z)+(2i-3)z^2}{(i-1)(i(1+z)-2)}\ {}_2F_1(3-i,2-i;2;z) \end{aligned} \tag{3.27}$$

Before moving on to the proof of Theorem 3.4.4, we need the following lemma:

Lemma 3.4.7.

1. *For all* $i,\ c > 0$,

$$\frac{{}_2F_1\,(i+2, i+1; c; z)}{{}_2F_1\,(i+1, i; c; z)} \leq \frac{1}{(1-\sqrt{z})^2}, \tag{3.28}$$

2. *For all* $i \geq 3$ *and* $c > 0$,

$$\frac{{}_2F_1\,(3-i, 2-i; c; z)}{{}_2F_1\,(2-i, 1-i; c; z)} \geq \frac{1}{(1+\sqrt{z})^2}. \tag{3.29}$$

Moreover, the left-hand side of each inequality converges to the right-hand side as $i \to \infty$.

Proof. The convergence to the right-hand side is a direct result of Theorem 2 of [31] on the asymptotic estimates of the large-parameter Gauss hypergeometric functions seen as solutions to given second-order recurrence relations, in particular on the $(+\ +\ 0)$ case. One can check that the inequalities hold for $i = 1$, and the inequalities for $i > 1$ comes from the monotonicity of the pointwise convergence. □

With (3.22)-(3.29), we can now prove Theorem 3.4.4.

Proof of Theorem 3.4.4. Since the log-concavity of $\pi(i,j)$ gives us that $\left\{\frac{\pi(i,j+1)}{\pi(i,j)}\right\}$ is nonincreasing, it is enough to show that $\frac{\pi(i,i-1)}{\pi(i,i-2)} > 1$ and $\frac{\pi(i,i+1)}{\pi(i,i)} < 1$. Using (3.22), (3.23), (3.26) and (3.28), we have the following:

$$\begin{aligned}
\frac{\pi(i,i+1)}{\pi(i,i)} &= i(\sqrt{z}-z)\frac{{}_2F_1\,(i+2,i+1;2;z)}{{}_2F_1\,(i+1,i;1;z)} \\
&= \frac{i}{2i+1}\frac{(1-z)(\sqrt{z}-z)}{z}\frac{{}_2F_1\,(i+2,i+1;1;z)}{{}_2F_1\,(i+1,i;1;z)} - \frac{i}{2i+1}\frac{\sqrt{z}-z}{z} \\
&\leq \frac{i}{2i+1}\frac{(1-z)(\sqrt{z}-z)}{z}\frac{1}{(1-\sqrt{z})^2} - \frac{i}{2i+1}\frac{\sqrt{z}-z}{z} \\
&= \frac{i}{2i+1}\left(\frac{(1-z)(\sqrt{z}-z)-(\sqrt{z}-z)(1-\sqrt{z})^2}{z(1-\sqrt{z})^2}\right) \\
&= \frac{i}{2i+1}\frac{2(\sqrt{z}-z)^2}{z(1-\sqrt{z})^2} \\
&= \frac{2i}{2i+1} < 1.
\end{aligned}$$

Also with (3.24), (3.25), (3.27) and (3.29), we get

$$\begin{aligned}
\frac{\pi(i,i-2)}{\pi(i,i-1)} &= \frac{i-1}{2}(\sqrt{z}+z)\frac{{}_2F_1\left(3-i,2-i;3;z\right)}{{}_2F_1\left(2-i,1-i;2;z\right)} \\
&= \frac{i-1}{i(1+z)-2}(\sqrt{z}+z) - \frac{(i-1)(1-z)+(2i-3)z^2}{2(i(1+z)-2)}(\sqrt{z}+z)\frac{{}_2F_1\left(3-i,2-i;2;z\right)}{{}_2F_1\left(2-i,1-i;2;z\right)} \\
&\leq \frac{i-1}{i(1+z)-2}(\sqrt{z}+z) - \frac{(i-1)(1-z)+(2i-3)z^2}{2(i(1+z)-2)}(\sqrt{z}+z)\frac{1}{(1+\sqrt{z})^2} \\
&= \frac{2(i-1)(\sqrt{z}+z)(1+\sqrt{z}) - ((i-1)(1-z)+(2i-3)z^2)\sqrt{z}}{2(i(1+z)-2)(1+\sqrt{z})} \\
&= 1 - \frac{(2i-4)(1-z)+(i-3)\sqrt{z}(1-z)+(2i-3)z^{5/2}}{2(i(1+z)-2)(1+\sqrt{z})} \\
&< 1.
\end{aligned}$$

□

We get a similar result as Theorem 3.4.4 for $\rho(i,j)$, although we must account for the generational intruder particle of the Galton-Watson process Z_t.

