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Structure and stability analysis of large scale systems using a new graph-theoretic approach

Wang Tang  
Iowa State University

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Structure and stability analysis of large scale systems using a new graph-theoretic approach

by

Wang Tang

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1. INTRODUCTION

Many existing large scale systems may be viewed as an interconnection of a number of relatively simple subsystems. In this dissertation we are interested in how these subsystems are interconnected and whether or not a system performs in a stable mode. The structure and stability of large scale systems and related topics are the main topics of this dissertation.

Lyapunov's direct method is a well-established and much studied technique for determining the stability of dynamical systems in a state space setting without any knowledge of the system solutions. This method requires an auxiliary function, called a Lyapunov function $V$, which can be thought of as the generalization of the total stored energy in a system, and its derivative $\dot{V}$ along the solutions of the system. The stability (asymptotic stability, instability, etc.) of the system equilibrium is then determined on the basis of the properties of $V$ and $\dot{V}$. A number of researchers have attempted to analyze the qualitative behavior of large scale systems using scalar Lyapunov functions consisting of a weighted sum of Lyapunov functions for individual subsystems. A recent monograph by Michel and Miller [15] deals with several techniques on this subject and contains some interesting and useful results.
Due to the sparsity of large scale systems (i.e., a subsystem is usually connected only to a few other subsystems), the structure of some large scale systems may often be used to advantage in either qualitative or quantitative analysis. A unique tool in examining the structure of large scale systems is the utilization of graph-theoretic methods. Roughly speaking, a graph may be thought of as a set of objects together with a set of relations among some pairs of these objects. Applying this concept to large scale systems, we may treat each subsystem as an object and if two subsystems are connected, then there exists a relation between these two objects. A graph is then made to correspond to a large scale system. Certain properties in such graphs reveal some advantageous features of large scale systems. Along these lines, several results have been established, involving matrix inversion [12], the solutions of system equations [13], controllability and observability of systems [14], input-output stability [5], and Lyapunov stability [16].

A shortcoming of the above approach is that a relation in a graph must exist between two objects. However this is not always the case in physical situations. Sometimes we must relate several objects to one object (e.g., the blood type of a child is a function of both father's blood type and mother's blood type). Therefore, a generalization of the existing graph-theoretic approach is desirable.
To overcome this limitation, the G-graph (generalized graph), the G-digraph (generalized digraph) and the M-digraph (modified G-digraph) are introduced and some results characterizing their properties are introduced. A G-graph may in a certain sense be viewed as a special case of a hypergraph (see Berge [3]) and as a generalization of the usual notion of graph. Also, a G-digraph is a generalization of the digraph and the M-digraph is a limited class of G-digraph. These concepts and their properties, which are of interest in their own right, can be used to describe and decompose large scale dynamical systems in a variety of useful equivalent forms by means of efficient computer algorithms. These decomposition techniques constitute generalizations (and also unification) of previous results [5, 14, 16]. Using these graph-theoretic results, we extend and improve the Lyapunov stability results developed in [16]. The present stability results are applicable to a larger class of problems than those considered in [16] and they yield results which in general are less conservative than corresponding ones reported in [16]. Furthermore, the results developed in [5] for input-output stability may be extended and generalized by the present method as well.

For stability analysis, the most useful of these equivalent decompositions is the so-called sequential form which may be viewed as an interconnection of semi-strong components,
called subsystems. (A semi-strong component is a generalization of the concept of strong component (see [6, 11])).

Roughly speaking, the present stability results state that the overall large scale system will be uniformly asymptotically stable (resp., exponentially stable, uniformly ultimately bounded) if all interconnections are stability preserving (see Hahn [10], Thomas [23]), if the subsystems (semi-strong components) are uniformly asymptotically stable (resp., exponentially stable, uniformly ultimately bounded) and some additional minor and reasonable requirements are met. Also, the overall large scale system will be unstable (completely unstable) if some (if all) subsystems (i.e., semi-strong components) are unstable (completely unstable). Thus, the present stability results are in the spirit of those established in [16] and also in [5].

The properties of G-graphs and M-digraphs allow the generation of several equivalent forms via efficient computer algorithms, including the "interconnected form", "lower block triangular form", "composite sequential form" and "sequential form". These forms can be used to advantage in investigations other than stability studies. For example, the composite sequential form can be used to advantage in studies of controllability and stabilization of large scale systems.

There are numerous results for Lyapunov stability and input-output stability of large scale systems which do not
use graph-theoretic decomposition techniques. A survey of these methods, which are applicable to finite dimensional systems as well as infinite dimensional systems, which may be deterministic or stochastic, can be found in Michel and Miller [15]. The main difference between the present method of analysis and the methods advanced in [15] are as follows:

(i) The present results make it possible to address the "largeness" of complex systems, since the semi-strong components can be identified by means of efficient computer algorithms. No generally accepted computer algorithms have been devised for the results in [15].

(ii) In [15] the analysis is accomplished in terms of the original system structure (which is frequently desirable in the case of decentralized systems), while the present method involves a transformation of the system.

(iii) The stability results in [15] require that the subsystems (in terms of the original system structure) be stable, while the present results do not explicitly require this.

(iv) The results in [15] are applicable to interconnected systems consisting of several or only one semi-strong component. When a system consists of only one semi-strong component, no advantage can be realized by the present method.

It is this author's view that the advantages of the present method can be combined with the advantages of the methods in [15] to generate an effective tool in the quali-
tative analysis of large scale systems. In such an approach, the subsystems (semi-strong components which may be complex and of high dimension in their own right) are analyzed by means of results of the type given in [15]. The overall interconnected system is then treated by the present approach.

The contents of this dissertation are as follows. In chapter 2 the conventional graph-theoretic decomposition techniques are introduced. New concepts of graphs along with their properties are presented in chapter 3. In chapter 4 five equivalent forms of a large scale system are described and algorithms which make decompositions possible using computers are developed. The main stability results along with some examples are presented in chapter 5. The dissertation is concluded in chapter 6.
2. GRAPHS AND DIGRAPHS

The primary objective of this chapter, which consists of three sections, is to show how graph theory has been applied in exploring the structure of large scale systems. In the first section we introduce a collection of definitions which are required in the subsequent discussions. Two algorithms for identifying strongly connected components and for topological sorting are presented in the second section. In the third section we discuss some applications of this graph-theoretic decomposition.

2.1. Some Basic Definitions

It is necessary to start with the definition of a graph.

**Definition 2.1.** A graph, \( G=(V,E) \), is a finite set of objects \( V=\{v_1, v_2, \ldots, v_n\} \), the elements of which are called vertices, and a finite family\(^1\) \( E=\{e_1, e_2, \ldots, e_m\} \) whose elements are called edges, such that each edge \( e_k \) is identified with an unordered pair \((v_i, v_j)\) of vertices.

**Definition 2.2.** A self-loop is an edge associated with a vertex pair \((v_i, v_i)\). Multiple edges are edges associated with a given pair of vertices. A graph that has no multiple edges and no self-loop is called a simple graph.

---

\(^1\)By family we have in mind a collection in which some member elements may be repeated.
Definition 2.3. If \( e_k = (v_i, v_j) \), then \( v_i \) (or \( v_j \)) and \( e_k \) are said to be incident with each other. The number of edges incident with a vertex \( v_i \), with a self-loop counted twice, is called the degree, \( d(v_i) \), of vertex \( v_i \).

Definition 2.4. A walk is a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. A walk which begins and ends at the same vertex is called a closed walk. A walk which is not closed is called an open walk. An open walk in which no vertex appears more than once is called a path. A closed walk in which no vertex appears more than once, except the initial and final vertex, is called a circuit. A walk in which no edge appears more than once is called a trail. A graph is said to be connected if there is at least one walk between every pair of vertices.

Definition 2.5. An Euler line is a closed trail that traverses every edge exactly once. A unicursal line is an open trail that traverses every edge exactly once. A Hamiltonian closed circuit is a closed path that traverses every vertex exactly once, except the starting vertex, at which the path also terminates. A Hamiltonian open path is an open path that traverses every vertex exactly once.

We now present two basic results concerning Euler lines.
and unicursal lines. The proofs of these results can be found in any standard textbook on graph theory (see, e.g., Deo [6]).

Proposition 2.1. A graph contains an Euler line if and only if every vertex is of even degree.

Proposition 2.2. A graph contains a unicursal line if and only if it has exactly two vertices of odd degree. If this is the case, a unicursal line must start and end at vertices with odd degree.

If in a graph each edge is assigned a direction we have a directed graph. More formally, we make the following definition.

Definition 2.6. A directed graph, or digraph for short, $D=(V,E)$, consists of a finite set of vertices $V=\{v_1,v_2,\ldots,v_n\}$, together with a finite family of edges $E=\{e_1,e_2,\ldots,e_m\}$, such that each edge $e_k$ is identified with an ordered pair $(v_i,v_j)$ of vertices. For the edge $e_k=(v_i,v_j)$, $v_i$ is called the \textit{initial vertex} and $v_j$ is called the \textit{terminal vertex}. The number of edges which take $v_i$ as their initial vertex is called the \textit{out-degree} of $v_i$ while the number of edges which take $v_i$ as their terminal vertex is called the \textit{in-degree} of $v_i$. Several distinct edges with same initial and terminal vertices are called \textit{multiple edges} in $D$. An edge
of the form \((v_i, v_i)\) is called a **self-loop** in \(D\). A digraph having no multiple edges and no self-loops is called a **simple digraph**.

**Definition 2.7.** For every directed graph \(D\), there is a graph \(G\), called the **undirected version** of \(D\), whose vertex set and edge family are the same as those in \(D\) except the directions of the edges are removed. A directed graph is **connected** if its undirected version is connected.

**Definition 2.8.** Two digraphs are said to be **isomorphic** if there exists a one-to-one correspondence between their vertices and a one-to-one correspondence between their edges such that corresponding edges have corresponding vertices as their initial vertices and corresponding vertices as their terminal vertices.

**Definition 2.9.** In a digraph a **diwalk** is a finite alternating sequence of vertices and edges, starting and terminating with vertices, such that the preceding and following vertices of each edge in the sequence are also the initial and terminal vertices of the edge. A diwalk is **closed** if the starting and terminating vertices of the sequence are the same. Otherwise it is **open**. A simple digraph that has no closed diwalk is called **acyclic**.

**Definition 2.10.** A digraph is **strongly connected** if there
is a diwalk between every pair of vertices. A digraph $D_s = (V_s, E_s)$ is a subdigraph of $D = (V, E)$ if $V_s \subseteq V$ and $E_s \subseteq E$. A maximal strongly connected subdigraph of a digraph is called a strongly connected component. The strong condensation of a digraph $D$ is a simple digraph in which each strongly connected component of $D$ is represented by a vertex and all edges of $D$ from vertices of a strongly connected component to vertices in another strongly connected component are replaced by one edge.

**Definition 2.11.** The vertices of an acyclic digraph are labeled in topological order if for every edge, the initial vertex is numbered lower than the terminal vertex. The process of labeling the vertices in topological order is called **topological sorting**.

**Remark 2.1.** Several facts should be noted: (1) A strongly connected subdigraph is not necessarily a strongly connected component due to the maximality requirement in Definition 2.10. (2) The strong condensation of a digraph is always an acyclic digraph. (3) If a digraph is strongly connected, then the only strongly connected component is itself and the strong condensation of the digraph contains only one vertex.

In applications, a graph is usually represented equivalently by a diagram in which the vertices are represented by small dots, while edges are represented by curves or
straight lines connecting vertices. For directed graphs (i.e., digraphs), we add an arrow to each edge to indicate ordering or direction. We conclude this section by the following example.

Example 2.1. To illustrate some of the preceding concepts we consider a digraph $D$ with five vertices and nine edges shown in Figure 2.1. As can be seen, $e_7$ and $e_8$ are multiple edges and $e_9$ is a self-loop. This digraph will result in a simple digraph if $e_8$ and $e_9$ are removed. The in-degree of $v_4$ is one and the out-degree of $v_4$ is three. This digraph is not strongly connected since we can not find a diwalk from $v_5$ to $v_1$. \{$v_4,e_4,v_3,e_7,v_2,e_2,v_4,e_4,v_3$\} is an open diwalk while \{$v_4,e_3,v_1,e_1,v_2,e_2,v_4$\} is a closed diwalk. The strongly connected components are shown in Figure 2.2. The strong condensation of $D$ is shown in Figure 2.3.

2.2. Decomposition Algorithms

We now present two algorithms for finding all strongly connected components of a digraph and for relabeling the vertices in topological order. Both algorithms can easily be programmed on a digital computer.

In the following, we find the notion of the adjacency matrix useful. This matrix can be used to input a digraph into a digital computer.
Figure 2.1. Digraph D

Figure 2.2. Two strongly connected components of D

Figure 2.3. Strong condensation of D
Definition 2.12. The adjacency matrix \( A = (a_{ij}) \) of a simple \( n \)-vertex digraph is a binary \( n \times n \) matrix given by

\[
a_{ij} = \begin{cases} 
1 & \text{if } (v_j,v_i) \in E \\
0 & \text{elsewhere.}
\end{cases}
\]

Clearly, if the vertices of a digraph are labeled in topological order, then the adjacency matrix will be in lower triangular form (i.e., all super-diagonal elements will in this case be zero).

In identifying the strongly connected components of a digraph, we will also require the notion of transitive closure.

Definition 2.13. The simple digraph \( D^* = (V^*,E^*) \) is called the transitive closure of a simple digraph \( D = (V,E) \) if (i) \( V = V^* \) and (ii) \((v_i,v_j) \in E^* \) if there is a dipath from \( v_i \) to \( v_j \) in \( D \).

