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Theory of flux-flow voltage noise in type-II superconductors

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THEORY OF FLUX-FLOW VOLTAGE NOISE IN TYPE-II SUPERCONDUCTORS

Iowa State University

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Theory of flux-flow voltage noise in type-II superconductors

by

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I. INTRODUCTION

A. Type-I and Type-II Superconductors

It is well-known that superconductors can be classified into two types, type-I and type-II, according to their magnetic properties (1). The magnetization curve of a long cylindrical, homogeneous type-I superconductor subjected to a magnetic field parallel to its axis is shown in Fig. la. At fields below the critical field $H_c$, the induction inside the specimen is zero, and the superconductor is said to be in the Meissner state (2). This is caused by a current flowing in a surface layer which screens out the applied field. The current decays exponentially from the surface with a decay length $\lambda$, the penetration depth. At $H_c$, there is a first-order phase transition and the specimen reverts to the normal state with magnetic flux penetrating throughout the specimen. If the applied field is not parallel to the axis of the cylinder or if the specimen is of some other shape, demagnetization effects can occur. In the range $H_c(1 - D) < H < H_c$, where $D$ is the demagnetization coefficient, the interior of the superconductor splits into normal and superconducting domains, and the specimen is said to be in the intermediate state. The magnetization curve of a sphere, which exhibits this effect, is also shown in Fig. la.

Figure lb shows the magnetization curve of a long cylinder of homogeneous type-II material with a longitudinal applied field. For $H < H_{c1}$, the lower critical field, the specimen is in the Meissner state. At intermediate fields $H_{c1} < H < H_{c2}$, flux partially
Figure 1. The magnetization of a type-I (a) and a type-II (b) superconductor
penetrates into the superconductor. Finally at $H_{c2}$, the upper critical field, there is a second-order phase transition and most of the specimen changes to the normal state. As in type-I materials, demagnetization effects can occur for other specimen shape/field direction combinations. The case of a sphere is also shown in Fig. 1b. A thermodynamic critical field can be defined for type-II materials by

$$-\int_0^{H_{c2}} MdH = \frac{H_{c2}^2}{8\pi}.$$  \hspace{1cm} (1.1)

B. Ginzburg-Landau Theory and Structure of the Mixed State

The magnetic properties of a type-II superconductor in the mixed state can be described very well by the Ginzburg-Landau theory (3). This theory assumes that the free energy density can be expanded in terms of a complex order parameter $\psi(\vec{r})$. $|\psi(\vec{r})|^2$ represents the local density of superconducting electrons. The expansion consists of powers of $|\psi|^2$ and $|\nabla \psi|^2$. A variational method was applied to the expansion, which results in two differential equations for $\psi(\vec{r})$ and the vector potential $A(\vec{r})$. The Ginzburg-Landau theory was shown by Gor'kov (4) to be derivable from the microscopic Bardeen-Cooper-Schrieffer (5) (BCS) theory in the limit of temperature close to the superconducting transition temperature and small spatial variation of $\psi(\vec{r})$. In 1957 Abrikosov (1) showed the existence of the mixed state and the difference between the two kinds (type-I and type-II) of superconductors from the Ginzburg-Landau theory. The theory of the
mixed state has since been extended to the whole temperature-induction plane for certain types of materials (6,7).

In the mixed state, the magnetic flux inside the superconductor is of the form of vortices, each carrying a quantum unit of flux

\[ \psi_0 = \frac{hc}{2e}, \quad (1.2) \]

where \( h \) is Planck's constant, \( c \) is the speed of light in vacuum, and \( e \) is the electronic charge. An isolated vortex consists of screening currents that flow around the axis of the vortex out to a distance approximately \( \lambda \). In the Ginzburg-Landau theory, \( \lambda \) is given by the relation

\[ \lambda^2 = \frac{m^* c^2}{4\pi n_s^* e^*}, \quad (1.3) \]

where \( m^* \) is the effective mass, \( n_s^* \) the effective number density, and \( e^* \) the effective charge of the charge carriers. Experimental data and the BCS theory indicate that \( m^* = 2m, e^* = 2e, \) and \( n_s^* = n_s/2 \), with the unstarred quantities being those of the superconducting electrons. \( \lambda \) is also the length over which the magnetic field decays away from the vortex axis. At the vortex axis, \( \psi(r) \) drops to zero. Over a distance \( \xi \), the coherence length, away from the vortex axis, \( \psi(r) \) rises to its equilibrium value at infinity, which is identified with \( n_s^* : \)

\[ |\psi_0|^2 \equiv n_s^* \quad . \quad (1.4) \]
The coherence length is given by

\[ \xi = \frac{\phi_0}{\sqrt{2\pi} H_c} \]  \hspace{1cm} (1.5)

For fields higher than \( H_{c1} \), when a number of vortices are present, the number density \( n(\mathbf{r}) \) of the vortices is related to the local induction \( \mathbf{B}(\mathbf{r}) \) by

\[ \mathbf{B}(\mathbf{r}) = n(\mathbf{r}) \phi_0 \]  \hspace{1cm} (1.6)

The decay length of the current and magnetic field away from the vortex axis is now given by \( \lambda_B \propto \lambda(1 - B/H_{c2})^{-1/2} \) (8,9).

The difference between a type-I and type-II superconductor lies in the surface energy between a normal region and a superconducting region. For a type-I material, this energy is positive, whereas for a type-II material, it is negative. This can be characterized by the Ginzburg-Landau parameter \( \kappa \), given by

\[ \kappa = \frac{\lambda}{\xi} \]  \hspace{1cm} (1.7)

For \( \kappa \) less (larger) than \( 2^{-1/2} \), the material is type-I (II). The values of the upper critical field is given by

\[ H_{c2} = \sqrt{2} \kappa H_c \]  \hspace{1cm} (1.8)

The value of the lower critical field for arbitrary \( \kappa \) can be obtained only through numerical calculations involving the Ginzburg-Landau equations. In the limit \( \kappa \gg 1 \),
\[ H_{cl} \approx \frac{Hc}{\sqrt{2}} \ln \kappa \]  

(1.9)

The supercurrent density is given by

\[ j(r) = \frac{e}{n_s} \varepsilon \frac{\psi'}{\psi} \]  

(1.10)

where \( \varepsilon \) is the supercurrent velocity, is

\[ \frac{\psi'}{\psi} = \mathcal{H} \gamma a \]  

(1.11)

with \( \gamma \) being the phase of the order parameter. The exact values of \( \psi'(\vec{r}) \) and hence \( \mathcal{H}(\vec{r}) \), the local field, for the mixed state can only be obtained by numerical calculations. For an isolated vortex, Clem (10) presented a variational method with which accurate results can be obtained with relatively simple calculations. Recently, Brandt proposed approximate results in the entire field range for many vortices (11,12).

If there are many vortices in the specimen, they arrange themselves to form a triangular lattice (13,14) with nearest-neighbor distance

\[ a = 1.075 \left( \frac{\phi_0}{B} \right)^{1/2} \]  

(1.12)

This is the case for homogeneous, isotropic materials. In some materials the atomic crystal lattice symmetry dominates, and the vortices may then be in some other, such as square, or even rectangular array. Inhomogeneities in forms of defects may also produce a fluid-like structure in the vortex array.
C. Flux Flow and Flux Pinning

Each vortex inside a type-II superconductor in the mixed state experiences a force per unit length given by

$$\mathbf{f} = \mathbf{j}_s \times \frac{\mathbf{\hat{\phi}}_0}{c}, \quad (1.13)$$

where \( \mathbf{j}_s \) is the total supercurrent density at the core, and \( \mathbf{\hat{\phi}}_0 = \phi_0 \mathbf{\hat{z}} \), where \( \mathbf{\hat{z}} \) is the direction of the local flux density. This implies that without any externally applied current, the vortices will be stationary if they are in a symmetrical array. If there is any externally applied current, it follows the net force per unit length on a vortex is

$$\mathbf{f}_L = \mathbf{j}_T \times \frac{\mathbf{\hat{\phi}}_0}{c}, \quad (1.14)$$

where \( \mathbf{f}_L \) is called the Lorentz or driving force, and

$$\mathbf{j}_T = \frac{c}{4\pi} \nabla \times \mathbf{H} \quad (1.15)$$

is the coarse-grained macroscopic current density, more commonly called the transport current density. Therefore, if the only force the vortices feel is the driving force, they will move under the application of a transport current. An electric field is then induced by the moving vortices. It is given by

$$\mathbf{E} = \frac{1}{c} \mathbf{B} \times \mathbf{\hat{v}}, \quad (1.16)$$

where \( \mathbf{\hat{v}} \) is the velocity of the vortices. Because \( \mathbf{E} \) is parallel to \( \mathbf{j}_T \),
power is dissipated. There is also a Hall voltage perpendicular to $J_\tau$, but usually the contribution is small. Therefore, an ideal type-II superconductor in the mixed state cannot carry any resistanceless current. Inhomogeneities in the materials, however, can act as pinning centers, which tend to pin the vortices stationary. Flux pinning, as this is called, is therefore very important in practical applications of superconductors. The vortices will move if the average pinning force acting on them is smaller than the driving force. The theory of flux pinning is very complicated and will be described in a later section.

The origin of the dissipation that occurs when a vortex moves has been investigated by various investigators. One theory which works quite well and is relatively simple is the Bardeen-Stephen model (15). The assumption here is that the superconductor is local, and that a fully normal core of radius $\xi$ is situated at the center of each vortex. This dissipation comes from the current and electric field inside the core, and from the normal current immediately outside the core. The result is

$$W = \eta v^2$$

where $W$ is the dissipation, and

$$\eta = \frac{\phi_0 H c^2}{\rho_n c^2}$$

is the viscous drag coefficient, with $\rho_n$ being the normal state
resistivity. Therefore, a viscous drag force equal to \(-\eta \dot{\vec{V}}\) acts on the vortex when it is moving. From the above, we see that we can write a force balance equation for a vortex

\[-\eta \dot{\vec{V}} + \vec{F}_L - \vec{F}_p = 0\]  \hspace{1cm} (1.19)

where \(\vec{F}_p\) is the pinning force.

Using Eq. (1.16) we can write

\[\vec{E} = \frac{\vec{E}^\parallel}{\eta c} \times (\vec{F}_L - \vec{F}_p)\]  \hspace{1cm} (1.20)

In a simple geometry, when \(\vec{B}\) is perpendicular to \(\vec{J}_c\), we can write

\[\vec{E} = \frac{B\phi_0}{\eta c^2} (\vec{J} - \vec{J}_c)\]  \hspace{1cm} (1.21)

where the critical current \(\vec{J}_c\) is defined by

\[\vec{F}_p = \vec{J}_c \times \frac{\phi_0}{c}\]  \hspace{1cm} (1.22)

The quantity

\[\rho_f \equiv \frac{B\phi_0}{\eta c^2}\]  \hspace{1cm} (1.23)

is called the flux-flow resistivity. From Eq. (1.18) we see that

\[\frac{\rho_f}{\rho_n} \approx \frac{B}{H_{c2}}\]  \hspace{1cm} (1.24)

It should be noted that Eqs. (1.19) and (1.20) are time-averaged equations, with \(\vec{F}_p\) being an average pinning force, as opposed to the
elementary pinning force exerted by the pinning centers directly on the vortices. The time-averaged voltage-current characteristic of a type-II superconductor in the mixed state with pinning, according to Eq. (1.21), will then be like that in Fig. 2. Actually, the voltage shows a noisy component superimposed on the d.c. part given by Eq. (1.21). This noisy part, flux-flow noise, involves the detailed motion of the vortices inside the superconductor and is the main phenomenon under investigation in this thesis. There may also be additional structure in the curve. The curve usually deviates from a straight line for $I_T$ slightly above $J_c$. All these features reflect the interaction between the pinning centers and the vortices. There is still not a satisfactory theory for flux motion in superconductors in the presence of flux pinning, nor is there a satisfactory theory for flux pinning itself. In a later section, a brief review of the theories of flux pinning in superconductors will be provided. Since the interactions between vortices are important in these theories, we shall first give a brief description in the next section of these interactions.

D. Interactions Between Vortices and the Elastic Constants

Since the hexagonal lattice of vortices constitutes a stable equilibrium, a vortex displaced slightly from its equilibrium position will experience a restoring force. We have already seen an example of such a force in Eq. (1.13). In general, such a restoring force can be calculated for small displacements by doing a differentiation of the free energy with respect to displacements.
Figure 2. Idealized voltage versus current density curve for a type-II superconductor with pinning.
The interactions between the vortices in general consist of two contributions. There is a repulsive electromagnetic interaction and an attractive core interaction. For application in flux pinning, the displacement of the vortex due to a local force is needed, and it is convenient to use elasticity theory in such a case. Following the usual Voigt's notation, the stresses $\sigma_i$ relate to the strains $\epsilon_j$ by Hooke's law

$$\sigma_i = C_{ij} \epsilon_j \quad .$$

If we take the z-axis to be parallel to the vortices, we see that properties in the x-y plane are isotropic (16), since the lattice is hexagonal if the superconductor is assumed to be isotropic. The fact that displacements parallel to the vortices do not effect a restoring force lead to the following matrix equation

$$\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{yz} \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} \\
C_{12} & C_{11} \\
& & C_{44} \\
& & & C_{44}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\epsilon_{yz} \\
\epsilon_{xy}
\end{bmatrix},$$

with $C_{66} = (C_{11} - C_{12})/2$.

The modulus for uniform compression is given by (17)

$$\frac{1}{2} \left( C_{11} + C_{12} \right) = \frac{B^2}{4\pi} \frac{dH}{dB},$$

following a simple thermodynamic argument. Since the shear modulus $C_{66}$ is small we see that
The tilting modulus \( C_{44} \) is given by (17)

\[
C_{44} = \frac{B H}{4\pi} \quad ,
\]

as can be found by considering the work done when tilting a bundle of vortices from the z-axis with the cross section remaining constant.

To calculate \( C_{66} \) we must consider the change in free energy when the lattice is being sheared in the x-y plane. The structure of the vortex lattice plays an important role, especially at high fields. At low fields, \( C_{66} \) is given by (17)

\[
C_{66} = \frac{1}{8\pi} \int_0^B x^2 \frac{d^2 H(x)}{dx^2} dx \quad , \quad B \ll H_{c2} \quad .
\]

At higher fields one must calculate the change in free energy with respect to a change in the structure of the vortex lattice. The result is (17)

\[
C_{66} = 0.038 \frac{H_c^2 \kappa^2 (2\kappa^2 - 1)}{(1 + \beta_A (2\kappa^2 - 1))^2} \left( 1 - \frac{B}{H_{c2}} \right)^2 \quad , \quad B \ll B_{c2} \quad (1.31)
\]

where \( \beta_A = \frac{|\psi|^4}{(\psi^2)^2} \) with \( |\psi| \) being the magnitude of the order parameter averaged out one unit cell, is a measure of the structure of the vortex lattice. \( \beta_A = 1.16 \) for a hexagonal lattice and 1.18 for a square lattice.
For high-\(\kappa\) materials and for high fields we have

\[
C_{11} \equiv C_{12} \equiv C_{44} \equiv \frac{H_c2}{4\pi} b^2 \tag{1.32}
\]

\[
C_{66} \equiv 0.01\left(\frac{H_c2}{\kappa}\right)^2(1 - b)^2 \tag{1.33}
\]

where \(b \equiv B/H_c2\).

The above treatment assumes that local elasticity theory, that is, the elastic energy is a space integral over a bilinear expression of strains at the same point, can be used. Recently (18,19) it was shown that at large inductions nonlocal effects may play a role. A recalculation of the elastic moduli allowing for nonlocal effects shows that they are wavevector-dependent. The displacements are now represented by superposing waves with wavevectors up to the zone-boundary. It is found that the elastic moduli not only depend on the magnitude of the wavevectors, but also on their directions, especially close to the zone boundary. \(C_{11}\) decreases by a factor \(2\kappa^2/(1-B/H_c2)^2\) and \(C_{44}\) by a factor \(2\kappa^2/(1-B/H_c2)\) when the wavevector increases from zero to the zone boundary. \(C_{66}\) is relatively less affected by the magnitude of the wavevector.

For thin films in a perpendicular applied field, the vortices also interact with each other via the space outside the film. This gives rise to a magnetic monopole-type long range interaction, which decreases as the inverse of the distance between the vortices. The
restoring force per unit displacement per unit length of vortex for B << H_{c2} and d_{f} << \lambda is given by (20,21)

\[ D_{q} = \frac{B\phi_{0}}{2\pi d_{f}}q \]  

(1.34)

\[ D_{qt} = \frac{\phi_{0}}{B} C_{66} q^{2} , \text{ for } a << \lambda , \]  

(1.35)

where d_{f} is the thickness of the film, \( a = (\phi_{0}/\pi B)^{1/2} \), and \( \Lambda = 2\lambda^{2}/d_{f} \).

The subscript \( \ell(t) \) denotes longitudinal (transverse) modes. q is the magnitude of the wavevector, and \( C_{66} \) is the bulk shear modulus for low field. For a >> \( \lambda \), the RHS of Eq. (1.35) has to be multiplied by \( \lambda/a \).

Since the longitudinal (compressional) mode force constant is proportional to q, but not \( q^{2} \), a compressional modulus in the usual sense cannot be defined. Recently Clem calculated the electromagnetic interaction energy between vortices for film of arbitrary thickness, taking both the bulk interaction and the interaction via the space outside into account (22). The resulting \( D_{q\ell} \) for B << H_{c2}, is given by (23)

\[ D_{q\ell} = \frac{B\phi_{0}q^{2}}{4\pi} \left[ 1 + \frac{2}{q d_{f}} \frac{1}{1 + q\lambda \coth(d_{f}/2\lambda)} \right] . \]  

(1.36)

This reduces to Eq. (1.34) when d_{f} << \lambda and q << 1/\( \lambda \). For intermediate film thickness, d_{f} \sim 2\lambda, q << 1/\( \lambda \),

\[ D_{q\ell} = \frac{B\phi_{0}q}{4\pi} \left( 2/d_{f} + q \right) . \]  

(1.37)
This is just the sum of the bulk result (Eq. (1.28)) and the thin film result (Eq. (1.34)).

We are now in a position to discuss the theory of flux pinning.

E. Theories of Flux Pinning

1. Static and dynamic pinning force

There are generally two ways of defining a volume pinning force.

We define the static volume pinning force to be the value of the Lorentz (driving) force when continuous flux movement starts to occur. This of course can be defined only when the distribution of pinning centers is uniform so that the situation in which only some vortices move continuously does not occur. In fact, in experiments a certain small-voltage criterion is used to establish $J_c$. Depending on the geometry of the measuring circuit, this may not indicate that all the vortices are moving.

We can also define a dynamic volume pinning force by the average force the vortices experience while they are moving. Experimentally, this is obtained by the intercept of the tangent at a point on the V-I characteristic with the current axis.

2. Pinning interactions

The first step in the calculation of the pinning force is to calculate the interaction between a pinning center and a vortex. This interaction depends on the type of pinning centers and the material. Since expressions for the interactions are not used later, they are not given in the following discussion.
If there is a void in the center of a vortex, then the condensation energy will be lowered (24,25). For the vortex to move away from the void, some electrons have to be driven normal. Therefore, there will be a force which tends to pin the vortex at the void. The magnitude of the force is approximately the change in condensation energy divided by a characteristic length, which depends on the size and orientation of the void with respect to the vortex (26).

If the pinning center is of the form of a boundary such that the region at both sides of the surface is large by comparison with the intervortex spacing, and that the free energy is different across the boundary, flux pinning can occur. This is the case at the surface of a superconductor. In this case the interaction is magnetic and can be calculated by the image method (27,28,29).

The specific volume of a metal in the superconducting state is larger than that in the normal state (30). Since the core of a vortex is normal, a strain field is present. If there is a defect nearby, this strain field will interact with the stress field of the defect. The interaction is linear in stress and is called the first-order or parelastic interaction (31).