Theorem 3.4.8. *Fix $\lambda \in \left(\frac{1}{2},1\right)$ and $i \geq 1$. Suppose there is an integer k such that $\rho(i,j) \leq \rho(i,k)$ for all $j \geq 0$. Then $k \in \{i,i+1\}$.*

3.4.4 Upper bound for $\frac{\pi(i,j-1)}{\pi(i,j)}$ for $i<j$

While Theorem 3.4.4 provides a lower bound for $\frac{\pi(i,j-1)}{\pi(i,j)}$ when $i<j$, we now seek an upper bound for the ratio. To achieve this, we continue our analysis of hypergeometric functions, but now in the context of an existing result in the statistical study of contingency tables. We define the *noncentral hypergeometric distribution* $\text{Hyper}(M_1,M_2,N_1,N_2,\theta)$ with the following formula for $\max(0,M_1-N_2) \leq x \leq \min(N_1,M_1)$:

$$P(X=x) = \frac{\binom{N_1}{x}\binom{N_2}{M_1-x}\theta^x}{\sum_{u=\max(0,M_1-N_2)}^{\min(N_1,M_1)}\binom{N_1}{u}\binom{N_2}{M_1-u}\theta^u},$$

where $X \sim \text{Hyper}(M_1,M_2,N_1,N_2,\theta)$. While the noncentral hypergeometric distribution has no direct application in our model, we utilize an upper bound for its expectation given in line (5.2) in [40]:

$$E(X) \leq \frac{-c+\sqrt{c^2+4\theta(1-\theta)i(i-1)}}{2(1-\theta)}, \tag{3.30}$$

where $c := N_1 + N_2 - (N_1 + M_1)(1-\theta)$. Using this inequality, we have the following lemma:

Lemma 3.4.9. *Let $i < j$. Let $X \sim \text{Hyper}(M_1 = i-1, M_2 = j, N_1 = i, N_2 = j-1, \theta = \left(\frac{1-\lambda}{\lambda}\right))$. Then $E(X) < (1-\lambda)(j-1)$.*

Proof. By (3.30), $E(X) \leq \frac{-c+\sqrt{c^2+4\theta(1-\theta)i(i-1)}}{2(1-\theta)}$, where $c = i+j-1-(2i-1)(1-\theta)$. Note that $c > (2i-1)\theta$, $i-1 < 2(1-\lambda)(i-1/2) + (2\lambda-1)(j-1)$ and $i \leq j-1$ So

$$\begin{aligned} \theta i(i-1) &< \theta(j-1)((1-\lambda)(2i-1) + (2\lambda-1)(j-1)) \\ &= (1-\lambda)(j-1)(2i-1)\theta + (1-\lambda)^2 \frac{2\lambda-1}{\lambda^2}(j-1)^2 \\ &< (1-\lambda)(j-1)c + (1-\theta)(1-\lambda)^2(j-1)^2 \end{aligned}$$

Thus, $c^2 + 4\theta(1-\theta)i(i-1) < c^2 + 4(1-\theta)(1-\lambda)(j-1)c + 4(1-\theta)^2(1-\lambda)^2(j-1)^2 = (c + 2(1-\theta)(1-\lambda)(j-1))^2$. Therefore, $E(X) \leq \frac{-c+\sqrt{c^2+4\theta(1-\theta)i(i-1)}}{2(1-\theta)} < (1-\lambda)(j-1)$. □

We use this lemma to get an upper bound for $\frac{\pi(i,j-1)}{\pi(i,j)}$ when $i < j$.

Lemma 3.4.10. *$\frac{\pi(i,j-1)}{\pi(i,j)} < \frac{1}{1-\lambda}\frac{j-i}{j-1} + 1$ for all $i < j$.*

Proof.

$$\begin{aligned} \frac{\pi(i,j-1)}{\pi(i,j)} &= \frac{\sum_{k=0}^{i-1}\binom{i}{k}(1-\lambda)^k\binom{j-2}{i-k-1}\lambda^{2(i-k)}(1-\lambda)^{j-1-(i-k)}}{\sum_{k=0}^{i-1}\binom{i}{k}(1-\lambda)^k\binom{j-1}{i-k-1}\lambda^{2(i-k)}(1-\lambda)^{j-(i-k)}} \\ &= \frac{1}{1-\lambda}\frac{\sum\binom{i}{h}\binom{j-1}{i-k-1}\left(\frac{j-i+k}{j-1}\right)\left(\frac{1-\lambda}{\lambda}\right)^{2k}}{\sum\binom{i}{h}\binom{j-1}{i-k-1}\left(\frac{1-\lambda}{\lambda}\right)^{2k}} \\ &= \frac{1}{1-\lambda}\frac{j-i}{j-1} + \frac{1}{1-\lambda}\frac{1}{j-1}\frac{\sum k\binom{i}{k}\binom{j-1}{i-k-1}\left(\frac{1-\lambda}{\lambda}\right)^{2k}}{\sum\binom{i}{k}\binom{j-1}{i-k-1}\left(\frac{1-\lambda}{\lambda}\right)^{2k}} \\ &= \frac{1}{1-\lambda}\frac{j-i}{j-1} + \frac{1}{1-\lambda}\frac{1}{j-1}E(X) \\ &< \frac{1}{1-\lambda}\frac{j-i}{j-1} + 1, \end{aligned}$$

where $X \sim \text{Hyper}(M_1 = i-1, M_2 = j, N_1 = i, N_2 = j-1, \theta = \left(\frac{1-\lambda}{\lambda}\right))$. The last inequality above comes from Lemma 3.13. □

3.5 Proof of Lemmas

We now go on to prove Lemmas A-E, which will complete the proof of Theorem 3.0.1 through Proposition 3.2.1.

