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Spider walk in a random environment

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Spider walk in a random environment

by

Tetiana Takhistova

A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Applied Mathematics

Program of Study Committee:
Alexander Roitershtein, Major Professor
Arka P. Ghosh
Scott W. Hansen

The student author and the program of study committee are solely responsible for the content of this thesis. The Graduate College will ensure this thesis is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University
Ames, Iowa
2017

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DEDICATION

I would like to dedicate this thesis to my brother Volodymyr and my mother Svetlana without whose support I would not have been able to complete this work. I would also like to thank my friends and family for their loving guidance and financial assistance during the writing of this work.
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ABSTRACT

We analyze a random process in a random media modeling the motion of DNA nanomechanical walking devices. We consider a molecular spider restricted to a well-defined one-dimensional track and study its asymptotic behavior in an i.i.d. random environment. The spider walk is a continuous time motion of a finite ensemble of particles on the integer lattice with the jump rates determined by the environment. The particles mutual location must belong to a given finite set of configurations $L$, and the motion can be alternatively described as a random walk on the ladder graph $\mathbb{Z} \times L$ in a stationary and ergodic environment. Our main result is an annealed central limit theorem for this process. We believe that the conditions of the theorem are close to necessary.
CHAPTER 1. DNA NANOMECHANICAL DEVICES AND MOLECULAR SPIDERS

1.1 Overview of the thesis

The organization of the thesis is as follows. In Chapter 1 we review the literature on molecular spiders. In Section 1.2 we survey applications of biological molecular motors and in Section 1.3 we present a general mathematical framework to model molecular spiders. In Sections 1.4 and 1.5 we discuss one-dimensional models in more details. In Section 1.4 we discuss basic techniques that have been developed to study the asymptotic behavior of one-dimensional spiders. The main topic of this thesis, namely spider random walk in a random environment (SRWRE) on the integer lattice $\mathbb{Z}$ is presented in Section 1.5. The model was introduced in [37], where recurrence and transience criteria along with a law of large numbers for the model have been obtained. A formal description of the model is given Section 1.5.1, and previous results, obtained in [37], are presented in Section 1.5.2. In Section 1.5.3 we state the main result of this thesis, a central limit theorem for the one-dimensional SRWRE model in the ballistic regime. The result is stated as Theorem 1.5.4. The proof of Theorem 1.5.4 is deferred to Chapter 4. In Chapter 2 we review basic limit theorems for the classical random walk in a random environment (RWRE) on $\mathbb{Z}$. Chapter 3 contains a necessary background on random walks in a random environment on strips, a model which is more general than and includes as a particular case both the classical RWRE and SRWRE. Although the dependence of the asymptotic behavior of the general model on the random environment is not known in an explicit form needed to deduce our main result, abstract contractions introduced in Chapter 4 serve as major ingredients of our proofs in Chapter 4. The thesis is concluded with Chapter 5 which includes a brief discussion of possible directions for the future research.
1.2 Introduction

Biological molecular motors are of fundamental importance for a variety of cell and tissue level processes [74]. Nanomotors such as polymerases move along DNA templates in assembling messenger RNA macromolecules, while micromotors such as proteins of the myosin family are responsible for actin-based cell motility and the transport of cargo inside cells [39]. Identifying the biochemical control mechanisms regulating such biological motor activities is the subject of current research activity in cellular and molecular biology [94], and different mathematical approaches have been employed in elucidating possible mechanisms at work [67, 93].

Research in biological motors in conjunction with recent advances in DNA nanofabrication technology have spurred a lot of interest in biomimetic nanomotor design and DNA-based devices, such as nanomechanical switches and DNA templates for the growth of semiconductor nanocrystals to name a few [96]. Research activity in this area has been focused on designing and controlling dynamic DNA nanomachines that can be activated by and respond to specific chemical signals in their environment, thus expanding on the biochemical paradigm of eukaryotic and prokaryotic cells [3, 31, 47, 60, 66, 69, 81]. Potential applications of such synthetic molecular machinery include DNA-based computing and engineered DNA motors designed for intelligent drug delivery among other in vivo therapeutic applications [2, 10, 40, 71, 86].

Currently, there exist two types of molecular designs implementing DNA-based walking devices. Both designs are based on control mechanisms that rely on nucleic acid hybridization, and the corresponding molecular constructs are often referred to as molecular spiders [62, 71, 75]. In the first implementation approach, devised by Sherman & Seeman [83], the walking device consists of two double helical domains (the device legs) connected by flexible linker regions. The construct is held on a self-assembled, one-dimensional path by DNA set strands with nucleic acid domains complementary to molecular imprints on the device legs and the substrate. In this context, the detachment of a leg from the path during a walk cycle is mediated by the removal of the set strand through a hybridization reaction [81, 83].

A more recent molecular design by Pei et al. [71] does not require the presence of interface strands in that it allows for each device leg to be directly attached to the substrate through
Watson-Crick base pair formation. Leg detachment during the walk cycle is controlled by the cleavage activity of nucleic acid domains imprinted on the leg. This latter attribute of the system leads to a random walk of the device on the substrate, dictated by the stochastic events of leg detachment and relocation. Experimental investigations of the asymptotic properties of such random walks are of great interest in the experimental community [70, 71, 95]. In what follows we refer to this class of random walks as a \textit{(molecular) spider random walk}.

1.3 Spider random walk: general mathematical framework

The long-term dynamics of spider random walks has been first investigated mathematically by Antal \textit{et al.} [7, 9], who have derived explicitly the mean velocity and the diffusion coefficient of the walker under specific assumptions on leg transition rates. The underlying random walk in [7] is the \textit{excited random walk} [14], which allows the authors to model a molecular spider with memory. The underlying random walk in [9] is a simple random walk on the integer lattice. The one-dimensional biped model of [9] was generalized by Ben-Ari \textit{et al.} in [12] to include asymmetric transition rates. For a general discussion on asymmetric spiders see, for instance, [69] and references therein. Gallesco \textit{et al.} introduced in [36] a mathematical model of molecular spider random walk on a general graph. In [37] the authors considered a spider \textit{random walk in random environment}. Various aspects of two-dimensional spiders were investigated in [31]. The work of [62, 69, 75] is concerned with collective behavior of several spider walkers. Two-dimensional spider walk is the topic of [8, 67, 82].

The following description of a general spider random walk is adopted from [36] and [12]. Spider random walks are systems of weakly interacting particles moving together on a given connected graph $G = (E, V)$ in continuous time. The particles move independently according to the law of an underlying random walk on $G$, provided the movement does not violate certain specific restrictions on their relative position. We refer to these particles as \textit{spider legs} and to the entire particle system as a \textit{spider}. The typical constraint on the configuration of the legs is a fixed upper bound for the maximal possible distance between any two legs of the spider. If a move of a particle attempts to violate the constraint, it is ignored and the spider stays put at its current position until one of the legs performs a feasible move.
The underlying random walk is a continuous-time process, which in general may be controlled, self-interacting or in random media, and thus it does not have to be Markovian. In [7] and [12] the underlying random walk is a usual nearest-neighbor random walk on the integer lattice \( \mathbb{Z} \), in [9] it is a 1-excited random walk (ERW) on \( \mathbb{Z} \) introduced in [14], and in [37] it is a random walk in random environment (RWRE) on \( \mathbb{Z} \) (see for instance [97] for a survey on RWRE). In [36] the authors considered a general graph \( G \) and a spider with leg relocation dynamics evolving according to the law of a nearest-neighbor symmetric random walk on \( G \). Most of the work in [36] focuses on recurrence and transience criteria and on comparing the spider random walk to the simple nearest-neighbor random walk on the same tree. Some interesting partial results regarding the speed of the spider on transient graphs are also proved in [36].

In [12] the authors considered an asymmetric bipedal spider, sometimes referred to as a molecular biped in the following, with in general different transition mechanisms (underlying random walks) associated with each leg. The main results in [12] are a strong law of large numbers and a functional central limit theorem for the location of the molecular biped with explicit expressions for the asymptotic velocity and limiting variance.

The existence of a law of large numbers and a functional central limit theorem for the model follows from the general theory of regenerative processes. However, the shortcoming of this purely probabilistic approach lies in the quantitative analysis. The expression it provides for the variance in the central limit theorem does not appear to be useful for the actual computation or estimation of the variance, which is crucial for applications (see, e.g., [70, 95]). Hence, in [12] the focus is on the explicit computation of the asymptotic variance and generalize the results of Antal et al. [9]. The latter are based on the existence of various symmetries in the definition of the underlying random walk, whereas our analysis relies on a different approach and is not restricted by such assumptions. In the following, we discuss in some detail some aspects of the mathematical content of [12].

1.4 Molecular biped in dimension one

Throughout the rest of the thesis we are concerned with spider random walks on \( \mathbb{Z} \). In this section we consider a continuous time random walk modeling the motion of a DNA molecular
biped on a one-dimensional walking path [7, 9, 12]. The legs of the biped move on the integer lattice representing the nucleic acid binding domains imprinted on the path. The waiting time for each leg is exponentially distributed. In general, the system is characterized by the following parameters: (a) the transition rate probabilities corresponding to the relocation of each leg (two legs and two possible movement directions) and (b) feasible (allowed) configurations of the legs. The existence of the feasible (and hence, in general, also prohibited) configurations is a reflection of mechanical constraints imposed by the design of the molecular construct, whereas the transition rate probabilities for leg movements encode information on the interactions of the legs with the substrate path.

To illustrate the general model, throughout the rest of this section we will consider the (asymmetric) one-dimensional biped studied in [12]. Let $\alpha$ denote the transition rate for the left leg moving to the left and $\beta$ be the corresponding transition rate for the left leg moving to the right. Similarly, let $\lambda$ and $\mu$ be the transition rate probabilities for the right leg moving to the left/right, respectively. The mechanical constraint is that the right leg is always between 0 and $L$ units to the right of the left leg, where $L \in \mathbb{N}$ is some fixed parameter. Whenever a clock ticks, an attempt to move is made by the corresponding leg in the corresponding direction. The attempt succeeds if the new leg configuration satisfies the mechanical constraint. We denote the position of the left and right leg of the biped at time $t \in \mathbb{R}_+$ by $X^{(1)}(t)$ and $X^{(2)}(t)$,
respectively. Note that neither $X^{(1)}$ nor $X^{(2)}$ is a Markov chain, as they interact weakly through the mechanical constraint $L$. However, the pair $(X^{(1)}, X^{(2)})$ is a Markov chain.

Figures 1.2 and 1.3 show computer simulations of the random walk above, performed using the Gillespie algorithm. Figure 1.2 shows individual realizations (for different transition rates) of the barycenter trajectory of a DNA biped, whereas Figure 1.3 shows the collective dynamics of a population of independent and identical DNA bipeds in different cases representing different transition rates.

1.4.1 The regenerative viewpoint

Let $Y(t) = X^{(2)}(t) - X^{(1)}(t)$ denote the process corresponding to the distance between the left and the right leg. Then $Y$ is a pure birth and death Markov chain on $\{0, \ldots, L\}$ with rates $x = \alpha + \mu$ to the right and $y = \beta + \lambda$ to the left. A significant amount of information for the asymptotic behavior of $X^{(2)}(t)$ can be derived using a renewal structure induced by $Y$, defined
Figure 1.3  Figure showing the collective dynamics of a population of independent and identical DNA bipeds in four different cases representing different molecular parameters. The asymptotic velocity and effective diffusion apparent in the figure was rigorously computed in [12].
by successful return times of this chain to a distinguished state, say $L$. More precisely, let $\tau_0 = 0$ and $\tau_k = \inf\{ t > \tau_{k-1} : Y(t) = L \}$ for $k \in \mathbb{N}$. Let $N_t = \sup\{ k \in \mathbb{N} : \tau_k < t \}$ be the number of returns to $L$ prior time $t > 0$. The following theorem is derived in [12] and establishes the existence of an asymptotic speed and an asymptotic variance (effective diffusion coefficient) for the molecular biped.

**Theorem 1.4.1.** [12] The following hold:

(i) (Strong Law of Large Numbers)

$$v = \lim_{t \to \infty} \frac{X_1^{(2)}(t)}{t} = \frac{E(X_{\tau_2}^{(2)} - X_{\tau_1}^{(2)})}{E(\tau_2 - \tau_1)} \in (-\infty, \infty), \text{ a.s.}$$

(ii) (Recurrence/Transience Dichotomy)

(a) If $v > 0$, then $\lim_{t \to \infty} X^{(2)}(t) = \infty$, a.s.

(b) If $v = 0$, then $\lim \inf_{t \to \infty} X^{(2)}(t) = -\infty$ and $\lim \sup_{t \to \infty} X^{(2)}(t) = \infty$, a.s.

