Extending the lag parameter approach of turbulence modeling

Rajarshi Biswas
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Extending the lag parameter approach of turbulence modeling

by

Rajarshi Biswas

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Aerospace Engineering

Program of Study Committee:
Paul Durbin, Major Professor
Shankar Subramaniam
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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

Iowa State University
Ames, Iowa
2019

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DEDICATION

To my parents and Atreyi.
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I thank Atreyi for the positive energy and enthusiasm for which I am very grateful. Last and most important, I thank my parents, Dr. Sudeb Biswas and Mrs. Sanghamitra Biswas for everything they have taught me and for their love & encouragement that constantly kept me motivated with this endeavour.
The work presented in this dissertation extends upon the elliptic-blending lag parameter approach for linear eddy-viscosity model (LVM). The model aims at preserving the numerical simplicity of an LVM while incorporating important features of an underlying Reynolds Stress Model (RSM). One important phenomenon in a non-equilibrium turbulent flow that the proposed model attempts to represent is the stress-strain misalignment. To this end, the model includes an additional transport equation of a field variable $\varphi^*$ (coined as a “lag” parameter) in addition to the conventional two equations ($k$ & $\omega$ for the current formulation). $\varphi^*$ is then used to suitably scale the eddy viscosity which turns out to be critical for improved predictions of non-equilibrium, separated flows in particular. Similar to the precursor models of the lag family, the proposed model preserves capability to project the six equations of an RSM onto a single equation through the definition of $\varphi^*$ as $\varphi^* = -\frac{a_{ij} S_{ij} \omega}{S}$. 

A review of the family of lag models are first presented in the dissertation. In particular, derivation of the Lag $k-\varepsilon$ model is discussed. The present formulation is a derivative of the underlying framework use for the Lag $k-\varepsilon$ version. The derivation of the new model, Lag $k-\omega$ is then discussed in detail. Two new non-dimensional variables, namely $\varphi^*$ and $\alpha$ provides the backbone for the proposed formulation.

Finally the lag models are compared for a number of flow problems, both 2D and 3D, complexity of which range from canonical to industrial in nature. The predictions are then compared to commonly used RANS models, namely Wilcox’s $k-\omega$ and Menter’s $k-\omega$ SST formulations. The advantages and shortcomings of the lag approaches compared to the other models are discussed in detail for each of the studied cases. One key benefit of introducing the lag parameter is a better scaling of the turbulent eddy viscosity particularly in the near-wall region. This is critical for improved predictions of separated flows.
CHAPTER 1. INTRODUCTION

1.1 Brief Background

Fluids tend to follow seemingly haphazard, unsteady paths when subjected to a driving force that would cause the flow velocity (often expressed in terms of Reynolds number) to go above a certain threshold. This non-linear nature of the fluid has been given the name “Turbulence” and has been a subject of decades of research. In mathematical terms, $U \cdot \nabla U$ from the famous Navier-Stokes (NS) equation can be primarily held accountable as the root cause for the “non-linearity” issue. An unfortunate outcome of this non-linearity is that a complete analytical description of a turbulent flow field is impossible to obtain. To quote the famous physicist and Nobel Laureate Dr. Richard Feynman, “Turbulence is the most important unsolved problem of classical physics”.

This motivated researchers to find alternate approaches in order to understand turbulence. Such attempts would include mathematical analysis of linearised Navier Stokes equations, carefully designed set of experiments and high-fidelity numerical simulations. From an engineering standpoint, most of the commonly encountered flow problems are highly turbulent in nature. While a typical engineer would not be concerned about the finer intricacies of why turbulence behaves the way it does, a major concern would certainly be as to how it affects the performance of a system at a macroscopic level. To that end, Computational Fluid Dynamics (CFD) has served as an indispensable mathematical tool to understand the effects of turbulence.

CFD employs the use of numerical algorithms and mathematical models to seek an approximate solution of the NS equation for a flow problem. From an industrial perspective, owing to the complexity of the problems, CFD is a standard tool for the design and development process. However owing to the purely predictive nature of CFD, the algorithms and models are in constant need of revisions in order to obtain superior accuracy for catering to the ever increasing complexity. Two
major facets that affect the performance of CFD predictions can be identified as accuracy of the numerical algorithms and the accuracy of the turbulence models implemented.

An important aspect of turbulence is the multi-scale nature. By that it means a fully turbulent flow field consists of a spectrum of scales of motion ranging over several orders of magnitude difference between the smallest and the largest eddies. Considering the unsteady micro-scale motions of turbulence is crucial to understand the physical processes at a fundamental level. However, for an engineer, the major concerns are the behaviour of the integral length scales and thus rises the need for mathematical formulations or turbulence models that cater to the issue.

Researchers have developed several ways to numerically solve the NS equations over the last few decades. These can be loosely categorized into the following: 1. Direct numerical simulations (DNS), 2. Large-eddy simulations (LES) & 3. Reynolds Averaged Navier Stokes (RANS) simulations. Although in recent years, there has been wide usage of even more sophisticated methods termed as Hybrid methods that aim to utilise the benefits of LES and RANS approaches. More on such methods will discussed later in the thesis. DNS resorts to solving the discretized NS equations on a highly resolved computational domain and therefore can represent even the minuscule scales of a turbulent flow field. This approach therefore eliminates the errors associated with modeling approximations. However, owing to the cost of computation for this approach particularly for highly inhomogeneous flow fields of high Reynolds numbers ($Re$) which typically includes a majority of the industrial problems of interest, this remains largely of academic interest. Nonetheless DNS serves as a great tool for understanding fundamental mechanisms of turbulence and for generating large data-sets that are often used for the development of turbulence models.

LES provides a better alternative in terms of computational expenses, where only the large scales of motions are resolved thereby the relaxing the domain resolution requirements. Velocity fields are decomposed into resolved and filtered components which when applied to the NS equations gives rise to an unclosed sub-grid stress tensor. The unclosed term is then modelled using some function that uses the grid resolution as an input. Argument in support of this approach is that the large scales are most dominant in determining phenomenons such heat transfer or turbulent
diffusion of momentum or scalar transport etc. Arguably the most notable contribution to this field was by Deardorff (1974), which initiated the idea of LES and since then a huge amount of work has been dedicated to the development and application of this approach even to problems of industrial relevance, e.g., Michelassi et al. (2002), Feng et al. (2015) etc. Some of the most commonly used LES models are the Smagorinsky model Smagorinsky (1963), WALE model Nicoud et al. (1999) etc., only to name a few.

Ideally, DNS or LES would be the preferred choice of an engineer provided the feasibility of computational resources. However, such is seldom true which necessitates alternatives routes to solve the NS equations. Reynolds decomposition of velocity field into a mean and instantaneous component, an approach which dates back to the early days of turbulence research provides such an alternative. The decomposition yields an extra term to the NS equations, $u'_i u'_j$, also known as the Reynolds stresses ($u'$ represents the velocity fluctuations), finding the best suited closure for which has been an area of extensive research. These classes of models known as the Reynolds Averaged Navier Stokes (RANS) models do not attempt to resolve all the scales of turbulent motion and only focuses on the integral length and time scales, thereby significantly reducing the grid resolution requirements. Also, contrary to the DNS/LES approaches, RANS formulations allow steady state computations thereby alleviating the temporal computation expenses. The trade-off being, poorer accuracy compared to DNS/LES predictions and a heavy reliance on the limitations of the RANS model used.

Finding the solution of the unclosed term $u'_i u'_j$ has multiple approaches. A highly significant idea was put forward by Joseph Valentin Boussinesq in 1877 based on the idea that the effect of turbulence on a fluid is analogous to the physical viscosity of the fluid. He introduced the “eddy-viscosity” concept, wherein the turbulent mixing was represented by a turbulent (eddy) viscosity which was a function of the flow properties. For simple shear flows, this can be mathematically represented as:

$$
\rho \frac{d\bar{U}}{dy} - \rho \bar{u}'v' = (\mu + \mu_\varepsilon) \frac{d\bar{U}}{dy},
$$

(1.1)

where $\mu_\varepsilon$ is the eddy viscosity.
Equation 1.1 can be extended to be written as:

\[-\rho u_i' u_j' = \mu_t \left( \frac{dU_i}{dx_j} + \frac{dU_j}{dx_i} \right) \]  

(1.2)

One important and the most fundamental limitation of this approach immediately becomes clear that, one single function for \( \mu_t \) cannot be sufficient to accurately predict all the components of \( u_i' u_j' \). However, despite this fundamental limitation eddy-viscosity models (EVM) where \( \mu_t \) is obtained by solving additional one, two and sometimes three additional transport equations, EVMs are of primary choice for engineers and researchers have put substantial amount of effort to improve accuracies of eddy viscosity models over the last few decades (Durbin (2018)).

Other approaches to the “closure” problem include algebraic stress modelling that involves use of non-linear higher order stress-strain terms to close \( u_i' u_j' \), such as the formulations by Wallin et al. (2000), Gatski et al. (2000) etc. Further sophisticated approaches include solving transport equations for the \( u_i' u_j' \) and using some form of closure model for the higher order unclosed terms. This approach, also known as “Reynolds stress modelling (RSM)” or “Second moment closure modelling (SMC)” increases the number of equations (one for each component of \( u_i' u_j' \) and more importantly, numerical complexity. Therefore, even though the prediction accuracy of RSMs are often higher than EVMs, they are usually not preferred. Some of the popular RSM models in use are the LRR model (Launder et al. (1975)), SSG model (Speziale et al. (1991)) and in recent years the Elliptic Blending Reynolds Stress Model (EBRSM)(Manceau et al. (2002)).

Steady state computations using one or two equation linear EVMs are by and large the most preferred choice for simulations of complex flows of industrial relevance. However, recent years have seen the rise of DES as a tool for design and development cycles. DES models rely on a RANS formulation in the near wall attached flow regions and resolve the eddies far from the wall. Therefore the accuracy of the RANS model still plays a critical part in the performance of a DES formulation. Most of the RANS models involve empiricism of some degree in their formulations which limits their use in engineering flow computations. The limitation however provides a productive area of research with the outcome being more sophisticated RANS models.
1.2 Study Objectives

Turbulent stresses are assumed to be linearly proportional to the mean strain rate in a linear eddy viscosity approach. This however is not true in most non-equilibrium or evolving flows where the misalignment can have severe effects, such as overestimation of turbulent kinetic energy (TKE). RSMs are inherently capable of representing the misalignment phenomena, however as has been mentioned before, the numerical complexities associated with them inhibits their large scale usage.

Revell et al. (2011) judiciously designed a variable $C_{as}$ that aimed to present the stress-strain misalignment issue. The definition of the parameter invokes the use of a Reynolds stress transport type equation. This enables the model to capture at least some if not all, the transport effects that a full scale RSM would otherwise be capable of. More on this is discussed in chapter 4. Building upon the success of the $C_{as}$ lag model, Lardeau et al. (2016) combined it with an underlying $k-\varepsilon$ linear eddy viscosity model (LEVM). Their formulation demonstrated significant improvement in prediction for separated flows and flows with rotation and curvature effects compared to predictions by commonly used models such as $k-\omega$ shear stress transport (SST).

For the current work, the approach of Lardeau et al. (2016), is extended to an underlying $k-\omega$ framework. The primary objective was to alleviate the limitations of the existing $k-\omega$ class of models, for example, inferior prediction of adverse pressure gradient flows, by improving on the near wall scaling of the eddy-viscosity (more on which will be discussed later).

1.3 Outline of the thesis

Chapter 2 outlines the basic fundamentals of turbulence and the predictive approaches. RANS modelling approaches being the focus of this work are discussed here. Various avenues in the RANS framework such as one and two equation models, that has evolved over the years are explained. Popular formulations under the linear eddy viscosity branch such $k-\varepsilon$ and $k-\omega$ variants are discussed.

Chapter 3 describes the issues concerned with near wall turbulence and modelling considerations. Background of an important concept, elliptic relaxation approach; a key component in the
proposed new model is discussed in detail in this chapter. Details of a novel Reynolds stress model implementing the elliptic relaxation is also mentioned.

Chapter 4 summarises the family of “lag” models. The derivation and calibration of the proposed formulation is described in this chapter.

Chapter 5 focuses on the numerical aspects of the simulations. Details of the discretization schemes, solvers used and boundary conditions are discussed. Computational expenses of the proposed formulation are also discussed.

Chapter 6 discusses in detail the application of the model on canonical flow cases, both 2D and 3D. The advantages and limitations of the model are scrutinised through these cases. The results are investigated thoroughly.

In chapter 7, certain test cases, accurate predictions of which proved to be challenging with the lag approach are covered. Finally chapter 8 concludes the work and proposes future investigations.
CHAPTER 2. THEORY OF TURBULENCE MODELLING

The work presented in this thesis deals with the development and assessment of a Reynolds averaged turbulence model. A brief theoretical background for turbulence modelling is therefore necessary.

2.1 Navier-Stokes equations

Fluid motion is governed by the Navier-Stokes equations. Conservation of mass and momentum are the underlying principles behind the NS equations. For an incompressible flow, i.e., considering the fluid density ($\rho$) to be constant, NS equations are written as:

$$\frac{\partial U_i}{\partial x_i} = 0 \quad (2.1)$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} \quad (2.2)$$

where $U$ denotes the velocity field and $p$ denotes pressure. Analytical solutions for the above equations exist for simplistic laminar flows (Couette, Poiseuille etc.). However, such solutions are not possible for turbulent flows. To this end, statistical approaches are necessary to find approximate solutions.

