Magnetohydrodynamic flow in closed channels

Monalisa Munsi

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Magnetohydrodynamic flow in closed channels

by

Monalisa Munsi

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:
Paul Sacks, Co-major Professor
Alric P. Rothmayer, Co-major Professor
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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University
Ames, Iowa
2017

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DEDICATION

I would like to dedicate this work to my parents, my brother and my husband Suman for being pillars of support and encouragement. I would also like to thank my friends for their loving guidance and support.
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GLOSSARY

\( B_x \) Magnetic field component in \( x \) direction. 6

\( B_y \) Magnetic field component in \( y \) direction. 6

\( B_z \) Magnetic field component in \( z \) direction. 50

\( \text{Ha} \) Hartmann number. 6

\( p \) pressure. 6

\( \text{Re} \) Reynolds number. 6

\( \text{Rm} \) Magnetic Reynolds number. 6

\( u \) Velocity component in \( x \) direction. 6

\( v \) Velocity component in \( y \) direction. 6

\( w \) Velocity component in \( z \) direction. 49
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ABSTRACT

The flow of an electrically charged fluid through a channel with asymmetric wall distortions and a cross-channel pressure interaction in the presence of a constant, transverse magnetic field is considered. It is shown that the basic nature of the hydrodynamic interaction is retained on a shorter stream-wise length scale, with gradually increasing magnetic field strength. An algebraic relation between the magnetic field strength and the Reynolds number is obtained. A new flow structure is obtained as the magnetic field strength becomes sufficiently large, where the stream-wise length of the cross-channel pressure interaction is proportional to the channel width. Linear and non-linear solutions along with linear free interactions are used to examine the structural properties.
CHAPTER 1. INTRODUCTION

1.1 Channel flow

Channel flow is the branch of fluid dynamics which deals with flow through containments like channels and pipes. There are two types of channel flows: open channel flow is the type of fluid flow through conduits with a free surface, while closed channel flow has no free surface. This work is about closed channel flows which have extensive applications in industrial, engineering, biological, and environmental fields.

In this work, the focus is on medium to high Reynolds numbers where asymptotic methods are used to provide insights. Substantial amount of research has been done in the area of asymptotic analysis of channel flows with wall disturbances in the form of constrictions, dilations and corners in both symmetric and asymmetric senses. The presence of a wall distortion in the form of dilation or constriction affects the fluid flow in a channel or pipe remarkably, particularly in the region very close to the distortion. The oncoming Poiseuille flow in a straight channel has been observed to undergo nonlinear changes in the neighborhood of the distortion for large Reynolds number. The methodology used to study the effect of wall disturbances on the flow is to examine the flow response for gradually increasing size of the disturbance. Eventually the critical size of the disturbance is determined when a non-linear response with unknown pressure is registered for the first time. Non-linear responses are usually associated with flow separations. For external flows, the triple-deck structure is known to be the most critical size as shown by Smith in [25, 26, 27]. For internal flow, he [16] showed that even a shallow indentation can cause big changes in the local pressure and lead to flow separation. He found a relation between the angle of inclination of the distortion and Reynolds number for such flow separation ahead of the indentation, for both symmetric as well as asymmetric channel and pipe. As the size is gradually increased, a separation is observed in the upstream zone which is pushed further upstream with increasing size as shown by Smith in [19, 26, 27]. This
breakaway separation is then followed by a long re-circulation zone leading to re-attachment of the separation streamline. In the downstream another separation occurs, as demonstrated by Lee and Fung [8], which is also followed by a long re-circulation zone that stretches till the final re-attachment. The length of the downstream separation zone is found to be directly proportional to the Reynolds number [26, 27]. On the other hand, the length of the upstream separation zone is found to increase slowly, like natural logarithm of the Reynolds number in symmetric channels [19] and one-seventh power of the Reynolds number in asymmetric channels [18]. Details of symmetric channels with sizes varying from 'fine' [16, 17] through 'moderate' [19] to 'severe' [20] have been discussed in [26, 27]. Numerical solutions for Navier-Stokes system have been compared with the theory of symmetric channels in [22], while comparison between numerical solution and experiments for axi-symmetric pipe flow are given in [20]. Corresponding studies for asymmetric channels have been done in [16] with an introduction to upstream free interaction in [17]. A detailed discussion of the upstream influence in asymmetric channels is done in [18]. Further studies on separation through asymmetric channels are done in [23, 26, 27].

As a natural extension, three dimensional constricted pipe flows have been analyzed using a similar procedure of finding the first critical height [26, 27]. A thorough study of linearized flow for small height has been done in [17]. Flow study for constrictions of finite height is done in [31]. Upstream influences for disturbances with large height has been examined in [24]. As mentioned previously, channel flow is known to have considerable amount of applications. A sub-branch of channel flow is magnetohydrodynamics which will be discussed in the next section.

1.2 Magnetohydrodynamics

Magnetohydrodynamics, MHD in short, is the field dealing with flows of electrically charged fluids through channels in the presence of an external magnetic field. It also includes the study of electric current driven by external voltage. This field was established by Hartmann [4] through his pioneering work in liquid metal flow through ducts in the presence of a strong applied magnetic field and Alfvén [1] on cosmic magneto gas dynamics. MHD has a considerable amount of applications
in various fields. From the study of blood through vessels in a human body using an MRI machine, fusion reactors, MHD pumps, to use in casting industry, MHD is used extensively.

Ample amount of studies have been done in MHD flow through channels and ducts. Hartmann [4] started by studying flow between two parallel plates with an applied, transverse magnetic field. Analytical solutions for this model for channels with both insulating and conducting walls were obtained by Chang and Lundgren [3]. Experimental validations were done by Murgatroyd [11] for MHD flow through rectangular channel with insulated channel walls. Branover et al. [2] obtained a good approximation for flow through rectangular channels with perfectly conducting walls. MHD flow through rectangular ducts with different conductivities of walls and sides have been studied as well. Shercliff [15] first considered the case with all walls being insulated. Corresponding study for conducting boundary walls and insulated side walls was done by Hunt [6]. For MHD flows with moderate or strong magnetic fields, as in many applications using liquid metals with high electrical conductivities, finding analytical solutions is generally quite difficult. Numerical solutions are also often costly. In such cases, asymptotic methods are used. Chang and Lundgren [3] found asymptotic solutions for rectangular ducts. Further studies on asymptotic approaches for MHD flows were done in [32, 33, 34].

An interesting feature of MHD flows is the suppression of flow separation by magnetic field, a phenomenon observed by Hartmann and Lazarus [5] and Murgatroyd [11]. This has been validated in many numerical studies [7, 12, 35] and others. Rothmayer [14] showed this property holds when an external magnetic field is applied to the flow structure discussed in [16, 17]. The search for an algebraic relation between the external magnetic field strength and flow separation characteristics, particularly for the flow structure described in [18], has driven the study done in this thesis. Asymptotic methods have been used to conduct the theoretical investigation.
1.3 Summary

In this section, a brief description for every chapter is presented.

Chapter 2: Preliminaries. In this chapter, a review of the governing equations using the non-dimensionalization, the scaling analysis and the flow structure which have motivated the study will be discussed.

Chapter 3: Solutions of hydrodynamic flow. Methodologies to obtain linear and non-linear solutions for the flow structure considered in chapter 2 will be discussed in detail. Non-linear solutions will be obtained, which has not been done before. Effects of introduction of an external magnetic field of finite strength on the flow structure will be explored.

Chapter 4: MHD channel flow with moderate magnetic strength. Analysis of the flow structure and properties with the gradual increase of the magnetic field strength will be discussed. Linear and non-linear solutions for this structure will be obtained and compared with the solutions obtained in previous chapters. The linear part of the upstream influence will be used as an important property to examine how this MHD flow is different from the previous hydrodynamic flow property.

Chapter 5: MHD channel flow with strong magnetic field. The magnetic field strength will be further increased to examine any changes in the flow structure and properties compared to MHD flow considered in the fourth chapter. Linear solutions and upstream influence will be used to examine the structure.

Chapter 6: Three dimensional channel flow. In this chapter, three dimensional flows corresponding to the flow structures discussed in chapter 3, 4 and 5 will be considered. Linear solutions will be used to analyse the structures.

Chapter 7: Conclusion. This chapter will list the results obtained in this study and future work to be done.
CHAPTER 2. PRELIMINARIES

2.1 Governing equations

A steady flow of an electrically conducting, incompressible fluid through a two dimensional channel of width $L$ under the action of a constant pressure gradient $g$ in the horizontal direction is considered. The channel is assumed to be long compared to its width. The channel walls are insulated. A constant magnetic field of strength $B_0$ is applied transversely to the flow. The fluid has a constant density $\rho$, constant viscosity $\mu$, permittivity constant $\varepsilon_0$, permeability $\mu_0$ and constant conductance $\sigma$. In a straight channel the flow would be a fully developed Poiseuille velocity profile, as given in equation (2.12) below. The spatial coordinates are non-dimensionalised by $L$, velocities by the characteristic velocity $V_0 = gL^2/\mu$ [18], and pressure by $gL$. The magnetic field is non-dimensionalised by $B_0$, electric field by $V_0B_0$, the current density by $\sigma V_0B_0$, and the charge density by $\varepsilon_0 V_0 B_0/L$.

Figure 2.1: Two dimensional channel flow
The $x$-axis is in the flow direction and aligned with the wall and the $y$-axis is normal to the wall. The governing equations are the incompressible mass conservation equation and the Navier-Stokes equations, coupled with the magnetic induction equation and are controlled by the Reynolds number $Re = \rho g L^3/\mu^2$, the Hartmann number $Ha = B_0 L \sqrt{\sigma/\mu}$ and the magnetic Reynolds number $Rm = \mu_0 \sigma V_0 L$. The non-dimensionalised governing equation are: the mass conservation equation:

$$\nabla \cdot \mathbf{V} = 0,$$  \hspace{1cm} (2.1)

the Navier-Stokes equation:

$$\mathbf{V} \cdot \nabla \mathbf{V} = Re^{-1} \left[ -\nabla p - Ha^2 Rm^{-1} \mathbf{B} \times \nabla \times \mathbf{B} + \nabla^2 \mathbf{V} \right],$$  \hspace{1cm} (2.2)

and the magnetic induction equation:

$$\mathbf{V} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V} + Rm^{-1} \nabla^2 \mathbf{B}.$$  \hspace{1cm} (2.3)

In two dimensions, the magnetic field has two components, $B_x$ and $B_y$. The subscripts for the magnetic field components indicate the direction of the components and are not partial derivatives. A constant transverse magnetic field implies that $B_x$ is equal to zero and the non-dimensional applied field $B_y$ is equal to unity. The governing equations in two dimensions take the form

$$u_x + v_y = 0,$$  \hspace{1cm} (2.4)

$$uu_x + vv_y = Re^{-1} \left[ -p_x - Ha^2 Rm^{-1} B_y \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) + u_{xx} + u_{yy} \right],$$  \hspace{1cm} (2.5)

$$uv_x + vv_y = Re^{-1} \left[ -p_y - Ha^2 Rm^{-1} B_x \left( \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} \right) + v_{xx} + v_{yy} \right],$$  \hspace{1cm} (2.6)

$$u \frac{\partial B_x}{\partial x} + v \frac{\partial B_x}{\partial y} = B_x u_x + B_y u_y + Rm^{-1} \left( \frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_x}{\partial y^2} \right),$$  \hspace{1cm} (2.7)

$$u \frac{\partial B_y}{\partial x} + v \frac{\partial B_y}{\partial y} = B_x v_x + B_y v_y + Rm^{-1} \left( \frac{\partial^2 B_y}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} \right).$$  \hspace{1cm} (2.8)

$Rm$ is assumed to be finite. No-slip conditions are applied at the upper and lower channel walls $u(x, f(x)) = v(x, f(x)) = 0$ where the upper and lower wall shapes are $f = f_L(x)$ and $f = 1 - f_U(x)$,
respectively. For finite values of the Hartmann number, $Ha$, the flow far upstream of the wall geometry in a straight portion of the channel is given by the Hartmann solution, which is $v = 0$ and

$$u = U_0(y; Ha) = \frac{1}{2Ha} \left( \frac{\exp(Hay) - 1}{1 - \exp(Ha)} + \frac{1 - \exp(-Hay)}{1 - \exp(-Ha)} \right).$$

This equation is derived from (2.5), with its left hand side being zero, and the pressure given by $p = P_0 - x$, where $P_0$ is an $O(1)$ constant. The derivation of the Hartmann solution has been discussed in Appendix, section 2.1.

### 2.2 Hydrodynamic flow

In pure hydrodynamic flow, no magnetic field is present. Hence the system of governing equations reduces to:

$$\nabla \cdot \mathbf{V} = 0, \quad \mathbf{V} \cdot \nabla \mathbf{V} = Re^{-1} \left[ -\nabla p + \nabla^2 \mathbf{V} \right].$$

Far upstream, the flow is a fully developed Poiseuille flow

$$u = \frac{1}{2} y (1 - y) \equiv U_0(y), \quad v = 0, \quad p = P_0 - x,$$

which is the limit of the Hartmann flow (2.9) as $Ha$ tends to zero.

![Figure 2.2: Wall distortion scale](image)

The fluid flow is essentially controlled by a competition between the inertial and the viscous terms which often leads to flow separation from the channel walls, especially for large $Re$. In case of laminar flows, the first flow response or separation generally happens when wall geometries are shallow. Thus, for a wall distortion of long stream-wise length scale $\Delta \gg 1$ and height scale $y \sim \delta$, ...
it is assumed that the viscous layer thickness $\delta$ is much smaller than the channel width. So the coordinates $(x,y)$ become equal to $(\Delta X, \delta Y)$, where $X, Y$ are $O(1)$.

In the viscous wall layer of thickness $\delta$, the $u$-velocity must match with the leading order Poiseuille flow. Hence

$$u = \frac{1}{2} y(1-y) \sim \frac{1}{2} \delta Y(1 - \delta Y) \sim \lambda \delta Y, \quad (2.12)$$

where $\lambda = \frac{\partial U_0}{\partial Y}(X,0) = 1/2$ is the wall shear. The pressure scale is determined from the $x$-momentum equation (2.5) by assuming a Bernoulli type balance between pressure and velocity:

$$uu_x \sim Re^{-1} p_x. \quad (2.13)$$

This results in pressure scaling like $O(Re \delta^2)$. From the continuity equation

$$v \sim uy/x \sim \delta^2/\Delta. \quad (2.14)$$

This mass balance preserves a balance between all convective terms in both momentum equations. Substituting the scales into the $x$-momentum equation (2.5) gives:

$$\frac{\delta^2}{\Delta} (UU_X + UV_Y) = Re^{-1} \left( -\frac{Re \delta^2}{\Delta} P_X + \frac{\delta}{\Delta^2} U_{XX} + \frac{\delta}{\delta^2} U_{YY} \right). \quad (2.15)$$

Given that $\delta \ll 1 \ll \Delta$, the stream-wise viscous term $\delta/\Delta^2$ is, therefore, smaller than $1/\delta$. So the term $U_{XX}$ in (2.15) is negligible. The remaining viscous term can balance the convective and pressure terms when

$$\frac{\delta^2}{\Delta} = Re^{-1} \frac{\delta}{\delta^2}, \quad (2.16)$$

which gives the thickness of the wall layers with $\delta$ being equal to $Re^{-1/3} \Delta^{1/3}$. Substituting the scales into the $y$-momentum equation (2.6), we obtain the leading order equation $P_Y = 0$, which shows the pressure is constant across the wall layers.

The stream-wise length scale $\Delta$ determines the basic properties of the flow. For

$$O(Re^{1/7}) \ll \Delta \ll O(Re),$$

Smith in [16, 17] showed that the flow is controlled by a prescribed displacement, $-A(X)$, of the core, given by the average of the wall shapes. In this long scale structure, there is no pressure
variation across the channel. For this to happen, the core induces a small slip velocity \((u_1, v_1)\) at the walls which creates a boundary condition difference which takes care of the different upper and lower wall shapes. So, the stream-wise velocity in the core is given by \(U_0 + \epsilon u_1\). In order to match with the wall layer and create the slip velocity, the perturbation parameter \(\epsilon\) must be equal to \(Re^{-1/3} \Delta^{-1/3}\). The velocity and pressure in this structure are

\[
(\mathbf{u}, \mathbf{v}, p - P_0) \sim (U_0, 0, 0) + (Re^{-1/3} \Delta^{1/3} u_1, Re^{-1/3} \Delta^{-2/3} v_1, Re^{1/3} \Delta^{2/3} p_1) + \ldots
\]  

(2.17)

Substituting these scales into (2.10) and neglecting terms of order \(Re^{-1/3} \Delta^{-2/3}\), the solutions for the perturbation velocities \(u_1, v_1\) are found to be [29]

\[
(u_1(X, y), v_1(X, y)) = (A(X) U_0'(y), -A'(X) U_0(y)),
\]  

(2.18)

with \(A(-\infty) = 0\) in order to merge with the upstream Poiseuille flow as \(X\) tends to \(-\infty\). The \(y\)-momentum equation in the core is:

\[
Re^{-1/3} \Delta^{-5/3} U_0 v_1x + Re^{-2/3} \Delta^{-4/3} v_1 v_1 y = -Re^{-2/3} \Delta^{2/3} p_1 y + Re^{-1} \left(Re^{-1/3} \Delta^{-8/3} v_1 x x + Re^{-1/3} \Delta^{-2/3} v_1 y y\right).
\]  

(2.19)

When compared to the pressure gradient, most of the terms are small provided \(Re \gg 1, \Delta \gg 1\). The stream-wise convective term can be neglected compared to the normal pressure gradient as long as \(Re^{-1/3} \Delta^{-5/3} \ll Re^{-2/3} \Delta^{2/3}\), i.e., when \(\Delta\) is much larger than \(Re^{1/7}\). For this reason, when \(\Delta\) decreases to \(O(Re^{1/7})\), the cross-channel pressure gradient becomes non-zero, and proportional to \(uv_x\).

2.2.1 Smith’s ’77 structure

In this three-layer structure, as discussed by Smith in [18], \((x, y)\) is equal to \((Re^{1/7}X, Re^{-2/7}Y)\) where \(X, Y \sim O(1)\). The flow in the viscous layers is driven by a pressure gradient across the channel which is induced by the core displacement. This pressure-displacement interaction is a self-sustaining process. If the pressure perturbation is positive in one layer, then the velocity in that layer decreases, leading to the thickening of the wall layer. The mass flux is maintained
through the compression of the opposite viscous wall layer which happens via displacement of the core. This results in the development of a transverse pressure gradient causing a negative pressure in the opposite wall layer thus retaining the pressure-displacement interaction. The expansions and equations corresponding to each layer are given below.

### 2.2.1.1 Channel core

In the core of the channel, the normal co-ordinate $0 < y < 1$ is of order 1. The expansions obtained by substituting $\Delta$ equal to $Re^{1/7}$ in (2.17) are:

$$
(u, v, p - P_0) \sim (U_0(y), 0, 0) + (Re^{-2/7}u_1, Re^{-3/7}v_1, Re^{3/7}p_1) + \ldots
$$

The leading order $y$-momentum equation produces a pressure gradient across the channel that is given by:

$$
p_{1y} = -U_0v_{1X} = U_0^2 A''(X),
$$

with the solution

$$
p_1(X, y) = P(X) + A''(X) \int_0^y U_0^2(s) \, ds,
$$

where $P(X) = p_1(X, 0)$ is unknown with $P(-\infty) = 0$. This is known as the pressure-displacement interaction.

![Smith’s ’77 structure](image-url)
2.2.1.2 Wall layers

In the lower wall layer, the normal coordinate, \( y \), scales like the wall layer thickness and is, thus, equal to \( Re^{-2/7}Y \) where \( Y \sim O(1) \). The expansions are:

\[
(u, v, p - P_0) \sim (Re^{-2/7}U(X,Y), Re^{-5/7}V(X,Y), Re^{3/7}P(X)) + ...
\]

These expansions are substituted in the continuity and momentum equations and every term is tracked. The leading order equations, thereby obtained, are:

\[
\begin{align*}
U_X + V_Y &= 0, \\
UU_X + UV_Y &= -P_X + U_{YY}, \\
P_Y &= 0,
\end{align*}
\]

with the boundary conditions

\[
\begin{align*}
U = V = 0 \text{ at } Y = 0, \\
U \to \lambda(Y + A(X)) \text{ as } Y \to \infty, \\
U \to \lambda Y \text{ as } X \to -\infty.
\end{align*}
\]

In the upper wall layer, the \( y \)-coordinate is pointed into the core of the channel as shown in Figure 2.3 and is equal to \( 1 - Re^{-2/7}\tilde{Y} \). The expansions have the same form as the lower wall layer but with the \( v \)-velocity expansion replaced by \(-Re^{-5/7}V(X,\tilde{Y})\). The negative sign in the \( v \)-velocity expansion negates the negative sign in the \( y \)-coordinate of the upper wall layer leading to a system of equations similar to (2.23)

\[
\begin{align*}
\tilde{U}_X + \tilde{V}_{\tilde{Y}} &= 0, \\
\tilde{U}\tilde{U}_X + \tilde{U}\tilde{V}_{\tilde{Y}} &= -\tilde{P}_X + \tilde{U}_{\tilde{Y}\tilde{Y}}, \\
\tilde{P}_Y &= 0,
\end{align*}
\]
with the boundary conditions
\[
\tilde{U} = \tilde{V} = 0 \text{ at } \tilde{Y} = 0,
\]
\[
\tilde{U} \to \lambda (\tilde{Y} - A(X)) \text{ as } \tilde{Y} \to \infty,
\]
\[
\tilde{U} \to \lambda \tilde{Y} \text{ as } X \to -\infty.
\]
A matching at the edge of the upper wall layer fixes the pressure \( \tilde{P} \) to the value of \( p_1(X,1) \).
Therefore, the wall layer pressures are connected through the core via (2.22):
\[
\tilde{P}(X) = P(X) + \kappa A''(X),
\]
where \( \kappa = \int_0^1 U_0^2(s) \, ds = 1/120. \)

### 2.3 Linear free interaction

An important feature of this structure is the free interaction. Smith in [18] showed that for a wall perturbation with length of order \( Re^{1/7} \), there would be an upstream influence which extended a distance upstream of order \( Re^{1/7} \) times the channel width. For very large \( Re \), he argued that any distance would be significantly far upstream which will lead the local effect of the wall shape to lose relevance and any change in the main flow would have to develop as a free interaction in a straight channel [28]. In order to check whether this is possible, (2.23) - (2.27) have to be solved simultaneously in a straight channel by applying a linear perturbation to the displacement function in the far upstream of the wall distortion.

Let the lower wall layer system after perturbation of magnitude \( \epsilon \) be:
\[
(U, V, P) \sim (\lambda Y, 0, 0) + \epsilon (u_p(Y), v_p(Y), p_p) e^{\theta X} + O(e^{2\theta X}),
\]
\[
A(X) \sim \epsilon a_p e^{\theta X} + O(e^{2\theta X}),
\]
where \( \theta \) is constant and the amplitude for the displacement function, \( a_p \) is equal to \( \pm 1 \). The linear system obtained by substituting the above expansions in (2.23) and (2.24) is:
\[
v'_p = -\theta u_p,
\]
\[
\lambda \theta Y u_p + \lambda v_p = -\theta p_p + u''_p,
\]
(2.28)
with no-slip condition \( u_p(0) = v_p(0) = 0 \) and far-field condition \( u_p(Y) \rightarrow \lambda a_p \) as \( Y \rightarrow \infty \).