(c) If $v < 0$, then $\lim_{t \to \infty} X^{(2)}(t) = -\infty$, a.s.

(iii) (Central Limit Theorem) The random variable $\frac{X^{(2)}(t) - vt}{\sqrt{t}}$ converges in distribution, as $t \to \infty$, to a normal distribution with the variance

$$\sigma_{\text{eff}}^2 = \frac{E((X_{\tau_2}^{(2)} - X_{\tau_1}^{(2)} - v(\tau_2 - \tau_1))^2)}{E(\tau_2 - \tau_1)}. \quad (1.1)$$

We remark that in fact the CLT is derived in [12] in a more general, functional form.

One of our main contribution in [12] is the analytical derivation of explicit expressions for $v$ and $\sigma_{\text{eff}}$ in terms of the transition rates that dictate the movement of the biped. We describe some of the key components of this derivation in the following section.

### 1.4.2 The analytic viewpoint

It is shown in [12] that the moment generating function for the position of an individual (non-Markovian) leg of the biped depends only on the distance between legs (i.e., it does not depend on the specific positions of the legs). This allows to define a specific perturbation $A(\eta)$ of the generator $A = A(0)$ of $Y$ that, in turn, is used to derive explicit estimates for $v$ and
\( \sigma_{eff} \). More precisely, if \( \Lambda(\eta) \) denotes the Perron root of \( A(\eta) \), we show in [12] that the following result on characteristic functions holds.

**Theorem 1.4.2.** There exists \( \varepsilon > 0 \) such that for all complex numbers \( \theta \) with \( |\theta| < \varepsilon \),

\[
\lim_{t \to \infty} E \exp \left( \theta \frac{X^{(2)}(t) - \Lambda'(0)t}{\sqrt{t}} \right) = e^{\frac{1}{2} \Lambda''(0) \theta^2}.
\]

In view of Theorem 1.4.1, Theorem 1.4.2 readily implies the following.

**Corollary 1.4.3.**

1. *(Central Limit Theorem)* \( \frac{X^{(2)}(t) - \Lambda'(0)t}{\sqrt{t}} \to N(\Lambda''(0)) \) in distribution, where \( N(\sigma^2) \) denotes the centered normal distribution with variance \( \sigma^2 \).

2. *(Strong Law of Large Numbers)* \( \frac{X^{(2)}(t)}{t} \to \Lambda'(0) \), \( P \)-a.s.

Corollary 1.4.3, in turn, implies that the asymptotic velocity of the biped is given by \( v = \Lambda'(0) \), whereas the asymptotic variance is provided by the second derivative at zero of the Perron root of \( A(\eta) \), i.e., \( \sigma_{eff}^2 = \Lambda''(0) \). Let \( \pi \) denote the invariant distribution of \( Y \) and \( A^\# \) denote the pseudo-inverse of \( A \).

**Theorem 1.4.4.** [12]

1. \( \Lambda'(0) = \mu(1 - \pi_L) - \lambda(1 - \pi_0) \).

2. Let \( \rho = \frac{\beta + \lambda}{\alpha + \mu} \) and

\[
Q = \begin{pmatrix}
\rho \pi_0 A^\#_{0L} & -\frac{1}{2}(\rho \pi_0 A^\#_{00} + \rho^{-1} \pi_S A^\#_{LL}) \\
-\frac{1}{2}(\rho \pi_0 A^\#_{00} + \rho^{-1} \pi_S A^\#_{LL}) & \rho^{-1} \pi_S A^\#_{L0}
\end{pmatrix}.
\]

Then

\[
\Lambda''(0) = \mu(1 - \pi_L) + \lambda(1 - \pi_0) + 2\langle Q \left( \begin{array}{c} \mu \\ \lambda \end{array} \right), \left( \begin{array}{c} \mu \\ \lambda \end{array} \right) \rangle.
\]
1.4.3 Ladder representation for the spider random walk

The ladder representation is a useful tool for a qualitative analysis of one-dimensional spiders, for instance with correlation between the legs or in random environment. According to our general definition, spider random walks are continuous-time stochastic processes that describe the evolution in time of the pair \( S_t = (X_t, Y_t) \), where \( Y_t \) is the configuration of the spider at time \( t \geq 0 \) (encoding the relative placement of its legs) and \( X_t \) is the vector of coordinates of its distinguished characteristic, for instance the placement of the right-most leg or the location of the spider’s barycenter. Recall that \( S_t \) is a Markov process which completely specifies the spatial location of the spider at time \( t \). Thus, whenever we deal with a finite number of available configurations and with \( X_t \) moving along the one-dimensional lattice \( \mathbb{Z} \), we can think of the spider as a random walk on the ladder \( \mathbb{Z} \times \{0, 1, \ldots, L\} \) for some \( L \in \mathbb{N} \).

**Definition 1.4.5.** Let \( L \in \mathbb{N} \) and \( \mathbb{L} = \mathbb{Z} \times \{0, 1, \ldots, L\} \). A simple (possibly biased) random walk on the ladder \( \mathbb{L} \) is a time-homogeneous Markov chain \((S_t)_{t \geq 0}\) with the state space \( \mathbb{L} \), such that

\[
Q(S^{(1)}, S^{(2)}) > 0 \text{ for } S^{(1)} = (X^{(1)}, Y^{(1)}), S^{(2)} = (X^{(2)}, Y^{(2)}) \text{ only if } |X^{(1)} - X^{(2)}| \leq 1,
\]

where \( Q \) is the generator of the Markov process \( \xi_t \).

We mention this representation here since it provides easy arguments for existence of such fundamental qualitative characteristics as, for instance, recurrence and transience dichotomy, law of large numbers, and diffusive behavior for most of the variants of the spider random walk model discussed in this proposal. Unfortunately, this approach typically fails to complement qualitative descriptions with their exact quantitative characteristics (such as precise characterization in terms of the basic parameters of recurrent and transient regimes, the asymptotic speed, and the correlation matrix of the limiting Gaussian process). From this point of view, those cases (as for instance the models of one-dimensional bipeds in [7, 12] and spiders in random environment in [37]), where explicit analytical characterization is possible, appear to be of a special interest in applications.
1.5 Molecular spiders in a random environment on \( \mathbb{Z} \)

The main topic of this thesis is spider random walk in a random environment on \( \mathbb{Z} \). The goal of this section is to introduce formally the model and state our main result.

1.5.1 The model

Spider random walks in a random environment (SRWRE) were introduced and studied in [37]. The model describes molecular motors on a non-homogeneous surface where single-stranded segments of DNA covering the surface are assumed to have random characteristics obeying the same law. Mathematically, SRWRE is a system of (finitely many) interacting particles (referred to as legs of the spider). The legs move on the integer lattice \( \mathbb{Z} \) representing the nucleic acid binding domains imprinted on the path. The waiting time for each leg follows an exponential distribution. Different legs being in general associated with different exponential clocks. The clocks are independent modulo certain restrictions on mutual location of the legs, and as long as these constraints are not violated the legs move independently according to a law of a random walk in random environment (RWRE). Moves that attempt to violate the constraints are ignored, and the spider stays put at its current position until one of the legs performs a feasible move. The rules describe feasible (and thus also forbidden) relative positions (configurations) of the legs. The number of feasible configurations is assumed to be finite. The spider walk is a continuous time stochastic process. Formally, the system is characterized by:

(a) Rates of the exponential variables

(b) Space of configurations.

Let \( N \geq 2 \) be a given integer parameter, the number of the legs of the spider. The particular case \( N = 1 \) corresponds to the regular RWRE on \( \mathbb{Z} \). Location of the spider is given by a vector

\[
S(t) = (S_1(t), \ldots, S_N(t)) \in \mathbb{Z}^N,
\]

where \( S_i(t) \in \mathbb{Z} \) is the location of leg \( i \) at time \( t \).
Spiders position can be thus characterized by location of the fist leg \(S_1(t)\) and the configuration
\[
Y(t) := (S_2(t) - S_1(t), \ldots, S_N(t) - S_1(t)) \in \mathbb{Z}^{N-1} \quad \forall \ t \geq 0.
\] (1.2)

We will assume that the number of feasible (allowed) configurations for the spider is finite and denote by \(L\) the set of such configurations. The spiders’ random walk can be thought as a random motion on the ladder graph (or strip) \(\mathbb{Z} \times L\).

The random environment \(\omega = (\omega^+_x)_{x \in \mathbb{Z}}\) is a sequence of i.i.d. random variable with with \(\omega^+_x \in (0, 1)\). Denote \(\omega^-_x = 1 - \omega^+_x\). Transition rate \(x \mapsto y\) for \(x, y \in \mathbb{Z}^N\) is given by
\[
q^\delta(x, y) = \begin{cases} 
\omega^+_y, & \text{if } y_i = x_i + 1, \\
\omega^-_y, & \text{if } y_i = x_i - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

We denote by \(P\) the distribution of \(\omega\) and by \(E_P\) the corresponding expectation operator. The law of spider’s motion in a given environment \(\omega\) is called quenched law, we denote it by \(P_\omega\). We denote by \(E_\omega\) the corresponding expectation. Notice that \(S(t)\) is a non-homogeneous Markov chain under \(P_\omega\). The annealed (average) law of the motion is defined as follows:
\[
P(\cdot) := E_P(P_\omega(\cdot)) = \int_\Omega P_\omega(\cdot) \, dP(\omega).
\]

Remark that \(S(t)\) is not a Markov chain under \(P\) and the increments of \(S(t)\) are not stationary under \(P\).

1.5.2 Previous result

Following [37], consider the following spider graph:
\[
G = G(w) = (\mathbb{V}, \mathcal{E}(\omega)), \quad \mathbb{V} := \mathbb{Z} \times L,
\]
with \(e = (x, y) \in \mathcal{E}(\omega)\) if and only if \(q^\delta(x, y) > 0\). \(G\) is deterministic due to the following connectivity and ellipticity conditions assumed throughout this work:

**Assumption 1.5.1 ([37]).**
1. (connectivity and irreducibility) \( \{0\} \times L \) and \( \{0,1\} \times L \) are connected sub-graphs of \( G \) with probability one.

2. (ellipticity) \( \exists 0 < \delta < \frac{1}{2} \) such that \( P(\delta \leq \omega_0^+ \leq 1 - \delta) = 1 \)

3. \( E_P(\log \rho_0) < 0 \) (spider is transient to the right) with

\[
\rho_n := \frac{\omega_n^-}{\omega_n^+}, \quad n \in \mathbb{Z}. \tag{1.3}
\]

4. \( P(\rho_0 > 1) > 0 \) and \( P(\rho_0 < 1) > 0 \).

The following lemma is well-known, the observation goes back to [85].

**Lemma 1.5.2.** Suppose that Assumption 1.5.1 holds. Let \( \Lambda(s) := \log E_P(\rho_0^s) \). Then

- \( \Lambda(s) \) is convex, \( \Lambda(0) = 0 \), \( \Lambda'(0) = -E_P(\log \rho_0) < 0 \), \( \lim_{s \to \infty} \Lambda(s) = +\infty \).

- In particular, there exists unique \( \kappa > 0 \) such that

\[
E_P(\rho_0^\kappa) = 1. \tag{1.4}
\]

The speed of the spider will be denoted as following

\[
v_P := \lim_{t \to \infty} \frac{S_1(t)}{t}. \tag{1.5}
\]

Recall the configuration vector \( Y(t) \) from (1.2). Let

\[
T := \inf \{ s > 0 : S_1(s) > 0 \text{ and } Y(s) = Y(0) \}. \tag{1.6}
\]

The following theorem is the main result of [37].

**Theorem 1.5.3 ([37]).** Let Assumption 1.5.1 hold. Then the following is true \( \mathbb{P} \)-a.s.

- The spider walk is transient to the right, that is

\[
\lim_{t \to \infty} S_1(t) = +\infty
\]

- Furthermore, the speed \( v_P \) of the spider is well-defined and

\[
v_P = \begin{cases} 
\frac{E(S_1(T))}{E(T)} > 0 & \text{if } \frac{k}{N} > 1, \\
0 & \text{if } \frac{k}{N} < 1.
\end{cases} \tag{1.7}
\]
Remarkably, the above result is independent of a particular form of $L$ as long as Assumption 1.5.1 is satisfied. The law of large numbers in the case when $\kappa/N = 1$ remains an open problem.

1.5.3 Our main result

In this work we focus on a transient regime of the spider random walk. The following is the main result of this thesis. The proof is deferred to Chapter 4.