The elastic energy of a defect depends on the elastic constants of the crystal. The elastic compliance, however, is larger in the superconducting state. This leads to an interaction between the vortex core and a defect. This is the second-order or the dielastic interaction (32). First estimates of these interactions have been given by various authors (31,32,33).
At high fields or low $\kappa$, one has to use the Ginzburg-Landau (G-L) theory since the difference between core and magnetic pinning is not easy to define. Also, to treat the parelastic or dielastic interactions rigorously, one must use the G-L theory. Since the G-L free energy depends on $\kappa$ and $H_{c2}$, one can visualize a pinning center with different $\kappa$ and $H_{c2}$ from the surrounding medium. For a material with internal stress, the G-L free energy has to be minimized with respect to the stress, thus giving rise to a third G-L equation. This new equation is used to treat both parelastic and the dielastic interactions (34-39). For magnetic inclusion, additional variation in the magnetic part of the free energy has to be considered (40).

The above procedures have been applied to various types of pinning centers (41-45). However, an unambiguous comparison with experiments is not possible since the individual forces have to be summed up to give the total force. We shall see in the next section that this summation process is very complicated and is far from being solved. The case of grain boundary pinning has been treated using the G-L theory recently (46). The results compare favorably with experiments (47) on niobium bicrystals in which only one grain boundary is present, and a summation is not needed.

3. The summation of pinning forces

Consider a large number of identical, randomly distributed symmetric pinning centers. If the FLL is completely rigid, i.e., $C_{ij} \to \infty$, there will be no volume pinning force, since equal numbers
of pinning centers will be applying positive and negative forces to the FLL. If the FLL is completely soft, i.e., $C_{ij} \rightarrow 0$, the volume pinning force will be just the number of pinning centers per unit volume multiplied by the maximum force a pinning center can exert on a vortex, since the FLL can adjust to the pinning centers completely. This may be realized in high-$\kappa$ hard superconductors, as suggested by Dew-Hughes (48) where the pinning centers are strong enough to totally disrupt the lattice.

In the intermediate regime in which the pinning force is small compared to the interaction between vortices, the calculation of the volume pinning force is very difficult. Traditionally, there are two approaches to the problem. The static approach (49,50) calculates the largest non-dissipative current the superconductor can carry. The corresponding volume pinning force is equivalent to the static volume pinning force and the critical current static critical current. This involves a statistical summation of the elementary pinning force from all the pinning centers. The dynamic approach (51,52,53) calculates the dynamic volume pinning force. This is equivalent to the calculation of the work done when each vortex is released from a pinning center. The FLL are assumed to be moving with a uniform velocity $\vec{v}_0$. The vortices lie within range of a pinning center along the direction of $\vec{v}_0$ will be slowed down and then released when they, in turn, encounter the pinning center. Power dissipated is equated to $F_p v_0$ to obtain the dynamic volume pinning force $F_p$. 
For both approaches, there is a threshold below which the volume pinning force is zero. The idea here is that in order for the volume pinning force to be nonzero, an elastic instability must occur when a vortex breaks away from a pinning center. This gives rise to irreversible dissipation of energy in the dynamic approach. In the static approach, there must be an asymmetric region in the pinning center in which no vortex can attain an equilibrium position. The threshold is related to the relative strength of the pinning force and the interaction between vortices. The weaker the former or the stronger the latter, the higher is the threshold.

The idea of a threshold and the essence of the elastic theory of flux-pinning are easily illustrated with a theory due to Campbell (54). Although the theory is probably too simple, it contains most of the features in the more exact theory by Labusch. In addition to this, it allows the calculation of a large number of experimentally measurable quantities. Our main purpose here is to show how the vortex lattice moves within the present elastic theory.

The assumptions are: (i) pinning centers are small and widely spaced, so that the displacement of a vortex due to a pinning center is not affected by the other pinning centers; (ii) pinning centers are randomly distributed; and (iii) FLL can be treated as an elastic continuum.

The main idea here, in relation with flux motion, is that the force on the whole FLL due to a pinning center must be periodic in the relative distance between the pinning center and a coordinate frame.
fixed in the FLL. This follows directly from the periodicity of the FLL. The force on a single vortex, however, is not periodic. For simplicity, the total force $f$ versus distance $x$ curve is taken to be linear over half the period (Fig. 3b). The pinning potential then looks like that in Fig. 3a. If we assume all the force is concentrated on one vortex (the one closest to the pinning center) then the FLL behaves like an elastic spring with spring constant $C$. $C$ is a function of the elastic moduli $C_{11}$, $C_{44}$, $C_{66}$ depending on whether we have point, line, or planar forces.

Let us place a pinning center at a random position in the FLL, so that $x_0$ is the relative position of the pinning potential with respect to the FLL. The final position of the FLL, $x$, is given by the force balance equation

$$C(x - x_0) = f(x_0) + \frac{df}{dx} \bigg|_{x_0} (x - x_0).$$

(1.38)

Notice that when $|C| < \left| \frac{df}{dx} \right|$ there is a region which no vortex can occupy. If the Lorentz force is zero, the region in which stable positions exist is symmetric. For a large number of pinning centers with random initial positions $x_0$, there will be vortices occupying all the positions in the region (Fig. 3c). Now if we apply a Lorentz force to the left, the region of stable positions moves to the left, and becomes asymmetric. The critical state is when one end of the region reaches the maximum force, $f_p$ (Fig. 3d). If we increase the Lorentz force further, the vortices will start to move. A vortex at the
Figure 3. Pinning potential versus distance curve (a) and pinning force versus distance curve (b–e) showing the stable positions of a vortex when the pinning potential is at threshold (c), above threshold (d), and below threshold (e)
maximum force position A in Fig. 3d, will be released from the pinning center. The next vortex on the line will move into the pinning center, and attain a position at B. Thus the force on the FLL from this pinning center jumps from the maximum value at A to the value at B. Since the vortices at other positions in the region of stable positions will also move to the left, the net effect for all the pinning centers and the whole FLL is unchanged: the region of stable positions is still fully occupied. The total force on the FLL is the same as that at the critical state. This force can be shown to be

\[ F_p = n_p \frac{f_p (f_p - f_t)}{(f_p + f_t)} \]  

(1.39)

where \( n_p \) is the density of the pinning center, and \( f_t \) is the threshold force

\[ f_t = \frac{1}{4} Cd \]  

(1.40)

for the above pinning potential. \( d \) is the intervortex spacing.

For the purpose of flux-flow noise, it is important to note that at a given pinning center, the motion of the FLL is periodic. Each vortex moves exactly the same way when it moves through the pinning center.

Figure 4e shows a well below threshold. If we apply a Lorentz force again, the vortices will occupy the whole region between C and D all the time. There is no total force since the region is symmetric.

The main problem with the elastic theory of flux pinning is that the threshold is not observed. If there is a threshold, then
sufficiently weak pinning centers will not give rise to a volume pinning force at all. Experiments, however, show that weak pinning centers that are well below the threshold can pin the FLL. The observed magnitude of the pinning force is also higher than that predicted by Eq. (1.39) for point forces (45).

It should be pointed out that there would be a small contribution to the dissipation from the subthreshold pinning centers. Although there is no elastic instability, there is a small perturbation in the velocity of the vortices, which increases the dissipation. This effect has been treated recently and shown to be relatively small (55). The motion of the vortices is also periodic.

The above theory applies to a system of widely spaced pinning centers. For a dense system of pinning centers, Kerchner (56) recently proposed a theory which shows that the volume pinning force is proportional to \( n^{1/2} \). (This result is also obtained by Campbell and Evetts (26), Larkin and Ovchinnikov (57), and Fietz and Webb (58).) Exactly how the vortices move in this model is not clear, however.

The role of defects in the FLL with respect to the volume pinning force so far has not been taken into account quantitatively. It has been suggested that the presence of dislocations may lower the pinning threshold (45). The presence of defects in the FLL will certainly complicate the way the vortices move and may affect the noise much more than the volume pinning force.
II. PREVIOUS EXPERIMENTAL AND THEORETICAL WORK

A voltage is measured across a magnetic-flux containing superconductor when the magnetic flux is moving with respect to the measuring circuit. This voltage in general depends on the geometry of the specimen and the measuring circuit (59,60). As mentioned earlier, the voltage consists of a time-averaged part and a noisy part:

\[ V(t) = \langle V \rangle + \delta V(t) \]  \hspace{1cm} (2.1)

Since the voltage is due to the motion of magnetic flux, the average motion gives rise to the average voltage, and the details of the motion will contribute to the noisy part. Although flux-flow noise occurs in both the intermediate state of a type-I superconductor and the mixed state of a type-II superconductor, we will concentrate on the latter case from now on.

The quantities being measured are usually the autocorrelation function

\[ \psi_V(s) = \langle \delta V(t) \delta V(t + s) \rangle \]  \hspace{1cm} (2.2a)

with the brackets \( \langle \ldots \rangle \) being an average over time \( t \), or the power spectrum

\[ W_V(\omega) = 2 \int_{-\infty}^{\infty} ds \psi_V(s) e^{i\omega s} \]  \hspace{1cm} (2.2b)

The first experiment on flux-flow noise was done by van Ooijen and van Gurp (61). They assumed flux moves in bundles in a superconductor. For a foil of width \( w \) and separation between voltage contacts
£, with applied field perpendicular to the foil surface and flux moving across the width of the foil, the average voltage is

$$<v> = \frac{B_v \xi}{c}$$  \hspace{1cm} (2.3)

where \( v_0 \) is the constant velocity of the flux bundles. They further assumed that such movement of a flux bundle produces a square voltage pulse given by

$$V(t) = \left\{ \begin{array}{ll} \frac{\phi v_0}{c w} & 0 < t < \frac{w}{v_0} \\ 0 & t < 0, \ t > \frac{w}{v_0} \end{array} \right.$$  \hspace{1cm} (2.4)

with \( \phi \) being the amount of flux in the bundle. The power spectrum was then calculated using a shot noise approach. The resulting power spectrum has a characteristic roll-off frequency \( \omega_c \) given by

$$\omega_c = \frac{v_0}{w}$$  \hspace{1cm} (2.5)

Later van Gurp (62) did more elaborate experiments on flux-flow noise and interpreted the results using the same assumptions. Generally speaking, the observed characteristic frequency \( \omega_c \) is higher than that predicted by Eq. (2.5). A pinned flux fraction, \( p \), thus was incorporated into the theory in order to explain this effect. \( p \) is the fraction of vortices that is not moving at a given time and is assumed to be constant. The consequence is to increase \( v_0 \) by a factor \((1-p)^{-1}\), thus enhancing \( \omega_c \).
The zero-frequency limit of the power spectrum supposedly will give information about the bundle size \( \phi \):

\[
W_{V}(0) = 2 \phi \langle V \rangle .
\]  

(2.6)

From the experiments, \( \phi \) was determined to be very large at low \( V_0 \) \((\sim 10^5 \phi_0)\) and very small (approaches \( \phi_0 \)) at large \( V_0 \).

In van Gurp's experiments, he identified a source of voltage noise at temperatures above the lambda point of liquid helium. He showed that the noise is due to temperature fluctuations coming from bubble formation in the boiling liquid helium. This is manifested as a sharp peak at low frequencies. This is called flicker noise.

The first detailed theoretical work done on the subject was by Clem (59). He showed that for a measuring circuit as shown in Fig. 4, with leads of zero radii and negligible resistance, the measured voltage is

\[
V = \int_a^b \dot{e}' \cdot \hat{r} \, dr - \frac{1}{c} \frac{d}{dt} \phi_{MS} ,
\]  

(2.7)

where \( \dot{e}' \) is the effective electric field, \( C_S \) is a path inside the specimen from \( a \) to \( b \), and

\[
\phi_{MS} = \int_{[S_{MS}]} ds \, \hat{b} \cdot \hat{n}
\]  

(2.8)

is the magnetic flux through a surface \( S_{MS} \) bounded by \( C_S \) and \( C_M \), with \( C_M \) being a path from \( a \) to \( b \) through the leads. Note that the voltage is independent of \( C_S \), but dependent on the geometry of the measuring
Figure 4. Schematic voltage measuring circuit attached to a specimen at points a and b
circuit. Essentially all the experimental results after van Gurp's were interpreted directly (63-65) or indirectly using this formula.

Thompson and Joiner (66,67) and Habbal and Joiner (68,69) have done extensive work on flux-flow noise. They used a foil in the form of a strip of width $w$ and distance between contacts $\lambda$, with $\lambda \gg w$. The voltage pulse generated by one flux bundle for this geometry was originally calculated to be a rectangular pulse, as in Eq. (2.4) (70, 71). Since the characteristic frequency $\omega_c$ was again generally larger than that predicted by Eq. (2.5), they proposed that the flux bundles are being interrupted in their motion by pinning centers. Thus the square pulse is being broken up into a series of (square) sub-pulses with duration equal to the time the flux bundle takes to go from one pinning center to the next. A distribution function was used to include the effect of the spread of the distances between the pinning centers. By adjusting parameters inherent in the distribution function, they were able to fit most of their experimental data. The main objection to this approach is that the parameters are too arbitrary. For example, a square distribution was assumed for the sub-pulse duration $\tau$:

$$g(\tau) = \begin{cases} \frac{v_0}{(\alpha - \beta)w} & \frac{\beta w}{v_0} \leq \tau \leq \frac{\alpha w}{v_0} \\ 0 & \text{otherwise} \end{cases}$$

In principle, to have a meaningful fit, one has to have a direct correspondence between the crystal-lattice grain sizes (distances
between pinning centers) and the parameters. \( \beta = 0, \alpha = 1 \), say, means that there is an equal number of grains of sizes ranging from zero to \( w \). This can only be checked by direct comparison with micrograph of the foils, which was not done in Refs. 66-69. Although the data in these references reveal that a large \( \alpha \) generally occurs in specimens with larger average grain size, this is not sufficient evidence that a square distribution can be used.

The second objection to this theory is that the wrong voltage pulse shape was used. Clem (72) has recently shown that if all specimen and measuring circuit dimensions are small as compared to

\[ \lambda_B \equiv \lambda(1 - B/H_{c2})^{-1/2}, \]

the contribution of one vortex to the measured voltage can be expressed as

\[ V(t) = \frac{\Phi}{2e} \frac{d\theta}{dt}, \]

where

\[ \theta = \frac{c}{2I_M} \int_{b_i[C]}^{t_i} d\tau \cdot \vec{b}_M. \]

\( \vec{b}_M \) is the self-field produced by a virtual current \( I_M \) flowing through the measuring circuit. \( C \) is a path that goes from the bottom of the vortex \( b_i \) to the top of the vortex \( t_i \). The total voltage is then a superposition of contributions from individual vortices. In the case of a flux bundle of size \( \Phi = N\Phi_0 \), the RHS of Eq. (2.10) has to be multiplied by \( N \). Applying this to a strip, the voltage from one vortex is found to be (73)
for a vortex moving with constant velocity $v_0$ across the strip midway between the contacts, and $\ell \gg w$. This is not a square pulse. Moreover, the pulse shape would change when the vortex is closer to one contact than the other. Thus, the assumption that a square voltage pulse constitutes the contributions from all the vortices (or flux-bundles) in the part of the foil between the contacts is false. Specifically, a weight factor depending on the instantaneous position of the flux bundle with respect to the contacts should be attached to the contribution from each sub-pulse.

Thompson and Joiner (66) also measured what they called the generalized bundle size $n(0)$. This is defined as

$$n(0) = \frac{W_v(0)}{2\Phi_0 \langle V \rangle}.$$  

(2.13)

This quantity is no longer just the number of vortices in a bundle according to their theory. They found that for small average velocity $v_0$, $n(0)$ decreases exponentially. Above a certain value of average velocity, $n(0)$ decreases more slowly. As a function of flux density $B$, $n(0)$ decreases monotonically with $B$. It should be remembered that the significance of $n(0)$ should always be connected with $W_v(0)$ via Eq. (2.13), i.e., $B v_0 n(0)$ is proportional to $W_v(0)$.

Another objection to the concept of a flux bundle is that it is ill-defined. The idea that a flux bundle is a group of vortices that
moves in the superconductor, while the other vortices remain stationary, implies that the flux bundle pushes or squeezes its way through the vortices in front of it, which seems a highly unlikely process. Any other definition of flux bundle will significantly change the interpretation of the Habbal-Thompson-Joiner theory.

In spite of the above objections, Habbal, Thompson, and Joiner's idea of vortices being interrupted in their path by pinning centers, which gives rise to flux-flow noise, is an important one.

Heiden and co-workers (74-81) have also done numerous experiments on flux-flow noise. The basic geometry used was a foil with $w \gg l$. The voltage probes rise perpendicularly from the foil surface. If flux bundles are moving parallel to each other across the width of the foil, as assumed by van Gurp, the characteristic frequency will be

$$\omega_c = \frac{v_0}{l}$$

(2.14)

from Clem's theory.

The results showed that the observed characteristic frequency was again too high. Also, the magnitude of the power spectrum was not changed significantly when the direction of transport current flow is changed with respect to the contact separation (76). This fact cannot be explained by the idea that flux bundles move parallel to each other.

Heiden and co-workers also measured the cross-correlation function

$$\psi_{12}(s) \equiv \langle \delta V_1(t) \delta V_2(t + s) \rangle$$

(2.15)
between two pairs of contacts. Here $\delta V_{1(2)}$ is the noise voltage measured by the first (second) pair of contacts. With one pair of contacts closed to the edge of the specimen, there was a time-of-flight peak at $s = L/v_0$, where $L$ was the distance along the flux-flow direction between the pairs of contacts if there were few pinning centers in the specimen (77-79). This shows that vortex density disturbances produced at the edge travel along rigidly with the vortex lattice. If there were many pinning centers in the specimen, there was no time-of-flight peak (75). This means that vortex density fluctuations are broken up by pinning centers.

By doing experiments on a polycrystalline foil whose grain structure was observed by a microscope, Heiden et al. (74,80) have obtained more results on the effects of pinning on flux-flow noise. When the pair of contacts were near the pinning centers (grain boundaries), the flux-flow noise voltage was the largest. When they were within a large grain and far away from the grain boundaries, the flux-flow noise voltage was reduced. The critical current also showed similar behavior. The characteristic frequencies were of the order of the grain transit time $D_g/v_0$, where $D_g$ is the average size of the grains. For $D_g$ larger than a certain value, however, the characteristic frequency was independent of $D_g$ (80).

There were also other measurements on flux-flow noise. Jarvis and Park showed that the noise voltage does indeed depend on the geometry of the measuring circuit (82). They also observed some peaks in the power spectrum which could not be explained (60). Wade (83)
measured voltage noise power spectrums on a pair of magnetically
coupled superconducting thin films. Transport current was passed only
through one of the films. The noise spectrum measured for both films,
which are of different widths, nevertheless showed similar frequency
dependence. This shows that the transit time for the flux to cross
the specimen is not a controlling factor in the noise. In the same
paper Wade suggested a model in which vortices move in channels. All
vortices in a channel move in a coordinated way. This model, however,
is only qualitative. If the pinning force is uniform, then the defini-
tion of a channel will be quite arbitrary and the identity of a channel
will not be unique. Jarvis and Park also presented a similar model (60).

There are also general features that were observed in most experi-
ments on flux-flow noise. Over a range of frequencies, the power
spectrum usually decreases as $1/\omega$ (60,66,68,82,83). The noise is also
largest for currents only slightly above the critical currents (59).
This can be interpreted as the relative effect of pinning being largest
in this situation. Of course, when $v_0$ is zero, there is no noise,
therefore the noise level has to rise to a peak and then decreases
monotonically as a function of $v_0$ (67).

As we see in the above discussion, there is still no complete
theory for the flux-flow voltage noise. It is now reasonably certain
that the noise is produced at pinning centers. The theory of flux
pinning, especially that of the dynamic pinning force, should then be
related to the noise. Most theories of flux pinning have the inter-
action between vortices (FLL elasticity) playing a central role. No
The theory of flux-flow noise, however, has yet included FLL elasticity in its formalism. Hence the present theories of flux flow noise cannot be related to the present theories of flux pinning in a direct way.

The objective of the present work is to develop a theory of flux-flow noise which takes into account the interaction between vortices and is compatible with the elastic theory of flux pinning. From such a theory, a picture of how magnetic flux moves in a superconductor may be developed. Experimental results can also be interpreted unambiguously.
III. MEASURED VOLTAGE IN SUPERCONDUCTORS

A. The Resolution Function

It is well-known that a voltage is measured across a superconductor in the flux-flow regime of the mixed or intermediate state. The origin of the voltage is the dissipation of energy when the normal magnetic flux-containing regions move in the superconducting matrix. The exact expression for the voltage measured is dependent on the geometric configuration of the measuring circuit as well as the magnitude and velocity of the moving flux entities (59,60).

