The effect of hand held calculators on learning about: functions, functional notation, graphing, function composition, and inverse functions

Robert Loren Rule
Iowa State University
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THE EFFECT OF HAND HELD CALCULATORS ON LEARNING ABOUT:
FUNCTIONS, FUNCTIONAL NOTATION, GRAPHING, FUNCTION
COMPOSITION, AND INVERSE FUNCTIONS

Iowa State University

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The effect of hand held calculators on learning about:
Functions, functional notation, graphing, function composition, and inverse functions

by

Robert Loren Rule

A Dissertation Submitted to the
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INTRODUCTION

In recent years there has been a dramatic increase in the use of calculators in schools. The calculator is now recognized as an aid to instruction, especially if the instruction involves considerable numeric work.

Marilyn N. Suydam, director of the ERIC Calculator Information Center, writes

Almost 100 studies on the effects of calculator use have been conducted during the past four or five years. This is more investigations than almost any other topic or tool or technique for mathematics instruction during this century and calls attention to the intensive interest about this potentially valuable tool. Many of these studies had one goal: to ascertain whether or not the use of calculators would harm students' mathematical achievement. The answer continues to be "No". The calculator does not appear to affect achievement adversely (Suydam 1979, p. 3).

By contrast, there are few investigations involving curriculum designed to be used with the calculator. There are even fewer investigations whose goal was to ascertain whether or not the calculator, when integrated into the curriculum, effected higher achievement. In particular, there are few studies at the secondary or collegiate level which investigate the effect of integrating the use of the calculator into instruction about functions.
Need for the Study

It is generally conceded that hand-held calculators have become commonplace in our homes and businesses. They are inexpensive and widely available. Most investigations thus far have involved what Weaver (1976) has termed calculator-assisted (CA) context. Weaver (1976) in his position paper recognized three categories relating calculators to the curriculum and has diagramed the relation as in Figure 1:

![Diagram of CA, CM, and CB curricula]

Figure 1. The relation between CA, CM and CB curricula
In Figure 1, a Calculator Based (CB) curriculum is the most specialized of the three types of curricula being considered. The notable aspect of CB instruction is that every lesson is designed so that the calculator is involved in a pertinent way. Thus, the calculator is actually a basis for instruction and the name calculator based curriculum is appropriate. In a Calculator Modulated (CM) curriculum, the instruction does not necessarily involve the calculator. That is, not every lesson is designed to involve the calculator. However, in those lessons where it facilitates learning, the calculator is an integral part of the instruction. The calculator used in such a way actually tempers the instruction providing a consistency and regulation to the instruction. Such a curriculum is called a Calculator Modulated curriculum. In a CA curriculum, the calculator is not used as a central part of any lesson. The instruction involves the calculator as an adjunct. The calculator is viewed as a tool which can provide help. For example, in CA curricula the calculator might be used to check problems previously solved without the use of a calculator.

Weaver, while speaking to a group at the 1979 Annual NCTM meeting in Boston, was heard to suggest that a CB curriculum in school mathematics may not even be feasible.
However, Weaver felt there are topics into which the calculator should be incorporated as part of the learning and instruction. That is, there are topics in school mathematics where certain lessons should involve the calculator to facilitate learning. In short, CM curriculum should be used in our classrooms. The function unit may be an example. Pollack (1977) and Ockenga (1976) have asserted that the calculator should be very useful in helping a student to learn about functions.

In June 1976, a conference to explore needed research and development with calculators was jointly sponsored by the National Institute of Education and the National Science Foundation (1977). In the report from this conference, it is urged that materials be developed

... to exploit the calculator as a teaching tool at every point in the curriculum to test a variety of ideas and possibilities pending emergence of calculator-integrated curriculums (National Institute of Education 1977, p. 28).

The same report on page thirty mentions graphing and functions as important ideas which need to have materials developed.

The needed research has not been completed. Suydam writes that

In January 1979, a second conference was sponsored by the National Institute of Education to specify in more detail the points made at the first conference
as well as to explore additional needs. While a report of the conference has not yet been developed it appears that it will stress the need for:
• Research on how calculators can develop mathematical skills and concepts at all levels.
• Development of instructional materials which integrate the use of calculators.
• Continuing evaluation of the effects of calculators on achievement and attitudes.
• Continuing concern for the needs of all types of learners as they use calculators.
• Emphasis on providing preservice and in-service teachers with calculator experiences (Suydam 1979, p. 6).

Immerzeel, Ockenga and Tarr also see a need for more research when they write "... experimental studies with specific objects and with specific software need to be carried out at all levels" (Immerzeel, Ockenga and Tarr 1976, p. 210).

Purpose of the Study

This study examined the relative effectiveness of Calculator Modulated instruction in a unit on functions presented to three classes in Basic Collegiate Mathematics. Evidence needs to be presented concerning the effect of incorporating calculators into various instructional units. Teachers, researchers and curriculum planners would benefit from such data. It was a purpose of this study to determine the effect of integrating calculators into a unit on functions.
Statement of the Problem

This investigation was to determine if the incorporation of the calculator (i.e., Calculator Modulated instruction) into a unit on functions in Basic Collegiate Mathematics at the University of Northern Iowa (UNI) was more effective than Calculator Assisted instruction (i.e., the calculator being available but not mentioned during instruction) in the same unit. The criterion variables are achievement in learning the function concept, functional notation achievement, function graphing achievement, achievement in learning composition of functions, achievement in learning the concept of an inverse function, and achievement in the unit.

An objective of this study was to develop a Calculator Modulated unit on functions. Two questions this study was to answer were:

1. Can a Calculator-Modulated unit on functions cause more effective learning with Basic Collegiate Mathematics students than a Calculator-Assisted unit?

2. Do sub-units in the CM unit on functions cause more effective learning with Basic Collegiate Math students than similar CA sub-units?

A central interest was to determine the effectiveness of a CM functions unit with a Basic Collegiate Mathematics
class. Similarly, there was interest in determining the effectiveness of concept learning as well.

In order to examine main effects, the data were subjected to an analysis of covariance design which defines the student as the experimental unit.

The following set of six null hypotheses were tested under the above assumptions.

**Null Hypothesis 1.** There will be no significant difference in the sub-unit achievement of students concerning the idea of a function due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

**Null Hypothesis 2.** There will be no significant difference in sub-unit achievement of students concerning functional notation due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

**Null Hypothesis 3.** There will be no significant difference in the sub-unit achievement of students concerning graphing due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

**Null Hypothesis 4.** There will be no significant difference in the sub-unit achievement of students
concerning composition of functions due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

**Null Hypothesis 5.** There will be no significant difference in inverse functions sub-unit achievement of students due to treatment when the initial difference between students has been adjusted with respect to sub-unit achievement.

**Null Hypothesis 6.** There will be no significant difference in the functions unit achievement of students due to treatment when initial difference between students has been adjusted with respect to functions unit achievement.

In planning this study, it was not possible to randomly assign students to classes. The students registered for classes according to established university policy. There were cut-off numbers for each class and available cards were distributed to registrants. When the cards were gone registration ceased. Since this procedure has been used for years and since there was no indication to students that these three classes were different than other sections of Basic Collegiate Mathematics, it was decided that the classes contained typical Basic Collegiate Mathematics students. Randomization occurred in this study when treatments were randomly assigned to classes.
Definition of Terms

To make uniform the meanings of various terms used in this study the following definitions were adopted.

**Calculator.** A hand-held, battery-powered, portable device with at least +, -, x, ÷, log, \(y^x\), sin, cos and tan keys. It may contain other keys as well.

**Calculator Modulated Unit.** A unit within a course of curriculum designed to incorporate calculators into study and discussion whenever such incorporation is judged pertinent.

**Calculator Assisted Unit.** A unit within a course of curriculum in which calculators are welcome and may be used by the student but are not specifically referred to or incorporated into the study or discussion.

**Students.** The subjects for this study:

a) Were enrolled in 80:040 Basic Collegiate Mathematics at the University of Northern Iowa (UNI), Cedar Falls, Iowa during the Fall Semester 1979. 80:040 is a regularly offered mathematics course. The course may be used for general education credit. It is an intermediate algebra course. The topics usually included are: Arithmetic of integers, rational numbers and real numbers; linear equations; inequalities; systems of equations; polynomials; exponents, powers and roots; quadratic equations and functions; common logarithms; introduction to trigonometry.
b) Completed the pretest and the posttest for the functions unit.

Delimitations of the Study

The information contained in this study comes from three classes in Basic Collegiate Mathematics at UNI during a portion of the Fall semester 1979. There were 164 students that completed both pretest and posttest. It has been assumed that the three classes contained representative Basic Collegiate Mathematics students, so that achievement observed should be a fair indicator of achievement obtained by Basic Collegiate Mathematics students throughout the university. Any conclusions are based upon experimental methods limited to a functions unit in mathematics at the general education level. Conclusions do not necessarily apply to other units (chapters) in mathematics. Nor do the conclusions necessarily generalize to other levels of mathematics students. Finally, since the functions unit used was prepared by the investigator with the assistance of the mathematics staff at UNI and distributed in a mimeograph format, some students may have been influenced by this change in assignment "from the regular text". This could cause the conclusions to be, at best, tentative.
REVIEW OF LITERATURE

This chapter cites literature and research on the use of a calculator as an aid to learning about functions. In reviewing the literature relevant to this study two general categories were considered: (1) a review of written articles concerning the use of calculators, (2) a review of the research involving use of calculators.

Review of Calculator Use

Weaver (1976) relates calculator use to the curriculum by suggesting that any curriculum may employ the calculator as an adjunct. Such use is termed Calculator Assisted (CA). If a curriculum is modified in some way so that the calculator is used specifically by the teacher and/or every student as an aid to learning during a lesson, then the usage is termed Calculator Modulated (CM). Such use would qualify as being CA also, hence in Figure 1 (p. 2) CM is a subset of CA. Finally, Weaver suggests that a curriculum could be designed so that every lesson involved use of the calculator then usage would be termed Calculator Based (CB). Such use would qualify as CM also, hence in Figure 1 (p. 2) CB is a subset of CM. The diagram which shows the relationship between CA, CM and CB curricula is Figure 1 on page 2. Articles about
a Calculator Modulated curriculum, the treatment used in this study, are cited now.

Early articles on calculators were largely activity oriented and usually the context was elementary school. Etlinger (1974) in work with third and fourth graders pointed out that use of the calculator could be purely functional—a classroom device like an eraser or desk—to do quantitative chores. On the other hand, the calculator could be purely pedagogical—an aid to facilitate learning of facts and algorithms. He feels the calculator can probably be both functional and pedagogical.

Stultz (1975) outlining uses for the calculator in the elementary classroom suggests several activities but also a few possibilities for facilitating learning such as aid to understanding of exponents, aid to introducing the function idea and aid to graphing of functions.

Mendoza (1975) states that understanding of a newtonian orbit is better when numerical methods involving the calculator are used in the lessons. Similarly, understanding about large oscillations of a pendulum is enriched when the calculator is incorporated into the lesson. He relates that numerical methods have limitations and their own difficulties, but the procedures are similar to those in the
world of professional physics. Therefore, a calculator modulated lesson is both realistic and motivational.

Judd (1975) has said that teachers need to devise projects and problems using the calculator. He sees the calculator as an aid to skill development. The comments are indicative of the need for calculator modulated units.

Quinn (1976) relates that the evidence indicates that students suffer no loss in paper-and-pencil ability after using calculators in the classroom. Furthermore, since calculator usage is widely accepted and encouraged for quantitative work in disciplines other than mathematics, he feels it is time for calculators to be integrated into the mathematics curriculum.

Harrington (1976) seems to advocate calculator modulated curriculum when he says,

Calculators must be introduced into a carefully tailored curriculum. . . .Taught in conjunction with a properly oriented syllabus, calculators need not alter greatly the current objectives of mathematics instruction (Harrington 1976, p. 44).

The National Advisory Committee on Mathematical Education (NACOME) chaired by Shirley Hill (1976) had as its goal the examination of three questions:

(1) What are the predominant patterns in mathematics curriculum content and instructional style, and to what extent do these current practices represent a realization of the goals of recent innovative efforts?
(2) What do research and general achievement testing data say about the effectiveness of current programs in reaching their goals?
(3) What are the challenges facing mathematics education in the near future, and what research, development, and implementation activity is needed to meet those challenges (Hill 1976, p. 441)?

Out of this examination came a report (NACOME, 1976) containing two recommendations which are relevant here. Recommendation 3: Curriculum Content states in part,

(e) that beginning no later than the end of the eighth grade, a calculator should be available for each mathematics student during each mathematics class. Each student should be permitted to use the calculator during all his or her mathematical work including tests (NACOME 1976, p. 138).

Recommendation 1: Needed Research says, in part,

"(g) Research is urgently needed concerning the uses of computing and calculating instruments in curriculum at all levels and their relationship to a broad array of instructional objectives" (NACOME 1976, p. 144). The message conveyed in these recommendations is that CM units are appropriate as classroom procedure and definitely appropriate as a broad topic for research.

Taylor (1976) suggests that Recommendation 3, part (e), as quoted above is probably the most controversial recommendation to come from NACOME. He discusses calculators,

This recommendation on calculators in the NACOME report will stimulate further interest in solving
the instructional and logistical problems related to having calculators in the classroom. Furthermore, the recommendation should be influential in helping school systems decide to invest funds in calculators (Taylor 1976, p. 460).

Later, he states that teachers should be active as initiators. This implies classroom work with CM type units is appropriate and needed.

Maor (1976) has viewed the calculator as an influential device which can bring new insight, interest and even fun to learning. He says the purpose of his article is to offer several examples where this might be done. One of his examples is the topic, functions.

Bell (1976) declares a need to nurture the fundamentally new possibilities offered by calculators so that maximal learning may result. This is encouragement for use of the calculator with any topic where understanding is enhanced by involving the calculator. That is, CM units are appropriate in school.

Ockenga (1976) specifically suggests use of a calculator to help introduce the idea of a function to students. In particular, he advocates the machine analogy, where the calculator is interpreted as a function machine, as being helpful to students.

Gawronski and Coblentz (1976) see the calculator as an opportunity. Teachers could use calculators in ways
and with strategies which can help children to think, create and learn mathematics.

Pollack (1977) suggests calculators are helpful in learning about functions since the students, ". . . get hold of the idea that what matters is that when a given number is fed into the function, a single number always comes out" (Pollack 1977, p. 294). He feels the calculator should enhance understanding of an inverse function. For example, he states that using the calculator to experience

\[ y = \sin \left( \frac{x}{1 + x} \right) \quad \text{and} \quad x = \frac{\arcsin y}{1 - \arcsin y} \]

or

\[ y = 8x^3 - 36x^2 + 54x - 27 \quad \text{and} \quad x = \frac{1}{2}(y^{1/3} + 3) \]

on the hand-held calculator and see how it all comes out, they will get a good understanding of inverse functions (Pollack 1977, p. 295).

Hilton and Rising (1975) state, "We hope that the hand-held calculator will be used in the mathematics classroom predominantly to support and extend conceptual understandings of mathematics" (Hilton and Rising 1975, P. 39).

Review of the Research Involving Calculators

Two remarks illustrate the writer's feeling on the importance of research concerning CM curriculum:
1. F. H. Bell says,

During the next decade, many people should concentrate on finding answers to the question, How can students use computers or calculators to better learn those things in mathematics that are hard to learn and difficult to measure and evaluate (F. H. Bell 1978a, p. 433).

2. M. S. Bell says,

I believe we will succeed or fail according to how innovative and thoughtful many individual teachers are in accommodating their classrooms to the new realities of a calculator and computer age. If we fail, a substantial part of mathematics education may eventually go the way of instruction in handwriting, Greek or Latin. But, if we succeed, the advent of calculators and their microelectronic cousins can help us achieve results we have long hoped for but not attained (M. S. Bell 1978b, p. 410).

In considering the research involving calculators it was observed that there have been many empirical studies. Roberts (1980) mentions thirty-four calculator studies ranging from elementary school to college level during the period from 1973 to 1978. He categorized the studies as being computational and/or conceptual. He appears to have judged a study as being conceptual if the tests involved "... could be considered to emphasize concepts rather than sheer computations" (Roberts 1980, p. 84). He judged sixteen of the studies to be conceptual in nature. While speaking about the premise that calculator usage can have an impact on mathematical concept formation he states, "In fact, a strong case can be made that this hypothesis has not been
adequately tested since few studies made any real attempt to carefully integrate calculator use into the curriculum that would illustrate how calculators can facilitate concept learning" (Roberts 1980, p. 84). He says that only four of sixteen conceptual studies showed the calculator group as superior to the no calculator group.

Roberts' (1980) assessment of the twenty-five studies judged to be computational indicate that nineteen of the studies showed the calculator group superior to the no calculator group. The indication is that there is computational benefit in usage of calculators.

Roberts' (1980) consideration of the thirty-four studies had another dimension. He classified twenty of the studies as attitudinal studies and states that seven out of twenty studies showed the calculator group acquired attitudinal benefits. The indication is that calculators don't automatically effect a change in attitude.

The article by Roberts (1980) propounds that, although there are many empirical studies involving calculators, very few are concerned with integrating the calculator into the curriculum and using it to facilitate concept learning. In this review of the literature the writer could find no studies which specifically compared a CA versus a CM type curriculum with respect to learning about functions. A few
studies were located in which a CM type curriculum seemed to be involved. The remainder of this section will cite those studies individually.

A study by Hutton (1977) compared three pairs of ninth grade algebra classes during a unit on powers, roots and radicals. One class in each pair was allowed to use the calculator (CA) but the other class had the use of the calculator incorporated into the lecture (CM). No significant difference was noted.

Anderson (1977) compared three seventh grade classes in each of the four junior high schools in Sioux Falls, South Dakota. In each school one class was restricted to using the calculator to verify calculations, one class was allowed unrestricted use of the calculator, and one class was allowed no use of the calculator. Among the findings were that: "There was no effect on overall achievement. There was no effect on mastery of concepts. There was no effect on computation skill when calculators were not allowed."

O'Loughlin (1975) carried out a study in which he compared a class T of twenty calculus students with another calculus class S of seventeen students. Class T used no calculators and an ordinary text. Class S used experimental materials in which the programmable calculator was employed. Apparently, students were allowed to use the calculator only
in the lab, but the instructor used it as a teaching aid in class. The students were tested for achievement in six areas: (1) Limits, (2) Continuity, (3) Interrelationships between a function and its first two derivatives, (4) Local extrema of a function, (5) Solving verbal problems involving max/min and related rates, (6) Definite integrals. Class S, the experimental group, tested significantly higher in achievement on tests (3), (5) and (6) but no significant differences appeared between classes on tests (1), (2) and (4). O'Loughlin thinks that the calculator was helpful in allowing a somewhat broader understanding of function to develop. He reports that students made favorable comments when asked if the calculator was helpful as a teaching aid.