3.5.1 Lemma A

Proof of Lemma A. Assume that $h \geq 2$ and $k \geq 1$, and let $n = h - 2$ and $\ell = k - 1$. Note that $\pi(k, h-1-k)$ can be interpreted as the probability that the walk leaves an arbitrary site k times to the left before visiting it a total of h times, given that the site's h^{th} visit from the walk came from the left side. For each $m \in \mathbb{N}$, define K_m to be the number of downcrossings from the site given m visits to the site. We can write $K_m = \sum_{t=1}^{m} J_t$ such that for each t, $J_t = 1$ if the t^{th} visit to the site is immediately followed by a downcrossing and $J_t = 0$ otherwise. For the persistent walk, it is clear that $\{J_t\}_{t=1}^{\infty}$ is a Markov chain on the state space $\{0,1\}$ with $P(J_t = 0|J_{t-1} = 0) = P(J_t = 1|J_{t-1} = 1) = 1 - \lambda$ and $P(J_t = 0|J_{t-1} = 1) = P(J_t = 1|J_{t-1} = 0) = \lambda$. It can also be shown easily that the $\{J_t\}_{t=1}^{\infty}$ is stationary with uniform stationary distribution $\mu \equiv \frac{1}{2}$.

Using this new notation, we have the following:

$$\pi(k, h-1-k) = P\left(K_{n+1} = k | J_{n+1} = 1\right) = P\left(K_n = \ell\right).$$

Thus,

$$\sum_{k:|h-2k|>h^{1/2+\epsilon}} \pi(k, h-1-k) = P\left(|n - 2K_n| > (n+2)^{1/2+\epsilon}\right). \tag{3.31}$$

We now seek for an upper bound for the right-hand side of (3.31). To accomplish this, we use a functional central limit theorem for Markov chains from [38]. Let $f : \{0,1\} \to \mathbb{R}$ be defined as $f(x) = x$. Then $E_\mu f := \int_{\{0,1\}} f(x)\mu(dx) = \frac{1}{2}$ and $E_\mu f^2 = \frac{1}{2} < \infty$. Also, since $\{J_t\}_{t=1}^{\infty}$ is a finite Markov chain, it is uniformly ergodic. Thus, by Theorem 9 in [38], we get the following weak convergence as $m \to \infty$:

$$\sqrt{m}\left(\frac{1}{m}\sum_{t=1}^{m} J_t - E_\mu f\right) = \frac{K_m - \frac{m}{2}}{\sqrt{m}} \Rightarrow N(0, \sigma_f^2),$$

where $N(0, \sigma_f^2)$ is a normal distribution with mean 0 and variance $\sigma_f^2 > 0$. Using this central limit theorem, we can show that, for any $\gamma < 1/2$, $E\left(\exp\{\gamma\,(2K_m - m)^2\,/m\}\right)$ converges as $m \to \infty$. Hence,

$$\sup_m E\left(\exp\{\gamma\,(2K_m - m)^2\,/m\}\right) < \infty.$$

Using (3.31) and Markov's inequality, we finally arrive at our result.

□

3.5.2 Overshooting Lemma

Before proving the remaining four lemmas, we first want to establish a rather important result in the study of favorite sites of the simple walk for the case of the persistent walk. It was shown in [61] that the probabilities of the Markov chains Y_t and Z_t going past a threshold point by a given amount, conditioned on the processes reaching the threshold for the first time at this instant, is comparable to the probability that the Markov chain achieves the same amount of overshoot conditioned on the threshold being reached on the very first step. This allows one to find simple asymptotic bounds for the conditional moments of Y_t and Z_t, essential for the application of optional stopping theorems in the proofs ahead.

Here, we obtain the analogous result for the persistent case.

Lemma 3.5.1 (Overshooting Lemma)**.** *For any $0 \leq k < h \leq u$ the following overshoot bounds hold:*

$$P(Y_{\sigma_h} \geq u | Y_0 = k, \sigma_h < \infty) \leq P(Y_1 \geq u | Y_0 = h, Y_1 \geq h) = \frac{\sum_{v=u}^{\infty} \pi(h,v)}{\sum_{w=h}^{\infty} \pi(h,w)}$$

$$P(Z_{\tau_h} \geq u | Z_0 = k) \leq P(Z_1 \geq u | Z_0 = h, Z_1 \geq h) = \frac{\sum_{v=u}^{\infty} \rho(h,v)}{\sum_{w=h}^{\infty} \rho(h,w)}$$

Proof. For $1 \leq h \leq u$,

$$P(Y_{\sigma_h} \geq u | Y_0 = k, \sigma_h < \infty) = \sum_{l=0}^{h-1} P(Y_{\sigma_h - 1} = l | Y_0 = k, \sigma_h < \infty) \frac{\sum_{v=u}^{\infty} \pi(l,v)}{\sum_{w=h}^{\infty} \pi(l,w)},$$

$$P(Z_{\tau_h} \geq u | Z_0 = k) = \sum_{l=0}^{h-1} P(Z_{\tau_h - 1} = l | Z_0 = k) \frac{\sum_{v=u}^{\infty} \rho(l,v)}{\sum_{w=h}^{\infty} \rho(l,w)}.$$

Note that if the ratios of the right-hand side are bounded above by the case in which $l = h$, we'd get our desired inequalities, since the probabilities of the right-hand side partition their respective conditioned event. It is then enough to show that the ratios on the right-hand side are increasing in l.