Differentiating the second equation in (2.28) with respect to the normal coordinate gives

\[
\frac{\partial}{\partial Y} \left( \frac{\partial^2 u_p}{\partial Y^2} \right) - \lambda \theta Y \frac{\partial u_p}{\partial Y} = 0,
\]

which is an Airy equation for the derivative of the perturbation velocity \( u_p \). In general, the Airy function \( \text{Ai}(x) \) and the related function \( \text{Bi}(x) \) are the linearly independent solutions to the equation

\[
\frac{d^2 y}{dx^2} - xy = 0.
\]

subject to certain initial conditions. The velocity \( u_p \) is obtained by integrating the solution of the equation (2.29) and is given by

\[
u_p(Y) = C_1 \int_0^Y \text{Ai}(ms) \, ds + C_2 \int_0^Y \text{Bi}(ms) \, ds,
\]

where the coefficient \( m \) is equal to \((\lambda \theta)^{1/3}\) and the constants \( C_1, C_2 \) are evaluated from the boundary conditions. The far-field condition on \( u_p \) implies that it is finite in value. Hence the constant \( C_2 \) has to be set to zero to nullify the contribution of the \( \text{Bi}(Y) \) function which keeps growing. Hence \( u_p \) is equal to \( C_1 f_1(Y) \) where the function \( f_1(Y) \) is equal to \( \int_0^Y \text{Ai}(ms) \, ds \). The perturbation velocity \( v_p \) is evaluated from the first equation in (2.28) and is equal to \( -\theta C_1 f_1(Y) \). From the second equation in the system (2.28) at \( Y = 0 \), the pressure perturbation \( p_p \) is evaluated to be equal to \( \frac{C_1 m \theta}{\theta} \text{Ai}'(0) \).

Likewise, in the upper wall layer the expansions are given by

\[
(\tilde{U}, \tilde{V}, \tilde{P}) \sim (\lambda \tilde{Y}, 0, 0) + \epsilon \left( \tilde{C}_1 \hat{f}_1(\tilde{Y}), -\theta \tilde{C}_1 \hat{f}_1(\tilde{Y}), \frac{\tilde{C}_1 m}{\theta} \text{Ai}'(0) \right) e^{\theta X} + O(e^{2\theta X}),
\]

where \( \hat{f}_1(\tilde{Y}) \) is equal to \( \int_0^{\tilde{Y}} \text{Ai}(ms) \, ds \). The linearized cross-channel pressure relation of (2.27) and the far-field conditions of the upper and lower wall layer give the value of \( \theta \) as

\[
\theta = 2\left[ -45 \text{Ai}'(0) \right]^{\frac{2}{3}} \approx 5.727...
\]

The only constant which remains undetermined is \( C_1 \) which is set by a match with the non-linear solution, and is effectively an origin shift. Interestingly, the free interaction in this study is
independent of the sign of $C_1$. This is unlike the studies done by Stewartson & Williams [29] and Stewartson [30], where positive and negative values of upstream disturbances resulted in distinct downstream solutions. For, if $C_1 < 0$, the pressure increases in the lower wall layer leading to an increase in the displacement function $-A(X)$. The transverse pressure gradient, thus generated, compresses the upper wall layer. This results in a negative pressure in the upper wall layer which increases the compression further, hence accentuating the compression and sustaining the pressure-displacement interaction. For $C_1 > 0$ the situation gets interchanged. Therefore, the case $C_1 < 0$ covers both the situations.
CHAPTER 3. SOLUTIONS OF HYDRODYNAMIC FLOW

3.1 Flow with wall geometry

The three-layer structure solved by Smith in [18] was for a flat channel. If there is any wall geometry \((h_F(X), h_U(X))\), where \(h\) is the height, then the no-slip conditions become \(U(X, h_F(X)) = V(X, h_F(X)) = 0\). In such cases, the equations are simplified using a transformation due to Prandtl (see [16]) which transforms the normal co-ordinate. In the lower wall layer, the transformed normal co-ordinate is \(\eta = Y - h_F(X)\). Then the partial derivatives transform to \(\partial_X = \partial_X - h_F'(X)\partial_\eta\) and \(\partial_Y = \partial_\eta\). The velocity \(V\) gets transformed to \(\hat{V}\) which is equal to \(V - Uh_F'(X)\). Thus the equations take the form:

\[
\begin{align*}
U_X + \hat{V}_\eta &= 0, \\
UU_X + \hat{V}U_\eta &= -P_X + U_{\eta\eta}, \\
P_\eta &= 0.
\end{align*}
\] (3.1)

The boundary conditions are the no-slip conditions at the wall, given by \(U(X, 0) = \hat{V}(X, 0) = 0\), and the far-field condition given by

\[U \rightarrow \lambda(\eta + h_F + A(X)) \text{ as } \eta \rightarrow \infty.\] (3.2)

In the upper wall layer, the transformed co-ordinate is \(\tilde{\eta} = \bar{Y} - h_U(X)\) and the transformed \(\tilde{V}\) velocity is \(\hat{V}\) which is equal to \(\bar{V} - U h_U'(X)\). The equations are:

\[
\begin{align*}
\tilde{U}_X + \hat{V}_{\tilde{\eta}} &= 0, \\
\tilde{U}\tilde{U}_X + \hat{V}\tilde{U}_{\tilde{\eta}} &= -\tilde{P}_X + \tilde{U}_{\tilde{\eta}\tilde{\eta}}, \\
\tilde{P}_{\tilde{\eta}} &= 0.
\end{align*}
\] (3.3)
Similar to the lower wall layer, the boundary conditions in the upper wall layer are the no-slip conditions \( \tilde{U}(X,0) = \tilde{V}(X,0) = 0 \), and the far-field condition

\[
\tilde{U} \rightarrow \lambda(\tilde{\eta} + hF_U - A(X)) \text{ as } \tilde{\eta} \rightarrow \infty.
\]

(3.4)

The core of the channel is governed by the transformed pressure-displacement interaction:

\[
\tilde{P}_1(X) = P_1(X) + \kappa A_{XX}(X),
\]

(3.5)

with \( \kappa \) equal to 1/120.

### 3.2 Linearization for shallow wall geometry

One important tool that can be used to analyse the underlying structure of the flow is linearised theory which is obtained by introducing a small perturbation to the system and linearising the non-linear boundary layer system. This linearised system can be solved by numerical methods as well as by using the Fourier transform. The transform is applied to the linearised system and solution in Fourier space is obtained. The advantage of this procedure is that it provides insights into the structure of the solution without having to solve the more complicated non-linear equations [28].

A small perturbation of magnitude \( h \), the maximum distortion height, which is small compared to one is applied to the three-layer system. In the lower wall layer the perturbed variables are:

\[
(U, \hat{V}, P, A) \sim (\lambda \eta, 0, 0, 0) + h(u_L(X, \eta), v_L(X, \eta), p_L(X), a(X)) + O(h^2)
\]

and the linear system thus obtained by substituting these variables into (3.1) is:

\[
\begin{align*}
   u_{LX} + v_{L\eta} &= 0, \\
   \lambda \eta u_{LX} + \lambda v_L &= -p_L' + u_{L\eta},
\end{align*}
\]

(3.6)

with the no-slip conditions at the wall \( u_L(X,0) = v_L(X,0) = 0 \) and the matching condition at the lower wall layer-core interface \( u_L \rightarrow \lambda(F_L(X) + a(X)) \) as \( \eta \) tends to infinity. Similarly, in the upper wall layer the perturbed variables are given by

\[
(\tilde{U}, \tilde{V}, \tilde{P}) \sim (\lambda \tilde{\eta}, 0, 0, 0) + h(u_U(X, \tilde{\eta}), v_U(X, \tilde{\eta}), p_U(X)) + O(h^2),
\]
with the linear system obtained by substituting the variables into (3.3)

\[ u_{UX} + v_{U\tilde{\eta}} = 0, \]

\[ \lambda\tilde{\eta}u_{UX} + \lambda v_U = -p'_U + u_{U\tilde{\eta}}. \]  

(3.7)

The boundary conditions are no-slip conditions at the wall \( u_U(X,0) = v_U(X,0) = 0 \) and the far-field condition \( u_U \rightarrow \lambda(F_U(X) - a(X)) \) as \( \tilde{\eta} \rightarrow \infty \). The equation governing the core is obtained by substituting the pressure and the displacement variables into the equation (3.5):

\[ p_U = p_L + \kappa a''(X). \]  

(3.8)

### 3.2.1 Fourier transform solution

The definitions of the Fourier transform and inverse Fourier transform used here are given, respectively, by

\[ F^*(\alpha) = \int_{-\infty}^{\infty} F(X)e^{-i\alpha X} d\xi, \]

\[ F(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\alpha)e^{i\alpha X} d\alpha, \]  

(3.9)

where \( \alpha \) is the transformed variable corresponding to \( X \). The \('s'\) symbol represents the transformed function. The transform of the derivative of the function is given by

\[ (F')^*(\alpha) = i\alpha F^*(\alpha). \]  

(3.10)

The Fourier transform is applied to each of the three layers. The boundary layer system in the lower wall layer, after the transformation, is given by:

\[ i\alpha u^*_L + v^*_L = 0, \]

\[ i\alpha \lambda\eta u^*_L + \lambda v^*_L = -i\alpha p'_L + u^*_{L\eta\eta}. \]  

(3.11)

with the no-slip conditions at the wall \( u^*_L(\alpha,0) = v^*_L(\alpha,0) = 0 \) and the far-field condition \( u^*_L \rightarrow \lambda(F^*_L(\alpha) + a^*(\alpha)) \) as \( \eta \rightarrow \infty \). The system (3.11) is solved directly by differentiating the \( X \)-momentum equation with respect to \( \eta \) and using the mass conservation equation, thus giving a form of Airy equation satisfied by \( u^*_L \eta \eta \)

\[ u^*_{L\eta\eta\eta} - (i\alpha \lambda)\eta u^*_L = 0. \]  

(3.12)
The solution, when integrated with respect to \( \eta \) gives the transformed \( u \)-velocity perturbation
\[
\varepsilon_{UL} = \frac{3}{2}(L^{*} + a^{*}) \left( \frac{i\alpha}{2} \right)^{1/3} \int_{0}^{\eta} \text{Ai} \left[ \left( \frac{i\alpha}{2} \right)^{1/3} s \right] ds. \tag{3.13}
\]

The pressure perturbation is obtained from the \( X \)-momentum equation by using the no-slip conditions and the value of \( \varepsilon_{UL}^{*} \) at the wall
\[
\varepsilon_{UL}^{*} = \frac{3}{2}(L^{*} + a^{*}) \left( \frac{i\alpha}{2} \right)^{1/3} \text{Ai} \left[ \left( \frac{i\alpha}{2} \right)^{1/3} s \right] \tag{3.14}
\]

In a similar fashion, the solutions are obtained for the transformed system in the upper wall layer
\[
i\alpha \varepsilon_{U}^{*} + \varepsilon_{U}^{**} = 0, \tag{3.15}
i\alpha \lambda \varepsilon_{U}^{**} + \lambda^{*} \varepsilon_{U}^{*} = -\alpha \varepsilon_{U}^{*} + \varepsilon_{U}^{**},
\]
with the boundary conditions being the no-slip condition at the wall \( \varepsilon_{U}^{*}(\alpha, 0) = \varepsilon_{U}^{**}(\alpha, 0) = 0 \) and the far-field condition \( \varepsilon_{U}^{*} \rightarrow \lambda(L^{*}(\alpha) - a^{*}(\alpha)) \) as \( \eta \rightarrow \infty \). The transformed \( u \)-velocity and pressure perturbations in the upper wall layer are given by
\[
\varepsilon_{U}^{*} = \frac{3}{2}(L^{*} - a^{*}) \left( \frac{i\alpha}{2} \right)^{1/3} \int_{0}^{\eta} \text{Ai} \left[ \left( \frac{i\alpha}{2} \right)^{1/3} s \right] ds, \tag{3.16}
\]
\[
\varepsilon_{P}^{*} = \frac{3}{2}(L^{*} - a^{*}) \left( \frac{i\alpha}{2} \right)^{1/3} \text{Ai} \left[ \left( \frac{i\alpha}{2} \right)^{1/3} s \right] \tag{3.17}
\]

The pressure perturbations in the two wall layers are related through the core by
\[
\varepsilon_{P}^{*} = \varepsilon_{U}^{*} - \alpha^{2} \kappa a^{*} \tag{3.18}
\]

Solving the above equations gives the transformed displacement perturbation
\[
\gamma = \frac{3}{2}(L^{*} - a^{*}) \left( \frac{i\alpha}{2} \right)^{1/3} \text{Ai} \left[ \left( \frac{i\alpha}{2} \right)^{1/3} s \right].
\]

The velocity and pressure perturbations in each wall layer are obtained by using the fast Fourier transform or 'fft' and the inverse fast Fourier transform or 'ifft' routines of MATLAB. The lower wall is taken to be a Gaussian hump \( F_{L} = \exp \left( -\frac{24}{2.25}X^{2} \right) \), where the hump length is 1.5 units.
The upper wall is flat. The transformed variable $\alpha$ is taken to lie in the interval $(-15, 15)$ with the step length $2.3969 \times 10^{-4}$. A linear extrapolation has been used to calculate the value of $a^*$ and $\gamma$ at the origin. The length of padding is given by $n$ which is equal to $2^{19}$. The step length in the $X$-axis is calculated from the relation

$$dX d\alpha = \frac{2\pi}{n}.$$  

(3.19)

The functions obtained after performing the inverse fft routine are then normalised by the step-length $dX$.

### 3.2.2 Linear finite difference solutions

The linearized equation systems (3.6) - (3.8) are also finite differenced and solved using the Thomas algorithm discussed in the Appendix, section 1.2 as well as in the next section. The linear system is numerically solved to provide a foundation for the solution of the non-linear system. The linear solutions, represented by red and black lines in the plot, are compared with the solutions obtained through inverse fast Fourier transform, represented by blue and green lines, in Figure 3.1 and show good agreement through complete overlap.

![Figure 3.1: Comparison of linear finite difference and ifft solutions](image)

### 3.3 Non-linear solutions

To study how this three-layer system behaves, the equations (3.1) through (3.5) along with the no-slip conditions are solved. A fully implicit second-order accurate space marching scheme in
the X-direction with Newton linearization and a block tri-diagonal inversion in \( \eta \)-direction at each \( X \) gridline have been used. The two wall layers are coupled with the core (3.5) through the far-field conditions (3.2) and (3.4). Reverse flow regions are computed using a FLARE approximation [13], where \( \omega \) is set equal to 1 for positive \( U \)-velocity and zero for negative \( U \)-velocity. Repeated iterations in the \( X \)-direction are used to converge the solution. The step-length in the \( X \)-direction is \( \Delta X \) and in the \( \eta \)-direction is \( \Delta \eta \). The \( \Delta \) notation used here is different from the notation used for stream-wise length scale. The subscript \((i, j)\) correspond to the grid values \((X_i, \eta_j)\) where \( i \) ranges from 0 to \( M \), \( j \) ranges from 0 to \( N \) in lower wall layer and from \( N + 1 \) to \( N_t \) in the upper wall layer, respectively. The velocities \((U^g, \hat{V}^g)\) are the guessed values for \((U, \hat{V})\) which are set equal to the values of \( U \) and \( \hat{V} \) obtained in the previous iteration, except for the first iteration when the guessed values are set equal to the corresponding values at the previous \( X \) location. The upstream flow conditions are set to undisturbed Poiseuille flow with local pressure. The lower wall boundary condition for the lower wall layer is:

\[
U_{i,0} = \hat{V}_{i,0} = 0, \ -P_{i,0} + P_{i,1} = 0. \tag{3.20}
\]

The finite differenced equations in the lower wall layer take the form:

\[
\begin{align*}
\frac{3}{4\Delta X}U_{i,j-1} - \frac{1}{\Delta \eta} \hat{V}_{i,j-1} + \frac{3}{4\Delta X}U_{i,j} + \frac{1}{\Delta \eta} \hat{V}_{i,j} \\
= \frac{U_{i-1,j} + U_{i-1,j-1} - U_{i-2,j} + U_{i-2,j-1}}{4\Delta X}, \\
\left( \frac{\hat{V}^g}{2\Delta \eta} - \frac{1}{(\Delta \eta)^2} \right) U_{i,j-1} + \left( \omega \left( \frac{U^g}{X} + \frac{3U^g}{2\Delta X} \right) + \frac{2}{(\Delta \eta)^2} \right) U_{i,j} + U_{i,j} U_{i,j} \\
+ \frac{3}{2\Delta X}P_{i,j} + \left( \frac{\hat{V}^g}{2\Delta \eta} - \frac{1}{(\Delta \eta)^2} \right) U_{i,j+1} \\
= \omega U^g \left( \frac{4U_{i-1,j} - U_{i-2,j}}{2\Delta X} + \frac{U^g}{X} \right) + \hat{V}^g U_{i,j} \left( - \frac{4P_{i-1,j} - P_{i-2,j}}{2\Delta X} \right) \\
- P_{i,j} + P_{i,j+1} = 0.
\end{align*}
\tag{3.21}
\]

The matching condition \( U \rightarrow \lambda(\eta + hF_L(X) + A(X)) \) in the lower wall layer (3.2) couples with the core equation (3.5) via the displacement function. \( A(X_i) \) is expressed in terms of the upper and lower wall pressure as

\[
A_i = \frac{\Delta X^2}{2\kappa} \left( P_{i,N} - \tilde{P}_{i,N+1} \right) + \frac{(A_{i-1} + A_{i+1})}{2}. \tag{3.22}
\]
and is then substituted into the matching conditions \( U_{i,N} \rightarrow \lambda(\eta_N + hF_{L,i} + A_i) \). The finite-differenced equations for the interface of the lower wall layer and the core are then

\[
U_{i,N} - \frac{\lambda \Delta X^2}{2K} P_{i,N} + \frac{\lambda \Delta X^2}{2K} \tilde{P}_{i,N+1} = \lambda(\eta_N + hF_{L,i}) + \frac{(A_{i-1} + A_{i+1})}{2},
\]

\[
= \left( \frac{3U^g}{2\Delta X} + \frac{U^g_X}{X} \right) U_{i,N} + \lambda \dot{V}_{i,N} + \frac{3}{2\Delta X} P_{i,N},
\]

\[
= U^g \left( \frac{4U_{i-1,N} - U_{i-2,N}}{2\Delta X} + \frac{U^g_X}{X} \right) + \frac{4P_{i-1,N} - P_{i-2,N}}{2\Delta X}, \quad (3.23)
\]

\[
\frac{3}{4\Delta X} \dot{U}_{i,N-1} - \frac{1}{\Delta \eta} \ddot{V}_{i,N-1} + \frac{3}{4\Delta X} U_{i,N} + \frac{1}{\Delta \eta} \dot{V}_{i,N} = \frac{U_{i-1,N} + U_{i-1,N-1}}{\Delta X} - \frac{U_{i-2,N} + U_{i-2,N-1}}{4\Delta X}.
\]

In the upper wall layer, the finite-differenced equations are similar to (3.21) and (3.23) but with the \( j-1 \) index being replaced by \( j+1 \) and the displacement function changing its sign from positive to negative. Hence, the interface of the upper wall layer and the core is dictated by the equations:

\[
\tilde{U}_{i,N+1} - \frac{\lambda \Delta X^2}{2K} \tilde{P}_{i,N+1} + \frac{\lambda \Delta X^2}{2K} P_{i,N} = \lambda(\bar{\eta}_{N+1} + hF_{U,i}) - \frac{(A_{i-1} + A_{i+1})}{2},
\]

\[
= \left( \frac{3\bar{U}^g}{2\Delta X} + \frac{\bar{U}^g_X}{X} \right) \tilde{U}_{i,N+1} + \lambda \dot{\tilde{V}}_{i,N+1} + \frac{3}{2\Delta X} \tilde{P}_{i,N+1},
\]

\[
= \bar{U}^g \left( \frac{4\bar{U}_{i-1,N+1} - \bar{U}_{i-2,N+1}}{2\Delta X} + \frac{\bar{U}^g_X}{X} \right) + \frac{4\tilde{P}_{i-1,N+1} - \tilde{P}_{i-2,N+1}}{2\Delta X}, \quad (3.24)
\]

\[
\frac{3}{4\Delta X} \dot{\tilde{U}}_{i,N+1} - \frac{1}{\Delta \bar{\eta}} \ddot{\tilde{V}}_{i,N+1} + \frac{3}{4\Delta X} \tilde{U}_{i,N+2} + \frac{1}{\Delta \bar{\eta}} \dot{\tilde{V}}_{i,N+2} = \frac{\tilde{U}_{i-1,N+1} + \tilde{U}_{i-1,N+2}}{\Delta X} - \frac{\tilde{U}_{i-2,N+1} + \tilde{U}_{i-2,N+2}}{4\Delta X}.
\]
The upper wall layer is governed by the system:

\[
\begin{align*}
\frac{3}{4\Delta X} \tilde{U}_{i,j} + & \frac{1}{\Delta \tilde{\eta}} \hat{\tilde{V}}_{i,j} + \frac{3}{4\Delta X} \tilde{U}_{i,j+1} - \frac{1}{\Delta \tilde{\eta}} \hat{\tilde{V}}_{i,j+1} \\
= & \tilde{U}_{i-1,j} + \tilde{U}_{i-1,j+1} - \frac{\tilde{U}_{i-2,j} + \tilde{U}_{i-2,j+1}}{4\Delta X}, \\
\left( \hat{\tilde{V}}_{i,j} - \frac{1}{(\Delta \tilde{\eta})^2} \right) \tilde{U}_{i,j-1} + \left( \omega \left( \frac{\tilde{U}_{i,j}}{X} + \frac{3\tilde{U}_{i,j}}{2\Delta X} \right) + \frac{2}{(\Delta \tilde{\eta})^2} \right) \tilde{U}_{i,j} + & \\
\frac{3}{2\Delta X} \tilde{P}_{i,j} + \left( \frac{\hat{\tilde{V}}_{i,j}}{2\Delta \tilde{\eta}} - \frac{1}{(\Delta \tilde{\eta})^2} \right) \tilde{U}_{i,j+1} \\
= & \omega \tilde{U}_{i,j} \left( \frac{4\tilde{U}_{i-1,j} - \tilde{U}_{i-2,j}}{2\Delta X} + \tilde{U}_{i,j} \right) + \hat{\tilde{V}}_{i,j} \tilde{U}_{i,j} + \frac{4\tilde{P}_{i-1,j} - \tilde{P}_{i-2,j}}{2\Delta X}, \\
\tilde{P}_{i,j-1} - \tilde{P}_{i,j} = 0.
\end{align*}
\] (3.25)

The boundary conditions for the upper wall layer being the no-slip conditions and the pressure being constant across the layer, it follows that

\[
\tilde{U}_{i,0} = \hat{\tilde{V}}_{i,0} = 0, \quad \tilde{P}_{i,0} - \tilde{P}_{i,1} = 0.
\] (3.26)

To investigate the downstream flow, a similarity solution approach as discussed in Smith’s ’77 [18] has been taken. It is found that the displacement function \( A(X) \) behaves like \( A_0/X^2 \) where \( A_0 \) is a constant. Thus the second derivative of the displacement function is equal to \( A(X)/X^2 \).

The expression for \( A(X) \) is obtained from the pressure-displacement interaction at the downstream location \( X_M \)

\[
A_i = \frac{X_M^2}{\kappa} (P_{M,N+1} - P_{M,N}).
\] (3.27)

The coupling of the far-field condition \( U \rightarrow \lambda (\eta + hF_L(X) + A(X)) \) and the pressure-displacement interaction gives the interface condition

\[
U_{M,N} - \frac{\lambda X_M^2}{\kappa} P_{i,N} + \frac{\lambda X_M^2}{\kappa} P_{M,N+1} = \lambda (\eta_N + hF_{L,M}).
\] (3.28)

Similarly, in the upper wall layer-core interface the equation is

\[
\tilde{U}_{M,N+1} - \frac{\lambda X_M^2}{\kappa} P_{M,N+1} + \frac{\lambda X_M^2}{\kappa} P_{M,N} = \lambda (\eta_{N+1} + hF_{U,M}).
\] (3.29)
In Figure 3.2 a comparison has been shown between the wall layer solution of Smith’s long scale problem [16] and the non-linear solution generated using the numerical method used in this work for the hump \( F_U = F_L = X \exp \left( -X^2/32 \right) \) for \( X > 0 \), with \( h = 2.0 \). In order to compare, the displacement function in the non-linear method has been replaced by the average of the two wall shapes. First order backward difference method is used for the \( X \)-derivative. The grid used is \(-5 \leq X \leq 30\) with 401 points, \(0 \leq \eta \leq 50\) in the lower wall layer with 2500 points and \(0 \leq \tilde{\eta} \leq 55\) in the upper wall layer with 2751 points. Tolerance of \(10^{-7}\) is used for convergence at every \( X \) location.