Recently Clem (72) has derived a gauge-invariant expression for the voltage which takes the form of the Josephson relation. The derivation applies only to measuring circuits consisting of leads with vanishing radius and must be generalized for application to flux-flow noise in which the characteristic length scale for local velocity fluctuations is smaller than the lead radius. In this chapter, a corresponding expression is derived, which accounts for the finite radius of the leads.

Consider a type-II superconductor with two normal leads attached to it, as sketched in Fig. 5. The other ends of the leads are connected to terminals A and B of a sensitive voltmeter. The measured voltage generated by the motion of a vortex, which threads the superconductor, is (59)

\[ V = V_A - V_B = - \int_{A[C_M]}^{a} d\vec{z} \cdot \nabla \phi - \int_{a[C_M]}^{b} d\vec{z} \cdot \nabla \psi' - \int_{b[C_M]}^{B} d\vec{z} \cdot \nabla \phi . \]
Figure 5. Measuring circuit with thick leads, for which the measured voltage $V$ is described by Eq. (3.1)
Here \( \psi' \) is the electrochemical potential per unit charge in the superconductor, \( \phi \) is the same quantity in the normal leads, \( C_S \) is a path inside the specimen that connects two points, \( a \) and \( b \), one under each contact, and \( C_M \) is a path that connects \( a \) and \( b \) through the leads and the voltmeter. Assuming no contact voltage between the superconducting specimen and the leads, \( V \) is independent of \( C_S \) and \( C_M \). In the normal leads we can write

\[
\dot{\psi} = -\nabla \phi - \frac{1}{c} \dot{\hat{a}},
\]

where \( \dot{\psi} \) is the electric field, \( c \) is the speed of light in vacuum, and \( \dot{\hat{a}} \) is the vector potential. In the superconductor the electrochemical potential per unit charge can be written as

\[
\psi' = \frac{\phi_0}{2\pi c} \gamma,
\]

where \( \phi_0 \) is the flux quantum and \( \gamma \) is the phase of the order parameter.

Combining Eqs. (3.1) through (3.3) yields

\[
V = -\frac{\phi_0}{2\pi c} \int_a^{b_{[C_S]}} \mathbf{d} \mathbf{x} \cdot \nabla \gamma - \frac{1}{c} \int_a^{A_{[C_M]}} \mathbf{d} \mathbf{x} \cdot \dot{\hat{\mathbf{a}}} - \frac{1}{c} \int_{B^{[C_M]}}^{b} \mathbf{d} \mathbf{x} \cdot \dot{\hat{\mathbf{a}}}
- \int_a^{A_{[C_M]}} \mathbf{d} \mathbf{x} \cdot \dot{\mathbf{e}} - \int_{B^{[C_M]}}^{b} \mathbf{d} \mathbf{x} \cdot \dot{\mathbf{e}}.
\]

For leads with negligible resistance, the last two terms in Eq. (3.4)
are very small, and the equation is the same as Eq. (2.1) of Ref. 59 and Eq. (1) of Ref. 60, with \( \psi' \) replaced by \( (\phi_0/2\pi c)^{\gamma} \).

Next we look at the quantity \( I_M V \), where \( I_M \) is a virtual current flowing through the leads and the superconductor from A to B. Assuming that there is no accumulation of charge anywhere, we can visualize a continuous "current tube", through which a fixed amount of current, \( dI_M \), flows. Choosing the paths \( C_S \) and \( C_M \) to lie along a current tube, we have,

\[
dI_M V = -\frac{\phi_0}{2\pi c} dS_1 \hat{n} \cdot \hat{j}_M \frac{d\gamma_a}{dt} + \frac{\phi_0}{2\pi c} dS_2 \hat{n} \cdot \hat{j}_M \frac{d\gamma_b}{dt} - dS_M \hat{n} \cdot \hat{j}_M \\
\cdot \iint d\vec{z} \cdot (\hat{\alpha} + \hat{e}) - dS_2 \hat{n} \cdot \hat{j}_M \iint d\vec{z} \cdot (\hat{\alpha} + \hat{e}) ,
\]

(3.5)

where \( dS_1 \) and \( dS_2 \) are surface element in contacts 1 and 2 respectively, \( dS_M \) is the cross section of the current tube between A and a, \( dS_2 \) is the cross section of the current tube between b and B. In all terms on the right-hand side (RHS) of Eq. (3.5)

\[
dSn \cdot \hat{j}_M \equiv dI_M ,
\]

(3.6)

where \( dS \) is any surface element specified above. Integrating Eq. (3.5) over all current tubes yields

\[
I_M V = \frac{\phi_0}{2\pi c} \int_{\text{both contacts}} dSn \cdot \hat{j}_M \hat{\gamma} - \frac{1}{c} \int_{\text{leads}} d^3\tau \hat{j}_M \cdot \hat{a} - \int_{\text{leads}} d^3\tau \hat{j}_M \cdot \hat{e} .
\]

(3.7)

Integrating by parts yields
\begin{equation}
\frac{\phi_0}{2\pi c} \frac{d}{dt} \int_{\text{inside}} d^3r \mathbf{J}_M \cdot \mathbf{\nabla}_Y = \frac{\phi_0}{2\pi c} \frac{d}{dt} \int_{\text{both contacts}} d\mathbf{S}_n \cdot \mathbf{\hat{J}}_M
- \frac{\phi_0}{c} \frac{d}{dt} \int_{\text{cut}} d\mathbf{S}_n \cdot \mathbf{\hat{J}}_M .
\end{equation}

The integral on the left-hand side (LHS) is over the inside of the superconductor. The integral in the second term on the RHS is over a cut surface bounded by the closed curve formed by the vortex axis and an arbitrary curve C on the surface of the superconductor connecting the top (t) and bottom (b) of the vortex, as shown in Fig. 6.

We now consider the interaction energy

\begin{equation}
\Delta U = \int_{\text{all space}} d^3r \frac{1}{4\pi} \mathbf{\hat{b}}_M \cdot \mathbf{\hat{b}}_1 + \int_{\text{inside}} d^3r \frac{4\pi \lambda^2}{c f^2} \mathbf{\hat{J}}_M \cdot \mathbf{\hat{J}}_1 ,
\end{equation}

where f is the magnitude of the reduced order parameter, and the subscript 1 denotes contribution from the vortex. Writing \( \mathbf{\hat{b}}_M \cdot \mathbf{\hat{b}}_1 / 4\pi = \mathbf{V} \cdot \mathbf{\hat{a}}_M x \mathbf{\hat{a}}_1 / 4\pi + \mathbf{\hat{J}}_M \cdot \mathbf{\hat{a}}_1 / c \), we can show that \( \Delta U = 0 \). Next we evaluate \( \Delta U \) by writing \( \mathbf{\hat{b}}_M \cdot \mathbf{\hat{b}}_1 / 4\pi = \mathbf{V} \cdot \mathbf{\hat{a}}_M x \mathbf{\hat{b}}_1 / 4\pi + \mathbf{\hat{J}}_M \cdot \mathbf{\hat{a}}_1 / c \). This leads to the equation

\begin{equation}
\frac{\phi_0}{2\pi c} \int_{\text{inside}} d^3r \mathbf{\hat{J}}_M \cdot \mathbf{\nabla}_Y = \frac{1}{c} \int_{\text{leads}} d^3r \mathbf{\hat{J}}_M \cdot \mathbf{\hat{a}}_1 ,
\end{equation}

where we have used the fact that \( \mathbf{\hat{J}}_1 \) is parallel to the specimen surface.

Now we make the important assumption that \( \mathbf{\hat{J}}_M \) is independent of time both under the contacts and in the leads. This means that the skin depth associated with the characteristic frequency of the flux movements
Figure 6. View of specimen showing the cut surface appearing in Eq. (3.8); $t$ and $b$ denote the top and bottom of the vortex, respectively. The voltmeter and leads are not shown.
has to be large by comparison with the size of the leads, and the "backflow current" around the vortex core is being ignored.

With this assumption and the aid of Eqs. (3.8) and (3.9), we can rewrite Eq. (3.7) as

$$V = \frac{\phi_0}{I_M} \frac{d}{dt} \int_{\text{cut}} dS \hat{n} \cdot \vec{j} - \int_{\text{leads}} dV \vec{j} \cdot \hat{e} .$$

If we have low resistance leads, we can drop the 2nd term at the RHS. Ampere's law then enables us to write

$$V = \frac{\hbar}{2e} \frac{d\delta}{dt} ,$$

where

$$\theta = \frac{c}{2I_M} \int_{bC} \hat{d} \cdot \vec{b} .$$

The integral in Eq. (3.13) is along the boundary C of the cut from the bottom (b) to the top (t) of the vortex, as sketched in Fig. 6. This is the same as Eq. (4) in Ref. 72 for the case of vanishing lead radius.

Note that in the case of vanishing lead radius, the current in the leads and under the contacts is time-independent for all practical purposes.

Applying Ampere's law and Stokes' theorem, and ignoring the contribution in Eq. (3.13) from along the vortex axis, we can also write

$$V = \frac{\phi_0}{4\pi I_M} (\vec{b}_M(\vec{\rho}_t) \cdot \vec{\rho}_t - \vec{b}_M(\vec{\rho}_b) \cdot \vec{\rho}_b) ,$$

where $\vec{\rho}_t$ and $\vec{\rho}_b$ are the coordinates of the top and bottom of the vortex, respectively.
For the case of a flat slab or thin film of uniform thickness, we can usually assume the top and bottom of a vortex move at the same velocity. The voltage is then simply

$$V = \mathbf{g}(\mathbf{p}) \cdot \mathbf{v},$$  \hspace{1cm} (3.15)$$

where $\mathbf{p}$ is the position of the vortex, now a two-dimensional vector, $\mathbf{v} \equiv \dot{\mathbf{p}}$, and

$$\mathbf{g}(\mathbf{p}) = \frac{\phi_0}{4\pi I_M} \left[ \mathbf{b}_{M\ell}(\mathbf{p}) \mathbf{m}_{b} - \mathbf{b}_{M\ell}(\mathbf{p}) \mathbf{m}_{b} \right].$$  \hspace{1cm} (3.16)$$

$\mathbf{b}_{M\ell}$ and $\mathbf{b}_{M\ell}$ is the values of $\mathbf{b}_{M\ell}$ at the top and bottom of the slab (film), respectively.

**B. Measured Voltage and Noise**

For a specimen containing a number of vortices, the total measured voltage is a superposition of contributions from individual vortices, such that

$$V = \sum_{i} V_i.$$  \hspace{1cm} (3.17)$$

Here,

$$V_i = \frac{\pi}{2e} \frac{d\theta_i}{dt},$$  \hspace{1cm} (3.18)$$

where $\theta_i$ is defined as in Eq. (3.13), but according to the position of vortex $i$.

From now on, we shall only consider the case of a flat slab or thin film, so that we can extend Eq. (3.18) to
\[ V = \sum_i \hat{g}(\mathbf{r}_i) \cdot \mathbf{v}_i \quad (3.19) \]

According to Ref. 73, we can define a vortex-current density

\[ \mathbf{J}(\mathbf{r},t) = \sum_i \mathbf{v}_i(t) \delta_2[\mathbf{r} - \mathbf{r}_i(t)] \quad (3.20) \]

Conservation of the number of vortices leads to the continuity equation

\[ \frac{\partial n}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad , \]

where \( n \) is the vortex density:

\[ n(\mathbf{r},t) = \sum_i \delta_2[\mathbf{r} - \mathbf{r}_i(t)] \quad (3.22) \]

\( \mathbf{J}(\mathbf{r},t) \) contains all the information about vortex dynamics. The voltage then can be written as

\[ V(t) = \int d^2 \rho \, \hat{g}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r},t) \quad , \]

where the integral is over the whole specimen. The time-averaged voltage is

\[ \langle V(t) \rangle_t = \int d^2 \rho \, \hat{g}(\mathbf{r}) \cdot \langle \mathbf{J}(\mathbf{r},t) \rangle_t \quad , \]

and the noisy part of the voltage is

\[ \delta V(t) \equiv V(t) - \langle V(t) \rangle_t \]

\[ = \int d^2 \rho \, \hat{g}(\mathbf{r}) \cdot \delta \mathbf{J}(\mathbf{r},t) \quad , \]

where
\[ \delta \tilde{J}(\vec{r},t) = \tilde{J}(\vec{r},t) - \langle \tilde{J}(\vec{r},t) \rangle_t \quad . \] (3.26)

The usual measured quantities are the autocorrelation function and the power spectrum. The autocorrelation function is given by Eq. (2.14)

\[ c_V(s) \equiv \langle \delta V(t) \delta V(t + s) \rangle_t \]
\[ = \int d^2 \rho \int d^2 \rho' \sum_{\alpha, \beta} g_{\alpha}(\vec{\rho}) g_{\beta}(\vec{\rho}') K_{\alpha\beta}(\vec{\rho}, \vec{\rho}', s) \quad , \] (3.27)

where

\[ K_{\alpha\beta}(\vec{\rho}, \vec{\rho}', s) \equiv \langle \delta J_{\alpha}(\vec{\rho}, t) \delta J_{\beta}(\vec{\rho}', t+s) \rangle_t \] (3.28)

is the vortex-current correlation function. Here, \( \alpha \) and \( \beta \) refer to the Cartesian coordinates. The power spectrum, \( W_V(\omega) \), is twice the Fourier transform of the autocorrelation function.

We thus see that the voltage and its autocorrelation function consist of two parts, one depending only on the measuring circuit geometry, and the other on the vortex dynamics. This is true if we assume the measuring circuit does not influence the motion of the vortices.

The above formalism can be easily applied to the case of cross correlation measurements, where two pairs of leads, placed at different parts of the specimen, are used to measure the noise.

We now turn our attention to the vortex dynamics, keeping in mind that we want to take the interactions between vortices into account. Since the interaction force on a vortex in general depends on the position of all other vortices, a treatment in reciprocal space is much simpler.
mathematically than in real space. This is done in the next chapter and the results are applied to a special kind of noise, Johnson noise.
IV. THEORY INCLUDING THE INTERACTION BETWEEN VORTICES - APPLICATION TO JOHNSON NOISE

A. Expansion of Vortex Positions in Normal Modes

Consider a slab or thin film, with no bending of vortices allowed. The positions of vortices are described by two-dimensional vectors. If the vortices form a perfect lattice, each vortex can be uniquely identified by its equilibrium position within the lattice. We attach a reference frame to the flux-line lattice (FLL frame). The equilibrium position of a vortex is measured in this frame. There is also a laboratory frame (lab frame), which coincides with the FLL frame if the lattice is not moving as a whole. The position of the vortex with equilibrium position \( \vec{\ell} \) is, in the FLL frame,

\[
\rho_0(\vec{\ell},t) = \vec{\ell} + \vec{s}(\vec{\ell},t) ,
\]

where \( \vec{s}(\vec{\ell},t) \) is the deviation from the equilibrium position. The interaction energy between the vortices can be written in quadratic form in the harmonic approximation

\[
V\{\vec{s}(\vec{\ell},t)\} = \frac{1}{2} \sum_{\vec{\ell},\vec{\ell}'} \tilde{G}(\vec{\ell},\vec{\ell}') \vec{s}(\vec{\ell},t) \vec{s}(\vec{\ell}',t) ,
\]

where \( \tilde{G}(\vec{\ell},\vec{\ell}') \) is the elastic matrix. The interaction force on vortex \( \vec{\ell} \) is then

\[
\vec{f}_{\text{int}}(\vec{\ell},t) = -\frac{1}{\vec{s}(\vec{\ell},t)} V\{\vec{s}(\vec{\ell},t)\} = -\sum_{\vec{\ell}'} \tilde{G}(\vec{\ell},\vec{\ell}') \vec{s}(\vec{\ell}',t) .
\]
Since $\tilde{G}(\mathbf{z},\mathbf{z}')$ depends only on $\mathbf{z}-\mathbf{z}'$ and is real and symmetric,

$$\tilde{G}(\mathbf{z},\mathbf{z}') = \tilde{G}(\mathbf{z}'-\mathbf{z}) = \tilde{G}(\mathbf{z})$$ \hspace{1cm} (4.4)

we can define a dynamical matrix, as in a crystal lattice,

$$\tilde{D}(\mathbf{q}) = \sum_{\mathbf{h}} \tilde{G}(\mathbf{h}) \, e^{i\mathbf{q} \cdot \mathbf{h}}$$ \hspace{1cm} (4.5)

$\tilde{D}$ is also real and symmetric. The basis vectors $\hat{e}(\mathbf{q},\lambda)$ that diagonalize $\tilde{D}$ are called polarization vectors:

$$\tilde{D}(\mathbf{q})\hat{e}(\mathbf{q},\lambda) = D_{\mathbf{q},\lambda} \hat{e}(\mathbf{q},\lambda)$$ \hspace{1cm} (4.6)

There are two polarizations, which at long wavelengths can be identified as transverse ($\lambda=t$) or longitudinal ($\lambda=z$). As usual, the polarization vectors are orthogonal:

$$\hat{e}(\mathbf{q},\lambda) \cdot \hat{e}(\mathbf{q}',\lambda') = \delta_{\lambda\lambda'}$$ \hspace{1cm} (4.7)

$$\sum_{\lambda} \epsilon_i(\mathbf{q},\lambda)\epsilon_j(\mathbf{q},\lambda) = \delta_{ij}$$ \hspace{1cm} (4.8)

From $\sum_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{h}} = N\delta_{\mathbf{h},0}$, where $N$ is the number of vortices and the sum is over the first Brillouin zone (BZ), we obtain

$$\tilde{G}(\mathbf{h}) = \frac{1}{N} \sum_{\mathbf{q}}, 1st \ BZ \tilde{D}(\mathbf{q}) \, e^{i\mathbf{q} \cdot \mathbf{h}}$$ \hspace{1cm} (4.9)

The orthogonality of the polarization vectors yields further

$$G_{ij}(\mathbf{h}) = \frac{1}{N} \sum_{\mathbf{q},\lambda} D_{\mathbf{q},\lambda} \epsilon_i(\mathbf{q},\lambda)\epsilon_j(\mathbf{q},\lambda) \, e^{\pm i\mathbf{q} \cdot \mathbf{h}}$$ \hspace{1cm} (4.10)
In the following, all summations on \( \hat{q} \) are over the first BZ unless otherwise stated.

Now we expand \( \hat{s}(\hat{r},t) \) in the basis \( \hat{e}(\hat{q}_\lambda) \):

\[
\hat{s}(\hat{r},t) = \sum_{\hat{q},\lambda} e^{i\hat{q} \cdot \hat{r}} \hat{e}(\hat{q}_\lambda) \hat{Q}_{q_\lambda}(t).
\]  

(4.11)

The coefficients of expansion, \( \hat{Q}_{q_\lambda}(t) \), are the normal mode amplitudes. The inverse relation is

\[
\hat{Q}_{q_\lambda}(t) = \frac{1}{N} \sum_{\hat{r}} e^{-i\hat{q} \cdot \hat{r}} \hat{e}(\hat{q}_\lambda) \cdot \hat{s}(\hat{r},t). \]

(4.12)

All the above relations are derived for the FLL frame.

According to Eq. (3.22), the noise measured can be expressed in terms of the vortex-current correlation function. If the dimensions of the measuring circuit are large by comparison with the intervortex spacing, we can treat the vortex lattice as a continuum. This is equivalent to saying that we only consider modes with \( q \) much smaller than those at the zone boundary. To first order in small quantities, the change in the vortex-current density is, assuming the displacements of the vortices from their equilibrium positions to be small,

\[
\delta \hat{J}(\hat{r},t) = n_0 \delta \hat{v}(\hat{r}-\hat{v}_0 t,t) + \hat{v}_0 \delta n(\hat{r}-\hat{v}_0 t,t),
\]

(4.13)

where \( n_0 \) is the equilibrium vortex density, \( \delta \hat{J} \) and \( \delta \hat{r} \) are vectors in the lab frame, and \( \delta \hat{v} \) and \( \delta n \) are referred to the FLL frame, which is moving with velocity \( \hat{v}_0 \). We can identify \( \delta \hat{v} \) as just \( \hat{s} \), and \( \delta n \) as \( -n_0 \hat{v} \cdot \hat{s} \).