Observe the following relations for $\pi(l,v)$:

$$\begin{aligned}
\pi(l,v)\pi(l+1,v+1) - \pi(l+1,v)\pi(l,v+1) &= \pi(l,v)\cdot(\pi(1,\cdot)*\pi(l,\cdot))(v+1) \\
&\quad - \pi(l,v+1)\cdot(\pi(1,\cdot)*\pi(l,\cdot))(v) \\
&= (1-\lambda)\pi(l,v)\pi(l,v+1) + \sum_{j=1}^{v+1}\lambda^2(1-\lambda)^{j-1}\pi(l,v+1-j)\pi(l,v) \\
&\quad - (1-\lambda)\pi(l,v)\pi(l,v+1) - \sum_{j=1}^{v}\lambda^2(1-\lambda)^{j-1}\pi(l,v-j)\pi(l,v+1) \\
&= \lambda^2(1-\lambda)^v\pi(l,0)\pi(l,v) \\
&\quad + \sum_{j=1}^{v}\lambda^2(1-\lambda)^{j-1}(\pi(l,v+1-j)\pi(l,v) - \pi(l,v-j)\pi(l,v+1))
\end{aligned}$$

The terms in the sum of the last line are nonnegative, by the log-concavity of $\pi(l,\cdot)$. Thus, $\frac{\pi(l+1,v)}{\pi(l,v)} \leq \frac{\pi(l+1,v+1)}{\pi(l,v+1)}$. Similarly, $\frac{\rho(l+1,v)}{\rho(l,v)} \leq \frac{\rho(l+1,v+1)}{\rho(l,v+1)}$.
So, for all $v < w$,

$$\pi(l,v)\pi(l+1,w) \geq \pi(l+1,v)\pi(l,w)$$

$$\rho(l,v)\rho(l+1,w) \geq \rho(l+1,v)\rho(l,w)$$

Hence, for all $0 \leq l < h \leq u$,

$$\sum_{v=h}^{\infty}\pi(l,v)\sum_{w=u}^{\infty}\pi(l+1,w) \geq \sum_{v=h}^{\infty}\pi(l+1,v)\sum_{w=u}^{\infty}\pi(l,w),$$

$$\sum_{v=h}^{\infty}\rho(l,v)\sum_{w=u}^{\infty}\rho(l+1,w) \geq \sum_{v=h}^{\infty}\rho(l+1,v)\sum_{w=u}^{\infty}\rho(l,w).$$

Thus, $\left\{\frac{\sum_{w=u}^{\infty}\pi(l,w)}{\sum_{v=h}^{\infty}\pi(l,v)}\right\}_{l=0}^{h}$ is an increasing sequence, and so is $\left\{\frac{\sum_{w=u}^{\infty}\rho(l,w)}{\sum_{v=h}^{\infty}\rho(l,v)}\right\}_{l=0}^{h}$. This completes the proof. □

Using the Overshooting Lemma, we obtain the following set of inequalities:

Corollary 3.5.2. *There exist constants C_1, C_2, C_3 and C_4 such that for any $0 \leq k < h$,*

$$E\left(Y_{\sigma_h}|Y_0=k,\sigma_h<\infty\right) \leq \frac{\sum_{v=h}^{\infty}\pi(h,v)v}{\sum_{w=h}^{\infty}\pi(h,w)} \leq h + C_1 h^{1/2} \tag{3.32}$$

$$E\left(Y_{\sigma_h}^2|Y_0=k,\sigma_h<\infty\right) \leq \frac{\sum_{v=h}^{\infty}\pi(h,v)v^2}{\sum_{w=h}^{\infty}\pi(h,w)} \leq h^2 + C_2 h^{3/2} \tag{3.33}$$

$$E\left(Z_{\tau_h}|Z_0=k\right) \leq \frac{\sum_{v=h}^{\infty}\rho(h,v)v}{\sum_{w=h}^{\infty}\rho(h,w)} \leq h + C_3 h^{1/2} \tag{3.34}$$

$$E\left(Z_{\tau_h}^2|Z_0=k\right) \leq \frac{\sum_{v=h}^{\infty}\rho(h,v)v^2}{\sum_{w=h}^{\infty}\rho(h,w)} \leq h^2 + C_4 h^{3/2} \tag{3.35}$$

3.5.3 Lemma B

To begin the proof of Lemma B, we start with an application of Corollary 3.5.2.

Sublemma 3.5.3. *There exists a constant $C < \infty$ such that for any $h \geq 1$ and $k \geq 0$,*

$$E(\sigma_h \wedge \omega | Y_0 = k) < Ch^2.$$

Proof. Let $\mathcal{F}_t$ be the sigma algebra generated by the set $\{Y_s\}_{s=0}^t$. Then

$$E\left(Y_{t+1}^2 - 2\frac{1-\lambda}{\lambda}\sum_{s=0}^{t}Y_s - \left[Y_t^2 - 2\frac{1-\lambda}{\lambda}\sum_{s=0}^{t-1}Y_s\right] \middle| \mathcal{F}_t\right) = Var(Y_{t+1}|\mathcal{F}_t) - 2\frac{1-\lambda}{\lambda}Y_t = 0,$$

since $Var(\zeta_{t+1,i}) = 2\frac{1-\lambda}{\lambda}$ for any i. So $Y_t^2 - 2\frac{1-\lambda}{\lambda}\sum_{s=0}^{t-1}Y_s$ is a martingale. Thus, by the Optional Stopping Theorem, using stopping time $\sigma_h \wedge \omega$, we have