We are now in a position to present an algorithm to identify all strongly connected components of a digraph, based on the fact that two vertices, \( v_i \) and \( v_j \), are in the same strongly connected component if and only if both \( a_{ij} \) and \( a_{ji} \) of the adjacency matrix \( A \) of \( D^* \) are equal to one. The idea is then to identify the maximal set of elements having this property. The algorithm, which follows directly from the definition of transitive closure, may be summarized as
follows:

Step 1: Simplify a given digraph by deleting all self-loops and by replacing all multiple edges by one edge and denote the resulting simple digraph by \( D=(V,E) \).

Step 2: Label arbitrarily the vertices and edges of \( D \) as \( \{v_1,v_2,\ldots\} \) and \( \{e_1,e_2,\ldots\} \), if they are not already labeled in this fashion.

Step 3: Find the transitive closure of \( D \), denoted by \( D^* \), and find the adjacency matrix \( A=(a_{ij}) \) of \( D^* \).

Step 4: For each \( i \) and \( j \), if \( a_{ij}=0 \), let \( a_{ji}=0 \).

Step 5: For each row (or column), all nonzero elements correspond to a vertex set of a strongly connected component.

The most commonly used algorithm for the determination of the transitive closure of a digraph is due to Warshall [24]. The algorithm starts with the adjacency matrix \( A=(a_{ij}) \) of a digraph and for each nonzero element of \( A \), say \( a_{ij} \), we replace \( i \)th row by the Boolean union of \( i \)th row and \( j \)th row. More specifically, given the adjacency matrix of an \( n \)-vertex digraph, we have the following steps.

Step 1: Let \( i=1 \).

Step 2: Let \( j=1 \).

Step 3: If \( i=j \) or \( a_{ij}=0 \), go to step 5.

Step 4: For each \( k \), if \( a_{jk}=1 \), set \( a_{ik}=1 \).

Step 5: Let \( j=j+1 \). If \( j \leq n \), go to step 3.
Step 6: Let $i = i + 1$. If $i \leq n$, go to step 2.

Step 7: Stop.

The resulting matrix $A$ is then the adjacency matrix of the transitive closure of the original digraph.

After all strongly connected components of a digraph $D=(V,E)$ have been identified, it is a trivial matter to obtain the strong condensation $D_c=(V_c,E_c)$ of $D$. However when $D_c$ is endowed with a large number of vertices, the labeling of the vertices of $D_c$ (i.e., the strongly connected components of $D$) in topological order will generally require topological sorting. This can be accomplished by noticing that the condensation $D_c$ is acyclic and any acyclic digraph has at least one vertex with zero in-degree. In the following procedure, we first label a vertex with zero in-degree as one. Then we remove this vertex and all edges incident with this vertex. There results another acyclic digraph which also has a vertex with zero in-degree. We label this vertex two, and continue this process. A complete step-by-step procedure of topological sorting is as follows:

Step 1: Arbitrarily label the vertices and edges of $D_c$ as $\{v_1, v_2, \ldots, v_m\}$ and $\{e_1, e_2, \ldots\}$.

Step 2: Find the adjacency matrix $A$ and let all diagonal elements of $A$ be one.

Step 3: Set $I=1$ and reserve an $m$-element array called
Step 4: Find a row, say the $k$th, having only one element with value one. Let $\text{LABEL}(k)=I$ and $I=I+1$.

Step 5: Let the $k$th column be the zero column.

Step 6: If $I>m$, stop. Otherwise go to step 4.

After going through these steps, we obtain the mapping $\text{LABEL}$ which enables us to relabel the $k$th vertex by the value $\text{LABEL}(k)$. A computer program (written in Fortran WATFIV) for this topological sorting procedure is given in Appendix B.

We conclude this section with the following example.

Example 2.2. Consider a digraph with twelve vertices and twenty-one edges shown in Figure 2.4.

We first find the adjacency matrix and enter this matrix into the computer program provided in Appendix A for finding all strongly connected components. We obtain the following results:
THE CLOSURE MATRIX IS

\[
\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

THE ADJACENCY MATRIX IS

\[
\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

THE COMPONENT MATRIX IS

\[
\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

COMPONENT NO: 1

1

COMPONENT NO: 2

2

COMPONENT NO: 3

3

COMPONENT NO: 4

4

COMPONENT NO: 5

5

COMPONENT NO: 6

6
Here the closure matrix is the adjacency matrix of the transitive closure of this digraph and the component matrix is a matrix in which all nonzero elements of each row (or column) correspond to a strongly connected component. Note that the component matrix is symmetric and it contains some duplicated rows (or columns). Also, strongly connected component #1 is the section digraph defined by \( v_1 \) and \( v_8 \), strongly connected component #2 is a section digraph defined by \( v_2 \), and so forth.

The condensation of the digraph is as shown in Figure 2.5.

Next, we find the adjacency matrix of the strong condensation and enter this matrix into the computer program provided in Appendix B for topological sorting. We have the following results:

<table>
<thead>
<tr>
<th>THE ADJACENCY MATRIX IS</th>
<th>RELABEL VERTEX</th>
<th>3 AS 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 1 0 0 1</td>
<td>RELABEL VERTEX</td>
<td>5 AS 2</td>
</tr>
<tr>
<td>1 0 0 0 1 0</td>
<td>RELABEL VERTEX</td>
<td>1 AS 3</td>
</tr>
<tr>
<td>0 0 0 0 0 0</td>
<td>RELABEL VERTEX</td>
<td>6 AS 4</td>
</tr>
<tr>
<td>1 1 0 0 1 0</td>
<td>RELABEL VERTEX</td>
<td>4 AS 5</td>
</tr>
<tr>
<td>1 0 1 0 0 1</td>
<td>RELABEL VERTEX</td>
<td>2 AS 6</td>
</tr>
<tr>
<td>0 0 1 0 0 0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This enables us to relabel the vertices in topological order by relabeling strongly connected components 1, 2, 3, 4, 5, 6 as strongly connected components 3, 6, 1, 5, 2, 4, respectively.
Figure 2.4. Digraph considered in Example 2.2.

Figure 2.5. Strong condensation
2.3. Applications

In this dissertation we concern ourselves with dynamical systems characterized by ordinary differential equations. We begin by considering large scale systems given in aggregate form (A),

\[ \dot{x}_i = h_i(x_1, x_2, \ldots, x_n, t), \quad i=1,2,\ldots,n, \quad (A) \]

and we assume that if possible system (A) has been decomposed into the form

\[ \dot{y}_i = f_i(y_i, t) + \sum_{j=1, j \neq i}^{m} g_{ij}(y_j, t), \quad i=1,2,\ldots,m, \quad (B) \]

where \( x_i \in \mathbb{R} \), \( t \in J = [t_0, \infty) \), \( t_0 \geq 0 \), \( \dot{x}_i = dx_i/dt \), \( h_i : \mathbb{R}^n \times J \rightarrow \mathbb{R} \), \( y_i \in \mathbb{R}^{n_i} \), \( \dot{y}_i = dy_i/dt \), \( f_i : \mathbb{R}^{n_i} \times J \rightarrow \mathbb{R}^{n_i} \), \( g_{ij} : \mathbb{R}^{n_j} \times J \rightarrow \mathbb{R}^{n_i} \), and \( \sum_{i=1}^{m} n_i = n \). (As customary, \( \mathbb{R}^n \) denotes the Euclidean n-space and \( \mathbb{R}^n \times J \) denotes the product space of \( \mathbb{R}^n \) and \( J \)).

In studying the qualitative behavior of large interconnected systems, system (B) is frequently viewed as a nonlinear and time-varying interconnection of \( m \) isolated or free subsystems described by equations of the form

\[ \dot{u}_i = f_i(u_i, t). \]

In this case, the functions \( g_{ij} \) comprise the interconnecting structure of system (B).

The preceding algorithms may be used to establish the
lower block triangular form for system (\(\tilde{\mathcal{B}}\)), given by

\[
\dot{z}_i = F_i(z_i, t) + \sum_{j=1}^{i-1} G_{ij}(z_j, t), \quad i = 1, 2, \ldots, p, \quad (\tilde{\mathcal{C}})
\]

with maximal \(p\), where \(z_i \in \mathbb{R}^{P_i}\), \(\dot{z}_i = \frac{dz_i}{dt}\), \(F_i : \mathbb{R}^{P_i} \times J \to \mathbb{R}^{P_i}\), \(G_{ij} : \mathbb{R}^{P_j} \times J \to \mathbb{R}^{P_i}\), and \(\sum_{i=1}^{p} P_i = n\). Roughly speaking, our basic approach is to associate a large scale system in form (\(\tilde{\mathcal{B}}\)) with a digraph by assuming that each state vector \(y_i\) corresponds to a vertex and that there is an edge from \(y_j\) to \(y_i\) for each nonzero interconnection \(G_{ij}\). Once this is accomplished, we use the computer programs provided in Appendices A and B to find the strong condensation of the digraph with vertices labeled in topological order. By regrouping all state variables of (\(\tilde{\mathcal{B}}\)) which belong to the same strongly connected components, we obtain the state vectors of system (\(\tilde{\mathcal{C}}\)). It follows that system (\(\tilde{\mathcal{B}}\)) is now in its equivalent form (\(\tilde{\mathcal{C}}\)). In the trivial case when \(p = 1\), the digraph of system (\(\tilde{\mathcal{B}}\)) is strongly connected. In this case no advantage can be realized by the present method. On the other hand, when \(p \geq 2\), a system in form (\(\tilde{\mathcal{C}}\)) offers great advantages. Indeed, results given in [5] and [16] make use of form (\(\tilde{\mathcal{C}}\)).

We may also associate system (\(\tilde{\mathcal{A}}\)) with a digraph by assuming that each \(x_i\) corresponds to a vertex and that there is an edge from \(x_j\) to \(x_i\) if \(x_j\) is a variable of function \(h_i\). Following decomposition steps similar to those given above,
we obtain equivalent system (\( D \)) given by

\[ \dot{w}_i = H_i(w_1, w_2, \ldots, w_i, t), \quad i = 1, 2, \ldots, q, \quad (\tilde{D}) \]

with maximal \( q \), where \( w_i \in \mathbb{R}^{q_i} \), \( \dot{w}_i = dw_i/dt \), \( H_i: \mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \times \ldots \times \mathbb{R}^{q_i} \times J \rightarrow \mathbb{R}^{q_i} \), and \( \sum_{i=1}^{q} q_i = n \).

Kevorkian [10] considers a control system, similar to system (\( \tilde{A} \)), given by

\[ \dot{x}_i = h_i(x_1, x_2, \ldots, x_n, u(t)), \quad i = 1, 2, \ldots, n, \quad (\tilde{A}) \]

where \( u(t) \) is the control vector with components \( u_k(t) \), for \( 1 \leq k \leq m \). He then decomposes system (\( \tilde{A} \)) into system (\( \tilde{B} \)), which is similar to system (\( \tilde{D} \)),

\[ \dot{w}_i = H_i(w_1, w_2, \ldots, w_i, u(t)), \quad i = 1, 2, \ldots, p, \quad (\tilde{D}) \]

and takes advantage of it in analyzing the controllability of system (\( \tilde{A} \)). More specifically, he proves the following result in [10].

**Proposition 2.3.** The dynamical system (\( \tilde{A} \)) is completely uncontrollable (see [25]) if one of the functions \( H_i \) in (\( \tilde{D} \)) is of the form \( H_k(w_k) \), where \( H_k \) is a function of \( w_k \) only.

One disadvantage of equivalent form (\( \tilde{C} \)) is that it must start with a system having separable (i.e., additive) interconnecting structure (form (\( \tilde{B} \))) due to the limitations of conventional graph theory. If we start with the original
aggregate form ($\tilde{A}$), only equivalent system ($\tilde{D}$) can be obtained, in which free subsystems are not specified. To generalize these decomposition techniques, and to allow a system having a nonseparable (i.e., nonadditive) interconnecting structure, a new concept of a graph will be introduced in the next chapter.
3. GENERALIZED GRAPHS AND MODIFIED DIGRAPHS

The present chapter consists of five sections. In the first two sections we introduce a generalized concept of a graph (G-graph) and we examine some of the properties of such graphs. In the third and fourth sections we introduce a modified notion of a digraph (M-digraph) and we establish some of the properties of such digraphs. Finally in the fifth section we present some decomposition algorithms for M-digraphs.

3.1. G-graphs

We will find it useful to consider the following generalization of a graph.

**Definition 3.1.** A *generalized graph*, called a G-graph, is a pair G=(V,E), where

(i) V={v_1, v_2, ..., v_n} is a finite set, called the *vertex set*, and each element v_i ∈ V is called a *vertex*, and

(ii) E={E_1, E_2, ..., E_m} is a finite family, called the *edge family*, for which the elements are of the form \{E'_i, E''_i\} where the E'_i and E''_i are nonempty subsets of V and no ordering of these subsets is implied.

It should be noted that all graphs are G-graphs. Specifically, we may view a graph as a G-graph for which the elements of the edge family consist of pairs of singleton
subsets. On the other hand, the notion of a G-graph in general is not a special case of a hypergraph\(^1\) (see, e.g., Berge [3]).

**Definition 3.2.** A semi-loop of a G-graph is an edge \(E_i = (E_i', E_i'')\) having \(E_i' \cap E_i'' \neq \emptyset\). A loop is an edge \(E_i = (E_i', E_i'')\) having \(E_i' = E_i''\). Two edges, \(E_i = (E_i', E_i'')\) and \(E_j = (E_j', E_j'')\), are said to be semi-multiple edges if \(E_i' \cap E_j' \neq \emptyset\) and \(E_i'' \cap E_j'' \neq \emptyset\), or \(E_i' \cap E_j'' \neq \emptyset\) and \(E_i'' \cap E_j' \neq \emptyset\). Two edges, \(E_i\) and \(E_j\), are said to be multiple edges if \(E_i\) and \(E_j\) are characterized by identical sets. A G-graph is **simple** if it has no loops and no multiple edges. A G-graph is **strictly simple** if it has no semi-loops and no semi-multiple edges.