Theorem 1.5.4. Let Assumption 1.5.1 hold and suppose in addition that $\kappa/N > 2$. Then there exists $b > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{S_1(t) - tv_P}{b\sqrt{t}} \leq x \right) = \Phi(x)$$

for all $x \in \mathbb{R}$. 
CHAPTER 2. ASYMPTOTIC BEHAVIOR OF RWRE on $\mathbb{Z}$

In this chapter we recall recurrence and transience criteria, laws of large numbers (for $X_n$ and $T_n$), and limit laws in the transient regime for random walks in a stationary and ergodic environment on $\mathbb{Z}$. Remark that though the spider walk is defined in an i.i.d. environment, a more general theory extended to ergodic environments provide a better insight into the corresponding random walk on the spider graph (strip). The results for one-dimensional RWRE discussed in this chapter motivate the general theory of RWRE on a strip presented in Chapter 3. In particular, they elucidate the role of the critical exponent $\kappa$.

2.1 Random walks in a random environment in dimension one

Let $\Omega = [0,1]^\mathbb{Z}$ and let $\mathcal{F}$ be the Borel $\sigma$-algebra of subsets of $\Omega$. A random environment is a random element (sequence) $\omega = (\omega_i)_{i \in \mathbb{Z}}$ in $(\Omega, \mathcal{F})$. Denote by $P$ the probability measure associated with $\omega$ in $(\Omega, \mathcal{F})$ and use the notation $E_P$ for the corresponding expectation operator.

A random walk on $\mathbb{Z}$ in a random environment $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Omega$ is a Markov chain $X = (X_n)_{n \geq 0}$ on $\mathbb{Z}$ governed by the following transition law:

$$P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} 
\omega_i & \text{if } j = i + 1, \\
1 - \omega_i & \text{if } j = i - 1, \\
0 & \text{otherwise.}
\end{cases} \quad (2.1)$$

The law of the random walk in a fixed environment is usually referred to as a quenched law of the random walk. We will use the notation $P^x_\omega$ for the quenched probability law when $X_0 = x$ and $E^x_\omega$ for the corresponding expectation operator acting in the space of trajectories $\mathbb{Z}^{\mathbb{Z}^+}$.

The so called annealed or averaged distributions $\mathbb{P}^x$ of the random walk are obtained by averaging the quenched distributions over all possible environments, that is $\mathbb{P}^x(\cdot) = \int P^x_\omega(\cdot) P(d\omega)$. 
More precisely, let $\mathcal{G}$ be the cylinder $\sigma$-algebra of the space of trajectories $\mathbb{Z}^{\mathbb{Z}^+}$. Note that for each $G \in \mathcal{G}$, $P_\omega^x(G) : \Omega \to [0, 1]$ is a $\mathcal{F}$-measurable function of $\omega$. The joint probability distribution $\mathbb{P}^x$ of the random walk and the environment on the product space $(\Omega \times \mathbb{Z}^{\mathbb{Z}^+}, \mathcal{F} \otimes \mathcal{G})$ is defined by $\mathbb{P}^x(F \times G) = \int_F P_\omega^x(G) P(d\omega), F \in \mathcal{F}, G \in \mathcal{G}$. The projection of $\mathbb{P}^x$ on the space of trajectories $\mathbb{Z}^\mathbb{N}$ is the annealed law of the random walk. The expectation under the law $\mathbb{P}^x$ is denoted by $\mathbb{E}^x$. We will usually assume that $X_0 = 0$ and often omit the upper index $x$ in the notations for the underlying probability laws and expectation operators when $x = 0$. For instance, we will write $P_\omega$ for $P_0^\omega$ and $E$ for $E^0$.

For a general overview of RWRE, see the lecture notes of Zeitouni [97]. We remark that RWRE on $\mathbb{Z}$ are notably different from the usual nearest-neighbor random walks in some of their fundamental properties. In particular, it is possible that the speed of a transient random walk is zero. In general, RWRE on $\mathbb{Z}$ is slower than its appropriate usual counterpart, and in particular it has non-trivial scaling properties leading to limit theorems with normalization factors different from $n^{1/2}$ for nearest-neighbor walks [49, 84]. Random walks in random environment (RWRE) have been shown to exhibit an interesting and surprising behavior.

Solomon [85] identified a recurrent/transient criterion for one-dimensional RWRE and obtained an explicit formula for the limiting velocity $v_P = \lim_{n \to \infty} \frac{X_n}{n}$. From these results it is easy to construct explicit examples of RWRE that are transient to $+\infty$ even though $\mathbb{E}(X_1) < 0$ or that are transient with asymptotically zero velocity ($v_P = 0$).

In the regular RWRE model, $\omega$ is a stationary and ergodic sequence under $P$ [97]. Let

$$\rho_n = \frac{1 - \omega_n}{\omega_n}, \quad n \in \mathbb{Z}. \tag{2.2}$$

Asymptotic results for one-dimensional RWRE can often [6, 18, 26, 29, 33, 49, 73, 84, 89, 97] be stated in terms of certain averages of functions of $\rho_0$ and explained by means of typical “landscape features” (such as traps and valleys, cf. [89, 97]) of the random potential $(V_n)_{n \in \mathbb{Z}}$, which is associated with the random environment as follows: $V_0 = 0$ and

$$V_n = \begin{cases} \sum_{k=1}^{n} \log \rho_k & \text{if } n > 0, \\ \frac{|n|}{|n|} \sum_{k=1}^{n} \log \rho_k & \text{if } n < 0. \end{cases} \tag{2.3}$$
The role played by the process $V_n = \sum_{i=0}^{n} \log \rho_{-i}$ in the theory of one-dimensional RWRE stems from the explicit form of harmonic functions (cf. [97, Section 2.1]), which allows one to relate hitting times of the RWRE to those associated with the random walk $V_n$. This phenomenon provides a heuristic explanations to most of the results about random walks in random environment, including those discussed in this thesis.

### 2.2 Basic asymptotic results for the classical RWRE

Criteria for transience and recurrence of the one-dimensional RWRE were provided by Solomon [85] in the case where $\omega_n$ is an i.i.d. sequence and extended to stationary and ergodic environments by Alili [4] (see also [65, 97]). Let

$$R(\omega) = 1 + \sum_{n=1}^{\infty} \rho_0 \rho_{-1} \cdots \rho_{-n+1} = 1 + \sum_{n=1}^{\infty} \exp \left\{ \sum_{i=0}^{n-1} \log \rho_{-i} \right\},$$

(2.4)

Let $T_0 = 0$ and for $n \in \mathbb{N}$,

$$T_n = \inf\{k \geq 0 : X_k = n\} \quad \text{and} \quad \tau_n = T_n - T_{n-1}.$$

(2.5)

The walk $X_n$ is a.s. transient if $E_P(\log \rho_0) \neq 0$ and is a.s. recurrent if $E_P(\log \rho_0) = 0$. If $E_P(\log \rho_0) < 0$ then $\mathbb{P}(\lim_{n \to \infty} X_n = +\infty) = 1$, $T_n$ are a.s. finite, $\{\tau_n\}_{n \geq 1}$ is a stationary and ergodic sequence (but not i.i.d. since the walk returns to sites already visited with a positive probability), and

$$v_P := \lim_{n \to +\infty} \frac{X_n}{n} = \lim_{n \to +\infty} \frac{n}{T_n} = \frac{1}{2E_P(R) - 1}, \quad \mathbb{P} - \text{a.s.}$$

(2.6)

Thus, the transient walk $X_n$ has a deterministic speed $v_P = \lim_{n \to +\infty} X_n/n$ which may be zero. In the case of i.i.d. environments, Solomon’s result, $v_P = 0$ if $E(\rho_0) \geq 1$ and $v_P = \frac{1 - E_P(\rho_0)}{1 + E_P(\rho_0)}$ otherwise, is recovered from (2.6). It follows from (2.6), by Jensen’s inequality, that the random walk $X_n$ is always slower than the usual random walk with deterministic probability $p$ of moving to the right given by $\log \left( \frac{1-p}{p} \right) = E_P(\log \rho_0)$. The heuristic explanation [84, 89, 97] is that, in contrast to a homogeneous medium, random environments may create atypical segments (traps, valleys) which retain the random walk for abnormally long periods of time.
For the classical RWRE model, asymptotic limit theorem are known for both transient and recurrent regimes. These theorems measure the amplitude of the fluctuations in the deviation of $X_n$ from its asymptotic expected value $v_P \cdot n$.

In the transient regime, Solomon’s law of large numbers for random walks in i.i.d. environment was complemented by limit laws in [49]. The limit laws for the RWRE $X_n$ are deduced in [49] from stable limit laws for the hitting times $T_n$, and the index $\kappa$ of the stable distribution is determined by the condition

$$E_P(\rho_0^\kappa) = 1.$$  \hspace{1cm} (2.7)

In particular, if $E_P(\rho^2) < \infty$ then the central limit theorem holds with the standard normalization $\sqrt{n}$. The limit laws of [49] are extended in [63] to environments that are stochastic functionals of either Markov processes or so called chains of infinite order. For related quenched results in the transient regime see recent articles [29, 33, 73] and references therein.

Sinai [84] studied a recurrent RWRE on $\mathbb{Z}$ and obtained the following limit theorem:

$$\frac{\sigma^2}{(\log n)^2} X_n \Rightarrow b_\infty$$

where $\Rightarrow$ denotes convergence in law and $b_\infty$ is a non-degenerate random variable with the following distribution:

$$P(b_\infty \in dx) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{ -\frac{(2k+1)^2 \pi^2}{8} |x| \right\} dx$$

The random variable $b_\infty$ was characterized by Sinai as the limit of a sequence of the bottoms of “deepest valleys” associated with the random potential $V_n$. In the theory developed by Sinai, the unusual scaling $(\log n)^2$ (to be compared with $\sqrt{n}$ for the simple recurrent random walk on $\mathbb{Z}$) is explained by the fact that the RWRE in Sinai’s regime spends almost all time being localized in the “deep valleys” of the random potential.

### 2.3 Recurrence and transience criteria for RWRE

Recall $\rho_n$ from (2.2) and $T_n$ from (2.5). The fundamental role of the random potential $V_n$ in the theory of one-dimensional RWRE can be illustrated by recurrence and transience criteria.
Fix an environment \( \omega \) such that \(| \log \rho_z | < \infty \) for each \( z \in \mathbb{Z} \). For any \( m_-, m_+ \in \mathbb{N} \) and an integer \( z \in [-m_-, m_+] \), define

\[
\phi_{\omega, z}(m_-, m_+) := P_{\omega}^z(\{X_n \} \text{ hits } - m_- \text{ before hitting } m_+) = P_{\omega}^z(T_{-m_-} < T_{m_+}).
\]

Due to the Markov Property, \( \phi_{\omega, z}(m_-, m_+) \) as a function of \( z \) is harmonic function for the random walk. Namely, it satisfies the following equation

\[
\phi_{\omega, z}(m_-, m_+) = (1 - \omega_z)\phi_{\omega, z-1}(m_-, m_+) + \omega_z\phi_{\omega, z+1}(m_-, m_+)
\]

with boundary conditions \( \phi_{\omega, -m_-}(m_-, m_+) = 1 \) and \( \phi_{\omega, m_+}(m_-, m_+) = 0 \). This yields (see, for instance, [97, Section 2.1]) the following explicit formula for the hitting probabilities in a fixed environment:

\[
\phi_{\omega, z}(m_-, m_+) = \frac{\sum_{i=m_-+1}^{m_+} \prod_{j=z+1}^{i-1} \rho_j}{\sum_{i=z+1}^{m_-+1} \prod_{j=z+1}^{i-1} \rho_j + \sum_{i=-m_-+1}^{z} \prod_{j=i}^{z} \rho_j^{-1}}. \quad (2.8)
\]

The formula illuminates the role of the random potential \( V_n = \log \prod_{j=1}^n \rho_j \) in the theory of one-dimensional RWRE.

Let

\[
S(\omega) = \sum_{k=1}^{\infty} \rho_1 \rho_2 \cdots \rho_k \quad \text{and} \quad F(\omega) = \sum_{k=0}^{\infty} \rho_0^{-1} \rho_{-1}^{-1} \cdots \rho_{-k}^{-1}. \quad (2.9)
\]

The following two propositions are the key to the recurrence and transience criteria for RWRE [97]. The propositions are deduced from (2.8) which relates the asymptotic behavior of the random walk to the asymptotic behavior of the exit probabilities as \( m_\pm \to \infty \).