$$k^2 = E\left(Y_{\sigma_h\wedge\omega}^2 - 2\frac{1-\lambda}{\lambda}\sum_{s=0}^{\sigma_h\wedge\omega-1}Y \middle| Y_0 = k\right) \leq E(Y_{\sigma_h}^2|Y_0=k) - 2E(\sigma_h\wedge\omega|Y_0=k),$$

since $Y_s \geq 1$ for $s < \omega$ w.p.1. Therefore,

$$2E(\sigma_h\wedge\omega|Y_0=k) \leq E(Y_{\sigma_h}^2|Y_0=k,\sigma_h<\infty)P(\sigma_h<\infty|Y_0=k) - k^2 < C_2h^2,$$

by (3.33) of Corollary 3.5.2. This completes the proof. □

We state the following inequality on the tail probabilities of sums of i.i.d. random variables, which is proven in [65].

Sublemma 3.5.4 (Exponential Kolmogorov Inequality)**.** *Let ξ_j, $j \geq 1$, be i.i.d. random variables with $E\left(e^{\theta|\xi_j|}\right) < \infty$ for some $\theta > 0$ and $E(\xi_j) = 0$. Then for any $N > 0$ and $n \in \mathbb{N}$,*

$$P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\xi_j\right| > N\right) \leq e^{-\theta N}\left(E\left(e^{\theta\xi_j}\right)^n + E\left(e^{-\theta\xi_j}\right)^n\right)$$

Let $\xi_j = \zeta_j - 1$, where $P(\zeta_j = x) = \begin{cases} 1-\lambda & \text{if } x = 0 \\ \lambda^2(1-\lambda)^{x-1} & \text{if } x \geq 1 \end{cases}$. Then, for any t,

$$\begin{aligned} E(e^{t\xi_1}) &= (1-\lambda)e^{-t} + \frac{\lambda^2}{1-(1-\lambda)e^t} \\ &= \frac{(1-\lambda)e^{-t} - 1 + 2\lambda}{1-(1-\lambda)e^t} \\ &= \frac{2e^{-t} - e^{-2t} - 2(1-\lambda) + (1-\lambda)e^{-t} + e^{-2t} - 2e^{-t} + 1}{1-(1-\lambda)e^t} \\ &= 2e^{-t} - e^{-2t} + \frac{e^{-2t} - 2e^{-t} + 1}{1-(1-\lambda)e^t}. \end{aligned}$$

Note that for fixed t, the formula in the final line decreases with an increase in λ, so for $\lambda \in [\frac{1}{2}, 1)$, $E(e^{t\zeta_1})$ is maximized at $\lambda = \frac{1}{2}$, which is the simple walk case. So, by [61], assuming that $\lambda \in \left[\frac{1}{2}, 1\right)$, there is a constant $\theta_0 > 0$ such that for all $\theta \in [0, \theta_0)$, $E(e^{\theta\xi_1}) < e^{2\theta^2}$ and $E(e^{-\theta\xi_1}) < e^{2\theta^2}$. Using the Exponential Kolmogorov Inequality and choosing $\theta = N/(4n)$, we obtain the following:

Sublemma 3.5.5. *There is a constant such that for any $N > 0$ and $n \in \mathbb{N}$ satisfying $N/(4n) < \theta_0$,*

$$P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(\zeta_i - 1)\right| > N\right) \leq 2\exp(-N^2/(8n)).$$

Proof of Lemma B. We prove the first inequality here in detail. The proof of the second inequality is similar and is left for the reader. Choose $0 < \gamma < \frac{1}{16}$.

$$\begin{aligned} P(M_h > h^{1/2+\epsilon}|Y_0 = k) \leq & P(M_h > h^{1/2+\epsilon}, \sigma_h \wedge \omega \leq h^2\exp(\gamma h^{2\epsilon})|Y_0 = k) \\ & + P(\sigma_h \wedge \omega > h^2\exp(\gamma h^{2\epsilon})|Y_0 = k) \end{aligned}$$

For the first term on the right-hand side, we represent the Markov chain Y_t as the sum of i.i.d. random variables,

$$Y_{t+1} = \sum_{j=1}^{Y_t}\zeta_{t+1,j},$$

in order to obtain the following:

$$\begin{aligned}
&P(M_h > h^{1/2+\epsilon}, \sigma_h \wedge \omega \le h^2 \exp(\gamma h^{2\epsilon}) \\
&\le P\left(\max\left\{\max_{1\le j\le h}\left|\sum_{i=1}^{j}(\zeta_{t,i}-1)\right| : 1 \le t \le h^2\exp(\gamma h^{2\epsilon})\right\} > h^{1/2+\epsilon}\right) \\
&= 1 - \left(1 - P\left(\max_{1\le j\le h}\left|\sum_{i=1}^{j}(\zeta_{t,i}-1)\right| > h^{1/2+\epsilon}\right)\right)^{h^2\exp(\gamma h^{2\epsilon})} \\
&\le h^2\exp(\gamma h^{2\epsilon}) P\left(\max_{1\le j\le h}\left|\sum_{i=1}^{j}(\zeta_{t,i}-1)\right| > h^{1/2+\epsilon}\right).
\end{aligned}$$