**Example 3.1.** To illustrate the preceding concepts we consider a specific case of a G-graph,

\[
G = (\{v_1, v_2, \ldots, v_5\}, \{E_1, E_2, \ldots, E_6\}),
\]

\(^1\)A hypergraph \(H=(V,E)\) is a vertex set \(V\) together with an edge family \(E\), for which the elements of \(E\) are nonempty subsets of \(V\). A G-graph may be associated or identified with a hypergraph \(H=(V,E)\) having the property that for every \(E_i \in E\), there is a function \(f_i : E_i \to Y = \{y_1, y_2, y_3\}\) such that two nonempty subsets of \(V\), denoted by \(E_i\) and \(E_i\), may be formed having the property:

- \(v_j \in E_i\) if and only if \(f_i(v_j) = y_1\) or \(y_3\);
- \(v_j \in E_i\) if and only if \(f_i(v_j) = y_2\) or \(y_3\).

However, strictly speaking, as noted before, the notions of G-graph and hypergraph are two distinct concepts.
where \( E_1 = (\{v_1, v_2\}, \{v_1, v_2\}) \), \( E_2 = (\{v_1, v_4\}, \{v_5\}) \), \( E_3 = (\{v_5\}, \{v_1, v_4\}) \), \( E_4 = (\{v_2\}, \{v_3, v_4, v_5\}) \), \( E_5 = (\{v_2, v_3\}, \{v_3\}) \), and \( E_6 = (\{v_3\}, \{v_4\}) \). This G-graph may be represented pictorially as shown in Figure 3.1a, where each edge is represented by a linkage of two subsets of \( V \) (e.g., if \( E_k = (\{v_1, v_j\}, \{v_i, v_j\}) \), \( E_k \) is represented as shown in Figure 3.1b). Note that \( E_1 \) is a loop, \( E_5 \) is a semi-loop, \( E_2 \) and \( E_3 \) are multiple edges, and \( E_4 \) and \( E_5 \) are semi-multiple edges.

3.2. Properties of G-graphs

The notions of connectivity and traversability give rise to many practical applications in conventional graph theory. Among the properties of traversability, Euler line, unicursal line, Hamiltonian circuit, and Hamiltonian path are of special interest. In this section we extend these notions to G-graphs and generate a class of graphs associated with a G-graph. The existence of the traversal properties in a G-graph can be related to the existence of such properties in a corresponding class of graphs. A proposition on which this relationship is based is also provided.

**Definition 3.3.** Consider a G-graph, \( G=(V,E) \). A walk is a finite alternating sequence of vertices and edges, \((\bar{v}_1, \bar{E}_1, \bar{v}_2, \bar{E}_2, \ldots, \bar{E}_n, \bar{v}_{n+1})\), beginning and ending with vertices such that for all \( i=1,2,\ldots,n \), and \( \bar{E}_i = (\bar{E}'_i, \bar{E}''_i) \) in the sequence, we have
Figure 3.1a. G-graph

Figure 3.1b. an edge $E_k$
either $v_i \in E'_1$ and $v_{i+1} \in E''_1$ or $v_i \in E'_1$ and $v_{i+1} \in E'_1$. If $v_1 = v_{n+1}$, a walk is closed. Otherwise it is open. A walk in which no edge is repeated is a trail. An open walk in which no vertex is repeated is a path. A closed walk in which no vertex is repeated except for the initial and final vertex, is a circuit. A G-graph is connected if there exists a walk between every pair of vertices.

**Definition 3.4.** In a connected G-graph, a closed trail traversing every edge exactly once is called an Euler line. An open trail traversing every edge exactly once is called a unicursal line. A circuit traversing every vertex exactly once, with the starting and terminating vertex excepted, is called a Hamiltonian circuit. A path traversing every vertex exactly once is called a Hamiltonian path.

**Definition 3.5.** A graph $G'_x = (V'_x, E'_x)$ is said to be in the linear class of a G-graph $G = (V, E)$ if $V'_x = V$ and if there is a one-to-one correspondence between $E'_x$ and $E$ such that for each $E_i = (E'_i, E''_i) \in E'_x$ and its corresponding edge $e_i = (v_{i1}, v_{i2}) \in E_x$, we have $v_{i1} \in E'_1$ and $v_{i2} \in E''_1$. The collection of all such possible graphs is called the linear class of $G$.

The following result enables us to deduce the existence of an Euler line, a unicursal line, a Hamiltonian circuit, or a Hamiltonian path in a G-graph from its
Proposition 3.1. A given connected G-graph contains an Euler line (resp., a unicursal line, a Hamiltonian circuit, a Hamiltonian path) if and only if at least one graph in the linear class of this G-graph has an Euler line (resp., a unicursal line, a Hamiltonian circuit, a Hamiltonian path).

Proof. Obvious.

Example 3.2. Consider eight cities connected by nine highways as shown in Figure 3.2. We wish to answer the following questions. Using each highway only once, is there a closed trail with any one of the indicated cities as a starting and terminating point? If the answer is no, are there one or more open trails with any pair of the indicated cities serving as starting and terminating points (using each highway only once)? If the answer is affirmative, determine all possible combinations of cities which serve as starting and terminating points. Subsequently, we rule out the following unreasonable (i.e. inefficient) cases: from Minneapolis to Albert Lea (or vice versa) by highway 9; from Milwaukee to Chicago (or vice versa) by highway 9; from Chicago to St. Louis (or vice versa) by highway 4; and from Chicago to Indianapolis (or vice versa) by highway 3.

Relabeling, we obtain from Figure 3.2 the G-graph shown
Figure 3.2. Eight cities with nine highways
Figure 3.3. G-graph of the transportation system
Figure 3.4. Linear class of a G-graph
in Figure 3.3. The preceding posed questions are equivalent to establishing the existence of an Euler line or a unicursal line in the G-graph. Totally, there are sixteen different graphs in the linear class of G, as shown in Figure 3.4.

Since none of the graphs $G_a - G_p$ has all vertices with even degree, this indicates that no Euler line can be found in $G_a - G_p$. It follows, by Proposition 3.1, that no Euler line can be found in the G-graph. Only four graphs, $G_a$, $G_c$, $G_d$ and $G_k$, have one pair of vertices with odd degree. Therefore there exist several open trails between Des Moines and either Milwaukee, Chicago, Indianapolis or St. Louis.

3.3. G-digraphs and M-digraphs

We begin with the following definition.

**Definition 3.6.** A G-graph with ordered elements in its edge family is called a generalized digraph (G-digraph) and is denoted by D. The first member of each edge $E_i$ is called the initial vertex set and is usually denoted henceforth by $E_i^j$, while the second member of $E_i$ is called the terminal vertex set and is usually denoted henceforth by $E_i^t$.

**Definition 3.7.** A G-digraph in which the terminal vertex set in each edge is a singleton is called a modified G-digraph or M-digraph.

Although the subsequent definitions and results are
stated for M-digraphs, most of these are applicable to G-
digraphs as well, with obvious modifications.

**Definition 3.8.** A simple digraph \( M_\ell = (V_\ell, E_\ell) \) is said to be
the linear version of an M-digraph \( M = (V, E) \) if \( V_\ell = V \) and if
for each \( E_\ell = (\{v_{i1}, v_{i2}, \ldots\}, \{v_j\}) \) in \( E \), there exist edges
\((v_{i1}, v_j), (v_{i2}, v_j), \ldots \) in \( E_\ell \), and for each \( e_\ell = (v_{j1}, v_{j2}) \in E_\ell \)
there exists an edge \((\ldots, v_{j1}, \ldots, v_{j2}) \) in \( E \).

**Example 3.3.** Consider an M-digraph given by
\[
M = (\{v_1, v_2, v_3, v_4, v_5\}, \{E_1, E_2, E_3, E_4, E_5, E_6\}),
\]
where \( E_1 = (\{v_1, v_4\}, \{v_5\}) \), \( E_2 = (\{v_1, v_2, v_3\}, \{v_3\}) \), \( E_3 = (\{v_2\}, \{v_1\}) \), \( E_4 = (\{v_4, v_5\}, \{v_3\}) \), \( E_5 = (\{v_3, v_5\}, \{v_5\}) \), and \( E_6 = (\{v_1\}, \{v_4\}) \). The linear version of \( M \) is then given by
\[
M_\ell = (\{v_1, v_2, v_3, v_4, v_5\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}),
\]
where \( e_1 = (v_1, v_5) \), \( e_2 = (v_4, v_5) \), \( e_3 = (v_5, v_3) \), \( e_4 = (v_1, v_3) \), \( e_5 = (v_2, v_3) \), \( e_6 = (v_2, v_1) \), \( e_7 = (v_4, v_3) \), \( e_8 = (v_3, v_5) \), and \( e_9 = (v_1, v_4) \).

\( M \) and \( M_\ell \) are depicted in Figure 3.5a and Figure 3.5b respectively.

**Definition 3.9.** For an M-digraph, a diwalk is a finite
alternating sequence of vertices and edges, \((v_1, E_1, v_2, E_2, \ldots, E_n, v_{n+1})\) beginning and ending with vertices such that \( v_1 \in E_1 \)
and \( v_{n+1} \in E_n \) for all \( i = 1, 2, \ldots, n \), where \( E = (E_1', E_2') \). If
\( v_1 = v_{n+1} \), a diwalk is open. Otherwise it is closed. An
Figure 3.5a. M-digraph

Figure 3.5b. Linear version
M-digraph is **acyclic** if it has no closed diwalk. An open diwalk in which no vertex appears more than once is called a **dipath**. If in an M-digraph there is a dipath from \( v_i \) to \( v_j \), \( i \neq j \), then \( v_j \) is said to be **reachable** from \( v_i \) and we write \( v_i \rightarrow v_j \). We admit the case that \( v_i \rightarrow v_i \). If \( v_i \rightarrow v_j \) and \( v_j \rightarrow v_i \), then \( v_i \) and \( v_j \) are **mutually reachable**. If all pairs of a set of vertices are mutually reachable, then the set is said to be **mutually reachable**.

**Definition 3.10.** A sub-M-digraph \( S = (V_s, E_s) \) of an M-digraph \( M = (V, E) \) is an M-digraph with \( V_s \subseteq V \) and \( E_s \subseteq E \). A sub-M-digraph \( S \) is called a **section M-digraph** defined by \( V_s \) if \( S \) is the maximal sub-M-digraph defined by the vertex set \( V_s \). (From the maximality of a section M-digraph the vertex set completely defines the section M-digraph).

**Definition 3.11.** We write \( v_i \sqsubset v_j \) if there exists an edge \( E_k = (E_k', E_k'') \) such that \( v_i \) and \( v_j \) are contained in \( E_k' \). Two vertices \( v_i \) and \( v_j \) are said to be in the **same level** if \( v_i \sqsubset v_j \) or if there is a finite sequence of vertices, \( (v_i = \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_k = v_j) \), such that \( \tilde{v}_p \sqsubset \tilde{v}_{p+1} \) for \( p = 1, 2, \ldots, k-1 \). \( v_j \) is said to be **quasi-reachable** from \( v_i \), denoted by \( v_i \not\sqsubset v_j \), if there is a finite sequence of vertices, \( (v_i = \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_k = v_j) \), such that for \( q = 1, 2, \ldots, k-1 \), we have either \( \tilde{v}_q \sqsubset \tilde{v}_{q+1} \) or \( \tilde{v}_q \not\sqsubset \tilde{v}_{q+1} \). If \( v_i \not\sqsubset v_j \) and \( v_j \not\sqsubset v_i \), then \( v_i \) and \( v_j \) are **mutually quasi-reachable**. If all pairs of a set of vertices are mutually
Definition 3.12. A strong component (SC) is a section M-digraph defined by a maximal set of mutually reachable vertices. A strong component SC is said to be between strong components SC and SC if there is a diwalk from \( v_i \in SC \) to \( v_j \in SC \) traversing a vertex in SC. Two strong components SC and SC are in the same level if there exist two vertices \( v_i \in SC \) and \( v_j \in SC \) which are in the same level. A level component (LC) is a section M-digraph defined by a maximal set of vertices in the same level. A level-strong component (LSC) is a section M-digraph defined by a maximal set of mutually quasi-reachable vertices.

Definition 3.13. A condensation of an M-digraph \( M=(V,E) \) is a simple M-digraph, denoted by \( M^*=(V^*,E^*) \), in which each vertex \( C_i \in V^* \) is a section M-digraph defined by a subset of \( V \), called \( V_i \), where \( \cup V_i = V \) and \( V_i \cap V_j = \emptyset \), and there is an edge from \( \{C_p,C_q,\ldots\} \) to \( C_r \) if there is an edge from \( \{v_i,v_j,\ldots\} \) to \( v_k \) in \( M \) where \( v_i \in C_p \), \( v_j \in C_q \), \ldots, and \( v_k \in C_r \). The strong (resp., level, level-strong) condensation of an M-digraph has all strong (resp., level, level-strong) components as its vertices. The semi-strong condensation of an M-digraph is a maximal acyclic M-digraph in which each vertex is a section M-digraph defined by a union of some vertex sets.
of strong components. Each vertex in the semi-strong condensation of an M-digraph is called a semi-strong component. An edge is called a strong (resp., level, level-strong, semi-strong) interconnecting edge if it is not contained in any strong (resp., level, level-strong, semi-strong) component. If \( v_i \) and \( v_j \) are in the same strong (resp., level, level-strong, semi-strong) component, we write \( v_i \leftrightarrow v_j \) (resp., \( v_i \equiv v_j \), \( v_i \not\equiv v_j \), \( v_i \lessdot v_j \)).

