Negligibility in non-locally convex spaces

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Negligibility in non-locally convex spaces

by

Charles Allen Riley

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2. NEGLIGIBILITY PROPERTIES AND APPLICATIONS</td>
<td>4</td>
</tr>
<tr>
<td>3. AN ISOTOPY THEOREM</td>
<td>26</td>
</tr>
<tr>
<td>4. ADDITIONAL PROPERTIES OF SHRINKABLE NEIGHBORHOODS AND SOME FACTS CONCERNING WEAKENING TOPOLOGIES</td>
<td>34</td>
</tr>
<tr>
<td>5. BIBLIOGRAPHY</td>
<td>43</td>
</tr>
<tr>
<td>6. ACKNOWLEDGMENT</td>
<td>45</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

A set $A$ in a topological space $X$ is negligible if $X$ is homeomorphic to $X\setminus A$. The problem of topological classification of linear topological spaces has led to an interest in negligibility in these spaces. For example, Anderson and Bing in [3] use negligibility properties in $s$ and $l_2$ to show these spaces homeomorphic. Negligibility investigations in linear topological spaces include [2], [3], [4], [5], [9], and [10]. Early results of Klee pertain to normed linear spaces, while Anderson and Bing in [3] have shown countable unions of compact sets negligible in $s$. Results of Anderson [1], Bessaga and Pelczynski [6], and Kadec [8] show that all separable Frechet spaces (locally convex complete metrizable linear topological spaces) are mutually homeomorphic, and negligibility of countable unions of compact sets in these spaces follows. Bessaga and Klee [4] work in a space admitting a countable collection of convex bodies whose intersection is a point, and Anderson in [2] obtains a result in an $\aleph_0$ space, i.e., an infinite dimensional space admitting a Schauder basis $\{b_n\}$ with continuous coordinates, and an open neighborhood $U$ of 0 such that $b_n \notin U$ for each $n$. Bessaga [5] shows an extensive class of sets are negligible in spaces admitting continuous incomplete norms. Each space above admits a continuous linear functional, so that the spaces $L^p$, $0 < p < 1$, among others, are not covered. Except
for theorems on s, and in certain cases in \([4]\), the spaces treated admit weaker linear topologies. Bessaga and Klee \([4]\) show that each infinite dimensional normed linear space admits a weaker norm topology. In view of (2.2) below, it would be nice to extend this to a larger class of spaces, but this seems a difficult problem. The main results of this paper are (2.1) and (2.12). Theorem (2.1) shows negligibility of compact sets in a wide class of linear topological spaces, including \(L^p\), \(0 < p < 1\). For \(L^p\), \(0 < p < 1\), negligibility results are obtainable indirectly by way of a modification of Mazur's proof \([12]\) that \(L^p\) is homeomorphic to \(L'\), \(p > 1\), which extends this to \(0 < p < 1\). Theorem (2.12) treats a more restricted class of spaces, and generalizes Bessaga's theorem that if \(X\) is a linear topological space and \(K \subseteq X\) is such that \(X\) admits a continuous incomplete norm \(w\) with \(K\) closed in the completion of \(w\), then \(K\) is negligible. Theorem (2.12) requires \(X\) admit a continuous paranorm with certain properties, and includes a result on \(L^p\), \(0 < p < 1\). In Chapter 3 an isotopy theorem is established which settles a question in \([10]\). Chapter 4 lists properties of shrinkable sets useful in obtaining negligibility results before the general statement of (2.1) was found. One of these proofs is sketched to illustrate a method of proof using shrinkable sets and cones, which might be useful in another context. Some simple facts on weakening topologies are also listed in this chapter.
Shrinkable neighborhoods of Ives [7] and Klee [11] are used throughout, and many proofs are based on ideas of Bessaga and Klee.
2. NEGLIGIBILITY PROPERTIES AND APPLICATIONS

We prove the following Theorem and Corollaries. All linear spaces are real.

(2.1) THEOREM. Suppose \((X, \tau')\) is a linear topological space admitting a linear topology \(\tau_2 \subset \tau'\) such that \((X, \tau_2)\) is metrizable and incomplete, and that \(K\) is \(\tau_2\)-compact, \(U\) is \(\tau_2\)-open with \([0,1]\) \(\subset U\). Then there is a \(\tau'\) homeomorphism \(h: X \to X \setminus K\) with \(h|X \setminus U = \text{Id}\).

(2.2) COROLLARY. Suppose \((X, \tau')\) is a metrizable complete linear topological space, and \(\tau_2\) is metrizable and strictly weaker than \(\tau'\). If \(K, U\) are as in (2.1), the conclusion holds.

(2.3) COROLLARY. Let \(M[0,1]\) be the space of a.e. finite Lebesgue measurable functions, \(S[0,1]\) the simple functions, and \(C[0,1]\) the continuous functions, all on \([0,1]\). If \(S[0,1] \subset X \varsubsetneq M[0,1]\) or \(C[0,1] \subset X \varsubsetneq M[0,1]\), and if convergence in \(X\) implies convergence in measure, then subsets of \(X\) which are compact in \(M\) (with the topology of convergence in measure) are negligible in \(X\).

(2.4) COROLLARY. Suppose the hypotheses of (2.1) hold, except \(U \in \tau'\), \(K\) is \(\tau'\)-compact, and \(\tau_2\) contains a linearly bounded set. Then the conclusion of the Theorem holds.
It follows from (2.3) that $\mathcal{M}$-compact sets are negligible in $L^p$, $p > 0$. (Suppose $\int_{[0,1]} |f_\alpha|^p \to 0$, and take $\epsilon > 0$. Let $E_\alpha = \{ x \in [0,1] \mid |f_\alpha(x)| \geq \epsilon \}$. $\int_{[0,1]} |f_\alpha|^p \geq \int_{E_\alpha} |f_\alpha|^p \geq \epsilon \mu E_\alpha$, so that $\mu E_\alpha \to 0$.) Corollary (2.3) also implies negligibility of compact sets in $\{ f \mid f : [0,1] \to \mathbb{R}, f$ has finitely many discontinuities $\}$ with convergence in measure. This space is not covered by earlier results since it admits no continuous linear functional [13, page 114]. A case of Anderson's result on $\mathcal{A}$ spaces follows from (2.1). If $(X, \mathcal{T})$ is an $\mathcal{A}$ space, then $X$ may be regarded as a linear subspace of $s$. If $X \neq s$, then $\mathcal{T} |_X \subset \mathcal{T}$, where $\mathcal{T}$ is the topology of coordinatewise convergence, and $(X, \mathcal{T})$ is dense in $(s, \mathcal{T})$, since if $x \in s$, $(x_1, \ldots, x_n, 0, 0, \ldots) \in X$, and approaches $x$. Thus $(X, \mathcal{T})$ is incomplete, and the theorem applies to show $\mathcal{T}$-compact sets negligible. If $X = s$, and $X$ is metrizable, again $\mathcal{T} \subset \mathcal{T}$ and, in the notation of Chapter 1, $b_\alpha \to 0 (\mathcal{T})$, but $b_\alpha \not\to 0 (\mathcal{T})$, since $b_\alpha \notin U$. By the open mapping theorem, $(X, \mathcal{T})$ is incomplete, and again $\mathcal{T}$-compact sets are negligible in $(X, \mathcal{T})$.

If $U$ is a set in a linear topological space, and $p \in \text{Int } U$, then $U$ is shrinkable at $p$ if $[0,1) \cup U - p \subset \text{Int } (U - p)$. Notice that each ray $r$ from $p$ meets $\text{Bd } U$ at most once, and that $r \cap \overline{U}$ is closed and connected. If $U$ is shrinkable at $p$, the gauge functional $\mathcal{V}_U(x, p)$ is defined by
\[ x = p + \gamma_U(x,p)(\pi_U(x,p) - p) \] in case ray \( \overrightarrow{px} \) meets \( \text{Bd} \ U \) in \( \pi_U(x,p) \). If \( \overrightarrow{px} \subset U \) or \( x = p \), then \( \gamma_U(x,p) = 0 \). Ives [7] has shown that if \( U \) is open, then \( \gamma_U(x,p) \) is continuous in \( x \). It follows that \( \pi_U(x,p) \), as a function of \( x \), is continuous on its domain. According to Klee [11], each Hausdorff linear topological space has a basis at 0 of open sets, shrinkable at 0. A set \( A \) in a linear space is star-shaped at \( a \in A \) if \( tx + (1-t)a \in A \) whenever \( x \in A \), \( t \in [0,1] \). In the following, "\( f:X \to Y \) is an embedding" will mean \( f \) is a homeomorphism from \( X \) onto a subset of \( Y \), while "\( f:X \to Y \) is a homeomorphism" will imply the mapping is onto \( Y \). [A] denotes the convex hull of \( A \).

(2.5) LEMMA. Let \( Y \) be a linear subspace of a linear topological space \( X \), \( y \in Y \), and \( U \) open, shrinkable at \( y \). Then \( U \cap Y \) is shrinkable at \( y \) in \( Y \).

PROOF. Put \( [0,1)(U \cap Y-y) \subset [0,1)(U-y) \subset U-y \), and \( [0,1)(U \cap Y-y) \subset Y \).

(2.6) LEMMA. Let \((X,\tau)\) be a linear topological space, and \( \tau_2 \subset \tau \). If \( U \in \tau_2 \) is \( \tau_2 \)-shrinkable at 0, then \( U \) is \( \tau \)-shrinkable at 0, \( \text{Bd}_{\tau}U = \text{Bd}_{\tau_2}U \), and \( U \cap _\tau = U \cap _{\tau_2} \).

PROOF. Put \( [0,1)U \subset [0,1)U \subset U \).

\( \text{Bd}_{\tau}U = \overline{U} \cap \text{Bd}_{\tau_2}U \subset \overline{U} \cap _{\tau_2}U = \text{Bd}_{\tau_2}U \). If \( x \in \text{Bd}_{\tau_2}U \) and \( V \) is a \( \tau \)-neighborhood of \( x \), then there exists \( \lambda \in (0,1) \) such that \( \lambda x \in V \). Since \( U \) is \( \tau_2 \)-shrinkable at 0, \( \lambda x \in U \).
Since $x \notin U$, this shows $x$ is a $\gamma'$-boundary point of $U$.

(2.7) LEMMA. Suppose $X$ is a metrizable linear topological space and $K, W \subset X$ with $K$ compact, $W$ open and shrinkable at 0. If $x \in X$, $k \in K$ and $r > 0$, then \( \{k + \lambda(x-k) \mid \lambda > 0\} \subset K + rW \) if and only if \( \{\lambda(x-k) \mid \lambda > 0\} \subset W \).

PROOF. Assume the first inclusion, and take $\lambda > 0$. For each $n$, $k + nr(x-k) \in K + rW$, so that $k + nr(x-k) = k_n + rw_n$ with $k_n \in K$, $w_n \in W$, and $w_n = (1/r)(k - k_n) + n(x-k) \in W$. \( \{k_n\} \) has a convergent subsequence $k_n \to k \in K$. If $l$ is large, \( \frac{\lambda+1}{n_l} < 1 \), and \( \frac{\lambda+1}{n_l} \left( \frac{1}{r}(k - k_{n_l}) + n_l(x-k) \right) \in W \). Letting $l \to \infty$, $(\lambda+1)(x-k) \in \overline{W}$. Therefore $\lambda(x-k) \notin W$. The converse is clear.

(2.8) LEMMA. If $X$ is a linear space, and $A, B \subset X$ with $A$ star-shaped at $x \in A$, then $C = \bigcup \{\lambda A + (1-\lambda)B \mid \lambda \in [0,1]\}$ is star-shaped at $x$.

PROOF. Take $y \in C$, $t \in [0,1]$. If $t = 1$, the result is trivial; if $t \neq 1$, $ty + (1-t)x = t(\lambda a + (1-\lambda)b) + (1-t)x = (1-(t-t\lambda)) \frac{t\lambda a + (1-t)x}{1-(t-t\lambda)} + (t-t\lambda)b \in C$.

The next lemma, due to Klee [11], is central in what follows, so we offer a proof.
(2.9) LEMMA. If \( U \) is open, shrinkable at 0, and \( K \) is compact, star-shaped at \( k_0 \in K \), then \( U+K \) is shrinkable at \( k_0 \). In particular, if \( K \) is convex, \( U+K \) is shrinkable at each point of \( K \).

PROOF. Take \( y \in \overline{U+K-k_0} \), \( t \in [0,1) \). \( y+k_0 \in \overline{U+K} = \overline{U+K} \) so that \( y+k_0 - k \in \overline{U} \) for some \( k \in K \). \( t(y+k_0 - k) \in U \), since \( U \) is shrinkable at 0, and thus \( ty+k_0 \in U+tk+(1-t)k_0 \subseteq U+K \).

(2.10) LEMMA. If \( K \) is compact and \( U \) open with \( K \subseteq U \) in a metrizable linear topological space, and \( \{W_n\} \) is a local base at 0, then for some \( n \), \( \overline{K+W_n} \subseteq U \).

PROOF. If for each \( m \), \( K+W_m \nsubseteq U \), take \( k_m \in K \), \( w_m \in W_m \), with \( k_m+w_m \notin U \). \( \{k_m\} \) has a convergent subsequence \( k_{m_n} \rightarrow k \). Then \( k_{m_n}+w_{m_n} \rightarrow k \notin U \), a contradiction. Thus \( K+W_m \subseteq U \) for some \( m \). Take \( n \) such that \( W_n \subseteq W_{m_n} \). Then \( \overline{K+W_n} = \overline{K+W_{m_n}} \subseteq K+W_m \subseteq U \).