Therefore, in terms of normal modes in the FLL frame,
\[ \delta J(q, t) = n_0 \sum_{q, \lambda} \left[ \frac{\hat{c}(q\lambda) \hat{Q}_q(t)}{\hat{v}_0} - \frac{i q \cdot \hat{c}(q\lambda) Q_q(t)}{\hat{v}_0} \right] e^{i q \cdot (\vec{v}_0 - \vec{v}_0) t} \] (4.14)

The measured noise voltage is, from Eq. (3.25)

\[ \delta V(t) = \sum_{q, \lambda} \delta V_{q\lambda}(t) \] , (4.15)

with

\[ \delta V_{q\lambda}(t) = \left[ F_{q\lambda} \dot{Q}_q(t) + G_{q\lambda} Q_q(t) \right] e^{i q \cdot \vec{v}_0 t} , \quad (4.16) \]

where

\[ F_{q\lambda} \equiv n_0 \int d^2 \rho \frac{\text{g}(\rho) \cdot \varepsilon_q(\rho)}{\varepsilon_q(\rho)} e^{i q \cdot \vec{\rho}} , \]

\[ G_{q\lambda} \equiv -n_0 \int d^2 \rho \frac{\text{g}(\rho) \cdot \vec{v}_0}{\varepsilon_q(\rho)} e^{i q \cdot \vec{\rho} \cdot (i q \cdot \varepsilon(q\lambda))} . \quad (4.17) \]

The Doppler factor \( e^{i q \cdot \vec{v}_0 t} \) in Eq. (4.16) follows from the fact that the normal modes are defined with respect to the moving FLL frame, whereas the measuring circuit is fixed in the lab frame.

There are two terms in the expression of the noise voltage. The first term is proportional to \( \delta \nu \), or \( \dot{Q}_{q\lambda} \), while the second term is proportional to \( \delta n \), or \( Q_{q\lambda} \). This is another way of saying that flux-flow noise can be produced in two ways: by local velocity fluctuations or by density fluctuations being carried along with the FLL (79-81).

We now specify the measuring circuit geometry and calculate the resolution function. We consider an infinite film of type-II superconducting material with thickness \( d_f \). A perpendicular magnetic field generates a flux density \( B \) in the film. The measuring circuit consists
of two identical low resistance leads of radius $R$ attached to the film. The leads rise perpendicularly to the film and connect to a voltmeter far away from the film. The distance between the two contacts, $\rho_{ab} \equiv |\vec{\rho}_a - \vec{\rho}_b|$ is much larger than $R$ (Fig. 7). From Eq. (3.16) we obtain the resolution function,

$$g(\rho) = -\frac{\phi_0}{4\pi} [\hat{b}'_{Ma}(\rho) - \hat{b}'_{Mb}(\rho)], \quad (4.18)$$

where $\rho$ is the position of the top of the vortex and

$$\hat{b}'_{Ma}(\rho) = \frac{2}{c|\vec{\rho} - \vec{\rho}_a|} \hat{z} \times (\vec{\rho} - \vec{\rho}_a) \quad (4.19)$$

for $|\rho - \rho_a| > R$ and $z \equiv \hat{B}/B$. $\hat{b}'_{Mb}$ is also given by Eq. (4.19) with subscript $a$ replaced by $b$. We assume that the resistivity of the normal leads is much smaller than the flux-flow resistivity of the superconductor, so that

$$\hat{b}'_{Ma}(\rho) \sim 0 \quad (4.20)$$

for $|\rho - \rho_a| < R$. The same applies to $\hat{b}'_{Mb}$. Equation (4.17) then yields

$$F_{q\lambda} = \frac{B}{c} \left[ z \cdot e(q\lambda) \times \hat{q} \right] \left( \frac{iq \cdot \rho_a}{iq} - \frac{iq \cdot \rho_b}{iq} \right) J_0(qR) \delta_{\lambda,t}, \quad (4.21)$$

$$G_{q\lambda} = \frac{B}{c} \left[ z \cdot \hat{q} \times \hat{v}_0 \right] (e^{iq \cdot \rho_a} - e^{iq \cdot \rho_b}) J_0(qR) \delta_{\lambda,t}, \quad (4.22)$$

where $J_0$ is the zeroth order Bessel function. For this measuring circuit, where the two leads run perpendicularly from the film up to a large distance away, the only contribution to the noise voltage is from the
Figure 7. Measuring circuit assumed for calculation of Johnson noise
transverse modes if $\dot{v}_0$ is zero. For other measuring circuit configurations, however, this will not in general be the case.

B. Vortex Dynamics

The phenomenological force balance equation for a vortex is

$$M \ddot{\xi}(\xi, t) = -\eta \dot{\rho}(\xi, t) - \sum \tilde{G}(\xi, \xi') \dot{s}(\xi', t) + \tilde{f}_{\text{ext}}(\xi, t) \quad (4.23)$$

Here $M$ is the mass per unit length of a vortex, as defined by Bardeen and Stephen (15), $\eta$ is the viscous drag coefficient per unit length, and $\tilde{\rho}(\xi, t)$ is the position in the lab frame of the vortex whose equilibrium position in the FLL frame is $\tilde{\xi}$. $\tilde{f}_{\text{ext}}(\xi, t)$ is the external force per unit length on the vortex, excluding the viscous drag forces and the interaction force. It includes, for instance, the elementary pinning forces and the Lorentz force from the transport current. Assuming that the whole FLL is moving with an average velocity $\dot{v}_0$, which is time-independent, we obtain with the help of Eq. (4.23),

$$M \ddot{s}(\xi, t) + \eta \dot{v}_0 + \eta \dot{s}(\xi, t) + \sum \tilde{G}(\xi, \xi') \dot{s}(\xi', t) = \tilde{f}_{\text{ext}}(\xi, t) \quad (4.24)$$

Taking the time average of Eq. (4.24) yields

$$\eta \dot{v}_0 = \tilde{f}_d + \langle \tilde{f}(\xi, t) \rangle \quad , \quad (4.25)$$

and

$$M \ddot{s}(\xi, t) + \eta \dot{s}(\xi, t) + \sum \tilde{G}(\xi, \xi') \dot{s}(\xi', t) = \delta \tilde{f}(\xi, t) \quad . \quad (4.26)$$
Here, \( <\ldots> \) denotes an average over time, and \( \mathbf{\vec{f}} = \mathbf{\vec{f}}_{\text{ext}} - \mathbf{\vec{f}}_{d} \), where \( \mathbf{\vec{f}}_{d} \) is the constant Lorentz force due to the constant transport current, and \( \delta \mathbf{\vec{f}} = \mathbf{\vec{f}} - <\mathbf{\vec{f}}> \). We have assumed that \( <\mathbf{\vec{s}}(\mathbf{\vec{r}}, t)> \) and \( <\mathbf{\vec{s}}(\mathbf{\vec{r}}, t)> = 0 \) for all \( \mathbf{\vec{r}} \), and that \( \mathbf{\vec{f}}(\mathbf{\vec{r}}, t) \) is distributed such that \( <\mathbf{\vec{s}}(\mathbf{\vec{r}}, t)> \) is independent of \( \mathbf{\vec{r}} \). \( \int_{\mathbf{\vec{r}}} G(\mathbf{\vec{r}}, \mathbf{\vec{r}}') = 0 \) has also been used.

Equation (4.25) is just the familiar dynamic force balance equation. Applied to flux pinning where \( \mathbf{\vec{f}} \) is the elementary pinning force, this equation states that the dynamic pinning force is just the time average of the elementary pinning force on a vortex. Equation (4.25) also shows that the deviation of the positions of the vortices from equilibrium is determined by the deviation of the applied force from its time-averaged value. Therefore, for flux-flow noise induced by pinning, the fluctuating part of the pinning force is the governing factor. There are also other sources of the applied force. A thermal gradient will generate a force on the vortices. If the thermal gradient fluctuates with time, voltage noise can be produced. Even with no thermal gradient, at nonzero temperature each vortex is in thermal equilibrium with its surroundings and undergoes thermally-induced random motion about its equilibrium position. If the time-averaged velocity of the vortex, \( \mathbf{\vec{v}}_{0} \), is not zero, then the dissipation can be separated into two parts. The part that is proportional to \( \mathbf{\vec{v}}_{0}^{2} \) is associated with the viscous drag coefficient \( \eta \), with a corresponding viscous drag force, \( -\eta \mathbf{\vec{v}}_{0} \). The other part of the dissipation is associated with a random force, which is called the Langevin force, as in the case of the Brownian movements of
colloidal particles. The associated noise is called Johnson noise. It is present at nonzero temperatures, whether $\nu_0$ is zero or not.

Expanding Eq. (4.26) in normal modes, we obtain

$$\ddot{Q}_{q\lambda}(t) + \eta \dot{Q}_{q\lambda}(t) + D_{q\lambda} Q_{q\lambda}(t) = \delta f_{q\lambda}(t) ,$$

(4.27)

where

$$\delta f_{q\lambda}(t) = \frac{1}{N} \sum_{\lambda} \int \frac{d\omega}{2\pi} \gamma_{\lambda}(q,\omega) A_{\lambda}(q,\omega) \exp(-i\omega t) .$$

(4.28)

The solution of Eq. (4.27) is

$$Q_{q\lambda}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \gamma_{\lambda}(q,\omega) A_{\lambda}(q,\omega) \exp(-i\omega t) ,$$

(4.29)

where

$$A_{\lambda}(q,\omega) = \int_{-\infty}^{\infty} dt A_{q\lambda}(t) \exp(i\omega t) ,$$

(4.30)

$$A_{q\lambda}(t) \equiv \frac{1}{M} \delta f_{q\lambda}(t) ,$$

(4.31)

$$\gamma_{\lambda}(q,\omega) = -\left(\omega^2 + i\omega/\tau - \frac{2\omega}{\omega_{q\lambda}}\right)^{-1} ,$$

(4.32)

$$\omega_{q\lambda} \equiv \frac{D_{q\lambda}}{M} ,$$

(4.33)

$$\tau \equiv \frac{M}{\eta} .$$

(4.34)

Hence, if $\delta f$ is known, we can calculate $Q_{q\lambda}$ and $\dot{Q}_{q\lambda}$, and the associated noise voltage.
C. Johnson Noise

1. Formalism

As an example, we shall apply the above procedure to Johnson noise in an ideal film with no transport current in constant temperature. In this case, the FLL is stationary and \( \dot{v}_0 = 0 \). The force we are concerned with is the Langevin force and has the property that it is uncorrelated in direction, space, and time, and has zero time average:

\[
\langle f_i(z,t)f_i(z',t') \rangle = \delta_{ii} \delta_z \delta(t-t') \, , \quad (4.35)
\]

\[
\langle f(t) \rangle = 0 \, . \quad (4.36)
\]

Since \( \dot{v}_0 = 0 \), if we consider the measuring circuit shown in Fig. 7, the only contribution to the noise voltage is from the transverse modes.

The autocorrelation function is then

\[
\psi_V(s) = \sum_{\hat{q}, \hat{q}'} F_{\hat{q}t} F_{\hat{q}'t}^* \langle \dot{q}_{\hat{q}t}(t) \dot{q}_{\hat{q}'t}(t+s) \rangle \quad . \quad (4.37)
\]

By using Eq. (4.29), this can be shown to be

\[
\psi_V(s) = \frac{\varphi}{M^2 N} \sum_{\hat{q}} |F_{\hat{q}t}|^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^2 |g_t(\hat{q},\omega)|^2 e^{-i\omega s} \quad , \quad (4.38)
\]

Equation (4.35) has been used to eliminate the cross terms with \( \hat{q} \neq \hat{q}' \).

The power spectrum then follows directly:

\[
W_V(\omega) = \frac{\varphi}{M^2 N} \sum_{\hat{q}} |F_{\hat{q}t}|^2 \omega^2 |g_t(\hat{q},\omega)|^2 e^{-i\omega s} \quad . \quad (4.39)
\]

The constant \( \varphi \) can be shown from equipartition to be given by
\[ \beta = \frac{2k_B T_n}{d_f} \]  

where \( k_B \) is the Boltzmann constant. Using Eq. (4.21) to calculate \( |F_{qT}|^2 \), the complete expression for the power spectrum is found to be

\[ W_q(\omega) = \frac{4k_B T_B}{N_d f n \tau_c^2} \sum \frac{\sin \frac{1}{2} q^* p_{ab}}{q} \frac{J_0^2(qR)}{(q/2)^2} \frac{\omega^2}{(\omega^2 - \omega_q^2)^2 + (\omega/\tau)^2} \]  

Since the power spectrum depends on \( D_{qT} \) only through \( \omega_{qT} \), and \( D_{qT} \) is proportional to \( q^2 \) for both a bulk and a thin film superconductor (20,21) we can write

\[ \omega_{qT} = s_T q \]  

where

\[ s_T = (K_t / \mu)^{1/2} \]  

and

\[ K_t = \phi_0 c_{66} / B = D_{qT} / q^2 \]  

The summation in Eq. (4.41) is over the first BZ. We can approximate the summation by an integral over a circle of equal area as the first BZ in reciprocal space. The radius of the circle, and thus the upper limit of integration, is of the order of the inverse of the inter-vortex spacing \( d \). The factor \( J_0^2(qR) \), however, provides a much lower cutoff in \( q \), of the order of \( R^{-1} \), if \( R >> d \). The upper limit of integration can therefore be extended to infinity.
With the above substitutions, and the angular integrations done, we obtain

\[ W_V(\omega) = \frac{4k_B T \Phi_0}{\pi d_T \eta T^2 c^2} \int_0^\infty dq \frac{1 - J_0(q \rho_{ab})}{q} J_0^2(qR) \frac{\omega^2}{(\omega^2 - s_T^2 q^2)^2 + (\omega/\tau)^2}. \]

(4.45)

The power spectrum has the following limiting expressions:

\[ W_V(\omega) = C[\text{ker}(\sqrt{\omega T}_{ab}) + \ln \left( \frac{1}{2} \gamma \sqrt{\omega T}_{ab} \right)] , \quad \omega \ll \tau_{ab}^{-1}, \quad (4.46a) \]

\[ = C \ln(\rho_{ab}/R)[1 + (\omega T)^2]^{-1} , \quad \omega \gg \tau_R^{-1} , \quad (4.46b) \]

where

\[ C = \frac{4k_B T \Phi_0}{\pi d_T \eta c^2} . \quad (4.47) \]

\[ \tau_{ab}^{-1} = \tau(s_T/\rho_{ab})^2 = k_T/\eta_{ab}^2 , \quad (4.48) \]

\[ \tau_R^{-1} = \tau(s_T/R)^2 = k_T/\eta R^2 . \quad (4.49) \]

ker is a Kelvin function and \( \gamma = 1.78107 \ldots \) \( \tau_{ab} \) and \( \tau_R \) are the decay times for transverse modes of wavevector \( \rho_{ab}^{-1} \) and \( R^{-1} \), respectively. The expressions for the power spectrum simplify further in the following limits:

\[ W_V(\omega) = C \frac{\pi}{16} \omega \tau_{ab} , \quad \omega \ll \tau_{ab}^{-1} \quad (4.50a) \]

\[ = C \ln(\frac{1}{2} \gamma \sqrt{\omega T}_{ab}) , \quad \tau_{ab}^{-1} \ll \omega \ll \tau_R^{-1} , \quad (4.50b) \]
For \( \omega \gg \tau_R^{-1} \), \( W_V(\omega) \) has the frequency dependence of the normal state Johnson noise power spectrum, except that \( \tau \) is not in general equal to the normal collision time \( \tau_n \). We can write, in this frequency regime,

\[
W_V(\omega) = \frac{4k_B T}{\pi d_f} \ln(\rho_{ab}/R) \left(1 + (\omega \tau)^2\right)^{-1},
\]

where

\[
R_f = \frac{\rho_f}{\pi d_f} \ln(\rho_{ab}/R),
\]

and \( \rho_f = \frac{B \phi_0}{\eta c^2} \) is the flux-flow resistivity. Equation (4.51) is analogous to the Nyquist formula for the Johnson noise power spectrum of a passive resistor at absolute temperature \( T \). As the flux density approaches \( H_{c2} \), \( \rho_f \) approaches \( \rho_n \), and \( R_f \) approaches \( R_n \), the normal state resistance. At \( H_{c2} \), the power spectrum is identical to the Nyquist formula except that according to the Bardeen-Stephen theory (15) \( \tau = 2\tau_n \) at \( H_{c2} \).

The power spectrum was calculated numerically from the 110 Å thick oxygen-doped aluminum film used in Ref. 21. The relevant parameters for the film are shown in Table 1. The result is shown in Fig. 8. The quantity plotted is \( R(\omega)/R_n \), where \( R(\omega) \) is the real part of the ac impedance, which is proportional to the Johnson noise power spectrum. Also shown are the limiting expressions.
LIMITING EXPRESSIONS FOR

\[ \omega \gg \tau_{\text{ab}}^{-1} \quad \cdots \cdots \]

\[ \omega \ll \tau_{\text{R}}^{-1} \quad \cdots \cdots \]

Figure 8. Calculated Johnson noise power spectrum, expressed in terms of the equivalent frequency-dependent normalized resistance. Parameters are chosen corresponding to the granular aluminum film of thickness 110 Å in Ref. 21, whose properties are listed in Table 1.
Table 1. Characteristics of superconducting film and measuring circuit used for numerical calculations in the text (from Ref. 21)

<table>
<thead>
<tr>
<th>Material</th>
<th>Oxygen-doped Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness</td>
<td>110 Å</td>
</tr>
<tr>
<td>((K_t/\eta)^a)</td>
<td>12.7 cm² sec⁻¹</td>
</tr>
<tr>
<td>(\rho_{ab})</td>
<td>1 cm</td>
</tr>
<tr>
<td>(R)</td>
<td>(10^{-2}) cm</td>
</tr>
<tr>
<td>(\tau) (at 1.6K, 10G)</td>
<td>(1.98 \times 10^{-15}) sec</td>
</tr>
</tbody>
</table>

\(^a\)Independent of temperature and field at low fields.

2. Impedance of the flux-line lattice in an ideal type-II superconducting film in the mixed state

We consider the same geometry as in the preceding section. An ac current \(I e^{-i\omega t}\) is fed into the film via the leads. It can be shown that the driving force on a vortex at \(\vec{z}\) is given by

\[
\vec{f}(\vec{z}, t) = \frac{I \phi_0}{4\pi d_f} \left[ \vec{\delta}_M^a (\vec{z}) - \vec{\delta}_M^b (\vec{z}) \right] e^{-i\omega t},
\]

with \(\vec{\delta}_M^a\) given by Eq. (4.19). Substituting this into Eq. (4.29) to obtain \(\dot{Q}_q\), and using Eq. (4.16) to calculate the voltage, we obtain the ac impedance as

\[
Z(\omega) = \frac{1}{MN_d} \sum_{q} \left| P_{qt} \right|^2 \left[ -i\omega g_e(q, \omega) \right].
\]

The real and imaginary parts, \(Z = R + iX\), are given explicitly by

\[
R(\omega) = \frac{1}{\pi d_f n \tau^2 \eta} \sum_{q} \left| P_{qt} \right|^2 \frac{\omega^2}{(\omega^2 - \omega_{qt}^2)^2 + (\omega/\tau)^2},
\]

\[
X(\omega) = \frac{1}{\pi d_f n \tau^2 \eta} \sum_{q} \left| P_{qt} \right|^2 \frac{\omega}{(\omega^2 - \omega_{qt}^2)^2 + (\omega/\tau)^2}.
\]
R(ω), which corresponds to the resistance, is equal to \( W(ω)/4k_B T \) for a fixed temperature. This fact can also be derived independently from the fluctuation-dissipation theorem (84). \( X(ω) \) has the following limiting expression at low frequencies.

\[
X(ω) = \frac{1}{d_f n T N} \sum_q |F_{qt}|^2 \frac{ω(ω^2 - ω^2_{qt})}{(ω^2 - ω^2_{qt})^2 + (ω/τ)^2}.
\]

(4.56)

where \( kei \) is a Kelvin function. A plot of \( X(ω) \) is shown in Fig. 9.