Remark 3.1. The motivation for considering M-digraphs is their applicability to systems of nonseparable equations of the form

\[
x_i = f_i(x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n,
\]

while usual graphs normally apply to separable systems of equations of the form

\[
x_i = \sum_{j=1}^{n} g_{ij}(x_j), \quad i = 1, 2, \ldots, n.
\]

To fix some of the above ideas, we consider the specific set of equations

\[
x_1 = g_1(x_2), \quad x_5 = g_5(x_4, x_6),
\]

\[
x_2 = g_2(x_1), \quad x_5 = g_5(x_4, x_6),
\]

\[
x_3 = g_3(x_1), \quad x_6 = g_6(x_5).
\]
The M-digraph of this system may be represented pictorially as shown in Figure 3.6a. The strong components, the semi-strong components, the level components, and the level-strong components of this M-digraph are shown in Figures 3.6b-e, respectively. The strong, semi-strong, level, and level-strong condensations are shown in Figures 3.6f-i, respectively.

Remark 3.2. The maximal strongly connected section digraph is the proper definition of strong component in a digraph; however, it is not applicable for M-digraphs since some strong components of an M-digraph may not be strongly connected. For example, SC4 in Figure 3.6b is not strongly connected.

In the fifth section of this chapter we present decomposition algorithms which make it possible to identify the semi-strong components of M-digraphs. In chapter 4 we will use these algorithms in the decomposition of large scale dynamical systems described by ordinary differential equations. This in turn will enable us to obtain some very useful stability results for large scale systems. First however, we need to establish some of the properties of M-digraphs.
Figure 3.6. Properties of an M-digraph. (a) M-digraph. (b) Strong components. (c) Semi-strong components. (d) Level components. (e) Level-strong components. (f) Strong condensation. (g) Semi-strong condensation. (h) Level condensation. (i) Level-strong condensation.
3.4. Properties of M-digraphs

We now establish some of the properties of M-digraphs.

Proposition 3.2. "\( \leftrightarrow \)" , "\( \leftrightarrow ^* \)" , "\( \rightarrow \)" , and "\( \rightarrow ^* \)" are equivalence relations.

Proof. Obvious.

Proposition 3.3. In a digraph, "\( \leftrightarrow \)" , "\( \leftrightarrow ^* \)" , and "\( \rightarrow ^* \)" are the same equivalence relation.

Proof. Obvious.

Proposition 3.4. The level condensation of an M-digraph, denoted by \( M^*(\overline{M}) \), is a digraph.

Proof. For purposes of contradiction, assume that \( M^*(\overline{M}) = (V^*, E^*) \) is not a digraph. Then there exists at least one edge \( E^*_i = (E^*_i, E^*_i') \subseteq E^* \) where \( E^*_i \) is not a singleton. Without loss of generality, assume that \( E^*_i = (V^*_{i1}, V^*_{i2}, \ldots) \subseteq E^* \) and \( V^*_{i1} \not\subseteq V^*_{i2} \). This implies that \( V^*_{i1} \cap V^*_{i2} = \emptyset \), and contradicts that \( V^*_{i1} \) and \( V^*_{i2} \) are two different level components.

Proposition 3.5. Let \( D \) be the linear version of an M-digraph \( M \). Then \( v_i \leftrightarrow v_j \) in \( M \) if and only if \( v_i \leftrightarrow v_j \) in \( D \).
Proof. Assume that there is a dipath from $v_i$ to $v_j$ in $M$, i.e., $(v_i = \bar{v}_1, \bar{e}_1, \bar{v}_2, \bar{e}_2, \ldots, \bar{e}_n, \bar{v}_{n+1} = v_j)$. Since $D$ is the linear version of $M$, there exist $\bar{e}_k = (\bar{v}_k, \bar{v}_{k+1})$ for $k = 1, 2, \ldots, n$. We may construct a dipath in $D$, $(v_i = \bar{v}_1, \bar{e}_1, \bar{v}_2, \bar{e}_2, \ldots, \bar{e}_n, \bar{v}_{n+1} = v_j)$ so that $v_j$ is reachable from $v_i$ in $D$. For the same reason, $v_i$ is also reachable from $v_j$ in $D$. Therefore $v_i \leftrightarrow v_j$ in $D$.

Conversely, assume that $v_i \leftrightarrow v_j$ in $D$. Then there is a dipath $(v_i = \bar{v}_1, \bar{e}_1, \ldots, \bar{e}_n, \bar{v}_{n+1} = v_j)$ in $D$. For each edge $\bar{e}_k$ in the sequence, there exists a corresponding edge $\bar{E}_k$ in $M$ so that $\bar{v}_k \in \bar{E}_k$ and $\bar{v}_{k+1} \in \bar{E}_k$. Therefore a dipath $(v_i = \bar{v}_1, \bar{E}_1, \ldots, \bar{E}_n, \bar{v}_{n+1} = v_j)$ exists in $M$. For similar reasons, a dipath also exists from $v_j$ to $v_i$ in $M$. Therefore $v_i \leftrightarrow v_j$ in $M$.

Remark 3.3. Proposition 3.5 enables us to find all strong components of an $M$-digraph via its linear version. Therefore all existing techniques for finding strong components of digraphs (see section 2.2 or see [21]) are also accessible to $M$-digraphs.

Proposition 3.6. The level-strong condensation $M^*(\mathcal{A})$ of an $M$-digraph $M$ is isomorphic to the strong condensation of the level condensation of the $M$-digraph.

Proof. It suffices to show that $v_i \mathcal{A} v_j$ if and only if $v_i$ and $v_j$ are in the same vertex of the strong condensation of the level condensation of the $M$-digraph.
Assume that $v_i \not\equiv v_j$ and also assume that $v_i$ is in the level component $LC_i$ and $v_j$ is in the level component $LC_j$. Since $v_i$ and $v_j$ are mutually quasi-reachable, there exist two finite sequences of vertices, $(v_i=\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_m=v_j)$ and $(v_j=\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n=v_i)$, such that for $k=1,2,\ldots,m-1$, $\bar{v}_k \equiv \bar{v}_{k+1}$ or $\bar{v}_k \not\equiv \bar{v}_{k+1}$, and for all $l=1,2,\ldots,n-1$, $\bar{v}_l \equiv \bar{v}_{l+1}$ or $\bar{v}_l \not\equiv \bar{v}_{l+1}$. This implies that $LC_i$ and $LC_j$ are mutually reachable in the level condensation. Therefore $v_i$ and $v_j$ are in the same vertex of the strong condensation of the level condensation.

Conversely, if $v_i$ and $v_j$ are in the same vertex of the strong condensation of the level condensation and if $v_i \in LC_i$ and $v_j \in LC_j$, then $LC_i$ and $LC_j$ are mutually reachable in the level condensation. It follows that $v_i$ and $v_j$ are mutually quasi-reachable implying $v_i \not\equiv v_j$.

Remark 3.4. The level condensation of the strong condensation of an $M$-digraph is not necessarily isomorphic to the level-strong condensation of the $M$-digraph. For example, consider the $M$-digraph shown in Figure 3.7a, $M=\{v_1,v_2,v_3\}$, $\{E_1,E_2,E_3\}$ with $E_1=\{v_1\}, E_2=\{v_2\}, E_3=\{v_1,v_3\}$. The level condensation of the strong condensation of the $M$-digraph has two vertices while the level-strong condensation has only one vertex (see Figure 3.7).

Proposition 3.6 states that a systematic way of finding all level-strong components is to find first the level condensation, $M^*(\overline{\pi})$, and then the strong condensation of $M^*\overline{\pi}$. 
Figure 3.7a. M-digraph

Figure 3.7b. Level condensation of the strong condensation

Figure 3.7c. Level-strong condensation
Proposition 3.7. Each semi-strong component of an M-digraph \( M \) is a semi-strong component of one of the level-strong components of \( M \).

Proof. Note that the level-strong condensation is a maximal acyclic digraph and the semi-strong condensation is a maximal acyclic M-digraph. It is clear that the level-strong condensation is a condensation of the semi-strong condensation. This implies that each semi-strong component of \( M \) is contained in one of the level-strong components of \( M \), and since any semi-strong component defined by a subset of the vertex set \( V_s \subseteq V \) is also a semi-strong component of any section M-digraph defined by \( V_r \), where \( V_s \subseteq V_r \subseteq V \), the proof is completed.

Proposition 3.8. An M-digraph \( M \) has only one level-strong component if and only if it has only one semi-strong component.

Proof. Assume that an M-digraph has only one semi-strong component. Then by Proposition 3.7, the M-digraph has only one level-strong component.

Conversely, for purposes of contradiction, assume that an M-digraph having only one level-strong component contains two or more semi-strong components. It follows that the semi-strong condensation has two or more vertices and at least one of the vertices will never be the terminal vertex.
of any edge in the semi-strong condensation. This vertex is then a level-strong component and the M-digraph has at least two level-strong components which is a contradiction.

Remark 3.5. The semi-strong condensation of an M-digraph M may be constructed from the strong condensation by successively replacing the initial vertices of every semi-loop and every vertex (strong component of M) between any pair of initial vertices of the semi-loop, by one vertex. We will refer to this procedure as the **direct method**.

Proposition 3.7 and Proposition 3.8 enable us to find, indirectly, all semi-strong components by first obtaining all level-strong components of M, then obtaining all level-strong components of each level-strong component of M, and so forth. This process is terminated with a class of section M-digraphs each of which contains only one level-strong component. Each member of this class will also be a semi-strong component of the underlying M-digraph.

3.5. Decomposition Techniques

We now present some algorithms to identify the level, strong, level-strong, and semi-strong components of M-digraphs. A complete computer program to identify semi-strong components is provided in Appendix C.

For computer usage, it is necessary to represent the incidence properties of such graphs in numerical form.
Definition 3.14. The incidence array $C=(c_{ij})$ of an M-digraph is an $n \times m$ array with $c_{ij}=0,1,2,$ or $3$, where $n$ denotes the number of vertices and $m$ is the number of edges, so that for each edge $E_j=(E'_j,E''_j)$ in the edge family,

$$c_{ij} = \begin{cases} 
0 & \text{if } v_i \notin E'_j \text{ and } v_i \notin E''_j \\
1 & \text{if } v_i \in E'_j \text{ and } v_i \notin E''_j \\
2 & \text{if } v_i \notin E'_j \text{ and } v_i \in E''_j \\
3 & \text{if } v_i \in E'_j \text{ and } v_i \in E''_j.
\end{cases}$$

Clearly, each column of an incidence array of an M-digraph can have at most one 2 or one 3, and not both, because all terminal vertex sets in an M-digraph are singletons.

3.5.1 Level components

If a given M-digraph is a digraph, each vertex is a level component, otherwise the following steps are used.

Step 1: Given an M-digraph $M$ with $n$ vertices, determine its incidence array $C=(c_{ij})$.

Step 2: Reserve an $n$-element array, called $LC$, and let $LC(i)=i$ for $i=1,2,\ldots,n$.

Step 3: Check each column one time. If in the $j$th column, there are two or more nonzero $c_{ij} \neq 2$, e.g., $c_{pj}=1$, $c_{qj}=1$, $\ldots$, $c_{rj}=3$, we let $LC(s)=\min\{LC(p),LC(q),\ldots,LC(r)\}$
for each $s \in \{1, 2, \ldots, n\}$ where $s \neq p$, $s \neq q$, \ldots, $s \neq r$, and where $\text{LC}(s) = \text{LC}(p)$, $\text{LC}(s) = \text{LC}(q)$, \ldots, or $\text{LC}(s) = \text{LC}(r)$. Then let $\text{LC}(p) = \text{LC}(q) = \ldots = \text{LC}(r) = \min\{\text{LC}(p), \text{LC}(q), \ldots, \text{LC}(r)\}$.

Step 4: For each $k$ where $\text{LC}(k) = k$, there exists a level component which is the section $M$-digraph defined by vertex set $V_s = \{v_i : \text{LC}(i) = k\}$.

### 3.5.2. Strong components

Existing techniques for digraphs can be used to find the strong components of an $M$-digraph by Proposition 3.5.

**Step 1:** Given an $M$-digraph $M$, find the adjacency matrix, $A = (a_{ij})$, of the linear version of $M$.

**Step 2:** Enter matrix $A$ into a digital computer and find the strong components of the linear version of $M$, using one of several existing algorithms (see section 2.2 or see [14, 21]).

**Step 3:** Each section $M$-digraph defined by all the vertices in a strong component of the linear version of $M$ represents a strong component of $M$.

### 3.5.3. Level-strong components

We may find all level-strong components of an $M$-digraph by determining all section $M$-digraphs defined by the maximal mutually quasi-reachable sets of vertices. However, Proposition 3.6 indicates that the level-strong components of an $M$-digraph can also be determined by finding all strong...
components of the level condensation. In the following, we use the latter method.

Step 1: Find all level components of $M$.
Step 2: Find the level condensation $M^*(\overline{\mathbb{N}})$.
Step 3: Find all strong components of $M^*(\overline{\mathbb{N}})$.
Step 4: Each strong component determined in step 3, in its original form, corresponds to a level-strong component of $M$.

3.5.4. Semi-strong components

Two methods for finding the semi-strong components of an $M$-digraph have been alluded to, called the direct method and the indirect method (see Remark 3.5). In the former method, the strong components are determined first and then combined in an appropriate fashion, while in the latter method, an $M$-digraph is decomposed into a class of level-strong components where each member of the class contains only one level-strong component.

3.5.4.1. Direct method

Step 1: Find all strong components of a given $M$-digraph with $n$ vertices, $M$.