Non-linear solutions for the three-layer system, represented by (3.21) - (3.29), are generated with a Gaussian hump \( F_L(X) = 5 \exp \left( -(24/2.25)X^2 \right) \) in the lower wall and a flat upper wall. The grid used to produce grid converged solutions is \(-4 \leq X \leq 4\) with 161 points, \(0 \leq \eta \leq 14\) with 71 points in the lower wall layer and \(0 \leq \tilde{\eta} \leq 15\) with 145 points in the upper wall layer. So \( M \) equals to 160, \( N \) equals to 70 and \( N_t \) equals to 146. The hump height \( h \) is increased from 0 to 2.9.
in 40 increments to ensure stability of the solutions at larger hump heights. The tolerance $10^{-7}$ between successive iterates sets the criterion for convergence to the solution at every $X$ location.

![Figure 3.3: Non-linear wall layer solutions of Smith’s ’77 structure](image)

The pressure difference between the two wall layers can be clearly seen in Figure 3.3a. The small spike in the lower wall pressure where the hump starts, shows an upstream effect. The flow detaches from the hump in the lower wall layer, as shown by the negative value of wall shear in Figure 3.3b and re-attaches again after a small distance. The flow remains attached throughout the upper wall layer. Its clear from these two plots that the flow tends to regain its original fully developed Poiseuille far downstream.

A comparison between the ifft, linear and non-linear solutions for the lower wall shear and pressure is shown in Figure 3.4. It is clear that as the height $h$ of the hump is decreased, the non-linear solutions tend to the linear solution. The horizontal range used in the computations is $(-4, 4)$. The range has been adjusted during post-processing to obtain a better idea of the behavior of solutions.

### 3.4 Introduction of magnetic field: finite $Ha$ flow

In the previous chapter, it was seen that fluid flow in presence of a constant, transverse magnetic field is governed by equations (2.4) through (2.8) along with no-slip conditions at the channel walls. The changes in the three-layer hydrodynamic flow structure due to gradual increase in the strength of the magnetic field, i.e, change of Hartmann number $Ha$ from zero to some finite values, are
studied. In order to do this, the scale analysis of the upstream flow is done. The upstream flow is given by the Hartmann solution of equation (2.9), which is

\[ u = U_0(y; Ha) = \frac{1}{2Ha} \left( \frac{\exp(Hay) - 1}{1 - \exp(Ha)} + \frac{1 - \exp(-Hay)}{1 - \exp(-Ha)} \right), \quad v = 0. \]

For a wall distortion of stream-wise length \( \Delta \) which is of order \( Re^{1/7} \), the height scale as well as the wall layer thickness is denoted by \( \delta \) where \( \delta \) is quite small as compared to \( \Delta \). Substitution of the wall layer coordinate into the Hartmann solution followed by expansion of the exponential terms in their corresponding Taylor series shows that the \( u \)-velocity in the wall layer scales like \( \delta \) with the wall shear remaining unchanged. The \( v \)-velocity scale is obtained from the mass conservation equation and is equal to \( \delta^2/\Delta \). The pressure scales like \( Re \delta^2 \). Hence it is seen that the scales are similar to the corresponding velocity and pressure scales in the wall layers for the hydrodynamic flow. To find the leading order equations in the wall layers, it is required to find the expansions for the components of the magnetic field.

The magnetic field is divided into two contributions [9], one due to the external field and the other due to the magnetic field induced by the current

\[ B = \hat{y} + \frac{Rm}{Ha} b(y) \hat{x}, \quad (3.30) \]

where \((\hat{x}, \hat{y})\) are the unit vectors in the \( x \) and \( y \)-directions, respectively. The induced field \( b \) satisfies the equation

\[ \frac{\partial^2 b}{\partial y^2} + Ha \frac{\partial u}{\partial y} = 0, \quad (3.31) \]
which implies that $b$ scales like $Ha \delta^2$. Component-wise the magnetic field in (3.30) is equal to $(Rm \delta^2 \tilde{b}, 1)$ where $\tilde{b}$ is the scaled induced magnetic field.

Substituting scales of all the variables into the $x$-momentum equation (2.5), it is seen that there has to be a balance between the two leading order terms $p_x$ and $Re^{-1}u_{yy}$ to preserve the near-wall interaction. This balance sets $\delta$ equal to $Re^{-2/7}$. The pressure scales like $Re^{3/7}$. Therefore, no changes are seen in the wall layer.

Similarly, the expansions in the core also remain unchanged. The only change happens to the coefficient of the displacement term $\kappa$ which now takes the form [10]

\[
\kappa = \int_0^1 (U_0(y; Ha))^2 \, ds = \text{csch}^2 \left( \frac{Ha}{2} \right) \left[ \frac{Ha(2 + \cosh (Ha)) - 3 \sinh (Ha)}{8Ha^3} \right].
\]

Therefore, the pressure-displacement interaction takes the form:

\[
\tilde{P} = P + \kappa(\text{Ha})A'(X).
\]

\[
\text{(3.32)}
\]

(a) Constant hump length

(b) Variable hump length

Figure 3.5: Pressure variation with increasing $Ha$

In order to examine the effect of gradual increase of $Ha$ on the flow, the pressure in both the wall layers are plotted. In Figure 3.5a, it is observed that as $Ha$ is increased, the difference between
the wall pressure in the two wall layers decreases. This implies that the flow structure is tending to the long-scale structure where the hump length is greater than $O(Re^{1/7})$ and less than $O(Re)$. The pressure in the two wall layers tending to equality suggests that the $\kappa(Ha)A''(X)$ term is becoming small. Anticipating a hump-length change in order to retain Smith’s flow structure, computations were done varying the hump-length with increasing $Ha$. Figure 3.5b verifies that if the hump length is decreased from $O(Re^{1/7})$, with increasing $Ha$, then the overall flow structure is preserved. This led to the need for a relation between $Re$ and $Ha$ to analyse the MHD flow structure with more definiteness.
CHAPTER 4. MHD CHANNEL FLOW WITH MODERATE MAGNETIC STRENGTH

For finite values of $Ha$, as was seen in the last chapter, the three-layer flow structure does not change. The flow properties seem to tend to long scale properties as $Ha$ is increased. On the other hand, the flow properties seem to be preserved as the stream-wise length scale $\Delta$ is shortened with increasing $Ha$. In this chapter, a quantitative relation between $Ha$ and $\Delta$ for increasing $Ha$ is obtained and the effect of that change in the length scale on the flow structure and properties is studied.

4.1 Flow as $Ha \to \infty$

With increase in the strength of the magnetic field from its finite values, i.e., as $Ha \to \infty$, the Hartmann solution (2.9) tends to $1/(2Ha)$. Thus the flow profile $U_0(y; Ha)$ flattens in the core with increasing $Ha$, as shown in Figure 4.1.

Figure 4.1: $U_0(y; Ha)$ profile through a straight channel
This change in the upstream flow affects the displacement coefficient \( \kappa(Ha) \) given by (3.32). At leading order [10]

\[
\kappa(Ha) = \int_0^1 \left( \frac{1}{2Ha} \right)^2 ds \sim \frac{1}{4} Ha^{-2},
\]

which shows that the displacement term in the pressure-displacement interaction equation (2.27):

\[
\bar{P}(X) = P(X) + \kappa A''(X)
\]

is becoming small compared to the pressure terms. This behaviour is shown in Figure 3.5a as well.

In order to preserve this interaction the stream-wise length scale of the wall distortion is shortened. As a consequence, the height scale of the wall layer also needs to be decreased to maintain the near-wall interaction, thereby, changing the velocity and pressure scales. The \((x, y)\) co-ordinates in the wall layer become equal to \( \left( \frac{Re^1}{7} \Delta X, \frac{Re^{-2}}{7} \Delta^{1/3} Y \right) \) where \( \Delta \) tends to zero. The \( u \)-velocity is equal to \( Re^{-2/7} \Delta^{1/3} U \), \( v \)-velocity is equal to \( Re^{-5/7} \Delta^{-1/3} V \) and the pressure is equal to \( Re^{3/7} \Delta^{2/3} P \). The far-field condition \( U \rightarrow \lambda(Y + A(X)) \) is maintained by letting the displacement function \(-A(X)\) scale like the wall \( y \)-coordinate. In order to maintain the pressure-displacement interaction, the pressure scale must balance the displacement term

\[
\Delta^{2/3} \sim Ha^{-2} \frac{\Delta^{1/3}}{\Delta^2},
\]

which implies the length scale \( \Delta \) must scale like \( O(Ha^{-6/7}) \). Hence at a length scale of \( Re^{1/7} Ha^{-6/7} \), it is anticipated that the hydrodynamic flow structure can be retained, as shown in Figure 3.5b.

### 4.1.1 Wall layers

The expansions in the lower wall layer are therefore:

\[
(x, y) = (Re^{1/7} Ha^{-6/7} X, Re^{-2/7} Ha^{-2/7} Y),
\]

\[
(u, v, p - P_0) \sim (Re^{-2/7} Ha^{-2/7} U, Re^{-2/7} Ha^{-2/7} V, Re^{3/7} Ha^{-4/7} P) + ...
\]

The result that \( A(X) \) becomes small means that the \( u_1 \) and \( v_1 \) perturbation velocities as given by (2.18) also become small in the core. On scales equal to and shorter than \( Re^{1/7} \), a balance must occur between \( uv_x \) and \( Re^{-1} p_y \) in the \( y \)-momentum equation of the core flow. Using the pressure
scale from equation (4.3), and setting \( y \sim O(1) \) and \( u \) equal to \( 1/(2Ha) \) at the leading order, sets the \( v \)-velocity scale to be

\[ v \sim Re^{-3/7}Ha^{-3/7}v_1. \] (4.4)

The perturbation \( u \)-velocity is set from the mass conservation and is of the order of \( Re^{-2/7}Ha^{-9/7} \).

Thus the expansions in the core become

\[
(x, y) = (Re^{1/7}Ha^{-6/7}X, y),
\]
\[
(u, v, p - P_0) \sim \left( \frac{1}{2Ha}, 0, 0 \right) + (Re^{-2/7}Ha^{-9/7}u_1, Re^{-3/7}Ha^{-3/7}v_1, Re^{3/7}Ha^{-4/7}p_1) + \ldots
\] (4.5)

In order to obtain the leading order equations in the wall layer and the channel core, the expansions for the magnetic field in each of these layers are needed as well. In the lower wall layer, it is assumed that the \( x \)-component of the magnetic field \( B_x \) has a perturbation of order \( \epsilon_1 \) to zero strength and the \( y \)-component \( B_y \) has a perturbation of order \( \epsilon_2 \) to the constant strength of unity.

The parameter \( \epsilon_1 \) is evaluated from the balance between the terms \( B_y \partial u / \partial y \) and \( Rm^{-1} \partial^2 B_x / \partial y^2 \) in the \( x \)-induction equation (2.7). If these two terms do not balance each other, then the leading order \( x \)-induction equation in the wall would be \( U_Y = 0 \) which is not correct. Hence the magnetic field \( B_x \) goes like \( Re^{-4/7}Ha^{-4/7}B_X + \ldots \) where \( B_X \sim O(1) \) and satisfies the equation

\[ Rm^{-1} \frac{\partial^2 B_X}{\partial Y^2} + U_Y = 0. \] (4.6)

This equation has a solution \( B_X = Rm \left[ (c_1 - \lambda A)Y - \lambda \frac{Y^2}{2} - \int_0^Y (U - \lambda s - \lambda A) \, ds \right] \), where the constant \( c_1 \) has to be evaluated from the matching condition with the adjacent layer. Similarly, the perturbation to the \( y \)-component \( B_y \) is evaluated from the balance of the terms \( B_y \partial u / \partial y \) and \( Rm^{-1} \partial^2 B_y / \partial y^2 \) in the \( y \)-induction equation (2.8) and is found to scale like \( Re^{-1}B_Y + \ldots \) where \( B_Y \sim O(1) \). The component \( B_Y \) satisfies the equation

\[ Rm^{-1} \frac{\partial^2 B_Y}{\partial Y^2} + V_Y = 0, \] (4.7)

with a solution given by \( B_Y = Rm \left[ (d_1 - V_0(X))Y + \lambda A'(X)\frac{Y^2}{2} - \int_0^Y (V + \lambda A'(X)s - V_0(X)) \, ds \right] \), with the velocity \( V_0(X) \) being obtained from the \( x \)-momentum equation \( \lambda^2 AA' + \lambda V_0 = -P_X \). After
substituting all the expansions in the governing equations and tracking every term, the leading order equations in the lower wall layer are obtained and are given by:

\[ U_X + V_Y = 0, \]
\[ UU_X + UV_Y = -P_X + U_{YY}, \]  
(4.8)
\[ P_Y = 0, \]

along with the no-slip conditions at the wall \( U(X, hF_L(X)) = V(X, hF_L(X)) = 0 \). Clearly, the wall layer equations are similar to (2.23) of the hydrodynamic structure.

The equation system corresponding to the upper wall layer is:

\[ \tilde{U}_X + \tilde{V}_Y = 0, \]
\[ \tilde{U}\tilde{U}_X + \tilde{U}\tilde{V}_Y = -\tilde{P}_X + \tilde{U}_{YY}, \]  
(4.9)
\[ \tilde{P}_Y = 0, \]

with the no-slip conditions at the wall \( \tilde{U}(X, hF_U(X)) = \tilde{V}(X, hF_U(X)) = 0 \).

The matching condition between the wall layer and the core requires the \( U \)-velocity in the wall layer and the \( u \)-velocity in the core to tend to the same value at the interface of the two layers. But the \( U \)-velocity in the lower wall layer tends to \( \lambda Y \) and the \( u \)-velocity in the core tends to \( 1/(2Ha) \) as they approach the interface. This mismatch of values is accommodated by the introduction of a new layer in between the wall layer and core. This new layer is known as the Hartmann layer and has a thickness \( O(Ha^{-1}) \) [9]. This layer is thicker than the wall layer as long as \( Ha \) is smaller than \( Re^{2/5} \).

4.1.2 Hartmann layers

The \( u \)-velocity in the Hartmann layer scales to the thickness of the Hartmann layer at the leading order. The layer creates a slip velocity at the walls given by \( \epsilon \tilde{u}_1 \). Hence the stream-wise flow in the Hartmann layer is a small perturbation to the Hartmann flow scaled as the thickness
of this layer. In order to match with the wall layer and create the slip velocity, \( \epsilon \) must be equal to \( Re^{-2/7}Ha^{-2/7} \) and the following \( u \)-velocity expansion near the lower wall is

\[
u \sim Ha^{-1}\bar{U}_0(\bar{y}) + Re^{-2/7}Ha^{-2/7}\bar{u}_1 + ..., \tag{4.10}\]

where the upstream velocity is

\[
\bar{U}_0(\bar{y}) = \frac{1}{2}(1 - \exp(-\bar{y})). \tag{4.11}\]

Anticipating a pressure match between the wall and Hartmann layer so that the pressure-displacement interaction (2.27) can be maintained in the core, the pressure in this layer scales similar to that in the wall layer. Mass conservation sets the \( v \)-velocity to be \( Re^{-3/7}Ha^{-3/7}\bar{v}_1 \). The co-ordinates and the final expansions in the lower Hartmann layer become

\[
(x, y) = (Re^{1/7}Ha^{-6/7}X, Ha^{-1}\bar{y}), \tag{4.12}\]

\[
(u, v, p) \sim (Ha^{-1}\bar{U}_0(\bar{y}), 0, 0) + (Re^{-2/7}Ha^{-2/7}\bar{u}_1, Re^{-3/7}Ha^{-3/7}\bar{v}_1, Re^{3/7}Ha^{-4/7}\bar{P}) + ... \]

The magnetic field components are obtained from the magnetic induction equations (2.7) and (2.8). The scale of the \( x \)-component \( B_x \) is obtained from the balance between the \( B_y u_y \) and \( \partial B_x / \partial y \) terms and that for the \( B_y \) is obtained from the balance between the \( B_y v_y \) and the \( \partial B_y / \partial y \) terms. The magnetic fields are given by:

\[
B_x \sim Ha^{-2}\bar{B}_X + ..., \tag{4.13}\]

\[
B_y \sim 1 + Re^{-3/7}Ha^{-10/7}\bar{B}_Y + ..., \tag{4.14}\]

where \( \bar{B}_X \) is equal to \( Rm \left[ -\lambda \bar{y} - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right] \). The field \( \bar{B}_Y \) is equal to negative of \( Rm \left[ \int_0^\bar{y} \bar{v}_1 \, ds \right] \).

The leading order equation system in the lower Hartmann layer:

\[
\bar{u}_{1X} + \bar{v}_{1y} = 0, \tag{4.14}\]

\[
\bar{U}_0\bar{u}_{1X} + \bar{U}'_0\bar{v}_1 = 0, \tag{4.14}\]

\[
\bar{P}_{\bar{y}} = 0, \tag{4.14}\]

is similar to the main deck equations of the triple-deck structure with the solution

\[
(\bar{u}_1, \bar{v}_1) = (\bar{U}'_0A(X), -\bar{U}_0A'(X)), \tag{4.14}\]
equivalent to (2.18) in the core of Smith’s ’77 [18] hydrodynamic structure. This sets the matching condition \( U \to \lambda (Y + A(X)) \) between the lower wall layer and the lower Hartmann layer.

Similarly, in the upper Hartmann layer, the normal coordinate is \( y = 1 - Ha^{-1} \hat{y} \) and the leading order equation system is:

\[
\begin{align*}
\hat{u}_{1X} + \hat{v}_{1y} &= 0, \\
\hat{U}_0 \hat{u}_{1X} + \hat{U}'_0 \hat{v}_1 &= 0, \\
\hat{P}_y &= 0,
\end{align*}
\]

with the solution

\[
(\hat{u}_1, \hat{v}_1) = (\hat{U}'_0 B(X), -\hat{U}_0 B'(X)),
\]

where the upstream velocity \( \hat{U}_0(\hat{y}) \) has the same form as (4.11). The displacement function \( B(X) \) is the mirror image of \( A(X) \), that is, \( B(X) = -A(X) \). The matching condition \( \tilde{U} \to \lambda (\tilde{Y} + B(X)) \) between the upper wall layer and Hartmann layer sets the far-field condition for the upper wall layer.

### 4.1.3 Core

In the channel core, the normal coordinate \( y \) is in the interval \((0, 1)\) and is of order one. The leading order \( u \)-velocity in the core as well at the interface of the core and Hartmann layer is the constant \( 1/(2Ha) \). This implies that the \( X \)-derivative of the \( u \)-velocity is zero, which in turn makes the \( y \)-derivative of the \( v \)-velocity zero from the continuity equation. This sets the scale of the \( v \)-velocity in the core which is equal to \( Re^{-3/7} Ha^{-3/7} \). The scale of the perturbation \( u \)-velocity is obtained from the continuity equation. The pressure scale remains same as that in the wall and the Hartmann layers to preserve the pressure-displacement interaction. The scales of the magnetic field components are obtained in the similar method as was done in the wall and Hartmann layers. Hence the expansions in the core are:

\[
(u, v, p - P_0) \sim \left( \frac{1}{2Ha}, 0, 0 \right) + (Re^{-2/7} Ha^{-9/7} u_1, Re^{-3/7} Ha^{-3/7} v_1, Re^{3/7} Ha^{-4/7} p_1) + \ldots,
\]

\[
B_x \sim Ha^{-1} B_{X0} + Re^{-2/7} Ha^{-9/7} B_{X1} + \ldots,
\]

\[
B_y \sim 1 + Re^{-3/7} Ha^{-3/7} B_{Y1} + \ldots,
\]

\[(4.16)\]
where $B_{X0}$ is equal to $Rm(-\lambda y)$ and $B_{X1}$ is equal to $Rm \left[ c_1 y - c_2 - \int_0^y u_1 ds \right]$ obtained from the matching between core and lower Hartmann layer. The scaled component $B_{Y1}$ is equal to $Rm \left( \frac{A'(X)}{2} \right)$.

The expansions are substituted into the continuity equation (2.4) to obtain the leading order continuity equation in the core:

$$u_{1X} + v_{1y} = 0. \tag{4.17}$$

Substitution of the expansions into the $x$-momentum equation (2.5) and tracking every term leads to two leading terms: $Re^{-3/7} Ha^{-10/7} \frac{u_{1X}}{2}$ and $Re^{-5/7} Ha^{2/7} p_{1X}$. The velocity term is bigger than the pressure term as long as

$$Re^{1/6} \gg Ha. \tag{4.18}$$

Hence the leading order $x$-momentum equation is

$$u_{1X} = 0. \tag{4.19}$$

The upstream perturbation condition that $u_1(-\infty, y)$ is zero sets $u_1$ to zero. From the continuity equation (4.17), $v_{1y}$ is found to be equal to zero. This sets the boundary condition for the $v$-velocity at the interface with the two Hartmann layers

$$v_1(X, 0) = \tilde{v}_1(X, \infty) = -\frac{A'(X)}{2},$$

$$v_1(X, 1) = \hat{v}_1(X, \infty) = -\frac{B'(X)}{2} = \frac{A'(X)}{2}. \tag{4.20}$$

The leading order $y$-momentum equation obtained after substituting the expansions and tracking every term is:

$$\frac{1}{2} v_{1X} = -p_{1y}, \tag{4.21}$$

which on integration with respect to $y$ and the condition that the core pressure is equal to the wall layer pressure at the core-Hartmann layer interfaces gives the pressure-displacement interaction equation

$$\tilde{P}(X) = P(X) + \frac{1}{4} A''(X). \tag{4.22}$$
Therefore, it is seen that the pressure-displacement interaction in this structure with finite $Ha$ is similar in form to (2.27) except for the value of $\kappa$ which is equal to $1/4$ instead of $1/120$ as in the hydrodynamic structure.

4.2 Flow with wall geometry

Wall shapes $hF_L(X)$ and $hF_U(X)$ with height $h$ are considered on the lower wall and upper wall, respectively. After application of Prandtl’s transformation, the equations obtained in each layer are same as equations (3.1) - (3.5) with $\kappa$ equal to $1/4$.

4.3 Linearization for shallow wall geometry

The same methods of Fourier transform and linear solutions using numerical methods are followed, as done in the hydrodynamic structure. The solutions in the Fourier space are similar to
the hydrodynamic flow:

\[
\begin{align*}
    u^*_L &= \frac{3}{2}(F^*_L + a^*) \left(\frac{i\alpha}{2}\right)^{1/3} \int_0^\eta \text{Ai} \left[\left(\frac{i\alpha}{2}\right) \frac{1}{s}\right] \, ds, \\
p^*_L &= \frac{3A\alpha'(0)}{2^{2/3}} (i\alpha)^{-1/3}(F^*_L + a^*), \\
    u^*_U &= \frac{3}{2}(F^*_U - a^*) \left(\frac{i\alpha}{2}\right)^{1/3} \int_0^\eta \text{Ai} \left[\left(\frac{i\alpha}{2}\right) \frac{1}{s}\right] \, ds, \\
p^*_U &= \frac{3A\alpha'(0)}{2^{2/3}} (i\alpha)^{-1/3}(F^*_U - a^*),
\end{align*}
\]

(4.23)

with the transformed displacement function

\[
a^* = \frac{\gamma(F^*_U - F^*_L)}{2\gamma - \kappa\alpha^2},
\]

(4.24)

where \(\gamma\) is equal to \(\frac{3A\alpha'(0)}{2^{2/3}} (i\alpha)^{-1/3}\) and \(\kappa = 1/4\).