3. Discussion

If we compare the frequency dependence of the Johnson noise power spectrum with that of the normal state expression (Nyquist formula), we see that there is a suppression of the power spectrum at low frequencies (\( ω < \frac{1}{τ_R} \)) in the superconducting state. This is a result of the shear interaction between the vortices. As the flux density \( B \) approaches \( H_{c2} \), the shear modulus and the parameter \( K_t \) decrease to zero, and there is less and less suppression, as shown in Fig. 10. Finally, when \( B = H_{c2} \), \( K_t = 0 \) and there is no suppression. For \( B < H_{c2} \), if \( ω \) is less than the inverse of the decay time of a normal mode, that mode will not contribute to the power spectrum or the real part of the impedance; hence, a suppression occurs at low frequencies.

It is important to note that the resolution function \( g(ρ) \) (and hence \( |F_{qt}|^2 \)) plays a large part in determining the range of important
Figure 9. Calculated imaginary part of the ac impedance for the granular aluminum film of thickness 110 Å in Ref. 21, whose properties are listed in Table 1.
Figure 10. Calculated Johnson noise power spectrum, expressed in terms of the equivalent frequency-dependent normalized resistance, for various values of the magnetic induction. Parameters are chosen corresponding to the granular aluminum film of thickness 110 Å in Ref. 21, whose properties are shown in Table 1.
wavevectors. For the present geometry, the important wavevectors are those which satisfy \( \rho_{ab}^{-1} \lesssim q \lesssim R^{-1} \). For \( \omega > \tau_R^{-1} \), the effect of the measuring circuit dominates and the power spectrum "saturates" to a constant value. For \( \omega \lesssim \tau_R^{-1} \), the effect mentioned in the previous paragraph dominates. At even higher frequencies \( \omega > \tau^{-1} \), inertial effects come in, and power spectrum starts to decrease with frequency.

Comparing our expression for the impedance with those obtained earlier (85-88), the major difference is that we have included the effects of the interactions among vortices. Instead of assuming a restoring force of the form,

\[
F(\vec{\xi}) = -k\vec{s}(\vec{\xi}) \quad ,
\]

(4.58)

where \( k \) is a scalar force constant, as was done by previous authors, we have used the interaction tensor, Eq. (4.3). The force constant is now wavevector-dependent. As a consequence, the frequency dependence of \( Z(\omega) \) is modified at low frequencies: instead of \( R(\omega) \propto \omega^2 \) we now have \( R(\omega) \propto \omega \) for \( \omega < \tau_{ab}^{-1} \).
V. FLUX-FLOW NOISE

A. General Formula for the Power Spectrum

In the preceding chapter a theory is presented by which the noise voltage can be calculated, once the force per unit length acting on the vortices as a function of time is known. In the case of flux-flow noise, when pinning is involved, this force is extremely difficult to calculate. It is possible, however, to obtain information about the voltage noise power spectrum without knowing the detailed time dependence of the force.

Since $M$, the mass per unit length of the vortices is very small, we shall ignore it from now on. Then Eq. (4.27) becomes

$$\eta \dot{Q}_{q,\lambda}(t) + D_{q,\lambda} Q_{q,\lambda}(t) = \delta f_q(t). \quad (5.1)$$

This is a good approximation because $\tau^{-1} \equiv \eta/M$ is much larger than the frequencies at which flux-flow noise experiments are conducted. Instead of solving Eq. (5.1) for $\dot{Q}_{q,\lambda}$ and $Q_{q,\lambda}$, we instead try to obtain the voltage noise power spectrum directly from the equation. The following derivation is similar to the one in Ref. 84.

We notice that,

$$\int_{-\infty}^{\infty} ds \ e^{-i\omega s} \left< e^{\pm i(q - q') \cdot \vec{v}_0 t} \dot{Q}_{q,\lambda}(t) Q_{q',\lambda}^*(t + s) \right> \tag{5.2}$$

$$= \int_{-\infty}^{\infty} ds \ e^{-i\omega s} \frac{d}{ds} \left< e^{\pm i(q - q') \cdot \vec{v}_0 t} \dot{Q}_{q,\lambda}(t) Q_{q',\lambda}^*(t + s) \right> \tag{5.2}$$

$$= i\omega \int_{-\infty}^{\infty} ds \ e^{-i\omega s} \left< e^{\pm i(q - q') \cdot \vec{v}_0 t} \dot{Q}_{q,\lambda}(t) Q_{q',\lambda}^*(t + s) \right>. \tag{5.2}$$
The last line is obtained through integration by parts. The stationary property of the modes enables us to write

\[
0 = \frac{d}{dt} \langle e^{-i(q-q')\cdot v_0 t} Q_{q\lambda}(t) Q_{q',\lambda}^*(t+s) \rangle
\]

\[
= -i(q-q') \cdot v_0 \langle e^{-i(q-q')\cdot v_0 t} Q_{q\lambda}(t) Q_{q',\lambda}^*(t+s) \rangle + \langle e^{-i(q-q')\cdot v_0 t} \hat{\dot{Q}}_{q\lambda}(t) Q_{q',\lambda}^*(t+s) \rangle + \langle e^{-i(q-q')\cdot v_0 t} Q_{q\lambda}(t) \hat{\dot{Q}}_{q',\lambda}^*(t+s) \rangle.
\]

(5.3)

Combining Eqs. (5.2) and (5.3) and integrating by parts once more yields

\[
\int_{-\infty}^{\infty} ds e^{-i\omega s} \langle e^{-i(q-q')\cdot v_0 t} \hat{\dot{Q}}_{q\lambda}(t) \hat{Q}_{q',\lambda}^*(t+s) \rangle
\]

\[
= (\omega^2 - \omega(q-q') \cdot v_0) \int_{-\infty}^{\infty} ds e^{-i\omega s} \langle e^{-i(q-q')\cdot v_0 t} Q_{q\lambda}(t) Q_{q',\lambda}^*(t+s) \rangle.
\]

(5.4)

These equations enable us to express the power spectra of the mode displacements and mode velocities in terms of the power spectrum of \(\delta f_{q\lambda}'\). Using Eq. (4.16) and the definition of the power spectrum, we finally obtain the following expression:

\[
W_V(\omega) = 2 \int \frac{1}{q_{q\lambda}^2 \omega^2 + \text{in} \omega q_{q\lambda} (D_{q\lambda} - D_{q',\lambda}) - \frac{iD_{q',\lambda}}{\text{in} q_{q\lambda}} (q-q') \cdot v_0 + D_{q\lambda} D_{q',\lambda}}
\]

\[
\times \{ \omega q_{q\lambda} F_{q\lambda}^* F_{q',\lambda} + G_{q\lambda} G_{q',\lambda}^* + i\omega q_{q\lambda} F_{q\lambda} F_{q',\lambda}^* - i\omega q_{q\lambda} G_{q\lambda} G_{q',\lambda}^* \}
\]

\[
\times \int_{-\infty}^{\infty} ds e^{-i\omega s} \langle e^{-i(q-q')\cdot v_0 t} \delta f_{q\lambda}(t) \delta f_{q',\lambda}^*(t+s) \rangle,
\]

(5.5)

where \(\omega_q \equiv \omega - q \cdot v_0\).
In Eq. (5.4), the power spectrum depends not on $\omega$ alone, but on the Doppler combinations $\omega_q$ and $\omega_{q'}$. This follows from the fact that the measuring circuit is fixed with respect to the moving FLL frame, in which the modes are defined.

**B. Johnson Noise With Flux Flow**

If $\delta f$ is the Langevin force, and $v_0 = 0$, Eq. (5.5) reduces to the Johnson noise power spectrum Eq. (4.39). When there is continuous flux flux, $v_0 \neq 0$, the situation is more complicated. There is now a contribution from the longitudinal modes, which is the term proportional to $G_q^* G_{q'}^*$, in Eq. (5.5). This arises from density fluctuations being carried along rigidly with the moving FLL. To eliminate the cross terms with $q \neq q'$, $\lambda \neq \lambda'$, we have to assume that the Langevin force is uncorrelated between different points in the lab frame:

$$f_i(\hat{\rho},t)f_i'(\hat{\rho}',t') = \beta \delta_{i,i'}\delta_{\rho,\rho'}\delta(t-t'), \quad (5.6)$$

where $\hat{\rho}$ and $\hat{\rho}'$ are now position vectors in the lab frame (c.f., Eq. (4.36)).

The complete expression for the power spectrum is

$$W_V(\omega) = W_{\nu_\ell}(\omega) + W_{\nu_l}(\omega), \quad (5.7)$$

where

$$W_{\nu_\ell} = \frac{2\beta}{N} \sum_{q} |F_{q\ell}|^2 \omega_q^2 |g_{\ell}(\hat{q},\omega)|^2, \quad (5.8)$$
\[ W_{VL}(\omega) = \frac{2 \beta}{N} \sum_{q} |G_{q\lambda}|^2 |g_{q\lambda}(q,\omega)|^2, \quad (5.9) \]

with \( \beta \) and \( g_{q\lambda}(q,\omega) \) defined in the last chapter. If we ignore inertial effects, we have the following expressions for the two parts of the power spectrum:

\[ W_{VL}(\omega) = \frac{4k_B T B^2}{Nd_x \eta c^2} \sum_{q} \sin^2 \frac{1}{2} q \cdot \tau_{ab} \frac{J_0^2(qR)}{(q/2)^2} \frac{(\omega - \dot{\omega})^2}{(\omega - \dot{\omega})^2 + \tau_{qL}^2}, \quad (5.10) \]

\[ W_{VL}(\omega) = \frac{4k_B T B^2}{Nd_x \eta c^2} \sum_{q} \sin^2 \frac{1}{2} q \cdot \tau_{ab} \frac{J_0^2(qR)}{(q/2)^2} \frac{(q \cdot \dot{\omega})^2}{(q \cdot \dot{\omega})^2 + \tau_{qL}^2}, \quad (5.11) \]

with \( \tau_{qL}^{-1} \equiv D_{qL}/\eta \).

In general \( D_{qL} \gg D_{qT} \), and thus \( W_{VL} \gg W_{VL} \). Hence, we have

\[ W_V \approx W_{VL}. \]

The behavior of \( W_{VL} \) differs from the case when \( \omega = 0 \) in that the zero-frequency limit of the power spectrum is no longer zero.

The power spectrum remains essentially constant for \( \omega \ll \max(\tau_{qL}^{-1}, q\omega/v_0/\rho_{ab}) \)

where \( \tau_{qL} = K_{qL}/\eta q\omega^2 \). For larger \( \omega \), the power spectrum increases with \( \omega \), but flattens and approaches a constant for \( \omega \gg \max(\tau_{qL}^{-1}, q\omega/v_0/R) \).

Since Johnson noise arises from thermal fluctuations, it is always present at nonzero temperatures. Even when there is no dc voltage, there is still a noise voltage. We have seen this in the case without flux flow. When flux flow is present, the dc voltage is zero when \( \dot{\omega}_0 / \rho_{ab} \). The Johnson noise voltage in this case can still be calculated with Eqs. (5.10) and (5.11).
Figure 11 shows plots of $\mathcal{W}_{\nu_t}$ for $\nu_0 \perp \hat{p}_{ab}$ and $\nu_0 \parallel \hat{p}_{ab}$ for the film described in Table 1. It is seen that they do not differ significantly.

As $\nu_0$ is increased, the low frequency part of $\mathcal{W}_{\nu_t} (\omega < \tau_1^{-1})$ also increases. When $\nu_0$ is such that $\tau_1^{-1} > \tau_R^{-1}$, the power spectrum is just a constant. This is shown in Fig. 12, where the power spectra for different $\nu_0$'s are plotted.

Although the longitudinal contribution $\mathcal{W}_{\nu_L}$ is much smaller than the transverse contribution $\mathcal{W}_{\nu_t}$ in general, it is interesting to examine its properties. At low frequencies $\omega \ll \max(\tau_{ab,2}^{-1}, \nu_0/\rho_{ab})$, where $\tau_{ab,2}^{-1} \equiv \tau_{q^2}^{-1} \mid_{q=\rho_{ab}^{-1}}$, $\mathcal{W}_{\nu_L}$ is approximately a constant. This is because $|G_{q^2}|^2$ limits the smallest important $q$ to be $\rho_{ab}^{-1}$, therefore the effect of increasing $\omega$ in the factor $(\omega - \nu_0)^2 + \tau_{q^2}^{-2}$ at the RHS of Eq. (5.11) is first apparent when $\omega$ exceeds the larger of $q\omega_0$ and $\tau_{q^2}^{-1}$ evaluated at this $q$. When $\omega$ is bigger than $\max(\tau_{R,2}^{-1}, \nu_0/R)$, where $\tau_{R,2}^{-1} \equiv \tau_{q^2}^{-1} \mid_{q=R^{-1}}$, with $R^{-1}$, the inverse of the contact radius, being the largest important $q$ allowed by $|G_{q^2}|^2$, $\mathcal{W}_{\nu_L}$ decreases as $\omega^{-2}$, where $R \ll \rho_{ab}$, the difference between these two characteristic frequencies can span several decades. Therefore, for intermediate frequencies, $\max(\tau_{ab,2}^{-1}, \nu_0/\rho_{ab}) \ll \omega \ll \max(\tau_{R,2}^{-1}, \nu_0/R)$, $\mathcal{W}_{\nu_L}$ is proportional to $\omega^n$, with $-2 < n < 0$. It is thus possible to have a $\omega^{-1}$ behavior over a small range of frequencies. This behavior arises naturally from the distribution of relaxation times from modes with different wavevectors, even though the power spectrum of the Langevin force is frequency independent. Figure 13 shows plots of $\mathcal{W}_{\nu_L}$ with $\nu_0 \perp \hat{p}_{ab}$ and $\nu_0 \parallel \hat{p}_{ab}$. These graphs are calculated with $D_{q^2}$ given by Eq. (1.37), i.e., both
Figure 11. Calculated power spectrum due to transverse modes for Johnson noise with flux flow. The average velocity of the flux-line lattice is perpendicular (a) or parallel (b) to the line joining the voltage contacts.
Figure 12. Calculated power spectrum due to transverse modes for Johnson noise with flux flow, for various velocities of the flux-line lattice, $v_0$. $v_0$ is perpendicular to the line joining the voltage contacts.
Figure 13. Calculated power spectrum due to longitudinal modes for Johnson noise with flux-flow. Both long and short range interaction between the vortices are included. The average velocity of the FLL is perpendicular (a) or parallel (b) to the line joining the voltage contacts.
long and short range interactions are taken into account. Usually $D_{q\xi}$ is sufficiently large that $qv_0 < \tau^{-1}_{q\xi}$ for $\rho_{ab}^{-1} \leq q \leq R^{-1}$. For example, with $v_0 \sim 100 \text{ cm/sec}$, $v_0/\rho_{ab} \sim 100 \text{ sec}^{-1}$, $v_0/\tau_{q\xi} \sim 10^8 \text{ sec}^{-1}$, $v_0/R \sim 10^4 \text{ sec}^{-1}$, $v_{q\xi}/q=10^2 \text{ sec}^{-1}$, $v_0/\tau_{q\xi} \sim 10^2 \text{ sec}^{-1}$ at 10 gauss for the film described in Table 1. The frequency dependence of $W_{V\xi}$ is therefore insensitive to changes in $v_0$, but the magnitude of $W_{V\xi}$ is proportional to $v_0^2$.

C. Relating Mean Square Noise Voltage to Mean Square Velocity Fluctuations and Density Fluctuations

In Chapter III we have shown that to first order

$$\delta V(t) = \delta v_0 \cdot \int d^2 \rho \int d^2 \rho' g(\rho) \delta n(\rho) + n_0 \int d^2 \rho \int d^2 \rho' g(\rho) \cdot \delta \bar{v}(\rho) , \tag{5.12}$$

where $\delta n = n - n_0$ and $\delta \bar{v} = \bar{v} - \bar{v}_0$, and the resolution function $g$ is assumed to vary slowly over an intervortex spacing. Assuming velocity fluctuations and density fluctuations to be uncorrelated with each other, the mean square voltage is

$$\langle \delta V^2 \rangle = \int d^2 \rho \int d^2 \rho' g_\gamma(\rho) g_\gamma(\rho') v_0^2 \langle \delta n(\rho) \delta n(\rho') \rangle$$

$$+ n_0^2 \sum_{\alpha, \beta} \int d^2 \rho \int d^2 \rho' g_\alpha(\rho) g_\beta(\rho') \langle \delta v_\alpha(\rho) \delta v_\beta(\rho') \rangle , \tag{5.13}$$

where $\alpha, \beta$ refers to a component of the Cartesian coordinates, and $g_\gamma = \hat{g} \cdot \bar{v}_0/v_0$. To crudely account for the correlations in velocity and density fluctuations at $\hat{\rho}$ and $\hat{\rho}'$, we can write

$$\langle \delta n(\hat{\rho}) \delta n(\hat{\rho}') \rangle \approx \langle \delta n^2 \rangle \frac{N}{n_0} \delta_2(\hat{\rho} - \hat{\rho}') \tag{5.14}$$
The above equations state that the density (velocity) fluctuations at each point are correlated with the \(N_{cn}(N_{cv})\) number of vortices around that point, and that the correlation lengths are small by comparison with the characteristic length scale of \(\hat{g}\). Combining Eqs. (5.13), (5.14) and (5.15) yields

\[
<\delta v^2> = N_{cn} <\delta n^2> \frac{v_0^2}{n_0} \int d^2 \rho [g_{\gamma}(\hat{\rho})]^2 + \frac{1}{2} n_{cv} <\delta v^2> n_0 \int d^2 \rho [g(\hat{\rho})]^2 .
\]  

(5.16)

We have assumed that \(<\delta v^2_x> = <\delta v^2_y> = \frac{1}{2} <\delta v^2>\). If the velocity and density fluctuations are caused by pinning, it can be shown that

\[
<\delta v^2> = \frac{J_{c} \phi_0}{n_c} v_0 ,
\]  

(5.17)

so that

\[
<\delta v^2> = N_{cn} <\delta n^2> \frac{v_0^2}{n_0} \int d^2 \rho [g_{\gamma}(\hat{\rho})]^2 + \frac{1}{2} n_{cv} J_{c} \frac{E}{n_0} \int d^2 \rho [g(\hat{\rho})]^2 ,
\]  

(5.18)

where we have written \(E = n_0 \phi_0 v_0 / c\).

Applying Eq. (5.16) to Johnson noise measured with the circuit shown in Fig. 7, we find that the density and velocity fluctuations that contributed to the measured voltage noise are indeed uncorrelated since they are of different polarizations. Equation (5.16) then gives \(N_{cv} = 1\). The situation for \(N_{cn}\) is more complicated since density fluctuations cannot be related to thermal energy directly. However, if \(<\delta n^2>\) is
independent of position, we can show that

\[ \langle \delta n^2 \rangle = n_0^2 \sum_{\mathbf{q}} q^2 \langle |Q_{\mathbf{q}}|^2 \rangle \quad . \tag{5.19} \]

Now the longitudinal contribution to the mean square voltage, \( \langle \delta V_L^2 \rangle \), is given by

\[ \langle \delta V_L^2 \rangle = \frac{B^2}{c^2 n_0^2} \sum_{\mathbf{q}} \frac{\sin^2 \frac{1}{2} q \cdot \mathbf{R}_{ab}}{(q/2)^2} J^2_0(qR) (\mathbf{q} \times \mathbf{v}_0)^2 \langle |Q_{\mathbf{q}}|^2 \rangle \quad . \tag{5.20} \]

Here, we have set \( M = 0 \), and have used the fact that \( \langle \delta V_L^2 \rangle = \frac{1}{2} \int \frac{d\omega}{2\pi} W_L(\omega) \), and equipartition \( k_B T/\text{df} = k_B T/\text{df} = N q^2 \delta V_L^2 \langle |Q_{\mathbf{q}}|^2 \rangle \). For bulk materials, when \( D_q \ll q^2 \) so that \( q^2 \langle |Q_{\mathbf{q}}|^2 \rangle \) is a constant, we can write Eq. (5.20) exactly as

\[ \langle \delta V_L^2 \rangle = \frac{\langle \delta n^2 \rangle}{n_0} \mathbf{v}_0^2 \int d^2 \mathbf{p} [g_{\gamma}(\mathbf{p})]^2 \quad , \tag{5.21} \]

which upon comparison with Eq. (5.18), gives \( N_{cn} = 1 \). There is no simple expression for \( N_{cn} \) in the case of a thin film where \( D_q \ll q \).