Step 2: Reserve an $n$-element array, called SSC. Let $SSC(i)=SSC(j)$ if and only if $v_i \leftrightarrow v_j$. Clearly there are $m$ different values for SSC if there are $m$ strong components.
Step 3: For each strong interconnecting edge $E = (\{\bar{v}_1, \ldots, \bar{v}_j, \ldots, \bar{v}_k\}, \{\bar{v}_k\})$, if $SSC(j) = SSC(k)$, then add edges $(\bar{v}_k, \bar{v}_1), \ldots, (\bar{v}_k, \bar{v}_j), \ldots, (\bar{v}_k, \bar{v}_k)$ to $M$. Find all strong components of $M$ with added edges.

Step 4: Repeat step 3 until no additional edge can be added.

Step 5: Each strong component found in step 4 corresponds to a semi-strong component of $M$.

3.5.4.2. Indirect method

Step 1: Given an $M$-digraph $M$ with $n$ vertices, find the level-strong components of $M$, called $LS_1, \ldots, LS_m$.

Step 2: If $m=1$, there is only one semi-strong component. Stop.

Step 3: Find all level-strong components of the $LS_i$, $i=1,2,\ldots,m$, and denote these by $LS_{11}, LS_{12}, \ldots, LS_{21}, LS_{22}, \ldots$.

Step 4: Find all level-strong components of the $LS_{pq}$ determined in step 3 and denote these $LS_{111}, LS_{112}, \ldots, LS_{121}, LS_{122}, \ldots, LS_{221}, LS_{222}, \ldots$.

Step 5: The above procedure is terminated when every $LS$ contains only one level-strong component, which is then a semi-strong component of $M$.

Both methods have advantages and disadvantages. The direct method uses less memory storage since it determines the vertex set of each semi-strong component without a
complete section M-digraph description. Also in this case, existing well-established techniques for digraphs can be used to advantage. On the other hand, the indirect method contains a procedure of checking whether an M-digraph contains only one semi-strong component. If an M-digraph has only one level component (or one level-strong component) then we conclude that there is only one semi-strong component and there is no need to proceed further, thus requiring less computer time. A principal drawback of the indirect method is that it requires much more memory storage than the direct method since it is necessary to obtain the complete description of each level-strong component in order to decompose it further, if possible. A modified algorithm may be suggested which takes advantage of both methods provided that the computer memory is large enough. In this procedure, the first three or four steps of the indirect method are used to determine a class of level-strong components each of which may have two or more semi-strong components. Then the direct method is used to determine the semi-strong components of each member in the class.

The complete computer program of the direct method, coded in Fortran WATFIV, is given in Appendix C.

We conclude this section with the following example.

Example 3.4. Consider the M-digraph shown in Figure 3.6a. We first find the incidence array and enter this matrix into
the computer program provided in Appendix C for finding all semi-strong components. We have the following results.

\[
\begin{array}{cccccccc}
1 & 2 & 1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
\end{array}
\]

SEMI-STRONG COMPONENT NO: 1
\[
1 \quad 2 
\]
SEMI-STRONG COMPONENT NO: 2
\[
3 
\]
SEMI-STRONG COMPONENT NO: 3
\[
4 
5 
6 
\]

Note that we obtain three semi-strong components, i.e., the section M-digraphs defined by \(\{v_1, v_2\}\), \(\{v_3\}\) and \(\{v_4, v_5, v_6\}\), respectively.
4. STRUCTURE OF LARGE SCALE SYSTEMS

The present chapter consists of three sections. In the first and second sections we present and discuss several equivalent forms of large scale dynamical systems. These forms are generated by means of the results developed in chapter 3. In the third section we present an example to illustrate these decomposed equivalent systems.

4.1. Equivalent Forms

We will consider in the following several equivalent forms of representing large scale systems, including (i) the aggregate form (where the system structure can be associated with an M-digraph), (ii) interconnected form (where each vector-valued variable corresponds to the vertex set of a level component), (iii) lower block triangular form (where each vector-valued variable corresponds to the vertex set of a level-strong component), (iv) composite sequential form (where each vector-valued variable corresponds to the vertex set of a strong component), and (v) sequential form (where each vector-valued variable corresponds to the vertex set of a semi-strong component).

4.1.1. Aggregate form

We consider systems which can appropriately be described by ordinary differential equations of the form
\[ \dot{x}_i = h_i(x, t), \quad i = 1, 2, \ldots, n, \quad (A) \]

or equivalently, by

\[ \dot{x} = h(x, t), \quad (A') \]

where \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \), \( t \in J = [t_0, \infty) \), \( t_0 > 0 \), \( \dot{x} = dx/dt \), \( h_i : B(r) \times J \rightarrow \mathbb{R} \) for some \( r > 0 \) and for all \( i = 1, 2, \ldots, n \), where \( B(r) = \{ x \in \mathbb{R}^n : |x| < r \} \), \( \mathbb{R} \) denotes the set of real numbers, and \( |\cdot| \) represents the Euclidean norm. Henceforth we assume that \( h(\cdot) = (h_1(\cdot), h_2(\cdot), \ldots, h_n(\cdot))^T \) is continuous in all variables and that it is locally Lipschitz continuous in \( x \).

Under these assumptions system \((A)\) possesses for every \( x_0 \in B(r) \) and for every \( t_0 \in \mathbb{R}^+ = [0, \infty) \) a unique solution \( x(t; x_0, t_0) \) where \( x(t_0; x_0, t_0) = x_0 \). We also assume that Eq. \((A')\) admits the trivial solution \( x = 0 \) so that \( h(0, t) = 0 \) for all \( t \in J \). In fact, we assume that \( x = 0 \) is an isolated equilibrium.

The preceding assumptions pertain to local results. In the case of global results we assume that \( h : \mathbb{R}^n \times J \rightarrow \mathbb{R}^n \), that \( h \) is continuous in \((x, t)\) and locally Lipschitz continuous in \( x \) and that \( x = 0 \) is the only equilibrium for \((A')\). In this case system \((A')\) has for every \( x_0 \in \mathbb{R}^n \) and for every \( t_0 \in \mathbb{R}^+ \) a unique solution \( x(t; x_0, t_0) \).

We associate an M-digraph with the aggregate form \((A)\) as follows. First, each variable \( x_i \) is made to correspond
to a vertex, which we also denote by $x^1_i$, in the associated M-digraph. Next, if possible, each function $h_i$ is written as the sum of several simpler functions of fewer variables. For example, if $h_i(x,t)=h_i(x_1,x_3,x_5)=x_1x_3+x_3x_5$ then we write $h_i(x,t)=h_{i1}(x_1,x_3)+h_{i2}(x_3,x_5)$ where $h_{i1}(\cdot)$ and $h_{i2}(\cdot)$ are defined in the obvious way. Next, each simple function $h_{ij}(x_1^1,x_1^2,\ldots,x_1^\ell)$ is made to correspond to an edge $(\{x_1^1,\ldots,x_1^\ell\},\{x_1\})$. By construction, we arrive at an M-digraph which is not necessarily a simple M-digraph (see Definition 3.2). Finally, to simplify things later (when searching for level components, etc.), we replace all multiple edges by one edge and delete all loops. This results in a simple M-digraph, denoted $M$, associated with the aggregate form. The resulting M-digraph can be used to obtain different equivalent forms of system (A) by searching for its level, strong, level-strong, and semi-strong components. This is accomplished in the subsequent discussion.

4.1.2. Interconnected form

After obtaining all level components of the associated M-digraph of system (A), we let $y_i$ represent the vertex set of each component, where $1 \leq i \leq m$ if there are $m$ level components in the M-digraph $M$. This results in an equivalent description for system (A) given by
\[ \dot{y}_i = f_i(y_i, t) + \sum_{j=1}^{m} g_{ij}(y_j, t), \quad i = 1, 2, \ldots, m, \quad (B) \]

where \( m \) is maximal, where \( y_i \in \mathbb{R}_i, \ t \in J, \ \sum_{i=1}^{m} m_i = n, \) and where the functions \( f_i \) and \( g_{ij} \) are defined in the appropriate way.

4.1.3. Lower block triangular form

By determining the level-strong components of the \( M \)-digraph \( M \), or by determining the strong components of the digraph associated with system (B), we obtain the system of equations which is equivalent to system (A) and system (B), given by the lower block triangular form,

\[ \dot{z}_i = \tilde{f}_i(z_i, t) + \sum_{j=1}^{i-1} \tilde{g}_{ij}(z_j, t), \quad i = 1, 2, \ldots, p, \quad (C) \]

where \( p \) is maximal, where \( z_i \in \mathbb{R}_i, \ t \in J, \ \sum_{i=1}^{p} p_i = n, \) and where the functions \( \tilde{f}_i \) and \( \tilde{g}_{ij} \) are defined in an appropriate way.

4.1.4. Composite sequential form

By identifying all its strong components, we can transform system (A) into an equivalent form given by

\[ \dot{p}_i = F_i(p_1, p_2, \ldots, p_i, t), \quad i = 1, 2, \ldots, r, \quad (D) \]

where \( r \) is maximal, where \( p_i \in \mathbb{R}_i, \ t \in J, \ \sum_{i=1}^{r} r_i = n, \) and where the functions \( F_i \) are defined in an appropriate way. We call (D) the composite sequential form of (A).
4.1.5. Sequential form

In stability analysis of large scale dynamical systems (by decomposition techniques as well as other techniques (see Michel and Miller [15])) the usual approach is to decompose a large scale system into several smaller subsystems which are appropriately interconnected. The stability properties of the overall system are then deduced from the qualitative properties of the subsystems and the system interconnecting structure. This approach fails when system (A) is in the composite sequential form, since in this case the notion of isolated or free subsystem (see [11,12]) is not defined. To overcome this drawback while still maintaining some of the advantages of this form, we introduce the sequential form next.

By determining all semi-strong components of the M-digraph of system (A), we obtain the sequential form given by

\[ \dot{u}_i = \tilde{F}_i(u_1, t) + \tilde{G}_i(u_1, u_2, \ldots, u_{i-1}, t), \quad i=1,2,\ldots,s, \tag{E} \]

where \( s \) is maximal, where \( \tilde{G}_i \neq 0 \), where \( u_i \in R^{s_i}, \quad t \in J, \quad \sum_{i=1}^{s} s_i = n \), and where the functions \( \tilde{F}_i \) and \( \tilde{G}_i \) are defined in the appropriate way. Subsequently, we assume that \( u = (u_1^T, u_2^T, \ldots, u_s^T)^T = 0 \) is an isolated equilibrium of system (E).

System (E) may be viewed as a nonlinear and time-varying interconnection of \( s \) free subsystems described by
\[
\dot{w}_i = F_i(w_i, t) \quad (S_i)
\]

with interconnecting structure specified by the functions \( \tilde{G}_i \).

4.2. Discussion

We now discuss some features of the equivalent forms of large scale systems established in the preceding section.

Remark 4.1. The decomposition techniques in chapter 2 (see [5],[16]) start with interconnected systems of the form (B) and proceed to transform (B) into the equivalent lower block triangular form (C). This is necessitated by the fact that these results use the usual concept of graph. In contrast, the present results start with the aggregate form (A) and provide a systematic way of transforming this system into any of the equivalent forms (B) - (E). In the important work by Kevorkian [14], the aggregate form (A) is transformed into form (D). In the work by Caines and Prints [4], the starting point of the analysis is system (D) (with minor modifications). In Figure 4.1 we provide a summary of the various equivalent forms discussed above. It is clear from this figure that the present development makes possible, (i) the systematic decomposition of system (A) into systems (B) and (C) (via generation of all level and level-strong components), and (ii) two alternate ways of determining form (E) from (A) (see section 3.5).
\[ \dot{x}_i = h_i(x_1, x_2, \ldots, x_n, t) \quad i=1,2,\ldots,n \]  
\[ \dot{y}_i = f_i(y_1, t) + \sum_{j=1}^{m} g_{ij}(y_j, t) \quad i=1,2,\ldots,m \]
\[ \dot{p}_i = F_i(p_1, p_2, \ldots, p_i, t) \quad i=1,2,\ldots,r \]
\[ \dot{z}_i = \tilde{f}_i(z_1, t) + \sum_{j=1}^{i-1} \tilde{g}_{ij}(z_j, t) \quad i=1,2,\ldots,q \]
\[ \dot{u}_i = \tilde{F}_i(u_1, t) + \tilde{G}_i(u_1, u_2, \ldots, u_{i-1}, t) \quad i=1,2,\ldots,s \]

Legend:
LC = identification of level components
SC = identification of strong components
LSC = identification of level-strong components
SSC = identification of semi-strong components

Figure 4.1. Five equivalent forms
Remark 4.2. Kevorkian [14] was one of the first to use form (D) (adjoined by control functions and measurement equations) to advantage in studying controllability and observability of control systems (see section 2.3). The technique which he used to transform system (A) into system (D) is still within the scope of standard graph theory (as opposed to extensions of the type considered herein). He was able to affect the indicated transformation because of the equivalence of strong connectedness of digraphs and M-digraphs (see Proposition 3.5). It turns out that the digraph associated with a system similar to system (A), considered in [14], is the linear version of an M-digraph associated with system (A). Thus, by Proposition 3.5, form (D) and the corresponding form obtained in [14] will be the same.

The composite sequential form can also be used in determining the solutions of system (A) sequentially. In this approach one solves first the system of equations given by

\[ \dot{p}_1 = F_1(p_1, t). \]

Next, one uses the obtained solution \( p_1(t) \) in solving the equation

\[ \dot{p}_2 = F_2(p_1(t), p_2, t), \]

and so forth.
Remark 4.3. In stability analysis (using decomposition techniques) sequential form (E), which thus far does not appear to have been used, seems to offer quite a few advantages. We will use this form to generalize the stability results obtained in [16].

We note that form (E) does not require a separable interconnecting structure (which is given by the functions $G_i$) while form (C) (which is the basis of all results in [16]) does require this property. We also note that in general there are more (and never fewer) subsystems when system (A) is in the equivalent form (E) rather than in the equivalent form (C). Thus, the stability analysis using form (E) rather than form (C) may be advantageous since it is usually easier to ascertain the qualitative properties of lower dimensional system components.