**Proposition 2.3.1.** Let \( X = (X_n)_{n \geq 0} \) be a random walk in a stationary and ergodic environment \( \omega \). Assume that \( E_\rho(\log \rho_0) \) is well-defined, possibly infinite. Then

(a) \( P(S(\omega) < \infty) = 1 \Rightarrow \lim_{n \to \infty} X_n = +\infty, \ P - a. s., \)

(b) \( P(F(\omega) < \infty) = 1 \Rightarrow \lim_{n \to \infty} X_n = -\infty, \ P - a. s., \)

(c) \( P(S(\omega) = F(\omega) = \infty) = 1 \Rightarrow \limsup_{n \to \infty} X_n = +\infty \text{ and } \liminf_{n \to \infty} X_n = -\infty, \ P - a. s. \)

**Proposition 2.3.2.** Let \( X = (X_n)_{n \geq 0} \) be a random walk in a stationary and ergodic environment \( \omega \). Assume that \( E_\rho(\log \rho_0) \) is well-defined, possibly infinite. Then
(a) \( P(S(\omega) < \infty) = 1 \iff E_p(\log \rho_0) < 0, \)

(b) \( P(F(\omega) < \infty) = 1 \iff E_p(\log \rho_0) > 0, \)

(c) \( P(S(\omega) = F(\omega) = \infty) = 1 \iff E_p(\log \rho_0) = 0. \)

This leads to the following recurrence and transience criteria of [85] and [4]:

**Theorem 2.3.3.** Let \( X = (X_n)_{n \geq 0} \) be a random walk in a stationary and ergodic environment \( \omega \). Assume that \( E_P(\log \rho_0) \) is well-defined, possibly infinite. Then

(a) If \( E_P(\log \rho_0) < 0 \) then, \( \lim_{n \to \infty} X_n = +\infty, \mathbb{P} - a.s. \)

(b) If \( E_P(\log \rho_0) > 0 \) then, \( \lim_{n \to \infty} X_n = -\infty, \mathbb{P} - a.s. \)

(c) If \( E_P(\log \rho_0) = 0 \) then, \( \limsup_{n \to \infty} X_n = +\infty \) and \( \liminf_{n \to \infty} X_n = -\infty, \mathbb{P} - a.s. \)

### 2.3.1 Law of large numbers. Asymptotic speed of the random walk

In this section we review the laws of large numbers for \( X_n \) and \( T_n \) for classical RWRE. The proofs of these statements outlined below are due to [85] and [4]. Let

\[
\bar{S} := \sum_{i=1}^{\infty} \frac{1}{w_i} \prod_{j=0}^{i-1} \rho_{-j} + \frac{1}{w_0} = 1 + 2 \sum_{i=0}^{\infty} \prod_{j=0}^{i} \rho_{-j} \tag{2.10}
\]

and

\[
\bar{F} := \sum_{i=1}^{\infty} \frac{1}{(1-w_i)} \prod_{j=0}^{i-1} \rho_{j-1} + \frac{1}{(1-w_0)}.
\]

It is well-known (and straightforward to verify) that

\[
E_P(\bar{S}) = 2E_P(R) - 1 = 2E_P(\Lambda) - 1, \tag{2.11}
\]

where \( R \) and \( \Lambda \) are defined in (2.4) and (3.8) respectively. Similar identities hold for \( \bar{F} \).

For the asymptotic speed of the random walk, we have:

**Theorem 2.3.4.** [85, 4] Let \( X = (X_n)_{n \geq 0} \) be a random walk in a stationary and ergodic environment \( \omega \). Assume that \( E_P(\log \rho_0) \) is well-defined, possibly infinite. Then

(a) If \( E_P(\bar{S}) < \infty \), then \( \lim_{n \to \infty} \frac{X_n}{n} = \frac{1}{E_P(\bar{S})}, \mathbb{P} - a.s. \)
(b) If \( E_P(\overline{F}) < \infty \), then \( \lim_{n \to \infty} \frac{X_n}{n} = -\frac{1}{E_P(\overline{F})}, \) \( \mathbb{P} - \text{a.s.} \)

(c) If \( E_P(\overline{S}) = \infty \) and \( E_P(\overline{S}) = \infty \), then \( \lim_{n \to \infty} \frac{X_n}{n} = 0, \) \( \mathbb{P} - \text{a.s.} \)

Furthermore, the three cases listed above are exhausting all the possibilities.

**Corollary 2.3.5.** Let the conditions of Theorem 2.3.4 hold. If, in addition, \( \omega \) is an i.i.d. environment then:

(a) If \( E_p(\rho_0) < 1 \), then \( \lim_{n \to \infty} \frac{X_n}{n} = \frac{1-E_p(\rho_0)}{1+E_p(\rho_0)}, \) \( \mathbb{P} - \text{a.s.} \)

(b) If \( E_p(\rho_0^{-1}) < 1 \), then \( \lim_{n \to \infty} \frac{X_n}{n} = -\frac{1-E_p(\rho_0^{-1})}{1+E_p(\rho_0^{-1})}, \) \( \mathbb{P} - \text{a.s.} \)

(a) If \( E_p(\rho_0)^{-1} \leq 1 \leq E_p(\rho_0^{-1}) \), then \( \lim_{n \to \infty} \frac{X_n}{n} = 0, \) \( \mathbb{P} - \text{a.s.} \)

In general \( \{X_n\} \) is not a Markov chain under the annealed measure \( \mathbb{P} \) and the increments \( \{X_n - X_{n-1}\} \) are not even stationary. Thus one cannot apply the ergodic theorem to \( X_n/n \) directly. The key idea of the proof of Theorem 2.3.4 is to look first at the hitting time \( T_n \) and then deduce the claim for the random walk from the corresponding result for \( T_n \). More precisely, the proof is based on the following three auxiliary results (see [97] for the details):

**Lemma 2.3.6.** Let the conditions of Theorem 2.3.4 hold. Then:

(a) \( \mathbb{E}(T_1) = E_P(\overline{S}) \)

(b) \( \mathbb{E}(T_{-1}) = E_P(\overline{F}) \)

**Lemma 2.3.7.** Let the conditions of Theorem 2.3.4 hold. Suppose that \( \lim_{n \to \infty} X_n = +\infty \), \( \mathbb{P} - \text{a.s.} \). Then the sequence \( \{\tau_n\}_{n \geq 1} \) defined in (2.5) is stationary and ergodic. In particular,

\[
\lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \tau_i = \mathbb{E}(\tau_1) = E_P(\overline{S}), \quad \mathbb{P} - \text{a.s.}
\]

**Lemma 2.3.8.** Let the conditions of Theorem 2.3.4 hold and let \( \alpha \in (0, +\infty] \). Then

(a) If \( E_P(\log \rho_0) < 0 \) and \( \lim_{n \to \infty} \frac{T_n}{n} = \alpha \), then \( \lim_{n \to \infty} \frac{X_n}{n} = \frac{1}{\alpha}, \) \( \mathbb{P} - \text{a.s.} \)

(b) If \( E_P(\log \rho_0) > 0 \) and \( \lim_{n \to \infty} \frac{T_n}{n} = \alpha \), then \( \lim_{n \to \infty} \frac{X_n}{n} = -\frac{1}{\alpha}, \) \( \mathbb{P} - \text{a.s.} \)

(c) If \( E_P(\log \rho_0) = 0 \) and \( \lim_{n \to \infty} \frac{T_n}{n} = \lim_{n \to \infty} \frac{T_{-n}}{n} = \infty \), then \( \lim_{n \to \infty} \frac{X_n}{n} = 0, \) \( \mathbb{P} - \text{a.s.} \).
2.4 Non-Gaussian limit laws for general random partial sums

A non-degenerate random variable $W$ is said to have a stable distribution if it has a domain of attraction, i.e. if there is a sequence of i.i.d. random variables $(W_n)_{n \geq 1}$ and sequences of positive numbers $(b_n)_{n \geq 1}$ and real numbers $(a_n)_{n \geq 1}$ such that $\lim_{n \to \infty} b_n = \infty$ and

$$ S_n/b_n - a_n \Rightarrow W \quad \text{where} \quad S_n = \sum_{n=1}^{n} W_n. \tag{2.12} $$

Such random variables $W_n$ are said to be in the domain of attraction of the stable law. For equivalent definitions of stable laws and, in particular, for an explicit form of their characteristic functions see for instance [78, Chapter 1].

**Definition 2.4.1.** Let $(W_n)_{n \geq 1}$ be a sequence of (not necessarily i.i.d.) random variables. A stable limit theorem (SLT) is said to hold for the sequence $(W_n)_{n \geq 1}$ if (2.12) holds for some numbers $b_n \nearrow \infty$, $a_n \in \mathbb{R}$, and random variable $W$ with a stable distribution.

It is well-known (see e.g. [21, Chapter 9] or [38]) that a SLT holds for an i.i.d. sequence $(W_i)_{i \geq 1}$ if and only if either

$$ \lim_{t \to \infty} \frac{P(W_1 > t)}{P(|W_1| > t)} = \theta \in [0,1] \quad \text{and} \quad \lim_{\lambda \to \infty} \frac{P(|W_1| > \lambda t)}{P(|W_1| > t)} = \lambda^{-\kappa} \quad \forall \ t > 0, \tag{2.13} $$

for some $\kappa \in (0, 2)$, which is the index of the limit law, or

$$ \lim_{t \to \infty} \frac{t^2 P(|W_1| > t)}{E(W_1^2 \mathbb{1}_{|W_1| \leq t})} = 0. \tag{2.14} $$

In the latter case the limit random variable $W$ has a non-degenerate normal law, to which we refer as a stable law of index 2. Condition (2.14) is satisfied if $E(W_1^2) < \infty$, and this leads to the usual central limit theorem for i.i.d. variables.

Non-Gaussian stable limit theorems are known for many dependent sequences (see e.g. [1, 27, 28, 41, 42, 46, 45, 53, 61, 58, 92, 79]). Loosely speaking, these theorems state that if a stationary sequence $(W_n)_{n \geq 1}$ is “mixing enough” and its marginal distribution is such that an i.i.d. sequence with such common distribution is in the domain of attraction of a stable law, then (2.12) holds with the same $a_n$ and $b_n$ as in the i.i.d. case (for the proper choice of the normalizing constants see e.g. [32, p. 153] or [38, p. 175]). Typically, “mixing” conditions of general SLT’s with $\kappa \in (0, 2)$ include not only assumptions of “asymptotic independence” but also some limitations on the bivariate correlations.
2.5 Limit theorems for transient RWRE on $\mathbb{Z}$

Transient random walks in i.i.d. environments have a natural renewal structure which is defined (for $X_n$ transient to the right) by the sequence of sites $z_n \in \mathbb{N}$ where the random walk never moves to the left. When the environment is an i.i.d. sequence, the pieces of the trajectory between times $T_{z_n}$ and $T_{z_{n+1}} - 1$ are independent. In particular, the random variables $T_{z_n} - T_{z_1}$ can be represented as partial sums of an i.i.d. sequence. This renewal structure has been exploited by many authors and also can be carried over to higher dimensions but has a drawback which makes it difficult to use it for exact computations of the parameters of RWRE. Namely, $T_{z_n}$ are not stopping times with respect to the natural filtration of the random walk. The existence of the renewal structure for $X_n$ can nevertheless serve as a “strong supporting evidence” for the stable limit theorem for the random walk.

In [49] the limit laws for RWRE are derived from stable limit laws for the hitting times $T_n$, and the index $\kappa$ of the stable distribution is determined by the condition

$$E_P(\rho_0^\kappa) = 1. \quad (2.15)$$

The following is a summary of the main results of [49]. For $\kappa \in (0, 2]$ and $b > 0$ we denote by $\mathcal{L}_{\kappa, b}$ the stable law of index $\kappa$ with the characteristic function

$$\log \hat{\mathcal{L}}_{\kappa, b}(t) = -b|t|^{\kappa} \left(1 + \frac{t}{|t|} f_\kappa(t)\right), \quad (2.16)$$

where $f_\kappa(t) = -\tan \frac{\pi}{2} \kappa$ if $\kappa \neq 1$, $f_1(t) = 2/\pi \log t$. With a slight abuse of notation we use the same symbol for the distribution function of this law. If $\kappa < 1$, $\mathcal{L}_{\kappa, b}$ is supported on the positive reals, and if $\kappa \in (1, 2]$, it has zero mean but is not symmetric [78, Chapter 1].

**Theorem 2.5.1.** [49] Let the environment $(\omega_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence such that

(i) There exists a constant $\kappa > 0$ such that (2.15) holds and $E_P(\rho_0^\kappa \log^+ \rho_0) < \infty$, where we denote $\log^+(x) = \max\{\log x, 0\}$.

(ii) The distribution of $\log \rho_0$ is non-lattice (its support is not contained in any proper sublattice of $\mathbb{R}$).

Then the following hold for some $b > 0$ (recall the definition (2.6) of the asymptotic speed $v_P$):
(i) If \( \kappa \in (0, 1) \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-\kappa} X_n \leq \hat{z}) = 1 - \mathcal{L}_{\kappa, \hat{b}}(\hat{z}^{-1/\kappa}) \),

(ii) If \( \kappa = 1 \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-1}(\log n)^2(X_n - \delta(n)) \leq \hat{z}) = 1 - \mathcal{L}_{1, \hat{b}}(-\hat{z}) \), for suitable \( A_1 > 0 \) and \( \delta(n) \sim (A_1 \log n)^{-1} n \),

(iii) If \( \kappa \in (1, 2) \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-1/\kappa}(X_n - n\nu_P) \leq \hat{z}) = 1 - \mathcal{L}_{\kappa, \hat{b}}(-\hat{z}) \).