The fact that both \( N_{cn} \) and \( N_{cv} \) are unity should not come as a surprise since the Langevin force fluctuates rapidly and is uncorrelated between different vortices. As far as the mean square voltage is concerned, the FLL behaves exactly the same whether there are interactions between vortices or not.

It should be emphasized that the number \( N_{cv} \) and \( N_{cn} \) as defined above have no connection with the "bundle size" \( n(0) \) defined by
Van Gurp (62) and others. The interpretation of $n(0)$ as the size of a moving flux entity or disturbance is limited only to the "shot noise" models used in the flux-flow noise, and is not applicable, for example, to Johnson noise and the model of flux-flow noise presented in the next chapter.

D. Application to Flux-Flow Noise

We can apply Eq. (5.5) to flux-flow noise, when $\delta F$ is the time-varying part of the elementary pinning force. In general, the expression for the power spectrum is very complicated, involving cross terms between different wavevectors and polarizations. To see under what circumstances can we simplify the expression, we examine

$$W_\varepsilon(q, q'; \omega) = \int_{-\infty}^{\infty} ds e^{-i \omega s} \langle e^{-i(q-q')\cdot\varepsilon_0 t} \delta F_{q\lambda}(t) \delta F^*_{q',\lambda},(t+s) \rangle$$

$$= \int_{-\infty}^{\infty} ds e^{-i \omega s} \frac{1}{N^2} \sum_{i'\rho'} \varepsilon_{i}(q\lambda) \varepsilon^*_{i',}(q',\lambda') e^{-i(q-q')\cdot\varepsilon_0 t} e^{i\varepsilon_{i'} \cdot \rho'}$$

$$\times \langle \delta F_{0,i}(\rho,t) \delta F^*_{0,i'}(\rho',t+s) \rangle. \quad (5.22)$$

Here, $\rho$ and $\rho'$ are coordinates in the lab frame and

$$\delta F_{0}(\rho,t) = \delta F(\rho - \varepsilon_0 t,t) \quad (5.23)$$

is the fluctuating part of the elementary pinning force in the lab frame, while $\delta F$ is the same quantity in the FLL frame.

The force-force correlation function $\langle \delta F_{0,i}(\rho,t) \delta F_{0,i'}(\rho',t+s) \rangle$ vanishes unless both $\rho$ and $\rho'$ are within range of some pinning centers.
In such cases we can write

$$<\delta f_{0,i}(\rho, t)\delta f_{0,i}(\rho', t+s)> = <\delta f_{0,i}(\rho + \delta, t)\delta f_{0,i}(\rho + \delta', t+s)>, \quad (5.24)$$

where $\rho = \rho + \delta$, with $\rho$ the position of a pinning center, $\delta$ a vector connecting $\rho$ to each vortex that is within range of the pinning center, and $\delta'' = \delta' - \delta$. The power spectrum is then

$$W_f(q, \lambda, \lambda', \omega) = \int_{-\infty}^{\infty} ds \, e^{-i\omega s} \frac{1}{N^2} \sum_{\rho} \frac{1}{N} \sum_{\rho_q} e^{i(q-q') \cdot \rho} \left\{ \sum_{i,i'} \epsilon_i(q\lambda) \epsilon_{i'}(q\lambda') e^{-i(q-q') \cdot \delta} e^{iq' \cdot \delta''} \right\} \left\{ \epsilon_i(q\lambda) \epsilon_{i'}(q\lambda') e^{i(q-q') \cdot \delta} e^{iq' \cdot \delta''} \right\} \left< \delta f_{0,i}(\rho + \delta, t)\delta f_{0,i}(\rho + \delta', t+s) > \right\}. \quad (5.25)$$

If all pinning centers are identical and are distributed evenly, the quantity inside the braces in Eq. (5.25) is independent of $\rho$.

Assuming the density of pinning centers to be uniform over a distance large by comparison with the characteristic length of the measuring circuit, we can sum over $\rho$ and obtain

$$W_f(q, \lambda, \lambda'; \omega) = \delta_q \delta_{\rho} \int_{-\infty}^{\infty} ds \, e^{-i\omega s} \frac{N}{\delta} \sum_{i,i'} \epsilon_i(q\lambda) \epsilon_{i'}(q\lambda') e^{i(q-q') \cdot \delta} e^{iq' \cdot \delta''} \left< \delta f_{0,i}(\rho + \delta, t)\delta f_{0,i}(\rho + \delta', t+s) > \right\}. \quad (5.26)$$

where $\rho$ now stands for the position of any pinning centers. Equation (5.26) is only valid for modes with wavevectors less than the inverse
of the distance between pinning centers. For modes with larger wave-vectors cross terms have to be included.

To simplify the expression further, we assume \( \langle \delta f_{0,i} \mid \delta (\hat{\rho}_p + \hat{\omega}_t), t \rangle \)
\( \delta f_{0,i', \delta (\hat{\rho}_p + \hat{\omega}_t), t+s} \rangle \) to be proportional to \( \delta_{i,i'} \), and independent of \( i \).
This allows us to eliminate the cross terms with \( \lambda \neq \lambda' \) so that
\( W_p(q\lambda, q\lambda'; \omega) = \delta_{q,q}, \delta_{\lambda, \lambda'} W_f(q, \omega) \). The voltage power spectrum now can be written as

\[
W_V(\omega) = 2 \sum_{q, \omega} \frac{1}{N} \frac{1}{\omega q^2 + D_{q\lambda}} \left[ |\omega|^2 F_{q\lambda} |^2 + |C_{q\lambda}|^2 + 2 \omega q \text{Im}(C_{q\lambda}^* F_{q\lambda}) \right] W_f(q, \omega) ,
\]

(5.27)

where

\[
W_f(q, \omega) = \sum_{N} \frac{p}{N^2} e^{i q \cdot \rho} \int_{-\infty}^{\infty} ds e^{-i \omega s} \langle \delta f_{0,i} \mid \delta (\hat{\rho}_p + \hat{\omega}_t), t \rangle \langle \delta f_{0,i} \mid \delta (\hat{\rho}_p + \hat{\omega}_t), t+s \rangle .
\]

(5.28)

For the measuring circuit of Fig. 7, this can be simplified to

\[
W_V(\omega) = 2 \sum_{q} \frac{\omega^2}{q^2 + D_{q\lambda}} \left[ |F_{q\lambda}|^2 + \frac{1}{q^2 + D_{q\lambda}} |C_{q\lambda}|^2 \right] W_f(q, \omega) .
\]

(5.29)

There are two contributions to the power spectrum, one from transverse modes and one from longitudinal modes. However, in general \( D_{q\lambda} \) is much larger than \( D_{q\lambda} \) for a given \( q \) for bulk materials and even more so for thin films. Therefore, the most important contribution to the voltage power spectrum is from the transverse modes. The contribution from cross terms between the two polarizations is also small.
For the time being, let us assume that the elementary pinning forces from different pinning centers are independent of each other, and that each pinning center acts only on one vortex at a time. Then the voltage power spectrum is, taking the transverse contribution only,

\[
W_V(\omega) = \frac{2 N}{n D} \sum_{q} |F_q|^2 \frac{\omega^2}{\omega^2 + q^2 - \tau_{qt}^2} \int_{-\infty}^{\infty} ds \, e^{-i\omega s} <\delta f^0_{Q\mu}(\vec{\rho}_p, t) \delta f^0_{Q\nu}(\vec{\rho}_p, t+s)> ,
\]

(5.30)

where \( \tau_{qt} = n/D \) is the transverse mode relaxation time.

At high frequencies, where \( \omega >> \frac{k_q^2}{\ln \frac{2 \pi D}{q_{\max}}} \), being the inverse of the contact radius in this geometry, the characteristic roll-off frequency \( \omega_c \) is determined by the power spectrum of the elementary pinning force at each pinning center. The interaction between vortices only provides a suppression in the low-frequency regime of the power spectrum.

The part of the power spectrum due to longitudinal modes has a characteristic frequency determined by the elastic constants of the vortex lattice. Under the present assumptions, this contribution is small. However, it is conceivable that for some types and distributions of pinning centers, the longitudinal contribution may be important. The behavior of this contribution is described in Chapter VII.

Because of the coupling of temporal and spatial dependences in the power spectrum, the dimensions of the measuring circuit and the mean flux-flow velocity will affect the power spectrum in certain frequency regimes. This feature is also present in the earlier "shot noise" models.
The low-frequency behavior of Eq. (5.30) is the same as the transverse part of the power spectrum in Johnson noise with flux flow, Eq. (5.10). At high frequencies the behavior of $W_v(\omega)$ is governed by the power spectrum of the elementary pinning force. The power spectrum rolls off at $\omega_{CF}$, the characteristic frequency of the elementary pinning force power spectrum, which is at least as large as the inverse of the time for which a pinning center acts on a vortex, for randomly distributed, independent pinning centers acting on one vortex at a time. The latter quantity is smaller than $d/v_0 \sim 10^{-6} \text{ sec}$, where $d$ is the intervortex spacing. Hence $\omega_{CF}$ is larger than $v_0/d$, which in turn is much larger than the characteristic frequencies observed experimentally.

To investigate the subject further, we need to understand how the FLL moves under the influence of a pinning center. This is done in the next chapter for a simple model pinning force.
VI. RESPONSE OF THE FLUX-LINE LATTICE TO A LOCAL FORCE

In this chapter the motion of the FLL through an array of pinning centers under the influence of a constant driving force is investigated. The result is used to calculate a voltage noise power spectrum. There are a few assumptions to be made:

(i) the geometry is two-dimensional,
(ii) the pinning centers are parallel to the vortices,
(iii) the pinning centers are widely spaced,
(iv) surface effects are not important,
(v) heating effects are not important,
(vi) the FLL is perfect and infinite and undergoes no plastic deformations,
(vii) linear elastic theory is applicable.

None of these assumptions is strictly valid in practical real-life situations. However, these assumptions enable us to treat the problem with a minimal amount of mathematics.

We define the primed FLL frame to be the frame that moves with the center of mass of the FLL when there are no operating pinning centers. The displacement, \( \mathbf{s}(\mathbf{r},t) \), from equilibrium of the vortex whose equilibrium position in the primed FLL frame is \( \mathbf{i} \) is defined to be

\[
\mathbf{s}(\mathbf{r},t) = \mathbf{p}(\mathbf{r},t) - \mathbf{v}_L t - \mathbf{i},
\]

where \( \mathbf{p}(\mathbf{r},t) \) is the position of the vortex in the lab frame, and \( \mathbf{v}_L = \mathbf{F}_L/\eta \) is the velocity of the primed FLL frame in the lab frame, with \( \mathbf{F}_L \) being the constant driving force.
In the presence of pinning centers, the force balance equation in the lab frame is

\[ - \eta \dot{\mathbf{v}}_L - \eta \dot{s}(\mathbf{\tilde{l}}, t) - \sum_{\mathbf{\tilde{l}}'} G(\mathbf{\tilde{l}}, \mathbf{\tilde{l}}') \dot{s}(\mathbf{\tilde{l}}', t) + \dot{\mathbf{f}}_L + \mathbf{f}_p(\mathbf{\tilde{l}}, t) = 0, \quad (6.2) \]

where \( \mathbf{f}_p(\mathbf{\tilde{l}}, t) \) is the elementary pinning force on vortex \( \mathbf{\tilde{l}} \) at time \( t \).

Summing over \( \mathbf{\tilde{l}} \) gives

\[ - N \eta \dot{\mathbf{v}}_L - \eta \sum_{\mathbf{\tilde{l}}} \dot{s}(\mathbf{\tilde{l}}, t) + N \mathbf{f}_L + \sum_{\mathbf{\tilde{l}}} \mathbf{f}_p(\mathbf{\tilde{l}}, t) = 0, \quad (6.3) \]

where \( N \) is the total number of vortices and we have used the fact that

\[ \sum_{\mathbf{\tilde{l}}} G(\mathbf{\tilde{l}}, \mathbf{\tilde{l}}') = 0. \]

Taking the time average then yields

\[ - \eta \dot{\mathbf{v}}_0 + \dot{\mathbf{f}}_L + \frac{1}{N} \sum_{\mathbf{\tilde{l}}} \mathbf{f}_p(\mathbf{\tilde{l}}, t) = 0, \quad (6.4) \]

where

\[ \dot{\mathbf{v}}_0 \equiv \dot{\mathbf{v}}_L + \frac{1}{N} \sum_{\mathbf{\tilde{l}}} \mathbf{f}_p(\mathbf{\tilde{l}}, t). \quad (6.5) \]

We recognize \( \dot{\mathbf{v}}_0 \) as the average flux-flow velocity and

\[ \frac{1}{N} \sum_{\mathbf{\tilde{l}}} \mathbf{f}_p(\mathbf{\tilde{l}}, t) \]

as the average pinning force. For widely spaced pinning centers we can use superposition and write
\[ \mathbf{s}^{\dagger}(\mathbf{x},t) = \sum_{i} \mathbf{s}_{i}^{\dagger}(\mathbf{x},t), \quad (6.6) \]

\[ \mathbf{F}_{p}^{\dagger}(\mathbf{x},t) = \sum_{i} \mathbf{F}_{p,i}^{\dagger}(\mathbf{x},t) \quad , \quad (6.7) \]

where the subscript \( v \) denotes the contribution from the \( v \)-th pinning center. Then we can say that the \( v \)-th pinning center contributes an amount

\[ \frac{1}{N} \sum_{\mathbf{x}} \langle \mathbf{F}_{p,v}^{\dagger}(\mathbf{x},t) \rangle = \frac{1}{N} \sum_{\mathbf{x}} \langle \mathbf{s}_{v}^{\dagger}(\mathbf{x},t) \rangle \quad (6.8) \]

to the total average pinning force per vortex. Near the \( u \)-th pinning center we can write

\[ \sum_{v \neq u} \mathbf{s}_{v}^{\dagger}(\mathbf{x},t) = \sum_{v \neq u} \langle \mathbf{s}_{v}^{\dagger}(\mathbf{x},t) \rangle = \frac{1}{N} \sum_{\mathbf{x}} \sum_{v \neq u} \langle \mathbf{s}_{v}^{\dagger}(\mathbf{x},t) \rangle \quad . \quad (6.9) \]

The last equality is true because there is no plastic shear in the vortex lattice. Thus we can say that for the vortices near the \( u \)-th pinning center, the effect of all other pinning centers is to apply a constant force per vortex given by

\[ \frac{1}{N} \sum_{\mathbf{x}} \sum_{v \neq u} \langle \mathbf{F}_{p,v}^{\dagger}(\mathbf{x},t) \rangle \quad , \]

so that the vortices are moving with a uniform velocity

\[ v_{L} + \frac{1}{N} \sum_{\mathbf{x}} \sum_{v \neq u} \langle \mathbf{s}_{v}^{\dagger}(\mathbf{x},t) \rangle \quad . \]

The displacement due to the \( u \)-th pinning center is superimposed on this uniform motion, and is governed by the equation
\begin{equation}
- \eta s_{\mu}(\vec{x},t) - \sum_{\vec{x}'} G(\vec{x},\vec{x}')s_{\mu}(\vec{x}',t) + F_{p,\mu}(\vec{x},t) = 0 . \tag{6.10}
\end{equation}

If there is a large number of identical pinning centers, the contribution to the total pinning force by each individual pinning center is small, and the solution for Eq. (6.10) gives the time-varying part of the motion of the vortices to a high degree of accuracy.

We assume a constant force
\begin{equation}
F_{p,\mu}(\vec{x},t) = F_0 \delta_{\mu,0} \theta(t) \chi_k , \tag{6.11}
\end{equation}
where \( \theta(t) \) is the Heaviside step function. We drop the subscript \( \mu \) from Eq. (6.10), with the understanding that we are considering vortices close to a pinning center, and far away from all other pinning centers.

Following the procedure and notation in Chapters IV and V, we obtain
\begin{equation}
- \eta \dot{q}_\lambda(t) - D_{q\lambda} Q_{q\lambda}(t) + F_{p,q\lambda}(t) = 0 , \tag{6.12}
\end{equation}
where
\begin{equation}
F_{p,q\lambda}(t) = \frac{1}{N} F_0 \varepsilon_k(q_{\lambda}) \theta(t) . \tag{6.13}
\end{equation}

The solution is
\begin{equation}
Q_{q\lambda}(t) = \frac{F_0}{N} \varepsilon_k(q_{\lambda}) \left( \frac{1 - e^{-t/\tau_{q\lambda}}}{D_{q\lambda}} \right) , \tag{6.14}
\end{equation}
and
\begin{equation}
\dot{s}(\vec{x},t) = \frac{F_0}{N} \sum_{q\lambda} \varepsilon_k(q_{\lambda}) \varepsilon(\vec{x}) \frac{1 - e^{-t/\tau_{q\lambda}}}{D_{q\lambda}} e^{i q \cdot \vec{x}} . \tag{6.15}
\end{equation}
Since $\tau_q \ll \tau_{qt}$, we have

$$s(\mathbf{r},t) = \sum_{q} e^{i\mathbf{q} \cdot \mathbf{r}} \int_{-\tau/\tau_{qt}}^{\tau/\tau_{qt}} e^{i\mathbf{q} \cdot \mathbf{r}'} \phi(t) dt' .$$

For $\ell \equiv |\mathbf{\ell}| \gg d$, the intervortex spacing, and $\lambda_t^2 \equiv k t / \eta \gg d^2$, the result after replacing the summation with an integral is

$$s(\mathbf{r},t) = \frac{F_0^2}{4\pi B k} \left\{ \frac{1}{2} \right\}.$$

Note that these are independent of $k$. This is the same as the displacement field produced by rigid motion of the origin at a speed $F_0 / \eta$, with the other vortices backflowing around the circle of radius $F_0 / 2\pi B$. For vortices farther away from the pinning center than $\lambda_t$, the displacement field decreases as $\ell^{-2}$. 

For $\ell \gg \lambda_t$,

$$s(\mathbf{r},t) \approx \frac{F_0^2}{4\pi B k} \cos 2\theta$$

$$s(\mathbf{r},t) \approx \frac{F_0^2}{2\pi B n} \sin 2\theta .$$
It is possible to obtain the voltage power spectrum by substituting expressions for $Q_{q\lambda}$ and $\dot{Q}_{q\lambda}$ to the expressions derived earlier. The result agrees with that obtained by calculating the force-force correlation function.

The length $\lambda_t$ is analogous to a shear diffusion length with $K_t/\eta$ as the diffusion constant. Shear disturbances diffuse out from the source according to the force balance equation, and the displacements are more or less localized within a region of radius $\lambda_t$. The reason why the displacements outside $\lambda_t$ are not zero is that we are considering only transverse (shear) displacements. There are also longitudinal displacements representing density fluctuations. If we assume that the longitudinal force constant, $D_{qz}$, is proportional to $q^2$, as in the bulk, we then have a diffusion-type equation for the longitudinal modes too. There will be a corresponding longitudinal diffusion, $\lambda_z$. Since $D_{qz} \gg D_{qt}$ in general, $\lambda_z >> \lambda_t$. If we calculate the displacements of vortices far outside the circle of radius $\lambda_z >> \lambda_t$, we indeed find that they are zero. The situation for thin films is much different. The transverse mode force constant is proportional to $q^2$ and is the same as that of the bulk. The longitudinal mode force constant, however, is proportional to $q$. This is due to the interaction between vortices via the empty space outside the specimen. This interaction is of infinite range since it can be viewed essentially as a magnetic monopole interaction. As a consequence, the longitudinal modes no longer obey a diffusion-type equation. In a thin film, the displacements due to longitudinal modes are given by, with $\Lambda_z \equiv H_z t/\eta$, $D_{qz} = H_z q$, 

\[
\begin{align*}
\dot{s}_k(\vec{r}, t) & = \frac{F_0 \phi_0}{4 \pi B \hbar} \left[ \frac{1}{\lambda} \left( \frac{1}{\sqrt{1 + \lambda^2 \hbar^2}} \right) - \cos 2\theta \left( \frac{1}{\sqrt{1 + \lambda^2 \hbar^2}} \right) \right] , \\
\dot{s}_j(\vec{r}, t) & = -\frac{F_0 \phi_0}{4 \pi B \hbar} \sin 2\theta \left[ \frac{1}{\lambda} \left( \frac{1}{\sqrt{1 + \lambda^2 \hbar^2}} \right) \right] .
\end{align*}
\]

(6.21)

\[
\begin{align*}
\dot{s}_k(\vec{r}, t) & = \frac{F_0 \phi_0}{4 \pi B \hbar} \sin 2\theta \left[ \frac{1}{\lambda} \left( \frac{1}{\sqrt{1 + \lambda^2 \hbar^2}} \right) \right] ,
\end{align*}
\]

(6.22)

for \( \lambda \gg d, \xi \gg d \).