In 1895, Osborne Reynolds introduced the concept of decomposition of the velocity field into a mean and fluctuating part:

$$U = \overline{U} + u', \quad (2.3)$$
where $\overline{U}$ is interpreted as the ensemble-averaged and $u'$ as the fluctuating component. This decomposition when used in Eqns. 2.1 & 2.2, the following set of equations known as the Reynolds Averaged NS equations are obtained:

$$\frac{\partial \overline{U}_i}{\partial x_i} = 0 \quad (2.4)$$

$$\frac{\partial \overline{U}_i}{\partial t} + \overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 \overline{U}_i}{\partial x_j^2} - \frac{\partial u'_i u'_j}{\partial x_j} \quad (2.5)$$

The extra term appearing in Eqn. 2.5 are called Reynolds stresses. Dimensionally and behaviourally the term is analogous to the fluid viscous stress term $(\nu \frac{\partial^2 \overline{U}_i}{\partial x_j^2})$. $u'_i u'_j$ forms a symmetric tensor, finding a closure for which is the crux of RANS turbulence models.

### 2.2 Turbulence

As mentioned earlier, turbulence is manifestation of non-linear term $\overline{U}_j \frac{\partial \overline{U}_i}{\partial x_j}$ from Eqn. 2.5. Beyond a critical Reynolds number, instabilities start to evolve in a laminar flow and upon further increase in $Re$, eventually develops into a fully turbulent flow. Looking at a turbulent flow field of high $Re$, the primary observation that can be made is the presence of a plethora of structures with a wide dimensional spectrum. These structures, commonly known as “eddies” characterize a turbulent flow field and govern the fundamental mechanisms such as transfer of momentum & heat, transport of a scalars etc., inside a flow domain. While the size of the largest eddies are typically controlled by the size of the domain under consideration, the dimensions of the smallest eddies however, are governed by the molecular viscosity and the dissipation rate $(\varepsilon)$. Larger turbulent eddies are short lived in nature and have a finite life-time before they transfer their energy content into smaller eddies and the process continues till the smallest of the smaller eddies disappear due to the action of fluid viscosity. This transfer of energy is also known as “cascading”. Therefore, turbulent eddies have a characteristic length and time scale. The larger eddies have velocities in the order of $\sqrt{k}$, where $k$ denotes the energy content, while the smallest eddies’ velocity is in the order of $(\varepsilon r)^{1/3}$, with $r$ being the size of the eddies in the inertial range. In 1941, Kolmogorov
proposed a series of theories that elaborates on the nature of *eddies* in great detail. Key points from Kolmogorov's theories are (as quoted from Pope(2000)),

"1. At sufficiently high Reynolds number, the small-scale turbulent motions are statistically isotropic.
2. In every turbulent flows, at sufficiently high Reynolds number, the statistics of the small-scale motions have universal form that is uniquely determined by $\nu$ and $\varepsilon$.
3. In every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale $l$ in the range $l_0 \geq l \geq \eta$ have a universal form that is uniquely determined by $\varepsilon$, independent of $\nu$."

Mathematically these translate to a definition of the smallest scales or Kolmogorov length and time scales:

$$\eta \equiv (\nu^{1/3}/\varepsilon)^{1/4} \quad (2.6)$$

$$\tau_{\eta} \equiv (\nu/\varepsilon)^{1/2} \quad (2.7)$$

Another important outcome of the theories is the famous "-5/3" law that states that in the inertial range,

$$E(\kappa) \approx \varepsilon^{2/3} \kappa^{-5/3}, \quad (2.8)$$

where, $\kappa$ represents the wave number of the scales of motion and is defined as $\kappa = \frac{2\pi}{l}$, $l$ being the size of the scale. Eqn. 2.8 is universal in nature and often serves as a checkpoint to assess the validity of high-fidelity simulations.

### 2.3 Reynolds Averaged Navier Stokes Formulations

Unsteady computations using DNS/LES provide a great depth of understanding of a turbulent flow field. However, design engineers are often most concerned about the “mean” effect of turbulence
on a system. Fortunately, the chaotic nature of turbulence can be interpreted as oscillations about a mean value when a sufficiently long temporal statistic is recorded. RANS formulations are designed to predict such mean values of flow properties, such as velocity($U$), pressure($P$) etc. The mean and fluctuating component of a variable $\phi$ can be written as,

$$\phi = \bar{\phi} + \phi', \quad \text{(2.9)}$$

with,

$$\bar{\phi}(x) = \lim_{T \to \infty} \int_0^T \phi(x, t) dt, \quad \text{(2.10)}$$

where, $T$ denotes the total time interval for which the statistics are collected. The fluctuating component has the property, $\bar{\phi} = 0$. As mentioned earlier, decomposition of this kind when introduced into Eqn. 2.2 creates extra term $u'_i u'_j$ that represent the effect of turbulence on the mean flow. If a similar decomposition method is carried out for a velocity field obtained using experiments or DNS, one would find that that other than homogeneous box turbulence, the fluctuations are usually anisotropic in nature and often not proportional to the mean strain rate even for the simplest of flows. However, Bousinessq’s turbulence-viscosity hypothesis models the stress term as proportional to the mean rate of strain, with the proportionality constant being an eddy-viscosity. One key implication of the hypothesis is that the anisotropy tensor ($a_{ij}$) is aligned with the mean strain rate tensor, i.e.,

$$a_{ij} = \frac{u'_i u'_j}{3} - \frac{2}{3} k \delta_{ij} \equiv -\nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \equiv -2\nu_t \overline{s_{ij}} \quad \text{(2.11)}$$

This assumption forms the basis for eddy viscosity models, the impact of which will be covered in further detail in the following sections.
2.3.1 Reynolds Stress Models

Reynolds stress models do not follow the Boussinesq approximation and solve transport equations for each component of the $\overline{u_i'u_j'}$ tensor. The generic transport equation for the components can be written as,

$$
\frac{D u'_i u'_j}{Dt} = -\frac{1}{\rho} \left( u_j \frac{\partial p}{\partial x_i} + u_i \frac{\partial p}{\partial x_j} \right) + \frac{\nu}{\rho} \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} - 2 \nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \frac{\epsilon_{ij}}{\rho} - \frac{\partial (u_k u_i u_j)}{\partial x_k} \frac{\partial U_i}{\partial x_k} + \frac{\partial U_j}{\partial x_k} + \nu \nabla^2 u_i u_j, \tag{2.12}
$$

where, $\Pi_{ij}$, $\epsilon_{ij}$, $D_{ij}^T$, $P_{ij}$ and $D_{ij}^\nu$ are the pressure-velocity correlation, dissipation, turbulent transport, production and molecular diffusion terms respectively. $P_{ij}$ being a closed term do not require any modelling assumptions, whereas the other terms except for $D_{ij}^\nu$ require some form of closure.

- **Modelling $\Pi_{ij}$**: For modelling purposes, this term is split into two components,

$$
\frac{1}{\rho} \left( u_j \frac{\partial p}{\partial x_i} + u_i \frac{\partial p}{\partial x_j} \right) = \frac{1}{\rho} \left( u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{\rho} \frac{\partial}{\partial x_i} (\overline{u_i p \delta_{jk}} + \overline{u_j p \delta_{ik}}), \tag{2.13}
$$

where, $\phi_{ij}$ is called the pressure-strain term since it takes the role of redistribution of the energy amongst the stresses. The tendency of this term is to bring the turbulence to an isotropic state and therefore is given the name return to isotropy term. $D_{ij}^p$ is diffusive in nature and is often modelled with the turbulent transport term since it is most dominant only in close proximity of the wall.

$\phi_{ij}$ is a key term that requires careful closure for which the Poisson equation is used,

$$
\nabla^2 p = -2 \rho \frac{\partial \overline{U_i}}{\partial x_j} \frac{\partial u_j}{\partial x_i} - \rho \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j - \overline{u_i u_j}) \tag{2.14}
$$
An integral form of the pressure-strain rate correlation term can be obtained by multiplying the solution of Eqn. 2.13 with the gradient of the fluctuating velocity,

\[ \rho \phi_{ij}(x) = - \int_{\Omega} \nabla^2 p(x') \left( \frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right) \frac{\partial V(x')}{{4\pi||x' - x||}} \]

\[ - \int_{\partial \Omega} p(x') \left( \frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right) \frac{\partial}{\partial n'} \left( \frac{1}{{4\pi||x' - x||}} \right) dS(x') \]

\[ + \int_{\partial \Omega} \frac{\partial p}{\partial n}(x') \left( \frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right) \frac{\partial}{\partial n'} \left( \frac{S(x')}{{4\pi||x' - x||}} \right) \]

The surface terms are neglected since \(||x' - x||\) becomes large away from wall, therefore leaving the volume term to be split into “slow” (\(\phi_{ij}^{\text{slow}}\)) and “rapid” (\(\phi_{ij}^{\text{rapid}}\)) parts as,

\[ \phi_{ij}^{\text{slow}} = \int_{\Omega} \frac{\partial^2 \left( u_i u_m - u_i u_m \right)}{\partial x_j \partial x_m}(x') \left( \frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right) \frac{\partial V(x')}{{4\pi||x' - x||}} \]

\[ \phi_{ij}^{\text{rapid}} = \int_{\Omega} \frac{\partial U_i}{\partial x_m}(x') \frac{\partial u_m}{\partial x_i}(x) \left( \frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) \right) \frac{\partial V(x')}{{2\pi||x' - x||}} \]

Use of the terminology “rapid” only refers to the term responding instantaneously to the changes in mean flow contrary to the “slow” term. \(\phi_{ij}^{\text{slow}}\) is modelled considering a scenario where the mean strain is removed. This would lead to the turbulence gradually settling to an isotropic state. Rotta (1951) proposed a linear relaxation model,

\[ \phi_{ij}^{\text{slow}} = -C_1 \varepsilon a_{ij}, \]

where, \(a_{ij} = \frac{\overline{u_i' u_j'}}{k} - \frac{2}{3} \delta_{ij}\) is the anisotropy tensor. Development of models for the \(\phi_{ij}^{\text{rapid}}\) term have seen a lot of contribution from researchers and is based on the assumption of quasi-homogeneity away from the wall. Near the wall however, this assumption fails therefore necessitating alternative routes which will be discussed later in chapter 3. Simplest form of such model uses the concept of Isotropisation of production (IP), i.e., the production tensor returns to an isotropic state. However, more complex models such as the LRR (Launder et al.}
implement $\phi_{ij}^{\text{rapid}} = f(a_{ij}, S_{ij}, \Omega_{ij})$, therefore having terms of higher orders. For example, $\phi_{ij}$ from SSG model reads,

$$\begin{align*}
\phi_{ij} &= -C_1 a_{ij} + C_2 \varepsilon (a_{ik} a_{kj} - \frac{1}{3} A_2) + C_3 k S_{ij} \\
&\quad + C_4 k (a_{ik} S_{jk} + a_{jk} S_{ik} - \frac{2}{3} a_{mn} S_{mn} \delta_{ij}) \\
&\quad + C_5 k (a_{ik} \Omega_{jk} + a_{jk} \Omega_{ik})
\end{align*}$$

(2.19)

- **Modelling $D_{ij}^T$:** Most common closure of this term is done using the gradient diffusion theory (Daly et al. (1970)) and takes the form,

$$D_{ij}^T = -\frac{\partial}{\partial x_k} \left( C_s \frac{k}{\varepsilon} \frac{u_{ik} u_{lj}}{\partial x_l} \right).$$

(2.20)

$C_s \approx 0.22$ Launder (1989). As mentioned earlier, $D_{ij}^P$ is usually modelled along with this term and therefore takes the form,

$$-\frac{\partial}{\partial x_k} \left( \frac{u_{ik} u_{lj}}{\partial x_l} - \frac{2}{3} \frac{u_{ik} P_l}{\rho} \delta_{ij} \right) = -\frac{\partial}{\partial x_k} \left( C_s \frac{k}{\varepsilon} \frac{u_{ik} u_{lj}}{\partial x_l} \right).$$

(2.21)

- **Modelling $\varepsilon_{ij}$:** To avoid a term by term modelling of the complex exact transport equation of $\varepsilon_{ij}$, the term is usually modelled as,

$$\varepsilon_{ij} = \frac{2}{3} \varepsilon \delta_{ij} + \left( \varepsilon_{ij} - \frac{2}{3} \varepsilon \delta_{ij} \right)$$

(2.22)

The second term on the right is combined with pressure-strain rate term. The solution of $\varepsilon$ is obtained by solving an empirically formulated transport equation of the form,

$$\frac{D\varepsilon}{Dt} = C_{\varepsilon 1} P_k \varepsilon - C_{\varepsilon 2} \frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_k} \left( \nu + C_{\varepsilon} \frac{k}{\varepsilon} \frac{u_{ik}}{\partial x_l} \frac{\partial \varepsilon}{\partial x_l} \right),$$

(2.23)

where, $P_k = \frac{1}{2} P_{ii}$. 
2.3.2 Eddy viscosity Models

Eddy viscosity models seek to find the solution for the scalar $\nu_t$. $\nu_t$ is dependent upon the local flow conditions and is defined as the product of the a time scale and velocity scale which makes it dimensionally consistent with kinematic viscosity. Usually, solution of the parameters that are used to define the time & velocity scales are obtained by solving transport equations. These parameters represent essential features of turbulence such as the energy content, dissipation rate etc. In the crude versions of eddy viscosity models, $\nu_t$ was simply defined without taking into consideration the intricacies of turbulence mechanisms.