Szetela (1979) did a study on teaching trigonometric ratios to 131 ninth and tenth grade students of low to average ability. He developed special lessons and materials involving using the calculator as an aid for learning trigonometric ratios. Each of the four classes was subdivided into two groups. One group in each class was taught about ratios using no calculators. The other group used the special materials and calculators every day. After eighteen days of instruction an achievement test constructed by Szetela was administered. Differences between groups were not significant.
Summary

The foregoing review suggests the following pertinent points.

1. Many authors and researchers tacitly promote use of the calculator as a learning device in mathematics to develop better understanding of concepts. Several explicitly mention functions and related concepts as being among those for which CM curriculum is appropriate.

2. NACOME, the Conference on Basic Mathematical Skills and Learning and the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics all stress a need for calculator research.

3. Two studies, O'Loughlin (1975) and Szetela (1979) were located in which CM units appeared to be used.

4. There is a need for studies concerning the effect of the calculator on concept learning. Particularly in a context where the calculator has been incorporated into the curriculum.

5. No studies were located where there was a comparison of CM versus CA units.

6. The review failed to find any study where the central purpose was to use the calculator as an aid to learn about functions.
METHODS AND PROCEDURES

The purpose of this study was to examine the relative effectiveness of a CM unit versus a CA unit on functions in Basic Collegiate Mathematics at the University of Northern Iowa (UNI). The study also involved testing of five sub-units to determine if significant differences in achievement existed at the sub-unit level. The areas of interest investigated were:

1. student achievement in learning the idea of a function
2. student achievement in learning functional notation
3. student achievement in learning graphing of functions
4. student achievement in learning composition of functions
5. student achievement in learning inverse of a function
6. student achievement in the functions unit.

This chapter describes the methods and procedures used to gather and analyze the data for this study. The chapter is divided into five sections: (1) selection of the sample, (2) preparation of the materials, (3) schedule
and procedure used in the experiment, (4) testing, and (5) treatment of the data.

Selection of the Sample

The students surveyed were undergraduates at UNI, located in Cedar Falls, Iowa. UNI enrolls about 12,000 students. The university offers a broad curriculum at both the undergraduate and graduate levels. During the fall semester of 1979 the students involved in the study were enrolled in 80:040 Basic Collegiate Mathematics, a regularly offered mathematics course.

In the fall of 1979 the investigator arranged with the Department of Mathematics to conduct this study with three classes in Basic Collegiate Mathematics. The three classes were taught by the investigator to avoid differences due to teacher variable. It was arranged to have the classes meet in geographically separate buildings at two hour intervals in an attempt to minimize interaction and/or collaboration between classes. On the first day the students were told they would need a calculator for the course. All students indicated they owned a calculator or that they would get one to use for the duration of the course. The students were not initially told of the study. The functions unit was presented as an integral part of the course. Upon completion of the functions posttest it was announced that this unit was
part of a curriculum study. The students were told their scores would be used in a statistical analysis if they concurred. They were assured that as subjects they would remain anonymous, but, if they desired, they could have their scores removed from the analysis. No one asked to remove their score.

Initial enrollment was 207, but about seven weeks later 191 were left to begin the unit on functions. At this time there were eighty-eight students in Section 1, meeting at 8:00 AM. There were forty-seven in Section 2, meeting at 10:00 AM. There were fifty-six in Section 3, meeting at noon. All sections met on Monday, Tuesday, Wednesday and Thursday for a fifty minute period each day. A self-information form was administered (Appendix A). Information was gathered about age of student, sex of student, number of semesters of math completed during grades nine through twelve, number of years since last math course, quartile rank in high school graduating class (1 = lowest quartile, 4 = highest quartile), self rating of calculator experience (1 = least, 4 = highest) and years of calculator experience. These data are displayed in Table 1.

University registration procedure did not allow random assignment of students to classes but, as Table 1 indicates, the three classes are rather similar with respect to age,
Table 1. Descriptive data for sections 1, 2 and 3 in Basic Collegiate Mathematics

<table>
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<th>Section</th>
<th>Mean age years</th>
<th>Sex</th>
<th>Mean semesters of math</th>
<th>Mean years since last math</th>
<th>Mean rank</th>
<th>Calculator self rate</th>
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semesters of math experience, mean rank in high school, self rating in calculator experience and years of calculator experience. Notice that Section 3 was composed of mostly male students and the mean age is higher than that of Sections 1 or 2. The age difference is partially explained by the presence of eight students over twenty-five years of age (one was forty-eight) while in the other sections there were respectively one and two students over twenty-five. This also indicates why the mean number of years since last math course is higher for Section 3.

The study involved two treatments. One section, the control group, was treated with a CA unit on functions, that is the students had calculators to assist them whenever and however they wished. Two sections, the experimental group, were treated with a CM unit on functions, that is the calculator was incorporated into a discussion or demonstration whenever the idea or topic allowed. Section 1 and Section 3 were randomly assigned to be treated as CM sections. Section 2 was a CA section.

Preparation of the Materials

The initial preparation of the functions unit began in the spring of 1978. The unit was a self-contained introduction to functions. Emphasis was given to the fact that a function is a correspondence of a particular kind
between two sets of elements. Care was taken to dissuade the student from thinking of a function as an equation. It was emphasized that some functions are not described by equations and that some equations do not describe functions.

The unit consisted of ten sequential lessons, each designed for one class period (Appendix B). The lessons in the unit are sequential. Every lesson was written (for the student) to be read by every student, no matter which section; the conceptual content was to be equivalent in all sections. The way the calculator is involved with the concept is the variable, not the content of the lesson. An exercise set was prepared for each lesson (Appendix C). A pilot unit was taught during the spring semester of 1979 to three sections of Basic Collegiate Mathematics. This experience instigated a revision of several lessons in the unit. The pilot study was done over a period of about nine or ten class days and indicated a larger block of time was needed. The pilot experience indicated that probably solutions to exercises should be posted. These modifications were accomplished for the study.

Schedule and Procedure Used in the Experiment

All sections of Basic Collegiate Mathematics used the text: Essential Mathematics, A Modern Approach, Second Edition, Keedy and Bittinger, Addison-Wesley, 1976. This
text provides scant discussion on functions. Since a modicum of algebraic skill was needed in the functions unit, it was decided to begin the study after completion of chapters one through seven in the text. Every student was given reading material and exercise sets according to the schedule. Starting the third day model solutions were posted for the exercise set given the preceding day.

The study lasted thirteen school days as follows:

Oct. 9    Day 1    Administer pretest. Distribute Lesson 1 reading materials.

Oct. 10   Day 2    Discuss Lesson 1 (25-35 min.) Distribute Lesson 2, Exercise 1.


Oct. 15   Day 4    Discuss Lesson 3 (25-35 min.) Distribute Lesson 4, Exercise 3.

Oct. 16   Day 5    Discuss Lesson 4 (25-35 min.) Distribute Lesson 5, Exercise 4.

Oct. 17   Day 6    Discuss Lesson 5 (25-35 min.) Distribute Lesson 6, Exercise 5.

Oct. 22   Day 7    Discuss Lesson 6 (25-35 min.) Distribute Lesson 7, Exercise 6.

Oct. 23   Day 8    Discuss Lesson 7 (25-35 min.) Distribute Lesson 8, Exercise 7.
Oct. 29  Day 11  Discuss Lesson 10 (25-35 min.)  Distribute Exercise 10.
Oct. 30  Day 12  Review (50 min.)
Oct. 31  Day 13  Administer Posttest.

During each non-test day of the study some time (15-25 min.) was devoted to inquiries and/or solving exercises relating to the previous day's work. In the CM sections, the experimental group, the instructor made a pointed effort to relate a concept to the calculator where appropriate. In the CM sections on most days the instructor demonstrated how the calculator pertained to the concept under discussion. An overhead calculator which allowed projection of the calculator results was used in such demonstrations. A record of the approximate time spent in these demonstrations is displayed in Table 2.

In Appendix D a brief description for each day is written. The lesson plan for the CA class is sketched and then comments are given to indicate how the lesson was modified in the CM sections.
Table 2. Time in minutes used for calculator demonstration/activity

<table>
<thead>
<tr>
<th></th>
<th>Day 1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
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<tbody>
<tr>
<td>Section 1</td>
<td>-</td>
<td>20</td>
<td>5</td>
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<td>10</td>
<td>30</td>
<td>5</td>
<td>5</td>
<td>15</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>-</td>
</tr>
<tr>
<td>Section 2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>Section 3</td>
<td>-</td>
<td>20</td>
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<td>10</td>
<td>5</td>
<td>20</td>
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<td>20</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>-</td>
</tr>
</tbody>
</table>

Testing

The testing consisted of two similar forms of an achievement test constructed by the investigator with the assistance from several members of the Mathematics Department at UNI (Appendix E). Form 1 and Form 2 were constructed in such a way that each item has a parallel item of the same number in the other form. For example, 10 in the pretest is \[ h(t) = 5t^3 - 200 \]. Find \( h(h^{-1}(x)) \). In the posttest, 10 is \[ h(s) = 6s^3 - 240 \]. Find \( h(h^{-1}(x)) \). By making items on the two forms "similar", the forms are interchangeable. Form 1, the pretest, was administered on Day 1. A make-up was provided on Day 2. Twenty-seven students did not take the pretest and were not included in the study. Total enrollment in the three sections was 191. 164 students took both the pretest and posttest.
Form 1 and Form 2 of the achievement test on functions consist of thirty items. The items were chosen from a pool of items to satisfy the criteria:

1. The item was about functions.
2. The item implied a definite, precise answer.
3. The item could be placed into one of the categories:
   (a) The idea of a function
   (b) Functional notation
   (c) Graphing
   (d) Composition of Functions
   (e) Inverse of a function
4. Approximately the same number of items would be in each category so that each category could provide a measure for a sub-unit.
5. The item did not call for extensive numeric calculation.
6. The items were nonsequential.
7. The score obtained by a student in a category would be a measure of achievement in that category.
8. The total score obtained by a student would be a measure of achievement in the functions unit.

To verify the validity of the test and the subtests the investigator submitted the thirty selected items to five members of the UNI Mathematics Department. The
individuals were instructed to privately read an item then (1) judge if the item is appropriate to the unit achievement test on functions, (2) if appropriate to the unit place the item into one of the six categories:

   A. The idea of a function
   B. Functional notation
   C. Graphing
   D. Composition of functions
   E. Inverse of a Function
   F. None of the above

   It was decided that if fewer than four of the five rated an item to be appropriate the item would be dropped from the unit test. The five judges unanimously accepted all thirty items as appropriate for a unit test on functions. It was decided that if four or more of the five placed an item into a particular category then that item would be appropriate to measure achievement of a student in that sub-unit. Results were that:

   A. Items 1, 2, 16, 21, and 25 were appropriate to measure achievement in the idea of a function sub-unit.
   B. Items 3, 4, 5, 6, and 7 were appropriate to measure achievement in the functional notation sub-unit.
   C. Items 11, 12, 14, 15, 17 and 20 were appropriate to measure achievement in the graphing sub-unit.
D. Items 22, 26, 27, 28, 29, and 30 were appropriate to measure achievement in the composition of a function sub-unit.

E. Items 8, 9, 10, 13, and 24 were appropriate to measure achievement in the inverse of a function sub-unit. Items 18, 19, and 23 did not get placed by four of the judges into a category so they were not used to measure achievement of a student in any sub-unit but were used to measure achievement in the functions unit. The pretest and the posttest are the content of Appendix E.

The reliability of the subtests and the unit test is indicated by the data in Table 3. This table shows the mean scores for each of six sections on each pretest and posttest. On any given line of the table the dispersion of the mean scores is small which indicates the average student performs about at the same level each time the test was administered.
Table 3. Summary of mean scores for sections

<table>
<thead>
<tr>
<th></th>
<th>Pilot study section</th>
<th>Study section</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Preidea</td>
<td>1.64</td>
<td>1.70</td>
</tr>
<tr>
<td>Prenote</td>
<td>2.78</td>
<td>3.39</td>
</tr>
<tr>
<td>Pregraph</td>
<td>1.24</td>
<td>1.11</td>
</tr>
<tr>
<td>Precomp</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Preinv</td>
<td>.01</td>
<td>--</td>
</tr>
<tr>
<td>Pretot</td>
<td>5.78</td>
<td>6.27</td>
</tr>
<tr>
<td>Postidea</td>
<td>3.35</td>
<td>3.72</td>
</tr>
<tr>
<td>Postgraph</td>
<td>2.82</td>
<td>3.14</td>
</tr>
<tr>
<td>Postcomp</td>
<td>1.44</td>
<td>1.99</td>
</tr>
<tr>
<td>Postinv</td>
<td>1.33</td>
<td>1.71</td>
</tr>
<tr>
<td>Posttot</td>
<td>13.33</td>
<td>15.25</td>
</tr>
</tbody>
</table>

Letting $\frac{r_{1'}}{2'} = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{n\Sigma X^2 - (\Sigma X)^2} \sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$, then the Spearman-Brown formula for reliability is $r_{11} = \frac{2r_{1'}}{1 + r_{1'}}$.

Using this formula, the reliability for the posttest is displayed in Table 4.
Table 4. Reliability of the posttest for sections

<table>
<thead>
<tr>
<th>Pilot study section</th>
<th>Study section</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( r_{11} )</td>
<td>.98</td>
</tr>
<tr>
<td></td>
<td>.89</td>
</tr>
<tr>
<td></td>
<td>.90</td>
</tr>
<tr>
<td></td>
<td>.89</td>
</tr>
<tr>
<td></td>
<td>.93</td>
</tr>
<tr>
<td></td>
<td>.87</td>
</tr>
</tbody>
</table>

These data suggest that the reliability for the unit test is about \(.90\).

Treatment of Data

The goal of this study was to determine the relative effectiveness of a CM unit on functions versus a CA unit on functions as measured by a post-treatment-test on achievement. The research also investigated effectiveness of five CM sub-units versus five CA sub-units.

The statistical analysis of the data was done using analysis of covariance. This technique provided data regarding the main effect. Students were statistically equated with respect to the covariant pretest score. The student was assumed to be the experimental unit in this study. Since the investigator instructed both the control group and the experimental group, teacher effect was not treated as a variable.
The main effect in the analysis of covariance was curriculum. The criterion variables were the posttest achievement scores. The analysis of covariance was done using the Statistics Package for the Social Sciences (SPSS) at the Iowa State University Computation Center.
FINDINGS

The findings of this study were based upon the results obtained by testing 164 students in three classes of Basic Collegiate Mathematics at UNI.

To report the findings in this study two subdivisions were used:

1. analysis of the measurement instrument
2. analysis of covariance on the criterion variables.

Analysis of the Measurement Instrument

Table 5 displays the correlation coefficients of the subtests and the complete thirty item test. The correlations of the prenotation subtest to the postnotation subtest was .283. The correlation of the preinv of a function subtest to the postinv of a function subtest was .036. The other correlations for pre subtest to a similar post subtest are in between. This indicates that the scores on the pre subtests did not relate well to the scores on the post subtests. However, the pre subtest scores when correlated with the pretotal test scores ranged from .873 to .298 which shows that the pre subtest scores relate better with the pretotal test scores. This indicates some internal consistency. A similar but slightly better relation was
Table 5. Correlation of measurement instruments

<table>
<thead>
<tr>
<th>Subtest</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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<tbody>
<tr>
<td>1. Preidea</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Postidea</td>
<td>.039</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Prenote</td>
<td>.323</td>
<td>.133</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Postnote</td>
<td>.161</td>
<td>.277</td>
<td>.283</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Pregraph</td>
<td>.158</td>
<td>.073</td>
<td>.309</td>
<td>.160</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Postgraph</td>
<td>.151</td>
<td>.457</td>
<td>.309</td>
<td>.390</td>
<td>.175</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Precomp</td>
<td>.282</td>
<td>.129</td>
<td>.165</td>
<td>.063</td>
<td>.090</td>
<td>.157</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Postcomp</td>
<td>.141</td>
<td>.397</td>
<td>.294</td>
<td>.362</td>
<td>.100</td>
<td>.475</td>
<td>.166</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Preinv</td>
<td>.204</td>
<td>-.074</td>
<td>.220</td>
<td>.061</td>
<td>.074</td>
<td>.093</td>
<td>-.020</td>
<td>.083</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Postinv</td>
<td>.158</td>
<td>.378</td>
<td>.213</td>
<td>.345</td>
<td>.156</td>
<td>.516</td>
<td>.215</td>
<td>.523</td>
<td>.036</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Pretot</td>
<td>.647</td>
<td>.115</td>
<td>.873</td>
<td>.294</td>
<td>.558</td>
<td>.309</td>
<td>.298</td>
<td>.277</td>
<td>.315</td>
<td>.249</td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>
seen where the post subtest scores when correlated with posttotal test scores ranged from .790 to .631. This indicates subtest scores related more closely to total test scores after the students have taken the unit.

The summary in Table 6 indicates that all sections showed gains on the idea of a function subtest (5 items). The mean gains on the idea of a function subtest from pre to posttest were: Section 1, 1.85; Section 2, 1.45; Section 3, 2.11. It was observed that Section 3 showed less diligence on the pretest and thus their gain or difference score might be expected to be higher. The standard deviations are comparable which indicates the sections are not too different. The fact that the standard deviation on the post subtest was greater indicates a larger variance in post subtest scores.

Table 6. Summary of Idea of a function scores for pre and posttest

<table>
<thead>
<tr>
<th>Classification</th>
<th>Mean Pretest</th>
<th>Mean Posttest</th>
<th>Standard Deviation Pretest</th>
<th>Standard Deviation Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1</td>
<td>1.37</td>
<td>3.22</td>
<td>.79</td>
<td>.97</td>
</tr>
<tr>
<td>Section 2</td>
<td>1.76</td>
<td>3.21</td>
<td>.90</td>
<td>1.19</td>
</tr>
<tr>
<td>Section 3</td>
<td>.98</td>
<td>3.09</td>
<td>.83</td>
<td>.95</td>
</tr>
<tr>
<td>Total</td>
<td>1.34</td>
<td>3.18</td>
<td>.87</td>
<td>1.02</td>
</tr>
</tbody>
</table>

Reliability of pretest = .501 Reliability of posttest = .402
The summary in Table 7 shows that all sections made gains in the notation subtest (5 items). The mean gains on the notation of a function subtest from pre to posttest were: Section 1, 2.64; Section 2, 2.25; Section 3, 2.99. The greater gain by Section 3 was conjectured to be related to the low pretest scores. The standard deviations are quite comparable. The standard deviations on the post subtest were smaller than on the pre subtest.