Note that (2.7) and (2.10) can be stated and proved in a non-metric setting. The following convergence criterion is similar to that found in Anderson and Bing [3].

(2.11) LEMMA. Suppose \( X,Z \) are topological spaces, \( Y \subseteq Z \), \( h_n:X \rightarrow Z \) is a homeomorphism for each \( n \in \mathbb{N} \), and for each \( x \in X \), there exist open \( W_x \), \( m \in \mathbb{N} \) such that \( h_{m+k} \cdots h_{k+1} h_i |_{W_x} = h_m \cdots h_{i+1} h_i |_{W_x} \) for each \( k \geq 1 \). If \( h(x) \) is defined as \( \ldots h_2 h_1 (x) \), and \( h \) is onto \( Y \), then \( h:X \rightarrow Y \) is a homeomorphism.

PROOF. \( h \) is clearly 1-1. If \( x_n \rightarrow x \), then
$S \Rightarrow x_S \in W_x \Rightarrow h(x_S) = h_m \ldots h_1 (x_S) \Rightarrow h_m \ldots h_1 h_i (x) = h(x)$. Thus $h$ is continuous. Take $U$ open in $X$. We will show $h(U)$ open in $Y$. Take $y \in h(U)$. $y = h(x)$ for some $x \in U$. $h|W_x = h_m \ldots h_1 h_i |W_x$, so $h(W_x \cap U) = h_m \ldots h_1 h_i (W_x \cap U)$, open in $Z$ and thus in $Y$. $y \in h(W_x \cap U) \subset h(U)$.

We now prove Theorem (2.1). Let $\mathcal{K}$ be a linear metric completion of $(X, \tau_K)$, and $\{\mathcal{W}_n\}$ a basis of open sets, shrinkable at 0, with $\mathcal{W}_n \subset \mathcal{W}_{n+1}$. Unspecified closures are either with respect to $\tau_K$ or the completion of $\tau_K$. $U = \mathcal{U} \cap X$ for some open $\mathcal{U}$. There exists $x \in \mathcal{K} \setminus X$ such that $\bigcup \{\lambda \mathcal{K} + (1-\lambda)k | \lambda \in [0,1]\} \subset \mathcal{U}$, since otherwise we may choose $x_n \in \mathcal{W}_n \setminus X$, and find $\lambda_n \in [0,1], k_n \in K$ such that $\lambda_n x_n + (1-\lambda_n) k_n \notin \mathcal{U}$. There exists $n_0$ such that $\lambda_n \to \lambda$, $k_n \to k \in K$. Letting $\lambda \to 0$, $(1-\lambda)k \notin \mathcal{U}$, contradicting $[0,1]k \subset \mathcal{U}$. Let $y_n \to x$ with $y_n \in U$. There exists $n_0$ such that $\bigcup \{\lambda [x, y_n] + (1-\lambda)k | \lambda \in [0,1]\} \subset \mathcal{U}$, since otherwise, for each $n$, there exist $\mu_n, \rho_n, k_n$ such that $\lambda_n [\mu_n x + \rho_n y_n] + (1-\lambda_n) k_n \notin \mathcal{U}$, and taking convergent subsequences $\lambda_n \to \lambda, \mu_n \to \mu, \rho_n \to \rho, k_n \to k$, we get $\lambda x + (1-\lambda)k = \lambda (\mu x + \rho y) + (1-\lambda)k \notin \mathcal{U}$, a contradiction. Let $x_i = y_n$, and $\mathcal{K}_i = \bigcup \{\lambda [x, x_i] + (1-\lambda)k | \lambda \in [0,1]\}$, a compact subset of $\mathcal{U}$. By (2.10), there exists $\lambda_i$ such that $\mathcal{K}_i + 3\mathcal{K}_i \subset \mathcal{U}$.
Let $\tilde{A}_i = [\tilde{x}, x_i] + \tilde{w}_i$, $A_i = \tilde{A}_i \cap X$
$\tilde{B}_i = [\tilde{x}, x_i] + 2\tilde{w}_i$, $B_i = \tilde{B}_i \cap X$
$\tilde{C}_i = \tilde{C}_i + 2\tilde{w}_i$, $C_i = \tilde{C}_i \cap X$
$\tilde{D}_i = \tilde{C}_i + 3\tilde{w}_i$, $D_i = \tilde{D}_i \cap X$

By (2.8), (2.9), (2.5) and (2.6), $A_i$, $B_i$, $C_i$ and $D_i$ are $\tilde{T}_i$-shrinkable at $x_i$. $\tilde{A}_i \subset \tilde{A}_i = A_i \cap X = [\tilde{x}, x_i] + \tilde{w}_i \cap X = ([\tilde{x}, x_i] + \tilde{w}_i) \cap X \subset ([\tilde{x}, x_i] + 2\tilde{w}_i) \cap X = B_i$. Similarly, we get $\tilde{A}_i \subset B_i \subset C_i \subset D_i$. The statements $\{x_i + \lambda(x-x_i) \mid \lambda \geq 0\}$ $\subset A_i$, $\subset B_i$, $\subset C_i$, $\subset D_i$ are equivalent by (2.7). Now we define $h_i : X \to X$. $h_i|\tilde{A}_i \cup (X \setminus D_i) = Id$.

$h_i|\tilde{B}_i \setminus A_i : \tilde{B}_i \setminus A_i \to \tilde{C}_i \setminus A_i$ is defined as follows. If $y \in \tilde{B}_i \setminus A_i$, then $\pi_{\tilde{A}_i}(y, x_i)$ is defined, and by the above remark, so are $\pi_{\tilde{B}_i}(y, x_i)$, $\pi_{\tilde{C}_i}(y, x_i)$ and $\pi_{\tilde{D}_i}(y, x_i)$. If $r$ is a ray from $x_i$, intersecting $B \cap \tilde{A}_i$, $h_i$ maps $(\tilde{B}_i \setminus \tilde{A}_i) \cap r$ linearly onto $(\tilde{C}_i \setminus \tilde{A}_i) \cap r$. Thus

$h_i(x) = \frac{\pi_{\tilde{A}_i}(y, x) + \pi_{\tilde{C}_i}(y, x) - 1}{\pi_{\tilde{A}_i}(y, x) - \pi_{\tilde{A}_i}(y, x)}$, $h_i|\tilde{B}_i \setminus A_i$ has an inverse of the same form, so it is a homeomorphism. Similarly, we define $h_i|\tilde{D}_i \setminus B_i : \tilde{D}_i \setminus B_i \to \tilde{D}_i \setminus C_i$. $h_i$ is then a homeomorphism.

Another compactness argument shows there exists $n_2 > n_i$ such that $[\tilde{x}, y_{n_2}] \subset \tilde{A}_i$ and $\bigcup \{[\tilde{x}, y_{n_2}] + (1-\lambda)K \mid \lambda \in [0,1]\}$ $\subset \tilde{C}_i$. Let $x_2 = y_{n_2}$, $K_2 = \bigcup \{[\tilde{x}, x_2] + (1-\lambda)K \mid \lambda \in [0,1]\}$. Then $[\tilde{x}, x_2] \subset \tilde{A}_i$, $K_2 \subset \tilde{C}_i$ and there exists $\lambda_2 > \lambda_i$ such that $[\tilde{x}, x_2] + 2\tilde{w}_2 \subset \tilde{A}_i$, $K_2 + 3\tilde{w}_2 \subset \tilde{C}_i$. 

Let \( \tilde{A}_2 = [\tilde{x}, x_2] + \tilde{w}_2 \) and \( A_2 = \tilde{A}_2 \cap X \)
\( \tilde{B}_2 = [\tilde{x}, x_2] + 2\tilde{w}_2 \) and \( B_2 = \tilde{B}_2 \cap X \)
\( \tilde{C}_2 = \tilde{C}_2 + 2\tilde{w}_2 \) and \( C_2 = \tilde{C}_2 \cap X \)
\( \tilde{D}_2 = \tilde{D}_2 + 3\tilde{w}_2 \) and \( D_2 = \tilde{D}_2 \cap X \)

As before, \( A_1, B_1, C_1, D_1 \) are \( \gamma \)-shrinkable at \( x_1 \),
\( \overline{A}_1 \subseteq B_1 \subseteq C_1 \subseteq \overline{C}_1 \subseteq D_1 \), and the statements
\( \{ x_1 + \lambda (x-x_1) \mid \lambda > 0 \} \subseteq A_2, \subseteq B_2, \subseteq C_2, \subseteq D_2 \) are equivalent.

Also \( \overline{B}_2 \subseteq A_2 \) and \( \overline{D}_2 \subseteq C_2 \). Define \( h_2 : X \rightarrow X \) so that
\( h_2|_{A_2 \cup (X \setminus D_2)} = \text{Id} \), \( h_2|_{B_2 \setminus A_2 : B_2 \setminus A_2 \rightarrow \overline{C}_2 \setminus A_2} \),
\( h_2|_{D_2 \setminus B_2 : D_2 \setminus B_2 \rightarrow \overline{D}_2 \setminus C_2} \). Note \( h_2|_{X \setminus C_1} = \text{Id} \). Continue,

obtaining sets
\( D_1 \supseteq C_1 \supseteq B_1 \supseteq A_1 \)
\( D_2 \supseteq C_2 \supseteq B_2 \supseteq A_2 \)
\( D_3 \supseteq C_3 \supseteq B_3 \supseteq A_3 \)

and homeomorphisms \( h_n : X \rightarrow X \) with
\( h_n|_{\overline{A}_n \cup (X \setminus D_n)} = \text{Id} \),
\( h_n|_{\overline{B}_n \setminus A_n : \overline{B}_n \setminus A_n \rightarrow \overline{C}_n \setminus A_n} \),
\( h_n|_{D_n \setminus B_n : D_n \setminus B_n \rightarrow \overline{D}_n \setminus C_n} \),
\( h_n|_{X \setminus C_{n-1}} = \text{Id} \).

We claim \( \bigcap \overline{A}_n = \bigcap \overline{B}_n = c \). Since \( \overline{A}_n \subseteq \overline{B}_n \subseteq A_{n-1} \subseteq \overline{A}_{n-1} \),
it is sufficient to show \( \bigcap \overline{B}_n = c \). If \( y \in \bigcap \overline{B}_n \), then
\( y \in X \), and \( y \in [\tilde{x}, x_n] + 2\tilde{w}_n \), so that \( y = \lambda \tilde{x} + \mu x_n + 2\tilde{w}_n \), and
taking convergent subsequences, we see \( y = \tilde{x} \), a contradiction. Also \( \bigcap \overline{C}_n = \bigcap \overline{D}_n = K \). Suppose \( y \in \bigcap \overline{D}_n \). Then \( y \in X \)
and \( y \in \mathbb{K} + 3 \mathbb{W} \). \( y = \lambda (\mu \mathbb{X} + \nu \mathbb{x}_n) + (1 - \lambda) \mathbb{k} + 3 \mathbb{w}_n \), so that
\[ y = \lambda \mathbb{X} + (1 - \lambda) \mathbb{k}. \]
If \( \lambda \neq 0 \), \( \mathbb{X} = (1/\lambda) (y - (1 - \lambda) \mathbb{k}) \in \mathbb{X} \), a contradiction. Therefore \( y = k \in \mathbb{K} \). Thus \( \mathbb{K} \subset \cap \mathbb{C}_n \subset \cap \mathbb{D}_n \subset \cap \mathbb{D}_n = \mathbb{K} \), since \( \mathbb{D}_n \subset \mathbb{C}_n \subset \mathbb{D}_{n-1} \). We trace the motion of a point \( x \in \mathbb{X} \) under the successive homeomorphisms \( h_1, h_2, \ldots \).

If \( x \notin B_1 \), then \( h_1(x) \notin C_1 \) and for each \( n \), \( h_n \ldots h_2 h_1(x) = h_1(x) \). If \( x \in B_1 \setminus B_2 \) and \( x \notin A_1 \), \( h_1(x) = x \), and \( h_2 h_1(x) = h_2(x) \notin C_2 \), since \( x \notin B_2 \). Thus \( h_n \ldots h_2 h_1(x) = h_2 h_1(x) \) if \( n \geq 2 \). If \( x \in B_1 \setminus B_2 \) and \( x \notin A_1 \), then \( h_1(x) \notin A_1 \), so \( h_1(x) \notin B_2 \). Thus \( h_2 h_1(x) \notin C_2 \), and \( h_n \ldots h_2 h_1(x) = h_2 h_1(x) \) if \( n \geq 2 \). Otherwise \( x \in B_n \setminus B_{n+1} \) for some \( n > 1 \). \( h_n \ldots h_2 h_1(x) = x \), since \( x \in B_n \subset A_{n-1} \). If \( x \in A_n \), then \( h_n \ldots h_2 h_1(x) = h_n(x) = x \), and \( h_n \ldots h_2 h_1(x) = h_n(x) \notin C_{n+1} \), since \( x \notin B_{n+1} \).