Zero equation model

Prandtl proposed a mixing length model in 1925 based on the kinetic theory of gases. Eddy viscosity was defined as $\nu_t = l_m^2 |\frac{\partial \overline{U}}{\partial y}|$, where $\overline{U}$ was the streamwise velocity component, $y$ was the wall normal distance and $l_m$ was an empirical length scale. The rationale behind this model was not completely justifiable, and the predictions depended heavily on the choice of $l_m$. The mixing length model also did not account for the effect of pressure gradients and wall effects on turbulence. Van Driest proposed a viscous damping function to account for the wall effects on turbulent mixing. The Baldwin and Lomax model (Baldwin et al. (1978)) used rate of rotation in the definition of $\nu_t (\nu_t = l_m^2 |\omega|)$ and the formulation was targeted for thin-layer NS equations. Predictions were however not accurate particularly for massively separated flows and certain modifications were necessary to predict mildly separated boundary layer flows. Using the a priori definition of velocity and time scales in these approaches, it was often difficult to assess the appropriately prescribe those quantities for challenging flow problems (e.g., highly non-parallel flows, flows with curvature effects, shear layers intersecting at angles etc.). These limitations motivated researchers to formulate transport equations that would first find solutions of the turbulent scales using the flow conditions and then compute a $\nu_t$.

One equation model

Obtaining solution of eddy viscosity directly through a transport equation is an efficient approach in terms of computations expenses. The Spalart-Allmaras (SA) model (Spalart et al. (1992))
is one such formulation that is widely used in industries. The model solves a transport equation for a field variable $\tilde{\nu}$ and $\nu_t$ is defined as a function of $\tilde{\nu}$. The model is capable of producing accurate predictions for situations involving rotation-curvature effects, strong adverse pressure gradients and so on. Detailed explanation of SA model is however beyond the scope of this thesis.

**Two equation models**

Additional transport equations aim to extract more information from a flow field and therefore can be loosely expected to make more accurate predictions. To that end, two equation models serve as an optimum balance between accuracy and numerical complexity. A vast amount of work has been dedicated into the development of two equation models over decades and such models are perhaps the most popular formulations employed in industrial CFD codes. Two equation turbulence models usually employ the usage of a velocity scale that is defined as $\sqrt{k}$, where $k$ represents the TKE and a time scale that is a function of the turbulent dissipation.

TKE is defined as $k = \frac{1}{2}u_i u'_i$. Starting from Eqn. 2.5, one can derive the transport equation for $k$:

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = -\frac{1}{\rho} \frac{\partial}{\partial x_i} \left( \frac{u_i}{\rho} \right) - \nu \frac{\partial}{\partial x_k} \left( \frac{u_i}{\partial x_k} \right) - \frac{1}{2} \frac{\partial}{\partial x_k} \left( \frac{u_i u'_i}{\partial x_k} \right) - \nu_t u_k \frac{\partial U_i}{\partial x_k} + \nu \nabla^2 k \quad (2.24)$$

The production term ($P_k$) is a closed term and from RANS perspective is defined as $P_k = 2\nu_t S_{ij} S_{ij}$. A simplified transport equation is usually solved to determine the dissipation term ($\varepsilon$), although strictly speaking a rigorous transport equation can be derived from the dissipation term in Eqn. 2.24. Leschziner (2015) covers in great detail the intricacies involved with deriving a transport equation for $\varepsilon$. Pressure-diffusion and turbulent transport terms are grouped together and a simple gradient diffusion hypothesis is used for their closure:

$$-\frac{1}{\rho} \frac{\partial}{\partial x_i} \left( \frac{u_i}{\rho} \right) - \frac{1}{2} \frac{\partial}{\partial x_k} \left( \frac{u_i u'_i}{\partial x_k} \right) = \frac{\partial}{\partial x_k} \left( \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_k} \right), \quad (2.25)$$

where $\sigma_k$ is usually chosen as 1.
A second equation is solved in conjunction with the $k$ equation for either the dissipation rate ($\varepsilon$) or specific dissipation rate ($\omega$). Two prominent formulations that are in use are the $k-\varepsilon$ (Jones et al. (1972)) & $k-\omega$ (Wilcox (1988)).

### 2.3.3 $k-\varepsilon$ model

At sufficiently high $Re$, for a shear flow, the magnitude of the dissipation rate and production ($P$) of TKE are of the same order, therefore giving rise to the following relation (Durbin et al. (2010)):

$$\nu_t \varepsilon \approx \nu_t P = \nu_t \left( -\overline{u'v'} \frac{\partial \bar{U}}{\partial y} \right) = (\overline{u'v'})^2 \approx 0.09k^2$$ (2.26)

From experimental observations, in the log layer, $\frac{\overline{u'v'}}{k} \approx 0.3$. Therefore, on rearranging Eqn. 2.26, $\nu_t$ can be defined as,

$$\nu_t = C_\mu \frac{k^2}{\varepsilon},$$ (2.27)

where, $C_\mu = 0.09$. The time-scale therefore can be defined as $\tau = \frac{k}{\varepsilon}$. The full set of transport equations for $k-\varepsilon$ model are,

$$\frac{Dk}{Dt} = P - \varepsilon + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right],$$ (2.28)

$$\frac{D\varepsilon}{Dt} = \frac{C_{\varepsilon 1} P - C_{\varepsilon 2} \varepsilon}{\tau} + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} \right],$$ (2.29)

Analytical solutions of the $k-\varepsilon$ model are used to determine the values of the coefficients. For example, $C_{\varepsilon 2}$ is determined using the experimental data from a decaying homogeneous turbulence and usually takes a value of 1.83 or 1.92. $C_{\varepsilon 1}$ is determined using the growth rate relation of a homogeneous shear flow (Eqn. 2.30),

$$\frac{P}{\varepsilon - 1} = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon 1} - 1}$$ (2.30)

and takes a value of 1.44.
In the log region, under equilibrium, production is equal to the dissipation. Using this argument, one can derive the following relation between the coefficients,

$$\kappa^2 = (C_{\varepsilon 2} - C_{\varepsilon 1})\sigma_{\varepsilon}\sqrt{C_{\mu}},$$

(2.31)

where, $\kappa = 0.41 \pm 0.2$ is the Von Karman constant. Eqn. 2.31 is used to determine the value of $\sigma_{\varepsilon}$ and is usually $\approx 1.3$.

Certain issues needs to be addressed with the $k - \varepsilon$ formulation before a successful implementation into CFD codes. First, the boundary condition for $\varepsilon$ can be obtained using a limiting behaviour analysis of Eqns. 2.28 and 2.29. The wall value for dissipation is,

$$\varepsilon_{\text{wall}} = \lim_{y \to 0} \frac{2\nu k}{y^2}$$

(2.32)

Second, since the boundary condition for $k$ at wall is $k_{\text{wall}} = 0$, it can be easily deduced that the $\tau$ definition in Eqn. 2.29 would lead to a singularity issue. To avoid this, Durbin (1991), used the Kolmogorov time scale near the wall and reformulated $\tau$ as,

$$\tau_{\text{lim}} = \max\left(\frac{k}{\varepsilon}, 6\sqrt{\frac{\nu}{\varepsilon}}\right)$$

(2.33)

This wall scaling was backed by the DNS data from Antonia et al. (1994) and the coefficient '6' was calibrated based on the data. Irrespective of this corrections, the near-wall predictions of the $k - \varepsilon$ model were inaccurate owing to the fact $k$ is not the correct velocity scale in the near-wall region. To alleviate this, ad hoc damping functions to damp the eddy-viscosity were proposed by researchers (e.g., Launder et al. (1974)) In close proximity of the wall, turbulent transport is mostly dominated by the $\nu^2$ component of the fluctuations and Durbin (1991) proposed in detail a correction for the deficiencies. Brief description of such near wall scaling will be discussed in chapter 3. In addition to the damping functions, the epsilon equation also needed to modified and in the work by Launder et al. (1974), an additional term of the form $2\nu m\left(\frac{\partial^2 U}{\partial y^2}\right)^2$ was introduced to a modified epsilon ($\tilde{\varepsilon}$) equation. This additional term is active in the buffer region and acts as a low-Reynolds number correction for the $k - \varepsilon$ model.
2.3.4 $k-\omega$ model

The deficiencies associated with the $\varepsilon$ equation can be circumvented by defining a parameter as inverse of the turbulent time scale ($\omega$) and solving a transport equation for the same. Wilcox (Wilcox (1988)) chose the definition $\omega = \frac{\varepsilon}{C_\mu k}$. The transport equations for the $k-\omega$ model are:

$$\frac{Dk}{Dt} = 2\nu_t |S|^2 - C_\mu k \omega + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right], \quad (2.34)$$

$$\frac{D\omega}{Dt} = 2C_{\omega 1} |S|^2 - C_{\omega 2} \omega^2 + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right], \quad (2.35)$$

with $\nu_t = \frac{k}{\omega}$. The primary advantage of this approach is the dissipation equation gets rid of the singularity issue without any usage of wall functions or damping. However, upon a limiting behaviour analysis of the transport equations, one can see that $k$ does not behave as $y^2$ near the wall, and does not represent TKE in a strict sense. Also, the near-wall behaviour of the dissipation ($C_\mu k \omega$) is different from $\varepsilon$ from the $k-\varepsilon$ model, due to the fact that the $\omega$ equation lacks a source term which would otherwise remain prevalent particularly in the viscous sub layer provided a direct substitution of variables was made in Eqn. 2.29. The coefficients $C_{\omega 1}$, $C_{\omega 2}$ & $\sigma_\omega$ are calibrated using the same approach as $k-\varepsilon$ model and takes values of $\frac{5}{9}$, $\frac{3}{40}$ and 2 respectively. $\sigma_k = 2$ for this case. Certain shortcomings of this formulation was noted, particularly free-stream condition dependence of the $\omega$ equation and inaccurate prediction of shear stress when the flow is subjected to adverse pressure gradients.

2.3.5 $k-\omega$ SST model

Menter (1994) attempted to fix the deficiencies of Wilcox’s $k-\omega$ formulation through what he called the Shear stress transport (SST) model. The overestimation of the stress-intensity ratio ($\frac{\bar{u}' \bar{v}'}{k}$) was addressed by applying a bound of $\nu_t$,

$$\nu_t = \min \left( \frac{k}{\omega}, \frac{\sqrt{C_\mu k}}{2\Omega} \right), \quad (2.36)$$
where $|\Omega| = \frac{1}{2} |\frac{\partial U}{\partial y}|$, is the magnitude of the mean rotation-rate tensor. This essentially makes the definition $\frac{u'v'}{k}$ as,

$$\frac{u'v'}{k} = \min\left(\frac{|\partial U|}{\omega}, 0.3\right)$$

(2.37)

By circumventing this overestimation issue, predictions for adverse pressure gradient and separated flows were significantly improved by the SST model. However, this approach was detrimental for free shear flows, and to counter that Menter introduced a blending function of the form,

$$F_2 = \tanh(\text{arg}_2^2)$$

$$\text{arg}_2 = \max\left(\sqrt{\frac{k}{C\mu\omega y}}, \frac{500\nu}{\omega y^2}\right)$$

(2.38)

This blending function was introduced in the denominator of the second term of Eqn. 2.36.

Menter also addressed the free stream sensitivity issue by proposing yet another blending function to blend between the $k-\omega$ in near-wall region and $k-\varepsilon$ in far-from-wall regions. The blending formulation was given as,

$$F_1 = \tanh(\text{arg}_1^4)$$

$$\text{arg}_1 = \min\left(\max\left(\sqrt{\frac{k}{C\mu\omega y}}, \frac{500\nu}{\omega y^2}\right), \frac{2k\omega}{y^2\max(\Delta k, \Delta \omega, 10^{-20}D)}\right)$$

(2.39)

The above formulation takes a value of unity till the inner half of boundary layer and then gradually drops to zero at the outer edge.
CHAPTER 3. TURBULENCE NEAR A SOLID WALL

The presence of a solid wall in a flow domain has influences the turbulence mechanisms significantly. In the proximity of the wall, viscosity has a strong influence on the transport of turbulence. For a boundary layer under zero pressure gradient, “near-wall” region would include the viscous sublayer and the buffer layer. In a broad sense, the presence of the wall suppresses the wall-normal fluctuations therefore having substantial ramification on the skin friction (thereby the drag forces) and heat transfer. Therefore, from an engineering view point, understanding the near-wall turbulence is critical for improving turbulence models. This chapter first presents the different regions in a parallel shear flow along a direction normal to the wall and then briefly explains the non-local wall effects and approaches to tackle such phenomenons from a modelling perspective.

3.1 Regions in a parallel shear flow

A flow bounded by smooth parallel walls driven by a pressure gradient, commonly known as the channel flow is the most basic non-homogeneous turbulent shear flow. Non-homogeneity of turbulence only in the wall-normal direction along with the 2D parallel flow assumption \( U(y) = f(y) \) simplifies Eqn. 2.5,

\[
-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \frac{uv - \nu \frac{\partial U}{\partial y}}{\partial y} \right) \quad (3.1)
\]

\[
-\frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial v^2}{\partial y} \quad (3.2)
\]

Since, \( \frac{\partial v^2}{\partial y} = 0 \), therefore, \( \frac{\partial p}{\partial y \partial x} = 0 \), which implies \( \frac{\partial p}{\partial x} = constant \). Pressure gradient is then linked with the skin friction by integrating Eqs. 3.1, 3.2,

\[
-\frac{2H}{\rho} \frac{\partial p}{\partial x} = 2\nu \frac{\partial U}{\partial y} \bigg|_{y=0} = \frac{2\tau_{wall}}{\rho} \equiv 2u_r^\tau, \quad (3.3)
\]
where $H$ is the channel height.