Table 7. Summary of Notation of a function scores for pre and posttest

<table>
<thead>
<tr>
<th>Classification</th>
<th>Mean Pretest</th>
<th>Mean Posttest</th>
<th>Standard Deviation Pretest</th>
<th>Standard Deviation Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1</td>
<td>1.29</td>
<td>3.93</td>
<td>1.38</td>
<td>1.10</td>
</tr>
<tr>
<td>Section 2</td>
<td>1.89</td>
<td>4.14</td>
<td>1.91</td>
<td>1.31</td>
</tr>
<tr>
<td>Section 3</td>
<td>0.84</td>
<td>3.83</td>
<td>1.22</td>
<td>1.23</td>
</tr>
<tr>
<td>Total</td>
<td>1.29</td>
<td>3.95</td>
<td>1.42</td>
<td>1.19</td>
</tr>
</tbody>
</table>

Reliability of pretest = .736 Reliability of posttest = .536

The summary in Table 3 indicates that all sections made gains on the graphing subtest (6 items). The mean gains on the graphing subtest from pre to posttest were: Section 1, 1.78; Section 2, 1.87; Section 3, 1.85. The mean scores
are quite comparable as are the standard deviations. The sections performed similarly on this measure.

Table 8. Summary of Graphing of a function scores for pre and posttest

<table>
<thead>
<tr>
<th>Classification</th>
<th>Mean Pretest</th>
<th>Mean Posttest</th>
<th>Standard Deviation Pretest</th>
<th>Standard Deviation Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1</td>
<td>.76</td>
<td>2.54</td>
<td>.68</td>
<td>1.31</td>
</tr>
<tr>
<td>Section 2</td>
<td>.86</td>
<td>2.73</td>
<td>.74</td>
<td>1.52</td>
</tr>
<tr>
<td>Section 3</td>
<td>.59</td>
<td>2.45</td>
<td>.68</td>
<td>1.32</td>
</tr>
<tr>
<td>Total</td>
<td>.73</td>
<td>2.56</td>
<td>.70</td>
<td>1.36</td>
</tr>
</tbody>
</table>

Reliability of pretest = .009  Reliability of posttest = 4.79

The summary in Table 9 shows that all sections exhibited gains on the Composition of functions subtest (6 items). The mean gains on the composition of a function subtest from pre to posttest were: Section 1, 1.36; Section 2, 1.91; Section 3, 1.52. The table indicates that the students had no previous experience in this topic as evidenced by the near zero scores on the pretest.
Table 9. Summary of Composition of functions scores for pre and posttest

<table>
<thead>
<tr>
<th>Classification</th>
<th>Mean Pretest</th>
<th>Mean Posttest</th>
<th>Standard Deviation Pretest</th>
<th>Standard Deviation Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1</td>
<td>.04</td>
<td>1.40</td>
<td>.26</td>
<td>1.42</td>
</tr>
<tr>
<td>Section 2</td>
<td>.00</td>
<td>1.91</td>
<td>.00</td>
<td>1.78</td>
</tr>
<tr>
<td>Section 3</td>
<td>.00</td>
<td>1.52</td>
<td>.00</td>
<td>1.42</td>
</tr>
<tr>
<td>Total</td>
<td>.02</td>
<td>1.56</td>
<td>.17</td>
<td>1.52</td>
</tr>
</tbody>
</table>

Reliability of pretest = .479  Reliability of posttest = .714

The summary in Table 10 indicates that all sections showed gains on the inverse function subtest (5 items). The mean gains on the inverse function subtest from pre to posttest were: Section 1, 1.09; Section 2, 1.22; Section 3, 1.16. The near zero scores on the pretest indicate that this was a new topic for the students. The scores and the standard deviations are quite comparable.
Table 10. Summary of Inverse functions scores for pre and posttest

<table>
<thead>
<tr>
<th>Classification</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pretest</td>
<td>Posttest</td>
</tr>
<tr>
<td>Section 1</td>
<td>.03</td>
<td>1.12</td>
</tr>
<tr>
<td>Section 2</td>
<td>.08</td>
<td>1.30</td>
</tr>
<tr>
<td>Section 3</td>
<td>.02</td>
<td>1.18</td>
</tr>
<tr>
<td>Total</td>
<td>.04</td>
<td>1.18</td>
</tr>
</tbody>
</table>

Reliability of posttest = .684

The summary in Table 11 shows that all sections made gains on the function unit test (30 items). The mean gains on the functions unit test of thirty items from pre to posttest were: Section 1, 9.57; Section 2, 9.63; Section 3, 10.32. The larger gain by Section 3 may be due to the lower score on the pretest by that section.

It was noticed that on three out of five of the pretests and posttests the mean score of Section 2 was the highest among the sections. That section also appeared to be somewhat better motivated than the other sections.
Table 11. Summary of Unit test scores for pre and posttests

<table>
<thead>
<tr>
<th>Classification</th>
<th>Pretest Mean</th>
<th>Posttest Mean</th>
<th>Pretest Std. Dev.</th>
<th>Posttest Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section 1</td>
<td>3.50</td>
<td>13.07</td>
<td>3.26</td>
<td>4.61</td>
</tr>
<tr>
<td>Section 2</td>
<td>4.68</td>
<td>14.31</td>
<td>3.00</td>
<td>6.41</td>
</tr>
<tr>
<td>Section 3</td>
<td>2.45</td>
<td>12.77</td>
<td>1.72</td>
<td>4.72</td>
</tr>
<tr>
<td>Total</td>
<td>3.45</td>
<td>13.28</td>
<td>2.43</td>
<td>5.06</td>
</tr>
</tbody>
</table>

Reliability of unit pretest = .694
Reliability of unit posttest = .850

The reliabilities of the pre subtests ranged from no measure (due to zero scores) to .736. The low ratios may be due to the shortness of each subtest, for when the thirty item pretest was checked the reliability was .694.

The reliabilities of the post subtests ranged from .402 to .714. Again the low ratios may be attributed to the few items on each subtest. The reliability of the unit posttest was .850.
Analysis of Covariance

The basic statistical design used in this study treated the student as the experimental unit (n = 164). This design enabled the investigator to examine the effect of CM curriculum versus CA curriculum on three sections of Basic Collegiate Mathematics students during a three week unit on functions.

Posttest scores on Idea of a function subtest, Notation of a function subtest, Graphing of a function subtest, Composition of a function subtest, Inverse of a function subtest and Functions unit test total were used as criterion variables.

The pretest scores for the criterion variables were used as covariates.

Null hypothesis 1 states: There will be no significant difference in the sub-unit achievement of students concerning the idea of a function due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement. Table 12 displays the data on Idea of a function subtest scores. Null hypothesis 1 was tested using data from this table. At the .05 level of significance null hypothesis 1 was not rejected. There was no significant difference in achievement of students on the Idea of a function sub-unit. The F value for the covariate
is .248. The covariate is not accounting for much of the variation here.

Table 12. Analysis of covariance: posttest idea of a function score is the criterion variable (n = 164)

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Sum of squares</th>
<th>Mean square</th>
<th>F</th>
<th>Significance of F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>2</td>
<td>.331</td>
<td>.165</td>
<td>.166</td>
<td>.847</td>
</tr>
<tr>
<td>Covariate</td>
<td>1</td>
<td>.247</td>
<td>.247</td>
<td>.248</td>
<td>.619</td>
</tr>
<tr>
<td>Error</td>
<td>160</td>
<td>159.288</td>
<td>.996</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Null hypothesis 2 states: There will be no significant difference in sub-unit achievement of students concerning functional notation due to treatment when initial differences between students has been adjusted with respect to sub-unit achievement. Table 13 displays the data on Functional notation subtest scores. Null hypothesis 2 was tested using data from this table. At the .05 level of significance null hypothesis 2 was not rejected. There was no significant difference in achievement of students on the Functional notation sub-unit. The F value for the covariate is 13.959.
Table 13. Analysis of covariance: posttest Functional notation score is the criterion variable (n = 164)

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Sum of squares</th>
<th>Mean square</th>
<th>F</th>
<th>Significance of F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>2</td>
<td>1.257</td>
<td>.628</td>
<td>.486</td>
<td>.616</td>
</tr>
<tr>
<td>Covariate</td>
<td>1</td>
<td>18.047</td>
<td>18.047</td>
<td>13.959</td>
<td>.000</td>
</tr>
<tr>
<td>Error</td>
<td>160</td>
<td>206.851</td>
<td>1.293</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Null hypothesis 3 states: There will be no significant difference in sub-unit achievement of students concerning graphing due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement. Table 14 displays the data on Graphing of functions subtest scores. Null hypothesis 3 was tested using data from this table. At the .05 level of significance null hypothesis 3 was not rejected. There was no significant difference in achievement of students on the Graphing of functions sub-unit. The F value for the covariate was (5.087).
Null hypothesis 4 states: There will be no significant difference in the sub-unit achievement of students concerning composition of functions due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement. Table 15 displays the data on Composition of functions subtest scores. Null hypothesis 4 was tested using data from this table. At the .05 level of significance null hypothesis 4 was not rejected. There was no significant difference in achievement of students on the Composition of functions sub-unit. The F value for the covariate was 4.603.
Table 15. Analysis of covariance: posttest Composition of function score is the criterion variable (n = 164)

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Sum of squares</th>
<th>Mean square</th>
<th>F</th>
<th>Significance of F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>2</td>
<td>5.016</td>
<td>2.508</td>
<td>1.105</td>
<td>.334</td>
</tr>
<tr>
<td>Covariate</td>
<td>1</td>
<td>10.451</td>
<td>10.451</td>
<td>4.603</td>
<td>.033</td>
</tr>
<tr>
<td>Error</td>
<td>160</td>
<td>363.263</td>
<td>2.270</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Null hypothesis 5 states: There will be no significant difference in inverse functions sub-unit achievement of students due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement. Table 16 displays the data on Inverse function subtest scores. Null hypothesis 5 was tested using data from this table. At the .05 level of significance null hypothesis 5 was not rejected. There was no significant difference in achievement of students on the Inverse functions sub-unit. The F value for the covariate was .205.
Table 16. Analysis of covariance: posttest Inverse function score is the criterion variable (n = 164)

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Sum of squares</th>
<th>Mean square</th>
<th>F</th>
<th>Significance of F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>2</td>
<td>.123</td>
<td>.062</td>
<td>.037</td>
<td>.964</td>
</tr>
<tr>
<td>Covariate</td>
<td>1</td>
<td>.342</td>
<td>.342</td>
<td>.205</td>
<td>.652</td>
</tr>
<tr>
<td>Error</td>
<td>160</td>
<td>267.235</td>
<td>1.670</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Null hypothesis 6 states: There will be no significant difference in the Functions unit achievement of students due to treatment when initial difference between students has been adjusted with respect to Functions unit achievement. Table 17 displays the data on Functions unit test scores. Null hypothesis 6 was tested using data from this table. At the .05 level of significance null hypothesis 6 was not rejected. There was no significant difference in achievement of students on the Functions unit. The F value for the covariate was 22.764.
Table 17. Analysis of covariance: posttest Functions unit score is the criterion variable (n = 164)

<table>
<thead>
<tr>
<th>Source of variation</th>
<th>Degrees of freedom</th>
<th>Sum of squares</th>
<th>Mean square</th>
<th>F</th>
<th>Significance of F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>2</td>
<td>.316</td>
<td>.158</td>
<td>.007</td>
<td>.993</td>
</tr>
<tr>
<td>Covariate</td>
<td>1</td>
<td>512.202</td>
<td>512.202</td>
<td>22.764</td>
<td>.000</td>
</tr>
<tr>
<td>Error</td>
<td>160</td>
<td>3600.539</td>
<td>22.500</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It was noted earlier (p. 25) that the three sections of students in this study were rather similar. Yet, there were noticeable differences between the sections. It was decided there was a need to determine if these apparent differences might have an effect. Five analyses of covariance were carried out where two covariates were used each time. Semesters of math in high school, years since last math course, rank in high school graduating class, calculator self-rating and years of calculator experience were used, respectively, along with pretest score as covariates. In each case there was no significant difference between groups, thus the apparent differences between sections were not important to this study. Since the data did not influence the findings of this study, they were not tabulated.
SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

The purpose of this investigation was to determine if a CM unit on functions taught to Basic Collegiate Mathematics students at the University of Northern Iowa could be established as being significantly more effective than a CA unit on functions. Analysis of covariance was used to analyze the data. The criterion variables were posttest scores on the Idea of a function subtest, the Functional notation subtest, the Graphing of a function subtest, the Composition of functions subtest, the Inverse function subtest and the Functions unit test. The covariates used were the pretest scores for the above criterion variables.

Two treatments were randomly assigned to three Basic Collegiate Mathematics classes. The treatments were:

1. a CM unit on functions
2. a CA unit on functions

One class was assigned to be CA; the other two were CM.

Six null hypotheses were used. Those null hypotheses were tested under the assumption that the experimental unit was the student. Treatments were selected to be the main effect in an analysis of covariance design.
The null hypotheses were:

1. There will be no significant difference in the sub-unit achievement of students concerning the idea of a function due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

2. There will be no significant difference in sub-unit achievement of students concerning functional notation due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

3. There will be no significant difference in the sub-unit achievement of students concerning graphing due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

4. There will be no significant difference in the sub-unit achievement of students concerning composition of functions due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.

5. There will be no significant difference in inverse functions sub-unit achievement of students due to treatment when initial difference between students has been adjusted with respect to sub-unit achievement.
6. There will be no significant difference in the functions unit achievement of students due to treatment when initial difference between students has been adjusted with respect to functions unit achievement.

The six null hypotheses above were found to be tenable. In no case was the data sufficient to reject a null hypothesis at the .05 level of significance.

Implications

The observations and commentary in this section are the result of the investigator's experiences during the study as well as post-experimental meditation.

The first day of the semester the investigator informed all three classes that the nature of the course was such that a calculator would be an asset. Furthermore, each student enrolled was expected to have a calculator and that calculator use was permissible in class, on homework or during tests. A poll indicated that no one would be without a calculator for at least that semester. Although this context was set, and in addition students in the CM sections were told to bring their calculators to class, there seemed to be a negative aura when , "Now take out your calculator and . . ." or similar such directives were voiced. Total participation in calculator activities was usually not accomplished. The calculator did not seem to motivate many
in the two CM classes. Informal consideration and recall indicate to the investigator that there was a widely held view, in all three classes, that the calculator "calculates" and that is all. For example, the typical student did not seem to accept the calculator as an actualized, hands-on, function machine of the sort we read about in lesson one and talked about and demonstrated in class. This state of mind impeded the incorporation of the calculator into discussion. Perhaps there was a motivation aspect—the students, by and large, just were not interested in functions and the incorporation of the calculator did not impress them.

The investigator observed that all three classes did poorly on the pretest, many students did not even try. Twenty-seven did not take the pretest so those students were not included in the study.

The students noticeably did not do their daily work well. Many waited until the solutions were posted then copied the offering into their notebooks. The investigator sees this practice as probably causing lower achievement and low reliability ratios. Several times the posted model solutions were stolen. This caused irritation and inconvenience; possibly it affected learning.

The investigator received about a dozen complaints from students who had learned that other Basic Collegiate
Mathematics classes did not have to take the unit on functions. On the other hand several students did voice firm approval of the functions unit to the instructor. It seems clear that there was some controversy about the functions unit. In particular there seemed to be some hesitance to accept the unit as part of the course. The hand out lessons and exercises were treated casually—not handled diligently. The students' diligence was needed to learn about functions. The change of format (from regular textbook to hand-out materials) may have partially caused this effect or perhaps the lack of diligence was already established but became noticeable when lessons were handed out instead of assignments in the text. Either way, it seems that lack of diligence could affect the study.

It should be stated that, although the investigator taught all three classes and made conscientious effort to teach content equivalently each day to each class, it is possible that teacher differences may have been coupled with the different presentations in the CM sections.

As indicated previously, Maor (1976), Bell (1976), Ockenga (1976), Pollack (1977) and O'Loughlin (1975) stated or implied that the calculator should be a positive influence in learning about functions. This study did not support that view. Concept learning is an involved process with many
variables. Roberts (1980) says, "Obviously, positive results related to conceptual benefits of calculator usage would not be expected to occur as often as simple computational benefits because conceptual acquisition is a more complex task" (Roberts 1980, p. 84). It may be that integration of the calculator into a unit on functions is a treatment which does not effect an improvement in achievement, at least not singlehandedly.

On the other hand, it is possible that there was an effect due to the integration of the calculator into a unit on functions, but that the effect was masked by the low reliability of the sub-unit tests. A second aspect which may possibly have affected the study is that the CM sections were called upon to abruptly adjust to using the calculator every day in class and to employ it as a learning aid. For example, in Lesson One, the CM sections were advised to consider their own calculator as an embodiment of a function -- a function machine. It was a machine which existed, not just a diagram on paper. It was a machine which took an input and generated an output; no reckoning with written numerals and/or symbols on paper was needed. It was a machine that could show how $f(x) = 4x + 2$ assigns 10 as an output when the input is 2. Most students seemed not to recognize this interpretation and did not employ their calculator to realize the idea of a
function. Perhaps if the classes had been accustomed to CM instruction where they were frequently called upon to use a calculator as an aid to concept learning then an effect might have been measured.

Conclusions

As previously indicated, the problem of this study was to answer two questions and test six null hypotheses. The first question was: Can a Calculator-Modulated unit on functions cause more effective learning with Basic Collegiate Mathematics students than a Calculator-Assisted unit? The investigator observed that the students in the CM classes did not enthusiastically receive integration of the calculator into classroom instruction. The data indicate that the students in the CM sections learned no more effectively than the students in the CA section.

The second question was: Do any sub-units in the CM unit cause more effective learning than similar CA sub-units? No sub-unit in the CM unit on functions was found to cause more effective learning than the similar sub-unit in the CA unit.