If \( x \notin A_n \), then \( h_n \ldots h_2 h_1(x) = h_n(x) \notin A_n \), so \( h_n \ldots h_2 h_1(x) \notin B_{n+1} \). Thus \( h_n \ldots h_2 h_1(x) \notin C_{n+1} \), so in either case \( h_n \ldots h_2 h_1(x) = h_n \ldots h_2 h_1(x) \) if \( k \geq 1 \). Thus we may define \( h \) on \( \mathbb{X} \) by \( h(x) = h_n \ldots h_2 h_1(x) \). Note that in each case above, \( h(x) \notin C_n \) for some \( n \), so \( h \) is into \( \mathbb{X} \setminus K \). If \( y \in \mathbb{X} \setminus K \), then \( y \notin D_n \) for some \( n \), and \( y = h_n \ldots h_2 h_1(x) \) for some \( x \).

Thus \( h(x) = h_n \ldots h_2 h_1(x) = y \). This shows \( h \) is onto \( \mathbb{X} \setminus K \).

If \( x \in \mathbb{X} \), then \( x \notin \overline{B}_n \) for some \( n \), and the above argument shows \( h | \mathbb{X} \setminus \overline{B}_n = h_n \ldots h_2 h_1 | \mathbb{X} \setminus \overline{B}_n \). Thus 2.11 applies to show \( h: \mathbb{X} \to \mathbb{X} \setminus K \) is a homeomorphism. Since \( \mathbb{X} \cup U \subset \mathbb{X} \setminus D_1 \), evidently \( h | \mathbb{X} \setminus U = \text{Id} \).

Corollary (2,2) is proved by noting \( I^2: (\mathbb{X}, \mathcal{D}_1) \to (\mathbb{X}, \mathcal{D}_2) \) is not an open map, so the open mapping theorem implies
is incomplete. For Corollary (2.3) note that $X$ with the topology of convergence in measure is incomplete, since it is dense and a proper subspace of $M$. Finally, we prove Corollary (2.4). Since $\tau_2$ contains a linearly bounded set, it contains a linearly bounded $\tau_2$-shrinkable neighborhood $V$ of $0$. If $V_t = V + [0,1]K$, then by (2.8) (with $A = [0]$, $B = K$), (2.9) and (2.7), each ray from $0$ meets $\text{Bd} \ V_t$ exactly once. Since $[0,1]K \subset U \in \tau_t$, we can find using (2.10) and (2.9) a $\tau_t$-open set $U_t$ such that $[0,1]K \subset U_t \subset U$ and $U_t$ is $\tau_t$-shrinkable at $0$. Let $U_2 = U_t \cap V \in \tau_t$. Then $U_2$ is $\tau_t$-shrinkable at $0$ and $[0,1]K \subset U_2$. Since $[0,1]K$ is $\tau_t$-compact, $[0,1]K \subset rU_2$ for some $r \in (0,1)$. Since $U_2$ and $V_t$ are $\tau_t$-shrinkable at $0$ and each ray from $0$ meets $\text{Bd} \ V_t$, we may find a $\tau_t$-homeomorphism $j: X \rightarrow X$ such that $j|_{rU_2 \cup (X \setminus 2V_t)} = \text{Id}$, $j|_{U_2 \setminus rU_2} : U_2 \setminus rU_2 \rightarrow \overline{V_t \setminus rU_2}$, and $j|_{2V_t \setminus U_2} : 2V_t \setminus U_2 \rightarrow 2\overline{V_t \setminus V_t}$. By the Theorem, there exists a $\tau_t$-homeomorphism $h: X \rightarrow X \setminus K$ such that $h|_{X \setminus \overline{V_t}} = \text{Id}$. Then $j^{-1}hj: X \rightarrow X \setminus K$ is a $\tau_t$-homeomorphism fixed on $X \setminus U$.

Next we give the generalization of Bessaga's theorem referred to in the Introduction. In his proof [5], Bessaga constructs a curve in a normed space with certain properties which seem to depend on the homogeneity of the norm. For the result here we have less than homogeneity to work with, and adopt another approach. All paranorms below are total.
(2.12) THEOREM. Suppose \( (X, \mathcal{T}) \) is a linear topological space and \( K \subset X \). If there exists a paranormed space \( (Y, \rho) \) such that

1. \( X \) is a \( \rho \)-dense linear subspace of \( Y \), but \( X \neq Y \),
2. \( \mathcal{T}_\rho \upharpoonright X \subset \mathcal{T} \),
3. For each \( t \in (0,1) \), there exists \( b_t \in (0,1) \) such that \( \rho(tx) \leq b_t \rho(x) \) whenever \( x \in Y \),
4. \( K \) is closed in \( Y \),
then \( X \) is \( \mathcal{T} \) homeomorphic to \( X \setminus K \).

(2.13) REMARK. Condition 1. can be replaced by "\( X \) is a linear subspace of \( Y \), but \( X \) is not closed in \( Y \)", since we may replace \( Y \) by the \( \rho \)-closure of \( X \) in \( Y \).

(2.14) COROLLARY. Suppose \( 0 < q < 1 \) and \( K \subset L^q \). If \( r < q \) is such that \( K \) is closed in \( L^r \), then \( L^q \) is homeomorphic to \( L^q \setminus K \).

(2.15) COROLLARY. Suppose \( (X, \mathcal{T}) \) is a linear topological space and \( (Y, \rho) \) satisfies 1, 2, 3 in (2.12). If \( K = \bigcup K_\omega \subset X \), where \( \{K_\omega\} \) is a \( \rho \)-locally finite collection, and each \( K_\omega \) is \( \mathcal{T} \)-compact or a closed subset of a finite dimensional space, then \( K \) is \( \mathcal{T} \)-negligible in \( X \).

We introduce a special notation to deal with the next lemma. Suppose \( (X, \mathcal{T}) \) is a linear topological space, and \( U \in \mathcal{T} \) is linearly bounded and \( \mathcal{T} \)-shrinkable at \( y \). If \( x \in X \) and \( x \neq y \),
then \( \Pi(x,y,U,X) = \frac{y^2}{x} \) and \( \mathcal{V}(x,y,U,X) \) is defined by \( x = y + \mathcal{V}(x,y,U,X)(\Pi(x,y,U,X) - y) \).

(2.16) LEMMA. Let \((X, \mathcal{T})\), \((Y, \mathcal{T}'')\) be linear topological spaces with \( X \) a linear subspace of \( Y \) and \( \mathcal{T}'|X \subseteq \mathcal{T} \), and suppose \( U_1, V_1 \in \mathcal{T}' \) and are \( \mathcal{T}' \)-shrinkable at \( x_i \in X \), for \( i = 1, 2 \). Suppose also \( V_2, U_2 \) are linearly bounded, \( \overline{V}_1 \subseteq U_1 \), and \( h: \partial U_1 \to \partial U_2 \) is a \( \mathcal{T}' \)-homeomorphism with \( h((\partial U_1) \cap X) = (\partial U_2) \cap X \), a \( \mathcal{T}' \)-homeomorphism. Then \( h \) extends to \( h: \overline{U}_1 \setminus V_1 \to \overline{U}_2 \setminus V_2 \), a \( \mathcal{T}' \)-homeomorphism.

PROOF. If \( r \) is a ray from \( x_i \), let \( h \) map \( r \cap (\overline{U}_1 \setminus V_1) \) linearly onto \( s \cap (\overline{U}_2 \setminus V_2) \) where \( s = x^2 \), \( h(r \cap \partial U_1) \). Thus for \( y \in \overline{U}_1 \setminus V_1 \),

\[
h(y) = \frac{\Pi(z,x_2,V_2,Y, \mathcal{T}'')}{\mathcal{V}(y,x_1,V_1,Y, \mathcal{T}'') \frac{y}{x}} (\Pi(z,x_2,V_2,Y, \mathcal{T}'')^{-1}(z - \Pi(z,x_2,V_2,Y, \mathcal{T}'')))
\]

where \( w = \Pi(y,x_1,U_1,Y, \mathcal{T}'') \) and \( z = h(w) \). \( h \) is \( \mathcal{T}' \)-continuous, and by symmetry, so is \( h' \). Clearly,

\[
h((\overline{U}_1 \setminus V_1) \cap X) = (\overline{U}_2 \setminus V_2) \cap X \) and for \( y \in X \), by (2.5), (2.6) and the definition of \( \mathcal{V} \) in terms of \( \Pi \),

\[
h(y) = \frac{\Pi(z,x_2,V_2 \cap X,X, \mathcal{T}'')}{\mathcal{V}(y,x_1,V_1 \cap X,X, \mathcal{T}'') \frac{y}{x}} (\Pi(z,x_2,V_2 \cap X,X, \mathcal{T}'')^{-1}(z - \Pi(z,x_2,V_2 \cap X,X, \mathcal{T}'')))
\]

where \( w = \Pi(y,x_1,U_1 \cap X,X, \mathcal{T}'') \), \( z = h(w) \). A similar statement
holds for $h^{-}$, so that $h|\left(\bar{U}, V_{1}\right) \cap X$ is a $\tau_{1}$ homeomorphism.

Clearly, $h|\text{Bd } V_{1} : \text{Bd } V_{1} \to \text{Bd } V_{2}$, a $\tau_{2}$ homeomorphism, and $h|\left(\text{Bd } V_{1}\right) \cap X : \left(\text{Bd } V_{1}\right) \cap X \to \left(\text{Bd } V_{2}\right) \cap X$, a $\tau_{3}$ homeomorphism.

(2.17) LEMMA. If $(X, p)$ is a paranormed linear space, $W_{n} = \left\{ x | p(x) < 2^{-n} \right\}$, and $K \subset X$, then $K + W_{n} \subset K + W_{n-1}$.

PROOF. Take $x \in K + W_{n}$. There exist $k_{\xi} \in K$, $w_{\xi} \in W_{n}$ such that $k_{\xi} + w_{\xi} \to x$. $p(x - k_{\xi}) \leq p(x - k_{\xi} - w_{\xi}) + p(w_{\xi}) \leq p(x - (k_{\xi} + w_{\xi})) + 2^{-n} < 2^{-1} \cdot 2^{-n} \leq 2^{-n}$ for large $\xi$. $x = k_{\xi} + (x - k_{\xi}) \in K + W_{n-1}$.

(2.18) LEMMA. Suppose $(X, p)$ is a paranormed space and for each $t \in (0, 1)$, there exists $\varepsilon_{t} \in (0, 1)$ such that $p(tx) \leq b_{t} p(x)$ whenever $x \in X$. Then $p$ restricted to a ray from 0 is monotone and unbounded. Each $p$-ball $\left\{ x | p(x) < \varepsilon \right\}$ is linearly bounded and shrinkable at 0.

PROOF. Since $p(y) \leq b_{\frac{1}{2}} p(2y) \leq \ldots \leq b_{\frac{1}{2}} p(2^{n} y)$, $p(2^{n} y) \geq b_{\frac{1}{2}} p(y)$. Thus $p$ restricted to a ray is unbounded. It is clearly monotone. Suppose $x \neq y$ and consider the line $x + \lambda (y - x)$. Since $p(\lambda (y - x)) \leq p(x + \lambda (y - x)) + p(-x)$, $p(x + \lambda (y - x)) \geq p(\lambda (y - x)) - p(x) > \varepsilon$, for large $|\lambda|$. To see that $\left\{ x | p(x) < \varepsilon \right\}$ is shrinkable at 0, take $y \in \left\{ x | p(x) < \varepsilon \right\}$, $t \in [0, 1)$. There exists $\left\{ x_{n} \right\}$ such that $p(x_{n}) < \varepsilon$ and $p(x_{n} - y) \to 0$. Now $p(y) \leq p(y - x_{n}) + p(x_{n}) < p(y - x_{n}) + \varepsilon$. Thus $p(y) \leq \varepsilon$ and $p(ty) \leq b_{t} p(y) < \varepsilon$.

(2.19) LEMMA. Suppose $(X, p)$ is as in (2.18). If $K$ is $p$-bounded, $k_{\lambda} \in K$ and $t_{\lambda} \to 0$, then $p(t_{\lambda} k_{\lambda}) \to 0$. 
PROOF. For \( k \in K \), \( p(k) < \varepsilon \). Take \( \delta > 0 \) and \( r \in (0, 1) \). For some \( n \), \( p(r^k k) \leq b_r p(r^{k-1} k) \leq \ldots \leq b_r^np(k) \leq b_r^n \varepsilon < \delta \) for each \( k \in K \). Then for large \( m \), \( |t_m| \leq r^n \) and 
\[ p(t_n k_m) = p(|t_m| k_m) \leq p(r^k k_m) < \delta. \]

Robert Neufeld noted in a conversation with the author that Klee's result (2.9) holds for non-compact \( K \) in a normed linear space. We extend this in the following lemma.

(2.20) LEMMA. Suppose \((X, p)\) is as in (2.18). If \( K \) is star-shaped at 0 and \( W = \{ x \mid p(x) < \varepsilon \} \), then \( K + W \) is shrinkable at 0. If \( K \) is \( p \)-bounded, \( K + W \) is linearly bounded.

PROOF. Take \( y \in K + W \), \( t \in [0, 1) \). \( p(k_n + w_n - y) \to 0 \) for some \( k_n \in K \), \( w_n \in W \) so that \( p(tk_n - ty) \leq b_t p(k_n - y) \leq b_t p(k_n + w_n - y) + p(w_n) \leq b_t p(k_n + w_n - y) + \varepsilon < \varepsilon \), if \( n \) is large.