4.3. Example

We consider an interconnected dynamical system whose M-digraph is identical to that of the algebraic system considered in Remark 3.1. Specifically, we consider the set of ordinary differential equations,

\[
\begin{align*}
\dot{x}_1 &= \bar{f}_1(x_1) + \bar{g}_1(x_2), \\
\dot{x}_2 &= \bar{f}_2(x_2) + \bar{g}_2(x_1), \\
\dot{x}_3 &= \bar{f}_3(x_3) + \bar{g}_3(x_1), \\
\dot{x}_4 &= \bar{f}_4(x_4) + \bar{g}_4(x_1, x_3), \\
\dot{x}_5 &= \bar{f}_5(x_5) + \bar{g}_5(x_4, x_6), \\
\dot{x}_6 &= \bar{f}_6(x_6) + \bar{g}_6(x_5).
\end{align*}
\]
Using the techniques discussed thus far, one may easily write this system in four different forms, namely,

(i) interconnected form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
f_1(x_1) \\
\bar{f}_3(x_3) + \bar{g}_3(x_1)
\end{bmatrix} + \begin{bmatrix}
\bar{g}_1(x_2) \\
0
\end{bmatrix},
\]

\[
\begin{bmatrix}
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
\bar{f}_2(x_2)
\end{bmatrix} + \begin{bmatrix}
\bar{g}_2(x_1)
\end{bmatrix},
\]

\[
\begin{bmatrix}
\dot{x}_4 \\
\dot{x}_6
\end{bmatrix} = \begin{bmatrix}
f_4(x_4) \\
\bar{f}_6(x_6)
\end{bmatrix} + \begin{bmatrix}
\bar{g}_4(x_1, x_3) \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
\bar{g}_6(x_5)
\end{bmatrix},
\]

\[
\begin{bmatrix}
\dot{x}_5
\end{bmatrix} = \begin{bmatrix}
\bar{f}_5(x_5)
\end{bmatrix} + \begin{bmatrix}
\bar{g}_5(x_4, x_6)
\end{bmatrix},
\]

or equivalently,

\[
\begin{align*}
\dot{y}_1 &= f_1(y_1) + g_{12}(y_2), \\
\dot{y}_2 &= f_2(y_2) + g_{21}(y_1), \\
\dot{y}_3 &= f_3(y_3) + g_{31}(y_1) + g_{34}(y_4), \\
\dot{y}_4 &= f_4(y_4) + g_{43}(y_3),
\end{align*}
\]

(ii) lower block triangular form:
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\end{bmatrix}
= \begin{bmatrix}
\tilde{f}_1(x_1) + \tilde{g}_1(x_2) \\
\tilde{f}_2(x_2) + \tilde{g}_2(x_1) \\
\tilde{f}_3(x_3) + \tilde{g}_3(x_1) \\
\tilde{f}_4(x_4) \\
\tilde{f}_5(x_5) + \tilde{g}_5(x_4, x_6) \\
\tilde{f}_6(x_6) + \tilde{g}_6(x_5) \\
\end{bmatrix}
\begin{bmatrix}
\tilde{g}_4(x_1, x_3) \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\]

or equivalently,

\[
\begin{align*}
\dot{z}_1 &= \tilde{f}_1(z_1), \\
\dot{z}_2 &= \tilde{f}_2(z_2) + \tilde{g}_2(z_1),
\end{align*}
\]

(iii) composite sequential form:

\[
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4 \\
\tilde{x}_5 \\
\tilde{x}_6 \\
\end{bmatrix}
= \begin{bmatrix}
\tilde{f}_1(x_1) + \tilde{g}_1(x_2) \\
\tilde{f}_2(x_2) + \tilde{g}_2(x_1) \\
\tilde{f}_3(x_3) + \tilde{g}_3(x_1) \\
\tilde{f}_4(x_4) + \tilde{g}_4(x_1, x_3) \\
\tilde{f}_5(x_5) + \tilde{g}_5(x_4, x_6) \\
\tilde{f}_6(x_6) + \tilde{g}_6(x_5) \\
\end{bmatrix},
\]

or equivalently,
\[ \dot{p}_1 = F_1(p_1), \]
\[ \dot{p}_2 = F_2(p_1, p_2), \]
\[ \dot{p}_3 = F_3(p_1, p_2, p_3), \]
\[ \dot{p}_4 = F_4(p_3, p_4), \]

(iv) sequential form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} =
\begin{bmatrix}
\bar{r}_1(x_1) + \bar{g}_1(x_2) \\
\bar{r}_2(x_2) + \bar{g}_2(x_1) \\
\bar{r}_3(x_3) \\
\bar{r}_4(x_4) + \bar{g}_5(x_4, x_6) \\
\bar{r}_5(x_5) + \bar{g}_6(x_5) \\
\bar{r}_6(x_6) + \bar{g}_6(x_5)
\end{bmatrix}
+ \begin{bmatrix}
\bar{e}_3(x_1) \\
\bar{e}_4(x_1, x_3) \end{bmatrix},
\]

or equivalently,

\[ \dot{u}_1 = \tilde{F}_1(u_1), \]
\[ \dot{u}_2 = \tilde{F}_2(u_2) + \tilde{G}_2(u_1), \]
\[ \dot{u}_3 = \tilde{F}_3(u_3) + \tilde{G}_3(u_1, u_2). \]
5. STABILITY OF LARGE SCALE SYSTEMS

The present chapter consists of four sections. In the first of these we make some statements concerning Lyapunov functions while in the second section we briefly discuss stability preserving mappings. In the third section we develop our stability results. Finally in the fourth section we present two examples to illustrate the applicability of these results.

5.1. Lyapunov Functions

Lyapunov stability and instability results as well as Lagrange stability results for system \( (A') \) involve the existence of functions \( v: D \rightarrow \mathbb{R} \), where in the case of local results \( D = B(r) \times J \) while in the case of global results \( D = \mathbb{R}^n \times J \). We will always assume that such \( v \)-functions are continuous on their respective domains and that they satisfy locally a Lipschitz condition with respect to \( x \). The upper right-hand derivative of \( v \) with respect to \( t \) along solutions of Eq. \( (A') \) is given by

\[
Dv(A')(x,t) = \lim_{h \to 0^+} \sup \left( \frac{1}{h} \right) \{v[x(t+h,x,t),t+h] - v(x,t)\}.
\]

If \( v \) is continuously differentiable with respect to all of its arguments, then the total derivative of \( v \) with respect to \( t \) along solutions of Eq. \( (A') \) is given by
where $\nabla v(x,t)$ denotes the gradient vector of $v(x,t)$ with respect to $x$. Whether $v$ is continuous or continuously differentiable will be clear from context or it will be specified.

We will find it convenient to characterize the properties of $v$-functions in terms of the following types of comparison functions. A continuous function $\psi: [0, r_1] \to \mathbb{R}^+$ (resp., $\psi: [0, \infty) \to \mathbb{R}^+$) is said to belong to class $K$ (i.e., $\psi \in K$) if $\psi(0) = 0$ and $\psi$ is strictly increasing on $[0, r_1]$ (resp., $[0, \infty)$). If $\psi: \mathbb{R}^+ \to \mathbb{R}^+$, if $\psi \in K$, and if $\lim_{r \to \infty} \psi(r) = \infty$, then $\psi$ is said to belong to class $KR$ (i.e., $\psi \in KR$). For a discussion of the properties of functions in class $K$ and class $KR$, refer to Hahn [9, pp.95-97]. For the usual definitions of uniform stability, uniform asymptotic stability, exponential stability, instability, complete instability (all in the sense of Lyapunov), and Lagrange stability (i.e., uniform boundedness and uniform ultimate boundedness of solutions) for system (A') and the principal Lyapunov stability results, refer also to Hahn [9].

5.2. Stability Preserving Mappings

We will find it useful to characterize the interconnections $\tilde{G}_i$ of system (E) in terms of stability preserving
mappings. Such functions are considered in Thomas [23] and Hahn [10].

Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. For every $t \in J$, let $x^0(t) \in X$ where $x^0(t)$ denotes some fixed reference motion (see [10]). Let $g : X \times J \to Y$. Then $g$ is said to be stability preserving if the image motion $y^0(t) = g(x^0(t), t)$ possesses the same stability properties as $x^0(t)$. (In [10] instability preserving mappings are also considered; however, we will not require this concept). Thomas has shown that $g$ is stability preserving if $g$ is a homeomorphism. Subsequently, Hahn [10] showed that $g$ preserves uniform stability as well as uniform asymptotic stability if there exists a function $\psi \in K$, and for any other motion $x(t)$ for which $y(t)$ is defined, we have

$$|y(t) - y^0(t)| < \psi(|x(t) - x^0(t)|)$$

for all $t \geq t_0 > 0$. Now if in particular $x^0(t) = 0$, the above inequality assumes the form

$$|y(t)| = |g(x(t), t)| < \psi(|x(t)|)$$

for all $t \geq t_0 > 0$. It can also be shown that if $g$ is stability preserving with $\psi \in K$ and if there exist constants $k, \alpha, \beta > 0$ such that $\psi(r) \leq kr^\alpha$ when $0 < r < \beta$, then $g$ preserves exponential stability of $x^0(t) = 0$. If in addition $\psi \in KR$, then $g$ preserves the global exponential stability of the trivial motion $x^0(t) = 0$ (see [16]).
Although the above results constitute only sufficient conditions, we will agree to the following definitions.

**Definition 5.1.** The mapping $g: \mathbb{R}^n \times J \rightarrow \mathbb{R}^m$ is said to be **stability preserving** (where $x^0(t)=0$ is the reference motion) if there exists a function $\varphi \in K$ such that

$$|y| = |g(x,t)| < \varphi(|x|)$$

for all $x \in B(r)$, resp., $x \in \mathbb{R}^n$ and for all $t \in J$.

**Definition 5.2.** The mapping $g$ in Definition 5.1 is said to be **exponential stability preserving** if it is stability preserving with $\varphi \in K$ and if there exist constants $k, \alpha, \beta > 0$ such that

$$\varphi(r) \leq kr^\alpha$$

when $0 < r < \beta$. If in addition $\varphi \in KR$, then $g$ preserves **exponential stability in the large** ($g$ preserves ESIL).

### 5.3. Stability Results

We now present some stability results.

**5.3.1. Uniform asymptotic stability**

We begin with a local result for system (E), or equivalently, for system (A).
Proposition 5.1. The equilibrium \( u=0 \) of system (E) is uniformly asymptotically stable if

(i) for each \( i=1,2,\ldots,s \), subsystem \( (S_i) \) is uniformly asymptotically stable, i.e., there exists a function \( v_i: R^{S_i} \times J \to R \) and three functions \( \psi_{i1}, \psi_{i2}, \psi_{i3} \in K \) such that

\[
\psi_{i1}(|w_i|) \leq v_i(w_i,t) \leq \psi_{i2}(|w_i|),
\]

\[
Dv_i(S_i)(w_i,t) \leq -\psi_{i3}(|w_i|)
\]

for all \( |w_i| \leq r_i, r_i > 0 \) and for all \( t \geq t_0 \);

(ii) there exists \( L_i > 0 \) such that

\[
|\nabla v_i(w_i,t)| \leq L_i \text{ on } [0, \infty) \times \{ w_i : |w_i| \leq r_i \}, i=1,\ldots,s; \text{ and}
\]

(iii) all interconnections \( \tilde{G}_i \) are stability preserving, i.e., there exists \( \psi_i \in K \) such that \( |\tilde{G}_i(q_i,t)| \leq \psi_i(|q_i|) \) for all \( |q_i| \leq r_i \) and for all \( t \geq t_0 \) where \( q_i^T=(u_1^T, u_2^T, \ldots, u_{i-1}^T) \).

Proof. Consider the first and second subsystems \((S_1)\) and \((S_2)\) for system (E), i.e.,

\[
\begin{align*}
\dot{w}_1 &= \tilde{F}_1(w_1,t) \quad (S_1) \\
\dot{w}_2 &= \tilde{F}_2(w_2,t). \quad (S_2)
\end{align*}
\]

Using the interconnection term \( \tilde{G}_2(u_1,t) \), we form the interconnected system

\[
\begin{align*}
\dot{u}_1 &= \tilde{F}_1(u_1,t) \\
\dot{u}_2 &= \tilde{F}_2(u_2,t) + \tilde{G}_2(u_1,t)
\end{align*}
\]

(1)
The equilibrium \((u_1,u_2)=(0,0)=0\) of this system is uniformly asymptotically stable, since all hypotheses of Theorem 1 in [16] are satisfied. System (1) can equivalently be written as

\[
\dot{u}^1 = F^1(u^1, t)
\]

where \(u^1\) and \(F^1\) are defined in the obvious way. Next, we consider the system described by the set of equations

\[
\begin{align*}
\dot{u}^1 &= F^1(u^1, t) \\
\dot{u}_3 &= \tilde{F}_3(u_3, t) + \tilde{G}_3(u^1, t)
\end{align*}
\]

and repeat the above argument, invoking Theorem 1 in [16], to show that the equilibrium \((u^1,u_2)=0\) of system (2) is also uniformly asymptotically stable.

Repeating the above argument \(s-1\) times, we conclude that the equilibrium \(u=0\) of system (E) (and hence, the equilibrium \(x=0\) of system (A)) is uniformly asymptotically stable.

Since the proofs of all subsequent results follow along similar lines as the proof of Proposition 5.1 (invoking appropriate results from [16], we will omit all subsequent proofs.

**Corollary 5.1.** The equilibrium \(u=0\) of system (E) is uniformly asymptotically stable if
(i) for each $i=1,2,\ldots,s$, $(S_i)$ is uniformly asymptotically stable;

(ii) the subsystems $(S_i)$ are all either autonomous or periodic; and

(iii) all interconnections $G_i$ are stability preserving.