Note that the last inequality above comes from the analytical fact that $1 - na \le (1-a)^n$ for $0 \le a \le 1$ and $n > 1$. From Sublemma 3.5.5, we get

$$P\left(\max_{1\le j\le h}\left|\sum_{i=1}^{j}(\zeta_{t,i}-1)\right| > h^{1/2+\epsilon}\right) \le 2\exp(-h^{2\epsilon}/8).$$

Hence, with $\gamma < \frac{1}{16}$, there is a constant $C > 0$ such that

$$P(M_h > h^{1/2+\epsilon}, \sigma_h \wedge \omega \le h^2\exp(\gamma h^{2\epsilon}) \le 2h^2\exp\left(\left(\gamma - \frac{1}{8}\right)h^{2\epsilon}\right) \le C\exp(-\gamma h^{2\epsilon}).$$

For the second term, we have from Sublemma 3.5.3 that there is a constant C such that

$$E(\sigma_h \wedge \omega)|Y_0 = k) \le Ch^2.$$

We get the following inequality after applying Markov's inequality:

$$P(\sigma_h \wedge \omega > h^2\exp(\gamma h^{2\epsilon})|Y_0 = k) \le C\exp(-\gamma h^{2\epsilon}).$$

Since both terms are bounded above by scalar multiples of $\exp(-\gamma h^{2\epsilon})$, the result follows. □

3.5.4 Lemma C

In order to prove Lemma C, we need the following sublemma:

Sublemma 3.5.6. *There exists a constant C s.t. for any $h \ge 1$ and $\ell \in \left[\frac{h-h^{1/2+\epsilon}}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$,*

$$\frac{\pi(\ell, h-\ell)}{\sum_{m\ge h-\ell}\pi(\ell, m)} < Ch^{-1/2+\epsilon}$$

Proof. We make use of the inequalities found in Corollary 3.4.5. We shall split this proof into two cases. First, assume $\ell \in \left[\frac{h}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$. Then $\ell > h-\ell$. Let $\{\zeta_k\}_{k=1}^{\ell}$ be a set of i.i.d. random variables with the same distribution as in (3.8). Recall that $E(\zeta_k) = 1$ and $\mathrm{Var}(\zeta_k) = 2\frac{1-\lambda}{\lambda}$. Let $\sigma^2 = \mathrm{Var}(X)$, and let Φ be the standard normal cdf. Then

$$\begin{aligned}
\pi(\ell, h-\ell) &\leq \pi(\ell, \ell-1) = P\left(\sum_{k=1}^{\ell} \zeta_k = \ell - 1\right) \\
&= \lim_{t\to 1^-} P\left(\sum_{k=1}^{\ell} \zeta_k \leq \ell - 1\right) - P\left(\sum_{k=1}^{\ell} \zeta_k \leq \ell - t\right) \\
&= \lim_{t\to 1^-} P\left(\frac{\sum_{k=1}^{\ell}(\zeta_k - 1)}{\sqrt{\ell}\sigma} \leq -\frac{1}{\sqrt{\ell}}\right) - P\left(\frac{\sum_{k=1}^{\ell}(\zeta_k - 1)}{\sqrt{\ell}\sigma} \leq -\frac{t}{\sqrt{\ell}}\right) \\
&\leq \lim_{t\to 1^-} \Phi\left(-\frac{1}{\sqrt{\ell}\sigma}\right) - \Phi\left(-\frac{t}{\sqrt{\ell}\sigma}\right) + \frac{C_1}{\sqrt{\ell}} \\
&= \frac{C_1}{\sqrt{\ell}}.
\end{aligned}$$

The last inequality above comes from the Berry-Esseen inequality. By the central limit theorem applied to $\{\zeta_k\}_{k=1}^{\infty}$, $\lim_{\ell\to\infty} \sum_{m\geq \ell} \pi(\ell, m) = \frac{1}{2}$. So there is a constant $C_2 > 0$ s.t.

$$\sum_{m\geq h-\ell} \pi(\ell, m) \geq \sum_{m\geq \ell} \pi(\ell, m) \geq C_2.$$

By these inequalities, we have our result for $\ell \in \left[\frac{h}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$.

We now continue with the $\ell \in \left[\frac{h-h^{1/2+\epsilon}}{2}, \frac{h}{2}\right]$ case for the proof of the lemma. Note that $\ell < h - \ell$. Let $k = \lfloor h - \ell + h^{1/2-\epsilon} \rfloor$. Then we get the following inequalities, which will be described in more detail below:

$$\begin{aligned}
\frac{\pi(\ell,h-\ell)}{\sum_{m\geq h-\ell}\pi(\ell,m)} &\leq (k-h+\ell+1)^{-1}\frac{\pi(\ell,h-\ell)}{\pi(\ell,k)} \\
&\leq (k-h+\ell+1)^{-1}\left(\frac{\pi(\ell,k-1)}{\pi(\ell,k)}\right)^{k-h+\ell} \\
&< (k-h+\ell+1)^{-1}\left(1+\frac{1}{1-\lambda}\frac{k-\ell}{k-1}\right)^{k-h+\ell} \\
&\leq h^{-1/2+\epsilon}\left(1+\frac{1}{1-\lambda}\frac{h^{1/2+\epsilon}+h^{1/2-\epsilon}}{h/2+h^{1/2-\epsilon}-1}\right)^{h^{1/2-\epsilon}} \\
&\leq h^{-1/2+\epsilon}\left(1+\frac{2}{1-\lambda}h^{-1/2+\epsilon}\right)^{h^{1/2-\epsilon}} \\
&\leq \exp\left(\frac{2}{1-\lambda}\right)h^{-1/2+\epsilon}.
\end{aligned}$$

The first inequality comes from Corollary 3.4.5. The second comes from the log-concavity of $\{\pi(\ell,j)\}$. The third is Lemma 3.4.10. The fourth used $k \leq h-\ell+h^{1/2-\epsilon}$ and $\ell \geq (h-h^{1/2+\epsilon})/2$. The fifth is due to the convergence of the base of the exponent above, as well as the monotonicity of that convergence. The final inequality relies on the exponential convergence of the power, as well as the monotonicity of that convergence.