**4.4 Non-linear solutions**

The method and the finite-differenced equations in wall layers and core are similar to the ones discussed in section 3.3. The lower wall has a Gaussian hump \(F^*_L(X) = 4 \exp\left(-\frac{24}{25}X^2\right)\) and the upper wall is flat. The grid used is \(-10 \leq X \leq 10\) with 241 points and \(0 \leq \eta \leq 30\) with 2401 points in both the wall layers. The hump height \(h\) is increased from 0 to 4 in 80 increments to ensure stability of the converged solutions. A tolerance of \(10^{-6}\) has been used between successive iterates for convergence at every \(X\) location.

In Figure 4.3a, an upstream effect can be seen in the lower wall layer. The difference between the pressure in the two wall layers has clearly increased in comparison to the difference in the hydrodynamic flow. This is due to the increase in the value of \(\kappa\) from 1/120 in the hydrodynamic structure to 1/4 in this new flow structure with moderate magnetic field strength. The negative values of the lower wall shear indicates flow separation along the lower wall. The flow is attached in the upper wall layer. The behaviour of the solutions show that the flow returns to its undisturbed form far downstream. It is clear from the linear and non-linear solutions of both the hydrodynamic flow and flow with moderate magnetic field strength that the overall behavior is retained.
A comparison between the ifft and non-linear solutions for the lower wall shear and pressure is shown in Figure 4.4. It is clear that for as the height $h$ of the hump is decreased, the non-linear solutions tend to the linear solution.

### 4.5 Linear free interaction

The methodology used to analyse the linear free interaction in this flow structure is similar to the one used in section 2.3. The only difference is the value of $\kappa$ which is $1/4$ in this case. The difference in the value of $\kappa$ leads to a change in the value of $\tilde{\theta}$ which is approximately 1.33... As $Ha$ becomes smaller, Smith’s $\theta$ value is recovered, as shown in Figure 4.5. The constant $\tilde{\theta}$ is the scaled growth exponent at the $Re^{1/7} Ha^{-6/7}$ order. On the order of $Re^{1/7}$, that is the length scale
of Smith’s ‘77 structure, the growth exponent \( \theta \) of Smith’s ‘77 linear free interaction is found to become similar to \( \tilde{\theta} H_a^{6/7} \) as \( H_a \) tends to infinity. It shows that the linear free interaction is also preserved but on a shorter length scale.

Therefore, it can be concluded that the overall flow structure and properties are retained in the flow with moderate magnetic field strength.
CHAPTER 5. MHD CHANNEL FLOW WITH STRONG MAGNETIC FIELD

5.1 Introduction

In the previous chapter, it was seen that magnetohydrodynamic flow for moderate magnetic field strength behaves similar to the hydrodynamic flow as long as $Ha$ is smaller than $Re^{1/6}$. A structural change when $Ha$ becomes of the order of $Re^{1/6}$ is anticipated. Any changes that might occur in the flow properties will also be studied.

5.2 Flow with $Ha \sim O(Re^{1/6})$

As $Ha$ is gradually increased from the values where the relation (4.18) holds, at some point it becomes comparable to $Re^{1/6}$

$$Ha \sim O(Re^{1/6}).$$  \hfill (5.1)

As the relation (5.1) is substituted into the $x$-coordinate in (4.3), it is seen that

$$x \sim O(1),$$  \hfill (5.2)

that is, the stream-wise length scale is comparable to the channel width. (5.1) can be rewritten as

$$Ha = Re^{1/6}H,$$  \hfill (5.3)

where $H$ is a scaled Hartmann number that is finite.

5.2.1 Wall layers

The expansions in the lower wall layer are therefore:

$$(x, y) = (X, Re^{-1/3}Y),$$

$$(u, v, p - P_0) \sim (Re^{-1/3}U, Re^{-2/3}V, Re^{1/3}P) + ...$$  \hfill (5.4)
The above expansions are used to calculate the scales of the magnetic field components $B_x$ and $B_y$, which are equal to some perturbations added to zero and unity strengths, respectively. The horizontal component $B_x$ is obtained from the balance between the terms $B_y \frac{\partial u}{\partial y}$ and $Rm^{-1} \frac{\partial^2 B_x}{\partial y^2}$ in the $x$-induction equation (2.7). $B_x$ is found to be equal to $Re^{-2/3} B_{X0}$, where $B_{X0}$ satisfies the equation:

$$Rm^{-1} \frac{\partial^2 B_{X0}}{\partial y^2} + U_Y = 0,$$

with a solution $B_{X0}$ equal to $Rm \left[ -\lambda \frac{Y^2}{2} - \int_0^Y (U - \lambda s - \lambda A) \, ds \right]$. Similarly, the normal component $B_y$ is evaluated from the balance of the terms $B_y \frac{\partial v}{\partial y}$ and $Rm^{-1} \frac{\partial^2 B_y}{\partial y^2}$ in the $y$-induction equation (2.8) and is found to scale like $Re^{-1} B_{Y0} + \ldots$. The scaled component $B_{Y0}$ satisfies the equation:

$$Rm^{-1} \frac{\partial^2 B_{Y0}}{\partial y^2} + V_Y = 0,$$

with the solution $B_{Y0}$ equal to $Rm \left[ \lambda A'(X) \frac{Y^2}{2} - \int_0^Y (V + \lambda A'(X) s - V_0(X)) \, ds \right]$ where the velocity $V_0(X)$ is evaluated from the $x$-momentum equation $\lambda^2 A A' + \lambda V_0 = -P_X$. Substituting all the expansions into the governing equations (2.4) - (2.6), the leading order equations in the lower wall layer are obtained

$$U_X + V_Y = 0,$$

$$UU_X + VU_Y = -P_X + U_{YY},$$

$$P_Y = 0,$$

along with the no-slip conditions at the wall $U(X, hF_L(X)) = V(X, hF_L(X)) = 0$. The equation system corresponding to the upper wall layer is

$$\hat{U}_X + \hat{V}_Y = 0,$$

$$\hat{U}\hat{U}_X + \hat{U}\hat{V}_Y = -\hat{P}_X + \hat{U}_{YY},$$

$$\hat{P}_Y = 0,$$

with the no-slip conditions at the wall $\hat{U}(X, hF_U(X)) = \hat{V}(X, hF_U(X)) = 0$. 
5.2.2 Hartmann layers

Substitution of the $Ha$ value given by (5.1) into the Hartmann layers (4.12) gives the expansions:

\[
(x, y) = (X, Re^{-1/6} \bar{y}),
\]

\[
(u, v, p - P_0) \sim (Re^{-1/6} \bar{U}_0(\bar{y}), 0, 0) + (Re^{-1/3} \bar{u}_1, Re^{-1/2} \bar{v}_1, Re^{1/3} \bar{P}) + ..., \tag{5.9}
\]

where the upstream flow velocity is:

\[
\bar{U}_0(\bar{y}) = \frac{1}{2H} (1 - \exp(-H \bar{y})). \tag{5.10}
\]

The upstream velocity profile is obtained from the equation (4.11) using (5.3) and re-scaling the $\bar{y}$ co-ordinate in (4.11) by Hartmann number $Ha = Re^{1/6} H$.

The magnetic field components are obtained from the magnetic induction equations (2.7) and (2.8). The scale of the $x$-component $B_x$ is obtained from the balance between the $B_y u_y$ and $\partial B_y / \partial y$ terms and that for the $B_y$ is obtained from the balance between the $B_y v_y$ and the $\partial B_y / \partial y$ terms. The magnetic fields are given by

\[
B_x \sim Re^{-1/3} \bar{B}_X + ..., \tag{5.11}
\]

\[
B_y \sim 1 + Re^{-2/3} \bar{B}_Y + ..., \tag{5.11}
\]

where $\bar{B}_X$ is equal to $Rm \left[ -\lambda \bar{y} - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right]$. The field $\bar{B}_Y$ is equal to negative of $Rm \left[ \int_0^\bar{y} \bar{v}_1 \, ds \right]$. The leading order equation system in the lower Hartmann layer is

\[
\bar{u}_{1X} + \bar{u}_{1\bar{y}} = 0,
\]

\[
\bar{U}_0 \bar{u}_{1X} + \bar{U}_0' \bar{v}_1 = 0, \tag{5.12}
\]

\[
\bar{P}_y = 0.
\]

These equations are same as (4.14). The solution is equivalent to (2.18) in the core of Smith’s ’77 [18] hydrodynamic structure:

\[
(\bar{u}_1, \bar{v}_1) = (\bar{U}_0' A(X), -\bar{U}_0 A'(X)).
\]

This sets the matching condition $U \rightarrow \lambda(Y + A(X))$ between the lower wall layer and the lower Hartmann layer.
Similarly, in the upper Hartmann layer, the normal coordinate is \( y = 1 - Re^{-1/6} \hat{y} \) and the leading order equation system is:

\[
\begin{align*}
\hat{u}_1X + \hat{v}_1\hat{y} &= 0, \\
\hat{U}_0\hat{u}_1X + \hat{U}'_0\hat{v}_1 &= 0, \\
\hat{P}_\hat{y} &= 0,
\end{align*}
\]  

(5.13)

with the solution

\[
(\hat{u}_1, \hat{v}_1) = (\hat{U}'_0B(X), -\hat{U}_0B'(X)),
\]

where the upstream velocity \( \hat{U}_0(\hat{y}) \) has the same expression as (5.10). The matching condition \( \hat{U} \rightarrow \lambda(\hat{Y} + B(X)) \) between the upper wall layer and Hartmann layer sets the far-field condition for the upper wall layer.

### 5.2.3 Core

The normal coordinate \( y \) in the channel core lines in the interval \((0, 1)\) and is of order one. The upstream \( u \)-velocity \((1/2Ha)\) becomes \( Re^{-1/6}/(2H) \) using the relation (5.3). Therefore, the expansions in the core are:

\[
(u, v, p - P_0) \sim \left( Re^{-1/6} \frac{1}{2H}, 0, 0 \right) + \left( Re^{-1/2}u_1, Re^{-1/2}v_1, Re^{1/3}p_1 \right) + ...
\]  

(5.14)

The magnetic field components \( B_x \) and \( B_y \) are perturbations to zero and unity strength magnetic fields, respectively. The scale for the horizontal component \( B_x \) is obtained from the balance between the terms \( B_yu_y\frac{\partial B_x}{\partial x} \) and \( \frac{\partial B_x}{\partial y} \) in the \( x \)-induction equation (2.7). Similarly, the scale normal component \( B_y \) is obtained from the balance between the terms \( B_yv_y\frac{\partial B_y}{\partial x} \) and \( \frac{\partial B_y}{\partial y} \) in the \( y \)-induction equation (2.8). Hence the magnetic field expansions are:

\[
\begin{align*}
B_x &\sim Re^{-\frac{1}{2}}B_X + ..., \\
B_y &\sim 1 + Re^{-\frac{1}{2}}B_Y + ..., 
\end{align*}
\]  

(5.15)

where the scaled components \( B_X \) and \( B_Y \) satisfy the equations

\[
\begin{align*}
Rm^{-1}\left[ \frac{\partial^2 B_X}{\partial X^2} + \frac{\partial^2 B_X}{\partial y^2} \right] + u_{1y} &= 0, \\
Rm^{-1}\left[ \frac{\partial^2 B_Y}{\partial X^2} + \frac{\partial^2 B_Y}{\partial y^2} \right] + v_{1y} &= 0,
\end{align*}
\]  

(5.16)
respectively. The leading order equations obtained in the core by substituting all the expansions into the governing equations (2.4)-(2.6) are:

\[ u_1X + v_1y = 0, \]

\[ \frac{H^{-1}}{2}u_1X = -p_1X, \]  \hfill (5.17)

\[ \frac{H^{-1}}{2}v_1X = -p_1y. \]

These equations when solved simultaneously give a Laplace equation for pressure [10]

\[ p_{1XX} + p_{1yy} = 0. \]  \hfill (5.18)

Simultaneously solving the continuity and the \( x \)-momentum equations in the core and using the fact that the \( v \)-perturbation velocity remains constant across the Hartmann layer-core interface give the boundary conditions for the \( y \)-derivative of the pressure

\[ p_{1y}(X,0) = \frac{A''}{4H^2}, \]

\[ p_{1y}(X,1) = \frac{-B''}{4H^2}, \]  \hfill (5.19)

where the displacement functions \( A(X) \) and \( B(X) \) are related by the relation

\[ A + B = 4H^2 \int_0^1 p_1(X,s) \, ds. \]  \hfill (5.20)

This relation is obtained by integrating the \( y \)-momentum equation and using the boundary conditions along with the upstream condition for the perturbation velocities and pressure.

Hence it is seen that the core flow has changed. Instead of the pressure being a linear function of the normal coordinate as in the hydrodynamic flow, and the MHD flow for moderate magnetic field strength, the core is governed by a Laplace equation for pressure in this case.

### 5.3 Flow with wall geometry

Figure 5.1 shows the five layer flow structure. The wall shapes \( hF_L(X) \) and \( hF_U(X) \) with height \( h \) are considered on the lower wall and upper wall, respectively. After the application of Prandtl’s transformation, \( \eta = Y - hF_L(X) \) in the lower wall layer and \( \tilde{\eta} = \tilde{Y} - hF_U(X) \) in the upper wall
Figure 5.1: MHD flow structure with very large $Ha$

layer, the equations obtained in the two wall layers are same as equations (3.1) - (3.3). The core is governed by the equation (5.18) with the boundary conditions (5.19) and the displacement relation (5.20).

5.4 Linearization for shallow wall geometry

To study the system at very small hump heights, the Fourier transform has been used. The solutions in the Fourier space for the wall layers are similar to the hydrodynamic flow:

\[
\begin{align*}
    u_L^* &= \frac{3}{2} (F_L^* + a^*) \left( \frac{i\alpha}{2} \right)^{1/3} \int_0^{\eta} \Ai \left( \frac{i\alpha}{2} \right)^{1/3} s \ ds, \\
    p_L^* &= \frac{3A\delta'(0)}{2^{5/3}} (i\alpha)^{-1/3} (F_L^* + a^*), \\
    u_U^* &= \frac{3}{2} (F_U^* + b^*) \left( \frac{i\alpha}{2} \right)^{1/3} \int_0^{\eta} \Ai \left( \frac{i\alpha}{2} \right)^{1/3} s \ ds, \\
    p_U^* &= \frac{3A\delta'(0)}{2^{5/3}} (i\alpha)^{-1/3} (F_U^* + b^*),
\end{align*}
\]

(5.21)

where $u_L^*$ and $u_U^*$ are the Fourier transform of the perturbation $u$-velocities in the lower and upper wall layers, respectively. In the core, the perturbed pressure is

\[ p_1(X, y) \sim hp_c(X, y) + ..., \]
where the height \( h \) is smaller than unity. The Fourier transform of \( p_c(X, y) \) is \( p_c^*(\alpha, y) \) which satisfies the equation

\[
p_{cyy} - \alpha^2 p_c^* = 0, \tag{5.22}
\]

with the solution

\[
p_c^*(\alpha, y) = c_1(\alpha)\exp(\alpha y) + c_2(\alpha)\exp(-\alpha y). \tag{5.23}
\]

The coefficient functions are given by

\[
c_1(\alpha) = \frac{\exp(-\alpha)p_L^* - p_U^*}{\exp(-\alpha) - \exp(\alpha)}, \quad c_2(\alpha) = \frac{p_U^* - \exp(\alpha)p_L^*}{\exp(-\alpha) - \exp(\alpha)}. \tag{5.24}
\]

The boundary conditions for \( p_c^* \) are

\[
p_{cy}^*(\alpha, 0) = -\frac{\alpha^2 a^*}{4H^2}, \quad p_{cy}^*(\alpha, 1) = \frac{\alpha^2 b^*}{4H^2}. \tag{5.25}
\]

Clearly, to evaluate the Fourier transform velocity and pressure variables, it is required to find the formulas for the displacement functions \( a^* \) and \( b^* \) obtained by solving the linear system:

\[
\begin{pmatrix}
a^* \\
b^*
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\alpha}{4H^2} - \gamma \coth \alpha & \frac{\gamma}{\sinh \alpha} \\
\frac{\gamma}{\sinh \alpha} & \frac{\alpha}{4H^2} - \gamma \coth \alpha
\end{pmatrix}
^{-1}
\begin{pmatrix}
\frac{\gamma}{\sinh \alpha} (F_L^* \cosh \alpha - F_U^*) \\
\frac{\gamma}{\sinh \alpha} (F_U^* \cosh \alpha - F_L^*)
\end{pmatrix}, \tag{5.26}
\]

where \( \gamma \) is equal to \( \frac{3 Ai'(0)}{2^{2/3}} (\iota \alpha)^{-1/3} \).

![Figure 5.2: Linear solutions of MHD flow structure with high \( Ha \)](image)

For the 'ifft' routine in MATLAB, a Gaussian hump \( F_L(X) = \exp(-24X^2) \) with a hump length of unity has been used in the lower wall and the upper wall is flat. \( 2^20 \) is the length of the padding.
used. The linear wall layer pressure and core pressure are shown in Figure 5.2a. The upper wall layer has considerably increased compared to upper wall layer pressure in the hydrodynamic flow and the flow with moderate magnetic strength. Likewise, Figure 5.2b shows significant flattening of upper wall shear compared to the other flow structures.

A comparative study has been done between the linear solutions of MHD flow with strong magnetic field and the MHD flow with moderate magnetic strength where $\kappa$ is equal to 1/4. For convenience, the MHD flow structure with moderate magnetic field strength is called "structure I" and with strong magnetic field is called "structure II". For the comparison, the physical length and height scales of the humps in the two structures are equated. Using the relation (5.3), the hump length scale in structure II is $D_{II} = H^{-6/7}D_I$ and the height scale is $h_{II} = H^{-2/7}h_I$. The solutions obtained using 'ifft' routine in MATLAB are then scaled back to that of the structure I. Hence the pressure solution is $P_I = H^{4/7}P_{II}$ and $X = X_I = H^{6/7}X_{II}$. The result is shown for the lower wall pressure for a Gaussian hump $h_I = 1$ and $F_L(X) = \exp \left(-24(X/D_I)^2\right)$ in Figure 5.3. This shows that the MHD flow structure with moderate $Ha$ can be recovered from the MHD flow with large $Ha = Re^{1/6}H$ as the strength $H$ is decreased.

![Figure 5.3: Connection between MHD flow structures with moderate and large strengths](image-url)
5.5 Linear free interaction

A change in the free interaction is anticipated and thus linear free interaction analysis is done. A perturbation of magnitude $\epsilon$ is introduced in all the layers in a flat channel. In the lower wall layer, the perturbed system is:

$$
(U, V, P_1) \sim (\lambda Y, 0, 0) + \epsilon(u_p(Y), v_p(Y), p_p)e^{\hat{\theta}X} + O(\hat{\theta}^2),
$$

$$
A(X) \sim \epsilon a_p e^{\hat{\theta}X} + O(\hat{\theta}^2),
$$

where $\hat{\theta}$ is constant and $a_p$ is equal to $\pm 1$. The linear system obtained after substitution of the expansions into (5.7) is:

$$
v_p' = -\hat{\theta}u_p,
$$

$$
\lambda \hat{\theta} Y u_p + \lambda v_p = -\hat{\theta} p_p + u''_p,
$$

with no-slip conditions $u_p(0) = v_p(0) = 0$ and far-field condition $u_p(Y) \to \lambda a_p$ as $Y \to \infty$.

Following the Airy equation method used in the hydrodynamic flow, as shown by equations (2.29)- (2.30), the solution obtained for the linear system (5.28) is given by:

$$
(u_p, v_p, p_p) = \left( C_1 f_1(Y), -\hat{\theta} C_1 f_1(Y), \frac{C_1 m}{\hat{\theta}} \right),
$$

where $m$ is equal to $(\lambda \hat{\theta})^{1/3}$ and $f_1'(Y)$ is equal to $\int_0^Y \text{Ai}(ms) ds$. The displacement perturbation $a_p$ is equal to $\frac{C_1}{3m\lambda}$. Similarly, in the upper wall layer the expansion is given by:

$$
(\tilde{U}, \tilde{V}, \tilde{P}_1) \sim (\lambda \tilde{Y}, 0, 0) + \epsilon \left( \tilde{C}_1 \tilde{f}_1(\tilde{Y}), -\hat{\theta} \tilde{C}_1 \tilde{f}_1(\tilde{Y}), \frac{\tilde{C}_1 m}{\hat{\theta}} \right) e^{\hat{\theta}X} + O(\hat{\theta}^2),
$$

$$
B(X) \sim \epsilon \frac{\tilde{C}_1}{3m\lambda} e^{\hat{\theta}X} + O(\hat{\theta}^2),
$$

where $\tilde{f}_1'(\tilde{Y})$ is equal to $\int_0^{\tilde{Y}} \text{Ai}(ms) ds$. In the core, the perturbed system is:

$$
p_1(X, y) \sim \epsilon \tilde{p}_c(y)e^{\hat{\theta}X} + O(\hat{\theta}^2),
$$

where the amplitude $\tilde{p}_c$ is obtained by substituting the above expansion into the equation (5.18), and solving it gives:

$$
\tilde{p}_c(y) = d_1 \cos (\hat{\theta} y) + d_2 \sin (\hat{\theta} y).
$$
Solving the constants $d_1$ and $d_2$ from the boundary conditions given by (5.19) and using the equation (5.20) gives the dispersion relation [10] between the growth factor $\hat{\theta}$ and the scaled Hartmann number $H$:

\[
2^{2/3}\hat{\theta}^{8/3} + 12H^2 \text{Ai}'(0)\hat{\theta}^{4/3} \cot(\hat{\theta}) = \frac{36}{2^{2/3}}H^4(\text{Ai}'(0))^2.
\] (5.33)

The details of the derivation has been discussed in Appendix, section 3. The dispersion relation (5.33) is a quadratic equation in the square power of the scaled Hartmann number $H$. Four solutions are obtained by solving this equation. The limit solution as $H \to \infty$ is $\pi$ as shown in Figure 5.4.

![Figure 5.4: $\hat{\theta}$ vs $H$ plot](image)

The constant value $\pi$ of the growth factor $\hat{\theta}$ shows that the linear free interaction is preserved but on a constant stream-wise length scale. As $H$ becomes smaller, it tends to zero at a rate of $H^{6/7}$. This shows that the growth exponent $\tilde{\theta}$ used in linear free interaction of flow with moderate magnetic field strength is related to $\hat{\theta}$ by a factor of $H^{6/7}$. As $H$ becomes smaller, the $\tilde{\theta}$ value can be recovered. The details of the connection between the two flow structures with magnetic field is discussed in Appendix, section 5.
CHAPTER 6. THREE DIMENSIONAL CHANNEL FLOW

In the last four chapters two dimensional structures and their properties have been discussed. As a logical extension, a start has been made in this chapter to investigate three dimensional structures. The aim is to make the analysis more relevant to the real world. The model used is that of two infinite plates separated by a very small distance. There is a wall distortion with length and width of same scale and the height being small compared to the length scale. The approach used here is similar to the one followed in the two dimensional study, that is, moving from the hydrodynamic flow to the flow with strong magnetic field.