The findings of this investigation indicate that the following conclusions seem reasonable:

1. Basic Collegiate Mathematics students do not learn any more effectively in a CM unit on functions than they learn in a CA unit on functions.
2. Basic Collegiate Mathematics students do not learn any more effectively in any sub-unit of the CM unit than they learn in a similar sub-unit of the CA unit.

Recommendations for Further Research

The experiment could be replicated. However, the investigator feels that an analogous study in which programmable calculators are used should be developed. It is felt that in such a study the CM treatment would be strengthened in the following ways:

1. The student could have more experience interpreting a function as a set of instructions because, in programming the calculator to evaluate a function, the student would be using a set of instructions.

2. The student could have a very realistic example of a function machine because, once programmed, all the calculator needs is an input and then the pressing of a single key would cause an output. The variety of the functions it is possible to exemplify in this manner would be large.

3. Graphing of functions could be more efficient. Once programmed, the calculator could quickly provide a chosen number of solutions to the equation \( y = f(x) \). These could then be graphed. If a pattern is apparent, the graph of \( f \) may be finished. If a pattern is not apparent, the
previously programmed calculator stands ready to provide more solutions which, when plotted, may indicate a pattern for the graph of \( f \).

The proposed study could be carried out so that all students would use identical calculators. This would remove some potential obstacles to classroom discussion and instruction. This would also provide some possible opportunities for intraclass activities and/or small group activities.

More generally, it is recommended that other concepts in the mathematics curriculum could be "Calculator Modulated" and then studied to determine if a CM unit could cause conceptual benefit. Such studies are needed to resolve the issue of the concept-formation benefits of calculators.
BIBLIOGRAPHY


Suydam, Marilyn N. The Use of Calculators in Education: A State-of-the-Art Review. Calculator Information Center, Ohio State University, Columbus, Ohio. April, 1979.


Appendix A:

INFORMATION FORM
Name______________________________

Section 1 3 4 (Circle one)

Age______________________________

Sex: Male Female (Circle one)

Number of units of math completed in grades 9-12.__________________________
(For example, 2 semesters of algebra and 1 semester of geometry would be 1.5 units.)

Please list any math courses you have taken after completing high school.

How long has it been since you completed your last high school math course?__________________________

Please indicate your rank in your high school graduating class.

_____ in top one-fourth (75% of class was below me)

_____ in top one-half (50% of the class was below me)

_____ in top three-fourths (25% of the class was below me)

_____ in the bottom one-fourth (75% of the class was above me)

I would rate my calculator experience as: (circle one)

none     beginner     intermediate     advanced

I have been using a calculator for ________ years.
Appendix B:

LESSONS
Lesson One

We are all acquainted with the arithmetic process of doubling a quantity. This process illustrates an idea that I want you to examine -- the idea is that every number is corresponded with a second number called its double in this process. You may think of a thread which ties each number with its double. This tie or correspondence is called a function. As another familiar example of a function we may note that in a retail store every salable item is corresponded with a price. (Sometimes the item is literally "tied" to its price tag.) In both examples we see that every element in one set is tied to one and only one element in a second set. Sometimes a diagram is used to visualize a function such as

\[
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 4 \\
3 & \rightarrow 6 \\
4 & \rightarrow 8 \\
5 & \rightarrow 10 \\
6 & \rightarrow 12
\end{align*}
\]

which indicates the doubling function on the set of the first six counting numbers. It is customary to think of a function as a tie between two sets of elements in such a way that a diagram may be drawn so that each element in the first set is tied by an arrow to only one element in the second set.

Some diagrams do not illustrate functions:

\[
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 5 \\
3 & \rightarrow 9 \\
0 & \rightarrow 0
\end{align*}
\]

are diagrams where every element in the first set is tied to one or more elements in the second set and do not represent functions. You should be able to invent many many functions by employing diagrams involving two lists connected by arrows. It should be recognized that not all of these functions will be important and valuable. In fact, it is probable that most of your invented functions may make no sense in terms of arithmetic or practical use; functions like

\[
\begin{align*}
7 & \rightarrow 3\overline{3} \\
8 & \rightarrow -4\overline{3}1 \\
9 & \rightarrow 15.7 \\
10 & \rightarrow -21.2
\end{align*}
\]

are invented and of little value except as personal creations which show you are able to diagram functions. However, you could invent
possibly useful functions like \[
\begin{align*}
1 & \rightarrow 3 & a & \rightarrow A \\
2 & \rightarrow 6 & b & \rightarrow B \\
3 & \rightarrow 9 & c & \rightarrow C \\
4 & \rightarrow 12 & d & \rightarrow D
\end{align*}
\]
ties between two sets which you recognize from experience. In particular these functions could be called the tripling function and the alphabet function. We will in most cases be interested in functions that make sense but for now you are only asked to notice and remember that a function, any function, is a tie that connects each element in one set (the domain) to one and only one element in a second set (the range). You are also asked to observe that a tie like 

is a function. Furthermore, it is a function which makes sense if we think of it as illustrating the assignment of Tom, Jane, Bill and Mary to Room 10 at school. This last function indicates that while a function must tie each element in the first set to only one element in the second set, that element in the second set may be an object to which more than one element in the first set is tied. We will say more about this later. It is customary to call the first set, the domain of the function and to refer to the second set as the range of the function.

Another way to interpret a function is to think of it as a kind of an operation which may take place in a sort of "machine" or black box. Under this interpretation an element from the first set is fed into the machine which is then activated to operate on that element and produce exactly one element in the second set. The machine (black box) is fiction, but, as fiction, dramatizes the idea that a function operates on one element from the first set and transforms it--"ties it"--into one and only one element of the second set. You may think of, and actually envision, a machine that operates on elements to manufacture related elements--an input-output machine. It will be a function if for every element input there is one and only one element output. Various pictorial aids shown shown below may help cement the idea into your mind. Remember, in the machine interpretation: an element from the first set goes in and then an element from the second set comes out.
Since most of the functions we will use have domains and ranges which are numbers, the machines in the drawings have been shown as having input numbers and output numbers. Note the input, the output, or both input and output could be non-numerical—the main idea is to intuitively interpret a function as being a machine which has been purposely designed so that whenever an input is made then a single output will result. Notice particularly that a function machine is reliable! Every time the same input is made the output will be the same. For example, if the machine accepts numbers, you put in 7 and the machine puts out 3, then the machine always puts out 3 when 7 is put in—you can depend on it! Now it may be possible that other inputs may cause 3 to be the output, but 3 will always be the output of this particular function when the input is 7.
Allow me to comment that the "machines" in the illustrations are more "artistic" than necessary. Their purpose is to catch your attention and allow you to entertain the idea that a function accepts inputs and produces one output for each input. Cement this idea into your understanding! Once you have done so, a simple figure like

\[ \text{Function} \]

will suffice as a schematic representation for a function.

We have seen that a function may be interpreted as a correspondence between the domain and the range. A diagram in which arrows "tie" domain elements to range elements is a good way to indicate a correspondence. We have seen a function may be interpreted as a machine having an input and output. A box-like figure is a good way to indicate a machine. Next we will consider another way to interpret a function.

A function may also be thought of as a set of instructions. Such instructions when followed carefully will produce exactly one element. For example, a set of instructions could be: Take a counting number and multiply it by two. Such instructions describe what we have called the doubling function with a domain of counting numbers. Sometimes instructions may be very involved. Special symbols and formulas are frequently devised to make the instructions easier to use. Whether the instructions are simple or complex, keep in mind that the instructions describe a function only if when carried out they produce a unique element.

When discussing a function we might say it sends 5 into -3, or 5 is the preimage of -3, or -3 is the image of 5. These are ways of noting that a diagram would have an arrow from 5 in the domain to -3 in the range. That is, the function ties 5 to -3. In our subsequent discussions you should become quickly acquainted with this and other terminology.
In Lesson One you learned that a function can be viewed or interpreted in several ways. We will now enlarge our interpretation and increase our efficiency by introducing some terms and symbols. Recall that a function always involves two sets of objects (perhaps numbers). The first set being input or pre-images and the second set being output or images. Recall we have a custom of naming the first set, the domain of the function, and we call the second set the range of the function. Frequently we use $x$ as a symbol to indicate or represent any element in the domain and we call $x$ an independent variable which leads us to think next of the image of $x$ as a dependent (on $x$) variable whose symbol is $y$, the next letter following $x$ in the alphabet. Now we can do something neat and powerful: instead of having to diagram a function by tying elements in the domain to elements in the range with arrows we may indicate a diagram by $x \mapsto y$. Here the $f$ may be thought of as the name of the function. This symbolic diagram is especially useful if we can formulate $y$. For example, if $f$ multiplies the natural number preimage by two (the doubling function), we could write $y = 2x$ so that, where $x$ represents a natural number, $x \mapsto 2x$ is a neat, short way to represent the function whose diagram is\[1 \mapsto 2, \quad 2 \mapsto 4, \quad 3 \mapsto 6, \quad 4 \mapsto 8, \ldots\] You should observe that this function has the natural numbers for its domain so that our diagram can not be completely written out but depends upon you to catch on to a pattern and realize that the pattern continues indefinitely (as the three dots signify). By employing an independent variable $x$ to represent any counting number and the dependent variable $y$, which in this case equals $2x$, we are able to symbolically duplicate the diagram $1 \mapsto 2, \quad 2 \mapsto 4, \quad 3 \mapsto 6, \quad 4 \mapsto 8, \ldots$ with the symbolic diagram $x \mapsto 2x$ where $x$ is a counting number.

Next observe that it needs to be clearly understood what is the domain in a given function, since to change the domain is to change the function. For example
if $x$ represents an even counting number then the function is represented by
$$
\begin{align*}
2 & \rightarrow 4 \\
4 & \rightarrow 8 \\
8 & \rightarrow 16
\end{align*}
$$
is not the same doubling function as that of the preceding paragraph.

The notation $x \rightarrow 2x$ where $x$ is an even natural number could be a symbolic diagram representing this different function. Note the name of this different function is $g$.

Not all functions allow us to find the image of every $x$ in the domain by doing a finite number of arithmetic operations on $x$. A familiar example is the principal square root function. Recall that this function sends $x$ into that positive $y$ such that $y \cdot y = x$. You have learned that if $x = 2$ then the positive number $y$ such that $y^2 = 2$ is an irrational number represented by $\sqrt{2}$. In symbols $x \xrightarrow{\text{psqr}} y$ for $x > 0$ or $x \xrightarrow{\text{psqr}} \sqrt{x}$. Here we have named the function psqr (principal square root abbreviated). The name of the function may be important, as in this case, because it tells us that we have given this function special recognition. Every reader should know (because of the name) that here is a function whose images are sometimes not rational. As noted above we can never determine in a finite number of arithmetic operation the exact principal square root of certain numbers ($x = 2$ for example). However, by writing the symbolism $x \rightarrow \sqrt{x}$, we use the notation $\sqrt{x}$ to represent the image of $x$, that is the number $y$ which has the characteristic that $y \cdot y = x$. In short, $y = \sqrt{x}$ where $\sqrt{}$ is a special symbol used to tell the reader a message. We will see several special symbols in our work and we will learn the message each special symbol conveys.

$f(\ )$ is another useful symbolism. $f(\ )$ is reminiscent of our machine interpretation of a function. You put $x$ into $f(\ )$ to get $f(x)$ which equals $y$—in short, $f(x) = y$. For example, if $f$ is the doubling function, $f(x) = 2x$; in particular $f(3) = 6$ or an input of 3 yields an output of 6 in the doubling function. Another example, $g(x) = 3x + 3$ tells us that an input of $x$ yields an output of three times $x$ plus 3; in particular $g(5) = 3 \cdot 5 + 3$ or $g(5) = 18$. That is if $x = 5$ then $g$ sends 5 into 18. This symbolism works very well as an alternative
to a diagram \( \frac{1}{2} \rightarrow 9 \) of \( x \rightarrow 3x + 3 \). In fact this very symbolism is so widely used that it has come to be called functional notation. It is customary to read \( f(x) = 3x + 3 \) as "\( f \) of \( x \) equals three times \( x \) plus 3." Read \( f(3) = 12 \) as "\( f \) of three equals twelve."

You may have studied polynomials. If so, you will recognize \( 3x + 3 \) as a first degree polynomial, \( 2x^2 - 3x + 1 \) as a second degree polynomial, and

\[
ax^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n
\]

as an nth degree polynomial. What we can now see is that polynomials are functions of the form \( x \mapsto ax^n + a_1x^{n-1} \ldots + a_{n-1}x + a_n \). You may be assured that these polynomial functions are among the most useful functions in algebra.
Lesson Three

You have already seen an example of special symbolism in \( y = \sqrt{x} \).

There are several functions which involve their own special symbolism. We will consider some of these here and meet more later in our course of study.

Recall that we ordinarily use letters such as \( f, g, h, F, G, H \), as names for functions so we can write \( f(x) = \sqrt{x} \) as an alternative to \( x \sqrt{x} \) or \( y = \sqrt{x} \) when describing the principal square root function.

Now we will consider a familiar question which involves a special function. When someone asks how old you are you usually respond with an answer such as eighteen or twenty-one or some similar natural number. A bit of thinking indicates that the response is accurate only if it happens to be your birthday. When you answer by stating a natural number (like 18) you are using a special function called the greatest integer function (sometimes also called the birthday function). We usually describe this function by stating: \( f(x) \) represents the greatest integer which is less than or equal to \( x \). Mathematics students use \( [ \ ] \) as a special symbol for this function and write \( f(x) = [x] \). Here \( x \) could represent your actual age, say 18 years and 219 days, \( x = 18 \frac{219}{365} \) then \( f(18 \frac{219}{365}) = [18 \frac{219}{365}] = 18 \). Observe that you are expected to see the special symbol \( [18 \frac{219}{365}] \) and make the connection (correspondence) with 18, the greatest integer which is less than or equal to \( 18 \frac{219}{365} \).

Another familiar function is indicated by the answer to: How far is a certain point \( x \) from the origin on a number line? The answer may be three or 4.7 or even zero but always non-negative. Such a response is called the absolute value of \( x \) or the absolute value function. We write (again a special symbolism) \( f(x) = |x| \) to represent the distance of \( x \) from the origin. Once again note that you are expected to see the symbol \( |x| \) and make the connection that this symbol
|x| means the distance of x from the origin. For example, if x = -4.7 then |x| = 4.7.

As another somewhat special symbolism we use y = 5^x to represent a special function. You may recall from studying fractional exponents that 5^{3/2} = \sqrt{5^3} similarly 5^{5/3} = \sqrt[3]{5^5} = \sqrt[3]{3125}. Here (once again) we are expecting you to see the special symbol 5^x and make the connection, that if x = \frac{5}{3} then 5^x means \sqrt[3]{5^5} (which is about 14.6). This is an example of an exponential function. In place of y = 5^x we sometimes use x \exp 5^x or we can write \exp_5(x) = 5^x (read exponential of x to the base 5 equals 5 to the x). Actually we could use any positive number, say, a which is not one (a \neq 1) as a base and write \exp_a(x) = a^x to describe an exponential function. Here again if \( x = \frac{2}{3} \) then \( a^x = a^{2/3} \) means \( \sqrt[3]{a^2} \) and we are expected to recognize this meaning.

We have seen that some functions may be described by a rather straightforward formula such as f(x) = 2x + 3 or g(x) = x^2 - 2. We have seen some functions which involve some rather special symbolism such as F(x) = \lfloor x \rfloor, G(x) = |x|, \exp_4 = 4^x. Now we need to recognize that an equation does not always describe a function. For example \( y^2 = x \) does not describe a function of x, because every positive number x may be tied to two numbers y. A partial diagram looks like

\[
\begin{array}{c}
4 \rightarrow \sqrt{2} \\
9 \rightarrow \sqrt{3}
\end{array}
\]

where we see that if x = 4 then y could be 2 or -2 (since \( 2^2 = (-2)^2 = 4 \)). This shows us that the diagram is not the diagram of a function.

In conclusion, there are many equations which do not describe functions. It is improper to think of a function as being an equation for two reasons:

1. Some functions cannot be described by an equation.
2. Some equations simply do not describe a function.
Lesson Four

In the earlier lessons we saw that a function may be interpreted in three ways: (1) a diagram which ties (connects) elements called pre-images in the domain to elements called images (or values) in the range, (2) a machine which operates on an input element and transforms it into a unique output element or value, (3) a set of instructions which when followed exactly will connect an element in the domain to exactly one element in the range.

Today we will consider a fourth way to interpret a function. Some of you will probably be already familiar with this idea which we call a graph. To learn about graphs we will need some background from geometry having to do with the plane. Take any plane and divide it into top and bottom half-planes by using a horizontal number line, that is a line which goes from left to right and has points labeled on it. The plane is next divided into left and right half-planes by introducing a vertical number line which intersects the horizontal number line at zero so that the zero on the vertical line and the zero on the horizontal line are exactly together. We call this intersection point the origin of the coordinate plane. By referring to these axis we are able to name any point in the plane uniquely. As indicated in the figure below we will call the horizontal number line the x-axis and the vertical number line the y-axis. You may want to think of the system as like a map, the x-direction being east and the y-direction as being north. For each pair of numbers, for example (3,2), we assign a point in the plane.
by moving from the origin in the x-direction the number of units indicated as
the first number in the pair and then moving in the y-direction the number of
units indicated by the second number in the pair. Therefore, (3,2) is 3 units
east (the x-direction), then 2 units north (the y-direction) located in the upper
right side of the plane as shown. It is practical to interpret a negative 1st
coordinate as in (-4,1) to mean move the opposite of the x-direction from the
origin and similarly if the second coordinate is negative as in (2,-3). If both
coordinates are negative as in (-3,-2) this means to move 3 units from the origin
opposite to the x-direction and then move 2 units opposite to the y direction.
Thus (-3,-2) is shown in the left half-plane and below the x-axis. The first
coordinate of a pair is called the x-coordinate while the second coordinate is
called the y-coordinate. This way of naming points is often called a rectangular
coordinate system.
Now let us recall that if an element \( a \) in the domain (or first set) is tied to an element \( b \) in the range (or second set) then we can see \((a,b)\) as a pair of numbers showing up as a point on our rectangular coordinate system. If we use the notation \((x,y)\) to denote \( x \) is tied to \( y \) by some function \( f \) then when \((x,y) = (2,4)\) it means \( f(2) = 4 \). Furthermore, if we are able to locate and block in all the points \((x,y)\) the resulting picture is a visual interpretation of the function \( f \). We call this picture the graph of \( f \). An example or two will help.