That is, \( tk_n - ty \in W \), so \( ty \in K + W \), since \( tk_n \in K \). If \( K \) is \( p \)-bounded and \( k + w \in K + W \), then \( p(k+w) \leq p(k) + p(w) < B + \varepsilon \).

Thus \( K + W \) is \( p \)-bounded, and by (2.18), linearly bounded.

(2.21) THEOREM. Suppose \((Y, \mathcal{T}_X)\) is a metrizable linear topological space and \( X \) a proper linear subspace of \( Y \). Suppose also \( \mathcal{T}_1 \) is a linear topology for \( X \) such that \( \mathcal{T}_1 | X \subseteq \mathcal{T}_1 \) and \( X \) is \( \mathcal{T}_1 \)-dense in \( Y \). If \( U \in \mathcal{T}_2 \) is linearly bounded and contains 0, then there exist \( x \in U \setminus X \) and \( j: Y \setminus x \to Y \setminus 0 \) such that \( j \) is a \( \mathcal{T}_2 \)-homeomorphism,

\[ j|X: X \to X \setminus 0 \] is a \( \mathcal{T}_1 \)-homeomorphism, and \( j|Y \setminus U = \text{Id.} \)
PROOF. There exists \( x \in Y \setminus X \) such that \([0, x] \subset U\).

Let \( y_n \to x \) (\( \gamma_2 \)) with \( y_n \in X \). There exists \( n_1 \), such that
\[ [0, x, y_n] \subset U, \]
since otherwise for each \( n \), there exist \( \lambda_n, \mu_n, \rho_n \) such that \( \lambda_n + \mu_n x + \rho_n y_n \not\in U \), with \( \lambda_n + \mu_n + \rho_n = 1 \). Taking convergent subsequences \( \lambda_n \to \lambda, \mu_n \to \mu, \rho_n \to \rho \), we have \( (\mu + \rho) x \not\in U \), with \( \mu + \rho \leq 1 \). Let \( \{ W_n \} \) be a \( \gamma \)-basis of open sets \( \mathcal{T}_2 \)-shrinkable at 0, with \( W_n \subset W_{n-1} \). Letting
\[ x_1 = y_1, \quad [0, x, x_1] \subset U \]
and, by (2.10), there exists \( \ell_1 \) such that \([0, x, x_1] + W_{\ell_1} \subset U \). Let \( Z = [0, x, x_1] + W_{\ell_1} \). There exists \( n_2 > n_1 \), such that \([x, x_1, y_{n_2}] \subset Z \), since otherwise
\[ \lambda_n x + \mu_n x_1 + \rho_n y_n \not\in Z, \]
so that \( (\lambda + \rho) x + \mu x_1 \not\in Z \). Letting
\[ x_2 = y_{n_2}, \quad [x, x_1, x_2] \subset Z \]
and there exists \( \ell_2 > \ell_1 \), such that \([x, x_1, x_2] + W_{\ell_2} \subset Z \). Let \( U_1 = [x, x_1, x_2] + W_{\ell_2} \). There exists \( n_3 > n_2 \), such that \([x, x_2, y_{n_3}] \subset U_1 \), and \( \ell_3 > \ell_2 \), such that \([x, x_2, x_3] + W_{\ell_3} \subset U_1 \), where \( x_3 = y_{n_3} \). Let \( U_2 = [x, x_2, x_3] + W_{\ell_3} \). Continue, obtaining \( U_n = [x, x_n, x_{n+1}] + W_{\ell_{n+1}} \) with \( U_n \) \( \mathcal{T}_2 \)-shrinkable at \( x_n, x_{n+1} \) (by (2.9)), and \( U_n \subset U_{n-1} \). We claim \( \bigcap U_n = \{ x \} \).

If \( y \in \bigcap U_n \), then \( y = \lambda_n x + \mu_n x_1 + \rho_n y_n \not\in U_n \), so that, taking convergent subsequences, \( y = \lambda x + \mu x + \nu x + 0 = x \). There exists \( m \) such that \( W_m \subset Z \). Let \( V_n = W_{m+n} \). Then \( U_n, V_n \subset Z \). \( U_n, V_n \) are \( \mathcal{T}_2 \)-shrinkable at \( x_n, y_n \), \( Z \) \( \mathcal{T}_2 \)-shrinkable at 0, so, by (2.16), we may extend \( \text{Id}: \text{Bd} Z \to \text{Bd} Z \) to a \( \mathcal{T}_2 \) homeomorphism
\[ j: \overline{Z} \setminus U_1 \to \overline{Z} \setminus V_1 \]
with \( j \mid (\overline{Z} \setminus U_1) \cap X: (\overline{Z} \setminus U_1) \cap X \to (\overline{Z} \setminus V_1) \cap X \), a \( \mathcal{T}_1 \) homeomorphism, \( j \mid \text{Bd} U_1 : \text{Bd} U_1 \to \text{Bd} V_1 \), a \( \mathcal{T}_2 \) homeomorphism, and \( j \mid (\text{Bd} U_1) \cap X: (\text{Bd} U_1) \cap X \to (\text{Bd} V_1) \cap X \), a \( \mathcal{T}_2 \)
homeomorphism. Since $U$, $U_2$ are $\gamma_2$-shrinkable at $x_2$, we use (2.16) again and extend $j|\text{Bd } U_i$ to $j:U_1\setminus U \to V_1 \setminus V_2$.
Continue the extension and define $j|Y \setminus Z = \text{Id}$, obtaining $j:Y\setminus x \to Y \setminus 0$ with the stated properties.

(2.22) LEMMA. Suppose $(X, \gamma_1)$, $(Y, \gamma_2)$ are linear topological spaces with $X$ a linear subspace of $Y$ and $\gamma_2|X \subset \gamma_1$.
Let $U$ be a linearly bounded $\gamma_2$-shrinkable neighborhood of 0 and $i:Y\setminus 0 \to Y \setminus 0$ be defined by $i(y) = (1/\gamma(y,0,U,Y,\gamma_2)) \pi(y,0,U,Y,\gamma_2)$, an inversion with respect to $\text{Bd } \gamma_2 U$. Then $i$ is a $\gamma_2$ homeomorphism and $i|X \setminus 0: X \setminus 0 \to X \setminus 0$ is a $\gamma_1$ homeomorphism.

PROOF. $i$ is $\gamma_2$ continuous and $i^{-1} = i$. If $y \in X \setminus 0$, $i(y) = (1/\gamma (y,0,U \cap X,\gamma_1)) \pi (y,0,U \cap X,\gamma_1)$ by (2.5) and (2.6), so $i|X \setminus 0, i^{-1}|X \setminus 0$ are $\gamma_1$ continuous.

We now prove Theorem (2.12). We may assume $0 \notin K$.
Let $W_n = \{y | p(y) < 2^{-n}\}$. For some $n_0$, $W_{n_0} \cap K = \emptyset$. By (2.21), there exist $x \in W_{n_0} \setminus X$ and $j:Y\setminus x \to Y \setminus 0$ such that $j$ is a $p$ homeomorphism, $j|X:X \to X \setminus 0$ a $\gamma$ homeomorphism, and $j|Y \setminus W_{n_0} = \text{Id}$. By (2.22), there exists an inversion $i:Y\setminus 0 \to Y \setminus 0$ with respect to $\text{Bd } W_{n_0}$ such that $i$ is a $p$ homeomorphism and $i|X \setminus 0: X \setminus 0 \to X \setminus 0$, a $\gamma$ homeomorphism.
Since $i(K)$ is closed in $Y \setminus 0$, $j^{-1}i(K)$ is closed in $Y \setminus x$ and contained in $W_{n_0} \cap X$. Let $p(y\setminus x) \to 0$ with $y \notin X$. Let $x_1 = y_1$, $K_1 = \bigcup \{x | [x,x_1] + (1-\lambda)j^{-1}i(K)| \lambda \in [0,1]\}$ and $U_1 = K_1 + W_{n_1}$. 
K, is p-bounded, since\( p(\lambda (\mu x + \nu x) + (1 - \lambda) j' i(k)) \leq p(x) + p(x_i) + p(j' i(k)) \leq p(x) + p(x_i) + (1/2)^n \). By (2.8), \( K_i \) is star-shaped at \( x_i \), so according to (2.20) \( U_1 = K_i + W_i \) is linearly bounded and p-shrinkable at \( x_i \). There exists \( n_2 > 1 \) such that\( \bigcup \{ \lambda [x, x_i, y_{n_i}] + (1 - \lambda) j' i(k) \mid \lambda \in [0, 1] \} \subseteq K_i + W_2 \), since \( p(\lambda (\mu x + \rho x_i + \sigma y_n) + (1 - \lambda) j' i(k)) - \begin{aligned} [\lambda ((\mu + \nu)x + \rho x_i) + (1 - \lambda) j' i(k)] \end{aligned} = p(\lambda (\sigma (y_n - x)) < 1/2 \) for large \( n \). Let \( x_2 = y_{n_2}, K_2 = \bigcup \{ \lambda [x, x_i, x_2] + (1 - \lambda) j' i(k) \mid \lambda \in [0, 1] \} \). Then \( K_2 \subseteq K_i + W_2 \) and by (2.17) \( K_2 + W_3 \subseteq K_2 + W_2 \subseteq K_1 + W_2 \subseteq U_1 \). Let \( U_2 = K_2 + W_3 \), \( V_2 = [x, x_i, x_2] + W_3 \). Then \( U_2 \subseteq U_1, \overline{V_2} \subseteq U_1 \). By the foregoing argument, there exists \( n_3 > n_2 \) such that \( \bigcup \{ \lambda [x, x_i, y_{n_3}] + (1 - \lambda) j' i(k) \mid \lambda \in [0, 1] \} \subseteq K_2 + W_4 \) and \([x, x_i, y_{n_3}] \subseteq [x, x_i, x_2] + W_4 \). Letting \( x_3 = y_{n_3}, K_3 = \bigcup \{ \lambda [x, x_i, x_3] + (1 - \lambda) j' i(k) \mid \lambda \in [0, 1] \} \), then \( K_i + W_5 \subseteq K_3 + W_5 \subseteq K_2 + W_4 \subseteq K_2 + W_3 = U_2 = U_2 \) and \([x, x_i, x_3] + W_5 \subseteq \[x, x_i, x_2] + W_4 \subseteq [x, x_i, x_2] + W_3 = V_2 \). Let \( U_3 = K_3 + W_5 \), \( V_3 = [x, x_i, x_3] + W_5 \). Continue, obtaining \( K_0 = \bigcup \{ \lambda [x, x_{n-1}, x_n] + (1 - \lambda) j' i(k) \mid \lambda \in [0, 1] \} \), \( V_0 = [x, x_{n-1}, x_n] + W_{2n-1} \), with \( U_n \subseteq U_{n-1}, V_n \subseteq V_{n-1} \), and \( U_n, V_n \) p-shrinkable at \( x_{n-1}, x_n \). We claim \( (\bigcap V_n) \cap K = j' i(K) \)

Take \( y \in (\bigcap V_n) \cap K \). Then \( y = \lambda_n (\mu_n x + \rho_n x_{n-1} + \sigma_n x_n) + (1 - \lambda_n) j' i(k_n) + W_n \). If \( \lambda_n \to 1 \), let \( y = x \), a contradiction. If \( \lambda_n \to 0 \), then \( \lambda_n \to 0 \), for some \( \{n_k\} \). \( j' i(k_n) = (1 - \lambda_n)^{-1} (y - \lambda_n (\mu_n x + \rho_n x_{n-1} + \sigma_n x_n) - W_n \to (1 - \lambda)^{-1} (y - \lambda x) \) which is in \( j' i(K) \) or equal to \( x \), since \( j' i(K) \)
is closed in $Y \setminus x$. In the first case $y - \lambda x = (1 - \lambda)j^{-1}(k)$, and if $\lambda \neq 0$, $x = (1/\lambda)(y - (1 - \lambda)j^{-1}(k)) \in X$, a contradiction. Thus $\lambda = 0$, and $y = j^{-1}(k)$. If $(1 - \lambda)j^{-1}(y - \lambda x) = x$, then $y = x$, a contradiction. Similarly, we may show

$$(\bigcap V_n) \cap X = \emptyset.$$ Noting $U_1$, $U_2$, $V_2$ are $\tau$-shrinkable at $x_1$, we use (2.16) to extend $\text{Id}: \text{Bd} U_1 \to \text{Bd} U_1$ to $h: \overline{U_1} \setminus U_2 \to \overline{U_1} \setminus V_2$, a $\tau$ homeomorphism with

$$h|((\overline{U_1} \setminus {U_2}) \cap X : (\overline{U_1} \setminus {U_2}) \cap X \to (\overline{U_1} \setminus V_2) \cap X;$$

a $\tau$ homeomorphism, $h|\text{Bd} U_2: \text{Bd} U_2 \to \text{Bd} V_2$, a $\tau$ homeomorphism, and

$$h|((\text{Bd} U_2) \cap X : (\text{Bd} U_2) \cap X \to (\text{Bd} V_2) \cap X, a \tau homeomorphism. Since $U_2$, $V_2$, $U_3$, $V_3$ are $\tau$-shrinkable at $x_2$, we may extend $h|\text{Bd} U_2$ to $h: \overline{U_2} \setminus U_3 \to \overline{V_2} \setminus V_3$ with these properties.