6.1 Governing equations

In three dimensions, the non-dimensional governing equations (2.1) - (2.3) along with the no-slip conditions at the walls take the form

\begin{align}
    u_x + v_y + w_z &= 0, \\
    uu_x + vu_y + wu_z &= Re^{-1}[-p_x - Ha^2 Rm^{-1} \left(B_y \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_z}{\partial y}\right) + B_z \left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right)\right) + u_{xx} + u_{yy} + u_{zz}], \\
    vw_x + vv_y + vw_z &= Re^{-1}[-p_y - Ha^2 Rm^{-1} \left(B_x \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) + B_y \left(\frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y}\right)\right) + v_{xx} + v_{yy} + v_{zz}], \\
    uw_x + vw_y + uw_z &= Re^{-1}[-p_z - Ha^2 Rm^{-1} \left(B_z \left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right) + B_x \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_y}{\partial x}\right)\right) + w_{xx} + w_{yy} + w_{zz}].
\end{align}
\[
\frac{u}{\partial x} + \frac{v}{\partial y} + \frac{w}{\partial z} = B_xu_x + B_yu_y + B_zu_z + Rm^{-1}\left(\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2}\right),
\]
(6.5)

\[
\frac{u}{\partial x} + \frac{v}{\partial y} + \frac{w}{\partial z} = B_xv_x + B_yv_y + B_zv_z + Rm^{-1}\left(\frac{\partial^2 B_y}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_y}{\partial z^2}\right),
\]
(6.6)

\[
\frac{u}{\partial x} + \frac{v}{\partial y} + \frac{w}{\partial z} = B_xw_x + B_yw_y + B_zw_z + Rm^{-1}\left(\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2}\right).
\]
(6.7)

Here \( B_z \) is the component of the magnetic field in the \( z \)-direction. The no-slip conditions \( u(x, f(x), z) = v(x, f(x), z) = w(x, f(x), z) = 0 \) are applied at the channel walls where the upper and lower wall shapes are \( f = f_L(x, z) \) and \( f = 1 - f_U(x, z) \), respectively. The Hartmann solution in three dimensions have the same form as in two dimensions given by (2.9) along with \( w \)-velocity being zero.

### 6.2 Hydrodynamic structure

In pure hydrodynamic flow, due to absence of magnetic field, the system of governing equations reduces to:

\[
\nabla \cdot \mathbf{V} = 0, \mathbf{V} \cdot \nabla \mathbf{V} = Re^{-1}\left[-\nabla p + \nabla^2 \mathbf{V}\right].
\]
(6.8)

The Poiseuille flow retains its two dimensional form given by (2.12) with \( w \)-velocity equal to zero. The wall distortion length and width scale are similar, hence \( x \) and \( z \) are equal to \( \Delta X \) and \( \Delta Z \), respectively. The height scale \( y \sim \delta \) where \( \delta \) is small compared to \( \Delta \). Following the procedure in section 2.2 to find out the distortion scales along with maintaining the mass conservation (6.1) and \( x \)-momentum balances, the height scale in the wall layer should be of the order \( Re^{-1/3} \Delta^{1/3} \). The velocities \( u, w \) in the wall layers should be of order \( Re^{-1/3} \Delta^{1/3} \), and the \( v \)-velocity is of the order \( Re^{-2/3} \Delta^{-1/3} \). Hence the co-ordinates in the wall layer are given by:

\[
(x, y, z) = (\Delta X, Re^{-1/3} \Delta^{1/3} Y, \Delta Z),
\]
(6.9)

where \( X, Y \) and \( Z \) are of order one. The velocities are given by:

\[
(u, v, w) \sim (Re^{-1/3} \Delta^{1/3} U, Re^{-2/3} \Delta^{-1/3} V, Re^{-1/3} \Delta^{1/3} W) + \ldots,
\]
(6.10)

where \( U, V, W \) are order one velocities. The no-slip conditions at the wall are \( U(X, 0, Z) = V(X, 0, Z) = W(X, 0, Z) = 0 \). The \( W \)-velocity decays to zero as it approaches the core. More
precisely, it is known to behave like $D/Y$ as $Y$ tends to infinity. Here $D$ is a function of $X$ and $Z$. This sets the condition for the slip-velocity component $w_1$ in the core. Since it has to be small compared to the slip-velocity component $u_1$, the term $w_1Z$ should become negligible compared to the other terms in the continuity equation (6.1) in the core. Hence in the core the $u$-velocity behaves like $U_0 + Re^{-1/3}Δ^{1/3}u_1$, $v$-velocity like $Re^{-1/3}Δ^{-2/3}v_1$ and the pressure $p - P_0$ like $Re^{1/3}Δ^{2/3}p_1$. In order to find the scale $ε$ of $w_1$, the known expansions are substituted into the equation (6.4). Anticipating a balance between the leading order term $U_0w_1X$ and the pressure term $Re^{-1}p_z$, the value of $ε$ is found to be equal to $Re^{-2/3}Δ^{2/3}$. Therefore, the final expansions in the core are:

\[(u, v, w, p - P_0) \sim (U_0, 0, 0, 0) \]

\[+ (Re^{-1/3}Δ^{1/3}u_1, Re^{-1/3}Δ^{-2/3}v_1, Re^{-2/3}Δ^{2/3}w_1, Re^{1/3}Δ^{2/3}p_1) + ... \]

(6.11)

Substituting these scales into the system (6.8), the leading order equations obtained are:

\[u_{1X} + v_{1y} = 0, \]
\[U_0u_{1X} + U_0'v_1 = 0, \]
\[U_0v_{1X} = -p_{1Y}, \]
\[U_0w_{1X} = -p_Z. \]

(6.12)

Solving the above equations, the solutions obtained for the perturbation velocities are given by:

\[(u_1(X, y, Z), v_1(X, y, Z), w_1(X, y, Z)) = (A(X, Z)U_0'(y), -A'(X, Z)U_0(y), \]
\[\frac{-1}{U_0} \int_{-∞}^{∞} p_Z(s, y, Z) ds), \]

(6.13)

with $A(-∞, Z) = 0$. These velocities merge with Poiseuille flow upstream as $X$ tends to $-∞$. The matching between the lower wall layer and the core gives the conditions $U \to λ(Y + A(X, Z))$ and $λD_1/Y$ equal to negative of $p_{1Z}(X, 0, Z)$ as $Y$ tends to infinity.

6.2.1 Smith’s ’77 structure

In this structure, the coordinates $(x, z)$ are equal to $(Re^{1/7}X, Re^{1/7}Z)$ with $X, Z$ being finite. The expansions and equations corresponding to each layer are given below.
6.2.1.1 Channel core

In the core of the channel, the normal co-ordinate $y$ is of order one and belongs to the interval $(0, 1)$. The expansions are:

$$(u, v, w, p - P_0) \sim (Re^{-2/7} u_1, Re^{-3/7} v_1, Re^{-4/7} w_1, Re^{3/7} p_1) + ...$$  \hspace{1cm} (6.14)

The leading order $y$ and $z$-momentum equations are

$$p_1(X, y, Z) = P(X, Z) + A_{XX}(X, Z) \int_0^y U_0^2(s) \, ds,$$  \hspace{1cm} (6.15)

$$U_0 w_{1X} = -p_{1Z},$$

where $P(X, Z) = p_1(X, 0, Z)$ is unknown with $P(-\infty, Z) = 0$.

6.2.1.2 Wall layers

In the lower wall layer $y$ scales like the wall layer thickness and is, thus, equal to $Re^{-2/7} Y$ where $Y \sim O(1)$. The expansions are:

$$(u, v, w, p - P_0) \sim (Re^{-2/7} U(X, Y, Z), Re^{-5/7} V(X, Y, Z), Re^{-2/7} W(X, Y, Z), Re^{3/7} P(X, Z)) + ...$$

The leading order equations are:

$$U_X + V_Y + W_Z = 0,$$  
$$UU_X + VU_Y + WU_Z = -P_{1X} + U_{YY},$$  \hspace{1cm} (6.16)

$$P_{1Y} = 0,$$

$$UW_X + VW_Y + WW_Z = -P_{1Z} + W_{YY},$$

with the boundary conditions

$$U = V = W = 0 \text{ at } Y = hF_{L}(X, Z),$$

$$U \sim \lambda(Y + A(X, Z)), \quad W \to D/Y \text{ with } \lambda D_X = -P_Z \quad \text{as } Y \to \infty,$$  \hspace{1cm} (6.17)

$$U \to \lambda Y \text{ as } X \to -\infty.$$
In the upper wall layer, the $y$-coordinate is pointed into the core of the channel as shown in Figure 2.3 and is equal to $1 - \text{Re}^{-2/7}\hat{Y}$. The expansions have the same form as the lower wall layer but with the $v$-velocity expansion replaced by $-\text{Re}^{-5/7}V(X,\hat{Y},Z)$. The negative sign in the $v$-velocity expansion negates the negative sign in the $y$-coordinate in the upper wall layer leading to a system of equation similar to (6.16)

\[
\begin{align*}
\tilde{U}_X + \tilde{V}_Y + \tilde{W}_Z &= 0, \\
\tilde{U} \tilde{U}_X + \tilde{U} \tilde{V}_Y + \tilde{W} \tilde{U}_Z &= -\tilde{P}_X + \tilde{U}_Y, \\
\tilde{P}_Y &= 0, \\
\tilde{U} \tilde{W}_X + \tilde{U} \tilde{W}_Y + \tilde{W} \tilde{W}_Z &= -\tilde{P}_Z + \tilde{W}_Y, \\
\end{align*}
\]

(6.18)

and the boundary conditions being

\[
\begin{align*}
\tilde{U} = \tilde{V} = \tilde{W} &= 0 \text{ at } \hat{Y} = h\hat{F}_U(X,Z), \\
\tilde{U} &\sim \lambda(\hat{Y} - A(X,Z)), \quad \hat{W} \to \hat{D}/\hat{Y} \text{ with } \lambda \hat{D}_X = -\tilde{P}_Z \quad \text{as } \hat{Y} \to \infty, \\
\hat{U} &\to \lambda\hat{Y} \text{ as } X \to -\infty.
\end{align*}
\]

(6.19)

The wall layer pressures are connected through the core via (6.15):

\[
\tilde{P}(X,Z) = P(X,Z) + \kappa A_X(X,Z)
\]

(6.20)

where $\kappa = \int_0^1 U_0^2(s) \, ds = 1/120$.

6.2.2 Linearization for shallow wall geometry

The wall layer equation systems (6.16)- (6.20) are linearized by introducing a small perturbation to the lower and upper wall layers, respectively. The perturbed variables in the lower wall layer are:

\[
(U, \hat{V}, W, P, A) \sim (\lambda \eta, 0, 0, 0)
\]

(6.21)

\[
+ h(u_L(X,\eta,Z), v_L(X,\eta,Z), X_L(X,\eta,Z), p_L(X,Z), a(X,Z)) + ..., \]

where $u_L$, $v_L$, $X_L$, $p_L$, and $a$ are functions of $X$, $Z$, and $\eta$. The negative sign in the $v$-velocity expansion negates the negative sign in the $y$-coordinate in the upper wall layer leading to a system of equation similar to (6.16).
where $h$ is smaller than unity. The linearised system in the lower wall layer becomes:

\[
\begin{align*}
    u_{LX} + v_{L\eta} + w_{LZ} &= 0, \\
    \lambda \eta u_{LX} + \lambda v_L &= -p_{LX} + u_{L\eta}, \quad (6.22) \\
    \lambda \eta w_{LX} &= -p_{LZ} + w_{L\eta},
\end{align*}
\]

with the no-slip conditions $u_L(X, 0, Z) = v_L(X, 0, Z) = w_L(X, 0, Z) = 0$ and the far-field conditions $u_L \to \lambda(F_L(X, Z) + a(X, Z))$, $w_L \to 0$ as $\eta \to \infty$.

In order to solve the above three dimensional system with ease, Smith’s transformation [21] is used which transforms the three dimensional system to two dimensional form by using the transformation:

\[
\tilde{u} = u_X + w_Z, \quad \tilde{v} = v_X.
\]  
(6.23)

Hence taking the $X$-derivative of the $X$-momentum and $Z$-derivative of the $Z$-momentum and adding them together gives the equation

\[
\lambda \eta \tilde{u}_{LX} + \lambda \tilde{v}_L = -Q_L + \tilde{u}_{L\eta\eta},
\]

where $Q_L$ is the Laplacian of the pressure $p_L(X, Z)$. Therefore, the transformed system in the lower wall layer becomes:

\[
\begin{align*}
    \tilde{u}_{LX} + \tilde{v}_{L\eta} &= 0, \\
    \lambda \eta \tilde{u}_{LX} + \lambda \tilde{v}_L &= -Q_L + \tilde{u}_{L\eta\eta}, \quad (6.24)
\end{align*}
\]

with boundary conditions $\tilde{u}_L(X, 0, Z) = \tilde{v}_L(X, 0, Z) = 0$, $\tilde{u}_L \to \lambda(F_{LX}(X, Z) + a_X)$ as $\eta \to \infty$.

Similarly, in the upper wall layer, the linearized and transformed system takes the form:

\[
\begin{align*}
    \tilde{u}_{UX} + \tilde{v}_{U\tilde{\eta}} &= 0, \\
    \lambda \tilde{\eta} \tilde{u}_{UX} + \lambda \tilde{v}_U &= -Q_U + \tilde{u}_{U\tilde{\eta}\tilde{\eta}}, \quad (6.25)
\end{align*}
\]

where $Q_U$ is equal to Laplacian of $p_U(X, Z)$. The boundary conditions are $\tilde{u}_U(X, 0, Z) = \tilde{v}_U(X, 0, Z) = 0$, $\tilde{u}_U \to \lambda(F_{UX}(X, Z) - a_X)$ as $\tilde{\eta} \to \infty$. The transformed Laplacian of the wall pressures are connected through the core via the equation:

\[
Q_U = Q_L + \kappa(a_{XXX} + a_{XZZ}).
\]  
(6.26)
Fourier transform is used to analyse the linear systems like in the two dimensional cases. The definitions used for Fourier transform and inverse Fourier transform are given, respectively by:

\[
F^*(\alpha, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(X, Z)e^{-i(\alpha X + \beta Z)} \partial X \partial Z,
\]

\[
F(X, Z) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^*(\alpha, \beta)e^{i(\alpha X + \beta Z)} \partial \alpha \partial \beta,
\]

where \(\alpha\) and \(\beta\) are the transformed variables corresponding to \(X\) and \(Z\), respectively. The \('^*'\) symbol represents the transformed function. The transformation of the partial derivatives of the function are given by

\[
\left(\frac{\partial F}{\partial X}\right)^* (\alpha, \beta) = i\alpha F^*(\alpha, \beta),
\]

\[
\left(\frac{\partial F}{\partial Z}\right)^* (\alpha, \beta) = i\beta F^*(\alpha, \beta),
\]

\[
\left(\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Z^2}\right)^* (\alpha, \beta) = -(\alpha^2 + \beta^2)F^*(\alpha, \beta).
\]

Applying the definitions of Fourier transform and the derivatives, the transformed system in the lower wall layer along with the no-slip and far field conditions become:

\[
i\alpha u^*_L + v^*_L \eta = 0,
\]

\[
i\alpha \lambda u^*_L + \lambda v^*_L = -Q^*_L + u^*_{L \eta \eta},
\]

\[
u^*_L(\alpha, 0, \beta) = v^*_L(\alpha, 0, \beta) = 0,
\]

\[
u^*_L \to i\alpha \lambda (F^*_L(\alpha, \beta) + a^*(\alpha, \beta)) \text{ as } \eta \to \infty.
\]

Likewise, the transformed, linearized system in the upper wall layer becomes:

\[
i\alpha u^*_U + v^*_U \tilde{\eta} = 0,
\]

\[
i\alpha \lambda \tilde{\eta} u^*_U + \lambda v^*_U = -Q^*_U + u^*_{U \tilde{\eta} \tilde{\eta}},
\]

\[
u^*_U(\alpha, 0, \beta) = v^*_U(\alpha, 0, \beta) = 0,
\]

\[
u^*_U \to i\alpha \lambda (F^*_U(\alpha, \beta) - a^*(\alpha, \beta)) \text{ as } \tilde{\eta} \to \infty.
\]

The core is governed by the pressure-displacement interaction:

\[
Q^*_U = Q^*_L + \kappa \alpha^2(\alpha^2 + \beta^2)a^*.
\]
These Fourier transformed systems are solved as in section 3.2. In the lower wall layer the solutions are:

\[ u_L^* = 3 \left( \frac{i\alpha}{2} \right)^{4/3} (F_L^* + a^*) \int_0^\eta \text{Ai} \left[ \left( \frac{i\alpha}{2} \right)^{1/3} s \right] ds, \]

\[ Q_L^* = u_{Lm}(0) = 3 \text{Ai}'(0) \frac{(i\alpha)^{5/3}}{2} (F_L^* + a^*), \]  

and in the upper wall layer are:

\[ u_U^* = 3 \left( \frac{i\alpha}{2} \right)^{4/3} (F_U^* - a^*) \int_0^\eta \text{Ai} \left[ \left( \frac{i\alpha}{2} \right)^{1/3} s \right] ds, \]

\[ Q_U^* = 3 \text{Ai}'(0) \frac{(i\alpha)^{5/3}}{2} (F_U^* - a^*). \]

Substituting the values of \( Q_L^* \) and \( Q_U^* \) into (6.31) the transformed displacement function is given by:

\[ a^* = \frac{\theta (F_U^* - F_L^*)}{2\theta + \kappa \alpha^2 (\alpha^2 + \beta^2)}, \]

where \( \theta \) is equal to \( 3 \text{Ai}'(0) \frac{(i\alpha)^{5/3}}{2} \). The transformed pressures in each layer are given by the relations:

\[ p_L^* = -Q_L^*/(\alpha^2 + \beta^2), \]

\[ p_U^* = -Q_U^*/(\alpha^2 + \beta^2). \]

### 6.3 MHD flow with moderate magnetic field strength

The procedure followed in section 4.1 has been used here. With increasing value of \( Ha \), the Hartmann solution (2.9) tends to \( 1/(2Ha) \). One major new feature is that the perturbation velocities in the core \( u_1 \) and \( v_1 \) become small while \( w_1 \) tends to infinity.

#### 6.3.1 Wall layers

The expansions in the lower wall layer are therefore:

\[ (x, y, z) \sim (Re^{1/7} Ha^{-6/7} X, Re^{-2/7} Ha^{-2/7} Y, Re^{1/7} Ha^{-6/7} Z), \]

\[ (u, v, w, p - P_0) \sim (Re^{-2/7} Ha^{-2/7} U, Re^{-5/7} Ha^{2/7} V, Re^{-2/7} Ha^{-2/7} W, Re^{3/7} Ha^{-4/7} P) + \ldots \]
The magnetic field components in the lower wall layer are:

\[
B_x \sim Re^{-4/7}Ha^{-4/7}Rm \left[ (c_1 - \lambda A)Y - \frac{\lambda Y^2}{2} - \int_0^Y (U - \lambda s - \lambda A) \, ds \right] + ..., \\
B_y \sim Re^{-1}Rm \left[ (d_1 - V_0(X))Y + \lambda A'(X) \frac{Y^2}{2} - \int_0^Y (V + \lambda A'(X)s - V_0(X)) \, ds \right] + ..., \tag{6.37}
\]

\[
B_z \sim Re^{-4/7}Ha^{-4/7}Rm \left[ - \int_0^Y W \, ds \right] + ...
\]

Therefore, the equations in the lower wall layer are:

\[
U_X + V_Y + W_Z = 0, \\
UU_X + VU_Y + UW_Z = -P_X + U_{YY}, \tag{6.38}
\]

\[
P_Y = 0, \\
UW_X + VW_Y + WW_Z = -P_Z + W_{YY},
\]

with the boundary conditions

\[
U = V = W = 0 \text{ at } Y = hF_L(X, Z), \]

\[
U \sim \lambda (Y + A(X, Z)), \quad W \rightarrow D/Y \text{ with } \lambda D_X = -P_Z \text{ as } Y \rightarrow \infty, \tag{6.39}
\]

\[
U \rightarrow \lambda Y \text{ as } X \rightarrow -\infty.
\]

In the upper wall layer, the equation system is:

\[
\tilde{U}_X + \tilde{V}_Y + \tilde{W}_Z = 0, \\
\tilde{U}\tilde{U}_X + \tilde{V}\tilde{V}_Y + \tilde{W}\tilde{W}_Z = -\tilde{P}_X + \tilde{U}\tilde{Y}_Y, \tag{6.40}
\]

\[
\tilde{P}_Y = 0, \\
\tilde{U}\tilde{W}_X + \tilde{U}\tilde{W}_Y + \tilde{W}\tilde{W}_Z = -\tilde{P}_Z + \tilde{W}\tilde{Y}_Y,
\]

with the boundary conditions

\[
\tilde{U} = \tilde{V} = \tilde{W} = 0 \text{ at } \tilde{Y} = hF_U(X, Z), \]

\[
\tilde{U} \sim \lambda(\tilde{Y} - A(X, Z)), \quad \tilde{W} \rightarrow \tilde{D}/\tilde{Y} \text{ with } \lambda \tilde{D}_X = -\tilde{P}_Z \text{ as } \tilde{Y} \rightarrow \infty, \tag{6.41}
\]

\[
\tilde{U} \rightarrow \lambda \tilde{Y} \text{ as } X \rightarrow -\infty.
\]
6.3.2 Hartmann layers

The expansions in the lower Hartmann layer are:

\[ (x, y, z) \sim (Re^{1/7} Ha^{-6/7} X, Ha^{-1} \bar{y}, Re^{1/7} Ha^{-6/7} Z) \]

\[ (u, v, w, p - P_0) \sim (Ha^{-1} \bar{U}_0(\bar{y}), 0, 0, 0) \]

\[ + (Re^{-2/7} Ha^{-2/7} \bar{u}_1, Re^{-3/7} Ha^{-3/7} \bar{v}_1, Re^{-4/7} Ha^{3/7} \bar{w}_1 Re^{3/7} Ha^{-4/7} \bar{P}) + ..., \]  

(6.42)

where \( \bar{U}_0(\bar{y}) = \frac{1}{2} (1 - e^{-\bar{y}}) \) is the scaled Hartmann velocity in the lower Hartmann layer. The magnetic field components are:

\[ B_x \sim Ha^{-2} Rm \left[ -\lambda \bar{y} - \int_0^{\bar{y}} (\bar{U}_0 - \lambda) \, ds \right] + ..., \]

\[ B_y \sim Re^{-3/7} Ha^{-10/7} Rm \left[ \int_0^{\bar{y}} \bar{v}_1 \, ds \right] + ..., \]

\[ B_z \sim Re^{-4/7} Ha^{-4/7} Rm \left[ c_1 \bar{y} - \int_0^{\bar{y}} \bar{w}_1 \, ds \right] + ... \]  

(6.43)

The leading order equation system in the lower Hartmann layer:

\[ \ddot{u}_1 X + \dot{v}_1 \bar{y} = 0, \]

\[ \bar{U}_0 \ddot{u}_1 X + \bar{U}_0' \dot{v}_1 = 0, \]

\[ \bar{P}_y = 0, \]

\[ \bar{U}_0 \ddot{w}_1 X = -\bar{P}_Z, \]  

(6.44)

with the solution

\[ (\ddot{u}_1, \dot{v}_1, \ddot{w}_1) = \left( \bar{U}_0' A(X, Z), -\bar{U}_0 A_X(X, Z), -\frac{1}{\bar{U}_0} \int_{-\infty}^{\infty} \bar{P}_Z(s, Z) \, ds \right). \]  

(6.45)

This sets the matching condition \( U \to \lambda (Y + A(X, Z)) \) and \( W \to D(X, Z)/Y \) with \( \lambda D_X = -P_X(X, Z) \) between the lower wall layer and the lower Hartmann layer.
Similarly, in the upper Hartmann layer, the normal coordinate is $y = 1 - Ha^{-1} \hat{y}$ and the leading order equation system is:

\[
\begin{align*}
\hat{u}_1 X + \hat{v}_1 + \hat{w}_1 X &= 0, \\
\hat{U}_0 \hat{u}_1 X + \hat{U}'_0 \hat{v}_1 &= 0, \\
\hat{P}_y &= 0, \\
\hat{U}_0 \hat{w}_1 X &= -\hat{P}_Z,
\end{align*}
\]  

(6.46)

with the solution

\[
(\hat{u}_1, \hat{v}_1, \hat{w}_1) = \left(\hat{U}_0 B(X, Z), -\hat{U}_0 B(X, Z), -\frac{1}{\hat{U}_0} \int_{-\infty}^{\infty} \hat{P}_Z(s, Z) ds\right),
\]  

(6.47)

where the upstream velocity $\hat{U}_0(\hat{y})$ has the same expression as (4.11). The displacement function $B(X, Z)$ is the mirror image of $A(X, Z)$. The matching condition $\tilde{U} \to \lambda (\tilde{Y} + B(X, Z))$ and $\tilde{W} \to \tilde{D}(X, Z)/\tilde{Y}$ with $\lambda \tilde{D}_X = -\tilde{P}_X(X, Z)$ between the upper wall layer and the Hartmann layer sets the far-field condition for the upper wall layer.