(iv) If \( \kappa = 2 \), then \( \lim_{n \to \infty} \mathbb{P} ((\log n)^{-1/2}(X_n - n\nu_P) \leq \hat{z}) = \mathcal{L}_{2, \hat{b}}(\hat{z}) \).

(v) If \( \kappa > 2 \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-1/2}(X_n - n\nu_P) \leq \hat{z}) = \mathcal{L}_{2, \hat{b}}(\hat{z}) \).

Since the function \( \lambda \to E_P(\rho_0^\lambda) \) is convex, the parameter \( \kappa \) is uniquely determined by the conditions of the theorem. By Jensen’s inequality, \( E_P(\log \rho_0) < 0 \) and hence \( X_n \) is transient to the right. The law \( \overline{\mathcal{L}}_\kappa \), defined precisely in Proposition 2.5.2, is closely related to \( \mathcal{L}_\kappa \), whose characteristic function is given in (2.16).

For the hitting times \( T_n \), we have:

**Proposition 2.5.2.** [49] Let the conditions of Theorem 2.5.1 hold. Then the following hold for some \( \hat{b} > 0 \):

(i) If \( \kappa \in (0, 1) \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-1/\kappa} T_n \leq t) = \mathcal{L}_{\kappa, \hat{b}}(t) \),

(ii) If \( \kappa = 1 \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-1}(T_n - nD(n)) \leq t) = \mathcal{L}_{1, \hat{b}}(t) \), for suitable \( c_0 > 0 \) and \( D(n) \sim c_0 \log n \),

(iii) If \( \kappa \in (1, 2) \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-1/\kappa}(T_n - n\nu_P^{-1}) \leq t) = \mathcal{L}_{\kappa, \hat{b}}(t) \).

(iv) If \( \kappa = 2 \), then \( \lim_{n \to \infty} \mathbb{P}((\log n)^{-1/2}(T_n - n\nu_P^{-1}) \leq t) = \mathcal{L}_{2, \hat{b}}(t) \).

(v) If \( \kappa > 2 \), then \( \lim_{n \to \infty} \mathbb{P}(n^{-1/2}(T_n - n\nu_P^{-1}) \leq t) = \mathcal{L}_{2, \hat{b}}(t) \).

We remark the scaling coefficients in Proposition 2.5.2 are chosen the same as in the corresponding limit theorems for partial sums of i.i.d. sequences. The proof that Theorem 2.5.1 follows from Proposition 2.5.2, and is based on the observation that for any positive integers \( \eta, \zeta, n \)

\[
\{T_\zeta \geq n\} \subset \{X_n \leq \zeta\} \subset \{T_{\zeta+\eta} \geq n\} \bigcup \{\inf_{k \geq T_{\zeta+\eta}} X_k - (\zeta + \eta) \leq -\eta\}.
\]  

(2.17)

Because the random variables \( \inf_{k \geq T_{\zeta+\eta}} X_k - (\zeta + \eta) \) and \( \inf_{k \geq 0} X_k \) have the same annealed distribution, the probability of the last event in (2.17) can be made arbitrary small uniformly
in $n$ and $\zeta$ by fixing $\eta$ large (since the RWRE $X_n$ is transient to the right). For $\kappa = 1$, the rest of the argument is detailed in [49, pp. 167–168], where no use of the i.i.d. assumption for $\omega$ is made at that stage, and a similar argument works for all $\kappa \in (0, 2]$.

The limit laws of [49] was extended in [63] to environments which are semi-Markov process.
CHAPTER 3. LIMIT LAWS FOR TRANSIENT RWRE ON A STRIP

In this chapter we review transient random walks on a strip in a random environment. The abstract model was introduced by Bolthausen and Goldsheid in [17]. We discuss a strong law of large numbers for the random walks in a general ergodic setup and an annealed central limit theorem in the case of uniformly mixing environments. The results presented here are borrowed from [77]. They are used to derive our theorems for the spider random walk in the next chapter.

3.1 Introduction

We consider here RWRE on the strip $\mathbb{Z} \times \{1, \ldots, d\}$ for some fixed $d \in \mathbb{N}$. In notations of Section 1.4, $d = L + 1$. Transition probabilities of the random walk are not homogeneous in the space and depend on a realization of the environment. The environment is a random sequence $\omega = (\omega_n)_{n \in \mathbb{Z}}$, and, given $\omega$, the random walk $S_n$ is a time-homogeneous Markov chain on the strip with transition kernel $H_\omega$ such that

(i) $H_\omega((n, i), (m, j)) = 0$ if $|n - m| > 1$, that is transitions from a site $(n, i)$ are only possible either within the layer $n \times \{1, \ldots, d\}$ or to the two neighbor layers;

(ii) For $n \in \mathbb{Z}$, the $d \times d$ matrices $H_\omega((n, \cdot), (m, \cdot))$ with $m = n - 1, n, n + 1$ are functions of the random variable $\omega_n$.

The RWRE on a strip were introduced by Bolthausen and Goldsheid in [17]. It is observed in [17] that the RWRE on $\mathbb{Z}$ with bounded jumps introduced by Key [50] can be viewed as a special case of this model. Other particular cases include directed-edge-reinforced random walks on graphs [48] as well as persistent RWRE on $\mathbb{Z}$ [87, 5] and, more generally, certain
RWRE on \( \mathbb{Z} \) with a finite memory. In the last case, the elements of the set \( \{1, \ldots, d\} \) represent states of the memory.

Due to the uniqueness of the escape direction in the state space of the random walk and to the finite range of the second coordinate, the model shares some common features with the well-studied nearest-neighbor RWRE on \( \mathbb{Z} \). However, due to the inhomogeneity of the kernel \( H_\omega \) in the second coordinate, there are important differences that make most of the standard one-dimensional techniques not directly available for the study of the RWRE on strips. For instance, in the contrast to the one-dimensional model, for a general RWRE on a strip 1) exit probabilities of the random walk are rather non-explicit functionals of the environment 2) excursions of the walker between layers are dependent of each other and do not form any probabilistic branching structure 3) the random times \( \tau_n = T_n - T_{n-1} \), where \( T_n \) is the first hitting time of the layer \( n \), are not independent in the quenched setting and are not stationary in the annealed setup.

The asymptotic speed of the random walk is expressed in terms of certain matrices (in particular \( a_n \)) introduced in [17] and we give a sufficient condition for the non-zero speed regime (cf. Corollary 3.2.2). To prove the law of large numbers and the limit theorem we introduce an annealed measure that makes \( \tau_n = T_n - T_{n-1} \) into a stationary ergodic sequence. The results that we obtain are similar to, although are less explicit, than the corresponding statements for nearest-neighbor RWRE on \( \mathbb{Z} \).

We turn now to a precise definition of the model. Let \( d \geq 1 \) be any integer and denote \( \mathcal{D} = \{1, \ldots, d\} \). The environment \( \omega = (\omega_n)_{n \in \mathbb{Z}} \) is a stationary and ergodic sequence of random variables taking values in a measurable space \( (\mathcal{S}, \mathcal{B}) \). We denote the distribution of \( \omega \) on \( (\Omega, \mathcal{F}) := (\mathcal{S}^\mathbb{Z}, \mathcal{B}^\otimes \mathbb{Z}) \) by \( P \) and the expectation with respect to \( P \) by \( E_P \).

Let \( \mathcal{A}_d \) be the set of \( d \times d \) matrices with real-valued non-negative entries and let \( \mathcal{P}_d \subset \mathcal{A}_d \) denote the set of stochastic matrices. Let three functions \( p = (p(i, j))_{i,j \in \mathcal{D}}, q = (q(i, j))_{i,j \in \mathcal{D}}, r = (r(i, j))_{i,j \in \mathcal{D}} : \mathcal{S} \to \mathcal{A}_d \) be given such that \( p + q + r \in \mathcal{P}_d \). For \( n \in \mathbb{Z} \), define the following random variables in \( (\Omega, \mathcal{F}, P) : p_n = p(\omega_n), q_n = q(\omega_n), r_n = r(\omega_n) \).

The random walk on the strip \( \mathbb{Z} \times \mathcal{D} \) in environment \( \omega \in \Omega \) is a time-homogeneous Markov chain \( X = (S_t)_{t \in \mathbb{Z}_+} \) taking values in \( \mathbb{Z} \times \mathcal{D} \) and governed by the quenched law \( P^\omega_t \).
transition probabilities (here \( n \in \mathbb{Z} \) and \( i, j \in \mathcal{D} \))

\[
H_\omega(x, y) := P_\omega^\mu(S_1 = y | S_0 = x) = \begin{cases} 
p_n(i, j) & \text{if } x = (n, i) \text{ and } y = (n + 1, j), 
r_n(i, j) & \text{if } x = (n, i) \text{ and } y = (n, j), 
q_n(i, j) & \text{if } x = (n, i) \text{ and } y = (n - 1, j), 
0 & \text{otherwise}, \end{cases}
\]

and initial distribution \( \mu = \mu_\omega \) of \( S_0 \). That is for any \( n \in \mathbb{N} \),

\[
P_\omega^\mu(S_0 = x_0, S_1 = x_1, \ldots, S_n = x_n) = \mu_\omega(x_0) H_\omega(x_0, x_1) H(x_1, x_2) \cdots H(x_{n-1}, x_n).
\]

Note that the initial distribution of the random walk might depend on \( \omega \). We shall specify below in this introduction the type of initial distributions in which we are actually interested in this chapter. We shall always assume that \( \mu_\omega \) is a measurable function of the environment.

Let \( \mathcal{G} \) be the cylinder \( \sigma \)-algebra on \((\mathbb{Z} \times \mathcal{D})^\mathbb{N}\), the path space of the walk. Given a collection of initial distributions \((\mu_\omega)_{\omega \in \Omega}\), the random walk on \( \mathbb{Z} \times \mathcal{D} \) in a random environment is the process \((\omega, X)\) in the measurable space \((\Omega \times (\mathbb{Z} \times \mathcal{D})^\mathbb{N}, \mathcal{F} \times \mathcal{G})\) with the annealed law \( \mathbb{P}^\mu = P \otimes P_\omega^\mu \) defined by

\[
\mathbb{P}^\mu(F \times G) = \int_F P_\omega^\mu(G) P(d\omega) = E_P(P_\omega^\mu(G); F), \quad F \in \mathcal{F}, \ G \in \mathcal{G}.
\]

That is, the annealed law of \( S_n \) is obtained by averaging its quenched law (i.e. given a fixed environment) over the set of environments.

Let us introduce some notations. We denote the first component of \( S_n \in \mathbb{Z} \times \mathcal{D} \) by \( X_n \) and the second by \( Y_n \). That is,

\[
S_n = (X_n, Y_n), \quad X_n \in \mathbb{Z}, Y_n \in \mathcal{D}.
\]

Further, we denote by \( P_\omega^\mu \) (respectively \( \mathbb{P}^\mu \)) the quenched (annealed) law of the random walk starting at site \((0, i)\). That is,

\[
P_\omega^\mu(\cdot) = P_\omega^\mu(\cdot) \text{ with } \mu_\omega((0, i)) = 1.
\]

We shall consider random walks \( S_n \) for which the starting position \( Y_0 \) may depend on the non-positive part of the environment \( \omega_- : = (\omega_n)_{n \leq 0} \). Let \( \mathcal{P}_{r_d} \) be the set of probability measures
on the finite set $D$. For any collection of distributions $\mu = \{\mu_\omega \in \mathcal{P}_d : \omega \in \Omega\}$ such that $\mu_\omega : \Omega \to \mathcal{P}_d$ is a measurable function of the environment and $\mu_\omega$ depends only on $(\omega_n)_{n \leq 0}$, we denote

$$P^\mu_\omega(\cdot) = \sum_{i \in D} P^\mu_i(\cdot) \mu_\omega(i), \quad \mathbb{P}^\mu(\cdot) = E_P(P^\mu_\omega(\cdot)).$$

We use the notation $M_d$ for the set of all such collections of distributions on $D$:

$$M_d = \{(\mu_\omega)_{\omega \in \Omega} : \omega \mapsto \mu_\omega \text{ is measurable from } \Omega \text{ to } \mathcal{P}_d \text{ and } \mu_\alpha = \mu_\beta \text{ if } \omega_-(\alpha) = \omega_-(\beta)\}$$

Throughout this thesis we use the notations $0 := (0, \ldots, 0) \in \mathbb{R}^d$, $1 := (1, \ldots, 1) \in \mathbb{R}^d$, and denote by $e_i$, $i = 1, \ldots, d$, the vectors $(0, \ldots, 1, \ldots, 0)$ of the canonical basis of $\mathbb{R}^d$. We denote by $O$ the zero $d \times d$ matrix and by $I$ the unit $d \times d$ matrix. The notions ">", "<", "positive", "negative", and "non-negative" are used for vectors and matrices in the usual way. For instance, a $d \times d$ matrix $A$ is said to be non-negative if $A \geq O$, i.e. $A(i,j) \geq 0$ for all $i, j \in D$. For a vector $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$ let $\|x\| := \max_{i \in D} |x^i|$ and for a $d \times d$ real matrix $A$ let $\|A\|$ denote the corresponding operator norm $\sup_{x \in \mathbb{R}^d, \|x\|=1} \|Ax\| = \max_{i \in D} \sum_{j \in D} |A_{i,j}|$.