Continuing the extension, and letting $h|Y \setminus U_1 = \text{Id}$, we have $h: (Y \setminus U_1) \cup \bigcup_{n=1}^{\infty}(\overline{U_n} \setminus U_{n+1}) \to (Y \setminus U_1) \cup \bigcup_{n=1}^{\infty}(\overline{V_n} \setminus V_{n+1})$ (with $V_1 = U_1$), a $\tau$ homeomorphism with

$$h|((\text{dom } h) \cap X: [(Y \setminus U_1) \cap X] \cup \bigcup_{n=1}^{\infty}(\overline{U_n} \setminus U_{n+1}) \cap X] \rightarrow [(Y \setminus U_1) \cap X] \cup \bigcup_{n=1}^{\infty}(\overline{V_n} \setminus V_{n+1}) \cap X], a \tau homeomorphism. (Note that the unions are over $\tau$-locally finite collections of $\tau$-closed sets.) This can be written

$$h|((\text{dom } h) \cap X: X: X \setminus [(\bigcap U_n) \cap X] \to X \setminus [(\bigcap V_n) \cap X]. That is,$$

$$h|((\text{dom } h) \cap X: X \setminus j^{-1}(K) \to X, a \tau homeomorphism. It is then easy to check that $h|j^{-1}j(K \setminus K \to X, a \tau homeomorphism: a) h|j^{-1}j is defined on $X \setminus K$, and into $X$. Take $y \in X \setminus K$. Clearly $j^{-1}j(y)$ is defined, and in $X$. We must show $j^{-1}j(y) \neq j^{-1}(k)$. If $j^{-1}j(y) = j^{-1}(k)$, then $j(y) = k$ and thus $y = k$, since $j|K = \text{Id}$.,
b) $h_j^{-1}j_k|X \setminus K$ is onto $X$. Take $y \in X$, and let

$z = j^{-1}j_k^{-1}(y) \in X$. Then $h_j^{-1}j_k(z) = y$. If $z \in K$, then

$h_j^{-1}(z) = h_k^{-1}(y)$ with $j(z) = z$, so that $j^{-1}(z) = h_k^{-1}(y)$, and

$h_k^{-1}(y) \in j^{-1}(K)$, a contradiction.

Next we prove Corollary (2.14). Define $p: L^r \to R$ by

$p(f) = \sum_{x \in J} |f(x)|^r$, and $p: L^q \to R$ by $p(f) = \sum_{x \in J} |f(x)|^q$. We must

show $L^q \subset L^r$. Suppose $\int_{\{x \in J\}} |f|^q < \infty$. If $E = \{x \in [0,1] \mid |f(x)| \leq 1\}$ and $F = \{x \in [0,1] \mid |f(x)| > 1\}$, then

$\int_{\{x \in J\}} |f|^r = \int_{E} |f|^r \chi_E + \int_{F} |f|^r \chi_F$. The first integral is

finite, and $|f|^r \chi_F \leq |f|^q \chi_F$ implies $\int_{\{x \in J\}} |f|^r \chi_F < \infty$. Now

let $f(x) = x^{-\frac{r}{q}} 1_F \setminus L^q$. Then $\chi_{E \setminus F} f \in L^q$ and

$\int_{\{x \in J\}} |f|^r \chi_{E \setminus F} = \int_{\{x \in J\}} x^{-\frac{r}{q}} \to 0$, so that the condition of (2.13)

holds. Next we show $p|L^q$ is weaker than $p|L$. Suppose

$p_n(f_n) \to 0$ and let $E_n = \{x \in [0,1] \mid f_n(x) \leq 1\}$.

$F_{n\varepsilon} = \{x \in [0,1] \mid |f_n(x)| > \varepsilon\}$. $\int_{\{x \in J\}} |f_n|^q \geq \varepsilon^q n F_{n\varepsilon}$, so

$\mu F_{n\varepsilon} \to 0$ for each $\varepsilon > 0$. $\int_{\{x \in J\}} |f_n|^r \chi_{E_n} = \int_{\{x \in J\}} |f_n|^r \chi_{E_n \setminus F_{n\varepsilon}} + \int_{\{x \in J\}} |f_n|^r \chi_{E_n \setminus F_{n\varepsilon}} \leq \mu F_{n\varepsilon} + \varepsilon^r$. This shows $\int_{\{x \in J\}} |f_n|^r \chi_{E_n} \to 0$.

Also, $\int_{\{x \in J\}} |f_n|^r \chi_{F_{n\varepsilon}} \leq \int_{\{x \in J\}} |f_n|^q \chi_{F_{n\varepsilon}} \leq \int_{\{x \in J\}} |f_n|^q \to 0$. It remains
to verify condition 3 of (2.12). 

$$p(tf) = \int_{[0,1]} |tf| \, \mu = \int_{[0,1]} |f| \, \mu$$, so we may take $b = t$. For Corollary (2.15) note $\overline{K} = \bigcup_{K_n} \rho = \bigcup_{K_n} \rho = \bigcup_{K_n} \rho = K$.

The final results of this chapter are applications of the sort found in Klee [9].

(2.23) THEOREM. Suppose $(X, \mathcal{T})$ is a linear topological space admitting a metrizable incomplete linear topology $\mathcal{T} \subset \mathcal{T}$ and $U \in \mathcal{T}$ is $\mathcal{T}$-shrinkable at 0. If $\mathcal{T}$ contains a linearly bounded set, then there exists a $\mathcal{T}$ homeomorphism $h:X \to X$ such that $h = h^{-1}$, $h|U$ maps $U$ onto $X \setminus U$, and $h|\text{Bd} U = \text{Id}$.

PROOF. By (2.4), with $K = \{0\}$, there exists a $\mathcal{T}$ homeomorphism $j:X \to X \setminus 0$ such that $j|X \setminus U = \text{Id}$. Let $i:X \setminus 0 \to X \setminus 0$ be defined by $i(x) = (1/\mathcal{N}(x,0)) \mathcal{N}(x,0)$, a $\mathcal{T}$ homeomorphism with $i^{-1} = i$. Then $h = j^{-1} i j$ has the desired properties.

(2.24) LEMMA. If $(X, \mathcal{T})$ is an incomplete linear topological space and $f$ a continuous linear functional on $X$, then $f$ is incomplete.

PROOF. Suppose $f$ is complete, and $\{x_s\}$ is Cauchy in $X$. $x_s = w_s + \lambda_s y$ where $w_s \in f^\perp$, $y \not\in f^\perp$. $x_s - x_r = (w_s - w_r) + (\lambda_s - \lambda_r) y$, so that $f(x_s - x_r) = (\lambda_s - \lambda_r) f(y)$. Since $x_s - x_r \to 0 \implies f(x_s - x_r) \to 0$, we see $\{\lambda_s\}$ is Cauchy. Thus
\[ \lambda_5 \rightarrow \lambda_6. \] It then follows that \( \{w_j\} \) is Cauchy, and so, under the supposition, \( w_5 \rightarrow w_6. \) But then \( x_5 \rightarrow w_0 + \lambda_6 y, \) showing \( X \) complete.

(2.25) LEMMA. If, for \( i = 1, 2, \ldots \), \( U_i \) is open, shrinkable at \( x_i \) and radially bounded at \( x_i, \) then \( \text{Bd } U_i \) is homeomorphic to \( \text{Bd } U_2. \)

PROOF. \( U_i - x_i \) is shrinkable at 0, so we may assume each \( U_i \) shrinkable at 0. \( \pi_{U_i} : X \setminus 0 \rightarrow \text{Bd } U_i \) is continuous for each \( i, \) and so are \( \pi_{U_i} |_{\text{Bd } U_i}, \pi_{U_2} |_{\text{Bd } U_1}. \) Thus \( \pi_{U_2} |_{\text{Bd } U_1} : \text{Bd } U_1 \rightarrow \text{Bd } U_2 \) is a homeomorphism with inverse \( \pi_{U_1} |_{\text{Bd } U_2}. \)

(2.26) THEOREM. Suppose \( (X, \tau_1) \) is a linear topological space admitting a metrizable incomplete linear topology \( \tau_2 \subset \tau_1 \) and \( U \in \tau_1 \) is linearly bounded and \( \tau_2 \)-shrinkable at 0. If \( \tau_2 \) contains a linearly bounded set and \( f \) is a non-zero \( \tau_2 \) continuous linear functional on \( X, \) then \( f^\perp \) is \( \tau_1 \) homeomorphic to \( \text{Bd } \tau_2 U. \)

PROOF. Note that \( f \) is also \( \tau_1 \) continuous. Take \( x \in X \) such that \( f(x) > 0. \) By (2.6) and (2.9), \( U_i = U + [-x, x] \) is \( \tau_1 \)-shrinkable at \(-x, 0 \) and \( x, \) and by (2.7), \( U_i \) is radially bounded at 0. If \( y \in \text{Bd } U_i \) and \( f(y) \leq 0, \) then \([x, y]\) meets \( f^\perp \) in exactly one point, \( h(y). \) This map \( h \) is from \( (\text{Bd } U_i) \cap \{y \mid f(y) \leq 0\} \) onto \( f^\perp \cap U_i, \) and is 1-1, \( \tau_1 \) bicontinuous. \( h(y) = y + (f(y)/(f(y)-f(x)))(x-y), \) and
$h(y) = \Pi_U(y, x)$. By (2.5), $U_i \cap f^\perp$ is $\tau_i$-shrinkable in $f^\perp$ at 0. $(f^\perp, \tau_i)$ is incomplete by (2.24) and since $\tau_i|f^\perp$ contains a linearly bounded set, (2.23) applies giving a homeomorphism $k:f^\perp \to f^\perp$ such that $k(U_i \cap f^\perp) = f^\perp \setminus (U_i \cap f^\perp)$, $k|\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = \text{Id}$. Since $U_i$ is $\tau_i$-shrinkable at 0, $\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = (\text{Bd}_{\tau_i} U_i) \cap f^\perp$. Map $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus \overline{U}$, by projection from $-x$. Calling this map $\lambda$, the function $k\lambda$ is a $\tau_i$ homeomorphism from $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus U_i$. Then $h \circ k\lambda$ is a $\tau_i$ homeomorphism from $\text{Bd } U_i$ onto $f^\perp$. Since $U_i$ is radially bounded at 0, we may use (2.25) and conclude $f^\perp \tau_i$ homeomorphic to $\text{Bd } U$.

(2.27) THEOREM. Under the hypotheses of Theorem (2.26), the half-space $H = \{y|f(y) \geq 0\}$ is $\tau_i$ homeomorphic to $U$.

PROOF. Take $x \in X$ such that $f(x) = 1$. By (2.4), with $K = \{0\}$, there exists a $\tau_i$ homeomorphism $j:X \to X \setminus 0$ such that $j|X \setminus U = \text{Id}$. There exists a $\tau_i$ homeomorphism $k:U \to f^\perp$ according to (2.26). Let $h:U \to H$ be defined by $h(y) = k \Pi_U(j(y), 0) + ((1/\gamma(j(y), 0)) - 1)x$. $h$ is $\tau_i$ continuous and has $\tau_i$ continuous inverse $h^{-1}(z) = j^{-1}(1/(\gamma(z) + 1))k^{-1}(q(z))$, where $q:X \to f^\perp$ is projection in direction $x$, onto $f^\perp$. 

h(y) = \Pi_U(y, x)$. By (2.5), $U_i \cap f^\perp$ is $\tau_i$-shrinkable in $f^\perp$ at 0. $(f^\perp, \tau_i)$ is incomplete by (2.24) and since $\tau_i|f^\perp$ contains a linearly bounded set, (2.23) applies giving a homeomorphism $k:f^\perp \to f^\perp$ such that $k(U_i \cap f^\perp) = f^\perp \setminus (U_i \cap f^\perp)$, $k|\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = \text{Id}$. Since $U_i$ is $\tau_i$-shrinkable at 0, $\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = (\text{Bd}_{\tau_i} U_i) \cap f^\perp$. Map $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus \overline{U}$, by projection from $-x$. Calling this map $\lambda$, the function $k\lambda$ is a $\tau_i$ homeomorphism from $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus U_i$. Then $h \circ k\lambda$ is a $\tau_i$ homeomorphism from $\text{Bd } U_i$ onto $f^\perp$. Since $U_i$ is radially bounded at 0, we may use (2.25) and conclude $f^\perp \tau_i$ homeomorphic to $\text{Bd } U$.

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h(y) = \Pi_U(y, x)$. By (2.5), $U_i \cap f^\perp$ is $\tau_i$-shrinkable in $f^\perp$ at 0. $(f^\perp, \tau_i)$ is incomplete by (2.24) and since $\tau_i|f^\perp$ contains a linearly bounded set, (2.23) applies giving a homeomorphism $k:f^\perp \to f^\perp$ such that $k(U_i \cap f^\perp) = f^\perp \setminus (U_i \cap f^\perp)$, $k|\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = \text{Id}$. Since $U_i$ is $\tau_i$-shrinkable at 0, $\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = (\text{Bd}_{\tau_i} U_i) \cap f^\perp$. Map $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus \overline{U}$, by projection from $-x$. Calling this map $\lambda$, the function $k\lambda$ is a $\tau_i$ homeomorphism from $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus U_i$. Then $h \circ k\lambda$ is a $\tau_i$ homeomorphism from $\text{Bd } U_i$ onto $f^\perp$. Since $U_i$ is radially bounded at 0, we may use (2.25) and conclude $f^\perp \tau_i$ homeomorphic to $\text{Bd } U$.