6.3.3 Core

In the channel core, the normal coordinate $y$ is in the interval $(0, 1)$ and is of order one. The expansions are:

\[
(x, y, z) \sim (Re^{1/7} Ha^{-6/7} X, y, Re^{1/7} Ha^{-6/7} Z),
\]

\[
(u, v, w, p - P_0) \sim \left(\frac{1}{2Ha}, 0, 0, 0\right)
\]

\[
+ (Re^{-2/7} Ha^{-9/7} u_1, Re^{-3/7} Ha^{-3/7} v_1, Re^{-4/7} Ha^{3/7} w_1, Re^{3/7} Ha^{-4/7} p_1) + ..., \\
B_x \sim Ha^{-1} Rm(-\lambda y) + Re^{-2/7} Ha^{-9/7} Rm \left[c_1 y - c_2 - \int_0^y u_1 ds\right] + ..., \\
B_y \sim 1 + Re^{-3/7} Ha^{-3/7} Rm \left(\frac{A'(X)}{2}\right) + ..., \\
B_z \sim Re^{-4/7} Ha^{3/7} Rm \left[c_1 y - \int_0^y w_1 ds\right] + ...
\]

(6.48)
The leading order equation system in the core is:

\[
\tilde{P}(X, Z) = P(X, Z) + \frac{1}{4} A_{XX},
\]

\[
p_1(X, 0, Z) = \tilde{P}(X, Z),
\]

\[
p_1(X, 1, Z) = \hat{P}(X, Z).
\]

(6.49)

6.3.4 Linearization for shallow wall geometry

The system of equations are identical to Smith’s 3D model, only difference being in the value of \( \kappa \). Hence the solutions are obtained in same manner as in the previous section with the \( \kappa \) value being replaced by \( 1/4 \).

6.4 MHD flow with strong magnetic field

Setting \( Ha = Re^{1/6} H \) as in section 5.2, \( x \) and \( z \) become equal to \( Re^{1/6} X \) and \( Re^{1/6} Z \), respectively.

6.4.1 Wall layers

The expansions in the lower wall layer are therefore:

\[
(x, y, z) = (X, Re^{-1/3} Y, Z),
\]

\[
(u, v, w, p - P_0) \sim (Re^{-1/3} U, Re^{-2/3} V, Re^{-1/3} W, Re^{1/3} P) + ...,\]

\[
B_x \sim Re^{-2/3} Rm \left[ -\frac{Y^2}{2} - \int_0^Y (U - \lambda s - \lambda A) \, ds \right] + ...,\]

(6.50)

\[
B_y \sim 1 + Re^{-1} Rm \left[ \lambda A'(X) \frac{Y^2}{2} - \int_0^Y (V + \lambda A'(X) s - V_0(X)) \, ds \right] + ...,\]

\[
B_z \sim Re^{-2/3} Rm \left[ - \int_0^Y W \, ds \right] + ...,\]

where the velocity \( V_0(X) \) is evaluated from the \( x \)-momentum equation \( \lambda^2 A A_X + \lambda V_0 = -P_X \).

Substituting all the expansions into the governing equations (6.1) - (6.3), the leading order equations
in the lower wall layer are obtained:

\[ U_X + V_Y + W_Z = 0, \]
\[ UU_X + VU_Y + WU_Z = -P_X + U_{YY}, \]
\[ P_Y = 0, \]
\[ UW_X + VW_Y + WW_Z = -P_Z + W_{YY}, \]

along with the no-slip conditions at the wall \( U(X, hF_L(X, Z), Z) = V(X, hF_L(X, Z), Z) = W(X, hF_L(X, Z), Z) = 0 \).

The equation system corresponding to the upper wall layer is:

\[ \tilde{U}_X + \tilde{V}_Y + \tilde{W}_Z = 0, \]
\[ \tilde{U}\tilde{U}_X + \tilde{U}\tilde{V}_Y + \tilde{W}\tilde{U}_Z = -\tilde{P}_X + \tilde{U}\tilde{V}_Y, \]
\[ \tilde{P}_Y = 0, \]
\[ \tilde{U}\tilde{W}_X + \tilde{U}\tilde{W}_Y + \tilde{W}\tilde{W}_Z = -\tilde{P}_Z + \tilde{W}\tilde{V}_Y, \]

along with the no-slip conditions at the wall \( \tilde{U}(X, hF_U(X, Z), Z) = \tilde{V}(X, hF_U(X, Z), Z) = \tilde{W}(X, hF_U(X, Z), Z) = 0 \).

### 6.4.2 Hartmann layers

Substitution of the \( Ha \) value given by (5.1) into the Hartmann layers (6.42) gives the expansions:

\[ (x, y, z) = (X, Re^{-1/6}\tilde{y}, Z), \]
\[ (u, v, w, p - P_0) \sim (Re^{-1/6}\tilde{U}_0(\tilde{y}), 0, 0, 0) + (Re^{-1/3}\tilde{u}_1, Re^{-1/2}\tilde{v}_1, Re^{-1/2}\tilde{w}_1, Re^{1/3}\tilde{P}) + ..., \]

where the upstream flow velocity is:

\[ \tilde{U}_0(\tilde{y}) = \frac{1}{2H} (1 - \exp(-H\tilde{y})). \]
The magnetic fields are given by:

\[ B_x \sim Re^{-1/3} Rm \left[ -\lambda \bar{y} - \int_0^y (\bar{U}_0 - \lambda) \, ds \right] + ..., \]
\[ B_y \sim 1 + Re^{-2/3} Rm \left[ \int_0^y \bar{v}_1 \, ds \right] + ..., \]  
(6.55)
\[ B_z \sim Re^{-2/3} Rm \left[ -\int_0^y \bar{w}_1 \, ds \right] + ... \]

Therefore, the leading order equation system in the lower Hartmann layer is:

\[ \bar{u}_{1X} + \bar{v}_{1Y} = 0, \]
\[ \bar{U}_0 \bar{u}_{1X} + \bar{U}_0' \bar{v}_1 = 0, \]  
(6.56)
\[ \bar{P}_y = 0, \]
\[ \bar{U}_0 \bar{w}_{1X} = -\bar{P}_Z, \]

with the same solution as (6.45), setting the matching conditions \( U \rightarrow \lambda(Y + A(X, Z)) \) and \( W \rightarrow D(X, Z)/Y \) with \( \lambda D_X = -P_X(X, Z) \) between the lower wall layer and the lower Hartmann layer.

The system in the upper Hartmann layer is same as (6.46) with solution (6.47).

### 6.4.3 Core

The normal coordinate \( y \) in the channel core belongs to the interval \((0, 1)\) and is of order one.

The upstream \( u \)-velocity \((1/2Ha)\) becomes \(1/(2H)\) with a scale factor of \(Re^{-1/6}\) by using the relation (5.3). Therefore, the expansion in the core are:

\[ (u, v, w, p - P_0) \sim \left( Re^{-1/6}, \frac{1}{2H}, 0, 0, 0 \right) + (Re^{-1/2}u_1, Re^{-1/2}v_1, Re^{-1/2}w_1, Re^{1/3}p_1) + ... \]  
(6.57)

The magnetic field components are:

\[ B_x \sim Re^{-1/2} B_X + ..., \]
\[ B_y \sim 1 + Re^{-1/2} B_Y + ..., \]  
(6.58)
\[ B_z \sim 1 + Re^{-1/2} B_Z + ..., \]
where the scaled components $B_X, B_Y$ and $B_Z$ satisfy the equations

\begin{align}
Rm^{-1} \left[ \frac{\partial^2 B_X}{\partial X^2} + \frac{\partial^2 B_X}{\partial y^2} + \frac{\partial^2 B_X}{\partial z^2} \right] + u_{1y} &= 0, \\
Rm^{-1} \left[ \frac{\partial^2 B_Y}{\partial X^2} + \frac{\partial^2 B_Y}{\partial y^2} + \frac{\partial^2 B_Y}{\partial z^2} \right] + v_{1y} &= 0, \\
Rm^{-1} \left[ \frac{\partial^2 B_Z}{\partial X^2} + \frac{\partial^2 B_Z}{\partial y^2} + \frac{\partial^2 B_Z}{\partial z^2} \right] + w_{1y} &= 0.
\end{align}

(6.59)

The leading order equations obtained in the core by substituting all the expansions into the governing equations (2.4)-(2.6) are:

\begin{align}
&u_{1X} + v_{1y} + w_{1Z} = 0, \\
&\frac{1}{2H} u_{1X} = -p_{1X}, \\
&\frac{1}{2H} v_{1X} = -p_{1y}, \\
&\frac{1}{2H} w_{1X} = -p_{1Z}.
\end{align}

(6.60)

These equations when solved simultaneously give a Laplacian of pressure:

\begin{align}
p_{1XX} + p_{1yy} + p_{1ZZ} &= 0.
\end{align}

(6.61)

Solving the continuity and the $x$-momentum equations in the core simultaneously and using the fact that the $v$-perturbation velocity remains constant across the Hartmann layer-core interface give the boundary conditions for the $y$-derivative of the pressure:

\begin{align}
p_{1y}(X,0,Z) &= \frac{A_{XX}}{4H^2}, \\
p_{1y}(X,1,Z) &= -\frac{B_{XX}}{4H^2},
\end{align}

(6.62)

where the displacement functions $A(X,Z)$ and $B(X,Z)$ are related by the relation:

\begin{align}
A + B = 4H^2 \int_0^1 p_1(X,s,Z) \, ds.
\end{align}

(6.63)

This relation is obtained by integrating the $y$-momentum equation and using the boundary conditions along with the upstream condition for the perturbation velocities and pressure.

Hence it can be concluded that the core flow has changed. Instead of the pressure being a linear function of the normal coordinate as in the hydrodynamic flow and the MHD flow for moderate magnetic field strength, the core is governed by a Laplace equation in pressure, as in the 2D flow.
6.4.4 Linearization for shallow wall geometry

The linearized wall layer systems are similar to (6.24) and (6.25). The perturbed core pressure becomes:

\[ P \sim h p_c(X, y, Z) + \ldots \]

and the core is controlled by the linear system:

\[ q_{XX} + q_{yy} + q_{ZZ} = 0, \]

\[ q_y(X, 0, Z) = \frac{a_{XXXX} + a_{XZZ}}{4H^2}, \]

\[ q_y(X, 1, Z) = -\frac{b_{XXXX} + b_{XZZ}}{4H^2}, \]

\[ q(X, 0, Z) = Q_L(X, Z), \quad q(X, 1, Z) = Q_U(X, Z), \]

\[ \Delta(a + b) = \frac{1}{4H^2} \int_0^1 q(X, s, Z) ds, \]

where \( q \) is the Laplacian of \( p_c \). The Fourier transformed equation system in the lower wall and upper wall layers are identical to (6.29) and (6.30). Thus the solutions are same as (6.32) and (6.33). The Fourier transformed equation system in the core is:

\[ q_{yy}^{*} - (\alpha^2 + \beta^2)q^* = 0, \]

\[ q_y^{*}(\alpha, 0, \beta) = (\alpha^4 + \alpha^2\beta^2) \frac{a^*}{4H^2}, \]

\[ q_y^{*}(\alpha, 1, \beta) = -(\alpha^4 + \alpha^2\beta^2) \frac{b^*}{4H^2}, \]

\[ q^*(\alpha, 0, \beta) = Q_L^*, \quad q^*(\alpha, 1, \beta) = Q_U^*, \]

\[ (\alpha^2 + \beta^2)(a^* + b^*) = \frac{1}{4H^2} \int_0^1 q^*(\alpha, s, \beta) ds, \]

where the solution \( q^* = c_1(\alpha, \beta)e^{\sqrt{\alpha^2 + \beta^2}y} + c_2(\alpha, \beta)e^{-\sqrt{\alpha^2 + \beta^2}y} \), the functions \( c_1 \) and \( c_2 \) being unknown. The procedure used to solve these functions and thereby solve the system is very much similar to the one used in the two-dimensional situation given in the section 5.4. The linear system
is given by:

\[
\left( \theta \coth \left( \sqrt{\alpha^2 + \beta^2} \right) + \frac{\alpha^2 \sqrt{\alpha^2 + \beta^2}}{4H^2} \right) a^* - \frac{\theta}{\sinh \left( \sqrt{\alpha^2 + \beta^2} \right)} b^* \\
= \frac{\theta}{\sinh \left( \sqrt{\alpha^2 + \beta^2} \right)} \left[ F_U^* - F_L^* \cosh \left( \sqrt{\alpha^2 + \beta^2} \right) \right], \\
- \frac{\theta}{\sinh \left( \sqrt{\alpha^2 + \beta^2} \right)} a^* + \left( \theta \coth \left( \sqrt{\alpha^2 + \beta^2} \right) + \frac{\alpha^2 \sqrt{\alpha^2 + \beta^2}}{4H^2} \right) b^* \\
= \frac{\theta}{\sinh \left( \sqrt{\alpha^2 + \beta^2} \right)} \left[ F_L^* - F_U^* \cosh \left( \sqrt{\alpha^2 + \beta^2} \right) \right],
\]

where the matrix

\[
M = \begin{pmatrix}
\left( \theta \coth \left( \sqrt{\alpha^2 + \beta^2} \right) + \frac{\alpha^2 \sqrt{\alpha^2 + \beta^2}}{4H^2} \right) & -\frac{\theta}{\sinh \left( \sqrt{\alpha^2 + \beta^2} \right)} \\
-\frac{\theta}{\sinh \left( \sqrt{\alpha^2 + \beta^2} \right)} & \left( \theta \coth \left( \sqrt{\alpha^2 + \beta^2} \right) + \frac{\alpha^2 \sqrt{\alpha^2 + \beta^2}}{4H^2} \right)
\end{pmatrix}
\]

on the left hand side of the equation is non-singular.
CHAPTER 7. CONCLUSION

In this thesis, it was shown that the MHD flow structure is similar to the hydrodynamic structure but on a shorter stream-wise length scale. A new flow structure, with stream-wise length scale proportional to the channel width, evolves as the magnetic field strength is increased.

In Chapter 3, linear solutions for the hydrodynamic structure have been obtained using a Fourier transform. Non-linear solutions have also been obtained using the Thomas algorithm, which is a modified form of Gaussian elimination used for tri-diagonal systems for equations. Comparison between the linear and non-linear solutions at very small heights shows good agreement. A magnetic field of small, constant strength is introduced into the system and it is found that the flow behavior remains essentially unchanged.

In Chapter 4, the magnetic field strength is gradually increased. It is seen that the flow structure is retained as the scales of the wall distortion as well as the velocity and pressure are decreased 4.3. Linear and non-linear solutions are shown to match at very small heights of the wall distortion. It is clear, even from the solutions, that the flow behaviour remains similar to the hydrodynamic flow on a shorter length scale. The linear free interaction, used to check flow properties, is found to match the linear free interaction of the hydrodynamic structure as the magnetic field strength is decreased.

In Chapter 5, a new flow structure is obtained when the magnetic field strength reaches the point 5.1. The main change is seen in the core of the flow which affects the way the two wall layers couple. In the previous two structures, the wall layer pressures are seen to interact through a linear pressure-displacement interaction. In this new structure, the coupling takes place via the core pressure, the Laplace equation of which governs the core. The flow change is further verified by linear solutions and linear free interaction.
In Chapter 6, a start has been made to extend the two dimensional analysis to three dimensions by including a span-wise length scale of same order as stream-wise length scale. It was seen that the flow structures overall remain similar to two dimensions. Linearization using Fourier transform have been discussed. Obtaining linear solutions and comparing with two dimensional solutions remain as a part of future work.
BIBLIOGRAPHY


APPENDIX. ADDITIONAL MATERIAL

In this section, we have detailed derivations of the relations and expressions discussed in the thesis.

1. Hydrodynamic flow

The perturbation velocities \( u_1 \) and \( v_1 \) in the hydrodynamic structure are obtained by substituting the core expansions

\[
(u, v, p - P_0) = (U_0, 0, 0) + (Re^{-1/3} \Delta^{1/3} u_1, Re^{-1/3} \Delta^{-2/3} v_1, Re^{1/3} \Delta^{2/3} p_1).
\]

(.1)

into the governing equations

\[
\nabla \cdot \mathbf{V} = 0, \quad \mathbf{V} \cdot \nabla \mathbf{V} = Re^{-1}[-\nabla p + \nabla^2 \mathbf{V}].
\]

(.2)

The leading order equations obtained are:

\[
\begin{align*}
  u_1X + v_1y &= 0, \\
  U_0u_1X + U'_0v_1 &= 0,
\end{align*}
\]

(.3)

Solving the two equations simultaneously, we obtain:

\[
\begin{align*}
  U'_0v_1 - U_0v_1y &= 0, \\
  \Rightarrow U_0^2 \frac{\partial}{\partial y} (\frac{v_1}{U_0}) &= 0, \\
  \Rightarrow v_1 &= -U_0(y)A'(X), \quad u_1 = U'_0(y)A(X),
\end{align*}
\]

(.4)

as shown in (2.18).
1.1. Linear free interaction

For the linear free interaction analysis, a perturbation of magnitude $\epsilon$ is introduced to the wall layers. In the lower wall layer, the expansions are:

$$
(U, V, P) = (\lambda Y, 0, 0) + \epsilon (u_p(Y), v_p(Y), p_p) e^{\theta X} + O(e^{2\theta X}),
$$

$$
A(X) = \epsilon a_p e^{\theta X} + O(e^{2\theta X}), \quad (5)
$$

where $a_p = \pm 1$. Substituting these expansions in the lower wall layer equation system given by

$$
U_X + V_Y = 0,
$$

$$
U U_X + V U_Y = -P_{1X} + U_{YY},
$$

$$
P_{1Y} = 0, \quad (6)
$$

the linear system obtained is:

$$
v'_p = -\theta u_p, \quad (7)
$$

$$
\lambda \theta Y u_p + \lambda v_p = -\theta p_p + u''_p,
$$

with boundary conditions: $u_p(0) = v_p(0) = 0$, $u_p(Y) \to \lambda a_p$ as $Y \to \infty$. Differentiating the momentum equation in (7) with respect to $Y$ we get the Airy equation in $u'_p$

$$
u''_p - \lambda \theta Y u'_p = 0 \quad (8)
$$

$$\implies u_p(Y) = C_1 \int_0^Y \text{Ai} (ms) \, ds + C_2 \int_0^Y \text{Bi} (ms) \, ds, \quad \text{where } m = (\lambda \theta)^{1/3}. \quad (9)
$$

Since $u_p$ is finite as $Y \to \infty$, the constant $C_2$ is set to zero to nullify the growing effect of Bi $(Y)$. Therefore the perturbation velocity $u_p(Y)$ is equal to $C_1 \int_0^Y \text{Ai} (ms) \, ds$. Let the integral $\int_0^Y \text{Ai} (ms) \, ds = f'_1(Y)$. Therefore, $u_p(Y)$ can be written as $C_1 f'_1(Y)$.

From (7) we get $v_p(Y) = -\theta C_1 f_1(Y)$ since $f_1(0) = 0$. 

From the far-field condition \( u_p(Y) \to \lambda a_p \) as \( Y \to \infty \) we get:

\[
C_1 \int_0^\infty \Ai(ms) \, ds = \lambda a_p \quad \Rightarrow \quad \lambda a_p = \frac{C_1}{3m}
\]

\[
\Rightarrow \quad \frac{\lambda a_p}{C_1} = \frac{1}{3m}.
\] (10)

From \((.7)\), at \( Y = 0 \)

\[
\theta p_p = u_{p}''(0) \quad \Rightarrow \quad p_p = \frac{C_1 m}{\theta} \Ai'(0), m = (\lambda \theta)^{1/3}.
\] (11)

Similarly, in the upper wall layer the expansions are:

\[
(\tilde{U}, \tilde{V}, \tilde{P}) = (\lambda \tilde{Y}, 0, 0) + \epsilon(\tilde{u}_p(\tilde{Y}), \tilde{v}_p(\tilde{Y}), \tilde{p}_p)e^{\theta X} + O(e^{2\theta X}),
\] (12)

with boundary conditions: \( \tilde{u}_p(0) = \tilde{v}_p(0) = 0, \tilde{u}_p(\tilde{Y}) \to -\lambda a_p \) as \( \tilde{Y} \to \infty \). The perturbation velocity and pressure are:

\[
\tilde{u}_p(\tilde{Y}) = \tilde{C}_1 \int_0^{\tilde{Y}} \Ai(ms) \, ds = \tilde{C}_1 \tilde{f}_1(\tilde{Y}),
\]

\[
\tilde{v}_p(\tilde{Y}) = -\theta \tilde{C}_1 \tilde{f}_1(\tilde{Y}),
\]

\[
\tilde{p}_p = \frac{\tilde{C}_1 m}{\theta} \Ai'(0),
\] (13)

where \( \int_0^{\tilde{Y}} \Ai(ms) \, ds = \tilde{f}_1(\tilde{Y}) \). The displacement function and the constant \( \tilde{C}_1 \) has the relation

\[
\frac{\lambda a_p}{\tilde{C}_1} = -\frac{1}{3m}.
\] (14)

From the core pressure-displacement interaction \( \tilde{P}(X, y) = P(X) + A''(X) \int_0^y U_0^2(s) \, ds \) we get the linear equation:

\[
\tilde{p}_p = p_p + \frac{\theta^2}{120} a_p \quad \Rightarrow \quad \frac{\tilde{C}_1 m}{\theta} \Ai'(0) = \frac{C_1 m}{\theta} \Ai'(0) + \frac{\theta^2}{120} a_p
\]

\[
\Rightarrow \quad \tilde{C}_1 = C_1 + \frac{\theta^3}{120 m \Ai'(0) 3\lambda m}
\]

\[
\Rightarrow \quad \frac{\tilde{C}_1}{C_1} = 1 + \frac{2\theta^{7/3}}{360 \lambda^{2/3} \Ai'(0)}.
\] (15)
From (.10) and (.14) we get:

\[
\frac{\theta^{7/3}}{360 \lambda^{2/3} A i'(0)} = -1
\]

\[
\Rightarrow \theta = 2[-45 A i'(0)]^{3/7} \approx 5.727...
\]

1.2. Thomas algorithm

The Thomas algorithm, also known as tri-diagonal matrix algorithm, is used to solve linear systems, often as part of an iterative scheme for non-linear systems of equations. It is a simplified form of Gaussian elimination and can be used on tri-diagonal system of equations. A tri-diagonal system for \( M \) unknowns can be written as:

\[
A_j X_{j-1} + B_j X_j + C_j X_{j+1} = D_j, \quad j = 1, ..., M - 1
\]

(.17)

with boundary conditions:

\[
A_L X_1 + B_L X_2 = D_L,
\]

\[
A_U X_M + B_U X_{M-1} = D_U.
\]

(.18)

The tri-diagonal system hence obtained is:

\[
\begin{bmatrix}
A_L & B_L & 0 & \cdots & 0 \\
A_2 & B_2 & C_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & C_{M-1} \\
\vdots & \vdots & \vdots & 0 & B_U & A_U
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_{M-1} \\
X_M
\end{bmatrix}
= \begin{bmatrix}
D_L \\
D_2 \\
\vdots \\
D_{M-1} \\
D_U
\end{bmatrix}
\]

The tri-diagonal system is solved using a recursion relation:

\[
X_j = R_j X_{j-1} + S_j,
\]

(.19)

which when substituted in (.17) gives the recursive relation

\[
R_j = -(B_j + C_{j-1} R_{j+1})^{-1} A_j, \quad S_j = (B_j + C_{j-1} R_{j+1})^{-1} (D_j - C_j S_{j+1}).
\]

(.20)
This recursive relation is started by using the values of $R_M$ and $S_M$ which are obtained from the second boundary condition in (.18)

$$
R_M = -AU^{-1}BU, \quad S_M = AU^{-1}DU.
$$

In order to solve for the unknowns $X_j$, the value of $X_1$ is used which is found using the relation:

$$
X_1 = (AL + BL \times R_2)^{-1}(DL - BL \times S_2).
$$

1.2.1. Linear solution method

The equation system in the lower wall layer after application of Prandtl’s transposition is:

$$
UX + \hat{V}\eta = 0,

UU_X + \hat{V}U_\eta = -P_X + U_{\eta\eta},

P_\eta = 0,
$$

with the boundary conditions: $U(X, 0) = \hat{V}(X, 0) = 0$, where $\hat{V} = V - hUF_L'$, $U \to \lambda(\eta + hF_L)$ as $\eta \to \infty$.