For \( \xi \gg \lambda \gg d \), we have

\[
\begin{align*}
\dot{s}_k(\vec{r}, t) & = \frac{F_0 \phi_0}{8 \pi B n} \frac{t}{\xi^2} \left( \frac{\lambda}{\xi} - 4 \cos 2\theta \right) , \\
\dot{s}_j(\vec{r}, t) & = -\frac{F_0 \phi_0}{2 \pi B n} \frac{t}{\xi^2} \sin 2\theta .
\end{align*}
\]

(6.23)

(6.24)

Therefore in a thin film,

\[
\dot{s}_j(\vec{r}, t) = 0 \quad \text{for} \quad \xi \gg \lambda \gg \xi_t ,
\]

(6.25)

but in the same limit

\[
\begin{align*}
\dot{s}_k(\vec{r}, t) & = \frac{1}{8 \pi} \frac{F_0}{n} \frac{\phi_0/B}{\xi^2} \frac{\lambda_t}{\xi} .
\end{align*}
\]

(6.26)

We emphasize, again, however, that since the longitudinal force constant is much larger than the transverse force constant, the displacements due
to density fluctuations are very small compared to shear displacements, and we only need to consider the latter. Another way of saying this is that the longitudinal modes decay much faster than the transverse modes, so that after a very short period of time the longitudinal modes have already decayed to magnitudes much smaller than those of the transverse modes.

In the last chapter, we arrived at the conclusion that the voltage power spectrum involves the power spectrum of the elementary pinning force. We can imagine a vortex being pinned stationary at a pinning center for a time $\tau_p$, when an elastic instability occurs, and the vortex is released from the pinning center. In the frame of the vortex lattice, this can be modeled by letting the elementary pinning force go to zero after a time $\tau_p$. In our simple model of a constant force, this means

$$\hat{F}_p(\hat{l},t) = F_0 \delta_{\hat{l}}^0 \delta(t) \delta(\tau_p - t) \hat{x}_k . \quad (6.27)$$

If we use $\dot{s}'(\hat{l},t)$ to denote the displacements for $t > \tau_p$, we can show that

$$\ddot{s}'(\hat{l},t) = \ddot{s}(\hat{l},t) - \ddot{s}(\hat{l},t-\tau_p) , \quad (6.28)$$

where $\ddot{s}(\hat{l},t)$ is given by Eq. (6.15). Thus we see that the displacements for $t > \tau_p$ consists of two contributions. As the effect of the elementary pinning force diffuses out from the point of application, so does the effect of the release of the vortex from the pinning center.

So far we have only concentrated on the pinning of one vortex. In real life we have a lattice of vortices. In the limit of pinning centers being separated by a distance much larger than
we can assume that the pinning centers do not affect each other. If we look at the vicinity of one pinning center, we see that the motion of the vortices must be periodic with respect to the pinning center. More specifically, the elementary pinning force at each pinning center is just a periodic sequence of pulses, with a period of the order of \( d/v_0 \). The power spectrum is a delta function centered at the inverse of the period. The movement of a perfect vortex lattice through widely and randomly spaced pinning centers cannot produce voltage noise. The overall voltage measured must be periodic.

As the vortex is being released from the pinning center, the higher velocity produces higher viscous loss, or heat, which in turn decreases the strength of the pinning. If the heat is not conducted away quickly enough, a thermal runaway can occur around a pinning center so that not all the vortices that hit the pinning center experience the same force. This may lead to a quasi-random process of flux movements around the pinning center. This process is being regulated by the velocity of the vortex lattice, the thermal conductivity and heat capacity of the material, the viscous drag coefficient, and the heat transfer coefficient between the specimen and the thermal bath. Mathematically, however, this problem is intractable even for a two-dimensional film.

We can see whether the above process can produce the observed flux-flow noise by noting that the experimental characteristic frequency is of the order of \( 10^2 \) Hz. If the thermal runaway is due to
one depinning event, the characteristic frequency would be of the order of \( \tau_F^{-1} \). Hence, \( \tau_F \approx 10^2 \) sec. With an average velocity of 10 cm/sec, this means that a vortex (or a group of vortices) is being pinned stationary at a pinning center while the vortices far away move a distance of \( 10 \cdot 10^{-2} \sim 10^{-1} \) cm before the vortices is depinned. This seems unlikely, unless plastic deformation occurs. We can see whether this is possible without plastic deformation as follows.

If a vortex is being held stationary at a pinning site for a period of time, while the other vortices are free to move, we can calculate the resulting displacement field. Mathematically, it is simpler to look at the static limit. For a force \( F_0 x_k \) applied to the vortex at the origin, the k-th component of the displacement of vortex \( x_k \) due to the transverse mode (shear) response, after the vortex lattice is allowed to relax, is according to Eq. (6.16),

\[
s_k(x_k, \infty) = \frac{F_0}{N} \sum_{q} \frac{\varepsilon_k(qt)e^{i\mathbf{q} \cdot \mathbf{x}_k}}{D(qt)}
\]  

(6.30)

With \( D(qt) = K_t q_k^2 \), this can be evaluated to give

\[
s_k(x_k, \infty) = \frac{\phi_0 F_0}{4 \pi K} \left\{ \int_{q_L}^{q_D} \frac{dq}{q} J_0(q \xi) + \cos 2\theta \left[ \frac{J_1(q_L \xi)}{q_L^2} - \frac{J_1(q_D \xi)}{q_D^2} \right] \right\}
\]  

(6.31)

where \( q_D \) is the upper cutoff wavevector defined earlier \( (\pi q_D^2 = \text{area of first BZ}) \), \( q_L \) is a lower cutoff wavevector, of the order of the inverse of the specimen size. \( \theta \) is the angle between \( x_k \) and \( \xi \).
Although $D_\ell = \mu_0 q^2$ cannot be valid up to the zone boundary, Eq. (6.31) gives a good order-of-magnitude estimate. For the vortex being pinned ($\ell = 0$), we have

$$s_k(0,\omega) = \frac{\phi_0^2}{4\pi BK} \log \frac{q_D}{q_L} \quad (6.32)$$

This equation is similar to the one derived by Good and Kramer (53) for line forces except the inner cutoff in their expression is replaced by $q_D^{-1}$ here. There is also a numerical factor difference of order unity which can be traced to the fact that the hexagonal symmetry is ignored in the present treatment.

With Eqs. (6.31) and (6.32), we can determine the condition for slip in the vortex lattice. Specifically, slip would occur if the difference in the $k$-th components of the displacements between the vortex at $\ell = 0$ and the one at $\ell = d$, $\theta = 60^\circ$ is larger than $d/2$. Actually, linear elastic theory requires that this difference be much less than $d$.

From Eqs. (6.31) and (6.32) we have

$$s_k(0,\omega) - s_k(\ell,\omega) = \frac{\phi_0^2}{4\pi BK} \left\{ \frac{q_D}{q_L} \int dq \frac{1 - J_0(q\ell)}{q} - \cos \theta \left[ \frac{J_1(q_L\ell)}{q_L\ell} - \frac{J_1(q_D\ell)}{q_D\ell} \right] \right\} \quad (6.33)$$

We can write this as
We require this to be less than \( d/2 \). Substituting \( \ell = d \) and \( \theta = \pi/3 \) into Eq. (6.34), we obtain the condition for no slippage around the origin:

\[
S_k(0,\omega) \leq 3.5d, \quad (6.35)
\]

where we have taken \( \log(q_D/q_L) = 10 \) and evaluate \( J_1(q_L\ell)/q_L\ell \) at the limit \( q_L\ell = 0 \). This also gives an approximate upper limit for \( \tau_p \), in order for slip not to occur:

\[
\tau_p \leq \frac{3.5d}{v_0}, \quad (6.36)
\]

For \( d \approx 10^{-5} \) cm, \( v_0 \approx 10 \) cm/sec, we have \( \tau_p \approx 3.5 \times 10^{-6} \) sec. This is much smaller than the observed time scale for flux-flow noise \((\omega_c^{-1} \approx 10^{-2} \text{ to } 10^{-3} \text{ sec})\). Note that long before the limits in Eqs. (6.35) and (6.36) are reached, linear elastic theory may fail to be valid.

Earlier we mentioned that the disturbance diffuses out from the origin with a characteristic diffusion length \( \lambda_t = (K_t t/\eta)^{1/2} \). This can be used to obtain a dynamic limit for slip to occur. Note that as long as \( t < \tau_p \), the displacement at the origin is \( v_0 t \) while the displacements of vortices with \( \ell > \lambda_t \) is approximately zero. (We have
neglected the contribution from the mode with \( q = 0 \), which is usually small). Therefore, for slip not to occur, we need approximately

\[
\frac{v_0 t}{\lambda t} < 1 \quad (6.37)
\]

This translates to

\[
v_0^2 < \frac{K_t}{\eta t} \quad (6.38)
\]

Taking the largest \( t = \tau_F \), we arrive at

\[
v_0^2 < \frac{K_t}{\eta \tau_F} \quad (6.39)
\]

For small \( v_0 \), this is usually a much less stringent condition than the static one, Eq. (6.36). As soon as \( \tau_F > (K_t/\eta d^2) \), the static condition would apply.

The above limits, although derived for the pinning of one vortex, can be applied to the pinning of more vortices. Actually, the corresponding limits would be even lower in most cases. We can see this by considering the static case, with a line of vortices being pinned. This can be viewed as a superposition of the pinning of individual vortices. The maximum strain, which occurs around the end of the line, is obviously larger than the case for the pinning of a single vortex since the strain field decays with distance.

From the above discussion, we conclude that a single pinning-depinning event of a localized region of the FLL cannot produce the observed power spectrum, within the framework of linear elastic theory,
in the case of randomly and widely spaced pinning centers. In the same limit, a perfect vortex lattice moving among such a distribution of pinning centers cannot even produce noise.

We may now ask what assumptions have to be given up in order that the observed voltage noise can be produced. In most experiments performed to date, grain boundaries were the most important pinning centers. Most grain boundaries are continuous and connected and therefore cannot strictly be independent of each other. Interaction between adjacent grain boundaries may lead to flux motions that are aperiodic and also on a larger time scale. If we allow the FLL to slip and undergo plastic deformations, nonperiodic flux motion may also occur. Finally, the presence of defects in the FLL may lead to many different types of flux motions which can produce noise. Any one of the above proposals makes the problem much more difficult and will not be dealt with in the present work.
VII. CONTRIBUTION OF DENSITY FLUCTUATIONS TO THE NOISE

In the last two chapters we see that the flux-flow noise power spectrum consists of two parts. One is from velocity fluctuations, which, for a certain special measuring circuit, contains contributions from only the transverse modes. The other is from density fluctuations being carried along by the moving FLL. Under certain assumptions, this contribution is very small as compared to the first one. The velocity fluctuations, however, cannot explain the observed flux-flow noise power spectrum if random, independent pinning centers are assumed. The problem is two-fold. Firstly, the voltage measured would be periodic, hence no noise would be produced. Secondly, even if there is noise, the characteristic frequency would be too high if no slippage is allowed to occur.

In Ref. 80 it was suggested that a picture in which vortex density disturbances are created at one pinning center, carried along with the FLL and destroyed at the next pinning center, may explain the observed frequency dependence of the measured noise. We can use the present formalism to treat such a picture and this is done in this chapter.

We focus our attention on the contribution to the power spectrum from the density fluctuations. We assume that there are random fluctuations in the vortex density near a pinning center which give rise to noise. We also assume that the elementary pinning force distribution is such that only the longitudinal force contributes to the noise. Then the voltage power spectrum is
\[ W_V(\omega) = 2 \sum_{q} \frac{1}{\omega^2_0} \left| C_{q} \right|^2 W_F(q_1^2, q_2^2; \omega), \quad (7.1) \]

where \( W_F(q_1^2, q_2^2; \omega) \) is defined in Eq. (5.22). We have dropped the cross terms with \( q \neq q' \). If the pinning centers produce density fluctuations independently of each other, \( W_F(q_1^2, q_1^2; \omega) \) is independent of \( q \), as is shown in Chapter V.

The characteristic frequency of \( W_V(\omega) \), unlike the transverse mode contribution, Eq. (5.30), is determined by both the elementary pinning force power spectrum and the elastic response. If we let \( W_F(q_1^2, q_2^2; \omega) \) to be a constant, the frequency dependence of \( W_V(\omega) \) is exactly that of the longitudinal contribution to the power spectrum for Johnson noise with flux flow. The characteristic frequency of \( W_V(\omega) \) is therefore given by

\[ \omega_c \overset{\sim}{=} \min(\tau_F^{-1}, \omega_{cJ}), \quad (7.2) \]

where \( \tau_F^{-1} \) is the characteristic frequency of \( W_F(q_1^2, q_1^2; \omega) \), and \( \omega_{cJ} \) is the characteristic frequency of the longitudinal mode contribution to the power spectrum for Johnson noise with flux flow:

\[ \omega_{cJ} \overset{\sim}{=} \max(\tau_{ab,1}^{-1}, \nu_0/\rho_{ab}), \quad (7.3) \]

with \( \tau_{ab,1}^{-1} \equiv \tau_q^{-1} \left| q = \rho_{ab} \right. \). If we assume that \( \tau_F^{-1} \) is very large so that \( W_F(q_1^2, q_1^2; \omega) \) is indeed a constant at the frequencies of interest, we see that \( \omega_c \overset{\sim}{=} \omega_{cJ} \). Therefore, although the elementary pinning force power spectrum may have a high characteristic frequency, the voltage noise power spectrum has a characteristic frequency that can be much lower if density fluctuations contribute primarily to the noise.
Using the bulk result for \( D_{q^2} \), namely

\[
D_{q^2} = \frac{\phi_0}{B} c_{11} q^2 \approx \frac{B\phi_0}{4\pi} q^2 ,
\]

we find that \( \tau_{ab} = 3800 \text{ sec}^{-1} \) for the niobium film in Ref. 80

\((B = 1760 \text{ gauss}, \rho_{ab} = 1 \text{ mm}, \eta = 7.6 \times 10^{-7} \text{ dyne sec/cm}^2)\). This is within an order of magnitude of the observed value, \( \omega_c \sim 10^3 \text{ sec}^{-1} \).

As in Johnson noise with flux flow, at frequencies higher than \( \omega_c \),

the power spectrum decreases approximately as \( \omega^{-1} \) until \( \omega \approx \max(\tau_{R}^{-1}, \frac{v_0}{R}) \), where \( \tau_{R}^{-1} = \tau_{q^2}^{-1} \bigg|_{q=R} \). At higher frequencies, the power spectrum decreases as \( \omega^{-2} \). These are the cases when \( \omega \) is still less than \( \tau_{R}^{-1} \). Again, we emphasize that the \( \omega^{-1} \) behavior arises naturally from the distribution of relaxation times of modes with different wavevectors. Plots of the longitudinal modes contributions to the power spectrum, with \( D_{q^2} \) calculated according to Eq. (7.4), are shown in Fig. 14, for \( \vec{v}_0 \perp \vec{\rho}_{ab} \) and \( \vec{v}_0 \parallel \vec{\rho}_{ab} \).

In order to incorporate the effect of the density distribution being destroyed at the next pinning center, we have to modify \( W_f(q^2, q^2; \omega) \). This leads to a non-Markov process because events at different pinning centers are now coupled together. The mathematics then become intractable and the present formalism is not very useful. We can however, approximate the effect by noting that \( \tau_{q^2} \equiv \eta/D_{q^2} \) is the relaxation time for a longitudinal mode with wavevector \( q^2 \). Hence a good approximation is to replace \( \tau_{q^2} \) by \( \min(\tau_{q^2}, D_p/v_0) \) in the expression for the power spectrum, Eq. (7.1). Here, \( D_p \) is the average distance between the pinning centers. The result is
Figure 14. Calculated power spectrum due to longitudinal modes for flux-flow noise. Only short range interaction between the vortices is included. The average velocity of the flux-flow lattice is parallel (a) or perpendicular (b) to the line joining the voltage contacts.
\[ W_V(\omega) = \frac{2}{\pi^2} \sum_q \frac{1}{\omega^2 + \max(\tau_{qz}^{-2}, \nu_0^2/D_p^2)} |G_{qz}|^2 W_f(qz, qz; \omega). \]  

(7.5)

This approximation is good as long as the buildup times of the disturbances are small compared to \( D_p/\nu_0 \). The characteristic frequency is now, for a constant \( W_f(qz, qz; \omega) \),

\[ \omega_c = \max(\omega_{cJ}, \nu_0/D_p) = \max(\tau_{ab, z}^{-1}, \nu_0/\nu_{ab}, \nu_0/D_p). \]  

(7.6)

We can see this by looking at the denominator at the RHS of Eq. (7.5). If \( \nu_0/D_p \) is bigger than \( \tau_{ab, z}^{-1} \) and \( \nu_0/\nu_{ab} \), it always dominates and the power spectrum will decrease with \( \omega \) only when \( \omega \) gets bigger than \( \nu_0/D_p \). Of course, when \( \tau_F^{-1} \) is smaller than the value given by Eq. (7.6), \( \tau_F^{-1} \) is the characteristic frequency.

The frequency dependence of the power spectrum is still approximately \( \omega^{-1} \) for frequencies higher than \( \omega_c \), unless \( \nu_0/D_p > \tau_{R, z}^{-1} \), when it will be \( \omega^{-2} \). Note that in the former case a single parameter \( D_p \) is sufficient to ensure a \( \omega^{-1} \) behavior, unlike the Habbal-Thompson-Joiner theory (66-69), in which a distribution of distances between pinning centers is needed.

For small distances between pinning centers, \( \nu_0/D_p \) is indeed the characteristic frequency as observed in Ref. 80. However, the independence of \( \omega_c \) on \( \nu_0 \) at larger distances between pinning centers was not observed. Figures 15 and 16 (curve a) shows the power spectra for \( \omega_cJ \) smaller than and larger than \( \nu_0/D_p \), respectively. The parameters are chosen so that \( \omega_cJ = \tau_{ab, z}^{-1} \).
Figure 15. Calculated power spectrum due to longitudinal modes for flux-flow noise assuming the density fluctuations are destroyed at the pinning centers. Only short range interaction is included. The average velocity of the flux-line lattice, $v_0$, is perpendicular to the line joining the voltage contacts. $v_0/D_p$ is larger than $\tau_{ab,1}^{-1}$.
Figure 16. Calculated power spectrum due to longitudinal modes for flux-flow noise assuming the density fluctuations are destroyed at the pinning centers. Only short range interaction is included. The average velocity of the flux-line lattice, $v_0$, is perpendicular (a), or parallel (b) to the line joining the voltage contacts. $v_0/D_p$ is smaller than $r_{ab,2}^{-1}$. 
The power spectrum calculated above, like the velocity fluctuation contribution (see Chapter V), changes little with the angle between the average flux-flow direction \( \vec{v}_0 \), and the vector joining the voltage probe contacts \( \vec{\rho}_{ab} \). In particular, when \( \vec{v}_0 \parallel \vec{\rho}_{ab} \), the dc voltage vanishes.

The power spectrum, however, only decreases slightly in magnitude, compared to that when \( \vec{v}_0 \perp \vec{\rho}_{ab} \). One such power spectrum is shown in Fig. 16 (curve b). The insensitivity of the power spectrum to the flux-flow direction was observed by Heiden (78).