Remark that if a $d \times d$ matrix $A$ is non-negative then $\|A\| = \|AI\|$.

We next introduce basic assumptions made in [12]. For $n \in \mathbb{Z}$, define the random times

$$T_n = \inf\{i : X_i = n\} \quad \text{and} \quad \tau_n := T_n - T_{n-1}, \quad (3.1)$$

with the usual convention that the infimum over an empty set is $\infty$ and $\infty - \infty = \infty$. Let $\eta_n(i,j)$ be the probability that the walker starting at site $(n,i)$ in environment $\omega$ reaches the layer $n + 1$ at point $(n + 1,j)$, i.e. (with some abuse of notations, in this and similar cases, we usually will not indicate explicitly the dependence on $\omega$)

$$\eta_n(i,j) = P^\theta_{\nu_\omega}(Y_{\tau_1} = j), \quad (3.2)$$

where $\theta : \Omega \to \Omega$ is the shift operator defined by

$$\theta_\omega(n) = \omega_{n+1}, \quad n \in \mathbb{Z}. \quad (3.3)$$

A “continued fractions” representation of $\eta_n$ in terms of the sequence $\omega_n$ is provided in [17, p. 437] (see especially identities (2.5) and (2.7) there).

The following Condition C is borrowed from the paper of Bolthausen and Goldsheid [17].
Condition C.

(C1) The sequence \((\omega_n)_{n \in \mathbb{Z}}\) is stationary and ergodic.

(C2) \(E_P(\log(1 - \|r_0 + p_0\|)^{-1}) < \infty\), \(E_P(\log(1 - \|r_0 + r_0\|)^{-1}) < \infty\).

(C3) For all \(j \in \mathcal{D}\),
\[
\sum_{i=1}^{d} q_0(i, j) > 0 \quad \text{and} \quad \sum_{i=1}^{d} p_0(i, j) > 0,
\]
P-almost surely.

(C4) \(P(\eta_0(i, j) > 0) = 1\) for all \(i, j \in \mathcal{D}\).

Note that since
\[
\min_{i \in \mathcal{D}} \sum_{j \in \mathcal{D}} q_0(i, j) = 1 - \|r_0 + p_0\| \quad \text{and} \quad \min_{i \in \mathcal{D}} \sum_{j \in \mathcal{D}} p_0(i, j) = 1 - \|r_0 + q_0\|,
\]
(3.4)
Condition C2 implies that for all \(i \in \mathcal{D}\), P-a.s.,
\[
\sum_{j=1}^{d} q_0(i, j) > 0 \quad \text{and} \quad \sum_{j=1}^{d} p_0(i, j) > 0,
\]
(3.5)
We focus here on random walks \(S_n\) which are transient to the right, i.e.
\[
P^i(\lim_{n \to \infty} X_n = \infty) = 1 \quad \text{for all} \quad i \in \mathcal{D}.
\]
We next recall the criterion for the transient behavior of \(S_n\) obtained by Bolthausen and Goldsheid in [17]. Let
\[
\gamma_n := (I - q_n \eta_{n-1} - r_n)^{-1},
\]
(3.6)
and introduce for \(n \in \mathbb{Z}\) the following random matrices:
\[
a_n := \gamma_n q_n = (I - q_n \eta_{n-1} - r_n)^{-1} q_n.
\]
(3.7)
As we shall see, the matrices \(a_n\) play for transient RWRE on a strip essentially the same role as the random variables \(\rho_n = P_\omega(\text{jump from } n \text{ to } n-1)/P_\omega(\text{jump from } n \text{ to } n+1)\) do for the nearest-neighbour RWRE on \(\mathbb{Z}\).
By the sub-additive ergodic theorem of [35], Condition C2 ensures that there exists a non-random number \( \lambda \) such that

\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \cdot E_P(\log \|a_n \cdots a_1\|) = \lim_{n \to \infty} \frac{1}{n} \cdot \log \|a_n \cdots a_1\|, \quad P - \text{a.s.} \quad (3.8)
\]

It is shown in [17] that under Condition C, \( S_n \) is transient to the right if and only if \( \lambda < 0 \). Therefore we will assume the following:

**Condition D.** \( \lambda < 0 \).

This condition is a generalization of the transience criterion \( E_P(\log \rho_0) < \infty \) for the RWRE on \( \mathbb{Z} \) (see e.g. [97, Section 2]).

The rest of the chapter is organized as follows. The main results are stated in Section 3.2. A law of large numbers for \( S_n \) (cf. Theorem 3.2.1) and a criterion for a positive asymptotic speed (cf. Corollary 3.2.2) are stated in Subsection 3.3. An annealed central limit theorem for a normalized random variable \( S_n \) in the case of uniformly mixing environments is given by Theorem 3.2.3 and is stated in Section 3.4. In particular, it is shown in [12] that the law of the Markov chain converges toward a limiting distribution.

### 3.2 Limit theorems for a RWRE on a strip

Let \( \pi = (\pi_\omega)_{\omega \in \Omega} \in \mathcal{M}_d \) be a collection of probability measures on \( \mathcal{D} \) defined \( P \)-a.s. by

\[
\pi_\omega = \lim_{n \to \infty} e^{i \eta_{-n}(\omega) \cdots \eta_{-1}(\omega)}. \quad (3.9)
\]

It can be shown [17] that the limit in (3.9) exists \( P \)-a.s., does not depend on \( i \), and defines a probability measure on \( \mathcal{D} = \{1, \ldots, d\} \) whose support is the whole \( \mathcal{D} \).

It is shown in Section 3.3 that the sequence \( (\tau_n)_{n \in \mathbb{N}} \) is stationary and ergodic under \( \mathbb{P}^{\pi} \) and the following law of large numbers holds.

**Theorem 3.2.1.** Let Conditions C and D hold. Then for every \( \mu \in \mathcal{M}_d \), the following limits exist \( \mathbb{P}^{\mu} \)-a.s. and the equality holds:

\[
v_p := \lim_{n \to \infty} X_n / n = \lim_{n \to \infty} n / T_n = 1 / E_P(\pi_\omega \cdot (b_0 + a_0 b_{-1} + a_0 a_{-1} b_{-2} + \ldots)),
\]

where \( b_n := \gamma_n 1 \).
The formula for the asymptotic speed yields the following characterization of the positive speed regime.

**Corollary 3.2.2.** Assume that

(i) Condition C holds.

(ii) There exists $\varepsilon > 0$ such that $P$-a.s., $\sum_{j=1}^{d} p_n(i,j) > \varepsilon$ for all $i \in D$ and all $n \in \mathbb{Z}$.

(iii) $\lim \sup_{n \to \infty} 1/n \log E_P(\|a_0 a_1 \cdots a_{n-1}\|) < 0$.

Then $P(\nu_P > 0) = 1$.

Assumption (iii) of Corollary 3.2.2 is a generalization of the corresponding condition

$$\lim \sup_{n \to \infty} \frac{1}{n} \log E_P(\rho_0 \rho_1 \cdots \rho_{n-1}) < 0$$

for RWRE on $\mathbb{Z}$ (cf. [97, Section 2]). Note that by Jensen’s inequality this assumption implies Condition D. It can be shown [12] that Condition C ensures $\nu_P > 0$.

In the case where $(\omega_n)_{n \in \mathbb{Z}}$ is a uniformly mixing sequence we have the following central limit theorem. Let

$$\phi(n) = \sup \{ P(A|B) - P(A) : A \in \sigma(\omega_i : i \geq n), B \in \sigma(\omega_i : i \leq 0), P(B) > 0 \}.$$

Recall that the sequence $\omega_n$ is called **uniformly mixing** if $\phi(n) \to_{n \to \infty} 0$. For uniformly mixing sequence it holds that (cf. [30, p. 9])

$$|E_P(fg) - E_P(f)E_P(g)| \leq \varphi(n)\|f\|_\infty\|g\|_1 \text{ with } \varphi(n) := 2\phi(n),$$

for all bounded $\sigma(\omega_i : i \geq n)$-measurable functions $f$ and all bounded $\sigma(\omega_i : i \leq 0)$-measurable functions $g$.

**Theorem 3.2.3.** Let Condition C hold and assume in addition that

(i) There exists $\varepsilon > 0$ such that $P$-a.s., $\sum_{j=1}^{d} p_n(i,j) > \varepsilon$ for all $i \in D$ and all $n \in \mathbb{Z}$.

(ii) $\lim \sup_{n \to \infty} n^{-1} \log E_P(\|a_0 a_1 \cdots a_{n-1}\|^2) < 0$.

(iii) $\sum_{n=1}^{\infty} \varphi(2^n) < \infty$, where the mixing coefficients $\varphi(n)$ are defined in (3.10).
(iv) \( \limsup_{n \to \infty} n^{-1} \log E_P(c_0 c_1 \cdots c_{n-1}) < 0 \), where \( c_n = 1 - \max_{i \in D} \min_{j \in D} \eta_n(i,j) \).

Then there exists a constant \( \sigma > 0 \) such that for any \( \mu \in \mathcal{M}_d \), \( n^{-1/2}(X_n - n \cdot v_{\mu}) \Rightarrow N(0, \sigma^2) \) under \( \mathbb{P}_\mu \) as \( n \to \infty \), where \( N(0, \sigma^2) \) stands for the normal distribution with variance \( \sigma^2 \).

Assumption (ii) of the theorem implies by Jensen’s inequality Condition D. Moreover, by Corollary 3.2.2, \( v_{\mu} > 0 \).

### 3.3 Law of large numbers for \( S_n \)

This section is devoted to the proof of Theorem 2.3.4. The law of large numbers for \( S_n \) is derived from the corresponding statement for the hitting times \( T_n \) introduced in (3.1). More precisely, we have:

**Lemma 3.3.1.** Let Conditions C and D hold and assume that \( \mathbb{P}_\mu(\lim_{n \to \infty} T_n/n = \alpha) = 1 \) for some \( \alpha \in (1, \infty] \) and \( \mu \in \mathcal{M}_d \). Then \( \mathbb{P}_\mu(\lim_{n \to \infty} X_n/n = 1/\alpha) = 1 \).

In the contrast to the nearest-neighbour RWRE on \( \mathbb{Z} \), the sequence \( \tau := (\tau_n)_{n \in \mathbb{N}} \) defined in (3.1) is in general not stationary for transient random walks on the strip. However, as the next lemma shows, the sequence is stationary and ergodic under \( \mathbb{P}_\pi \), where \( \pi = (\pi_\omega)_{\omega \in \Omega} \) is defined in (3.9).

**Lemma 3.3.2.**

(a) \( \tau = (\tau_n)_{n \in \mathbb{N}} \) is a stationary sequence under \( \mathbb{P}_\pi \).

(b) \( \tau = (\tau_n)_{n \in \mathbb{N}} \) is ergodic under \( \mathbb{P}_\pi \).

Before we turn to the proof of the lemma, let us have a closer look on the matrices \( \eta_n \) introduced in (3.2). Conditioning on the direction of the first step of the random walk we get that \( \eta_n \) verify the following equation:

\[
\eta_n = p_n + r_n \eta_n + q_n \eta_{n-1} \eta_n,
\]

which is equivalent to

\[
\eta_n = (I - q_n \eta_{n-1} - r_n)^{-1} p_n = \gamma_n p_n, \quad n \in \mathbb{Z}.
\] (3.11)
In fact, under Conditions C and D, \((\eta_n)_{n \in \mathbb{Z}}\) is the unique stationary sequence which satisfy recursion (3.11) (cf. [17]).