Eqn. 3.3 when substituted in Eqns. 3.1, 3.2 along with the symmetry conditions of a channel flow yields,

$$u_r^2 = H \frac{\partial}{\partial y} \left( \frac{u^2}{H} - \nu \frac{\partial U}{\partial y} \right)$$

(3.4)

Non-dimensionalizing all the variables using the friction velocity($u_r$) and kinematic viscosity ($\nu$),

$$y^+ = \frac{y u_r}{\nu}, \quad U^+ = \frac{U}{u_r}, \quad \overline{uv}^+ = \frac{\overline{u v}}{u_r^2} \quad & \quad Re_r = \frac{u_r H}{\nu},$$

the following relationship is obtained,

$$\frac{dU^+}{dy^+} - \overline{uv}^+ = 1 - \frac{y^+}{Re_r},$$

(3.5)

therefore suggesting that the total stress (viscous plus Reynolds stresses) decrease linearly as the distance from the wall increases. Eqn. 3.5 is particularly important for identifying the different regions(Figure 3.1a) in a channel flow as discussed below.

Assuming the flow to have a sufficiently high Reynolds number(i.e., $Re_r \gg 1$), for small values of $y^+$, Eqn. 3.5 simplifies to $\frac{dU^+}{dy^+} - \overline{uv}^+ = 1$. Combining this with continuity $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}$, the behaviour of $U^+$ can be seen to vary linearly with $y^+$. This behaviour is universal in nature for shear flows under equilibrium and the entire region is called as the “viscous sublayer”.

Turbulent fluctuations derive their energy content from the mean flow. The mean momentum is high in the channel center and diffuses as the wall is approached till viscosity reduces it to zero. The intermediate region between the wall and center is of particular interest and forms an important check point for development of turbulence models. In this region, also known as the “logarithmic region” $\overline{uv}^+ \approx 1$. Using the dimensional definition of eddy viscosity and using the velocity scale as $u_r$, eddy viscosity can be defined as $\nu_t = \kappa u_r y$. Also for a parallel shear flow, the following relationship holds,

$$\overline{uu} = -\nu_t \frac{\partial U}{\partial y}$$

(3.6)
Figure 3.1: $U^+$ vs $y^+$ for (a) channel flow for $Re_\tau = 2000$ (DNS data of Lee et al. (2015)), (b) Zero pressure gradient boundary layer at $Re_\theta = 2000$ (DNS data of Schlatter et al. (2010)).

Making the substitutions in Eqn. 3.6 and integrating, the famous relationship,

$$U^+ = \frac{1}{\kappa} ln(y^+) + B,$$

(3.7)
can be obtained. $\kappa$ usually takes a value of 0.41 while $B$ is $\approx 5.2$ and was calibrated using experimental data.

Viscous and log layer regions are universally applicable to boundary layers too (Fig. 3.1b) as long as no separation is involved. The difference occurs above the log region, also known as the “wake region”, the shape of which depends on the pressure gradient that the boundary layer is
subjected to. A wake parameter and function is often used to characterise this region (Kline et al. (2010)).

### 3.2 Behaviour of turbulence near the wall

Assuming the turbulence variables to be continuous functions of $y$ and then applying the no-slip conditions, Taylor-series expansions can be derived for the variables which provide knowledge about their behaviour. The following sets of equations summarise the same for $U, u', v'$ and $w'$ (Eqn. 3.8),

\[
\begin{align*}
U &= A_1 y + O(y^2) \\
V &= B_1 y^2 + O(y^3) \\
W &= C_1 y + O(y^2) \\
u' &= a_1 y + O(y^2) \\
v' &= b_1 y^2 + O(y^3) \\
w' &= c_1 y + O(y^2)
\end{align*}
\]

(3.8)

Important observation from the asymptotic analysis is that the wall-normal fluctuation ($v'^2$) is two orders of magnitude smaller than $u'^2$ and $w'^2$. Also, the shear stress component $u'v'$ has a behaviour if $O(y^3)$ near the wall. Both these conclusions are critical in nature from a RANS modelling perspective. The first one shows that the turbulence is highly anisotropic in the near wall region with $v'^2$ being the dominant component and thereby provides an important velocity scale to correctly dampen the eddy-viscosity in the near wall region, while the second one says what an outcome from the RANS model should be. Although in most cases the $u'v'$ returned by RANS
models show a departure from the theoretical behaviour of $O(y^3)$. Other important conclusions from the analysis are the summarised in Eqn. 3.9,

$$\begin{align*}
    k & \sim O(y^2) \\
    \lim_{y \to 0} \epsilon & = \lim_{y \to 0} \nu \frac{\partial^2 k}{\partial y^2} \\
    \omega & \sim O(y^{-2})
\end{align*}$$

(3.9)

Critical to note here that, the parameter $k$ defined in $k - \omega$ based RANS models do not follow the theoretical behaviour of $O(y^2)$ and in most circumstances do not represent TKE.

Billard (2011) covers in great detail an analysis of the near-wall behaviour of various quantities of interest from a RANS perspective. Important points from the analysis are, variables $U^+, y^+dU^+/dy^+$ and $\nu_t^+ = \nu v' \frac{dU^+}{dy^+}$ are dependent on $Re_\tau$ and need careful modelling for low values of $Re_\tau$. Also, the time scale and $\overline{v'^2}$ are weakly dependent on $Re_\tau$.

### 3.3 Near-wall modeling

Reynolds stress and eddy-viscosity models are ideally expected to accurately represent the behaviours of the turbulent variables as mentioned in the section above. However, careful modelling considerations are necessitated to ensure the same.

As seen already the wall normal fluctuations are suppressed more vigorously than the other two components, therefore it is imperative to focus on modelling the effects of this impairment. In this light, damping functions were proposed to replicate this phenomenon which relied mostly on ad-hoc functions that usually had $y^+$ or turbulent Reynolds number($Re_t = \frac{k^2}{\epsilon \nu}$) as inputs. For LEVMs, damping function was usually multiplied with $\nu_t$ and the goal was to match the RANS $\nu_t$ to the one calculated using DNS. Patel et al. (1984) has an extensive review on the different types of damping functions proposed for near-wall corrections. Modelling the pressure-
strain correlation term of Reynolds stress models follows similar approach. Models developed using quasi-homogeneous turbulence away from the wall fail near the wall and some formulations such as Chen (1995) or Gibson et al. (1978) aim to correct that using damping functions. Further sophisticated models (e.g., Craft (1998)) selectively damp the components of the Reynolds stresses.

3.3.1 Elliptic-relaxation

To tackle the non-local wall effects on the turbulence, Durbin (1993) proposed an approach of “elliptic relaxation”. Contrary to the wall-echo approach where an additive correction is made, elliptic relaxation introduces a non-homogeneous elliptic equation. The underlying concept of Durbin’s approach is that the terms of Eqns. 2.16 & 2.17 can be written as,

\[ f(x) = g(x') \exp\left(-\frac{|x_i - x'_i|}{L}\right) \]  

This formulation ensures that the correlation between turbulence phenomenons at \( (x'_i) \) and \( (x_i) \) exponentially decay with a rate governed by the length scale \( (L) \) which also dictates the region within which the functions \( f \) & \( g \) are correlated. This approximation when substituted into the sum of the slow and rapid parts (Eqns. 2.16, 2.17), and then by algebraically manipulating the velocity fluctuations and mean strains term, the following form can be obtained,

\[ \Phi_{ij} = \int_{\Omega} \left( A'_{ij} + B'_{ij} \right) \left( e^{-|x_i - x'_i|/L} \right) \frac{1}{L} dV(x'_i) \]  

In Eqn. 3.11, the term in the bracket is the Green’s function corresponding to the differential operator \( \frac{1}{L^2} - \nabla^2 \), therefore the equation can be re-writtien as,

\[ \Phi_{ij} - L^2 \nabla^2 \Phi_{ij} = L^2 \Phi^h_{ij} \]  

where, \( \Phi^h_{ij} \) is the homogeneous pressure-strain term. Essentially, Eqn. 3.12 can be treated as one with a source term \( \Phi^h_{ij} - \Phi_{ij} \) and \( \Phi_{ij} \) being sensitive to values in the neighbouring locations. The source term drives \( \Phi_{ij} \) to an average of the neighbouring values and the homogeneous values. This relaxation mechanism is collectively termed as elliptic relaxation.
To introduce this approach into the RSM framework, certain modifications were made by Durbin (1993). \( \varepsilon_{ij} \) is decomposed as,

\[
\varepsilon_{ij} = \left( \varepsilon_{ij} - \frac{\overline{u_iu_j}}{k} \varepsilon \right) + \frac{\overline{u_iu_j}}{k} \varepsilon
\]

(3.13)

Substituting this in Eqn. 2.12, the following is obtained,

\[
\frac{D\overline{u_iu_j}}{Dt} = P_{ij} + \left[ \Phi_{ij} - \left( \varepsilon_{ij} - \frac{\overline{u_iu_j}}{k} \varepsilon \right) \right] + D^r_{ij} + D^t_{ij} + D^p_{ij} - \frac{\overline{u_iu_j}}{k}
\]

(3.14)

\( \Phi_{ij} \) in Eqn. 3.12 is then replaced by \( \varphi_{ij} \),

\[
\varphi_{ij} - L^2 \nabla^2 \varphi_{ij} = \varphi^h_{ij},
\]

(3.15)

where,

\[
\varphi^h_{ij} = \Phi^h_{ij} - \left( \frac{2}{3} \delta_{ij} - \frac{\overline{u_iu_j}}{k} \right) \varepsilon
\]

(3.16)

Important concerns for the boundary conditions for Eqn. 3.16 now arise. Asymptotic analysis of Eqn. 3.14 gives insights to address the issue. Using the near-wall behaviour of the turbulent parameters as mentioned in Eqns. 3.8 and 3.9, it can be seen that,

\[
\varphi_{ij}|_{y \to 0} = \left( O(y), -10\nu \frac{v^2}{y^2}, O(y) - 4\nu \frac{uw}{y^2} \right)
\]

(3.17)

In order to ensure the correct behaviour of \( \overline{v^2} \) & \( \overline{uw} \) near the wall, \( \varphi_{22} \) can be normalised by \( k \) which yields \( \varphi_{22}|_{y \to 0} = -20\nu^2 \frac{y^2}{y^4} \varepsilon \), which is similar to imposing a Dirichlet type boundary condition. This idea led Durbin to finally frame Eqn. 3.15 as,

\[
f_{ij} - L^2 \nabla^2 f_{ij} = \frac{\Phi^h_{ij}}{k} + \left( \frac{2}{3} \delta_{ij} - \frac{\overline{u_iu_j}}{k} \right) \varepsilon
\]

(3.18)

The final issue to be taken care of was the definition of \( L \). Durbin addressed this issue by defining \( L \) as a combination of the turbulent eddy sizes and the Kolmogorov length scale and formulated \( L \) as,

\[
L = \max \left( \frac{k^{3/2}}{\varepsilon}, \frac{C_L \nu^{3/4}}{\varepsilon^{1/4}} \right),
\]

(3.19)
where $C_L$ and $C_\eta$ are model coefficients.

The elliptic relaxation combined with RSM, improved the predictions substantially owing to the physical rationale, however it posed a problem from a numerical standpoint. First, the relaxation equation introduced more equations into an already populated RSM thereby increasing computational expenses. Second, handling the boundary conditions for $f_{ij}$ was a challenge for CFD codes particularly while dealing with complex geometries. This issue was later addressed by numerous researchers, one of which will be discussed in the next section.

### 3.3.2 Elliptic blending Reynolds Stress Model (EBRSM)

Manceau et al. (2002) addressed the numerical issue relating to Eqn. 3.18 by replacing it with a scalar parameter ($\alpha$) that retains the ellipticity. A single elliptic equation for $\alpha$ was proposed,

$$\alpha - L^2 \nabla^2 \alpha = \frac{1}{k} \quad (3.20)$$

$\alpha$ was then used to blend the near-wall and homogeneous part of the pressure-strain correlation term as,

$$\varphi_{ij} = (1 - k\alpha)\varphi_{ij}^w - k\alpha\varphi_{ij}^h \quad (3.21)$$
Manceau et al. (2002) also proposed the wall values for $\varphi_{ij}^w$ in order to ensure the correct near-wall behavior of $\varphi_{ij} - \varepsilon_{ij}$ as,

$$\begin{align*}
\varphi_{11}^w &= \frac{1}{2} \varphi_{22}^w \\
\varphi_{22}^w &= -\frac{5}{k} \varepsilon_{uu}^2 \\
\varphi_{33}^w &= \frac{1}{2} \varphi_{22}^w \\
\varphi_{12}^w &= -\frac{5}{k} \varepsilon_{uv} \\
\varphi_{13}^w &= 0 \\
\varphi_{23}^w &= -\frac{5}{k} \varepsilon_{uw}
\end{align*}$$

(3.22)

Eqn. 3.22 was generalized as,

$$\varphi_{ij}^w \big|_{y \to 0} = -\frac{5}{k} \varepsilon \left[ \bar{u}_i \bar{u}_k n_j n_k + \bar{u}_j \bar{u}_k n_i n_k - \frac{1}{2} \bar{u}_k \bar{u}_l n_k n_l (n_i n_j - \delta_{ij}) \right],$$

(3.23)

where, $n$ is a unit vector designed to represent the wall normal direction as is defined as,

$$n = \frac{\nabla \alpha}{||\nabla \alpha||}$$

(3.24)

Manceau et al., reported that upon using the $\varphi_{ij}^h$ term from the SSG formulation, the predictions were better compared to when $\varphi_{ij}^h$ from Rotta-IP model. A low-Reynolds number version of the $\varepsilon$ equation is solved in conjunction with the Reynolds stress transport equations. As will be seen in the subsequent chapter, EBRSM plays an important role in the development of the lag models.