□

Proof of Lemma C. We provide details of the proof of the first inequality and leave the similar details of the second inequality for the reader. First, observe that

$$\begin{aligned}
&P(\widetilde{\sigma}_h<\infty, \widetilde{Y}_{\widetilde{\sigma}_h}=h|Y_0=k) \\
&\qquad = \sum_{\ell=0}^{\infty} P(\widetilde{\sigma}_h<\infty, Y_{\widetilde{\sigma}_h-1}=\ell, Y_{\widetilde{\sigma}_h}=h-\ell|Y_0=k)
\end{aligned}$$

We split the infinite sum above into two sums, one for values of ℓ inside the interval $\left[\frac{h-h^{1/2+\epsilon}}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$ and the other for values of ℓ outside the interval. For the first sum, we use Sublemma 3.5.6 to obtain

$$\sum_{\ell:|h-2\ell|\le h^{1/2+\epsilon}} P(\widetilde{\sigma}_h < \infty, Y_{\widetilde{\sigma}_h-1} = \ell, Y_{\widetilde{\sigma}_h} = h-\ell | Y_0 = k)$$
$$= \sum_{\ell:|h-2\ell|\le h^{1/2+\epsilon}} P(\widetilde{\sigma}_h < \infty, Y_{\widetilde{\sigma}_h-1} = \ell | Y_0 = k)\frac{\pi(\ell, h-\ell)}{\sum_{m=h-\ell}^{\infty}\pi(\ell, m)}$$
$$\le Ch^{-1/2+\epsilon}.$$

For the second sum, we use Lemma B to obtain

$$\sum_{\ell:|h-2\ell|>h^{1/2+\epsilon}} P(\widetilde{\sigma}_h < \infty, Y_{\widetilde{\sigma}_h-1} = \ell, Y_{\widetilde{\sigma}_h} = h-\ell | Y_0 = k)$$
$$P(M_h > h^{1/2+\epsilon} | Y_0 = k) < C\exp(-\gamma h^{2\epsilon}).$$

Therefore, we arrive at the result. □

3.5.5 Lemma D

For Lemma D, we require the following sublemma based on Corollary 3.5.2:

Sublemma 3.5.7. *There exists a constant $C < \infty$ such that for any $0 \le k < h$,*

$$P(\sigma_h = \infty | Y_0 = k) < \frac{h-k}{h} + Ch^{-1/2}.$$

Proof. Let $\mathcal{F}_t$ be the sigma algebra generated by the set $\{Y_s\}_{s=0}^t$. Then

$$E(Y_{t+1}|\mathcal{F}_t) = E(Y_{t+1}|Y_t) = 1 \cdot Y_t,$$

since $E(\zeta_{t+1,i}) = 1$ for any i. So Y_t is a martingale. Thus, by the Optional Stopping Theorem, using stopping time $\sigma_h \wedge \omega$, we have

$$k = E(Y_{\sigma_h\wedge\omega}|Y_0 = k) = E(Y_{\sigma_h}|Y_0 = k, \sigma_h < \infty)P(\sigma_h < \infty|Y_0 = k) \le (h+C_1h^{1/2})P(\sigma_h < \infty|Y-0 = k),$$

by (3.32) of Corollary 3.5.2. This completes the proof. □

Proof of Lemma D. Note that by Sublemma 3.5.7 and Lemma B, we get for two constants C_1, C_2,

$$P(\widetilde{\sigma}_h = \infty | Y_0 = k) \leq P(\widetilde{\sigma}_h = \infty, M_h \leq h^{1/2+\epsilon} | Y_0 = k) + P(M_h > h^{1/2+\epsilon} | Y_0 = k)$$
$$\leq P(\sigma_{(h+h^{1/2+\epsilon})/2} = \infty | Y_0 = k) + P(M_h > h^{1/2+\epsilon} | Y_0 = k)$$
$$\leq C_1 h^{-1/2} + C_2 \exp(-\gamma h^{2\epsilon})$$

for values of $k \in \left[\frac{h-h^{1/2+\epsilon}}{2}, \frac{h+h^{1/2+\epsilon}}{2}\right]$. □

3.5.6 Lemma E

In order to prove Lemma E, we first need upper bounds for the moments of τ_h.