In order to linearise it, a perturbation of magnitude $h << 1$ is introduced to the system. Hence the expansions are:

$$(U, V, P, A) = (\lambda\eta, 0, 0, 0) + h(u_L(X, \eta), v_L(X, \eta), p_L(X), a(X)) + ...$$

The linear system obtained by substituting the above expansions in (.23) is:

$$
u_{L,X} + v_{L,\eta} = 0

\lambda\eta u_LX + \lambda v_L = -p_L' + u_{L,\eta\eta},
$$

with no-slip conditions $u_L(X, 0) = 0$, $v_L(X, 0) = 0$ and far-field condition $u_L \to \lambda(F_L(X) + a(X))$ as $\eta \to \infty$.

Similarly, in the upper wall layer the linear system is:

$$
\tilde{u}_{U,X} + \tilde{u}_{U,\tilde{\eta}} = 0,

\lambda\tilde{\eta}\tilde{u}_{U,X} + \lambda\tilde{u}_U = -\tilde{p}_U' + \tilde{u}_{U,\tilde{\eta}\tilde{\eta}},
$$
with no-slip conditions \( \tilde{u}_U(X,0) = 0, \tilde{v}_U(X,0) = 0 \) and far-field condition \( \tilde{u}_U \to \lambda(F_U(X) - a(X)) \) as \( \tilde{\eta} \to \infty \).

The two wall layers are connected by the equation:

\[
\tilde{p}_U = p_L + \kappa a''.
\] (27)

A second-order finite differencing scheme is used. Any point in the rectangular grid \((X, \eta)\) is represented as \((X_i, \eta_j)\) where \(i = 0, ..., M\) and \(j = 0, ..., N_t\). The lower wall layer is indexed by \(j = 0, ..., N\), and upper wall layer by \(j = N + 1, ..., N_t\).

The \(u_{L,X}\) term in the continuity equation is finite differenced about \((i, j - \frac{1}{2})\) in a second order backward scheme. Therefore, the continuity equation is:

\[
u_{L,X} + v_{L,\eta} = 0
\]

\[
\implies \frac{3u_{L,i,j-1}}{4\Delta X} + \frac{3u_{L,i,j}}{4\Delta X} - \frac{v_{L,i,j-1}}{\Delta \eta} + \frac{v_{L,i,j}}{\Delta \eta} = \frac{u_{L,i-1,j} + u_{L,i-1,j-1}}{\Delta X} - \frac{u_{L,i-2,j} + u_{L,i-2,j-1}}{4\Delta X}.
\]

Hence the system in lower wall layer is:

\[
\frac{3u_{L,i,j-1}}{4\Delta X} - \frac{v_{L,i,j-1}}{\Delta \eta} + \frac{3u_{L,i,j}}{4\Delta X} + \frac{v_{L,i,j}}{\Delta \eta} = \frac{u_{L,i-1,j} + u_{L,i-1,j-1}}{\Delta X} - \frac{u_{L,i-2,j} + u_{L,i-2,j-1}}{4\Delta X},
\]

\[
\frac{1}{(\Delta \eta)^2} u_{L,i,j-1} + \left( \frac{3\lambda \eta_j}{2\Delta X} + \frac{2}{(\Delta \eta)^2} \right) u_{L,i,j} + \lambda v_{L,i,j} + \frac{3}{2\Delta X} p_{L,i,j} - \frac{1}{(\Delta \eta)^2} u_{L,i,j+1} = \lambda \eta_j \left( \frac{4u_{L,i-1,j} - u_{L,i-2,j}}{2\Delta X} \right) + \frac{4p_{L,i-1,j} - p_{L,i-2,j}}{2\Delta X},
\]

\[
-p_{L,i,j} + p_{L,i,j+1} = 0.
\]
1.2.2. Non-linear solution method

The non-linear equation system in the lower wall layer after application of Prandtl’s transposition is:

\[ U_X + \hat{V}_{\eta} = 0, \]
\[ UU_X + \hat{V} U_{\eta} = -P_X + U_{\eta\eta}, \]
\[ P_{\eta} = 0, \]

with the boundary conditions: \( U(X,0) = \hat{V}(X,0) = 0 \), where \( \hat{V} = V - hUF'_L \), \( U \rightarrow \lambda(\eta + hF_L) \) as \( \eta \rightarrow \infty \).

A second-order finite differencing scheme is used. Any point in the rectangular grid \((X,\eta)\) is represented as \((X_i,\eta_j)\) where \( i = 0, ..., M \) and \( j = 0, ..., N \). The lower wall layer is indexed by \( j = 0, ..., N \), and upper wall layer by \( j = N + 1, ..., N_t \).

The \( U_X \) term in the continuity equation is finite differenced about \((i, j - 1/2)\) in a second order backward scheme. Therefore, the continuity equation is:

\[ \frac{3U_{i,j-1}}{4\Delta X} + \frac{3U_{i,j}}{4\Delta X} - \frac{\hat{V}_{i,j-1}}{\Delta \eta} + \frac{\hat{V}_{i,j}}{\Delta \eta} = \frac{U_{i-1,j} + U_{i-1,j-1}}{\Delta X} \]
\[ - \frac{U_{i-2,j} + U_{i-2,j-1}}{4\Delta X}. \]

The \( UU_X \) term in the X-momentum equation is linearised using Newton linearization and becomes:

\[ UU_X \sim U^9U^9_X + (U - U^9)U^9_X + (U_X - U^9_X)U^9 \]
\[ = UU^9_X + U^9U_X - U^9U^9_X \]
\[ = U_{i,j}U^9_X + U^9 \frac{3U_{i,j} - 4U_{i-1,j} + U_{i-2,j}}{2\Delta X} - U^9U^9_X. \]
The \( X \)-momentum equation becomes:

\[
\omega \ast UU_X + \hat{V}U_\eta = -P_X + U_\eta X
\]

\[
\implies \omega \left[ U_{i,j}U'_{X} + U'g 3U_{i,j} - 4U_{i-1,j} + U_{i-2,j} - U'^g U_X \right] + \hat{V}_{i,j}U'_{\eta} + \hat{V}g U_{i,j+1} - U_{i,j-1} - \hat{V}g U'_{\eta}
\]

\[= - \frac{3P_{i,j} - 4P_{i-1,j} + P_{i-2,j} + U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{2\Delta \eta} \]

\[\implies \left( -\frac{\hat{V}g}{2\Delta \eta} - \frac{1}{(\Delta \eta)^2} \right) U_{i,j-1} + \left( \omega \left( U'^g + \frac{3U'^g}{2\Delta X} \right) + \frac{2}{(\Delta \eta)^2} \right) U_{i,j} + \hat{V}g \hat{V}_{i,j} + \frac{3}{2\Delta X} P_{i,j}
\]

\[
+ \left( \frac{\hat{V}g}{2\Delta \eta} - \frac{1}{(\Delta \eta)^2} \right) U_{i,j+1}
\]

\[= \omega U'^g \left( \frac{4U_{i-1,j} - U_{i-2,j}}{2\Delta X} + U'^g X \right) + \hat{V}g U'^g_{\eta} + \frac{4P_{i-1,j} - P_{i-2,j}}{2\Delta X}.
\]

The \( \eta \)-momentum equation is:

\[-P_\eta = 0
\]

\[\implies -P_{i,j} + P_{i,j+1} = 0.
\]

Writing these equations in matrix form we get:

\[
\begin{bmatrix}
\frac{3}{2\Delta \eta} & 0 & 0  \\
\frac{\hat{V}g}{2\Delta \eta} - \frac{1}{(\Delta \eta)^2} & 0 & 0  \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
U_{i,j-1}  \\
\hat{V}_{i,j-1}  \\
P_{i,j-1}
\end{bmatrix}
+ \omega \left( U'^g + \frac{3U'^g}{2\Delta \eta} \right) \begin{bmatrix}
0  \\
0  \\
0
\end{bmatrix}
\begin{bmatrix}
U_{i,j}  \\
\hat{V}_{i,j}  \\
P_{i,j}
\end{bmatrix}
+ \left( \frac{\hat{V}g}{2\Delta \eta} - \frac{1}{(\Delta \eta)^2} \right) \begin{bmatrix}
0  \\
0  \\
0
\end{bmatrix}
\begin{bmatrix}
U_{i,j+1}  \\
\hat{V}_{i,j+1}  \\
P_{i,j+1}
\end{bmatrix}
\]

\[
= \omega U'^g \left( \frac{4U_{i-1,j} - U_{i-2,j}}{2\Delta X} + U'^g X \right) + \hat{V}g U'^g_{\eta} + \frac{4P_{i-1,j} - P_{i-2,j}}{2\Delta X}
\]

where the unknown \( X_j \) is equal to \begin{bmatrix}
U_j  \\
\hat{V}_j  \\
P_j
\end{bmatrix}.
Hence in the recursion relation \( A_j X_{j-1} + B_j X_j + C_j X_{j+1} = D_j \) each of the matrices are given by:

\[
A_j = \begin{bmatrix}
\frac{3}{4\Delta X} & -\frac{1}{\Delta \eta} & 0 \\
-\frac{\hat{V}_g}{2\Delta \eta} - \frac{1}{(\Delta \eta)^2} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B_j = \begin{bmatrix}
\frac{3}{4\Delta X} & \frac{1}{\Delta \eta} & 0 \\
\omega \left( U_X^g + \frac{3U_Y^g}{2\Delta X} \right) + \frac{2}{(\Delta \eta)^2} U_Y^g & \frac{3}{2\Delta X} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
C_j = \begin{bmatrix}
0 & 0 & 0 \\
\left( \hat{V}_g - \frac{1}{(\Delta \eta)^2} \right) & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad D_j = \begin{bmatrix}
U_{i-1,j} + U_{i-1,j-1} \Delta X & -U_{i-2,j} + U_{i-2,j-1} \Delta X \\
\omega U_X^g \left( \frac{4U_{i-1,j} - U_{i-2,j} + U_X^g}{2\Delta X} \right) + \hat{V}_g U_Y^g & \frac{4P_{i-1,j} - P_{i-2,j}}{2\Delta X}
\end{bmatrix}
\]

The lower boundary condition \( U_{i,0} = \hat{V}_{i,0} = 0, -P_{i,0} + P_{i,0} = 0 \) written in matrix form becomes:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
U_{i,0} \\
\hat{V}_{i,0} \\
P_{i,0}
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
U_{i,1} \\
\hat{V}_{i,1} \\
P_{i,1}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

with \( AU = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \), \( BU = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \) and \( DU = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \).

The lower wall layer-core interface is governed by the continuity, \( X \)-momentum equations and the far-field condition coupled with the pressure-displacement interaction. The coupling between the far-field condition \( U \rightarrow \lambda(\eta + hF_L) \) and the pressure-displacement interaction \( \tilde{P}(X,y) = P(X) + A''(X) \int_0^y U_0^2(s) \, ds \) is done through the displacement function \( A(X) \). It is given by the formula

\[
A_i = \frac{\Delta X^2}{2\kappa} \left[ P_{i,N} - P_{i,N+1} \right] + \frac{A_{i-1} + A_{i+1}}{2}
\]
Hence the equations at the interface are:

\[
\begin{align*}
U_{i,N} - \frac{\lambda \Delta X^2}{2\kappa} P_{i,N} + \frac{\lambda \Delta X^2}{2\kappa} P_{i,N+1} &= \lambda \eta_N + h F_L(i) + \frac{A_{i-1} + A_{i+1}}{2}, \\
\left( \frac{3U^g}{2\Delta X} + U^g_X \right) U_{i,N} + \lambda \hat{V}_{i,N} + \frac{3}{2\Delta X} P_{i,N} &= U^g \left( \frac{4U_{i-1,N} - U_{i-2,N}}{2\Delta X} + U^g_X \right) + \frac{4P_{i-1,N} - P_{i-2,N}}{2\Delta X}, \\
\frac{3U_{i,N-1}}{4\Delta X} - \frac{\hat{V}_{i,N-1}}{\Delta \eta} + \frac{3U_{i,N}}{4\Delta X} + \frac{\hat{V}_{i,N}}{\Delta \eta} &= \frac{U_{i-1,N} + U_{i-1,N-1}}{\Delta X} - \frac{U_{i-2,N} + U_{i-2,N-1}}{4\Delta X}.
\end{align*}
\] (31)

In matrix form, these equations are given by:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{3}{4\Delta X} & -1 & 0
\end{bmatrix}
\begin{bmatrix}
U_{i,N-1} \\
\hat{V}_{i,N-1} \\
P_{i,N-1}
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 & -\frac{\lambda \Delta X^2}{2\kappa} \\
\frac{3U^g}{2\Delta X} + U^g_X & \lambda & \frac{3}{2\Delta X} \\
\frac{3}{4\Delta X} & \frac{1}{\Delta \eta} & 0
\end{bmatrix}
\begin{bmatrix}
U_{iN} \\
\hat{V}_{iN} \\
P_{iN}
\end{bmatrix}

+ \begin{bmatrix}
0 & 0 & \frac{\lambda \Delta X^2}{2\kappa} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
U_{iN+1} \\
\hat{V}_{iN+1} \\
P_{iN+1}
\end{bmatrix}

= \begin{bmatrix}
\lambda \eta_N + h F_L(i) + \frac{A_{i-1} + A_{i+1}}{2} \\
U^g \left( \frac{4U_{i-1,N} - U_{i-2,N}}{2\Delta X} + U^g_X \right) + \frac{4P_{i-1,N} - P_{i-2,N}}{2\Delta X} \\
\frac{U_{i-1,N} + U_{i-1,N-1}}{\Delta X} - \frac{U_{i-2,N} + U_{i-2,N-1}}{4\Delta X}
\end{bmatrix}
\].

In the upper wall layer, the equations look similar to the lower wall layer’s but the direction is inverted. Hence any point indexed as \((i, j - 1)\) in the lower layer becomes \((i, j + 1)\) and vice versa in the upper wall layer. So, the \(A_j\) and \(C_j\) matrices in the lower wall layer get interchanged in the upper wall layer. Therefore, equations in matrix form for the intermediate points in the upper wall
The upper wall layer-core interface is governed by the equations:

\[
\begin{align*}
\begin{bmatrix}
\frac{3}{4\Delta X} & -\frac{1}{\Delta \eta} & 0 \\
-\frac{3}{4\Delta X} & \frac{1}{\Delta \eta} & 0 \\
0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
U_{j-1} \\
\hat{V}_{j-1} \\
P_{j-1}
\end{bmatrix} + \omega \begin{bmatrix}
\frac{3}{4\Delta X} & \frac{2}{(\Delta \eta)^2} & U^g_{\eta} \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
U_j \\
\hat{V}_j \\
P_j
\end{bmatrix} \\
+ \begin{bmatrix}
\frac{3}{4\Delta X} & -\frac{1}{\Delta \eta} & 0 \\
-\frac{3}{4\Delta X} & \frac{1}{\Delta \eta} & 0 \\
0 & 0 & 0
\end{bmatrix} & \begin{bmatrix}
U_{j+1} \\
\hat{V}_{j+1} \\
P_{j+1}
\end{bmatrix} \\
\frac{u_{i-1,i}+u_{i-1,i+1}}{4\Delta X} - \frac{u_{i-2,i}+u_{i-2,i+1}}{4\Delta X} & U^g \begin{bmatrix}
\frac{3}{4\Delta X} \\
\lambda & \frac{3}{2\Delta X} \\
0 & 0
\end{bmatrix} \begin{bmatrix}
U_{i+1} \\
\hat{V}_{i+1} \\
P_{i+1}
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
0 & \frac{\lambda \Delta X^2}{2\kappa} \\
0 & 0 \\
0 & 0
\end{bmatrix} & \begin{bmatrix}
U_{i,N} \\
\hat{V}_{i,N} \\
P_{i,N}
\end{bmatrix} + \begin{bmatrix}
1 & 0 & -\frac{\Delta X^2}{2\kappa} \\
0 & \lambda & \frac{3}{2\Delta X} \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
U_{i,N+1} \\
\hat{V}_{i,N+1} \\
P_{i,N+1}
\end{bmatrix} \\
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{1}{\Delta \eta} & 0
\end{bmatrix} & \begin{bmatrix}
U_{i,N+2} \\
\hat{V}_{i,N+2} \\
P_{i,N+2}
\end{bmatrix} \\
\lambda(\eta_{N+1} + hF_u(i) - \frac{A_{i-1}+A_{i+1}}{2}) & U^g \begin{bmatrix}
\frac{4u_{i-1,N+1}+u_{i-2,N+1}}{2\Delta X} \\
\frac{u_{i-1,N+1}+u_{i-1,N+2}}{\Delta X} \\
\frac{u_{i-2,N+1}+u_{i-1,N+2}}{2\Delta X}
\end{bmatrix} \begin{bmatrix}
U_{i+1} \\
\hat{V}_{i+1} \\
P_{i+1}
\end{bmatrix}
\end{align*}
\]
2. MHD flow with magnetic field of moderate strength

2.1. Hartmann velocity derivation

The partial derivatives \( \frac{\partial u}{\partial x} \) and \( \frac{\partial w}{\partial z} \) are assumed to be equal to zero. Hence from the continuity equation (6.1), \( v = 0 \). To be consistent with the long scale boundary layer solution, pressure should drop in the direction of the flow, that is

\[
\frac{dp}{dx} = -1 \implies p = P_0 - x.
\]

Hence, the \( x \)-momentum equation (6.2) reduces to

\[
1 + Ha^2 Rm^{-1} \frac{\partial B_x}{\partial y} + u_{yy} = 0.
\] (32)

The \( y \)-momentum equation (6.3) reduces to

\[
- \frac{\partial p}{\partial y} - Ha^2 Rm^{-1} \left( B_x \frac{\partial B_x}{\partial y} + B_z \frac{\partial B_z}{\partial y} \right) = 0
\]

\[
\implies p = P_0 - x - Ha^2 Rm^{-1} \left( \frac{B_x^2}{2} + \frac{B_z^2}{2} \right).
\] (33)

The \( z \)-momentum equation (6.4) reduces to

\[
Ha^2 Rm^{-1} \frac{\partial B_z}{\partial y} + w_{yy} = 0.
\] (34)

The \( x \)-induction equation (6.5) becomes

\[
u_y + Rm^{-1} \frac{\partial^2 B_x}{\partial y^2} = 0.
\] (35)

The \( y \)-induction equation (6.6) becomes

\[
Rm^{-1} \frac{\partial^2 B_y}{\partial y^2} = 0.
\] (36)

Along with the boundary conditions \( B_y = 1 \) at \( y = 0, 1 \), we get \( B_y = 1 \) which is true.

The \( z \)-induction equation (6.7) becomes

\[
w_y + Rm^{-1} \frac{\partial^2 B_z}{\partial y^2} = 0.
\] (37)
The equations (.34) and (.37) are satisfied by \(w = 0\) and \(B_z = 0\). Thus (33) becomes:

\[
p = P_0 - x - Ha^2 Rm^{-1} \frac{B_x^2}{2}.
\]

Now, as (.35) is integrated we get

\[
u = C_1 - Rm^{-1} \frac{\partial B_x}{\partial y}.
\] (38)

Substituting this into (.32) we get

\[
1 + Ha^2 Rm^{-1} \frac{\partial B_x}{\partial y} - Rm^{-1} \frac{\partial^3 B_x}{\partial y^3} = 0
\]

\[
\Rightarrow Ha^2 B_x - \frac{\partial B_x}{\partial y} + Rmy = C_2,
\] (39)

solution of which is given by

\[
B_x = Ha^{-2} C_2 - Ha^{-2} Rm + C_3 \exp(Ha) + C_4 \exp(-Ha),
\] (40)

using particular solution method. Using the boundary conditions \(B_x = 0\) at \(y = 0, 1\), we get

\[
Ha^{-2} C_2 + C_3 + C_4 = 0,
\] (41)

\[
Ha^{-2} C_2 - Ha^{-2} Rm + C_3 \exp(Ha) + C_4 \exp(-Ha) = 0.
\] (42)

Substituting the value of \(B_x\) in (38) gives:

\[
u = C_1 + Ha^{-2} - C_3 Ha Rm^{-1} \exp(Ha) + C_4 Ha Rm^{-1} \exp(-Ha),
\] (43)

which on applying no-slip conditions gives

\[
C_1 + Ha^{-2} - C_3 Ha Rm^{-1} + C_4 Ha Rm^{-1} = 0,
\] (44)

\[
C_1 + Ha^{-2} - C_3 Ha Rm^{-1} \exp(Ha) + C_4 Ha Rm^{-1} \exp(-Ha) = 0.
\] (45)

Solving the above equations give:

\[
C_1 = -Ha^2 + \frac{Ha^{-1}}{2} \left( \frac{1}{1 - \exp(-Ha)} - \frac{1}{1 - \exp(Ha)} \right),
\]

\[
C_2 = \frac{Rm}{2} \left( \frac{1}{1 - \exp(-Ha)} + \frac{1}{1 - \exp(Ha)} \right),
\] (46)

\[
C_3 = -\frac{Ha^{-2} Rm}{2(1 - \exp(Ha))},
\]

\[
C_4 = -\frac{Ha^{-2} Rm}{2(1 - \exp(-Ha))}.
\]
Hence the Hartmann solution, thus obtained from (.43) is:

\[ u = \frac{1}{2Ha} \left( \frac{\exp(Hay) - 1}{1 - \exp(Ha)} + \frac{1 - \exp(-Hay)}{1 - \exp(-Ha)} \right). \]

2.2. Hartmann velocity for \( Ha \to \infty \)

The behaviour of the upstream Hartmann velocity as the Hartmann number \( Ha \to \infty \) is found by taking a limit of the velocity

\[
\lim_{Ha \to \infty} u = \lim_{Ha \to \infty} U_0(y) = \frac{1}{2Ha} \left[ \frac{e^{Hay} - 1}{1 - e^{Hay}} + \frac{1 - e^{-Hay}}{1 - e^{-Ha}} \right]
\]

\[
= \lim_{Ha \to \infty} \frac{1}{2Ha} \left[ (1 + e^{-Hay} - e^{-Hay(1-y)} - e^{-Hay})(1 - e^{-Hay})^{-1} \right]
\]

\[
= \lim_{Ha \to \infty} \frac{1}{2Ha} \left[ (1 + e^{-Hay} - e^{-Hay(1-y)} - e^{-Hay})(1 + e^{-Hay} + e^{-2Hay} + ...) \right]
\]

\[
= \lim_{Ha \to \infty} \frac{1}{2Ha} \left[ 1 - e^{-Hay} - e^{-Hay(1-y)} + 2e^{-Hay} + ... \right]
\]

\[
= \lim_{Ha \to \infty} \frac{1}{2Ha} - \frac{e^{-Hay} + e^{-Hay(1-y)}}{2} + ...
\]

This shows that \( U_0(y) \) is similar to \( 1/(2Ha) \) which tends to 0 as \( Ha \to \infty \). Behavior of its derivative is given by

\[
\lim_{Ha \to \infty} U'_0(y) = \lim_{Ha \to \infty} \frac{1}{2} \left[ \frac{e^{Hay}}{1 - e^{Hay}} + \frac{e^{-Hay}}{1 - e^{-Hay}} \right]
\]

\[
= \lim_{Ha \to \infty} \frac{1}{2} \left[ \frac{e^{-Hay(1-y)} - e^{-Hay}}{e^{-Hay} - 1 + e^{-Hay}} \right]
\]

\[
= \lim_{Ha \to \infty} \frac{1}{2} \left[ (e^{-Hay} - e^{-Hay(1-y)})(1 + e^{-Hay} + e^{-2Hay} + ...) \right]
\]

\[
= \lim_{Ha \to \infty} \frac{1}{2} \left[ e^{-Hay} - e^{-Hay(1-y)} + e^{-Hay(y+1)} - e^{-Hay(2-y)} + ... \right]
\]

\[
= 0
\]
and its square is

$$\lim_{Ha \to \infty} U_0^2 = \lim_{Ha \to \infty} \frac{1}{4Ha^2} \left[ 1 - 2e^{-Hay} - 2e^{-Ha(1-y)} + ... \right]$$

$$\therefore \int_0^y U_0^2 \, ds = \lim_{Ha \to \infty} \frac{1}{4Ha^2} \left[ y - \frac{2}{Ha} + \frac{2}{Ha} e^{-Hay} - ... \right]$$

2.3. Scaled Hartmann velocity

In the lower Hartmann layer, the normal co-ordinate scales like $O(Ha^{-1})$, which is its thickness. Hence $y = \frac{1}{Ha} \tilde{y}$. Therefore the upstream Hartmann velocity in the Hartmann layers is given by

$$u = U_0(y; Ha)$$

$$= \frac{1}{2Ha} \left[ e^{Hay} - 1 \right] + \frac{1}{1 - e^{Hay}}$$

$$= \frac{1}{2Ha} \left[ e^{\bar{y}} - 1 \right] + \frac{1}{1 - e^{\bar{y}}}$$

$$= \frac{1}{Ha} \left[ \frac{e^{\bar{y}} - 1}{e^{Ha}(e^{\bar{y}})^0 - 1} + (1 - e^{-\bar{y}}) \right]$$

$$= \frac{1}{2} (1 - e^{-\bar{y}})$$

$$= \frac{1}{Ha} \bar{U}_0$$

where $\bar{U}_0 = \frac{1}{2}(1 - e^{-\bar{y}})$.