### 3.3.3 Elliptic relaxation in LEVMs

It was previously mentioned in Section 2.3.3 that in close proximity of the wall, the usage of $k$ as a velocity scale fails to represent the turbulent mixing and owing to the two-component behaviour of turbulence in that region, the wall normal fluctuations are the correct velocity scale. Also, the previous section explains the importance of elliptic relaxation for representing the turbulence
anisotropy near the wall. These concepts are the building blocks for the \( v^2 - f \) model by Durbin (1991). Durbin proposed the definition of shear stress \( \overline{u'v'} \) as,

\[
-\overline{u'v'} = C_{\mu} \tau \overline{v^2} \frac{\partial U}{\partial y},
\]

(3.25)

where, \( \overline{v^2} \) is a scalar designed to represent the wall-normal fluctuations and \( \tau \) is the time scale chosen as \( \frac{k}{\varepsilon} \). The transport equation for \( \overline{v^2} \) is essentially the i,j = 2,2 component of Eqn. 3.14 without the production term and sum of turbulent and pressure diffusion being represented by an eddy-viscosity gradient diffusion hypothesis,

\[
\frac{D\overline{v^2}}{Dt} = \left( \Phi_{22} - \varepsilon_{22} \right) + \frac{\partial}{\partial x_k} \left( \nu_t \frac{\partial \overline{v^2}}{\partial x_k} \right) - \frac{\overline{v^2}}{k} \varepsilon
\]

(3.26)

The boundary condition for Eqn. 3.26 is \( \overline{v^2}_{wall} = 0 \). The elliptic equation also reduces to,

\[
f - L^2 \nabla^2 f = f^h,
\]

(3.27)

where, \( f = \frac{\varphi_{22}}{k} \).

The eddy-viscosity definition therefore takes the form,

\[
\nu_t = C_{\mu} \overline{v^2} \max \left( \frac{k}{\varepsilon}, 6 \sqrt{\frac{\nu}{\varepsilon}} \right)
\]

(3.28)

Another important feature of the \( v^2 - f \) model was the handling of the stagnation point anomaly(Durbin (1996)). From the definition of the production term as \( P = 2 \nu_t |S|^2 \) in a two equation RANS model, it can be immediately deduced that the production can become too high in stagnation regions due to the presence of high strain rates in the flow. In such scenarios, while production should grow linearly with strain, it grows quadratically with \( |S| \), thereby leading to an over-estimation of TKE. This is particularly detrimental for applications where heat transfer near the wall needs to be accurately computed. To alleviate this, Durbin, proposed a bound on the time scale reflected in the eddy viscosity formulation as,

\[
\nu_t \leq \frac{\alpha_s k}{\sqrt{6} |S|}
\]

(3.29)
Eqn. 3.29 inserted into Eqn. 3.28 gives the final formulation of the eddy viscosity as,

\[ \nu_t = C_\mu \bar{v}^2 \min \left[ \max \left( \frac{k}{\varepsilon}, 6\sqrt{\frac{\nu}{\varepsilon}} \right), \frac{\alpha_r k}{\sqrt{6|S|}} \right], \quad (3.30) \]

where \( \alpha_r \) is coefficient calibrated suitably.

The original \( \nu^2 - f \) formulation was often cumbersome for application in commercial CFD codes as the wall boundary condition \( f|_{y \to 0} = -20\nu^2 \frac{\bar{v}^2}{y^4 \varepsilon} \), often led to numerical instabilities. Several researchers proposed ideas to improve the numerical robustness, a prominent approach being by Lien et al. (1996), where a change of variable was made,

\[ f \rightarrow \bar{f} - 5\varepsilon \frac{\bar{v}^2}{k^2}, \quad (3.31) \]

which makes \( \bar{f}|_{wall} = 0 \).

In recent years, Billard (2011) have proposed even more robust versions of the \( \nu^2 - f \) model in the form of BL-\( \nu^2/k \) model. The concept is to define a variable \( \varphi_{BL} = \frac{\nu^2}{k} \) and then solve a transport equation for the same with straight forward boundary conditions. The transport equation for \( \varphi_{BL} \) can be readily derived by using chain rule of differentiation. The ellipticity is preserved by using similar equation as Eqn. 3.20. The additional transport equation of the BL-\( \nu^2/k \) model reads,

\[ \frac{D\varphi_{BL}}{Dt} = f - \frac{\varphi_{BL}}{k} P_k + \frac{\partial}{\partial x_k} \left[ \left( \nu + \nu_t \frac{\partial \varphi_{BL}}{\sigma_k \partial x_k} \right) + \frac{2}{k} \left( \nu + \nu_t \right) \frac{\partial k}{\partial x_k} \frac{\partial \varphi_{BL}}{\partial x_k} \right], \quad (3.32) \]

\( f \) is evaluated as a blending of the near-wall and homogeneous terms as,

\[ f = (1 - \alpha) f_w + \alpha f_h, \quad (3.33) \]

where, \( f_w = -10\nu \frac{\varphi_{BL}}{y^2} \) and \( f_h = -\frac{\varepsilon}{k} \left( C_1 - 1 + C_2 \frac{P_k}{\varepsilon} \right) \left( \varphi_{BL} - \frac{2}{3} \right) \).

The blending concept of the wall and homogeneous terms using the elliptic parameter \( \alpha \) will be further used in the derivation of the lag models.
CHAPTER 4. LAG MODELS

Parts of this section has been adapted from Biswas et al. (2019a).

The linear approximation of two equation eddy viscosity models has certain ramifications. An important one being the misrepresentation of the turbulent stresses on the mean flow. The linearity suggests that the mean strain rate tensor and the anisotropy tensor are aligned, i.e., the angle between the orthogonal principal axes of the two tensors are zero. However even for simple shear flows, such relationships do not hold and can be easily calculated using DNS data of flows like channel & boundary layer etc. This “misalignment” becomes particularly relevant for rapidly developing unsteady flows and leads to erroneous production of TKE in a RANS model, thereby adversely affecting the flow predictions. The mathematical description of this phenomenon is summarised below.

Carpy et al. (2006), Hadzic et al. (2006) reported that stress-strain misalignment plays an important role in non-equilibrium turbulent flows. From a RANS perspective, ignoring misalignment can lead to an over-estimation of the production of TKE ($P_k$). $P_k$ is given by,

$$P_k = -ka_{ij}S_{ij}, \quad (4.1)$$

$a_{ij}$ being the anisotropy tensor. In a two-dimensional, incompressible flow, this becomes

$$P = k\beta(\lambda_1 - \lambda_2)\cos2\theta, \quad (4.2)$$

where $\beta$ is the positive eigenvalue of $S_{ij}$, $\lambda_1$ and $\lambda_2$ are the eigenvalues of $a_{ij}$ and $\theta$ is the angle between the eigenvectors of the two tensors. In conventional LEVM’s, $\theta = 0$ and $P$ may be overestimated. This is a motivation for lag models.

Under a quasi-two dimensional assumption Revell et al. (2011) defined a “stress-strain lag” parameter to represent the misalignment by,

$$C_{as} = \frac{a_{ij}S_{ij}}{|S|}, \quad (4.3)$$
to represent the angle between the tensors $a_{ij}$ and $S_{ij}$ using a single scalar. However, in a
strict sense, three scalars (Euler angles) would be needed to fully represent in a three dimensional
flow. Such an approach increases numerical complexity and therefore does not serve the purpose
of numerical robustness, a necessary feature for LEVMs targeted at industrial applications.

The transport equation for $C_{as}$ is obtained by using the product rule of differentiation as,

$$\frac{DC_{as}}{Dt} = -\frac{1}{|S|} \left( S_{ij} \frac{Da_{ij}}{Dt} + a_{ij} \frac{DS_{ij}}{Dt} + C_{as} \frac{D|S|}{Dt} \right)$$  \hspace{1cm} (4.4)

$\frac{Da_{ij}}{Dt}$ term in Eqn. 4.4 provides a link to invoke a Reynolds stress model through,

$$\frac{Da_{ij}}{Dt} = \frac{1}{k} \left( \frac{Du_i'u_j'}{Dt} - \frac{u_i'u_j'}{k} \frac{Dk}{Dt} \right)$$  \hspace{1cm} (4.5)

A suitable RSM can be then substituted to replace $\frac{Du_i'u_j'}{Dt}$ in Eqn. 4.5. This substitution leads to
unclosed terms $P_{ij}, \Pi_{ij}, \varepsilon_{ij}$ and $a_{ij}$. Revell et al. (2011) closed the $P_{ij}$ term as $P_{ij} \equiv f(S_{ij}, \Omega_{ij}, a_{ij})$.
$\Pi_{ij}$ was closed using a quasi-linear pressure-strain model and $\varepsilon_{ij}$ was closed using an isotropic
formulation. Two approaches with one being a linear approximation to relate $a_{ij}$ with $S_{ij}$ and the
other being a non-linear relation of the same were chosen to close the $a_{ij}$ term. Introducing these
substitutions to Eqn. 4.4, the following transport equation for $C_{as}$ was proposed,

$$\frac{DC_{as}}{Dt} = \frac{\varepsilon}{k} C_{as} + \alpha_1 |S| C_{as}^2 + \alpha_2 \frac{S_{ij}a_{ik}a_{kj}}{|S|} + (\alpha_3 + \alpha_3^* \sqrt{A_2})|S| + \alpha_4 \frac{S_{ij}a_{ij}S_{jk}}{|S|} + \alpha_5 \frac{S_{ij}a_{ik}\Omega_{jk}}{|S|}$$  \hspace{1cm} (4.6)

Eqn. 4.6 did not include a near wall term. Thus, to ensure a correct near wall behaviour of $C_{as}$,
Revell et al., also proposed a damping function ($f(y^+)$) to be multiplied to the R.H.S. of Eqn. 4.6.
The transport equation was then combined with the $k – \omega$ SST formulation and TKE production
was defined as, $P_k = C_{as}k|S|$. 
4.1 Derivation

4.1.1 Lag $k - \varepsilon$ model

In their “Elliptic blending Lag $k - \varepsilon$” model, Lardeau et al. (2016) redefined the lag parameter as,

$$\varphi = - \frac{a_{ij}S_{ij}}{S} \frac{1}{\xi} \frac{\varepsilon}{kS}$$

(4.7)

and defined eddy viscosity as,

$$\nu_t = C_\mu \varphi k \min \left( \sqrt{\frac{k^2}{\varepsilon^2} + C_2^2 \frac{\nu}{\varepsilon}}, \frac{C_T}{C_\mu \sqrt{3} \varphi |S|} \right).$$

(4.8)

This allows $P_k$ to be scaled by $\varphi$ since $P_k = \nu_t |S|^2$. To close the $\frac{Da_{ij}}{Dt}$, stemming from $\frac{D\varphi}{Dt}$, they used the EBRSM model thereby addressing the near wall treatment of the additional transport equation. As will be seen later, the near wall term is critical for the prediction of separated flows. Their transport equation for $\varphi$ is,

$$\frac{D\varphi}{Dt} = -(1 - \alpha^3)C_\mu^* \frac{\varphi}{\tau} - \alpha^3 \left( \tilde{C}_1 + C_1^* \frac{P}{\varepsilon} \right) \frac{\varphi}{\tau} + \alpha^3 \frac{1}{S^2 \tau} (C_4^* a_{ik} S_{kj} - C_5^* a_{ik} W_{kj}) S_{ij} - C_{P1} \frac{\varphi P}{k} + \alpha^3 C_{P2} \varphi S + \alpha^3 C_{P3} \frac{\varphi}{\tau} + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu}{2} + \frac{\nu_t}{\sigma_\varphi} \right) \frac{\partial \varphi}{\partial x_j} \right].$$

(4.9)

Similar to EBRSM, $\alpha$ was obtained by solving an elliptic equation involving the turbulent length scale,

$$\alpha - L^2 \nabla^2 \alpha = 1.$$  

(4.10)

The boundary conditions are, $\varphi_{wall} = 0, \alpha_{wall} = 0$. The turbulent length scale has the definition,

$$L = C_L \sqrt{\frac{k^3}{\varepsilon^2} + C_9^2 \frac{\nu^3}{\varepsilon}}.$$  

(4.11)
Transport equations for $k$ and $\varepsilon$ were solved in conjunction with Eqn. 4.9 with a low Reynolds number term similar to that mentioned in Section 2.3.2 was used,

$$\frac{Dk}{Dt} = P - \varepsilon + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu}{2} + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]$$  \hspace{1cm} (4.12)

$$\frac{D\varepsilon}{Dt} = \frac{1}{\tau} (C_{\varepsilon 1} P - C_{\varepsilon 2} \varepsilon) + E + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu}{2} + \frac{\nu_t}{\sigma_{\varepsilon}} \right) \frac{\partial \varepsilon}{\partial x_j} \right]$$  \hspace{1cm} (4.13)

The coefficients for Eqn. 4.9 were taken directly from EBRSM while standard values were used for Eqns. 4.12 & 4.13,

<table>
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<tr>
<th>$\sigma_k$</th>
<th>$C_{\varepsilon 1}$</th>
<th>$C_{\varepsilon 2}$</th>
<th>$C_k$</th>
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4.1.2 Lag $k-\omega$ model

Defining the turbulent time scale as $1/\omega$ instead of $k/\varepsilon$ eliminates the requirement for low Reynolds number corrections, or wall functions. The wall value of $\omega$ can be judiciously chosen to incorporate surface roughness effects as well. The version of the $k-\omega$ model, herein referred to as $k-\omega 88^*$ (Wilcox (1988)), is chosen as the base model to which the lag approach is applied.