Example 4.1. Let \( f(x) = 2x \), \( x = -1, 0, 1, 2, 3 \). Clearly \(-1\) is tied to \(-2\), \(0\) to \(0\), \(1\) to \(2\), \(2\) to \(4\) and \(3\) to \(6\). We may write this as a diagram:

\[-1 \rightarrow -2 \]
\[0 \rightarrow 0\]
\[1 \rightarrow 2\]
\[2 \rightarrow 4\]
\[3 \rightarrow 6\]

or as a collection of "points" \((0,0)\), \((1,2)\), \((2,4)\), \((3,6)\). We could draw (blacken in) the set of "points" to get a graph of \( f \) as in the figure.
Example 4.2. Let \( g(x) = 3x - 2 \), \( x = -2, 0, 2, 4 \). We may interpret \( g \) as a collection of points \{(-2,-8), (0,-2), (2,4), (4,10)\} and blacken in or graph \( g \) accordingly as in the following figure.

In any graph of a function it is easy to identify the domain. All one needs to do is make a list of the \( x \)-coordinates of the points on the graph; that is, a list of the first coordinates of the blackened in points on the picture. The domain of \( g \), whose graph is above, is \(-2, 0, 2, 4\). Similarly the range is the set of values or \( y \)-coordinates. In the graph of \( g \) above the range is \(-8, -2, 4, 10\).
If a function may be described by an equation \( y = f(x) \) then the graph may be interpreted as a picture of all the points which solve the equation; that is, a visualization of all the points which make the equation \( y = f(x) \) true. For example, because the point \((0,-2)\) appears in the graph of \( g \), I can expect that \(-2 = g(0)\) will be a true sentence. Checking, I see that \( g(0) = 3(0) - 2 \) or equivalently the sentence \( y = 3x - 2 \) becomes true if \(-2 \) replaces \( y \) and \( 0 \) replaces \( x \). This is to say the point \((0,-2)\) is a solution to the equation \( y = 3x - 2 \). We are going to learn later that some equations which describe functions have many many solutions and thus their graph contains many many points.

It is most important that we be able to interpret a function by thinking of it as a graph or picture where the blackened in points are indicating that every element, say \( x \), in the domain is tied by the function to exactly one element in the range, say \( y \), where \( y = f(x) \).
Lesson Five

It is time to recognize that the functions we have met so far have been defined on integers or subsets of the integers. That is to say the domains (of definition) of the functions we have seen have been like $x = -4, -3, ..., 3, 4$ or $x = ..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...$. We intend to indicate that it makes sense to define a function on a subset of the real numbers. Toward that end, let us reconsider the function $f(x) = 2x$, $x = \text{any integer}$. Let us interpret this function as a graph—it will be an infinite set of points $(x, 2x)$. See graph below.
Now it seems agreeable that since we have called \( f(x) = 2x \) the doubling function, it should be possible to have \( f \) send 1/2 into 1 or 3/4 into 1.5 or 7/8 into 7/4. In fact, all this and more can be accomplished by stating that \( f(x) = 2x \) where \( x \) is a real number; i.e., the domain of \( f \) is the set of real numbers. The effect of this change in domain is most notable in the way the graph is changed. You probably observed the way that the points on the graph of \( y = 2^x \) "line up" as you plot the picture using integers as the domain. By enlarging the domain to include all real numbers we find that points like \((1/2,1), (1.21, 2.42), (\sqrt{2}, 2\sqrt{2})\) will solve the equation \( y = 2^x \) and therefore belong to the graph of \( f \). It is pleasing to see (literally see) these points appear along side and inbetween the points of the graph of \( y = 2^x \) where \( x \) is an integer. In fact, the points of the graph of \( y = 2^x \) where \( x \) is a real number fill out a line and we say the graph of \( y = 2^x \) is a line.

Next we will reconsider the function \( g(x) = x^2 - 1, x = -3, -2, -1, 0, 1, 2, 3, 4. \) The graph of \( g \) is the set of points \((-3, 8), (-2, 3), (-1, 0), (0, -1), (1, 0), (2, 3), (3, 8). (4,15). \) This graph is drawn below.
Now since \( g \) sends \( x \) into \( x^2 - 1 \) when \( x \) is an integer between -4 and 5 we could inquire what effect is caused by allowing \( x \) to take on non-integral values like -2.5, -1.25, -.5, .5, 1.5, \( \sqrt{2} \), 2.5, \( \sqrt{3} \), 3.5, etc. If we allow this enlargement of the domain of \( g \) it is satisfying to see that points like (-2.5, 5.25), (-1.5, 1.25), (-.5, -.75), (.5, -.75), (\( \sqrt{2} \), 1), (2.5, 5.25), (\( \sqrt{3} \), 2), (3.5, 11.25) fall into place (see figure) between the points of the graph of \( y = x^2 - 1 \), where \( x \) is an integer. In fact, the graph of \( y = x^2 - 1 \) where \(-3 \leq x \leq 4\), \( x \) a real number, fills in the points between the points of \( y = x^2 - 1 \), \( x = -3, -2, \ldots, 3, 4 \) to form a nice, smooth curved line (see figure) which is an arc of a parabola.
These two demonstrations of the effect of enlarging the domain of a function are examples of a general idea. A function will be changed whenever its domain is changed and since a graph is one of our interpretations of a function it (the graph) will be changed also. Usually, when a function is formulated, the instructions may be used for most of the real numbers on some interval of real numbers. For example, \( y = 2x - 3 \) where \( x \) is an integer, \(-2 \leq x \leq 5\) will yield a point graph. But \( y = 2x - 3, -2 \leq x \leq 5\) is taken to mean \( x \) is a real number in the interval between \(-2\) and \(5\) because no mention that \( x \) is an integer; this will yield a line segment graph. It is customary when working with functions to simply write the formula such as \( y = 2x - 3 \) or \( y = x^2 + 5 \) or the special symbolism such as \( y = \sqrt{x} \); the reader is then expected to recognize the domain of the function to be all those real numbers which are appropriate to the formula or to the special symbolism. For \( y = 2x - 3 \) the domain would be (recognized as) all real numbers. For \( y = x^2 + 5 \) the domain is recognized as all real numbers. For \( y = \sqrt{x} \) the domain is all non-negative real numbers. For \( y = \frac{1}{x} \) the domain is all non-zero real numbers. We customarily will indicate the domain of the function if it is a subset of the real numbers. For example, \( y = x + 2, 0 \leq x \leq 4\) is graphically a line segment from \((0,2)\) to \((4,6)\). \( y = x^2 + 1, -3 \leq x \leq 4\) is an arc of a parabola as we saw earlier (previous figure). Such functions are said to be defined on an interval.

If the interval of definition is finite in length the function usually has a graph which may be drawn completely. When the interval is infinite in length then the graph may never be drawn completely; however, it is generally satisfactory to draw a partial graph containing the most interesting parts of the graph while expecting the reader to recognize the situation; ie, the reader should recognize "less interesting" parts of the graph were not drawn. For example, in graphing \( y = f(x) = \sqrt{x}, x \geq 0 \) we note that the interesting part of the graph is for small, say less than 100, numbers.
On this finite interval the graph of \( f \) comes up rather quickly from the origin and then goes up rather less quickly as \( x \) increases; it (the graph) can be depended upon to go up more and more slowly as \( x \) gets bigger. The reader should know this so accordingly the graph of \( f(x) = \sqrt{x} \) is drawn (but not completely) as in the figure below.

For a second example consider \( y = x^2 \), the squaring function. Here the domain is all real numbers. As in the last example the graph of this function may never be drawn completely. However, we note that the interesting part of the graph occurs above a rather small interval about the origin. As \( x \) gets away from the origin—in both directions—the graph of the function \( y = x^2 \) goes up very steeply. Hence it is satisfactory to submit the drawing below as a graph of \( y = x^2 \) as long as the reader knows that this is not a complete graph. To help the reader recognize his responsibility, the writer can do at least two things: 1) label the graph carefully, 2) where the graph is incomplete indicate by using three dots ... or using an arrow. Practice both 1) and 2) when graphing. A properly labeled drawing will not be
It should be noticed that the size of the unit used on the x-axis and the y-axis may be chosen by the person drawing the graph. Indeed it is even useful to sometimes choose a unit of different size for one axis than that of the other. It is important that the units whatever their size be clearly and carefully marked so the reader may see the points on the graph. Only if the axes units are marked can the reader understand the graph! If the units are not labeled when graphing a function, the domain is not clear and hence the connection (tie, correspondence, function) from pre-image to image is not defined. Always choose what seem to be appropriate units and then label the axes accordingly!
In this lesson we narrow our discussion. We are now familiar with some general aspects of functions; it is time to look at two classes of functions in particular. The first is the class called linear functions. This class is characterized by the formulation $y = f(x) = ax + b$. It is a fact that any linear function has a graph which is a straight, non-vertical line. To convince yourself of this you might choose any linear function; say, $f(x) = 1.2x + (-1)$, and observe that by choosing easy values in the domain like $-15, -10, -5, 0, 5, 10, 15$ you get points $(-15, -19), (-10, -13), (-5, -7), (0, -1), (5, 5), (10, 11), (15, 17)$ and notice that as $x$ increases $5$, $f(x)$ increases $6$. The points located on an axis system do line up—as indicated below—to provide examples of points on the graph or solutions to the equation. We may simply fill in the rest of the points on the graph by connecting the existing points.
To widen our understanding of linear functions and their equations we will study the graph of a (any) linear equation. Observe that a graph of a linear function may slant down from left to right, slant up or not slant at all (horizontal). It is important to realize that amount of slant a given line possesses is a constant. If we look at a graph and choose a pair of points \( P_1(x_1, y_1), P_2(x_2, y_2) \) then choose any other point \( P(x, y) \) we can see that \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_2}{x - x_2} \) because of similar right triangles. This ratio of rise \( y_2 - y_1 \) to run \( x_2 - x_1 \) represents the slant a line possesses and is called the slope. In symbols, \( m \) is used for the slope and \( m = \frac{y_2 - y_1}{x_2 - x_1} \). In case \( m \) is positive the line goes upward from left to right. In case \( m \) is negative the line goes down from left to right. In case \( m = 0 \) the line is horizontal. Looking again at the graph of a line whose slope is \( m = \frac{y_2 - y_1}{x_2 - x_1} \) and contains \( (x_1, y_1) \) we see that for any \( (x, y) \) on the line \( \frac{y - y_1}{x - x_1} = m \) or \( y - y_1 = m(x - x_1) \). That is, we have described the line algebraically. It is interesting to note that if \( (x_1, y_1) \) is the point where the line intersects the y-axis then \( x_1 = 0 \) so that the equation becomes \( y - y_1 = m(x - 0) \) or \( y = mx + y_1 \), a familiar sight.

The upshot is that anytime we have an equation of form \( y = f(x) = ax + b \)
we can tell from experience that the slope is $a$ and the $y$ intercept is $b$.
For example, if $y = 1.2x + (-1)$ we see that the slope is $1.2 = \frac{12}{10} = \frac{6}{5}$ and
the $y$ intercept is $-1$. This agrees with our graphing of $y = 1.2x + (-1)$ as
discussed earlier in this lesson.

To end our discussion of linear functions we observe that any
equation of form $Ax + By + C = 0$ may be written as $y = -\frac{A}{B}x - \frac{C}{B}$ as long
as $B$ is not zero. Here again we have a familiar form, $y = ax + b$
$(a = -\frac{A}{B}, b = -\frac{C}{B})$, which is our linear function. In the special case, if
$B = 0$ we have an equation like $x - 3 = 0$ ($A = 1, C = -3$ in $Ax + Oy + C = 0$).
Such equations have graphs which are vertical lines. It should be
emphasized that neither this equation nor its graph represents a function.

The second class of functions are called quadratic functions.
This class is formulated by $y = f(x) = ax^2 + bx + c$, $a \neq 0$. These functions
have graphs that are bowl-shaped. The bowl is right side up if $a > 0$ and
upside down if $a < 0$. The squaring function $y = x^2$ is a good example to
recall; its graph is bowl-shaped and right side up, in this example
$a = 1, b = 0, c = 0$. Now if we wish to graph a quadratic function, say
$y = -3x^2 + 11x - 6$, we may proceed by hunting solutions and plotting them
repeatedly until we have enough points to see a pattern. Alternately (and
preferably) we can recognize the function as quadratic with $a < 0$ thus the
graph is bowl-shaped upside down. To continue, we could recognize that
by using some algebra to write the function as $y = -3\left(x^2 - \frac{11}{3}x + \_\right) - 6 + 3\cdot\_\_\_$
where the blanks are to be filled by $\left(\frac{1}{2}\cdot\frac{11}{3}\right)^2 = \frac{121}{36}$. Notice choosing
$\frac{121}{36}$ to fill the blanks will complete the square so that, after simplifying,
we have $y = -3\left(x - \frac{11}{6}\right)^2 + \frac{49}{12}$. Here we have an equation which informs us
that, as $x$ gets big, $y$ becomes very negative and, as $x$ goes negative,$y$ becomes more negative but, for $x$ near $\frac{11}{6}$, $y$ is positive. In fact, when
$x = \frac{11}{6}$, $y$ is the most positive it can be—in rough language when $x = \frac{11}{6}$ we are at the top or vertex of our upside down "bowl-shaped" graph at $(\frac{11}{6}, \frac{49}{12})$. It is useful to locate a couple of points to the right of the vertex. (Here $(3,0)$ and $(4,-10)$ are about 1 and 2 horizontal units respectively from $(\frac{11}{6}, \frac{49}{12})$. Now, since the graph of a quadratic function is "symmetrical" to a vertical line through its vertex, we see that $(\frac{4}{6}, 0)$ is symmetric to $(3,0)$ similarly $(\frac{-2}{6}, -10)$ is symmetric to $(4, -10)$. Hence, by plotting $(\frac{11}{6}, \frac{49}{12})$, $(3,0)$, $(4, -10)$ and using symmetry we have 5 nice points which "show us" the pattern (bowl-shaped) of the graph.
As another example we consider \( y = -0.5x^2 + 3x - 6.5 \) proceeding as before. \( y = -0.5 \left(x^2 - 6x + \_\right) - 6.5 + 0.5 \cdot (\_) \) and filling in the blanks with \( \left(\frac{6}{2}\right)^2 = 3^2 = 9 \), then after simplifying we have \( y = -0.5(x-3)^2 - \frac{13}{2} + \frac{1}{2} \) \( \left(9\right) \) or \( y = -0.5(x-3)^2 - \frac{4}{2} \). This last equation tells us that as \( x \) increases \( y \) gets more negative, as \( x \) decreases \( y \) gets more negative, but \( y \) will be least negative if \( x = 3 \). We call \((3, -2)\) the vertex of the graph. We locate \((5, -4)\) and \((7, -10)\) which by symmetry yield \((+1, -4)\) and \((-1, -10)\), giving 5 nice points to use. The graph is pictured below. Observe that the graph with \(|a| < 1\) is wider across than the graph with \(|a| > 1\)--this is true in general.
Lesson Seven

Remember how in algebra we use symbols for numbers. Symbols like b, c, s and t may be used and in essence treated as if they were numbers. For example, we learn to multiply a times x and add b to x by writing ax and b + x respectively. Now these symbols are treated as if they were numbers and so they may be subtracted, divided or appear in equations and lots more. What we propose to discuss next is rather similar—we suggest that functions may be considered as algebraic objects rather like numbers.

To begin let f and g be two functions whose domains are the same then we define the sum function f + g to be \((f+g)(x) = f(x) + g(x)\). The difference function \(f - g\) is defined by \((f-g)(x) = f(x) - g(x)\). The product function \(f \cdot g\) is defined by \((f \cdot g)(x) = f(x) \cdot g(x)\). The quotient function \(\frac{f}{g}\) is defined by \(\frac{f}{g}(x) = \frac{f(x)}{g(x)}\) for \(g(x) \neq 0\).

It is nice to be able to tell you that you have some acquaintance with these new functions. In algebra you worked with expressions like \(x + 11\) and \(x + c\). These may be viewed as sum functions: \(f(x) = x\) and \(g(x) = 11\) yield \((f+g)(x) = f(x) + g(x) = x + 11\) and \(f(x) = x\) and \(g(x) = c\) yield \((f+g)(x) = f(x) + g(x) = x + c\). Also expressions like \(6x\), \((x+1)(x-3)\), \(\frac{x+2}{x+1}\) are a product function of \(f(x) = 6\), \(g(x) = x\), a product function of \(f(x) = x + 1\), \(g(x) = x - 3\) and a quotient function of \(f(x) = x + 2\), \(g(x) = x + 1\). Indeed it is nice to notice that whether you add, subtract, multiply or divide one function by another you always end up with another function. In more advanced mathematics it is important to be able to recognize the various component functions which together make up a given function. For example, \(f(x) = 6x + 7\) is made up of a product function \(g(x) = 6x\) and a constant function \(h(x) = 7\).
added together. In fact all linear functions are made up of a product function \( g(x) = ax \) and a constant function \( h(x) = b \). Similarly quadratic functions are made up of a product function \( g(x) = ax^2 \) and a linear function \( h(x) = bx + c \) added together. We could proceed to illustrate that any polynomial function is made up of a number of more simple functions whose "sum" is the polynomial \( P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots \) 

\[ + a_{n-2} x^2 + a_{n-1} x + a_n \]. One benefit of viewing a function as an algebraic combination of other (simpler) functions is that graphing can be expedited. For example, the graph of \( y = 2x - 5 \) may be viewed as the sum of \( g(x) = 2x \) (the doubling function) and \( h(x) = -5 \) a constant function. The doubling function and the constant function are easy to graph. Now choose any \( x \),
find \( (x, g(x)) \) then locate \( (x, g(x) + h(x)) = (x, f(x)) \) by going up \( h(x) \) if \( h(x) > 0 \) or going down \( \lvert h(x) \rvert \) if \( h(x) < 0 \) as in this example. From examples like this it is easy to see that the slope of any linear function is the same as the coefficient or multiplier 2 in the \( g(x) = 2x \) part of the sum function while the \( h(x) = -5 \) part of the sum function causes a drop of 5 units below \( (x, 2x) \) to locate \( (x, 2x - 5) \) on the graph of the sum function \( g + h = f \).