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h(y) = \Pi_U(y, x)$. By (2.5), $U_i \cap f^\perp$ is $\tau_i$-shrinkable in $f^\perp$ at 0. $(f^\perp, \tau_i)$ is incomplete by (2.24) and since $\tau_i|f^\perp$ contains a linearly bounded set, (2.23) applies giving a homeomorphism $k:f^\perp \to f^\perp$ such that $k(U_i \cap f^\perp) = f^\perp \setminus (U_i \cap f^\perp)$, $k|\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = \text{Id}$. Since $U_i$ is $\tau_i$-shrinkable at 0, $\text{Bd}(f^\perp, \tau_i)(U_i \cap f^\perp) = (\text{Bd}_{\tau_i} U_i) \cap f^\perp$. Map $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus \overline{U}$, by projection from $-x$. Calling this map $\lambda$, the function $k\lambda$ is a $\tau_i$ homeomorphism from $(\text{Bd } U_i) \cap \{y|f(y) \geq 0\}$ onto $f^\perp \setminus U_i$. Then $h \circ k\lambda$ is a $\tau_i$ homeomorphism from $\text{Bd } U_i$ onto $f^\perp$. Since $U_i$ is radially bounded at 0, we may use (2.25) and conclude $f^\perp \tau_i$ homeomorphic to $\text{Bd } U$.
3. AN ISOTOPY THEOREM

In the proof of Theorem (2.1), the homeomorphisms \( \text{Id}, h_1, h_2h_1, h_3h_2h_1, \ldots \) may be regarded as successive stages of an isotopy whose final homeomorphism is \( h: X \to X \setminus K \).

There are obvious ways to fill the gaps, but the details are tedious. The statement of the isotopy theorem below is patterned after Klee's [9], and Corollary (3.4) answers Klee's question in [10], whether his isotopy theorem extends to an arbitrary normed linear space.

(3.1) THEOREM. Suppose \((X, \tau')\) is a linear topological space admitting a metrizable incomplete linear topology \( \tau_2 \subseteq \tau' \). If \( U \in \tau_2 \) and \( K \) is \( \tau'_2 \)-compact with \( [0,1]K \subseteq U \), then there exists a \( \tau'_2 \) embedding \( H: X[0,1] \to X[0,1] \) such that if \( h'_t(x) = p, H(x,t) \) for \( t \in [0,1] \), then \( \{ h'_t \} \) has the following properties.

1. \( h'_t: X \to X \) is a \( \tau'_2 \) homeomorphism for each \( t \in [0,1] \).
2. \( h'_0 = \text{Id} \).
3. \( h'_1: X \to X \setminus K \) is a \( \tau'_1 \) homeomorphism.
4. For each \( t \in [0,1] \), \( h'_t / X \setminus U = \text{Id} \).
5. \( \lim h'_t h'_t^{-1} = \text{Id}_{X}(\tau'_2) \) and \( \lim h'_t h'_t^{-1} = \text{Id}_{X \setminus K}(\tau'_2) \) with the convergence uniform on each \( \tau'_2 \)-compact set.

(3.2) COROLLARY. If \((X, \tau')\) is a metrizable complete linear topological space admitting a strictly weaker metrizable topology \( \tau'_2 \), then \((X, \tau')\) is incomplete and the
Theorem holds.

(3.3) COROLLARY. Suppose the hypotheses of the Theorem hold, except that \( U \in \mathcal{T}_1 \), \( K \) is \( \mathcal{T}_1 \)-compact and \([0,1]K \subset U\). Suppose also \( \mathcal{T}_2 \) contains a linearly bounded set. Then the conclusions of the Theorem hold (with limits uniform on \( \mathcal{T}_1 \)-compact sets).

(3.4) COROLLARY. If \( X \) is a normed linear space and \( K \subset U \), where \( K \) is compact and \( U \) is a bounded open convex body, then there is an isotopy with the properties stated in the Theorem (limits uniform on sets compact in the given norm).

To prove the Theorem, let \( A_\alpha, B_\alpha, C_\alpha, D_\alpha \) be defined as in the proof of (2.1) and recall the inclusion relations

\[
D_1 \supset C_1 \supset B_1 \supset A_1 \\
\cup \\
D_2 \supset C_2 \supset B_2 \supset A_2 \\
\cup \\
D_3 \supset C_3 \supset B_3 \supset A_3 \\
\vdots
\]

We want to define \( H_\alpha: X \times [0,1] \rightarrow X \times [0,1] \) so that for a fixed \( t \), \( H_\alpha(x,t) \) is fixed on \( \overline{A_\alpha} \cup (X \setminus D_\alpha) \) and \( H_\alpha(x,t) \) pushes \( \text{Bd} B_\alpha \) onto the dotted portion in Figure 1. For \( t = 0 \), the dotted portion coincides with \( \text{Bd} B_\alpha \); for \( t = \frac{1}{2} \), it coincides with \( \text{Bd} C_\alpha \). The details follow.
Figure 1. An isotopy stage

Define \( H_i : X \times [0, \frac{1}{2}] \rightarrow X \times [0, \frac{1}{2}] \) in three parts.

\[ H_{i_1} : [\bar{A}_i \cup (X \setminus D_i)] \times [0, \frac{1}{2}] \rightarrow [\bar{A}_i \cup (X \setminus D_i)] \times [0, \frac{1}{2}], \ H_{i_1} = \text{Id.} \]

\[ H_{i_2} : (\bar{B}_i \setminus A_i) \times [0, \frac{1}{2}] \rightarrow \]

\[ (x,t) \mid 1 \leq \frac{y_i(x,x_i)}{y_{i_1}(x,x_i)} \leq \frac{y_i(x,x_i) + 2t(y_i(x,x_i) - y_{i_1}(x,x_i))}{y_{i_1}(x,x_i)} \]

\[ \text{and } 0 \leq t \leq \frac{1}{2} \] where \( H_{i_2}(x,t) = (\pi_{A_i}(x,x_i) + \)}

\[ \frac{y_i(x,x_i) - 1}{y_{i_1}(x,x_i)} \left[ \pi_{B_i}(x,x_i) + 2t(\pi_{A_i}(x,x_i) - \pi_{B_i}(x,x_i)) - \pi_{A_i}(x,x_i) \right] , t). \]

\( H_{i_2} \) is a \( \pi_i \) homeomorphism with inverse.
\[ H_{i2}^{-1}(x, t) = \left( \frac{\nu_{A_i}(x, x_i) - 1}{\nu_{A_i}(\pi_{B_i}(x, x_i) + 2t(\pi_{C_i}(x, x_i) - \pi_{B_i}(x, x_i))), x_i) - 1}{\nu_{B_i}(\pi_{D_i}(x, x_i), x_i) - 1} \right) \]

\[ H_3: (D_i \cap B_1) \times [0, \frac{1}{2}] \rightarrow \left( (x, t) \mid \nu_{A_i}(\pi_{B_i}(x, x_i) + 2t(\pi_{C_i}(x, x_i) - \pi_{B_i}(x, x_i))), x_i) \leq \nu_{A_i}(x, x_i) \leq \nu_{A_i}(\pi_{D_i}(x, x_i), x_i) \right) \text{ and } 0 \leq t \leq \frac{1}{2}, \]

where

\[ H_3(x, t) = \left( \frac{\nu_{B_i}(x, x_i) + 2t(\nu_{D_i}(x, x_i) - \nu_{B_i}(x, x_i)))}{\nu_{B_i}(\pi_{D_i}(x, x_i), x_i) - 1} \right) \]

\[ H_3 \text{ is a } \gamma_i \text{ homeomorphism with inverse } H_3^{-1}(x, t) = \left( \frac{\nu_{B_i}(x, x_i) - 2t(\nu_{D_i}(x, x_i) - \nu_{B_i}(x, x_i))), x_i)}{\nu_{B_i}(\pi_{D_i}(x, x_i), x_i) - 1} \right). \]

Since the domains and ranges of the \( H_i \) are finite collections of closed sets and definitions agree on the intersections, \( H_3 = H_1 \cup H_2 \cup H_3 \) is a \( \gamma_i \) homeomorphism. Note that \( h^t_i : X \rightarrow X \), defined by \( h^t_i(x) = p_i H(x, t) \) (\( p_i = \) projection on the first coordinate) is a \( \gamma_i \) homeomorphism for each \( t \in [0, \frac{1}{2}] \), and that \( h^t_i = \) 'Id, \( h^t_i \cup (X \setminus D_i) = \) 'Id, and \( h^t_i = h_i \), where \( h_i \) is as in the proof of (2.1). Next use the shrinkability of \( A_2, B_2, C_2, D_2 \) at \( x_2 \) to define a homeomorphism \( K_2 : X \times [\frac{3}{2}, \frac{2}{3}] \rightarrow X \times [\frac{3}{2}, \frac{2}{3}] \) such that

\[ K_2 \left( (A_2 \cup (X \setminus D_2)) \times [\frac{3}{2}, \frac{2}{3}] \right) = \text{Id}, K_2 \left| XX [\frac{3}{2}, \frac{2}{3}] = \text{Id}, p_2 K_2(x, t) = t \right. \]

and \( K_2(x, \frac{3}{2}) = h_2(x) \), and let \( H_2: X \times [\frac{3}{2}, \frac{2}{3}] \rightarrow X \times [\frac{3}{2}, \frac{2}{3}] \) be defined by \( H_2(x, t) = K_2(h^t_i(x), t) = K_2(h_i(x), t). \) \( H_2 \) is a \( \gamma_i \) homeomorphism with inverse \( H_2^{-1}(y, t) = (h_i^{-1} \circ K_2^{-1}(y, t), t) \), and has the properties.
\[ H_2 \left( X \setminus D_i \right) x \left[ \frac{1}{2^n}, \frac{1}{2^{n+1}} \right] = \text{Id}, \]
\[ H_2 \left( x \right) x \left( \frac{1}{2^n} \right) = H_0 \left( x x \left( \frac{1}{2^n} \right) \right), \]
\[ H_2 \left( x, \frac{3}{4} \right) = \left( h_1, h_1 \left( x \right), \frac{3}{4} \right), \]
\[ H_2 \left( x, t \right) = \left( h, h_1 \left( x \right), t \right) = (h(x), t) \text{ if } x \notin B. \]

To see the last, note \( x \notin B_i \implies h_i \left( x \right) \notin C_i \implies h_i \left( x \right) \notin D_2 \implies K_2 \left( h_i \left( x \right), t \right) = (h, h_1 \left( x \right), t). \) It is shown in the proof of (2.1) that \( h(x) = h_i \left( x \right) \) for \( x \notin B_i. \)

Letting \( h_t' \left( x \right) = p_t H_2 \left( x, t \right) \) for \( t \in \left[ \frac{1}{2^n}, \frac{3}{4} \right], \) each \( h_t' \) is a \( \gamma \) homeomorphism of \( X, \) and \( h_t' = h_t h. \) We may continue the extension, obtaining at the \( n \)th stage \( K_n : X x [1-2^{n-1}, 1-2^{-n}] \to X x [1-2^{n-1}, 1-2^{-n}] \) with
\[ K_n \left[ A_n \cup \left( X \setminus \bar{A}_n \right) x [1-2^{n-1}, 1-2^{-n}] \right] = \text{Id}, \]
\[ K_n \left[ X x [1-2^{n-1}, 1-2^{-n}] \right] = \text{Id}, \]
\[ K_n \left( x, 1-2^{-n} \right) = h_n \left( x \right), \] and \( p_t K_n \left( x, t \right) = t. \) We define
\[ H_n : X x [1-2^{n-1}, 1-2^{-n}] \to X x [1-2^{n-1}, 1-2^{-n}] \] by \( H_n \left( x, t \right) = \left( h_n, \ldots, h_n, K_n \left( y, t \right), t \right), \) and properties
\[ H_n \left( X \setminus D_i \right) x [1-2^{n-1}, 1-2^{-n}] = \text{Id}, \]
\[ H_n \left[ X x [1-2^{n-1}, 1-2^{-n}] \right] = H_{n+1} \left[ X x [1-2^{n-1}, 1-2^{-n}] \right], \]
\[ H_n \left( x, 1-2^{-n} \right) = \left( h_n, \ldots, h_n, h_1 \left( x \right), 1-2^{-n} \right), \]
\[ H_n \left( x, t \right) = \left( h_n, \ldots, h_n, h_1 \left( x \right), t \right) = (h(x), t) \text{ if } x \notin B_{n-1}. \]

We let \( h_t' \left( x \right) = H_n \left( x, t \right) \) for \( t \in \left[ 1-2^{n-1}, 1-2^{-n} \right]. \) Since \( \left[ X x [1-2^{n-1}, 1-2^{-n}] \right] \) is a locally finite collection of closed sets, we may define a homeomorphism \( H : X x [0, 1] \to X x [0, 1] \)
by \( H = \bigcup_n H_n. \) If we let \( H(x, 1) = (h(x), 1) \) then
\( H : X x [0, 1] \to X x [0, 1] \) is 1-1 and, as we now show, bicontinuous. Evidently, it is sufficient to show continuity of \( H \) at \( (x, 1), \) \( x \in X, \) and continuity of \( H^{-1} \) at \( (y, 1), \) \( y \in X \setminus K. \)
Take \((x', t') \rightarrow (x, 1)\). For some \(n\), \(x \notin B_{n-1}\), so that \(h(x) = h_{n-1} \circ \cdots \circ h_1 \circ h_0(x)\). If \(S \geq S_0\), then \(x \notin B_{n-1}\) and \(1 - 2^{-m} \leq t \leq 1\), so that \(1 - 2^{-m} \leq t < 1 - 2^{-m-1}\) for some \(m > n\) (\(t = 1\) is easily taken care of), and \(H(x', t') = h_{\alpha_{-1}}(x, t') = (h_{\alpha_{-1}} \circ \cdots \circ h_1 \circ h_0(x), t') = (h(x'), t') \rightarrow (h(x), 1) = H(x, 1)\). Now take \((y', t') \rightarrow (y, 1)\) with \(y \in X \setminus K\). For some \(n\), \(y \notin B_n\), so that \(y = h(x) = h_{n-1} \circ \cdots \circ h_1(x)\) for some \(x \in X\). If \(S \geq S_0\), then \(y \notin B_n\) and \(1 - 2^{-m} \leq t \leq 1\). Then \(1 - 2^{-m} \leq t < 1 - 2^{-m-1}\) for some \(m > n\), and \(p_{\alpha_{-1}}(y', t') = y'\). Thus \(H_{\alpha_{-1}}(y', t') = H_{\alpha_{-1}}(y', t') = (h_{\alpha_{-1}} \circ \cdots \circ h_1 \circ h_0(y), t') = (h_{\alpha_{-1}}(y), t') \rightarrow (h_{\alpha_{-1}}(y), 1) = H_{\alpha_{-1}}(y, 1)\). 