The transport equations for $k$ and $\omega$ in $k-\omega 88$ read

$$\frac{Dk}{Dt} = P - \beta^* k \omega + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right]$$  \hspace{1cm} (4.14)

$$\frac{D\omega}{Dt} = \gamma \frac{P}{k} \omega - \beta \omega^2 + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right].$$  \hspace{1cm} (4.15)

Essentially, the underlying LEVM ($k-\varepsilon$ or $k-\omega$) determines the $\eta$ term in Eqn. 4.7. For $k-\omega$,

$$\eta = \omega/S$$ and

$$\varphi^* = -\frac{a_{ij} S_{ij} \omega}{S}.$$  \hspace{1cm} (4.16)
Taking the substantial derivative of Eqn. 4.16, the following transport equation for the modified lag parameter is obtained in the same manner as Eqn. 4.9:

\[
\frac{D\varphi^*}{Dt} = -(1 - \alpha^3)C_{\omega}\varphi^*\omega - \alpha^3 \left( \tilde{C}_1 + C_1^*\frac{P}{\beta^*k\omega} \right) \varphi^*k\omega - C_{p1}\frac{P}{k}\varphi^*
\]

\[+ \alpha^3 C_{p2}\varphi^*S + \alpha^3 \frac{C_{\omega}^2}{\varphi_h^2} + \alpha^3 \frac{\beta^*\omega}{\varphi_h} \left( C_4^*a_{ik}S_{kj} - C_5^*a_{ik}W_{kj} \right) S_{ij}\]

\[+ \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_{\varphi}} \right) \frac{\partial \varphi^*}{\partial x_j} \right].\]  

(4.17)

It is important to note here that the \(a_{ij}\) appearing in both Eqns. 4.9 & 4.17, is closed by the representation,

\[a_{ij} = -2\frac{\nu_t}{k} \left[ S_{ij} + \beta_2 \frac{2(S_{ik}W_{kj} - W_{ik}S_{kj})}{(S_{kl} + W_{kl})(S_{kl} + W_{kl})} \right].\]  

(4.18)

In the Lag EB \(k - \omega\) model, this definition of \(a_{ij}\) is only used to close the term; it is not used to represent the Reynolds stress tensor \(\overline{u'_i u'_j}\), which still uses the Boussinesq approximation. (Linear and non-linear algebraic stress models, in conjunction with the Lag EB \(k - \varepsilon\), were compared by Tunstall et al. (2016).) Hence, Eqn. 4.18 can be regarded as part of a pressure strain model.

The turbulent length scale becomes

\[L = \sqrt{\frac{k}{(\beta^*k\omega)^2} + C_\eta^2 \sqrt{\nu^3 \beta^*k\omega}}.\]  

(4.19)

The lag parameter (Eqn. 4.7) is scaled such that it reaches a value of unity in the free-stream. The wall boundary condition is \(\varphi^*_{wall} = 0\).

### 4.2 Model Calibration

Most of the coefficients did not require any recalibration from the values given in Lardeau et al. (2016), three modifications were needed:
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<table>
<thead>
<tr>
<th>$Re_\tau$</th>
<th>Lag EB $k - \omega$</th>
<th>DNS</th>
</tr>
</thead>
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</tr>
<tr>
<td>5200</td>
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<td>0.0415</td>
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</table>

Table 4.1: $u_\tau/U_b$ for channel flow from Lag EB $k - \omega$ and DNS data Lee et al. (2015).

Figure 4.1: Sensitivity of $\gamma$ on a flat plate zero pressure gradient boundary layer at $Re_\theta = 2500$, (a) $\varphi^*$ profile, (b) $\nu_t$ profile.

- The coefficient $C_w^*$, that appears in the near wall term in Eqn. 4.17, takes a value of 0.05, compared to the value 1.345, used in the k-ε lag model. The near wall limit $k \sim y^{2.23}$ from $k - \omega$ models, rather than $y^2$, is the primary reason for this change. The calibration ensures correct wall shear stresses. A comparison of the friction velocities predicted by the Lag EB $k - \omega$ and DNS data for channel flow is presented in Table 4.1.

- In Eqn. 4.15 $\gamma$ is replaced by

$$\gamma_{new} = (1 - \alpha^3)\gamma_1 + \alpha^3\gamma_2.$$  \hspace{1cm} (4.20)
Figure 4.2: Periodic Hill geometry, sensitivity of skin friction prediction to $C_{p1}$.

The blended formulation ensures no spurious production of $k$ beyond the wake region as is seen in Figure 6.3 (b). In addition to this, the blended formulation ensures the overshoot of $\varphi^*$ immediately before the wake region while maintaining the correct near wall eddy-viscosity levels as shown in Figure 4.1.

- Coefficient $C_{p1}$ takes a value of 0.4 compared to its $k$-$\varepsilon$ value of 0.56. The term involving $C_{p1}$ is a sink term and this value improves predictions of separated flows (Figure 4.2).

Predictions of free shear flows are standard tests to assess the validity of the coefficient values of the dissipation equation. For example, for a $k$-$\varepsilon$ model, it can be shown that the growth rate
of a mixing layer is determined by the production to dissipation of TKE ratio which relates on the coefficients $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ through the following relation,

$$\frac{P}{\varepsilon} - 1 = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon 1} - 1}$$  \hspace{1cm} (4.21)

$C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are usually calibrated such that they satisfy experimental values of $\frac{P}{\varepsilon}$ through the above relation. Coefficients of the $\omega$ equation namely $\gamma$ and $\beta$ satisfy similar relations, therefore any alteration from standard values should be verified for consistency. To that end, a mixing layer case was selected in which two shear layers of different velocities initially separated by a plate mix with each other. The experimental data used for comparison is by Delville et al. (1988) and the geometry details and velocity profile comparisons are presented in Figure 4.3. $Re_L$ is approximately 2900 and the momentum thickness for the faster flow is $\approx 1$ mm and for the slower flow is $\approx 0.73$ mm. Grid from the NASA Turbulence Modeling Resource (TMR) website with around 42000 computational nodes was used for simulation. The predicted velocity profiles were in very good agreement with the experimental data, thereby ensuring the validity of the coefficient adjustments.
Figure 4.3: (a) Mixing layer Domain, U upper = 41.54 m/s, U lower = 22.04 m/s. (b) U component of velocity at x = (a) 1, (b) 50, (c) 200, (d) 650, (e) 950.
CHAPTER 5. NUMERICS

The numerical results presented in the study were obtained using the open source finite volume CFD solver OpenFOAM (Weller et al. (1998)). The lag models as mentioned in the previous chapter were implemented to the OpenFOAM framework. In this chapter, the numerical set up for solving the incompressible NS equations is discussed.

5.1 Discretization

A general form of a transport equation of a variable \( \phi \) can be written as (Eqn. 5.1),

\[
\frac{\partial \phi}{\partial t} + \nabla \cdot (\vec{U} \phi) = \nabla \cdot (\Gamma \nabla \phi) + S_\phi,
\]

where, \( \vec{U} \) is the velocity of convection and \( \Gamma \) is the coefficient of diffusivity.

Selecting the appropriate discretization schemes for each of these terms is critical to the stability of the solution algorithm. Scheme selection is also dependent on the type of the flow problem whose solution is sought. Typically for turbulent flows, for handling the convective terms, schemes with less numerical viscosity are chosen, the most common being second order accurate central differencing scheme.

The test cases considered for testing the proposed RANS formulation are steady in nature. The \textit{simpleFoam} solver which is based on the Semi-implicit Method for Pressure Linked Equations (SIMPLE) algorithm is used for solving the steady state incompressible NS equations. The solver follows segregated solution methods, i.e., the momentum equations, pressure equation and the turbulence model equations are solved sequentially and the solution of the preceding equations is utilized in the succeeding equations. The SIMPLE algorithm solves the momentum equations first to create a velocity field that usually does not satisfy the continuity equation. The momentum and continuity equations are then utilised to form the pressure equation which is then solved. This
solution is then used to correct the velocity field by making it divergence free. The turbulence equations are then solved. This process is continued in a loop till convergence criteria is met. For the lag models, the sequence of equations solved for the turbulence variables is, $k$ followed by $\varepsilon$ or $\omega$, followed by $\varphi$ or $\varphi^*$, followed by $\alpha$.

The divergence scheme, \textit{bounded Gauss linear}, a second order accurate central differencing scheme that ensures boundedness of the solution particularly for steady-state flows, is used for the convective terms for all the transport equations. \textit{Gauss linear corrected} scheme is used for the diffusive term which includes a correction factor for non-orthogonal grids. For terms involving spatial gradients, \textit{Gauss linear} scheme is used.

\subsection{5.2 Solving the equation systems}

The discretized equations are converted into the linear algebraic form $Ax = b$, where $x$ is the desired solution. This equation is solved iteratively. Bi-conjugate gradient solver with a diagonal incomplete LU preconditioner ($PBiCG$) solver is used to solve the matrix systems for the momentum equations and the turbulence variables, whereas Preconditioned conjugate gradient ($PCG$) was chosen for pressure. The $\alpha$ equation was solved using a smooth solver with symmetric Gauss-Seidel smoother. Typically steady state solvers use an under-relaxation factor to increase the stability of the computations. An under-relaxation factor $\alpha_r$ is defined as,

$$U^{n+1,\text{used}} = U^n + \alpha(U^{n+1,\text{predicted}} - U^n) \quad (5.2)$$

For the test cases, the $\alpha_r$ values chosen were, 0.6 for the momentum and turbulence variable equations and 0.3 for the pressure equation.

\subsection{5.3 Boundary conditions}

Three types of boundary conditions, namely Dirichlet, Neumann and periodic boundary conditions were used for all the simulations. At no slip walls, the no-slip boundary conditions were used, whereas at the outlet boundary face the normal gradient of the velocity field (Neumann) was set
to 0. Neumann boundary condition for pressure was used at the walls. The variable $\alpha$ has a wall value equal to 0, similar to $\phi$ and $\phi^*$. For the turbulence parameters $k$, $\varepsilon$ and $\omega$, the following relationships were used to specify the initial conditions,

\[
\begin{align*}
k_{\text{init}} &= \frac{3}{2}(IU_{\text{ref}})^2 \\
\varepsilon_{\text{init}} &= C_\mu(k_{\text{init}})^{3/2}/L \\
\omega_{\text{init}} &= \sqrt{k_{\text{init}}}/L
\end{align*}
\]

(5.3)

where $I$ and $L$ denotes a suitable turbulence intensity and length scale respectively.

### 5.3.1 $\omega$ at wall

The inverse time-scale $\omega \to y^{-2}$ as $y \to 0$. At no-slip walls, Menter et al. (2003) suggested the use of

\[
\omega_{\text{wall}} = \sqrt{\omega_{\text{viscous}}^2 + \omega_{\text{log}}^2}
\]

(5.4)

where

\[
\omega_{\text{viscous}} = \frac{6\nu}{\beta y^2} \quad \text{and} \quad \omega_{\text{log}} = \frac{\sqrt{k}}{\sqrt[3]{\beta^3 k} y}
\]

(5.5)

to improve the results on under resolved grids. However, it is seen in Figure 5.1 that this condition shows some grid sensitivity unless the first grid point is at $y^+ \approx 1$. Due to the need for low $y^+$ values, the $\omega$ boundary condition for the current simulations is, $6\nu/\beta y^2$. 
Figure 5.1: Grid dependency analysis for channel flow, $u_\tau/U_b$ vs $Re_\tau$ for different $y^+$ values.

5.4 Convergence

In OpenFOAM, for a matrix system $Ax = b$, residuals are defined as,

$$ r = \frac{\sum |b - Ax|}{\sum (|Ax - A\bar{x}| + |b - A\bar{x}|)}, $$ (5.6)

where $\bar{x}$ denotes the average of the solution vector. Solutions are considered to be converged when the residual value falls below a prescribed tolerance level. Usually, the residual tolerance values are prescribed in terms of the pressure equation which is the slowest to converge. Pressure equation residuals reaching below $10^{-4}$ generally ensure a converged solution. Figure 5.2 shows the residuals plotted against the number of iterations for the Lag $k - \omega$ compared to $k - \omega$ 88, for the periodic hill case. For these computations, with around 40,000 computational nodes, Lag $k - \omega$ meets the convergence criteria with a fewer number of iterations than $k - \omega$ 88.
Figure 5.2: Convergence statistics from (a) Lag $k - \omega$, (b) $k - \omega$ 88 models.
CHAPTER 6. TEST CASES

*Parts of this section has been adapted from Biswas et al. (2019b).*\(^1\)

6.1 Channel Flow

A fully developed channel flow is a simple, non-homogeneous turbulent shear flow. Testing on this configuration provides the most basic validation of the formulation. Four different friction velocity Reynolds numbers \(Re_\tau = 550, 1,000, 2,000\) and 5,200 are chosen (Table 6.1). The mean velocity profiles, Figure 6.1, are in good agreement with the DNS data (Lee et al. (2015)). Predictions in the logarithmic region are improved when compared to \(k-\omega\) or \(k-\omega\) SST. The reason is a better near-wall scaling of the eddy viscosity (Figure 6.2), provided by the lag parameter. This trend is also observed in a zero pressure gradient boundary layer (ZPGBL), as seen in Figure 6.3(a).