As a second illustration, consider \( f(x) = x^2 + 3 \). Here we have the sum function \( g + h \) where \( g(x) = x^2 \) is the squaring function and \( h(x) = 3 \) is a constant function. Probably you can easily graph \( y = x^2 \) and \( y = 3 \); they are pictured below. The graph of \( f(x) = x^2 + 3 \) is in effect obtained by sliding the graph of \( y = x^2 \) three units up; i.e. adding 3 to \( x^2 \) for all \( (x, x^2) \).
In the next example we demonstrate how algebra may be used to simplify a function into a product function which is easier to work with and to graph. Let \( y = 1.5x^2 - 6x + 6 \) be the function. Now
\[
1.5x^2 - 6x + 6 = 1.5 \left( x^2 - 4x + 4 \right) = 1.5 \left( x - 2 \right)^2.
\]
Here is a place to think algebraically. Think of \( x - 2 \) as some variable which is to be squared (the squaring function) then multiplied by 1.5 (so \( y = g \cdot h(x) = g(x) h(x) \) where \( g(x) = 1.5 \), \( h(x) = (x-2)^2 \)). Viewing the function this way makes graphing easy—find points on the squaring function then multiply the second coordinate by 1.5 to get points on \( y = 1.5x^2 - 6x + 6 = 1.5 \left( x-2 \right)^2 \). For example, \((2,0),(3,1),(4,4)\) are points on \( y = (x-2)^2 \) so \((2,0),(3,1.5),(4,6)\) are points on \( y = 1.5 \left( x-2 \right)^2 \). Similarly \((1,1.5),(0,6),(-1,13.5)\) are points on \( y = 1.5 \left( x-2 \right)^2 \).
As a special case of a quotient function $\frac{f}{g}$ we need to mention the reciprocal function $\frac{1}{g}$ which is defined to be $\frac{1}{g}(x) = \frac{1}{g(x)}$, $g(x) \neq 0$. It is neat to see that $\frac{f}{g} = f \cdot \frac{1}{g}$ which is very like what we can do with ordinary fractions. $y = \frac{1}{x}$ describes a reciprocal function where $\frac{1}{f(x)} = \frac{1}{f(x)} = \frac{1}{x}$.

This particular function is not defined for $x = 0$ as the graph indicates.

It is worthwhile to notice how the graph tells us that the function is not defined at $x = 0$ by the way the points on the graph do not intersect the y-axis. Also notice how the graph visually says that for small but positive values of $x$, $y$ is quite large (eg. $x = \frac{1}{100} \Rightarrow y = 100$), but $y$ decreases in size quite rapidly as $x$ changes from $\frac{1}{100}$ to 1. This is just
what the symbol \( \frac{1}{x} \) indicates numerically. In the case of the basic functions (the functions we study in this course) you should work on mentally associating a graph of \( f \) with the formula for \( f \). This will give you a reservoir of functions always available for use.
Lesson Eight

We have seen how two functions may be added, subtracted, multiplied or divided to form a new function. Now we will observe that two functions may (possibly) form a new function by having some function operate upon the images of a first function. An illustration will convince you that this "combination process" is a natural way to form new functions or to interpret old functions. Suppose you add 3 to x and then square the sum; this would be a combination process. In symbols the first function g takes x to x + 3 then the second function f takes x + 3 to (x+3)² or x→g→x + 3→f→(x+3)². In other symbols g operates on x to get x + 3 or g(x) = x + 3, then f operates on g(x) to yield (x+3)² or f(g(x)) = f(x+3) = (x+3)². This "combination process" is called composition. Diagramatically, x→g→x + 3→f→(x+3)² or symbolically f(g(x)) = (x+3)² is the composition of g by f. Sometimes the symbol f o g is used to represent the composition function. Note f o g (x) = f (g(x)) so that the notation is "from the right first" or, "right-handed," g operates on some element x to yield g(x) then f operates on that image g(x) to yield an image f (g(x)) which is the image of x under f o g.

Some examples of composition follow.

Example 1. Let g(x) = x + 2 and f(x) = x + 5. Now f o g (x) = f (g(x)) but g(x) = x + 2 so f (g(x)) = f(x+2). But f(x+2) = (x+2) + 5 since f takes any pre-image and operates on it (by adding 5) to get the image. So f o g (x) = f (g(x)) = f(x+2) = (x+2) + 5 = x + 7 is the composition of g by f.

Example 2. Let g(x) = x - 5 and f(x) = √x. Now f o g (x) = f (g(x)) = f(x-5) = √(x-5) for x ≥ 5.

Example 3. Let g(x) = x + 7 and f(x) = |x|. Now f o g (x) = f (g(x)) = f(x+7) = |x+7|.
Example 4. Let \( g(x) = x^2 \) and \( f(x) = 2x - 1 \). Now \( f \circ g(x) = f(g(x)) = f(x^2) = 2x^2 - 1 \).

Example 5. Let \( g(x) = x - 3 \) and \( f(x) = [x] \). Now \( f \circ g(x) = f(x-3) = [x-3] \).

Example 6. Let \( g(x) = 3x - 7 \) and \( f(x) = x^2 \). Now \( f \circ g(x) = f(g(x)) = f(3x-7) = \frac{1}{3}(3x-7) + \frac{7}{3} = x - \frac{7}{3} + \frac{7}{3} = x \).

Perhaps you have observed that in order for composition of \( g \) by \( f \) to work it must be the case that all the images of \( x \) under \( g \) must belong to the domain of \( f \) so that \( f \) may operate and send these images of \( x \) under \( g \) into images of \( g(x) \) under \( f \). That is \( x \xrightarrow{g} y \xrightarrow{f} w \) or \( x \xrightarrow{f \circ g} w \). In practice one may always form a composition of \( f \circ g \) if there are some values of \( g \) which are also in the domain of \( f \). ie, \( D_f \cap R_g \neq \emptyset \). The reader should note that, once a composition has been formed, it is a bona fide function. Sometimes we will see functions which are actually composite functions. It will be a useful skill to be able to identify such functions and to rename them so that the composition is apparent. For example, \( f(x) = (3x-1)^2 \) is a composite function of \( h[h(x) = 3x - 1] \) by \( g[g(x) = x^2] \). To see this you might diagram \( x \xrightarrow{h} 3x - 1 \xrightarrow{g} (3x-1)^2 \) which indicates \( h \) sends \( x \) into \( 3x - 1 \) and \( g \) sends \( 3x - 1 \) into \( (3x-1)^2 \). A second example is \( y = |x^2 - 4| \). This is a composite function of \( f(x) = |x| \) and \( g(x) = x^2 - 4 \). In particular \( y = f(g(x)) = f(x^2-4) = |x^2-4| \). In diagramatic form \( x \xrightarrow{g} x^2 - 4 \xrightarrow{f} |x^2-4| \).

Consider \( y = \sqrt{x^2+3} \). One may interpret this as a composite function \( x \xrightarrow{g} x^2 + 3 \xrightarrow{f} \sqrt{x^2 + 3} \) or in other symbols \( f \circ g(x) = f(g(x)) = f(x^2+3) = \sqrt{x^2+3} \).

The reader should try "inventing" some composite functions. Use both diagrams and symbolic notation to show how the composite may be obtained.
Lesson Nine

Functions are individual; they are recognizable in that sense, but like many other entities they may be grouped, classified, according to certain identifiable characteristics. One such characteristic will be of interest to us now. We proceed by examples. Observe that if \( g(x) = x^2 \) then \( g(2) = 4 = g(-2) \) or in diagram \( \begin{array}{c}
-2 \\
\downarrow \\
2 \\
\rightarrow 4
\end{array} \). Similarly if \( h(x) = x^3 - x \) then \( h(0) = 0 = h(1) \) or in diagram \( \begin{array}{c}
0 \\
\downarrow \\
1 \\
\rightarrow 0
\end{array} \). In particular functions \( g \) and \( h \) have the characteristic that at least two domain elements have the same image or value under the function. In such a case the function is not one-to-one. In contrast, a function like \( f(x) = 5x \) has the characteristic that no two different elements in the domain of \( f \) will have the same image. Thus \( f \) is said to be a one-to-one function. In formal language a function \( f \) is one to one provided that \( f(x_1) = f(x_2) \) if and only if \( x_1 = x_2 \). It should be made clear that any function may be examined to determine whether it is one-to-one or not. We may use some of our geometric insight to see what it means for a function \( f \) to be one-to-one. Recall that a graph of \( f \) can be thought of as a visual description of \( f \) where the points on the graph represent the match or correspondence the function makes between the elements in the domain (\( x \)'s) and the elements in the range (\( y \)'s). Now imagine a horizontal line which is allowed to move up or down the \( y \)-axis (much in the same way a horizontal bar is allowed to move up or down in a track event called the high jump). If this horizontal line (or bar) cuts (or intersects) the graph of \( f \) more than once for any given value of \( y \) then \( (x_1, y) \) and \( (x_2, y) \) the points of intersection indicate that \( f \) is not one-to-one since \( y \) has more than one preimage.
For example consider \( f(x) = x^2 + 2x + 1, -6 \leq x \leq 4 \). You recognize this as a quadratic function whose vertex is \((-1,0)\) and whose graph is "bowl-shaped", opening upward. Now a horizontal line (bar) one unit above the origin is seen to intersect the graph of \( f \) at \((-2,1)\) and \((0,1)\). So you see two \( x \)'s \( x_1 = -2 \) and \( x_2 = 0 \) such that \( f(x_1) = f(x_2) \) and \( x_1 \neq x_2 \). Thus this \( f(x) = x^2 + 2x + 1 \) is not one to one.

As another example look at \( g(x) = x^3 - 2x^2 - x + 1 \). You can verify that if \( x_1 = -1, x_2 = +1, x_3 = 2 \) then \( f(x_1) = f(x_2) = f(3) = -1 \). Geometrically this means that a horizontal line one unit beneath the origin will intersect the graph of \( g \) three times at \((-1,-1), (1,-1) \) and \((2,-1)\). Thus \( g(x) = x^3 - 2x^2 - x + 1 \) is not one-to-one. Diagramatically this says \(-1 \rightarrow ^1 \rightarrow ^-1 \rightarrow ^2 \rightarrow ^1 \) so we have an image with three preimages here. Again, this shows that \( g \) is not one-to-one. Observe that in this geometric approach when a horizontal line intersects the graph of a function more than once, there will be a \( y \) value such that \( y = f(x_1) = f(x_2) \) and yet \( x_1 \neq x_2 \) so that our formal definition is consistent with our geometric interpretation.

In many cases when we wish to examine a function for one-to-oneness we may think algebraically as follows: If the equation \( y = f(x) \) when solved for \( x \) has two or more solutions (for a fixed \( y \)) then the function \( f \) is not one-to-one. As an example, we have learned that for any \( y = ax^2 + bx + c \) the equation has two solutions namely \( x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \) and \( x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \) hence a quadratic function will not be one-to-one.

In conclusion the reader should now be aware that any function may be examined to find it if is one-to-one. Some functions are one-to-one and others are not.
Lesson Ten

In this, our final lesson in this series, we introduce the idea of an inverse function. Recall that the function $f$ associates a value $f(x)$ with each $x$ in the domain of $f$. Now focusing upon these values of $f$ and inquiring if there exists a function $g$ (whose domain is the values of $f$) such that $g(f(x)) = x$ is to ask whether there is an inverse function of $f$. If there is such a $g$ function we call it the inverse of $f$ and denote it by $f^{-1}$.

Some care must be given to emphasizing that not every $f$ will have an inverse. Let us observe that if $f$ is not one to one, then there would have to be at least two numbers $x_1$ and $x_2$ such that $f(x_1) = f(x_2)$. This means that any function $g$ whose domain is the values of $f$ and such that $g(f(x)) = x$ would have to send $f(x_1)$ to $x_1$ thus $f(x_2)$ is sent to $x_1$ also. But this means $g(f(x_2)) \neq x_2$ and therefore there exists no $g$ function such that $g(f(x)) = x$ when $f$ is not one-to-one. In other words, a function $f$ will have an inverse $f^{-1}$ if and only if $f$ is one-to-one. For emphasis, if a function is one-to-one then it will have an inverse. Some students like to think of an inverse function as reversing or undoing whatever operation was performed by $f$. For example, if $x \rightarrow 3x$ is a diagram for $f$, then $f^{-1}$ exists (because $f$ is one-to-one) and $x \rightarrow \frac{x}{3}$, which is to say that since $f$ multiplied by 3 then $f^{-1}$ needs to divide by 3 to undo the work of $f$. In other symbols, $f(x) = 3x$ has an inverse $f^{-1}(x) = \frac{x}{3}$ since $f^{-1}(f(x)) = f^{-1}(3x) = \frac{3x}{3} = x$ as needed.

It is neat that, because $f$ is one-to-one, $f^{-1}$ will be one-to-one. Also this means we can write $f(f^{-1}(x)) = x$. Now this allows us to use algebra to "find" $f^{-1}(x)$ when we have a formula for $f$ and when $f$ is one-to-one. For example, if $f(x) = 3x$ then $f(f^{-1}(x)) = 3(f^{-1}(x))$, but $f(f^{-1}(x))$ also equals $x$, so $3(f^{-1}(x)) = x$ or $f^{-1}(x) = \frac{x}{3}$ as seen earlier. Another example, if $f(x) = x^3 - 2$, 

then \( f(f^{-1}(x)) = (f^{-1}(x))^3 - 2 \) and \( f(f^{-1}(x)) = x \) so that \( (f^{-1}(x))^3 - 2 = x \)

which yields \( f^{-1}(x) = \sqrt[3]{x + 2} \), a formula for \( f^{-1} \).

We have seen that a graph of \( f \) provides a visual record of the way \( f \) operates. Any point \((x, y)\) on the graph of \( f \) indicates that \( f \) sends \( x \) into \( y \) (which is a name for \( f(x) \), ie. \( y = f(x) \)). Now provided \( f \) is one-to-one the collection of points of the form \((y, x)\) constitutes a graph of \( f^{-1} \). Consider the diagram below which depicts a function \( f \) and its inverse \( f^{-1} \).

Notice that there is quite a bit of information which can be visualized. Note \( D_f \) is the interval of numbers from 9 to 14 inclusive and the range of \( f \), \( R_f \), is the interval of numbers from 3 to 13 while the \( D_{f^{-1}} \) is 3 to 13 and \( R_{f^{-1}} \) is 9 to 14. Also note \( x \rightarrow f(x) \rightarrow f^{-1} x \) for \( 9 \leq x \leq 14 \). For example, \( 11 \rightarrow 4 \rightarrow 11 \). Also note that the graph of \( f^{-1} \) is the mirror-image of the graph of \( f \) in the mirror line \( y = x \).

One might say that the graph of a one-to-one function "reveals" the graph of its inverse in that the reflection of \( f \) in the diagonal line \( y = x \) is the graph of the inverse, \( f^{-1} \). In other words, inverses are easy to graph if the graph of \( f \) is available.
Appendix C:

EXERCISE SETS
Which of the following are functions? The left list is the domain.

A

1 → 14
2 → 12
3 → 20
4 → 28
5 → 36
6 → 40

B

1 → 2
2 → 3
3 → 4
4 → 5
5 → 6
6 → 7

C

Coffee → bean
Corn → kernal
Lettuce → leaf
Potato → tuber

D

1 → 4
2 → 8
3 → 12
4 → 16
5 → 20
6 → 24

E

7 → 7
6 → 6
5 → 5
4 → 4
3 → 3
2 → 2
1 → 1
0 → 0

F

Coffee → black
Corn → hot
Potato → buttered
Lettuce → mashed
crisp

G

H

grounds → coffee
chips → potato
flakes → corn
slaw → cabbage

I

-4 → -1
-3 → -1
-2 → -1
-1 → 0
0 → 0
1 → 1
2 → +1
3 → 4

J

-7 → -7
-6 → -6
-5 → -5
-4 → -4
-3 → -3
-2 → -2
-1 → -1
0 → 0

K

-2
-1
0 → 1
1 → 2

L

-2
-1
-1
0 → 1
1 → 2
1. Diagram $x \xrightarrow{f} x - 2$ where $x$ is a counting number between 2 and 10.

2. Diagram $x \xrightarrow{g} x^2$ where $x$ is an integer between -5 and 4.

3. Diagram $x \xrightarrow{h} b - x$ where $x$ is any counting number.

4. Another way to write $f$ in problem 1 is $f(x) = \ldots$, $x$ as described.

5. Another way to describe $g$ in problem 2 is $g(x) = \ldots$, $x$ as described.

6. Another way to describe $h$ in problem 3 is $h(x) = \ldots$, $x$ as given.

7. When we read $g(x) = 3x - 1$, $x = -2, -1, 0, 1$ we are expected to understand that the domain of $g$ is the values of $x$ as listed. If the list were $x = 1, 2, 3, \ldots$ this would mean the domain is the set of natural numbers. Describe the functions in problems 1-3 using this technique.
8. It is generally agreed that any letter we choose may be used to denote an element in the domain of a function. Thus \( f(x) = 3x - 31, \) \( x \) integer and \( f(t) = 3t - 31, \) \( t \) integer both describe the same function \( f. \) Does it follow that \( r \rightarrow 3r - 31, \) \( r \) integer, describes \( f? \) Why or why not?

9. We read \( f(x) \) as "\( f \) of \( x\)." \( f(x) \) is called the value of \( f \) at \( x. \) \( x \rightarrow 7x, \) find \( f(2), f(-2), f(\frac{1}{3}), f(4), f(- \frac{1}{7}), f(t), f(b+1). \)

10. If gasoline sells for 73.9 cents a gallon then \( C(x) = .739x \) could be viewed as a cost function. Find \( C(10), C(5), C(1). \) What is the domain? Suppose we think of the gas pump as a function machine where the input is dollars. Call the function \( G \) and describe it using functional notation. Find \( G(10), G(5), G(1), G(7.39), G(1/4) \)

11. \( h(x) = x^2 + 4. \) Find \( h(0), h(2), h(20), h(b), h(t), h(x + 2). \)

12. Translate into a formula using functional notation: Take a nonzero real number. Square it. Multiply by 2. Add 1. Divide by 3 times the number. Add 1. Call it \( H. \)
Exercise Set 3

1. You have learned that not all functions may be described by an equation. (It is also a fact that not all equations describe a function.) As another example of such a function consider \( n \) (the number of primes less than or equal to \( n \)), where \( x \) is a non-negative real number. Here \( n(0) = 0 \), \( n(1) = 0 \), \( n(2) = 1 \), \( n(2.99) = 1 \), \( n(3) = 2 \). Find \( n(4) \), \( n(5) \), \( n(5.7) \), \( n(7) \), \( n(10) \), \( n(11) \).