2.4. Obtaining leading order equations

In the lower wall layer, the scales are:

$$(x, y) = (Re^{1/7} Ha^{-6/7} X, Re^{-2/7} Ha^{-2/7} Y),$$

$$(u, v, p) \sim (Re^{-2/7} Ha^{-2/7} U, Re^{-5/7} Ha^{2/7} V, Re^{3/7} Ha^{-4/7} P_1) + ..., $$

$$B_x \sim Re^{-4/7} Ha^{-4/7} B_X + ..., $$

$$B_y \sim 1 + Re^{-1} B_Y + ...$$
These scales are substituted in the governing equations, and every term is tracked to obtain the leading order terms. Thus the leading order equations are obtained. The method is illustrated below for $x$-induction equation.

\[ \frac{\partial B_x}{\partial x} + \frac{\partial B_x}{\partial y} = B_x u_x + B_y u_y + Rm^{-1} \left[ \frac{\partial^2 B_x}{\partial y^2} + \frac{\partial^2 B_x}{\partial x^2} \right] \]

\[ \Rightarrow \frac{Re^{-2/7} Ha^{-2/7} \ast Re^{-4/7} Ha^{-4/7}}{Re^{-1/7} Ha^{-6/7}} U \frac{\partial B_X}{\partial X} + \frac{Re^{-5/7} Ha^{-2/7} \ast Re^{-4/7} Ha^{-4/7}}{Re^{-2/7} Ha^{-2/7}} V \frac{\partial B_X}{\partial Y} \]

\[ = \frac{Re^{-4/7} Ha^{-4/7} \ast Re^{-2/7} Ha^{-2/7}}{Re^{-1/7} Ha^{-6/7}} B_X U_X + (1 + Re^{-1} B_Y) \ast U_Y \]

\[ + Rm^{-1} Re^{-4/7} Ha^{-4/7} \left[ Re^{-2/7} Ha^{12/7} \frac{\partial^2 B_X}{\partial X^2} + Re^{4/7} Ha^{4/7} \frac{\partial^2 B_X}{\partial Y^2} \right] \]

The above equation after simplification becomes

\[ Re^{-1} \left[ U \frac{\partial B_X}{\partial X} + V \frac{\partial B_X}{\partial Y} \right] = Re^{-1} B_X U_X + U_Y + Re^{-1} B_Y U_Y \]

\[ + Rm^{-1} \left[ Re^{-6/7} Ha^{8/7} \frac{\partial^2 B_X}{\partial X^2} + \frac{\partial^2 B_X}{\partial Y^2} \right] \]

Of all the terms, the leading order terms are the (IV) and (VI) terms which are of order one. The other terms are all neglected since they are smaller than one. Hence the leading order $x$-induction equation is:

\[ Rm^{-1} \frac{\partial^2 B_X}{\partial Y^2} + U_Y = 0 \]

\[ \Rightarrow B_X = Rm \left( c_1 Y - \int_0^Y U \, ds \right) \]

The far-field condition re-written becomes $U - \lambda A - \lambda Y \to 0$ as $Y \to \infty$. This implies

\[ B_X = Rm ((c_1 - \lambda A) Y - \lambda \frac{Y^2}{2} - \int_0^Y (U - \lambda A - \lambda Y) \, ds) \]

\[ \therefore B_x \sim Re^{-4/7} Ha^{-4/7} Rm \left[ (c_1 - \lambda A) Y - \lambda \frac{Y^2}{2} - \int_0^Y (U - \lambda s - \lambda A) \, ds \right] + ... \]
In the lower Hartmann layer, the leading order $x$-induction equation is
\[
Rm^{-1} \frac{\partial^2 \dot{B}_X}{\partial y^2} + \ddot{U}_0 = 0
\]
\[
\Rightarrow \dot{B}_X = Rm \left[ (\bar{c}_1 - \lambda)\bar{y} - \bar{c}_2 - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right]
\]
\[
\Rightarrow B_x \sim Ha^{-2} Rm \left[ (\bar{c}_1 - \lambda)\bar{y} - \bar{c}_2 - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right] + ...
\]
The unknowns $c_1, \bar{c}_1$ and $\bar{c}_2$ are obtained from the matching between the lower wall layer and Hartmann layer. Matching LWL - LHL gives:
\[
\lim_{Y \to \infty} Re^{-4/7} Ha^{-4/7} Rm \left[ (c_1 - \lambda A)Y - \lambda Y^2/2 - \int_0^Y (U - \lambda s - \lambda A) \, ds \right] + ...
\]
\[
= \lim_{\bar{y} \to 0} Ha^{-2} Rm \left[ (\bar{c}_1 - \lambda)\bar{y} - \bar{c}_2 - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right] + ...
\]
\[
\Rightarrow Re^{-4/7} Ha^{-4/7} \left[ (c_1 - \lambda A)Re^{2/7} Ha^{-5/7} \bar{y} - \lambda Re^{4/7} Ha^{-10/7} \bar{y}^2 \right] + ...
\]
\[
= Ha^{-2} \left[ (\bar{c}_1 - \lambda)\bar{y} - \bar{c}_2 - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right] + ...
\]
\[
\Rightarrow Re^{-2/7} Ha^{-9/7} (c_1 - \lambda A) \bar{y} - Ha^{-2} \bar{y}^2 4 + ...
\]
\[
= Ha^{-2} \left[ (\bar{c}_1 - \lambda)\bar{y} - \bar{c}_2 - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right] + ...
\]
\[
\Rightarrow \bar{c}_1 = 0 = \bar{c}_2
\]
\[
\therefore B_x \sim Ha^{-2} Rm \left[ -\lambda \bar{y} - \int_0^\bar{y} (\bar{U}_0 - \lambda) \, ds \right] + ... \text{ in lower Hartmann layer}
\]
\[
B_x \sim Re^{-4/7} Ha^{-4/7} Rm \left[ (c_1 - \lambda A)Y - \lambda Y^2/2 - \int_0^Y (U - \lambda s - \lambda A) \, ds \right] + ... \text{ in lower wall layer.}
\]

3. MHD flow with strong magnetic field strength

3.1. Free Interaction Analysis

The perturbed lower wall layer variables are
\[
(U, V, P_i) \sim (\lambda Y, 0, 0) + \epsilon(\upsilon_p(Y), \upsilon_p(Y), p_p)e^{\theta X} + O(\theta^2),
\]
\[
A \sim \epsilon a_p e^{\theta X} + O(\theta^2),
\]
where \( a_p = \pm 1 \). The perturbation solutions are

\[
\begin{align*}
u_p &= C_1 \int_0^Y \text{Ai}(ms) \, ds = C_1 f_1'(Y), \\
v_p &= -\theta C_1 f_1(Y), \\
p_p &= \frac{C_1m}{\theta} \text{Ai}'(0), \\
a_p &= \frac{C_1}{3m\lambda},
\end{align*}
\]

where \( m = (\lambda\theta)^{1/3} \). Similarly, in the upper wall layer, the solutions are

\[
\begin{align*}
\tilde{u}_p &= \tilde{C}_1 \tilde{f}_1'(\tilde{Y}), \\
\tilde{v}_p &= -\theta \tilde{C}_1 \tilde{f}_1(\tilde{Y}), \\
\tilde{p}_p &= \frac{\tilde{C}_1m}{\theta} \text{Ai}'(0), \\
\tilde{b}_p &= \frac{\tilde{C}_1}{3m\lambda}.
\end{align*}
\]

From the two systems, we get the relation between \( a_p \) and \( b_p \):

\[
\frac{a_p}{b_p} = \frac{C_1}{\tilde{C}_1}.
\]

In the core, the perturbed pressure is \( P \sim \varepsilon p_c(y)e^{\theta X} + O(\theta^2) \). The amplitude \( p_c \) satisfies the equation

\[
p''_c + \theta^2 p_c = 0,
\]

which has the solution \( p_c(y) = d_1 \cos(\theta y) + d_2 \sin(\theta y) \) with derivative \( p'_c(y) = -\theta d_1 \sin(\theta y) + \theta d_2 \cos(\theta y) \). From the boundary conditions, the constants \( d_1 \) and \( d_2 \) are obtained

\[
\begin{align*}
p_c(0) &= d_1 = p_p = \frac{C_1m}{\theta} \text{Ai}'(0), \\
p_c(1) &= d_1 \cos(\theta) + d_2 \sin(\theta) = \tilde{p}_p = \frac{\tilde{C}_1m}{\theta} \text{Ai}'(0), \\
p'_c(0) &= \theta d_2 = \frac{\theta^2}{4H^2} a_p, \\
p'_c(1) &= -\theta d_1 \sin(\theta) + \theta d_2 \cos(\theta) = -\frac{\theta^2}{4H^2} b_p.
\end{align*}
\]
Linearizing the equation $A + B = 4H^2 \int_0^1 p_1(X,s) \, ds$ and using the form of $p_c$, another relation between $a_p$ and $b_p$

$$a_p + b_p = \frac{4H^2}{\theta} [d_1 \sin(\theta) - d_2 (\cos(\theta) - 1)]$$

is obtained. Substituting the values of $d_1$ and $d_2$ in the boundary conditions at $y = 1$ and further simplification leads to the following equations

$$\cos(\theta) + \frac{\theta^2 a_p}{4H^2 C_1 m \text{Ai}'(0)} \sin(\theta) = \frac{b_p}{a_p}$$

$$\frac{4H^2 C_1 m \text{Ai}'(0)}{\theta^2 a_p} \sin(\theta) - \cos(\theta) = \frac{b_p}{a_p}$$

$$\frac{b_p}{a_p} = \frac{6H^2 m^2}{\theta^2} \text{Ai}'(0) \sin(\theta) - \cos(\theta)$$

Let $r = \frac{4H^2 C_1 m \text{Ai}'(0)}{\theta^2 a_p} = \frac{6H^2 m^2 \text{Ai}'(0)}{\theta^2}$. Hence the above two equations become

$$\cos(\theta) + \frac{1}{r} \sin(\theta) = \frac{b_p}{a_p}$$

$$r \sin(\theta) - \cos(\theta) = \frac{b_p}{a_p}$$

Solving the two equations simultaneously gives

$$2 \cos(\theta) = (r - \frac{1}{r}) \sin(\theta)$$

$$\implies \tan(\theta) = \frac{2r}{r^2 - 1}$$

Inserting the value of $r$ into the above equation leads to the dispersion relation

$$\tan(\theta) = \frac{12H^2 \text{Ai}'(0)\theta^4}{\frac{36}{2^5} H^4 (\text{Ai}'(0))^2 - 2^2 \theta^2}$$

$$\implies 2^7 \theta^5 + 12H^2 \text{Ai}'(0)\theta^4 \cot(\theta) = \frac{36}{2^5} H^4 (\text{Ai}'(0))^2$$

3.1.1. Connection between MHD flows with moderate magnetic fields strength and strong magnetic field

In order to find a connection between the flow structures, the growth constant $\theta$ is assumed to be proportional to $O(H^n)$ or in other words, $\theta = H^n \tilde{\theta}$. $\tilde{\theta}$ is of $O(1)$. This relation is substituted
into the dispersion relation to get

\[ 2^{2/3} H^{8n/3} \tilde{\theta}^{8/3} + 12 \text{Ai}'(0) H^{4n/3} + 2 \tilde{\theta}^{4/3} \left[ H^{-n} \tilde{\theta}^{-1} - \frac{H^n \tilde{\theta}}{3} - \frac{H^{3n} \tilde{\theta}^3}{45} - \ldots \right] = \frac{36}{2^{2/3}} H^4 (\text{Ai}'(0))^2 \]

\[ \Rightarrow 2^{2/3} H^{8n/3} \tilde{\theta}^{8/3} + 12 \text{Ai}'(0) H^{4n/3} + 2 \tilde{\theta}^{4/3} \left[ 1 - \frac{H^{2n} \tilde{\theta}^2}{3} - \frac{H^{4n} \tilde{\theta}^4}{45} - \ldots \right] = \frac{36}{2^{2/3}} H^4 (\text{Ai}'(0))^2 \]

The powers of \( H \) are compared to obtain the value of \( n \):

(I) \( \sim \) (II)

\[ \Rightarrow 8n \frac{3}{3} - 4 = n \frac{3}{3} - 2 \]

\[ \Rightarrow n = \frac{6}{7} \]

Substituting this value back in the dispersion relation gives:

\[ 2^{2/3} \tilde{\theta}^{8/3} H^{-12/7} + 12 \text{Ai}'(0) \tilde{\theta}^{1/3} H^{-12/7} - \ldots = \frac{36}{2^{2/3}} (\text{Ai}'(0))^2 \]

\[ \Rightarrow 2^{2/3} \tilde{\theta}^{8/3} + 12 \text{Ai}'(0) \tilde{\theta}^{1/3} = 0 \quad (\because H \to 0) \]

\[ \Rightarrow \tilde{\theta} = 2(-1.5 \text{Ai}'(0))^{3/7} \]

which matches the eigenvalue, \( \theta \), of MHD flow with moderate magnetic field strength.

4. Connection between hydrodynamic flow structure and MHD flow structure with finite and moderate \( Ha \)

In Smith’s 77 flow structure, the pressure-displacement interaction is \( \tilde{P}_1 = P(X) + \kappa A''(X) \)

where \( \kappa = \int_0^y U_0^2 ds \). In the MHD flow, the \( \kappa \) is given by

\[ \kappa = \int_0^y U_0^2 ds \]

\[ = \int_0^y \left( \frac{1}{2Ha} \left[ \exp(Has) - 1 - 1 - \exp(-Has) \right] + \frac{1 - \exp(-Has)}{1 - \exp(-Ha)} \right)^2 ds \]

\[ = \cosh \left( Ha \left[ Ha(2 + \cosh(Has)) - 3 \sinh(Has) \right] \right) \]
The limiting values of $\kappa$ are

$$\lim_{Ha \to 0} \kappa = \frac{1}{120},$$

$$\lim_{Ha \to \infty} \kappa \sim \frac{1}{4Ha^2} \to 0$$

Hence it is seen that as $Ha \to 0$, the pressure-displacement interaction in Smith’s 77 structure is restored. Similarly, this result can be shown for the linear free interaction as well. The perturbed variables in the wall layers are

$$(\tilde{P}_1, P, A) = \epsilon(\tilde{p}_p, p_p, a_p)e^{\theta x} + ...$$

Hence the cross-channel pressure-displacement equation becomes

$$\tilde{p}_p = p_p + \theta^2 \kappa a_p$$

From section 1., the interaction is simplified to

$$\theta^{7/3} = -\frac{3Ai'(0)}{2^{2/3} \kappa}$$

Let $\theta \sim \theta_1 Ha^p + ...$ for $Ha \to \infty$.

Substituting into the pressure-displacement equation gives

$$\tilde{p}_p = p_p + \frac{\theta^2 a_p}{4Ha^2}$$

$$\Rightarrow \theta^{7/3} = -\frac{3Ai'(0)}{2^{2/3} 4Ha^2}$$

$$\Rightarrow \theta_1^{7/3} Ha^{7p/3} + ... = -\frac{3Ai'(0)}{2^{2/3} 4Ha^2}$$

$$\Rightarrow \frac{7p}{3} = 2 \Rightarrow p = \frac{6}{7} and \ \theta_1 = 1.33...$$

$$\therefore \theta \sim \theta_1 Ha^{6/7}$$

which agrees with the eigenvalue of the MHD flow with moderate magnetic field strength.

5. $H \to 0$ limit of MHD flow with strong magnetic field strength

In the free interaction of MHD flow structure with strong magnetic field strength, it was found that $\theta \sim \tilde{\theta} H^2$ where $\tilde{\theta}$ matches with the corresponding $\theta$ value of MHD flow with moderate magnetic
field strength. Hence in the structure with strong magnetic field,

\[ e^\theta X = e^\theta H^{6/7}X = e^\theta \hat{X} \]

Here the stream-wise scale is

\[ \hat{X} = H^{6/7}X \]

\[ \implies X = H^{-6/7} \hat{X} \]

which means as \( H \to 0 \) in the structure with strong magnetic field, the \( X \)-scale has to expand as \( H^{-6/7} \) which is similar to the MHD flow with moderate magnetic field strength.

This deduction can also be found from the equation systems. In the lower wall layer, the equation system is

\[ U_X + V_Y = 0, \]

\[ UU_X + VU_Y = -P_{1X}(X) + U_{YY}, \]

with the no-slip conditions \( U(X,0) = V(X,0) = 0 \) and the far-field condition \( U \to \lambda(Y + A(X)) \) as \( Y \to \infty \). Let \( X = H^{-p} \hat{X} \) where \( p > 0 \). This implies scale lengthening in the stream-wise direction \( x = X = H^{-p} \hat{X} \). To maintain the wall layer interaction, the other scales also need to change:

\[ Y = H^{-p/3} \tilde{Y}, \quad U = H^{-p/3} \tilde{U}, \quad V = H^{p/3} \tilde{V}, \quad P_1 = H^{-2p/3} \tilde{P}_1, A = H^{-p/3} \tilde{A} \]

Substituting the expansions into the lower wall layer equation system gives

\[ \tilde{U}_X + \tilde{V}_Y = 0, \]

\[ \tilde{U}\tilde{U}_X + \tilde{V}\tilde{U}_Y = -\tilde{P}_{1X}(\tilde{X}) + \tilde{U}_{YY}, \]

with boundary conditions \( \tilde{U}(\tilde{X},0) = \tilde{V}(\tilde{X},0) = 0 \) and \( \tilde{U} \to \lambda(\tilde{Y} + \tilde{A}(\tilde{X})) \) as \( \tilde{Y} \to \infty \).

In the lower Hartmann layer, the perturbation velocities become \( \tilde{u}_1 = H^{-p/3} \tilde{u}_1, \quad \tilde{v}_1 = H^{2p/3} \tilde{v}_1 \). In the core, they are \( u_1 = H^{(p-3)/3} \tilde{u}_1, \quad v_1 = H^{(2p-3)/3} \tilde{v}_1, \quad P = H^{(2-p)/3} \). Hence, the core equation
becomes:

\[ P_{XX} + P_{yy} = 0 \]
\[ \implies \frac{H^{(2-p)/3}}{H^{-2p}} \dot{P}_{XX} + H^{-2p/3} \ddot{P}_{yy} = 0 \]
\[ \implies \ddot{P}_{yy} = 0 \]

The displacement relation becomes:

\[ A + B = 4H^2 \int_0^1 P(X,s) \, ds \]
\[ \implies H^{-p/3}(\ddot{A} + \ddot{B}) = 4H^{(6-2p)/3} \int_0^1 \ddot{P}(\ddot{X},s) \, ds \]
\[ \implies \dot{A} + \dot{B} = 0 \]

which holds in MHD flow with moderate magnetic field strength. In order to calculate the value of the exponent, \( p \), the interface condition is used

\[ P_y(X,0) = \frac{H^{-2}A''}{4} \]
\[ \implies H^{-2p/3} = \frac{H^{-2}H^{-p/3}}{H^{-2p}} \]
\[ \implies p = \frac{6}{7} \]

Therefore, the stream-wise length scale is \( x = H^{-6/7} \dddot{X} \) which matches the steam-wise scale of MHD flow with moderate magnetic field strength.

5.1. \( H \to \infty \) limit of MHD flow with strong magnetic field strength

The dispersion relation is a quadratic equation in \( H^2 \), and four values of \( H \) are found:

\[ \frac{36}{2^{2/3}} (A'(0))^2 H^4 - 12 A'(0) \dot{\theta}^{4/3} \cot (\dot{\theta}) H^2 - 2^{2/3} \dot{\theta}^{8/3} = 0 \]
\[ \implies H^2 = \frac{2^{2/3} \dot{\theta}^{5/3} (\cot (\dot{\theta}) \pm \csc (\dot{\theta}))}{6 A'(0)} \]
\[ \implies H = \pm \frac{2^{1/3}}{\sqrt{6 A'(0)}} \dot{\theta}^{2/3} \sqrt{\cot \left( \frac{\theta}{2} \right)} \pm \frac{2^{1/3} i}{\sqrt{6 A'(0)}} \dot{\theta}^{2/3} \sqrt{\tan \left( \frac{\theta}{2} \right)} \]
As $H \to \infty$ the solutions $\bar{\theta}^2/3 \sqrt{\cot (\theta/2)}$, $\bar{\theta}^2/3 \sqrt{\tan (\theta/2)} \to \infty$. This implies $\cot (\theta/2)$, $\tan (\theta/2) \to \infty$, that is, $\theta/2 \to n\pi/2$ or $\bar{\theta} \to n\pi$. 