This completes the proof that \(H\) is a \(\gamma\) embedding. We have noted \(h_{\alpha_{-1}}: X \rightarrow X\) is a homeomorphism whenever \(t \in [0,1)\), and defining \(h_{t_0}(x) = p_{\alpha_{-1}}H(x, 1) = h(x)\), we see \(h_{t_0}\) has the required properties. Since \(H_{\alpha_{-1}}[(X \setminus D) \times [1 - 2^{-n+1}, 1 - 2^{-n}] = Id, B = U\) and \(h_{t_0} = h\), we have \(h_{t_0}|X \setminus U = Id, each t \in [0,1]\). Next we show \(\lim h_{t_0}^{-1}h_{t_0}^{-1} = Id_{X} (\gamma_{t_0})\) uniformly on compact sets. If \(L \subset X\) and \(L\) is \(\gamma_{t_0}\)-compact, then \(L \cap B_{n} = \emptyset\) for some \(n\) (since \(\bigcap B_{n} = \emptyset\), and \(B_n \subset \overline{B_{n-1}}\)). It is easy to see \(h_{t_0}^{-1}(y) = p_{\alpha_{-1}}H_{t_0}^{-1}(y, t)\). By an argument of the sort above, if \(t\) is sufficiently near \(1\) and \(y \in L\), so that \(y \notin B_n\), then \(H_{t_0}^{-1}(y, t) = (h_{t_0}^{-1}(y), t)\), so that \(h_{t_0}^{-1}(y) = h_{t_0}^{-1}(y)\), and \(h_{t_0}^{-1}(y) = y\). The other uniform convergence statement is proved similarly.
The proof of Corollary (3.2) is the same as that of (2.2). To prove (3.3), begin as in the proof of (2.4) by finding \( U, V, r \in (0,1) \) with \([0,1]K \subset U \subset V, U \subset U,\) \( U \in \tau_1, V \in \tau_2 \) and radially bounded, and a \( \tau_1 \) homeomorphism \( j: X \to X \) such that \( j|_rU \cup (X \setminus 2V) = \text{Id}, j|_rU \setminus rU \to \overline{V} \setminus ru, \) and \( j|_rU \cup (X \setminus 2V) \to 2V \setminus U. \) By Theorem (3.1) there exists a \( \tau_1 \) isotopy, \( H:X \times [0,1] \to X \times [0,1] \) such that \( h_t^1: X \to X \) is a \( \tau_1 \) homeomorphism for \( t \in [0,1], h_0^1: X \to X \setminus K \) is a \( \tau_1 \) homeomorphism, \( \lim_{t \to 1} h_t h_t^{-1} = \text{Id}_{X \setminus K} (\tau_1), \lim_{t \to 1} h_t h_t^{-1} = \text{Id}_{X \setminus K} (\tau_1) \) with the convergence uniform on each \( \tau_2 \) compact set, and \( H((X \setminus V) \times [0,1]) = \text{Id}. \) Define \( L: X \times [0,1] \to X \times [0,1] \) by \( L(x,t) = (j p, H(j(x),t),t) \) and \( M: X \times [0,1] \cup (X \setminus K) \times [1] \to X \times [0,1] \) by \( M(y,t) = j p, H(j(y),t),t). \) Then \( M \) is an inverse for \( L. \) \( L \) and \( M \) are \( \tau_1 \) continuous, so \( L \) is a \( \tau_1 \) embedding. Letting \( \lambda_t^1: X \to X, \) \( \lambda_t^1(x) = p, L(x,t), \) we have \( \lambda_t^1(x) = j p, H(j(x),t) = j p, h_t^1(j(x)),t) = j h_t^1 j(x). \) That is, \( \lambda_t^1 = j h_t^1 j, \) so that each \( \lambda_t^1 \) is a \( \tau_1 \) homeomorphism, and \( \lambda_t^1 \) maps \( X \) onto \( X \) for \( t \in [0,1], \) maps \( X \) onto \( X \setminus K \) (note \( j|K = \text{Id}. \) If \( R \) is \( \tau_1 \)-compact, then as in the proof of the Theorem, there exists \( t_0 < 1 \) such that \( (t_0 \leq t \leq 1, y \in j(R) \Rightarrow h_t^1(y) = h_t^1(y)). \) Thus \( t_0 \leq t \leq 1, x \in R \Rightarrow \lambda_t^1(x) = j h_t^1 j h_t^1 j(x) = x, \) and so \( \lim_{t \to 1} \lambda_t^1 = \text{Id} \) uniformly on \( \tau_1 \)-compact sets. Similarly, for the other limit statement. If \( x \notin U, \) then \( j(x) \notin V, \) so that \( H(j(x),t) = (j(x),t), \) and \( L(x,t) = (j p, (j(x),t),t) = (x,t). \) Finally, we prove (3.4). We
may assume \( 0 \in U \), so that \([0,1] \subseteq U\). According to Bessaga and Klee \([4]\), \( X \) admits a weaker incomplete norm, so that Corollary (3.3) applies.
4. ADDITIONAL PROPERTIES OF SHRINKABLE NEIGHBORHOODS AND
SOME FACTS CONCERNING WEAKENING TOPOLOGIES

(4.1) LEMMA. If \( f \) is a non-zero continuous linear functional on a linear topological space \( X \), \( f(x) > 0 \), and \( U \) is open in the hyperplane \( x+f^\perp \), then \( V = \bigcup \{ \lambda U \mid \lambda > 0 \} \) is open in \( X \).

PROOF. \( V \) is the image of \( (0,\infty) \times (x+f^\perp) \) under the map
\[
h: (0,\infty) \times (x+f^\perp) \to [y \mid f(y) > 0],\ h(\lambda, z) = \lambda z.
\]
We show \( h \) is a homeomorphism. \( h \) is onto, since \( f(y) > 0 \) implies \( y = \frac{f(y)}{f(x)} [(f(x)/f(y))y] \). \( h \) is 1-1, since \( \lambda_1 z_1 = \lambda_2 z_2 \)
\[
\Rightarrow \lambda_1 f(z_1) = \lambda_2 f(z_2) \Rightarrow \lambda_1 f(x) = \lambda_2 f(x) \Rightarrow \lambda_1 = \lambda_2,
\]
which implies \( z_1 = z_2 \). To see that \( h^{-1} \) is continuous, suppose
\[
\lambda \lambda_0 z_0 \to \lambda z. \text{ Then } \lambda \lambda_0 f(x) \to \lambda f(x), \text{ so that } \lambda \lambda_0 \to \lambda. \ z_0 = (1/\lambda \lambda_0) (\lambda \lambda_0 z_0) \to (1/\lambda) (\lambda z) = z.
\]

A cone \( C \) in a linear space \( X \) is a subset of \( X \) such that \( c \in C, \lambda > 0 \Rightarrow \lambda c \in C \). All spaces below are metrizable.

(4.2) LEMMA. If \( X, f, x, U \) and \( V \) are as in (4.1), then
\( V, V \) and \( \text{Bd } V \) are cones, and \( V \cap (x+f^\perp) = U, \ V \cap (x+f^\perp) = U, \)\n\( \text{(Bd } V) \cap (x+f^\perp) = \text{Bd } U \).

PROOF. Proof of the cone statement is trivial.
\( y \in V \cap (x+f^\perp) \Rightarrow y = \lambda u \text{ with } u \in U, \lambda > 0 \text{ and } f(y) = f(x). \)
\( f(x) = f(y) = \lambda f(u) = \lambda f(x), \text{ so that } \lambda = 1, \text{ and } y = u \in U. \)
The reverse inclusion is clear. If \( y \in V \cap (x+f^\perp) \), then
there exist \( y_\alpha \in V \) such that \( y_\alpha \to y, \ y_\alpha = \lambda_\alpha u_\alpha, \text{ and } f(y_\alpha) = \lambda_\alpha f(u_\alpha) = \lambda_\alpha f(x). \lambda_\alpha = f(y_\alpha)/f(x) \to f(y)/f(x) = 1. \)
\[ u_n = (1/\lambda_n) y_n \rightarrow y. \] Thus \( y \in \overline{U}. \) The reverse inclusion is clear. Using (4.1), \((\text{Bd} V) \cap (x+f^\perp) = (V \setminus V) \cap (x+f^\perp) = [V \setminus (x+f^\perp)] \setminus [V \setminus (x+f^\perp)] = \overline{U} \setminus U = \text{Bd} \ U.\]

(4.3) LEMMA. Suppose \( f \) is a non-zero continuous linear functional on a linear topological space \( X, \) and \( f(x) > 0. \) If \( U \) is open in \( x+f^\perp, \) shrinkable at \( x \) in \( x+f^\perp, \) and \( V = \bigcup \{ \lambda U \mid \lambda > 0 \}, \) then \((y \in \overline{V}, \mu > 0 \Rightarrow y+\mu x \in V). \) Each line \([y+\lambda x \mid \lambda \in \mathbb{R}]\) meets \( \text{Bd} \ V \) exactly once.

PROOF. Take \( y \notin \text{Bd} \ V, \mu > 0, \) and \( y_n \in V \) such that \( y_n \rightarrow y. \) \( y_n = \lambda_n u_n \) with \( u_n \in U, \lambda_n > 0. \) \( f(y_n) = \lambda_n f(u_n) = \lambda_n f(x). \) \( \lambda_n = f(y_n)/f(x) \rightarrow f(y)/f(x). \) If \( f(y) > 0, \) then \( \lambda_o = f(y)/f(x) > 0 \) and \( u_n = (1/\lambda_o) y_n \rightarrow (1/\lambda_o) y \notin \overline{U}. \)

\( \frac{1}{\lambda_o} y-x \notin \overline{U}-x, \) and since \( \frac{1}{\mu + \lambda_o} \in [0,1), \) we have

\[ \frac{\lambda_o}{\mu + \lambda_o} \left( \frac{1}{\lambda_o} y-x \right) \in U-x. \] Thus \( (\mu + \lambda_o)x+y-\lambda_o x \notin (\mu + \lambda_o)U \) and \( y+\mu x \notin (\mu + \lambda_o)U \subset V. \) If \( f(y) = 0, \) then \( \lambda_o = 0. \) Choose \( \mu_i \) such that \( 0 < \mu_i < \mu. \)

\[ \frac{x+1}{\mu_i y_n - \lambda_o x} \notin U \Rightarrow x+\frac{1}{\mu_i} y \notin \overline{U} \Rightarrow x+\frac{1}{\mu_i} y \notin \overline{U} - x \Rightarrow \frac{1}{\mu_i} y \notin U - x \]

\[ \Rightarrow y+\mu_i x \notin \mu_i U \subset V. \] Since \( V \) is open, \([y+\lambda x \mid \lambda \in \mathbb{R}]\) meets \( \text{Bd} \ V \) at most once. To see that it meets \( \text{Bd} \ V \) at least once, take \( \lambda_o \) such that \( y+\lambda_o x \in x+f^\perp. \) If \( y+\lambda_o x \notin \overline{U} \subset \overline{V}, \) then it is clear that \([y+\lambda x \mid \lambda \in \mathbb{R}]\) meets \( \text{Bd} \ V, \) since
f(y + λx) < 0 for some λ. If y + λx ∉ U, then
x + λ(y + λx - x) ∈ Bd U for some λ > 0, since x ∈ U. Thus,
since Bd V is a cone and x + λ(y + λx - x) ∈ Bd U ⊆ Bd V,
y + (μ + λ - 1)x = μ(x + λ(y + λx - x)) ∈ Bd V.