Sublemma 3.5.8. *There exists a constant $C < \infty$ such that for any $0 \leq k < h$,*

$$E(\tau_h | Z_0 = k) < (h - k) + Ch^{1/2}.$$

Proof. Let $\mathcal{G}_t$ be the sigma algebra generated by the set $\{Z_s\}_{s=0}^t$. Then

$$E(Z_{t+1} - (t+1) | \mathcal{G}_t) = E(Z_{t+1} | Z_t) - (t+1) = Z_t - t,$$

since $E(\zeta^*_{t+1,i}) = 1$ for any i. So $Z_t - t$ is a martingale. Thus, by the Optional Stopping Theorem, using stopping time τ_h, we have

$$E(\tau_h | Z_0 = k) = E(Z_{\tau_h} | Z_0 = k) - k \leq h - k + C_3 h^{1/2},$$

by (3.34) of Corollary 3.5.2. This completes the proof. □

Sublemma 3.5.9. *There exists a constant $C < \infty$ such that for any $0 \leq k < h$,*

$$E(\tau_h^2 | Z_0 = k) < Ch^2.$$

Proof. Let $\mathcal{G}_t$ be the sigma algebra generated by the set $\{Z_s\}_{s=0}^t$. Then

$$E((t+1)^2 - 2(t+1)Z_{t+1} | \mathcal{G}_t) = (t+1)^2 - 2(t+1) - 2tZ_t - 2Z_t = t^2 - 2tZ_t + (1 - 2Z_t) \leq t^2 - 2tZ_t,$$

since $Z_t \geq 1$ for any t. So $t^2 - 2tZ_t$ is a supermartingale. Thus, by the Optional Stopping Theorem, using stopping time τ_h, as well as the Cauchy-Schwarz Inequality, we have

$$E(\tau_h^2 | Z_0 = k) \leq 2E(\tau_h Z_{\tau_h} | Z_0 = k) \leq 2\sqrt{E(\tau_h^2 | Z_0 = k)}\sqrt{E(Z_{\tau_h}^2 | Z_0 = k)}.$$

Therefore, by (3.35) of Corollary 3.5.2,

$$E(\tau_h^2|Z_0 = k) \leq 4E(Z_{\tau_h}^2|Z_0 = k) \leq 4C_4 h^2.$$

□

Proof of Lemma E.

$$E(\widetilde{\tau}_h|Z_0 = k) = E(\widetilde{\tau}_h \mathbb{I}_{(N_h \leq h^{1/2+\epsilon})}|Z_0 = k) + E(\widetilde{\tau}_h \mathbb{I}_{(N_h > h^{1/2+\epsilon})}|Z_0 = k).$$

Note that $\widetilde{\tau}_h \mathbb{I}_{(N_h \leq h^{1/2+\epsilon})} \leq \tau_{(h+h^{1/2+\epsilon})/2}$ and $\widetilde{\tau}_h \leq \tau_h$ w.p.1. Thus, with the Cauchy-Schwarz Inequality, we get

$$E(\widetilde{\tau}_h|Z_0 = k) \leq E(\tau_{(h+h^{1/2+\epsilon})/2}|Z_0 = k) + \sqrt{E(\tau_h^2|Z_0 = k)}\sqrt{P(N_h > h^{1/2+\epsilon}|Z_0 = k)}.$$

Through Lemma B and Sublemmas 3.5.8 and 3.5.9, we arrive at the result. □

CHAPTER 4. FUTURE DIRECTIONS IN RESEARCH

We close this thesis on a brief discussion of possible directions in future extensions of the preceding main results.

4.1 Range of the Frog Model

Conditions for recurrence and transience have been found for the frog model on a wide variety of graphs with certain underlying processes, see [18, 27, 36, 34, 59] for examples. When a frog model contains uninhabited regions, such as the case for transience in much of these models, a natural question to ask is what shape does the long-term range of the frog model take.

As an example of a possible direction in this line of inquiry, it was shown in [36] that the frog model on the infinite rooted d-ary tree, with the simple nearest-neighbor random walk serving as the underlying process, is transient for $d \geq 5$. The idea of the long-term range could be extended onto that model in the following way. Let $g_n(\omega)$ be the number of sites on the n^{th} generation of the tree that eventually get visited by an active frog. What happens with the sequence $d^{-n} g_n$ as $n \to \infty$?

A related problem to look into would be that of the short-term range of frog models, even those that are recurrent. For each time step, what is the distribution of the range? What is the limiting distribution of the range as time passes? How does this distribution compare to that of the long-term range for transient frog models?

4.2 Favorite Sites

As widely studied as the typical favorite site has been on various processes, not much is known about the set of favorite sites as a whole outside of the simple random walk case. One hope we have for our proof of Theorem 3.0.1 is that it would motivate study in the size of the set of favorite sites for other processes, even those with non-Markovian properties.

One possibility of extension is the persistent random walk on random environment, such as in [55] and [60]. Another possibility is that we keep the transition probabilities deterministic, but changing over time and possibly converging to extreme values such at $\frac{1}{2}$ or 1. The main problem with either extension is the possibility of losing the log-concavity and hypergeometric representations of the transition kernels for the case of the fixed parameter λ in Section 3.4.

While on the subject of Section 3.4, we believe that the results of the transition kernels in this section could prove useful in the local time analysis of persistent random walks beyond the topic of favorite sites. The parametric asymptotics of hypergeometric functions seen in [31] prove to be useful tools in understanding the distribution of the Markov chain associated with the local time.

Finally, for the persistent walk featured in this thesis, we wish to find the probability that $f(3) < \infty$, and thus either confirm or disprove the analogue of the conjecture in [21] for the persistent case. At this time in writing, it has been reported that $f(3) = \infty$ w.p.1 in the simple walk case ([17]), which would disprove the conjecture made by Erdős and Révész.

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