2. Remember \( f(x) = \left\lfloor x \right\rfloor \) is the way we write "\( f(x) \) is the greatest integer not greater than \( x \)". For example, \( \left\lfloor 2.9 \right\rfloor = 2 \), \( \left\lfloor 2.99 \right\rfloor = 2 \), \( \left\lfloor 2.999 \right\rfloor = 2 \), \( \left\lfloor 3 \right\rfloor = 3 \), \( \left\lfloor 1/2 \right\rfloor = 0 \). Find \( f(0) \): note this could be written \( \left\lfloor 0 \right\rfloor \), \( f(1/3) \), \( f(3/4) \), \( f(1) \), \( f(1.99) \), \( f(2) \), \( f(2.9999) \), \( f(-3) \), \( f(-3.1) \), \( f(-4) \), \( f(-4.01) \), \( [-0.001] \), \( [-1,000] \), \( [-2.0001] \).

3. Remember \( f(x) = |x| \) is the symbolism we use to denote "the absolute value function". Find \( f(0) \), \( f(1/2) \), \( f(3/4) \), \( f(8/9) \), \( f(1) \), \( f(2.3) \), \( f(20.1) \), \( f(-1.3) \), \( f(-1/2) \), \( f(-3.67) \), \( f(-19) \).

4. Remember \( f(x) = \sqrt{x} \) is the way we symbolize "the principal square root function". Find \( f(0) \), \( f(9/16) \), \( f(1) \), \( f(9/4) \), \( f(4) \), \( f(25/4) \), \( f(-4) \). Can you find \( f(2) \), \( f(5) \), \( f(13/2) \)?
5. What are the domains of the functions in problem 1-4 above?

6. Make up a function which may be described by a formula. Call it \( g \) and use functional notation to describe it. State its domain and range.

7. When we write \( g(x) = \begin{cases} -x, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases} \) we want the reader to know that this function involves two formulas. For \( x < 0 \) we have \( x \neq -x \), but for \( x \geq 0 \) we use \( x \neq \sqrt{x} \). Find \( g(-1) \), \( g(1) \), \( g(4) \). What is the domain of \( g \)? What is the range of \( g \)?

8. Suppose \( f(x) = \sqrt{x} \) and \( h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \), find \( f(-1) \), \( h(-1) \), \( f(-4) \), \( h(-2) \), \( f(-3) \), \( h(-3) \), \( f(1) \), \( h(1) \), \( f(2) \), \( h(2) \), \( f(3) \), \( h(3) \). Compare these values for \( f \) and \( h \). What can you say about \( f \) and \( h \)?
9. As problems 7 and 8 have indicated a function need not be described by a single equation. As another example let \( f \) be a function defined by

\[
 f(x) = \begin{cases} 
 x, & x \leq 0 \\
 x^2 + 2, & 0 < x \leq 4 \\
 x^2, & x > 4 
\end{cases}
\]

Observe this functional rule uses three equations.

It is worth stressing that you will need to work with functions like this one which is a single function in which the functional rule has more than one part. Find \( f(-4), f(-3), f(-2), f(0), f(1), f(2), f(4), f(5), f(6), f(10) \).

10. Write an equation for the function whose name is \( \exp_3 \). What is its domain? What is its range? Find the value of the function for \( x = -2, -1, 0, 1, 2, 3, 4 \).

11. Repeat problem 10 using \( \exp_{1/2} \).
Exercise Set 4

Use graph paper

1. Draw a rectangular coordinate system and locate number pairs \((1,5), (-1,4), (-5,-3), (3,-4)\). "Blacken in" the appropriate points. Is the picture you have drawn the graph of a function? Explain why or why not?

2. Draw another coordinate system and plot \((-8,-2), (-4,-1), (0,0), (4,1), (3,2)\). Is the graph a function? Explain why or why not.

3. Draw another coordinate system and plot \((-4,-2), (-2,0), (0,2), (2,4), (4,6)\). Is the graph a function? Add the points \((-3,0), (-1,2), (1,4), (3,6)\). Is the graph now a function?

4. Draw a coordinate system. Plot \((-6,3.5), (-4.5, 1.75), (-3.5, 1.25), (-1.5, .25), (0,-1/2), (1.5, -1/4), (3,1/2), (9/2,7/4), (6,7/2)\). Is the graph a function?

5. Draw a coordinate system. Locate \((-3,4), (-2,2), (-1,0), (0,-2), (1,-4), (-1,2)\). Is the graph a function?
6. Draw a coordinate system. Plot (-4,2) (-3,2) (-2,2) (-1,2) (0,2) (1,2) (2,2) (3,2) (4,2). Is the graph a function? Consider the diagram.

Does the diagram represent a function? How are the graph and the diagram related?

7. Draw a coordinate system. Make room for at least ten units in all directions from your origin. Graph the function

\[ f(x) = \begin{cases} 
  x, & x = -2, -3, -4, -5, \ldots \\
 -2, & x = -1, 0, 1 \\
 -x, & x = 2, 3, 4, 5, \ldots 
\end{cases} \]

What is the domain of \( f \)?
What is the range of \( f \)?

8. Draw a coordinate system (room for at least 10 units in all directions).

Graph

\[ f(x) = \begin{cases} 
  2x + 8, & x = -9, -8, -7, -6 \\
 -2x - 16, & x = -5, -4, -3 \\
 2x - 4, & x = -2, -1, 0 \\
 -2x - 4, & x = 1, 2, 3 \\
 2x - 16, & x = 4, 5, 6 \\
 -2x + 8, & x = 7, 8, 9 
\end{cases} \]

Do you see a "sawtooth" pattern? Figure out what formulas you could adjoin to the above list so that \( f \) will have "another tooth".
Exercise Set 5

Use graph paper

1. Graph

   \begin{align*}
   -6 & \rightarrow -3 \\
   -4 & \rightarrow -2 \\
   -2 & \rightarrow -1 \\
   0 & \rightarrow 0 \\
   2 & \rightarrow 1 \\
   4 & \rightarrow 2 \\
   6 & \rightarrow 3
   \end{align*}

   Call this function \( f \). Describe the function using functional notation. Describe the function using a symbolic diagram. What is the domain? What is the range?

2. Observe that \( f(x) = \frac{x}{2} \), \( x \) an even number and \(-6 \leq x \leq 6\). If we change the domain of \( f \) to any real number \(-6 \leq x \leq 6\), what is the image of \(-3\), \(-2.75\), \(-1.3\), \(-\sqrt{2}\), \(-\frac{1}{3}\), \(\frac{1}{2}\), \(2\), \(5.99\)? Graph the "new" \( f \). Compare with the graph in problem 1 above.

3. Graph \( g(x) = \lfloor x \rfloor \), \(-2 \leq x \leq 3\). Recall \( \lfloor x \rfloor \) denotes "the greatest integer less than or equal to \( x \)". Graph \( f(x) = |x| \), \(-3 \leq x \leq 4\).

4. Graph \( y = x^2 + 1 \), \( x = -3, -2.5, -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2, 2.5 \). Call this function \( h \). Write both the functional notation for \( h \) and the symbolic diagram for \( h \).
5. Extend the domain of h so that h is defined for any real in \(-3 \leq x \leq 3\). Graph this "new" h. What is its range?

6. Use functional notation and a formula to replace the word rule given.
   a) Each positive integer has an image of zero.
   b) Each positive integer is mapped onto the next largest integer.
   c) Each real number is mapped onto the sum of the number and its square.
   d) Each non-negative real number is corresponded with the sum of the number and its principle square root.
   e) Each non-zero number is mapped onto the sum of the number and its reciprocal.

7. Use a symbolic diagram to define the functions in problem 6.

8. State the domain and range of each function in problem 6.
Exercise Set 6

1. Let \( y = f(x) = 0.5x + 3 \). Graph \( f \) using graph paper and a scale of two squares per one unit.

2. On the same axis system used in problem 1 graph the five functions
   \( g(x) = x + 3 \), \( h(x) = 2x + 3 \), \( k(x) = -1/2x + 3 \), \( j(x) = -x + 3 \), \( l(x) = -2x + 3 \).

3. Explain how the "picture" formed by the graphs would be changed if in functions \( f \) through \( l \) you remove the 3 and put a 5 in its place.

4. Without actually graphing, explain how the graph of \( f(x) = 2x + b \) will vary if \( b \) can take on values of \(-2, -1, 0, 1, 2\) respectively. How many functions are involved in this problem?

5. Without actually graphing explain how the graph of \( g(x) = mx - 2 \) will vary if \( m \) can take on values of \(-2, -1, 0, 1, 2\) respectively. How many functions are involved in this problem?

6. Without actually graphing the line through \((-1,-2)\) and \((4,5)\), write an equation for the line.
7. The graph of a quadratic function \( f(x) = ax^2 + bx + c \) is called a parabola. You know how to complete the square. You may use this skill to easily and accurately identify the vertex of a parabola. For example, \( y = 2x^2 - 6x + 10 \) describes a parabola. Also \( y = 2(x^2 - 3x + \_\_) + 10 - \_\_ \) and if we complete the square by placing \( \frac{9}{4} \) in the first blank and \( \frac{9}{2} \) in the second blank (do you see why?) then we may write \( y = 2(x - \frac{3}{2})^2 + \frac{11}{2} \). This last equation says that \( \left( \frac{3}{2}, \frac{11}{2} \right) \) is the low point of the parabola--its vertex. The vertical line through the vertex is called the axis of symmetry.

Find the vertex and axis of symmetry for: \( y = 4x^2 - 5x + 10 \)
\[ y = 4x^2 + 5x - 10 \]

Without actually graphing the functions, see if you can explain how the picture would look when both graphs are placed on the same axis system.
1. Add \( x^2 + 2x - 3 \) and \(-2x^2 + x + 3\).

2. You could interpret problem 1 as forming a new function \( h(x) = -x^2 + 3x \) by adding \( f(x) = x^2 + 2x - 3 \) to \( g(x) = -2x^2 + x + 3 \). Note we could say \( h = f + g \) or \( h(x) = (f+g)(x) = -x^2 + 3x \). Use \( f \) and \( g \) to find \( f - g, f \cdot g, \frac{f}{g}, f + g, 2f + g, 2(f+g) \).

3. Use graphs of \( f(x) = -2x \) and \( g(x) = 3 \) to sketch a graph of \( f + g \).

4. Use graphs of \( f(x) = 2x^2 \) and \( g(x) = 5 \) to sketch a graph of \( f + g \).

5. Let \( f(x) = \sqrt{x-3}, g(x) = \frac{1}{x}, h(x) = x + 3 \). Find a formula for \( f + g + b, f \cdot f, \frac{f}{g}, g \cdot h, f \cdot f + g \).

6. What are the domains of the functions you found in problem 5?
7. Remember how we learned to solve equations such as A: $3x - 4 = 2x + 5$
and B: $2x^2 + 3x - 4 = x^2 + 2x + 2$? Now you may view problems like these in
terms of functions. For example, if $h(x) = 3x - 4$ and $g(x) = 2x + 5$ we
can write equation A to mean $g(x) = h(x)$ or equivalently $(g-h)(x) = 0$.
Since a number $r$ is called a zero of a function $f$ provided $f(r) = 0$, we
can say the solutions of equation A are zeros or roots of $(h-g)(x) = 0$.
Note that zeros of a function are, geometrically, just the $x$ intercepts of
the graph of the function—the places where the graph crosses, intercepts,
the $x$-axis.

Restate equation A as an equation involving the zero of a function $f$.
Graph $f$ and label the zeros. Check your work by solving equation A.

8. Restate equation B in problem 7 as an equation involving a function $f$.
Graph $f$ and label the zeros. Check your work by solving equation B.
Exercise Set 8

1. Consider the figure.

This drawing suggests that the composition function $f \circ g$ may be viewed as a machine which has been constructed by hooking two machines together, as indicated, to work in tandem thus forming a new machine or function. You should note that this machine will only have an output when the input $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$. In other words, the domain of $f \circ g$ is the set of all elements $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$.

As an example, let $g$ be the squaring machine and $f$ be the doubling machine when 3 is put in then $2(3)^2 = 18$ is the output. In detail, 3 is put in, squared by $g$, put out and in to $f$ where doubling takes place to put out 18. Explain what happens in detail when -4 is put in. What happens when 5 is put in? Write a formula for $f \circ g$.

2. Suppose we took apart the machine in problem 1 and we put it back together backward. That is, the input now goes directly into the doubling machine and, now to our surprise, an input of 3 causes an output of 36. What happens when a -4 is put in? A 5 is put in? It seems we have a different machine than that of problem 1; what should it be named? Write a formula for this function.
3. When forming a composite we must be somewhat careful to only hook two functions together when the second machine is capable of operating upon the output of the first machine. For example, if \( f(x) = -2x, \ 1 \leq x \leq 10 \) and \( g(x) = \sqrt{x} \) we should not expect \( g \circ f \) to work. Explain why. What about \( f \circ g \), would it work?

4. Let \( h(x) = 3x + 4, \ 0 < x \leq 3 \) and \( k(x) = 5x + 3, \ 0 \leq x \leq 3 \). Explain why \( k \circ h \) won't work. Will \( h \circ k \) work?

5. Is it possible to hook up more than two machines in tandem? Consider \( f(x) = 3x + 1, \ g(x) = 2x - 3 \) and \( h(x) = x + 4 \). Find a formula for \( g \circ h \) then write a formula for \( f \circ (g \circ h) \). Explain how this works.

6. Given that \( x \xrightarrow{f} x^2 - 1 \) and \( x \xrightarrow{g} x + 2 \). Find \( (f \circ g)(2), \ (g \circ f)(0), \ (g \circ g)(1), \ (f \circ f)(1), \ (f \circ g)(x), \ (g \circ f)(x) \).
7. Given that \( x \xrightarrow{f} 2x + 5 \) and \( x \xrightarrow{g} 3x + 4 \). Find \((f \circ g)(x)\), \((g \circ f)(x)\).

8. Let \( j(x) = x \) and \( f(x) = x + 2 \). Find a function \( g \) so that \( f \circ g = j \). Find a function \( h \) such that \( h \circ f = j \). Compare \( h \) and \( g \).

9. Suppose that \( f \) maps \( x \) onto \( x^2 \) and \( g \) maps \( x \) onto \( x^3 \). Find an expression for \( f \circ g \). For \( g \circ f \). For \( f \cdot g \).
Exercise Set 9

1. Examine each of the functions given and identify those which are one-to-one. (Recall that when a function is given in this style you are to understand that the domain is the maximal subset of the real numbers for which the rule makes sense.)

   a) \( f(x) = 2x^2 \)
   b) \( f(t) = 3t \)
   c) \( f(u) = \frac{2}{u} \)
   d) \( f(s) = s^3 \)
   e) \( f(x) = \sqrt{1-x} \)
   f) \( f(t) = 2t^2 + 1 \)
   g) \( f(u) = \frac{2}{u^2+1} \)
   h) \( f(s) = 1 - \frac{1}{s} \)

2. Consider \( f(x) = x^2 \) which is not one-to-one. Now if we make a new function \( g \) by restricting the domain of \( f \) to the positive real numbers, then \( g(x) = x^2 \) \( 0 < x \) is a one-to-one function. Graph \( g \).

3. Which of the figures below are graphs of one-to-one functions?
4. In your own words describe how you can determine if a graph represents a one-to-one function? Describe how to determine if a graph represents a function.

5. Let $x$ be the horizontal and $y$ the vertical axis as usual, then graph $y^2 + 2y + 1 = x$.

6. Starting with $F(x) = (x-2)^2$ make a new function $G$ by restricting the domain to $x \geq 2$. Write a formula for $G$. Graph $G$. 
Exercise Set 10

1. To help in understanding the idea of an inverse function we may view functions as machines and observe that some machines will invert or operate upside down. For example, the doubling machine has the property that each output is the result of just one input.

If we turn this machine upside down, reversing the machinery, we have, in effect, a halving machine.

Be aware that not all function machines will operate when inverted. The squaring machine will not! Recall that with the squaring function 9 is the output for both 3 and -3. The inverted machine will not work as a function machine because if 9 goes in both 3 and -3 will try to be output which "jams the machinery"—a function machine must have only one output. In formal language, when a function does not give the same output for two different inputs, then it is a one-to-one function and may be inverted. We will call the inverted machine the inverse function.

Can you invert the reciprocal function? Explain why or why not. You may wish to draw a diagram.

Can you invert the absolute value function? Explain why or why not.
2. Find the inverse of each of the following functions.
   a) \( x \rightarrow x - 5 \)  
   b) \( F(x) = 4x + 7 \)  
   c) \( G(x) = \frac{2}{3x} \)

3. Solve each of the following for \( x \), then compare your answers with the results in problem 2.
   a) \( y = x - 5 \)  
   b) \( y = 4x + 7 \)  
   d) \( y = \frac{2}{3x} \)

4. Let \( f(x) = 3x + 4 \). Show that the function described by \( y = \frac{1}{3x+4} \) is not \( f^{-1} \), the inverse of \( f \).

5. Sketch the graph of the inverse of each of the functions whose graphs are drawn below.
6. Sketch a graph of \( g(x) = -2x + 3, \ -2 \leq x \leq 2 \). Now sketch \( g^{-1} \) by reflecting \( g(x) = -2x + 3, \ -2 \leq x \leq 2 \) in the diagonal line \( y = x \). Read the domain and range of \( g^{-1} \) from your graph.
Appendix D:

LESSON PLANS
CA: Discuss the idea that a correspondence (match, tie) may be made in several ways between two sets. The matching may be diagramed using arrows. Stress and exemplify that if there is exactly one tie (arrow) between an object in the domain and an object in the range then the correspondence is a function. Exemplify diagrams which are not functions also.

Discuss the machine (input-output) analogy. Use a schematic such as \( f \) during discussion and demonstration.

Demonstrate instructions which describe a function. For example, take an integer, multiply by four then subtract three.

Discuss pre-image and image terminology.

CM: Same as above with these additions: (1) After discussing the machine analogy, use calculator as a hands on model of a function machine. Demonstrate and have class activity with functions like: \( x \to x + 5, \ x \to 3.5x, \ x \to \sqrt{x}, \ x \to x^2 \). (2) During discussion on instructions which describe a function elaborate on how the calculator allows us (in a sense) to "see" the instructions being carried out. For example put 2 in (2 indicates), push "times" key, push 5 key (5 indicates), push "add" key (10 indicates), push 3 (3 indicates) push "equals" key (13 indicates). 2 the
pre-image or input is seen to yield 13 the output or image of this function.
LESSON TWO

CA: Introduce notation such as $x \overset{f}{\rightarrow} y$ where $x$ represents any domain element and $y$ represents its image. Elaborate on the efficiency of such notation. Note that we may easily discuss functions whose domain and/or range is infinite. Stress that a change of domain causes the function to be altered.