Applying the ergodic theorem to the sequence \(\tau\), we obtain that \(\mathbb{P}^\pi (\lim_{n \to \infty} T_n/n = \alpha) = 1\), where \(\alpha := \mathbb{E}^\pi (T_1)\). Since \(P(\pi_\omega > 0) = 1\) [77], the a.s. convergence of \(T_n/n\) under \(\mathbb{P}^\pi\) implies the a.s. convergence under \(\mathbb{P}^\mu\) for any \(\mu \in \mathcal{M}_d\). In the next lemma we compute the limit \(\alpha = 1/\nu_\mu\). For any integer \(n \leq 0\) and \(j \in \mathcal{D}\) let

\[
u_j^i(\omega) = \mathbb{E}_{\omega}(T_1|S_0 = (n, j)),
\]

(3.12)

and denote by \(\nu_n\) the vector \((\nu_1^1, \ldots, \nu_d^n)\). Then \(\alpha = \mathbb{E}_P(\pi_\omega \nu_0(\omega))\) and we have:

**Lemma 3.3.3.** Let the conditions of Theorem 3.2.1 hold. Then \(P\)-a.s.,

\[
u_0 = b_0 + a_0 b_{-1} + a_0 a_{-1} b_{-2} + \ldots < \infty,
\]

where \(b_n := \gamma_n 1\) and the random matrices \(\gamma_n\) and \(a_n\) are defined in (3.6) and (3.7) respectively.

### 3.4 Annealed CLT for uniformly mixing environments

This sections is devoted to the proof of Theorem 3.2.3. The annealed CLT for \(S_n\) is derived from the corresponding result for the hitting times \(T_n\) by using the following observation.

**Lemma 3.4.1.** Let the conditions of Theorem 3.2.3 hold and assume that for some \(\mu \in \mathcal{M}_d\) and suitable constants \(\alpha > 0\) and \(\sigma > 0\), \((T_n - \alpha n)/\sqrt{n} \Rightarrow N(0, \sigma^2)\) under \(\mathbb{P}^\mu\). Then under \(\mathbb{P}^\mu\),

\[
(X_n - \alpha^{-1} n)/\sqrt{n} \Rightarrow N(0, \sigma^2 \alpha^{-3}).
\]

In order to show that the CLT holds for \(T_n\) we shall apply the following general central limit theorem for uniformly mixing sequences to \(Z_n = \tau_n - \nu_\mu\). The theorem is a combination of a central limit theorem for dependent variables of Ibragimov (cf. [43, Theorem 2.2]) and a result of Bradley (cf. [19, Theorem 1]).

**Theorem 3.4.2.** Let \((Z_n)_{n \in \mathbb{Z}}\) be a stationary random sequence such that \(\mathbb{E}^\pi (Z_0) = 0\) and \(\mathbb{E}^\pi (Z_0^2) < \infty\). For \(n \in \mathbb{N}\) let \(\mathcal{F}_n = \sigma(Z_i : i \geq n)\), \(\mathcal{F}_n = \sigma(Z_i : i \leq n)\),

\[
\beta(n) = \sup_{m \in \mathbb{N}} \{ \mathbb{P}^\pi (A|B) - \mathbb{P}^\pi (A) : A \in \mathcal{F}_n^{n+m}, B \in \mathcal{F}_m, \mathbb{P}^\pi (B) > 0 \}
\]

(3.13)
and suppose that \( \sum_{n=1}^{\infty} \beta(2^n) < \infty \). Further, let \( W_n = \sum_{i=1}^{n} Z_i \) and assume that \( \mathbb{E}^\pi(W_n^2) \to \infty \) as \( n \to \infty \). Then, \( W_n/\sqrt{n} \Rightarrow N(0,\sigma^2) \) for some \( \sigma > 0 \).

In our case, the condition \( \lim_{n \to \infty} \mathbb{E}^\pi(W_n^2) = \infty \) is fulfilled because otherwise the sequence \( W_n = T_n - n\nu_p^{-1} \) should be of the form \( S_n = J_n - J_0 \), where \( (J_n)_{n \geq 0} \) is a stationary (under \( \mathbb{P}^\pi \)) sequence of random variables with a finite support (cf. [44, Theorem 18.2.2], the mixing condition of the theorem is implied for the sequence \( Z_n = \tau_n - \nu_p \) by Proposition 3.4.3 stated below). This is impossible since \( T_n \) is unbounded from above for instance in virtue of the first inequality in (3.5). Therefore we need only to check that the the second moment condition and the mixing condition of Theorem 3.4.2 hold for \( Z_n = \tau_n - \nu_p \).

The next proposition shows that the sequence \( \tau_n - \nu_p \) is uniformly mixing under \( \mathbb{P}^\pi \) with a mixing rate fast enough to apply the general CLT. In the case of one-dimensional RWRE closely related results can be found in [59] (cf. Lemma 4) and [97] (cf. Lemma 2.1.10). The proof for RWRE on strips is more involved but is based on essentially the same idea. Namely, it holds with a large probability that \( \inf_{i \geq T_{m+n}} X_i \geq m + n/2 \), i.e. the \( \sigma \)-fields \( \mathcal{F}_m \) and \( \mathcal{F}^{m+n} \) are essentially well-separated enabling the use of the mixing properties of the environment. Note that asymptotically, \( P_\omega(Y_{T_{m+n}} = j|Y_{T_m} = i) \approx \pi_{\eta_{m+n}}(j) \) uniformly on \( i, j \in \mathcal{D} \) and \( m \geq 0 \) for large \( n \), that is the random walk “forgets” its starting position \( Y_0 \).

**Proposition 3.4.3.** Let the conditions of Theorem 3.2.3 hold and let \( Z_n := \tau_n - \nu_p, n \in \mathbb{N} \). Then \( \sum_{n=1}^{\infty} \beta(2^n) < \infty \) where the mixing coefficients \( \beta(n) \) are defined in (3.13).

We need the following lemma.

**Lemma 3.4.4.** Let the conditions of Theorem 3.2.3 hold. Then, there exist constants \( K_3 > 0 \) and \( K_4 > 0 \) such that

\[
E_P(\max_{i \in \mathcal{D}} P_i^\mu(X_k = -n \text{ for some } k \in \mathbb{N})) = E_P(\|\eta_0 \cdots \eta_{-n+1}\|) \leq K_3 e^{-K_4 n}, \quad n \in \mathbb{N}.
\]

The next lemma reduces the proof of Theorem 3.2.3 to the check that the second moment of the random variable \( \tau_1 \) is finite.

**Lemma 3.4.5.** Let the conditions of Theorem 3.2.3 holds and suppose that \( \mathbb{E}^\pi(\tau_1^2) < \infty \). Then there is a constant \( \sigma > 0 \) such that \( T_n \Rightarrow \sigma N(0,1) \) under \( \mathbb{P}^\mu \) for any \( \mu \in \mathcal{M}_d \).
CHAPTER 4. CLT FOR A SPIDER RWRE ON $\mathbb{Z}$

The chapter is devoted to the proof of Theorem 1.5.4 stated in the introduction (see Section 1.5.3). The organization of the chapter is as follows. In Section 4.1 we reduce Theorem 1.5.4 to the corresponding statement for hitting times of the spider random walk. The limit theorem for the hitting times is deduced from the general CLT stated above in Theorem 3.4.2. In Section 4.2 we verify necessary mixing conditions for the hitting times and in Section 4.3 we check a required second moment condition.

4.1 Reduction to the hitting times $T_n$

First we define the following stopping time for the SRWRE. Let $T_0 = 0$ and, for $n \in \mathbb{N}$,

$$T_n = \inf \{ t > 0 : S_1(t) = n \}. \quad (4.1)$$

The CLT for the spider random walk is reduced to a CLT for the hitting times $T_n$ in the following lemma.

**Lemma 4.1.1.** Let the conditions of Theorem 1.5.4 hold and assume that for some $x_0 \in L$ and suitable constants $\alpha > 0$ and $\sigma > 0$, $(T_n - \alpha n)/\sqrt{n} \Rightarrow N(0, \sigma^2)$ under $P^{x_0}$. Then under $P^{x_0}$, $(S_1(t) - \alpha^{-1}t)/\sqrt{t} \Rightarrow N(0, \sigma^2\alpha^{-3})$.

The proof is similar to that for RWRE on $\mathbb{Z}$ (see [49] or [4]) even though the sequence $x_{T_n}$ (i.e. the configuration of the spider at time $T_n$) is not necessarily stationary under $P^{x_0}$. In that, the spider RWRE is similar to a RWRE on a strip. In fact, under some extra technical assumption (necessary to formally fit into the framework which was introduced in [17]), Lemma 4.1.1 can be deduced from Lemma 5.1 in [77]. For the reader’s convenience we provide below a complete proof of the lemma.
Proof of Lemma 4.1.1. For $t \in \mathbb{R}_+$ let
\[
R_t := \inf_{u \geq t} S_1(u).\tag{4.2}
\]
Thus $R_t$ denotes the minimal location of the spider’s first leg after time $t$. Since the spider walk is transient to the right, $R_t$ is finite with probability one, that is $\mathbb{P}(R_t > -\infty) = 1$.

For any positive integers $a, b$ and a positive real $t$ we have:
\[
\{T_a \geq t\} \subset \{S_1(n) \leq a\} \subset \{T_{a+b} \geq t\} \bigcup_{t \geq T_{a+b}} \inf_{t \geq T_{a+b}} \{S_1(t) - (a + b) \leq -b\}.\tag{4.3}
\]
Since $S_t$ is transient to the right and
\[
\mathbb{P}^{x_0}(\inf_{t \geq T_{a+b}} S_1(t) \leq a) = \mathbb{E}_P\left(\sum_{x \in L} P_{x_0}^{x, a+b}(R_0 \leq -b) P_{x_0}^{x}(x(T_{a+b}) = j)\right) \leq \max_{x \in L} \mathbb{P}^x(R_0 \leq -b),
\]
the probability of the rightmost event in (4.3) can be made arbitrary small uniformly in $t$ and $a$ by fixing $b$ large.

It follows from (4.3) that
\[
\mathbb{P}^{x_0}(T_a \geq t) \leq \mathbb{P}^{x_0}(S_1(t) \leq a) \leq \mathbb{P}^{x_0}(T_{a+b} \geq t) + \max_{j \in L} \mathbb{P}^j(R_0 \leq -b).\tag{4.4}
\]
To complete the proof one can set $a(t) = tv_p + x\sqrt{3/2}\sqrt{t} + o(\sqrt{t})$, $x \in \mathbb{R}$, and then pass to the limit in (4.4), letting first $t$ and then $b$ go to infinity. The full argument is detailed in [4, pp. 344] and thus is omitted here. 

For $n \in \mathbb{Z}$ let
\[
\tau_n = T_n - T_{n-1} \quad \text{and} \quad Z_n = \tau_n - v_P.\tag{4.5}
\]
In the subsequent section we will verify the conditions of Theorem 3.4.2 for the sequence $Z_n$ defined in (4.5).

4.2 Mixing properties of the sequence $Z_n$

Let $\chi_0 = 0$ and $(\chi_n)_{n \in \mathbb{N}}$ be the sequence of jump times [57] of the Markov chain $(S_t)_{t \in \mathbb{R}_+}$. That is,
\[
\chi_n = \inf\{t > \chi_{n-1} : S^+(t) \neq S^-(t)\},\tag{4.6}
\]
where \( S^+(t) := \lim_{u \to t^+} S(u) \) and \( S^-(u) := \lim_{u \to t^-} S(u) \) are, respectively, the right and the left limits of the right-continuous process \((S_t)_{t \in \mathbb{R}^+}\).

For \( n \in \mathbb{N} \), denote
\[
h_n := \chi_n - \chi_{n-1}
\]
(4.7)
Assuming that the spider is transient to the right, denote sift of the configuration on \( \mathbb{Z} \) by \( \Theta_x \). Thus \( \Theta_x : \mathbb{Z}^N \to \mathbb{Z}^N \) such that
\[
\Theta_x S = (S_1 - x, S_2 - x, ..., S_N - x).
\]
(4.8)
Furthermore, denote by \( \theta_x \) the corresponding shift of the environment. That is, \( \theta_x : \Omega \to \Omega \) such that
\[
(\theta_x \omega^+)_n = \omega^+_{n+x}, \quad n \in \mathbb{Z}.
\]
Let \( \varpi_n \) denote the environment viewed from the position of the spider at time \( \chi_n \). That is,
\[
\varpi_n := \theta_{S_1(\chi_n)} \omega.
\]
By our assumption, both \((Y(\chi_n), \varpi_n)_{n \in \mathbb{Z}^+}\) and \((Y(\chi_{n-1}), h_n, \varpi_{n-1})_{n \in \mathbb{N}}\) are aperiodic Harris-recurrent Markov chains. Furthermore, each \( h_n \) is distributed as an exponential random variable with parameter that doesn’t exceed \( N : \mathbb{P}(h_n > t) = e^{-\lambda_n t} \) for all \( t \geq 0 \) and any \( n \in \mathbb{N} \), where \( \lambda_n = \lambda(Y(\chi_{n-1}), \varpi_{n-1}) \) is a random variable that depends on the configuration at time \( \chi_{n-1} \) and the environment viewed at that time from the position of the spider.