### 6.2 Zero Pressure Gradient Boundary Layer (ZPGBL)

Next, the flow over a flat plate in zero pressure gradient is examined. Statistics are computed at a momentum thickness based Reynolds number of \(Re_\theta = 2,500\). The ‘\(k\)’ obtained from \(k-\omega\) based models is not the actual turbulent kinetic energy; especially near walls, it behaves quite differently

<table>
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<th>(Re_\tau)</th>
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</table>

Table 6.1: \(u_\tau/U_b\) for channel flow from Lag EB \(k - \omega\) and DNS data Lee et al. (2015).

\(^1\)Lag EB \(k - \omega\) and Lag EB \(k - \varepsilon\) are same as Lag \(k - \omega\) and Lag \(k - \varepsilon\) formulations respectively.
Figure 6.1: U+ profiles in channel flow (a) $Re = 550$, (b) $Re = 1,000$, (c) $Re = 2,000$, (d) $Re = 5,200$.

However it gives a measure of the boundary layer thickness. From Figure 6.3(b), it is seen that, although the distribution of $k$ across the boundary layer obtained from Lag EB $k - \omega$ is different from $k - \omega$ and $k - \omega$ SST models, the thickness is consistent, thereby confirming the physicality of the solution. The $k$ from $k - \varepsilon$ is a different variable, as the figure shows.

The skin friction, displayed in Figure 6.4, goes through a laminar to turbulent transition then follows the experimental data, although the predicted $C_f$ is above the data.
Figure 6.2: Channel flow ($Re_\tau = 5,200$), comparison of eddy viscosity from different LEVMs with DNS data; semi-logarithmic scale highlights the near-wall region.

### 6.3 Backward facing step

The experiment of flow over a backward facing step is an example of reattachment of separated turbulent shear layers. The sudden expansion of the channel causes the flow to separate at the top of the step. This case provides an assessment of the ability of LEVMs to predict both reattachment location and the subsequent recovery of the velocity profiles. The inlet momentum thickness Reynolds number $Re_\theta$ is 5,000 and the step height Reynolds number $Re_H$ is 37,500 with an expansion ratio of 9:8.

Third most fine grid consisting of approximately 63000 nodes from the TMR website is used for the computations. As seen in Figure 6.5(a), all the models predict the reattachment location (where $C_f$ crosses zero) with reasonable accuracy. Lag EB $k-\omega$ predicts a separation bubble that extends to $6.35H$, $H$ being the step height. That lies within 2% of the experimental value of 6.26H. The velocity profiles(Figure 6.5(b)) recover more slowly than the experiments, a discrepancy shown by all models. The minimum $C_f$ is under-predicted by all models except the present. Figure 6.5(c) shows the shear stress ($-u'v'$) profiles, most of the models are seen to be successful in predicting the stresses compared to the experimental values.
Figure 6.3: ZPGFBL computations using three different models. Profiles of (a) $U^+$, (b) $k^+$ at $Re_\theta = 2,500$.

Figure 6.4: Skin friction coefficient for a ZPGBL from Lag EB $k-\omega$ compared to experimental data.
Figure 6.5: Flow over a backward facing step, $Re_H = 37500$. (a) Coefficient of skin friction, (b) $U$ component of velocity, (c) Shear stress profiles from different LEVMs.
6.4 Flow over curved geometry

A channel flow with a periodically repeated, hill shaped geometry on the lower wall is an attractive validation case. The domain of interest is periodic in the streamwise direction, which avoids errors due to inflow boundary conditions. The hills are spaced 9 hill heights apart and the Reynolds number based on the hill height is 10,595. LEVMs such as $k-\omega$ 2006 or $k-\omega$ SST perform poorly for this set up, under-predicting the turbulent mixing necessary to cause reattachment, hence, failing to predict the reattachment location. This failure is quite pronounced in the SST model, as is illustrated in the right panel of Figure 6.6.

Accurate prediction of separated flows from curved surfaces requires improved near wall modelling. The elliptic-blending, lag approach attempts to improve the near wall eddy-viscosity, thereby improving the skin friction prediction, as is seen in Figure 6.7(a). Lag EB $k-\omega$ shows a recirculation zone size that is approximately 10% larger than the LES data. The velocity profiles downstream of the separation bubble are noticeably improved relative to $k-\omega$ SST predictions (Figure 6.7(b)).

Comparison of $\nu_t$ serves as the best measure to assess the performances of RANS models, but owing to the ambiguity involved in computing eddy viscosity from LES/DNS data for non-equilibrium flows, shear stress ($-u'v'$) profiles can be chosen as an alternative. Figure 6.7(c) shows the comparison of the shear stress $-u'v'$ at different streamwise locations for the periodic hill case. Lag EB $k-\omega$ predicts the stresses in best agreement at reattachment, and immediate downstream of it. Far from the wall, deviations of the stresses returned by the models from LES have minimal effect on the mean flow.

A mesh sensitivity study for this geometry was carried out on a coarse, $120 \times 80$ grid point, mesh (Figure 6.8(a)) and a fine, $250 \times 160$, mesh (Figure 6.8(b)). The skin friction coefficient is seen to have negligible sensitivity to the grid resolution (Figure 6.8(c)).

Boundary layer separation over a gently curved surface finds application in various engineering applications such as compressor blades, curved ducts, diffusers, etc. An adverse pressure gradient caused by the geometry decelerates the boundary layer prior to separation. In the curved backstep of (Figure 6.9), the Reynolds number based on the step height $H$ and the inlet free stream velocity
Figure 6.6: Periodic Hill geometry, $Re_H = 10595$. Contours of $U$ mean from Lag EB $k - \omega$ (Left), $k - \omega$ SST (Right). Recirculation region (dashed lines) predicted by $k - \omega$ SST is significantly larger than Lag EB $k - \omega$.

is 13,700. A boundary layer with $Re_\theta = 1,190$ and boundary layer thickness of $\delta_{99} = 0.8H$ is prescribed at the inlet. Computational domain consists of 200 x 70 points in the XY plane. Pressure-velocity correlation plays a governing role in such flows. The $\varphi^*$ equation, being derived from a Reynolds stress equation, retains sensitivity to the closure of the redistribution tensor (e.g., the $\alpha^3 \left( \tilde{C}_1 + C_{1*} \frac{P}{\beta^* k \omega} \right)$ $\varphi^* k \omega$ term). This variation of $\varphi^*$ alters the eddy viscosity, and proves to be a crucial element for improving flow predictions for such geometries. Examining the shear-stress profiles (Figure 6.10) at streamwise locations of $x/H = 1.5$ and $x/H = 4$, it is seen that in the close proximity of the wall, the profile obtained from Lag EB $k - \omega$ is much closer to the LES data. This improvement in the near wall behavior of $\overline{-u'v'}$ is reflected in a better mean flow and quite accurate skin friction prediction (Figure 6.11(a)). The very large difference between model and experiment farther from the wall, fortunately, has a small effect on the mean flow (Figure 6.11(b)). Figure 6.11(c) shows the profiles of $k$ returned by the models at different streamwise locations.
Figure 6.7: Periodic Hill geometry (a) coefficient of skin friction, (b) $U$ component of velocity, (c) shear stress profiles at different streamwise locations from different LEVMs.

The effect of the coefficients $\beta_2$ and $C_{p2}$ on the skin friction coefficient is examined for this set-up in Figures 6.12(a) and (b). The term $\alpha^3 \frac{\beta^* \omega}{\varphi h} \left( C^*_4 a_{ik} S_{kj} - C^*_5 a_{ik} W_{jk} \right) S_{ij}$ is negative value, therefore acting as a sink term. A lower value of $\beta_2$ leads to lower values of $\varphi^*$, thereby lesser $\nu_t$; $\alpha^3 C_{p2} \varphi S$, being a source term, higher values of $C_{p2}$ lead to higher levels of $\nu_t$. 
Figure 6.8: Periodic Hill geometry, view of (a) Coarse mesh (120x80 grid points), (b) Fine mesh (250x160 grid points), (c) Skin friction coefficient.
Figure 6.9: Flow over a curved backstep, $Re_\theta = 1,190$. Contours of $k$ normalized by the freestream velocity.

Figure 6.10: Flow over a curved backstep, profiles of shear stress at (a) $x/H = 1.5$, (b) $x/H = 4$ from lag EB $k - \omega$ and $k - \omega$ SST compared to LES data.
Figure 6.11: Flow over a curved backstep, $Re_\theta = 1,190$. (a) Coefficient of skin friction, (b) U component of velocity, (c) ‘k’ profiles from different LEVMs.
Figure 6.12: Flow over a curved backstep, sensitivity of the coefficient of skin friction to (a) $\beta_2$, (b) $C_{p2}$.

### 6.5 NASA wall mounted hump

The motivation for this case is to test the accuracy of turbulence models to predict 2D flow separation from a smooth body at a high Reynolds number. $Re_c$ based on the chord length and freestream velocity is 936,000. The hump creates a favourable pressure gradient, accelerating the flow on the windward side, followed by a sudden expansion, resulting in a separation bubble on the leeward side (Figure 6.13). Mesh consisting of 817 x 217 grid points in XY plane from NASA TMR is used for the study.

Predictions in the accelerating part of the flow are improved by the lag models (Figure 6.14 (a)). Lag EB $k - \omega$ predicts a recirculation zone $\approx 8\%$ larger, whereas $k - \omega$ SST predictions are $\approx 14\%$ larger than the experimental results. The improvements are also reflected in the recovery of the velocity profiles downstream of the reattachment location (Figure 6.14 (c)).
Figure 6.13: NASA wall mounted hump, no flow control case. $Re_c = 936,000$. Contours of $k$ normalized by the freestream velocity and streamlines from Lag EB $k - \omega$.

Figure 6.14: NASA wall mounted hump, (a) Coefficient of skin friction, (b) Coefficient of static pressure on the wall, (c) U component of velocity, (d) Shear stress ($-u'v'$) profiles from different LEVMs.
6.6 Two dimensional bumps

A convex surface, mounted on a plate, perturbs a boundary layer flowing over it. While passing over the geometry, the boundary layer experiences changes in streamwise pressure gradients, viz., favorable on the windward and adverse on the leeward side. experimentally studied the evolution of such a flow and reported the features of the turbulent boundary layer for a momentum thickness based Reynolds number, $Re_\theta \approx 4,000$. used a similar geometry for an LES study at $Re_\theta$ of 2,500. In their study, the bump heights were successively increased from $H/c = 0.0656$ to 0.138, $c$ being the chord length (Figure 6.15). The incoming boundary layer separated on the adverse pressure gradient side of the highest bumps, while staying attached for the lowest bumps. For the current work, bumps with $H/c = 0.0656$ (B1), 0.0852 (B2) and 0.138 (B3) were chosen. The inlet boundary layer thickness was maintained same for all the bump heights. This corresponds to $\delta_{inlet}^{*}/h$ ($h$ being the bump height in mm) ratios of 0.19, 0.146 and 0.088 for B1, B2 and B3 respectively. For the second highest bump, LES data report the boundary layer to be on the verge of separation. The absence of equilibrium over the bump was noted both by Matai et al. (2019) and Webster et al. (1996) thereby making it an attractive case for lag parameter based LEVMs.

RANS models are prone to make poor predictions in adverse pressure gradient flows (Menter (1994, 1992)). The onset of separation and the recovery rate are often erroneous (Durbin et al.
Figure 6.16: (a) Computational domain for B1, (b) Skin friction coefficient of B3 using fine mesh (200 x 180) and coarse mesh (100 x 90).

(2010)). One reason for such failures is the insufficient levels of eddy-viscosity, returned by the formulations. Lardeau et al. (2016) and Biswas et al. (2019a) showed that the elliptic blending lag approach reduces this shortcoming.

Computations were done with $100 \times 90$ ($x$-$y$ plane) nodes with stretching in the wall normal direction to maintain a first node $y^+ \approx 1$ (Figure 6.16a). Mesh sensitivity was studied by increasing the number of grid points by a factor of 2 in both directions, for the highest bump. The skin friction coefficient showed very little grid dependence (Figure 6.16b).

Figure 6.17 shows the coefficient of static pressure on the wall for all three bumps. Immediately after the start of the bump, a favourable pressure gradient is imposed on the flow. Then it switches to a strong adverse pressure gradient on the downwind side of the bump. This is followed by
a recovery region. The lag models were observed to reproduce the pressure distribution more accurately than the $k - \omega$ 2006 and $k - \omega$ SST formulations.

Matai et al. (2019) reported a region of high turbulent kinetic energy (TKE) emerging from the surface on the adverse pressure gradient side of the bump, and argued that the failure of RANS models to capture the high TKE in close proximity to the wall, is the reason for certain inaccurate predictions. Figure 6.18 contains contours of the production of TKE for B3, comparing LES and different RANS models. The lag models were able to produce a higher level of TKE, closer to the wall, than the other formulations. This point will be further illustrated, through streamwise profiles of eddy viscosity. Toward that end, it is useful to extract an eddy viscosity from the LES data.

Parish et al. (2016) proposed an optimization algorithm to extract RANS features from data. Singh et al. (2017) used this method, with a cost function defined as the mean-squared discrepancy between the RANS prediction and data. It is important to note that the inverse solution requires an underlying EVM; they generated an inverse solution with the baseline $k - \omega$ model. The optimizer alters the eddy viscosity from its underlying value, only where such change has an impact on the cost function. In the present case, this is predominantly in the near wall region (Matai et al. (2019)). We use the same inverse eddy viscosity data, as in for the current comparisons. The optimal data
Figure 6.18: Contours of production of TKE normalized with inlet momentum thickness and $U_\infty$ for (a) LES, (b) Lag $k-\omega$, (c) Lag $k-\varepsilon$, (d) $k-\omega$ SST and (e) $k-\omega$ 2006. (labeled as ‘inverse’ in figures) most significant below $y/c$ of about 0.01; above $y/c \approx 0.01$, the eddy viscosity is simply that of the underlying model. More precisely, the eddy viscosity is regarded as optimal, only where it differs from the base, $k-\omega$ 2006 model.