The next theorem yields a global result.

**Proposition 5.2.** The equilibrium $u=0$ of system (E) is uniformly asymptotically stable in the large if

(i) for each $i=1,2,\ldots,s$, $(S_i)$ is uniformly asymptotically stable in the large, i.e., there exists a function $v_i: R^{S_i} \times J \rightarrow R$, functions $\psi_{11}, \psi_{12} \in KR$ and $\psi_{13} \in K$ such that

$$\psi_{11}(|w_i|) \leq v_i(w_i, t) \leq \psi_{12}(|w_i|)$$

$$Dv_i(S_i)(w_i, t) \leq -\psi_{13}(|w_i|)$$

for all $w_i \in R^{S_i}$ and for all $t \geq t_0$;

(ii) $\lim_{|w_i| \to \infty} \{ |\nabla v_i(w_i, t)| / \psi_{13}(|w_i|) \} = 0$, or equivalently, $|\nabla v_i(w_i, t)| = \sigma(\psi_{13}(|w_i|))$, uniformly in $t \in J$; and

(iii) all interconnections $\tilde{G}_i$ are stability preserving, i.e., for each $i$ there exists a $\varphi_i \in K$ such that $|\tilde{G}_i(q_i, t)| \leq \varphi_i(|q_i|)$ for all $q_i \in R^{S_1} \times R^{S_2} \times \ldots \times R^{S_{i-1}}$ and for all $t \geq t_0$.

5.3.2. Exponential stability

We begin with a local result.

**Proposition 5.3.** The equilibrium $u=0$ of system (E) is
exponentially stable if

(i) for each $i=1,2,\ldots,s$, $(S_i)$ is exponentially stable;

(ii) for each $i=1,2,\ldots,s$, $\tilde{F}_i$ in $(S_i)$ satisfies a Lipschitz condition which is local in $w_i$ and global in $t$; and

(iii) each interconnection $\tilde{G}_i$ is exponential stability preserving, i.e., $\tilde{G}_i$ is stability preserving with $\varphi_i \in K$ and there are constants $K_i, \alpha_i, \beta > 0$ such that $\varphi_i(r) \leq K_i r^{\alpha_i}$ for $0 < r < \beta$.

Corollary 5.2. The equilibrium $u=0$ of system $(E)$ is exponentially stable if

(i) the subsystems $(S_i)$ are all either autonomous or periodic;

(ii) for each $i=1,2,\ldots,s$, $(S_i)$ is exponentially stable; and

(iii) each interconnection $\tilde{G}_i$ is exponential stability preserving in the sense of Definition 5.2.

The next theorem yields a global result.

Proposition 5.4. The equilibrium $u=0$ of system $(E)$ is exponentially stable in the large if

(i) for each $i=1,2,\ldots,s$, $(S_i)$ is exponentially stable in the large;

(ii) for each $i=1,2,\ldots,s$, $\tilde{F}_i$ in $(S_i)$ is locally Lipschitz continuous in $w_i$, uniformly in $t \in J$; and

(iii) each interconnection $\tilde{G}_i$ is exponential stability
Corollary 5.3. The equilibrium \( u=0 \) of system (E) is exponentially stable in the large if

(i) the subsystems \((S_i)\) are all either autonomous or periodic;
(ii) for each \( i=1,2,\ldots,s \), \((S_i)\) is exponentially stable in the large; and
(iii) each interconnection \( \mathcal{G}_i \) is exponential stability in the large preserving.

5.3.3. Ultimate Boundedness

Next, we consider the uniform ultimate boundedness of system (E).

Proposition 5.5. The solutions of system (E) are uniformly ultimately bounded if

(i) the solutions of each \((S_i)\) are uniformly ultimately bounded, i.e., for each \( i=1,2,\ldots,s \), there exist \( v_i : \mathbb{R}^{S_i} \times J \rightarrow \mathbb{R} \), functions \( \psi_{i1}, \psi_{i2} \in \mathbb{K} \) and \( \varphi_{i3} \in \mathbb{K} \), and a constant \( R_i > 0 \) such that

\[
\psi_{i1}(|w_i|) \leq v_i(w_i,t) \leq \psi_{i2}(|w_i|)
\]

for all \( w_i \in \mathbb{R}^{S_i} \) and for all \( t \in J \) and

\[
Dv_i(S_i)(w_i,t) \leq -\varphi_{i3}(|w_i|)
\]

for all \( |w_i| \geq R_i \) and for all \( t \in J \).
(ii) \[ \lim_{|w_i| \to \infty} \left\{ \left| \nabla v_i(w_i,t) \right| / \psi_{i3}(|w_i|) \right\} = 0, \]
or equivalently, \[ |\nabla v_i(w_i,t)| = \sigma(\psi_{i3}(|w_i|)), \]
uniformly in \( t \in J; \) and
(iii) each interconnection \( \tilde{G}_i \) is stability preserving.

5.3.4. Instability

Finally, we consider the instability of system (E).

**Proposition 5.6.** The equilibrium \( u=0 \) of system (E) is **unstable** if for some \( k \in S=\{1,2,\ldots,s\} \), \((S_k)\) is unstable. The equilibrium \( u=0 \) of system (E) is **completely unstable** if for each \( k \in S=\{1,2,\ldots,s\} \), \((S_k)\) is completely unstable.

5.4. Examples

The purpose of the following two examples is to demonstrate the applicability of the present method to physical problems and to point out advantages of our results over existing ones.

**Example 5.1.** Consider an 8-bit bistable latch, SN74100 [22], depicted in Figure 5.1 consisting of two NOR gates (subsystems 4 and 7), two AND gates (subsystems 2 and 5), two buffers (subsystems 3 and 6) and one inverter (subsystem 1). The function table of this latch is shown in Table 5.1. Each of the subsystems may be represented by a nonlinear transistor-linear resistor model (see Sandberg [20], Mitra and So [17]) characterized by the set of differential equations
Figure 5.1. Functional block diagram

Table 5.1. Function table

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>G</td>
</tr>
<tr>
<td>L^a</td>
<td>H^b</td>
</tr>
<tr>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>X^c</td>
<td>L</td>
</tr>
</tbody>
</table>

^a_L = low level  
^b_H = high level  
^c_X = irrelevant  
^d_{Q_o} = the level of Q before the high-to-low transition of G
\[ \dot{z}_i + A_i f_i(z_i) + B_i g_i(z_i) = b_i(t), \quad i=1,2,\ldots,7 \quad (S_i) \]

where \( z_i \in \mathbb{R}^{n_i} \), \( A_i = (a_{i,k}) \) and \( B_i = (b_{i,k}) \) are constant square matrices with \( a_{i,k} \geq 0 \) and \( b_{i,k} > 0 \), where \( f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i} \) and \( g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i} \) are continuously differentiable in \( z_i \) and where \( f_i(z_i) = 0 \) and \( g_i(z_i) = 0 \) if and only if \( z_i = 0 \). Let \( f_i(z_i)^T = (f_i^1(z_i), \ldots, f_i^{n_i}(z_i)) \), \( g_i(z_i)^T = (g_i^1(z_i), \ldots, g_i^{n_i}(z_i)) \) and \( z_i^T = (z_i^1, \ldots, z_i^{n_i}) \). As in [17,20], it is assumed that \( f_k^i(z_i) = f_k^i(z_k^i), g_k^i(z_i) = g_k^i(z_k^i), \quad (f_k^i(z_k^i)/z_k^i) \geq \delta > 0 \) and \( g_k^i(z_k^i)/z_k^i \geq \delta > 0 \) for all \( z_k^i \neq 0 \), and that \( \left( \partial f_k^i(z_k^i)/\partial z_k^i \right)_{z_k^i=0} \geq \eta > 0 \) and \( \partial g_k^i(z_k^i)/\partial z_k^i \bigg|_{z_k^i=0} \geq \eta \). Since we are interested in studying the Lyapunov stability of the trivial solution, we consider henceforth the case \( b_i(t) = 0 \) for all \( t \in J \).

Since for each subsystem \((S_i)\) of Figure 5.1, we have typically \( n_i \geq 6 \), it follows that the overall dimension of this system is \( n = \sum_{i=1}^{7} n_i \geq 42 \). It is assumed that the output of subsystem \((S_i)\) is connected to the input of subsystem \((S_i)\), and that the state vector \( z_j \) affects the state vector \( z_i \) (but not vice versa) via the interconnection term \( C_{ij} g_j(z_j) \) where \( C_{ij} \) is a constant \( n_i \times n_j \) matrix. The circuit of Figure 5.1 may now be described by the set of equations (with \( b_i(t) = 0, i=1,2,\ldots,7 \)),
\[ \dot{z}_1 + A_1 f_1(z_1) + B_1 g_1(z_1) = 0 \]
\[ \dot{z}_2 + A_2 f_2(z_2) + B_2 g_2(z_2) + C_{21} g_1(z_1) = 0 \]
\[ \dot{z}_3 + A_3 f_3(z_3) + B_3 g_3(z_3) + C_{37} g_7(z_7) = 0 \]
\[ \dot{z}_4 + A_4 f_4(z_4) + B_4 g_4(z_4) + C_{42} g_2(z_2) + C_{43} g_3(z_3) = 0 \] (\( \Sigma_i \))
\[ \dot{z}_5 + A_5 f_5(z_5) + B_5 g_5(z_5) = 0 \]
\[ \dot{z}_6 + A_6 f_6(z_6) + B_6 g_6(z_6) + C_{64} g_4(z_4) = 0 \]
\[ \dot{z}_7 + A_7 f_7(z_7) + B_7 g_7(z_7) + C_{75} g_5(z_5) + C_{76} g_6(z_6) = 0 \]

which may be written as

\[ \dot{x} + Af(x) + Bg(x) = 0 \] (\( \tilde{S} \))

where \( x^T = (z_1^T, \ldots, z_7^T) \), \( f^T = (f_1^T, \ldots, f_7^T) \), \( g^T = (g_1^T, \ldots, g_7^T) \) and where the meaning of the \( n \times n \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) is clear.

Using previous results (see Sandberg [20], Mitra and So [17], Michel and Miller [15]) it can be shown that composite system (\( \tilde{S} \)) with decomposition (\( \Sigma_i \)) is exponentially stable in the large if there exist constants \( \lambda_i > 0, \eta_i > 0 \) such that

\[ a_{jj} - \sum_{i=1, i \neq j}^{n} \frac{\lambda_i}{\lambda_j} |a_{ij}| \geq \varepsilon > 0, \quad b_{jj} - \sum_{i=1, i \neq j}^{n} \frac{\eta_i}{\eta_j} b_{ij} \geq \varepsilon > 0 \] (3)

for all \( j = 1, 2, \ldots, n \).

We now transform system (\( \Sigma_i \)) into sequential form (or
lower block triangular form since they are equivalent in this case) and obtain (see Figure 5.2 and Figure 5.3)

\[ \tilde{y} + AF(y) + BG(y) = 0 \quad (S) \]

where

\[
A = \begin{bmatrix}
A_1 & A_5 & 0 \\
A_2 & A_3 & A_4 & A_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
B_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_3 & 0 & 0 & 0 & C_{37} \\
0 & 0 & 0 & C_{42} & C_{43} & B_4 & 0 \\
0 & 0 & 0 & 0 & C_{64} & B_6 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{76} & B_7
\end{bmatrix},
\]

\[
y^T = (y_1^T, y_2^T, v_3^T, v_4^T), \quad y_1 = z_1, \quad y_2 = z_5, \quad y_3 = z_2, \quad y_4^T = (z_3^T, z_4^T, z_6^T, z_7^T),
\]

\[
F^T = (F_1^T, F_2^T, F_3^T, F_4^T), \quad F_1 = f_1, \quad F_2 = f_5, \quad F_3 = f_2, \quad F_4^T = (F_3^T, F_4^T, F_6^T, F_7^T),
\]

\[
G^T = (G_1^T, G_2^T, G_3^T, G_4^T), \quad G_1 = g_1, \quad G_2 = g_5, \quad G_3 = g_2, \quad G_4^T = (G_3^T, G_4^T, G_6^T, G_7^T).
\]

Let \( \tilde{A}_1 = A_1, \quad \tilde{A}_2 = A_5, \quad \tilde{A}_3 = A_2, \quad \tilde{B}_1 = B_1, \quad \tilde{B}_2 = B_5, \quad \tilde{B}_3 = B_2, \) and
Figure 5.2. Digraph of a bistable latch

Figure 5.3. Strongly connected components of the digraph
System (S) may be viewed as an interconnection of the free subsystems

\[ \dot{y}_i + \tilde{A}_i F_i(y_i) + \tilde{B}_i G_i(y_i) = 0, \quad i = 1, 2, 3, 4, \quad (\tilde{S}_i) \]

where \( y_i \in \mathbb{R}^{p_i}, \quad \sum_{i=1}^{s} p_i = n, \quad s = 4 \). Let \( \tilde{A}_k = (\tilde{a}^k_{ij}) \) and \( \tilde{B}_k = (\tilde{b}^k_{ij}) \).

Using the results in [15, pp. 79], we can show that each \( (\tilde{S}_k) \) is exponentially stable in the large if there exist constants \( \tilde{\alpha}_i, \tilde{\eta}_i, \tilde{\varepsilon} > 0 \) such that

\[ \tilde{a}^k_{ij} - \sum_{i=1}^{p_k} \frac{\tilde{\alpha}_i}{\xi_j} |a^k_{ij}| > \tilde{\varepsilon}, \quad \tilde{b}^k_{ij} - \sum_{i=1}^{p_k} \frac{\tilde{\alpha}_i}{\xi_j} |b^k_{ij}| > \tilde{\varepsilon} > 0, \quad (4) \]

j=1,2,...,p_k. Since the functions \( G_i(y_i) \) are continuously differentiable and time invariant and since \( G_i(0) = 0 \), it follows that the interconnection terms \( C_{21} G_1(y_1), \quad C_{75} G_2(y_2) \) and \( C_{42} G_3(y_3) \) preserve exponential stability in the large. Hence, all conditions of Corollary 5.3 are satisfied.