Discuss $f(\ )$ notation. Emphasize that "putting in" $x$ yields $y$, that is $y = f(x)$. For example, if $f(x) = 4x - 2$ we may think of this as a set of instructions describing $f$; if you put in 3 then the machine multiplies by 4 and subtracts 2 to get an output of 10 ($f(3) = 4(3) - 2 = 10$). Stress that this $f(\ )$ notation is an alternative to $x \overset{f}{\rightarrow} 4x - 2$.

CM: Same as above with these additions: (1) Observe and demonstrate that when a formula like $f(x) = 4x - 2$ is known we may employ the calculator to easily find the image of a domain element. Demonstrate that the calculator has a few special keys which may be used to find output of the related special functions such as $f(x) = \sqrt{x}$, $f(x) = 2^x$, $f(x) = 1/x$. Demonstrate that for polynomial functions a calculator is quite helpful in finding output.
LESSON THREE

CA: Discuss \( f(x) = \sqrt{x} \). Review that \( \sqrt{9} \) means the positive number whose square is 9. Earlier we wrote \( x \text{\text{psqr} } \sqrt{x} \) or \( \text{psqr}(x) = \sqrt{x} \) but now we have another name (symbol) such as \( f \) or \( F \) for this function.

Discuss the greatest integer function \( G(x) = [x] \) as being another special function with special notation involved.

Discuss equations which do not describe a function where \( x \) represents a domain element. For example use \( y^2 = x + 2 \) to show \( 2 \rightarrow 2 \) and \( 2 \rightarrow -2 \) also so that by diagraming it becomes clear that this equation does not describe a function of \( x \). Not all equations represent functions of \( x \)!

Discuss \( \exp_3 = 3^x \) (or \( f(x) = 3^x \)) by diagraming \( 3^0 + 1, 3^{1/2} + \sqrt{3}, 3^1 + 3, 3^{3/2} + \sqrt{3^3}, 3^2 + 9, \ldots \ldots \)

CM: Same as above with this addition: When discussing \( f(x) = 3^x \) demonstrate using the \( y^x \) key. Stress that the output is usually a decimal approximation, for example, put in 3; press \( y^x \) key; put in 1/2; press equals key to get 1.732051 or approximately \( \sqrt{3} \).
LESSON FOUR

CA: Review coordinate system. Sketch \( f(x) = 3x \), \( x = -1, 0, 1, 2, 3 \). Elaborate on this being a graph of \( x \neq 3x \) where \( x = -1, 0, 1, 2, 3 \). Stress that the set of points constitutes a graph of \( f \). Do similarly with \( g(x) = 4x - 5 \), \( x = -2, 0, 1, 3 \). Stress that the graph may be thought of as a picture which shows all the points that make the equation true or as a picture of the points which solve the equation.

CM: Same as above but when making sketch(s) use the calculator to obtain images and stress how this use facilitates finding solutions, that is points on the graph.
LESSON FIVE

Discuss the idea of expanding the domain of a function. Note that this expansion will increase the number of points on the graph of f and hence literally change the graph which indicates the function is different. In effect we have a new function. Stress that if the domain is changed then the function is changed. In the case of expanding a domain the new function is called an extension of f. In the case of deletion from a domain the new function is called a restriction of f. Demonstrate: (1) A point graph with infinite points such as \( f(x) = x \) with \( x \) an integer. (2) An interval graph with infinite points such as \( g(x) = x^2 \) with \( x \) a real number such that \( 0 \leq x \leq 2 \). (3) A graph whose domain is all real numbers such as \( h(x) = -x \), \( x \) a real number. Note that in cases where the domain is infinite the graph contains an infinite number of points. Therefore the person graphing needs to look for a pattern. In our work we will see two kinds: (1) a discrete pattern or (2) a "continuous" pattern.

CM: Same as above only when demonstrating an extension of a function employ the calculator to find images. Assign each group of 5 or so students the task of finding a point on a graph. Note how the calculator can facilitate graphing the extension of a function.
LESSON SIX

Discuss linear functions $f(x) = ax + b$, $x$ is a real number. Review the idea that one looks for a pattern of points to emerge. Discuss slope. Discuss $y$-intercept. Emphasize that we have identified an infinite class of functions (linear) with a single formula $f(x) = ax + b$ and that this identification effects an insight about the graph of, say, $f(x) = 2x + 7$. The graph is a straight line with slope 2 and $y$-intercept of 7! A powerful insight.

Discuss quadratic functions $f(x) = ax^2 + bx + c$, $x$ is a real number, $a \neq 0$. Discuss vertex of the parabola. Involve completing the square in finding the vertex. Do demonstrations. Emphasize that we have identified an infinite class of functions (quadratic) with a single formula $f(x) = ax^2 + bx + c$ and that this identification effects a powerful insight about the graph of, say, $f(x) = -2(x - 1)^2 + 2$. The graph is a parabola which opens down and whose vertex is $(1, 2)$.

CM: Same as above only when demonstrating assign each group of 5 or so students to find points on the graph by employing the calculator. Note how the calculator facilitates graphing.
LESSON SEVEN

CA: Discuss the sum function \( f + g \), the difference function \( f - g \), the product function \( f \cdot g \), and the quotient function \( f/g \). Note how these definitions facilitate thinking of functions as elements to add, subtract, and so forth. We can use functions in ways somewhat similar to ways we have learned to use numbers.

CM: No applications.
LESSON EIGHT

CA: Discuss composition of functions $f \circ g$. Stress that $f \circ g \neq f \cdot g$ in general. $f \cdot g(x) = f(x)g(x)$ but $f \circ g(x) = f(g(x))$. Note that in order for $f \circ g$ to exist $g(x)$ must be in the domain of $f$. Note that $f \circ g$, when it exists, is a function so that if $g(x)$ is in the domain of $f$ we can form a new function $f \circ g$ from $f$ and $g$. Note that sometimes a composition function will be described conventionally as, say, $f(x) = \sqrt{x^2 + 9}$, $x \geq 0$ but as students we should recognize such composite functions and be able to rename them. For example $f(x) = \sqrt{x^2 + 9}$, $x \geq 0$ could be described as $F \circ G(x) = F(G(x)) = F(x^2 + 9) = \sqrt{x^2 + 9}$, that is, $F(x) = \sqrt{x}$ and $G(x) = x^2 + 9$.

Note and demonstrate that $f \circ g \neq g \circ f$, in general.

CM: Same as above with these additions: (1) Use the machine idea, actually two function machines tied in tandem, when demonstrating. For example $f \circ g(x) = (3x - 1)^2 = f(g(x)) = f(3x - 1) = (3x - 1)^2$ can be interpreted by putting in $x = 5$, push "times" key, push 3, push "subtraction" key, push 1 push equals (indicates $g(5) = 14$), push $x^2$ key (indicates $f(g(5)) = 196$) to get output of $f \circ g$. (2) Use the calculator to show $f \circ g \neq g \circ f$ in general. For example, if $g(x) = 3x - 1$ and $f(x) = x^2$ then $g \circ f$ can be interpreted by putting in $x = 5$, push $x^2$ key, push "times"
key, push 3, push "subtraction" key, push 1, push "equals" key (indicates \(g(f(5)) = 74\)) to get output of \(g \circ f\). (3) The calculator may be used to demonstrate how \(f \circ g\) "won't work", that is \(f \circ g\) is not defined, in some cases. For example, if \(g(x) = 3x - 7\) and \(f(x) = \sqrt{x}\) then \(f \circ g\) is not defined for \(x < 7/3\). To demonstrate that \(f \circ g\) is not defined when \(x = -1\) proceed as follows: put in \(x = -1\), push "times" key, enter 7, push "subtracts" key, push "equals" key (indicates \(g(-1) = -10\)) push \(\sqrt{}\) key (indicates error) to show -1 "jams up" the machine because \(g(-1)\) is not in the domain of \(f\).
LESSON NINE

CA: Discuss the one-to-one concept. Demonstrate. Discuss the characterization which says: a function $f$ is one-to-one if and only if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Discuss the geometric characterization which says: a function $f$ is one-to-one if and only if any horizontal line intersects the graph of $f$ at most one time. Discuss the algebraic observation that if the equation $y = f(x)$ describes $f$ and for any $y$ the equation has two or more distinct solutions then $f$ is not one-to-one.

CM: No application.
LESSON TEN

CA: Discuss the possible existence of a function $g$ such that $g(f(x)) = x$. If there is such a function $g$ it is called the inverse of $f$ and denoted by $f^{-1}$. Discuss that if the inverse function exists in a real sense it "undoes" whatever was "done" by the function. For example, if $f(x) = 5x$ we observe that since $f$ does multiplication $f^{-1}$ could be expected to un-do multiplication, that is $f^{-1}$ could be expected to do division. In particular note if $g(x) = x/5$ then $g(f(x)) = g(5x) = x$ so that $g$ is the inverse of $f$.

Discuss the fact that: a function $f$ will have an inverse $f^{-1}$ if and only if $f$ is one-to-one. Emphasize that this means any one-to-one function with a nice algebraic formula will satisfy the equation $f(f^{-1}(x)) = x$. One result is that in such cases we can use algebra to find a formula for $f^{-1}$. For example, $f(x) = 2x - 5$ is one-to-one so $f(f^{-1}(x)) = x$ and also $f(f^{-1}(x)) = 2f^{-1}(x) - 5$. It follows that $f^{-1}(x) = \frac{x + 5}{2}$ or $\frac{1}{2} x + \frac{5}{2}$.

Discuss the fact that if $f$ is one-to-one then the graph of $f$ when reflected through the diagonal line $y = x$ yields the graph of $f^{-1}$.

CM: Same as above only when discussing that $f^{-1}$ "undoes" what $f$ "did" one may use the calculator in
activities which help the student realize the process. For example use \( f(x) = 2x - 5 \). Put in 3, press "times" key, press 2, press "subtraction" key, press 5, press "equals" key (indicates \( f(3) = 1 \)). State this is the output of \( f \) and to "undo" what was done we might "sort of reverse the process." That is, since \( f \) first multiplied by 2 then subtracted 5, to "reverse the process" we could add 5 then divide by 2. In particular (with 1 in the x register of the calculator), push "addition" key, push 5, push "division" key, push 2, push "equals" key (indicates \( f^{-1}(1) = 3 \)). Observe this indicates \( 3 \xrightarrow{f} 1 \xrightarrow{f^{-1}} 3 \) to in general \( x \xrightarrow{f} f(x) \xrightarrow{f^{-1}} x \) so that \( f^{-1} \) "undoes" whatever \( f \) "did". Use several examples; each time mention \( f^{-1} \) takes the output of \( f \) and "undoes" what \( f \) "did" to yield as its output \( x \), the input of \( f \) (\( f^{-1}(f(x)) = x \)).
Appendix E:

PRETEST AND POSTTEST
1. Which of the following are functions? Choose by \( \square \) as appropriate.

<table>
<thead>
<tr>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-9</td>
</tr>
<tr>
<td>6</td>
<td>-2</td>
</tr>
<tr>
<td>7</td>
<td>-4</td>
</tr>
<tr>
<td>8</td>
<td>-3</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|c|c|c|c|c|c}
\text{Domain} & \text{Range} & \text{Domain} & \text{Range} & \text{Domain} & \text{Range} & \text{Domain} & \text{Range} \\
5 & \to -9 & -4 & \to 5 & -5 & \to 100 & \text{a} & \to \text{antelope} \\
6 & \to -2 & -2 & \to 4 & -6 & \to 200 & \text{b} & \to \text{bison} \\
7 & \to -4 & 0 & \to 3 & -7 & \to 400 & \text{c} & \to \text{camel} \\
8 & \to -3 & +2 & \to 2 & -8 & \to 500 & \text{d} & \to \text{donut} \\
\end{array}
\]

2. Which of the following are graphs of functions? Choose by \( \square \) as appropriate.

3. From this graph find \( h(-3) = \), \( h(0) = \), \( h(2) = \), \( h(5) = \)
4. \( f(x) = 5x^2 + 4x \). Find \( f(-1) \)

5. \( f(x) = 2|x| + 3x \). Find \( f(-4) \)

6. \( g(x) = 5x^2 - 2x \). Find \( g(a+k) \)

7. \( f(x) = 3\sqrt{x} - 7 \). Find \( f(16) \)

8. \( f(x) = 3x + 7 \). Find \( f^{-1} \)

9. \( g(x) = \frac{\sqrt{x}}{3} + 3 \). Find \( g^{-1} \)

10. \( h(t) = 5t^3 - 200 \). Find \( h(h^{-1}(x)) \).

11-12. Here is a graph of \( y = F(x) \).

11. Sketch \( y = \frac{1}{2} F(x) \)

12. Sketch \( y = F(x) - \frac{1}{2} \)
13. \( k(x) = x^3 + 3 \). Find \( k^{-1}(k(20)) \).

14. Sketch \( A(x) = |x| \). 

15. Use the axis system below. Plot \( A(-2,-5), B(2,-2), C(7,-2), D(3,-5) \).

16. The graph specified in problem 15 is the graph of a function. \( T \) \( F \) ?

17. Find the slope of the line segment \( \overline{GH} \) given \( G(3,-3) \) and \( H(4,-6) \).
18. Use functional notation to describe \(G\) the function whose graph is the straight line through \(A(227,542)\) and \(B(-223,-458)\).

19. \(f(x) = \begin{cases} x + 5, & x \leq 0 \\ (x-3)^2 - 4, & 0 < x < 6 \\ -5x + \frac{7}{5}, & x \geq 6 \end{cases}\) 

Find the zeros of \(f\).

20. \(H(x) = \frac{1}{2}x^2 - 2x + 1\). Find the vertex.

21. If \(f(x) = x^2\), find the domain of \(f\).

22. \(f(x) = x^2\), \(g(x) = \frac{1}{x + 1}\). Find \(g \circ f\) and state the domain.

23. Name the one-to-one functions, if any, in problems 4, 5, 6 and 7.

24. \(f(x) = \frac{5}{9}(x - 32)\) is a function which converts \(x^\circ\) Fahrenheit into degrees Celsius. Find \(f^{-1}\) the function which converts degrees Celsius to degrees Fahrenheit.
25. \( g(x) = x^2 - 4x + 4 \). Find the range of \( g \).

26. \( g(x) = 5x + 1 \). Find \( g \circ g \).

27. \( f(x) = -3x + 4 \), \( g(x) = 2x - k \). Find \( k \) so that \( (f \circ g)(x) = (g \circ f)(x) \).

28. Find \( (f \circ f \circ f)(3) \) if \( f(t) = 1 - \frac{1}{t} \).

29. \( f(x) = 2x + 1 \), \( g(x) = 3x + 2 \), \( h(x) = x^2 \). Find \( f \circ (g + h) \).

30. Suppose \( (f \circ g)(x) = x^3 + 3 \) and \( f(x) = x^3 \). Find \( g \).
1. Which of the following are functions? Choose by [ ] as appropriate.

-9 → 100  apple → a  6 → 1  5  6
-8 → 101  one → 1  3 → 2  10 → 11
-7 → 102  banana → b  two → 2  0 → 3  15 → 16
-6 → 103  cranberry → c  three → 4  -3  4  20 → 21
-5 → -5  donut → d  6 → 5

2. Which of the following are graphs of functions? Choose by [ ] as appropriate.

3. From this graph find h(-3) = ___, h(0) = ___, h(2) = ___, h(5) ___
4. \( f(x) = 4x^2 + 5x \). Find \( f(-1) \)

5. \( f(x) = 2x + 3|x| \). Find \( f(-4) \)

6. \( g(x) = 4^2 - 3x \). Find \( g(b+j) \)

7. \( g(x) = 4\sqrt{x-9} \). Find \( g(25) \)

8. \( f(x) = 7x + 3 \). Find \( f^{-1} \)

9. \( g(x) = \frac{\sqrt{x}}{2} + 2 \). Find \( g^{-1} \)

10. \( h(s) = 6s^3 - 240 \). Find \( h(h^{-1}(x)) \)

11. Sketch \( y = \frac{1}{2}G(x) \)

12. Sketch \( y = G(x) - \frac{1}{2} \)
13. \( k(x) = x^3 + 5 \). Find \( k^{-1}(k(20)) \)

14. Sketch \( M(x) = -|x| \). 

15. Use the axis system below.
Plot A(-3,-4), B(1,-1), C(6,-1), D(2,-4)

16. The graph specified in problem 15 is a graph of a function. True  False

17. Find the slope of line segment MN if given M(2,-1) and N(4,-4)

18. Use functional notation to describe \( H \) the function whose graph is the straight line through A(-4,-3) B(15,632).
19. \( f(x) = \begin{cases} 
-5x + \frac{7}{5}, & x \leq 0 \\
-(x-3)^2 - 4, & 0 < x < 6 \\
-x + 11, & x \geq 6 
\end{cases} \) Find the zeros of \( f \).

20. \( H(x) = \frac{1}{3}x^2 - 2x + 1 \). Find the vertex.

21. If \( f(x) = x^3 \), find the domain of \( f \).

22. If \( f(x) = x^3 \) and \( g(x) = \frac{1}{x+8} \), find \( g \circ f \) and state the domain.

23. Name the one-to-one functions, if any, in problems 4, 5, 6, and 7.

24. \( f(x) = \frac{9}{5}x + 32 \) is a function which converts \( x^\circ \) Celsius into degrees Fahrenheit. Find \( f^{-1} \), the function which converts degrees Fahrenheit to degrees Celsius.

25. \( h(x) = x^2 - 6x + 9 \). Find the range of \( h \).

26. \( g(x) = 2x + 5 \). Find \( g \circ g \).

27. \( f(x) = -2x + 3 \), \( g(x) = 3x - k \). Find \( k \) so that \( (f \circ g)(x) = (g \circ f)(x) \).
28. Find \((h \circ h \circ h)(2)\) if \(h(s) = 1 + \frac{1}{s}\)

29. \(f(x) = 3x + 1, g(x) = 2x + 3, h(x) = x^2\). Find \(f \circ (g \circ h)\)

30. Suppose \((f \circ g)x = x^4 + 4\) and \(f(x) = x^4\). Find \(g\).