As in [37], we introduce the discrete-time jump chain \((\Xi_n)_{n \in \mathbb{Z}^+}\) of the Markov chain \((S_t)_{t \in \mathbb{R}^+}\). That is [57], \( \Xi(0) = S(0) \) and for \( n \in \mathbb{N} \),
\[
\Xi(n) = S(\chi_n).
\]
(4.9)
Clearly, \( \Xi(n) \) is a RWRE on the strip \( \mathbb{Z} \times L \). Notice that the environment in the layer \( \{n\} \times L \) for any \( n \in \mathbb{Z} \) depends only on the finite interval \( \{\omega^+_k : n-d < k < n+d\} \), where \( d \) is the maximal allowed distance between spider’s legs:
\[
d := \max_{x \in \mathbb{L}} \max_{1 \leq i \leq N} |x_i - x_j|.
\]
Since the original environment \( \omega \) is an i.i.d sequence, we arrive to the following lemma:
Lemma 4.2.1. Consider the process $\Xi(n)$ as a RWRE on the strip $\mathbb{Z} \times L$. Then the environment in this process is a $d$-dependent random sequence. In particular, it is $\varphi$-mixing with exponential rate.

It is a routine check that under Assumption 1.5.1 the process $\Xi$ satisfies Conditions C and D introduced in Section 3.1. In particular, the latter condition is satisfied because the spider is transient to the right.

The ladder (RWRE on a strip) representation of the spider random walk allows us to use some of the results presented in Chapter 3. In particular, we can adopt the definition of the stationary measure $\pi$ from (3.9). We then have:

**Proposition 4.2.2.** Let Assumption 1.5.1 hold and suppose in addition that $\kappa/N > 1$. Then:

(a) $\tau = (\tau_n)_{n \in \mathbb{N}}$ is a stationary sequence under $\mathbb{P}^\pi$.

(b) $\tau = (\tau_n)_{n \in \mathbb{N}}$ is ergodic under $\mathbb{P}^\pi$.

(c) $\sum_{n=1}^\infty \beta(2^n) < \infty$ where the mixing coefficients $\beta(n)$ are defined in (3.13).

The proof of the proposition is nearly verbatim the same as in Lemma 3.3.2 (parts (a) and (b) of the above proposition) and Proposition 3.4.3 (part (c) of Proposition 4.2.2).

### 4.3 Second moment conditions

In view of Lemma 4.1.1, Theorem 3.4.2, and Proposition 4.2.2, in order to complete the proof of our main result stated in Theorem 1.5.4 it suffices to show that

$$\mathbb{E}^\pi(\tau_1^2) < \infty. \quad (4.10)$$

The analogue of this assertion for a general RWRE on a strip is stated in Lemma 3.4.5. The rest of this section is devoted to the proof that (4.10) holds true under the conditions of Theorem 1.5.3.

Following [37] we introduce the following regeneration times. Let $\zeta_0 = 0$, and for $n \in \mathbb{N},$

$$\zeta_n = \inf\{j > \zeta_{n-1} : \Xi_1(j) > \Xi_1(\zeta_{n-1}) \text{ and } \Xi = \Theta_{\Xi_1(j)}Y_0\} \quad (4.11)$$
Notice that according to this definition, $\Xi_1(\zeta_1) = S_1(T)$, where $T$ is the first time when the spider walk returns to the initial configuration while the first leg is located to the right of zero:

$$T := \inf\{t > 0 : Y(t) = Y(0) \text{ and } S_1(t) > 0\}.$$

Recall $\rho_n$ from (1.3) and $V_n$ from (2.3). Following the idea of [37], we make the following definition.

**Definition 4.3.1.** The potential $V = (V_n)_{n \in \mathbb{Z}}$ is said to be $(q,t)$-good if for fixed constant $K > 0$, fixed $t > 1$, $q > 0$, and $\varepsilon \in (0; 1)$ the following three inequalities hold:

1. $V(-K \ln t) \geq \frac{q + 1 + \varepsilon}{N} \ln t$,
2. $V(K \ln t) \leq -\frac{q + 1 + \varepsilon}{N} \ln t$,
3. $\max_{i \in [-K \ln t, K \ln t]} \max_{j \geq i} (V(j) - V(i)) \leq \frac{1 - \varepsilon}{N} \ln t$

We will denote the set of $(q,t)$-environments is denoted by $\Lambda_{q,t}$. In this thesis we are primarily concerned with $\Lambda_{2,t}$. Observe that:

$$P(T > t) = P(T > t, \Lambda_{q,t}) + P(T > t, \Lambda_{q,t}^c) \leq P(T > t, \Lambda_{q,t}) + P(\Lambda_{q,t}^c). \quad (4.12)$$

For the last term in (4.12) we have:

$$P(\Lambda_{q,t}^c) \leq P\left(V[-K \ln t] < \frac{q + 1 + \varepsilon}{N} \ln t\right) + P\left(V[K \ln t] > -\frac{q + 1 + \varepsilon}{N} \ln t\right) + P\left(\max_{i \in [0,2K \ln t]} \max_{j \geq i} (V(j) - V(i)) > \frac{1 - \varepsilon}{N} \ln t\right) \leq 2P\left(V[K \ln t] > -\frac{q + 1 + \varepsilon}{N} \ln t\right) + P\left(\max_{i \in [-K \ln t,K \ln t]} \max_{j \geq i} (V(j) - V(i)) > \frac{1 - \varepsilon}{N} \ln t\right). \quad (4.13)$$

Since $\varepsilon \in (0, 1)$, we have:

$$P(\Lambda_{q,t}^c) \leq 2P\left(V[K \ln t] > -\frac{q + 2}{N} \ln t\right) + P\left(\max_{i \in [0,2K \ln t]} \max_{j \geq i} (V(j) - V(i)) > \frac{1 - \varepsilon}{N} \ln t\right) \leq P\left(V[K \ln t] - EP(V(1)) \ln t \times K \ln t > a\right) + \sum_{i=0}^{2K \ln t} P\left(\max_{j \geq i} (V(j) - V(i)) > \frac{1 - \varepsilon}{N} \ln t\right)$$
if \(-q^2 > K(E_P(V(1)) + a)\). For instance, one can choose
\[
a = -\frac{1}{2} E_P(V(1)) \quad \text{and} \quad K > -\frac{2(q + 2)}{N E_P(V(1))}.
\]

Using the above upper bound for the measure of the event \(\lambda_{q,t}\), it is shown in [37, p. 141] that for \(K\) large enough and \(\varepsilon\) sufficiently small we have
\[
P(\Lambda_{q,t}^c) \leq C K \ln \frac{t}{t^\frac{\kappa(1-\varepsilon)}{N}}
\]
for some constant \(C > 0\). The upper bound is determined by the asymptotic behavior of the second term in the above decomposition. To establish this upper bound an estimate derived in [34] is used in [37]. The first term in the above is exponentially small due to the large deviation principle.

We now turn to an upper bound for the first term in the right-hand side of (4.12). For \(q = 1\) it is shown in [37] that, with probability one,
\[
\sup_{\omega \in \Lambda_{q,t}} P_\omega(T > t) \leq \frac{C_1}{t^{q+\varepsilon}}
\]
(4.15)
The proof can be carried over to a general \(q > 0\) verbatim. Combining (4.12), (4.14), and (4.15), and choosing \(\varepsilon\) small enough so that \(\frac{\kappa(1-\varepsilon)}{N} > 2\) in (4.14), we obtain that
\[
\mathbb{E}_x(T) = 2 \int_0^\infty t \mathbb{P}^x(T > t) \, dt < \infty
\]
for all initial configuration \(Y_0 = x\). Since the number of possible initial configurations is assumed to be finite and \(P_\omega(\tau_1 \leq T) = 1, \ P\text{-a.s.}\), the last inequality implies (4.10). The proof of Theorem 1.5.4 is complete.
CHAPTER 5. FUTURE WORK

We conclude the thesis with a brief discussion of some possible future research directions.

**Non-Gaussian limit theorems for SRWRE** By now, the RWRE on $\mathbb{Z}$ is a fairly well-understood object. The paper [37] settled one important basic question about spiders in random environment, but many others are left open. In particular, the main result in [37] prompts one to conjecture that the limit theorems for regular RWRE can be carried over to the spider random walk with the parameter $\rho_0 = \frac{1 - \omega_0}{\omega_0}$ (where $\omega_0$ is the probability of moving from 0 to 1 in a fixed environment) replaced by $\frac{\omega_0}{N}$, where $N$ is the number of legs of the spider. We expect that the difference between the continuous time model of spider random walks and the usual discrete-time RWRE will propagate into an additional scaling constant corresponding to the expected holding time of the spider in a site of $\mathbb{Z}$ (since holding times are exponentially distributed they cannot affect the exponent of power-laws). We believe that the limit theorem for a recurrent spider in random environment can be proved using the original methods of Sinai and in particular extending the notion of the *valleys* of the random potential (see Section 2.5 in [97]) to the spider random walk. The limit theorems in the transient regime appears to be a more involved question. We believe that the most promising approach to attack this problem is by adaptation of the method of [33] relying on an appropriate extension of the notion of the valleys to transient random walks in random environment.

**Large deviation principle and Berry-Esseen type estimates** The law of large numbers for the spider random walk, obtained in [12], can be complemented by the corresponding large deviation bounds (describing probabilities of rare events, i.e. *large deviations* from the law of large numbers). In fact, using the ladder representation (see Section 1.4.3 of this pro-
positional) and general results for additive functionals of Markov chains (see for instance [68]), large deviation estimates can be obtained for a much larger class of nanomechanical devices. Remarkably, an explicit form of the rate function for the biped model can be obtained based on the technique developed in [12].

With respect to the central limit theorem that is proved in [12], it is both of theoretical and practical interest to estimate the error in the normal approximation. The rate of convergence $O(n^{-1/2})$ of the (centered and normalized) sample mean in the classical CLT for IID variables is given by the Berry-Esseen theorem (see for instance Section 2.4 in [32]). A good control on the characteristic function of a certain Markov additive process associated with the biped, should allow one to obtain, in addition to the large deviation principle, the corresponding Berry-Esseen type estimates for the model studied in [12].

**Planar symmetric bipeds** An important particular case of the general model introduced in [36] is $G = \mathbb{Z}^d$, $d \geq 2$. This model seems to be especially well-motivated by possible applications for $d = 2$ (see, for instance, the experimental setup of [71]). The recurrence and transience criteria for the model with identically distributed jump kernels for all legs follow from general results of [36]. In a work in progress we prove a central limit theorem for the bipedal version of this model and also study implications of using different norms for the definition of the distance on the diffusion coefficient. The techniques that we use are adapted from [12]. So far, we have succeeded in getting explicit expressions for the correlation matrix of the limiting Brownian motion only for the most interesting case $d = 2$ and symmetric (equally distributed) legs. The issue of dependence on the underlying norm is of experimental interest for design and control problems pertaining to synthetic DNA molecular bipeds. (See also the experimental setups of [95] and [71].)

**Introducing correlation between transition kernels of the legs** An interesting direction we are interested in pursuing is relaxing the independence condition for the transition laws of the legs. This has been experimentally achieved for the first time for an actual synthetic DNA biped by Omabegho et al. [70]. At a very abstract level, this is part of an already
existing theory: the legs are additive functionals of Markov chains, and if we add dependence they remain as such for a high-dimensional process. Furthermore, fundamental results, such as the law of large numbers and the central limit theorem, with some rather non-explicit expressions for the parameters of the limiting Brownian motion, can be readily obtained from the ladder representation described below. It appears that obtaining explicit values for the limiting distribution for a general spider in higher dimensions requires developing new techniques and methods, beyond those of [7] and [12]. We remark that the interaction between the legs can be added to the model through introducing an interaction with a common non-homogeneous (in time or in space) environment.

**Some additional directions for future research** There are two interesting questions regarding the speed of transient spider random walks on a general graph posed in [36]. The first one concerns the existence of the limit of the asymptotic speed when the maximal possible diameter $S$ of the spider approaches infinity, while the second deals with the existence of the zero-speed regime for transient graphs. In addition, although some interesting initial results for spider random walks on trees are obtained in [36], this model clearly deserves additional study. It is of special interest because due to the branching structure of the tree one should expect that the spider random walk is much slower than its regular counterpart. It is actually shown in [36], that the speed of the spider on a homogeneous tree is smaller than the speed of the simple random walk on the same tree, but we are interested in a more detailed characterization of the gap between the two quantities.
BIBLIOGRAPHY