In Eq. (7.4), the elastic constant for the bulk is used to calculate \( \tau_{ab,\epsilon}^{-1} \), and is found to agree quite well with experiments. However, since in Ref. 80 a foil was used, the correct elastic constant to use should be that given by Eq. (1.37), i.e., the interaction between vortices via the space outside the foil should be taken into account. This is true even if we have a relatively thick foil, since according to Eq. (1.38), for \( q < d_f^{-1} \), where \( d_f \) is the thickness of the foil, the thin film result in Eq. (1.35) dominates. In the expression for \( \tau_{ab,\epsilon}^{-1} \), \( q = \rho_{ab}^{-1} < d_f^{-1} \) in usual situations. Hence \( \tau_{ab,\epsilon}^{-1} \) should be approximately given by

\[
\tau_{ab,\epsilon}^{-1} \approx \frac{B\Phi_0}{2\pi \eta d_f \rho_{ab}}.
\]  

This increases the estimate of \( \tau_{ab,\epsilon}^{-1} \) by a factor \( 2\rho_{ab}/d_f \) above that calculated using Eq. (7.4). With the niobium film in Ref. 80, the thickness is 30 \( \mu \)m; therefore, \( \tau_{ab,\epsilon}^{-1} \approx 2.5 \times 10^5 \) sec\(^{-1} \) according to Eq. (7.7). This value is larger than the experimentally observed
values for $\omega_c \sim 10^3 \text{ sec}^{-1}$. Moreover, Eq. (7.6) now demands that

$$\omega_c \geq \omega_c = \tau_{ab,\ell}^{-1}$$

most of the time since $\tau_{ab,\ell}^{-1}$ is now larger than $v_0/D_p$ in most cases.

In Ref. 80 what was measured was the autocorrelation function $\psi_V(s)$ which is the Fourier transform of the power spectrum. $\psi_V(s)$ was observed to decrease exponentially with $s$. This means that the power spectrum was a Lorentzian and for $\omega \gg \omega_c$, decays as $\omega^{-2}$. There was no $\omega^{-1}$ behavior observed as predicted by Eq. (7.6).

There is also the question of why do transverse velocity fluctuations not contribute. Notice that longitudinal velocity fluctuations are not measured by the measuring circuit shown in Fig. 7, which we are discussing. Jarvis and Park (60) measured flux-flow noise power spectra with different leads arrangements above the specimen surface. In some leads arrangements, there is a peak in the power spectrum at a frequency higher than the characteristic frequency of the power spectrum when no peak was observed using a different lead arrangement. In Appendix A it will be shown that this peak may be explained by the present model, and that it contains contributions from both longitudinal velocity and density fluctuations, and cross terms between the two.

In Chapter VI we saw that the FLL can be easily distorted so that slip occurs. Severe plastic deformations affect the shear modulus much more than the compressional modulus. If the elementary pinning force as a function of time remains the same, the decrease of the shear modulus would only lessen the suppression in the low frequency regime of the
transverse velocity contribution to the voltage power spectrum, Eq. (5.10). Changes in the elastic constants, however, would definitely change the way the vortices move with respect to the pinning center, thus changing the elementary pinning force as a function of time. The relative magnitudes of the transverse part and the longitudinal part of the elementary pinning force may change too. Whether the presence of plastic deformations leads to a reduction in the transverse velocity fluctuations contribution to the voltage power spectrum is not clear, however. This question may also be related to the origin of the randomness that leads to the noise.

In conclusion, the voltage power spectrum generated by longitudinal modes displacements (density fluctuations) seems to fit qualitatively the behavior of the observed characteristic frequency in some experiments. The value of the characteristic frequency can be calculated to within an order of magnitude of the observed values when only the bulk contribution of the elastic constant is used. The prediction of a constant characteristic frequency at low flux-flow velocity or large distances between pinning centers, however, has not been observed experimentally. The origin of the randomness that leads to the noise and a justification for the unimportance of the transverse modes contribution to the noise still remain to be found.
VIII. SUMMARY AND CONCLUSIONS

A theory for the voltage noise power spectrum in type-II superconductors is presented. The interaction between vortices is taken into account via an interaction matrix. The FLL is assumed to be perfect, and only two-dimensional displacements of the vortices are considered. A method is developed by which the measured voltage can be calculated when the measuring circuit consists of finite size leads. The voltage noise power spectrum is shown to be related to that of the elementary forces that give rise to the fluctuations. If there is no flux flow, only vortex velocity fluctuations contribute to the noise. Otherwise, both velocity and density fluctuations contribute.

The theory is applied to Johnson noise in an ideal type-II film. A special measuring circuit is assumed which is not sensitive to the longitudinal velocity contribution to the voltage. Without flux flow, only transverse velocity fluctuations contribute and the power spectrum shows a suppression at low frequencies from Nyquist formula. With flux flow, the longitudinal modes also contribute but the resulting power spectrum is much smaller than that of the transverse modes. It is found that the elastic response, the viscous drag force, the dimensions of the measuring circuit, and the flux-flow velocity all play a role in determining the characteristic time scales.

In the case of flux-flow noise, it is found that for isolated pinning centers that act on one vortex at a time, the flux motion is periodic and no noise can be produced. If randomness is assumed, the
power spectrum is related to that of the elementary pinning force. The contribution from the transverse modes again dominates and the characteristic roll-off frequency of the voltage power spectrum is the same as that of the elementary pinning force. For isolated pinning centers, this is much higher than those obtained experimentally. A predicted suppression at low frequencies has not been observed.

The contribution from the longitudinal modes can have a characteristic roll-off frequency independent of that of the power spectrum of elementary pinning force. It is, like its counterpart in Johnson noise with flux flow, determined by the elastic response, the viscous drag force, the dimensions of the measuring circuit, and the flux-flow velocity. This is again too high, however, if both the bulk interactions and the interaction via the space outside the specimen between the vortices are considered. Reasonable numbers are obtained, however, if only the bulk interactions are assumed. Further qualitative agreement to certain experiments can be obtained if the density fluctuations are assumed to be destroyed at the pinning centers.

Thus we see that, according to our theory, a picture of isolated pinning centers acting on one vortex at a time in a perfect FLL cannot explain the observed flux-flow noise. Flux-flow noise experiments with this kind of pinning structure and Johnson noise experiments on low pinning materials should provide a direct verification of the theory.

The present formalism is by no means restricted to Johnson noise and flux-flow noise. It can be applied to other kinds of voltage noise, e.g., flicker noise, temperature fluctuation-induced noise, etc., if the
appropriate power spectrum for the forces that generate the random flux movements can be calculated.

We have not considered the effect of defects in the FLL and the case when pinning centers are close to each other. The origin of the randomness in flux motion is not investigated. When pinning centers are close to and interacting with each other, randomness of flux motion can be produced as one depinning event triggers another. Interacting pinning centers may also hold the density fluctuations for a longer period of time before decaying, thus giving rise to a lower characteristic frequency. We believe that a picture of density fluctuations being created and broken up by interacting pinning centers is the most promising direction to be followed in future work on flux-flow noise.
IX. APPENDIX A: VOLTAGE MEASURED BY A LOOP

In this appendix we calculate the resolution function for a measuring circuit consisting of a loop placed above the superconductor. Consider an arbitrary measuring circuit consisting of two wires joining to a superconductor and a voltmeter (Fig. 17a). We call this circuit P. The radii of the wires are assumed to be vanishingly small. Neglecting the resistance of the leads, the voltage measured is

\[ V_P = \psi'_a - \psi'_b - \frac{1}{c} \frac{\partial}{\partial t} \int_{a[C_M]}^{b} \hat{a} \cdot d\hat{x}, \]  

(A.1)

where \( \psi \) is the electrochemical potential per unit charge, \( c \) is the speed of light in vacuum, \( \hat{a} \) is the vector potential, and \( C_M \) is the directed path adABcb along the leads and through the voltmeter. Now imagine two more measuring circuits: one is the circuit formed by discarding the parts of the leads bc and ad, and connecting c and d with a wire cd, thus forming a loop. We call this the circuit M and the corresponding voltage measured \( V_M \) (Fig. 17b). The other measuring circuit is shown in Fig. 17c. It is formed by discarding the wire cB from circuit P, joining a wire to c and lining the wire along cd as in circuit M, and tuning the wire up very close to dA until close to the voltmeter, then joining it to B (Fig. 17c). This is circuit PM and the corresponding voltage measured is \( V_{PM} \). It is readily shown that

\[ V_P = V_M + V_{PM}. \]  

(A.2)

Using Eq. (3.15) in the text we can show that the same relation exists
Figure 17. The three measuring circuits, for which the resolution functions are related by Eqs. (A.1) and (A.3)
for the resolution function $\hat{g}(\rho)$:

$$
\hat{g}_{p}(\rho) = \hat{g}_{m}(\rho) + \hat{g}_{pm}(\rho) \quad .
$$

We now turn to a specific measuring circuit. The circuit $P$ is the same as that shown in Fig. 7 in the text, except that here we consider the radii of the leads to be vanishing small. The circuit $M$ is shown in Fig. 18b with one side of the loop parallel to the surface of the superconductor and a distance $z_0$ above it. Circuit PM is shown in Fig. 18c.

The resolution functions can be easily calculated in the case $z_0 \gg \lambda$. The results are

$$
\hat{g}_{pm}(x,y) = \frac{\phi_0}{2\pi c} \frac{2z_0}{y^2 + z_0^2} \left[ \frac{x + s}{\sqrt{(x+s)^2 + y^2 + z_0^2}} - \frac{x - s}{\sqrt{(x-s)^2 + y^2 + z_0^2}} \right] 
$$

$$
- \frac{\phi_0}{2\pi c} \frac{-yx + (x+s)y}{(x+s)^2 + y^2} \frac{z_0}{\sqrt{(x+s)^2 + y^2 + z_0^2}} 
$$

$$
+ \frac{\phi_0}{2\pi c} \frac{-yx + (x-s)y}{(x-s)^2 + y^2} \frac{z_0}{\sqrt{(x-s)^2 + y^2 + z_0^2}}, \quad (A.4)
$$

$$
\hat{g}_{p}(x,y) = \hat{g}_{pm}(x,y) \bigg|_{z_0=\infty}, \quad (A.5)
$$

$$
\hat{g}_{m}(x,y) = \hat{g}_{p}(x,y) - \hat{g}_{pm}(x,y) \quad . \quad (A.6)
$$

Here, we have assumed that $\hat{\rho}_a = sx$, $\hat{\rho}_b = -sx$, and $z$ is the unit vector perpendicular to the surface of the superconductor.
Figure 18. The measuring circuits whose resolution functions are described by Eqs. (A.4) through (A.6)
The noise voltage is given by

\[ \delta V(t) = \sum_{q\lambda} \left[ F_{q\lambda} \dot{Q}_{q\lambda}(t) + G_{q\lambda} Q_{q\lambda}(t) \right], \quad (A.7) \]

where \( F_{q\lambda} \) and \( G_{q\lambda} \) are defined by Eq. (4.17) in the text. \( F_{q\lambda} \) and \( G_{q\lambda} \) are given by

\[ F_{q\lambda} = n_0 \hat{\varepsilon}(q\lambda) \cdot \hat{\sigma}(q), \quad (A.8) \]
\[ G_{q\lambda} = -n_0 i q v_0 \cdot \hat{\sigma}(q) \delta_{q,\lambda}, \quad (A.8) \]

where

\[ \hat{\sigma}(q) = \int d^2 \rho \hat{\varepsilon}(\rho) \ e^{iq \cdot \rho}. \quad (A.9) \]

For measuring circuit PM,

\[ \hat{\varepsilon}_{PM}(q) = \frac{-2z_0}{c} \frac{\sin q_x s}{q_x} e^{-z_0 q_y} - \frac{2z_0}{c} \frac{\sin q_x s}{q_x} (q \cdot \hat{\sigma}) (1 - e^{-z_0 q_y}) \quad (A.10) \]

For the measuring circuit M, we have

\[ F_{q\lambda}^M = \frac{2B}{c} \frac{\sin q_x s}{q_x} e^{-z_0 q_y} \delta_{q,\lambda}, \quad (A.11) \]
\[ G_{q\lambda}^M = \frac{2B}{c} \frac{\sin q_x s}{Tq_x} e^{-z_0 q_y} \delta_{q,\lambda}. \quad (A.11) \]

Note that only longitudinal modes contribute. This is expected since if \( z_0 \gg \lambda \), only vortex density fluctuations would be contributing to the voltage, and transverse modes do not contribute to density fluctuations.
If we now calculate the flux-flow noise power spectrum according to Eq. (5.5) in the text, assuming that \( W(q^0, q^*, u) \ll 6^2 \), we have

\[
W^M_V(\omega) = \frac{8 B^2}{n c^2} \sum_{q^0} \frac{\sin^2 q_x s}{q^2} e^{-2z_0 q} \frac{q^2}{q_x^2} \left\{ \omega^2 + (q^* v_0)^2 + 2 \omega (q^* v_0) \right\} \\
\times \left[ \frac{1}{\omega^2 + \tau^{-2}_q} \right] W_f(q^*, q^2, \omega)
\]

(A.12)

where \( \omega = \omega - q^* v_0 \). The first and second terms in the braces are from velocity fluctuations (\( \dot{q}q^* \)) and density fluctuations (\( q^*q^* \)), respectively. The third term is a cross term between velocity and density fluctuations. Note that Eq. (A.12) can be simplified to

\[
W^M_V(\omega) = \frac{8 B^2}{n c^2} \sum_{q^0} \frac{\sin^2 q_x s}{q^2} e^{-2z_0 q} \frac{q^2}{q_x^2} \frac{\omega^2}{(\omega - q^* v_0)^2 + \tau^{-2}_q} W_f(q^*, q^2, \omega)
\]

(A.13)

Therefore, \( W^M_V(0) = 0 \). At high frequencies, the main contribution is from the velocity fluctuations term, and the power spectrum is a constant, if we assume \( W_f(q^*, q^2, \omega) \) to be a constant. Note that this is a general result, independent of the measuring circuit geometry. We can see this by noting that \( \tilde{g}_M(\rho) = \tilde{b}_M(\rho) = -\psi^I_M(\rho) \), where \( \tilde{b}_M \) is the magnetic induction produced by the measuring circuit per unit current.

Hence, we have

\[
\tilde{g}_M(q) = \int d^2 \rho e^{\lambda(q^* \rho)} \phi^I_M(\rho) = \tilde{q} \phi^I_M(\tilde{q})
\]

(A.14)

since \( \phi^I_M \) is single-valued on the surface of the superconductor if the
measuring circuit is a loop. Using Eq. (A.8) we can show that

\[ |F_{q\lambda}^M|^2 = n_0^2 q^2 |\tilde{\phi}_M(q)|^2 \delta_{\lambda, \xi} \]

\[ |G_{q\lambda}^M|^2 = n_0^2 q^2 (\hat{v}_0 \cdot \hat{q})^2 |\tilde{\phi}_M(q)|^2 \delta_{\lambda, \xi} \]

\[ 2 \text{Im}(G_{q\lambda}^M F_{q\lambda}) = -2 n_0^2 q^2 (\hat{v}_0 \cdot \hat{q}) |\tilde{\phi}_M(q)|^2 \delta_{\lambda, \xi} \] (A.15)

Substituting into Eq. (5.5) in the text, again assuming \( W_f(q\lambda, q'^\lambda, \omega) = \delta_{q, q'} \), we obtain

\[ W^M_v(\omega) = \frac{n_0^2}{n^2} \sum_{q} q^2 |\tilde{\phi}_M(q)|^2 \frac{\omega^2}{\omega^2 + \frac{\tau^2}{q^2}} W_f(q\hat{x}, q\hat{z}, \omega) \] (A.16)

Jarvis and Park (60) observed a peak in the power spectrum when the noise voltage was measured by a loop. Their results can be qualitatively interpreted with the present model if we assume the decrease at the high frequency side of the peak to come from \( W_f(q\hat{x}, q\hat{z}, \omega) \). Indeed if \( W_f(q\hat{x}, q\hat{z}, \omega) \) were independent of \( \omega \), Eq. (A.16) indicates that the voltage power spectrum would attain a constant value only when \( \omega \) is much higher than the frequencies, \( \omega_p \), at which their peaks were observed. Also, at approximately \( \omega_p \) the power spectra measured by a circuit with contacts to the surface of the superconductor start to decrease faster with respect to \( \omega \). This is compatible with our results in Chapter VII of the text, if we take the decrease of the power spectra at frequencies higher than \( \omega_p \) to come from \( W_f(q\hat{x}, q\hat{z}, \omega) \).
In this appendix we apply the theory developed in the text to the "shot noise" model first proposed by van Ooijen and van Gurp (61). In this model, "flux bundles" are assumed to cross the specimen randomly with a constant velocity. In using the present theory, we assume that the "flux bundles" are randomly distributed density disturbances "frozen" in the FLL. No decay of these density fluctuations is allowed. We now show that with such a picture, the "shot noise" power spectrum can be derived from the present theory.

Frozen density disturbances contain only contributions from time-independent longitudinal modes. We can show that

$$\langle \delta n(\vec{r})^2 \rangle = \sum_{\vec{q}} q^2 \langle |Q_{\vec{q}L}|^2 \rangle$$

where $\delta n(\vec{r}) = n(\vec{r}) - n_0$, $\vec{r}$ is the position of a density disturbance, and we have assumed that $\langle \delta n(\vec{r})^2 \rangle$ is independent of $\vec{r}$. This equation is only valid for $q \lesssim n_d^{1/2}$ where $n_d$ is the density of the density disturbances. If each density disturbance is localized enough so that its spatial extension $L$ is small by comparison with $n_d^{-1/2}$, we can assume, to a good approximation,

$$q^2 \langle |Q_{\vec{q}L}|^2 \rangle = \text{constant}$$

with respect to $\vec{q}$, for $q \lesssim L^{-1}$. Also, the modes with $q \gtrsim L^{-1}$ are suppressed. Combining Eqs. (B.1) and (B.2) yields
\[ \langle |Q_{q'q}|^2 \rangle = \frac{\langle \delta n^2 \rangle}{n_0 n_d N q^2}, \quad (B.3) \]

where \( N \) is the total number of vortices.

The noise voltage is given by

\[ \delta V(t) = \sum_{q} G_{q'q} Q_{q'q} e^{-i\mathbf{q}' \cdot \mathbf{v}_0 t}, \quad (B.4) \]

so that the autocorrelation function is

\[ \psi_V(s) = \sum_{q, q'} G_{q'q}^* \langle Q_{q'q} \rangle e^{-i\mathbf{q}' \cdot \mathbf{v}_0 s - i\mathbf{q} \cdot \mathbf{v}_0 s}. \quad (B.5) \]

Under the present assumptions, \( \langle Q_{q'q} \rangle e^{-i\mathbf{q}' \cdot \mathbf{v}_0 s - i\mathbf{q} \cdot \mathbf{v}_0 s} \) is proportional to \( \delta_{q, q'} \), for \( q \approx L^{-1} \). The power spectrum is then

\[ W_V(\omega) = 2 \sum_{q} |G_{q}^{*} Q_{q} Q_{q}^{*} \rangle \langle \delta(\omega - \mathbf{q} \cdot \mathbf{v}_0) |. \quad (B.6) \]

This is a sum of delta functions and is difficult to analyze and compute. This is because we have treated the problem in reciprocal space. To convert to real space, we make use of Eq. (B.3) and the definition of \( G_{q}^{*} \), Eq. (4.17) in the text. Equation (B.6) can then be converted to

\[ W_V(\omega) = 2 \frac{\langle \delta n^2 \rangle}{n_d} \int d^2 \rho \int ds e^{-i\omega s} \hat{g} (\rho) \cdot \mathbf{v}_0 \hat{g} (\rho + \mathbf{v}_0 s) \cdot \mathbf{v}_0, \quad (B.7) \]

with \( \hat{g} (\rho) \) being the resolution function in real space. If we had treated the problem in real space, we would have
\[ <\delta n(\vec{\rho}, t)\delta n(\vec{\rho}', t+s)> \approx <\delta N^2>_{n_d} \delta_2(\vec{\rho} - \vec{\rho}' + \vec{v}_0 s) \]  \hspace{1cm} (B.8)

where \(<\delta N^2>\) is the mean square fluctuation number of vortices associated with each frozen density fluctuation. This leads to the vortex-current correlation function, Eq. (3.28), in the text,

\[ \kappa_{ab}(\vec{\rho}, \vec{\rho}', s) = v_0 a v_0 b \frac{<\delta n^2>}{n_d^2} \delta_2(\vec{\rho} - \vec{\rho}' + \vec{v}_0 s) \]  \hspace{1cm} (B.9)

where we have identified \(<\delta n^2>\) as \(<\delta N^2>n_d^2\). Equation (B.7) then follows directly.
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