Therefore, the trivial solution of system \( (\tilde{S}) \), and hence, of system \( (\tilde{S}) \), is exponentially stable in the large if the column dominance conditions (4) are true.

In comparing stability condition (4) with stability condition (1), note the following:
(i) Inequalities (3) may be easier to check than inequalities (4).

(ii) Condition (4) is less conservative than condition (3) since the interconnection matrices $C_{21}$, $C_{42}$, $C_{75}$ enter into column dominance condition (3) but do not appear in column dominance condition (4).

Example 5.2. Consider a direct control system with multiple feedbacks (see Michel and Miller [15, pp. 84-85]) described by the scalar equations

\[ \begin{align*}
\dot{x}_1 &= -x_1 + g_1(x_4) \\
\dot{x}_2 &= -x_2 + g_2(x_2 + x_4) \\
\dot{x}_3 &= -x_3 + g_3(x_1 + x_2 + x_6) \\
\dot{x}_4 &= -x_4 + g_4(x_2) \\
\dot{x}_5 &= -x_5 + g_5(x_3 + x_5) \\
\dot{x}_6 &= -x_6 + g_6(x_1 + x_6)
\end{align*} \tag{5} \]

where each function $g_i$, i=1,2,...,6, is continuously differentiable with respect to all of its variables and $g_i(\sigma_i)=0$ if and only if $\sigma_i=0$. These assumptions ensure that the $g_i$ are ESIL preserving.

In [15], the sector conditions

\[ 0 \leq \sigma_i g_i(\sigma_i) \leq k_i \sigma_i^2 \tag{6a} \]

for all $\sigma_i \neq 0$, i=1,2,...,6, are assumed as well, where $k_i \geq 0$.
is some constant. Using the technique in [15], it is shown that the trivial solution $x^T=(x_1,\ldots,x_6)=0$ of system (5) is exponentially stable in the large if

$$1-k_2 > 0, \ 1-k_2-k_4 > 0, \ 1-k_5 > 0, \ 1-k_6 > 0. \quad (6b)$$

To decompose system (5) into several equivalent forms, we first construct the associated M-digraph shown in Figure 5.4a. The decomposition techniques discussed in section 4.1 yield the following equivalent forms of system (5), which is in the aggregate form.

(i) **Interconnected form**, which in this case is also the lower block triangular form:

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} =
\begin{bmatrix}
-x_1+g_1(x_4) \\
-x_2+g_2(x_2+x_4) \\
-x_4+g_4(x_2) \\
-x_5+g_5(x_3+x_5) \\
-x_6+g_6(x_1+x_6)
\end{bmatrix}$$

$$\begin{bmatrix}
\dot{x}_3 \\
\dot{x}_5
\end{bmatrix} =
\begin{bmatrix}
g_3(x_1+x_2+x_6) \\
0
\end{bmatrix}
$$

(ii) **Composite sequential form**:
The M-digraphs of Eqs. (7), (8), and (9) are shown in Figures 5.4b–d, respectively.

The stability results developed in [16], which are based on the lower block triangular form (7) yield the following conditions for global exponential stability:

$$0 \leq \sigma_i g_i^\prime(\sigma_i) \leq k_1 \sigma_i^2$$

(10a)

for all $\sigma_i \neq 0$, $i=1,2,4,5,6$, and
Figure 5.4a. M-digraph of system (5)

Figure 5.4b. Level-strong condensation

Figure 5.4c. Strong condensation

Figure 5.4d. Semi-strong condensation
The present method (Proposition 5.4), which uses the sequential form (9) yields the following conditions for global exponential stability:

\[ 0 \leq \sigma_i g_i(\sigma_i) \leq k_i \sigma_i^2 \]  \hspace{1cm} (11a)

for all \( \sigma_i \neq 0 \), \( i=2,4,5,6 \), and

\[ 1-k_2 > 0, \ 1-k_2-k_2 k_4 > 0, \ 1-k_5 > 0, \ 1-k_6 > 0. \]  \hspace{1cm} (11b)

In arriving at stability conditions (10) and (11), we followed the techniques developed in [15] to establish the stability properties of the subsystems of systems (7) and (9). In any case, stability condition (11) is the least conservative one while stability condition (6) is the most conservative one, for the latter imposes sector conditions for \( i=1,2,3,4,5,6 \) while the former does not require any sector conditions for \( i=1,3 \). Also stability condition (10), which is less conservative than condition (6) and more conservative than condition (11), does not require any sector condition for \( i=3 \).
6. CONCLUDING REMARKS

New Lyapunov stability results for large scale systems having arbitrary interconnecting structures (i.e., not necessarily separable interconnecting structure) have been established. The present results are applicable to a larger class of problems than existing ones given in [16]. Indeed, the results in [16] (and also in [5]) are applicable only to systems with separable interconnecting structures while the present results apply to systems with nonseparable interconnections as well. (The input-output stability results in [5] may also be extended to systems with nonseparable structures, using the present graph-theoretic results). As has been demonstrated in the examples given in chapter 5, the present method can yield stability results which in general may be less conservative than corresponding ones in [16].

The graph-theoretic results for the decomposition of large scale systems given in chapter 4 generalize and unify several existing results (e.g., see [5, 14, 16]). They make possible the systematic decomposition of large scale systems into equivalent more useful forms than seemed possible before.

The concepts of a G-graph, a G-digraph, and an M-digraph were suggested by the work of Guardabassi and Sangiovanni-Vincentelli [8] who called the G-digraph a directed hyper-
graph. The properties of the G-graph and the M-digraph and the decomposition algorithms, developed herein, are new. G-graphs and M-digraphs appear to have potential for further application in several diverse areas (organic chemistry [19], mathematics [1], physics [7], economics [2], etc.).
7. REFERENCES


8. ACKNOWLEDGEMENTS

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9. APPENDIX A: PROGRAM FOR STRONG COMPONENTS


INTEGER A(50,50)
1 READ (5,10,END=1000) N
10 FORMAT (12)
WRITE (6,20)
20 FORMAT (1H1,23HTHE ADJACENCY MATRIX IS)
DO 40 I=1,N
READ (5,30) (A(I,J),J=1,N)
30 FORMAT (5011)
WRITE (6,50) (A(I,J),J=1,N)
A(I,I)=1
40 CONTINUE
50 FORMAT (1H0,50I2)

C FIND THE TRANSITIVE CLOSURE OF A

DO 80 I=1,N
DO 70 J=1,N
IF (I.EQ.J.OR.A(J,I).NE.1) GO TO 70
DO 60 K=1,N
IF (A(I,K).EQ.1) A(J,K)=1
60 CONTINUE
70 CONTINUE
80 CONTINUE
WRITE (6,90)
90 FORMAT (1H0,21HTHE CLOSURE MATRIX IS)
DO 100 I=1,N
A(I,I)=0
WRITE (6,50) (A(I,J),J=1,N)
A(I,I)=1
100 CONTINUE
WRITE (6,110)
110 FORMAT (1H0,23HTHE COMPONENT MATRIX IS)

C FIND THE COMPONENT MATRIX

DO 130 I=1,N
DO 120 J=1,N
IF (I.EQ.J) GO TO 120
IF (A(I,J).NE.0.AND.A(J,I).NE.0) GO TO 120
A(I,J)=0
A(J,I)=0
120 CONTINUE
WRITE (6,50) (A(I,J),J=1,N)
130 CONTINUE

C PRINT OUT
K = 1
DO 210 I = 1, N
IF (I .EQ. 1) GO TO 170
DO 140 J = 1, N
IF (A(I, J) .EQ. 1) GO TO 150
140 CONTINUE
150 I = I - 1
DO 160 I = 1, I
IF (A(I, J) .EQ. 1) GO TO 210
160 CONTINUE
170 WRITE (6, 180) K
180 FORMAT (1H0, 13HCOMPONENT NO:, I3)
DO 200 J = 1, N
IF (A(I, J) .EQ. 0) GO TO 200
WRITE (6, 190) J
190 FORMAT (1H0, 2X, I3)
200 CONTINUE
K = K + 1
210 CONTINUE
GO TO 1
1000 STOP
END
THIS PROGRAM IS USED TO LABEL THE VERTICES OF AN ACYCLIC SIMPLE DIGRAPH IN TOPOLOGICAL ORDER. "N" DENOTES THE NUMBER OF VERTICES AND "A" REPRESENTS THE ADJACENCY MATRIX. THE USER PROVIDES N IN THE FIRST TWO DIGITS OF THE FIRST DATA CARD, THEREAFTER, HE FURNISHES N CARDS TO DESCRIBE THE ADJACENCY MATRIX, ONE CARD FOR EACH ROW.

```
INTEGER A(50,50)
DIMENSION LABEL(50)
READ (5,10,END=1000) N
10 FORMAT(I2)
WRITE (6,20)
20 FORMAT(I1H,23HTHE ADJACENCY MATRIX IS)
DO 50 I=1,N
LABEL(I)=0
READ (5,30) (A(I,J),J=1,N)
30 FORMAT(S0I)
WRITE (6,40) (A(I,J),J=1,N)
A(I,I)=1
40 FORMAT(S0I)
CONTINUE
50 DO 90 I=1,N
DO 80 J=1,N
IF (LABEL(J).NE.0) GO TO 80
DO 60 K=1,N
IF (J.EQ.K) GO TO 60
IF (A(J,K).NE.0) GO TO 80
60 CONTINUE
LABEL(J)=I
DO 70 L=1,N
A(L,J)=0
70 CONTINUE
GO TO 90
80 CONTINUE
90 CONTINUE
DO 110 I=1,N
WRITE (6,100) LABEL(I),I
100 FORMAT(I1H,14HFLABEL VERTEX,2X,I2,2X,2HAS,2X,I2)
110 CONTINUE
1000 STOP
END
```
11. APPENDIX C: PROGRAM FOR SEMI-STRONG COMPONENTS

THIS PROGRAM IS USED TO FIND ALL THE SEMI-STRONG
COMPONENTS OF AN M-DIGRAPH. "N" DENOTES THE NUMBER
OF VERTICES AND "M" REPRESENTS THE NUMBER OF EDGES.
The user provides N and M in the first six columns.
Three each, of the first data card. Thereafter, he
furnishes N cards to describe the incidence array,
one card for each row.

INTEGER A[30,80], B[30,30], C[30]

READ (5,20) N, M
WRITE (6,30)
WRITE (6,30) THE INCIDENCE ARRAY IS
DO 60 I=1,N
READ (5,40) (A(I,J), J=1,M)
WRITE (6,50) (A(I,J), J=1,M)
60 CONTINUE

FIND THE LINEAR VERSION OF A

DO 70 I=1,N
DO 70 J=1,N
B(I,J) = 0
70 CONTINUE

DO 80 J=1,N
DO 100 I=1,N
B(I,I) = 1
IF (A(I,J) .EQ. 0 .OR. A(I,J) .EQ. 1) GO TO 130
DO 90 K=1,N
IF (A(K,J) .EQ. 1) B(I,K) = 1
90 CONTINUE
100 CONTINUE
110 CONTINUE

FIND ALL STRONG COMPONENTS OF B

DO 140 J=1,N
DO 130 I=1,N
IF (I.EQ.J .OR. B(J,I) .NE. 1) GO TO 130
DO 120 K=1,N
IF (B(I,K) .EQ. 1) B(J,K) = 1
120 CONTINUE
130 CONTINUE
140 CONTINUE

DO 160 I=1,N
DO 150 J=1,N
IF (I.EQ.J) GO TO 150
IF (B(I,J) .NE. 0 .AND. B(J,I) .NE. 0) GO TO 150
B(I,J) = 0
B(J,I) = 0
150 CONTINUE
160 CONTINUE
NC = 1
DO 220 I=1,N
DO 170 J=1,N
IF (B(I,J) .EQ. 1) GO TO 180
170 CONTINUE
180 I1=I-1
IF (I .EQ. 1) GO TO 200
DO 190 K=1,I1
IF (B(K,J) .EQ. 1) GO TO 220
190 CONTINUE
C(J)=NC
210 CONTINUE
NC=NC+1
220 CONTINUE
NC=NC-1
IF (NC .LT. 30) GO TO 290

C ADD ADDITIONAL EDGES TO A
C
IADD=0
DO 280 J=1,M
DO 270 I=1,N
IF (A(I,J) .NE. 3) GO TO 240
DO 230 K=1,N
IF (A(K,J) .NE. 1 .OR. C(K) .EQ. C(I)) GO TO 230
B(K,I)=1
B(I,K)=1
IADD=1
230 CONTINUE
GO TO 280
240 IF (A(I,J) .NE. 2) GO TO 270
DO 260 K=1,N
IF (A(K,J) .NE. 1 .OR. C(K) .EQ. C(I)) GO TO 260
DO 250 L=1,N
IF (A(L,J) .NE. 1 .OR. C(L) .EQ. C(I)) GO TO 250
B(L,I)=1
B(I,L)=1
IADD=1
250 CONTINUE
GO TO 280
260 CONTINUE
270 CONTINUE
280 CONTINUE
IF (IADD .LT. 0) GO TO 80

C PRINT OUT
C
290 DO 330 I=1,NC
WRITE (6,300) I
300 FORMAT (3H0,25HSemi-Strung Component No.: ,I3)
DO 320 J=1,N
IF (C(J) .NE. 1) GO TO 320
WRITE (6,310) J
310 FORMAT (1H0,13).
320 CONTINUE
330 CONTINUE
GO TO 10
1000 STOP
END