(4.4) LEMMA. Suppose X, f, x, U and V are as in (4.3) and ρ : X → Bd V is defined by ρ(y) = \{y + λx : λ ∈ \mathbb{R}\} ∩ Bd V.

Then ρ is continuous.

PROOF. Define q : X → x + f⊥ by q(y) =
\{y + λx : λ ∈ \mathbb{R}\} ∩ (x + f⊥) (projection in direction x onto x + f⊥),
y : x + f⊥ → \mathbb{R} by \mathcal{Y}(y) = \gamma_0(y, x), and π : (x + f⊥) \setminus \{y : \mathcal{Y}(y) = 0\} → Bd U by π(y) = π_0(y, x). Note that ρ(y) = \mathcal{Y}q(y)πq(y)
if \mathcal{Y}q(y) ≠ 0 and ρ(y) = q(y) - x if \mathcal{Y}q(y) = 0. See Figure 2.

Figure 2. Dependence of ρ on q, \gamma and π.
Since \( \pi \) is defined on an open subset of \( x+\mathbb{A} \), \( \rho \) is continuous at each point \( y \) for which \( \gamma q(y) \neq 0 \). If \( \gamma q(y) = 0 \), then clearly \( \rho(y_n) \to \rho(y) \) if \( y_n \to y \) and \( \gamma q(y_n) = 0 \). Now assume \( y_n \to y \) and \( \gamma q(y_n) \neq 0 \). We must show

\[
\gamma q(y_n) \pi q(y_n) \to q(y)-x.
\]

If \( \{ \gamma q(y_n) \} \) is unbounded, there exists \( \{ n_\ell \} \) such that

\[
\pi q(y_{n_\ell}) = \frac{1}{\gamma q(y_{n_\ell})} (q(y_{n_\ell})-x)+x \to x
\]

\( \in \) Bd U, since Bd U closed. This contradicts \( x \in U \). Thus \( \{ \gamma q(y_n) \} \) has a convergent subsequence \( \{ \gamma q(y_{n_\ell}) \} \). If

\[
\gamma q(y_{n_\ell}) \to r > 0,
\]

then \( \pi q(y_{n_\ell}) = \frac{1}{\gamma q(y_{n_\ell})} (q(y_{n_\ell})-x)+x \to
\]

\[
\frac{1}{r} (q(y)-x)+x \in \) Bd U, so that \( q(y) \neq x \) (since \( x \in U \)), and

\( \gamma q(y) = r \neq 0 \), a contradiction. Thus every convergent subsequence of the bounded sequence \( \{ \gamma q(y_n) \} \) approaches zero, and this proves \( \gamma q(y_n) \to 0 \). By definition of \( \gamma \),

\[
\gamma q(y_n) \pi q(y_n) = q(y_n)-x+\gamma q(y_n)x \to q(y)-x.
\]

(4.5) **LEMMA.** Suppose \( X \) is a metrizable linear topological space, and \( U \) is linearly bounded, open and shrinkable at each point of \( K \). Then the boundary and gauge functions

\( \pi_U : X \times K \to X, \ \gamma_U : X \times K \to R \) are continuous.

**PROOF.** Suppose \( (x_n, k_n) \to (x, k) \). Let \( u_n = \pi_U(x_n, k_n), \ u = \pi_U(x, k), \ \rho_n = \gamma_U(x_n, k_n) \) and \( \rho = \gamma_U(x, k) \). Then

\[
u_n = k_n + \frac{1}{\rho_n} (x_n-k_n) \text{ and } u = k + \frac{1}{\rho} (x-k).
\]

If \( \{ \rho_n \} \) has 0 as an accumulation point, then there exists \( \{ n_f \} \) such that \( \rho_{n_f} \to 0, \)
and \( \rho \kappa u \varphi = \rho k + (x - k) \rightarrow (x - k) = \frac{1}{\rho} (u - k) \). Thus

\[ \rho \kappa u \varphi \rightarrow u - k. \]

\( 2 \rho \kappa (u - k) \in U - k \) eventually, since \( U \) is shrinkable at \( k \), \( u - k \in U - k \), and \( 2 \rho \kappa < 1 \) for large \( \ell \).

\[ \rho \kappa (u - k) \in \frac{1}{2} (U - k), \]

and so \( u - k \in \frac{1}{2} (U - k) \subset U - k \), since \( U \) is shrinkable at \( k \). But \( u \in U \) contradicts \( u \in \text{Bd} \ U \). Thus \( \{ u \} \) is bounded away from 0, so that \( \{ \frac{u}{\rho} \} \) is bounded. If \( \sigma \) is an accumulation point of \( \{ \frac{u}{\rho} \} \) and \( \frac{1}{\rho} \rightarrow \sigma \), then

\[ u = k + \frac{x - k}{\rho} \rightarrow k + \sigma (x - k) \in \text{Bd} \ U. \]

Since \( k + \frac{x - k}{\rho} \in \text{Bd} \ U \), this shows \( \sigma = \frac{1}{\rho} \). \( \Rightarrow \quad \sigma \) is the only accumulation point of \( \{ \frac{u}{\rho} \} \), and \( \frac{1}{\rho} \rightarrow \frac{1}{\rho} \). Thus \( \gamma_U (x, k) \rightarrow \gamma_U (x, k) \). Finally, \( \gamma_U (x, k) = u = k + \frac{x - k}{\rho} \rightarrow k + \frac{x - k}{\rho} = \gamma (x, k) \).

To illustrate use of the foregoing, we sketch a proof of a special case of (2.1).

(1.6) If \((X, \tau)\) is a metrizable incomplete linear topological space admitting a non-zero continuous linear functional, \( f \), then \( X \) is homeomorphic to \( X \setminus 0 \).

Take \( x \) such that \( f(x) > 0 \), and begin the proof using (2.2) to show \((x + f^{\perp}, \tau)\) is incomplete. Complete \( x + f^{\perp} \), and take \( \widetilde{x} \) in the completion, but not in \( x + f^{\perp} \). We use the
technique of (2.21) to find $x_n \to \bar{x}$ with $x_n \in X$, and $\tilde{U}_n$ open in the completion (but not necessarily linearly bounded) such that if $U_n = \tilde{U}_n \cap X$, then $U_n$ is shrinkable at $x_n, x_n^+$, $\tilde{U}_n \subset U_n$, $\cap \tilde{U}_n = \emptyset$ and $\cap U_n = \emptyset$. The cones $V_n = \bigcup \{\lambda U/ \lambda > 0\}$ are open and $V_n \subset V_{n-1}$. We claim $\cap V_n = \emptyset$. Take $y \in \cap \tilde{V}_n$. If $f(y) \neq 0$, then $f(y) > 0$, and $(f(x)/f(y))y \in x + T$, so that for some $n$, $(f(x)/f(y))y \notin \tilde{U}_n$ (since $\cap \tilde{U}_n = \emptyset$). But $y \in \tilde{V}_n \Rightarrow (f(x)/f(y))y \in \tilde{V}_n \cap (x + T) = \tilde{U}_n$, a contradiction. If $f(y) = 0$ and $y \neq 0$, then for some $n$, $(\tilde{U}_n, y) \cap \tilde{U}_n = \emptyset$, since if $\tilde{U}_n$ is in both sets for each $n$, we have $\tilde{U}_n = u_n + y = u_n$, so that $y = u_n - u_n \to 0$. Thus $x_n + y \notin \tilde{U}_n$, and since $x_n + y \in x + T$, (4.2) shows $x_n + y \notin \tilde{V}_n$. Then by (4.3) and the shrinkability of $U_n$ at $x_n$, $y \notin \tilde{V}_n$. Thus we have $\cap \tilde{V}_n = \emptyset$. By (4.3) Bd $V_n$, Bd $\tilde{V}_n$ are each met by $\{y + \lambda x_n | \lambda \in \mathbb{R}\}$ in a single point. This line also meets $T^\perp$ and $\{x | f(x) = -1\}$ in single points, and using (4.4) and the continuity of $f$, we define a homeomorphism $h$ which is fixed on $\tilde{V}_n \cup \{x | f(x) \leq -1\}$ and flattens Bd $V_n$ onto $T^\perp$.

Figure 3. Definition of $h$. 
In Figure 3, \( h_1(a) = a, \ h_1(b) = c \) and \( h_1(d) = d \). The diagram is oversimplified in that portions of \( \text{Bd} V_1 \) and \( \text{Bd} V_2 \) may lie in \( f^\perp \). Next, using the fact that \( \{ y + \lambda x_3 \mid \lambda \in \mathbb{R} \} \) meets \( \text{Bd} V_2, \text{Bd} V_3, f^\perp \), and \( \{ x \mid f(x) = -\frac{1}{2} \} \) in single points, define \( h_2 \) as a homeomorphism fixed on \( \bar{V}_2 \cup \{ x \mid f(x) \leq -\frac{1}{2} \} \) and flattening \( \text{Bd} V_2 \) onto \( f^\perp \).

\[ \text{Figure 4. Definition of } h_2 \]

Again in Figure 4, \( h_2(a) = a, \ h_2(b) = c \) and \( h_2(d) = d \).

Define \( h_3, h_4, \ldots \) similarly and let \( h = \ldots h_2 h_1 \). It can be shown that \( h \) is well defined and that

\[ h|_{X \setminus 0:X \setminus 0} \to \{ x \mid f(x) < 0 \} \] is a homeomorphism.

We remark that the homeomorphism \( h \) obtained above when restricted to \( \{ y \mid f(y) \geq -1 \} \) maps this set onto

\[ \{ y \mid -1 \leq f(y) < 0 \} \cup \{ 0 \} \] with \( h|_{\{ y \mid f(y) = -1 \}} = \text{Id} \). A homeomorphism \( k: \{ y \mid f(y) \geq -1 \} \to \{ y \mid -1 \leq f(y) < 0 \} \) with \( k|_{\{ y \mid f(y) = -1 \}} = \text{Id} \) is easily found. Then

\[ kh^{-1}: \{ y \mid -1 \leq f(y) < 0 \} \cup \{ 0 \} \to \{ y \mid -1 \leq f(y) < 0 \} \] is a
homeomorphism fixed on \( \{ y \mid f(y) = -1 \} \). This sort of map has been used to show closed convex bodies in certain locally convex spaces are mutually homeomorphic. Possibly the extension here to non-locally convex spaces would be useful in showing that in certain situations closures of open shrinkable sets are mutually homeomorphic.

We finish with some elementary facts on weakening topologies.

(4.7) LEMMA. If \((X, \tau_1)\) is a metrizable linear topological space and \(Y\) is a closed linear subspace of \(X\) admitting a metrizable linear topology \(\tau_2\) strictly weaker than \(\tau_1|Y\), then \(X\) admits a metrizable linear topology \(\tau\) strictly weaker than \(\tau_1\).

PROOF. Let \(\{W_n\}\) be a \(\tau_2\) base at 0 with each \(W_n\) balanced and \(W_n + W_n \subset W_{n-1}\). Let \(\{U_n\}\) be a \(\tau_1\) base with these properties, and \(V_n = U_n + W_n\). Then \(V_n + V_n \subset V_{n-1}\) and \(V_n\) is balanced, so \(\{V_n\}\) gives a first countable linear topology \(\tau\) for \(X\). Since \(V_n \in \tau\), \(\tau\) is weaker than \(\tau_1\). Since \(\tau_2\) is strictly weaker than \(\tau_1|Y\), there exist \(x_n \in Y\) such that \(x_n \to 0 (\tau_2)\) and \(x_n \not\to 0 (\tau_1|Y)\). Then \(x_n \not\to 0 (\tau)\), but \(x_n \to 0 (\tau_1)\). We must show \(\tau\) is Hausdorff. If \(x \in \bigcap V_n\), then \(x = u_n + w_n\) with \(u_n \in U_n\) and \(w_n \in W_n\). Since \(w_n = x - u_n \to x (\tau_1), x \in Y\). Thus \(u_n = x - w_n \in Y\), so \(u_n \to 0 (\tau_2)\) (since \(u_n \to 0 (\tau_1|Y)\)). It follows that \(x = u_n + w_n \to 0 (\tau_2)\).
That is, \( x = 0 \).

The next lemma was mentioned in the text of Chapter 2.

(4.8) LEMMA. If \((X, \mathcal{T})\) is a metrizable \(\alpha\) space, i.e., one admitting a Schauder basis \(\{b_n\}\) and an open set \(U\) such that the coordinate functionals are continuous and \(b_n \notin U\) for each \(n\), then \(X\) admits a strictly weaker metrizable linear topology.

PROOF. We may regard \(X\) as a linear subspace of \(s\). If \(\mathcal{T}_x\) is the topology of coordinatewise convergence, then \(\mathcal{T}_x|X \subset \mathcal{T}\), and \(b_n \to 0 (\mathcal{T}_x), b_n \not\to 0 (\mathcal{T})\).

(4.9) COROLLARY. If \((X, \mathcal{T})\) is a metrizable linear topological space having a closed subspace \(Y\) such that \(Y\) is an \(\alpha\) space, then \(X\) admits a strictly weaker metrizable linear topology.

(4.10) COROLLARY. If \((X, \mathcal{T})\) is a metrizable \(\alpha\) space, and \((Y, \mathcal{T}_y)\) is any metrizable linear topological space, then \((X \times Y, \mathcal{T}_x \times \mathcal{T}_y)\) admits a strictly weaker metrizable linear topology.
5. BIBLIOGRAPHY


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