Comparisons at three locations, $x/c = 0.7$, 1 and 1.5 and are shown in Figure 6.19. The key observation is the elevated levels of eddy-viscosity returned by the lag models in the vicinity of the surface, in the adverse pressure gradient and recovery regions. The shear stress ($-\overline{u'v'}$) profiles, Figure 6.20, are examined at locations $x/c = -0.32$, 0.25, 0.7, 1 and 1.5. Predictions by both the lag models were in closer agreement to the LES data than the other models, particularly in the separated and recovery regions.
Figure 6.19: $\nu$ for (a) B1 at x/c = 0.7, (b) B1 at x/c = 1, (c) B1 at x/c = 1.5, (d) B3 at x/c = 0.7, (e) B3 at x/c = 1, (f) B3 at x/c = 1.5.

Predictions by both the lag models were in closer agreement to the LES data than the other models, particularly in the separated and recovery regions (Figure 6.21).

To conclude this bump example, it is informative to consider the behavior of the different terms of Eqn. (4.17) in different pressure gradients. B3 was chosen for such analysis, since the flow separates on the leeward side of the bump; therefore, it permits consideration of the behaviour in a separated, zero, favourable, and adverse pressure gradients. Figure 6.23(a) shows the streamlines computed using Lag $k - \omega$. Five locations of interest, indicated by the dashed lines, represent the zero-pressure gradient (ZPG), favourable pressure gradient (FPG), adverse pressure gradient (APG),
separation and recovery regions. Figure 6.23(b) shows the levels of $\varphi^*$ at these locations. It can be observed that the levels of $\varphi^*$ are lowest at the ZPG location.

To investigate this further, the contribution of each of the source terms of the $\varphi^*$ in Eqn. (4.17) was evaluated. Streamwise behavior of the terms was quantified by defining, $f_i = \frac{T_i}{\sum T_i}$, where $T_i$ refers to the individual terms, per Eqn. 6.1.

$$
\begin{align*}
T_1 &= C_w \varphi^* \omega \\
T_2 &= \left( \tilde{C}_1 + C_{1}^{*} \frac{P}{\beta^* k \omega} \right) \varphi^* k \omega \\
T_3 &= \frac{C_p}{k} \varphi^* \omega \\
T_4 &= C_{p2} \varphi^* S \\
T_5 &= \frac{C_p}{\varphi_h} \beta^* \omega \\
T_6 &= \frac{\beta^* \omega}{\varphi_h} \left( C_{4}^{*} a_{ik} S_{kj} - C_{5}^{*} a_{ik} W_{kj} \right) S_{ij}
\end{align*}
$$

(6.1)

Figure 6.24 overlays the velocity profiles and $f_i$. The contribution from $f_1$s is almost negligible at all locations, except in the close vicinity of the wall. Although in the near-wall regions $T_4$ is large, note that it is multiplied by $\alpha^3$, so it is less dominant than $T_1$, which is multiplied by $(1 - \alpha^3)$. Substantial reduction of the fraction, $f_2$, is observed for FPG, APG, separation and recovery locations compared to ZPG. $f_2$, being a destruction term, leads to an increase in the levels of $\varphi^*$ where it is small; and thereby, $\nu_t$ is increased (recall $\nu_t = k \min\left( \frac{\varphi^*}{\omega}, \frac{\alpha_s}{|S|} \right)$). $\nu_t$ is also augmented by the increase of $f_4$ at all locations, compared to ZPG. $f_6$ has negative values and is similar to $f_2$. In the FPG regions, $\nu_t$ should be lower near the wall than in ZPG, to accurately capture an accelerating boundary layer. Unfortunately, Lag $k - \omega$ fails to produce such a trend, with a resulting departure from the LES data in the skin friction plot (Figure 6.22(c)).
Figure 6.20: $\frac{-u'v'}{U_{inf}^2}$ for (a) B1, (b) B2, and (c) B3.
Figure 6.21: $\frac{U}{U_{inf}}$ for (a) B1, (b) B2, and (c) B3.
Figure 6.22: Coefficient of skin friction for (a)B1, (b)B2, (c)B3.
Figure 6.23: (a) Streamlines predicted by Lag $k - \omega$ for B3, (b) Levels of $\varphi^*$ at locations of interest.
Figure 6.24: Behaviour of the source terms at different streamwise locations. Solid lines show $f_i = \frac{T_i}{\sum T_i}$, symbols show $\frac{U}{U_{ref}}$ and black dashed lines show $\alpha$. Color maps are same as Figure. 6.23(b).
6.7 Swept bump

Webster et al. (1996) investigated the passage of a turbulent boundary layer over a swept bump, the bump height being the same as B1. The idea of this test was to generate a flow field with spanwise as well as streamwise velocity gradients, while maintaining a simple geometry. The inlet boundary layer had $Re_{\theta} = 3,800$ and the flow did not separate at the end of the bump. The pressure gradient regimes are similar to the two dimensional, non swept bumps, described in the previous section.

The bump is swept at an angle of $45^\circ$ to the horizontal direction. The flow is homogeneous along an axis rotated by $45^\circ$ to the bump chord (Figure 6.25). This makes the flow computation 2-D, with 3 velocity-components (Parneix et al. (1998)). The inlet is at a distance $c/2$ upstream of the bump, in the non-rotated frame. The inlet velocities are $U' = U \cos 45^\circ$ and $W' = U \sin 45^\circ$, in the rotated frame.

![Figure 6.25: Schematic of the two dimensional swept bump.](image)
Figure 6.26: Coefficient of (a) skin friction & (b) static pressure at wall for swept bump configuration.

Comparisons of the coefficients of static pressure and skin friction with the experimental data are shown in Figures 6.26a and 6.26b. While Lag $k - \omega$ and Lag $k - \varepsilon$ predict the recovery region downstream of the bump, the $k - \omega$ SST predicts a too slow recovery. This is further illustrated by a close look at the near wall region of the streamwise velocity profiles (Figure 6.27). Spanwise velocity profiles were predicted by all the models with reasonable accuracy (Figure 6.28). Note, however, that the spanwise velocity is considerably smaller than the streamwise velocity.

Contours of the eddy viscosity from the different formulations were examined. Focusing on the region from $x/c = 0.8$ to 1.2, it can be seen, qualitatively, that the lag models return higher levels of eddy viscosity compared to $k - \omega$ SST or $k - \omega$ 2006. Such observations are consistent with those made for the non-swept bumps.
Figure 6.27: U profiles at different streamwise locations.

Figure 6.28: W profiles at different streamwise locations.
Figure 6.29: Contours of $\frac{\nu_t}{\nu}$ (a) Lag $k - \omega$, (b) Lag $k - \varepsilon$, (c) $k - \omega$ SST (d) $k - \omega$ 2006.
6.8 Three dimensional Diffusers

The diffuser is a three dimensional, rectangular duct with two orthogonal, planar walls, and two outwardly sloping walls. The area expands, generating an adverse pressure gradient, often resulting in an internal flow separation.

Data for a parameterized set of diffusers was created by Durbin et al. (2016). The parameter is the inlet aspect ratio, $AR$. All members of the set have the same distribution of area versus $x$. As $AR$ increases, the separation bubble moves from the top wall to the side wall. The diffuser series provides data for assessing the performance of RANS models.

Figure 6.30 shows side and top views of the geometry under consideration. Further details are summarized in Table 6.2. The inflow is a fully developed channel flow with Reynolds number of 20,000, based on the hydraulic diameter and bulk velocity. The inlet extends 2 units upstream of the start of the expansion. The expanding walls extend for 15 units along the streamwise direction. The unit of length is the inlet height.

Figure 6.30: Geometry of the diffuser, (a) Side view, (b) Top view.
The full data set consists of 6 diffusers. Durbin et al. (2016) selected four aspect ratios for the purpose of testing a hybrid simulation model. For the current study two diffusers, with \( AR = 1.5 \) and 4 were chosen as representative. For \( AR = 1.5 \), the LES data show separation on the upper, flared wall; whereas for \( AR = 4 \), the data show separation on the side, flared wall.

Prediction of three dimensional separated flow fields are often a challenge for linear eddy viscosity models. The present diffuser geometry is an example. In addition to incorrect pressure distribution, RANS models may even predict separation on the wrong wall. These faults were observed in the current study, where the pressure distribution on the lower non-flared wall predicted by the \( k - \omega \) SST model deviated substantially from the LES data for both \( AR \)'s (Figure 6.31). The lag models show superior predictions, with the Lag \( k - \omega \) version being quite accurate. The \( k - \omega \) 2006 model is more accurate than the SST model.

Figure 6.32 shows a tendency for the SST model to produce separation on the wrong wall for \( AR=1.5 \). \( k - \omega \) 2006 does not have that fault, but it predicts a stronger backflow than the LES data. A similar trend was observed for the diffuser with \( AR=4 \) (Figure 6.33). The Lag \( k - \omega \) produces the correct pattern of separation and a more accurate magnitude of backflow.

Streamwise velocity profiles were examined in the \( y \) and \( z \)-planes that bisect the inlet face. Profiles are provided at various downstream locations in Figure 6.34. The velocity near the lower flat wall was over predicted by both the \( k-\omega \) SST and \( k-\omega \) 2006 models. The Lag \( k-\omega \) formulation provides an accurate prediction of the velocity profiles in close proximity to both the lower and side, flat walls.

<table>
<thead>
<tr>
<th>( AR )</th>
<th>( \theta_{\text{top}} )(deg)</th>
<th>( \theta_{\text{side}} )(deg)</th>
<th>( Z_{\text{inlet}} \times Y_{\text{inlet}} )(unit²)</th>
<th>( Z_{\text{outlet}} \times Y_{\text{outlet}} )(unit²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>3.13</td>
<td>9.27</td>
<td>1.64 \times 1.09</td>
<td>2.46 \times 3.56</td>
</tr>
<tr>
<td>4</td>
<td>5.11</td>
<td>5.7</td>
<td>2.67 \times 0.67</td>
<td>4.02 \times 2.17</td>
</tr>
</tbody>
</table>

Table 6.2: Geometry details for diffuser configurations.
Figure 6.31: Coefficient of static pressure based on bulk velocity on the bottom non-flaring wall for (a) $AR = 1.5$, (b) $AR = 4$. 
Figure 6.32: Contours of mean streamwise velocity at different locations along the diffuser of AR=1.5 for (a) LES, (b) Lag $k-\omega$, (c) Lag $k-\varepsilon$, (d) $k-\omega$ SST and (e) $k-\omega$ 2006.
Figure 6.33: Contours of mean streamwise velocity at different locations along the diffuser of $AR=4$ for (a) Lag $k - \omega$, (b) Lag $k - \varepsilon$, (c) $k - \omega$ SST and (d) $k - \omega$ 2006.
Figure 6.34: (a) Measurement planes, red marks the constant Z plane, blue marks the constant Y plane. Profiles of mean U along (b) constant Z plane for diffuser of $AR=1.5$, (c) constant Y plane for diffuser of $AR=1.5$, (d) constant Z plane for diffuser of $AR=4$ and (e) constant Y plane for diffuser of $AR=4$. 
6.9 NACA 0020 Wing Body Junction

Three dimensional junction flows are common in aerodynamic applications, such as along axial compressor/turbine end walls, aircraft wing/fuselage junctions, etc. A simple junction flow (Figure 6.35), that has been extensively studied, consists of an obstacle with a 3:2 elliptical nose, joined at the maximum thickness \( T \) to a NACA 0020 tail. It is mounted on a plate, to create a junction flow. Experimental data are available from Devenport et al. (1990); Fleming et al. (1993); Olcmen et al. (1990). The inlet condition is a two dimensional boundary layer with \( Re_\theta = 5,940 \), at a distance 0.75\( c \) upstream of the nose, \( c \) being the chord length. The Reynolds number based on the free-stream velocity and the maximum thickness of the wing is \( 1.17 \times 10^5 \).

![Figure 6.35: Schematic of the wing body junction configuration.](image-url)
As it flows past the wing, the boundary layer is skewed, due to spanwise pressure gradients. An adverse pressure gradient is also created by the wing in the symmetry plane, causing a recirculation zone upstream of the nose. Vortical structures, in the shape of a horseshoe vortex, are formed in front of the nose, and convect downstream along the wing. For the current setup, oil flow visualization by Olcmen et al. (1990) shows a separated flow with a ‘fish-tail’ structure at the trailing edge of the wing.

The complexity of the flow field poses a challenge to turbulence models, particularly linear eddy viscosity models. An assessment by Apsley et al. (2001) documents the performance of linear and non linear eddy viscosity models, and of Reynolds stress models. It was reported that most of the models were unable to reproduce the flow near the nose, although the Reynolds stress models were in closer agreement to experimental data than the other approaches. Similar observations were made by Chen (1995). Parneix et al. (1998) used the \( \nu^2 - f \) model and were able to predict the strength and position of the horseshoe vortex with accuracy.

Quantitative comparisons were made at several locations for this configuration, including the symmetry plane in the vicinity of the nose, seven different stations around the nose of the wing (Olcmen et al. (1990)) and spanwise planes at \( x/T = 0.76, 2.72, 3.95 \). The streamwise evolution of the vortices was examined qualitatively, by comparing contours of TKE as computed by the models, to experimental data at \( x/T = 4.46 \) and 6.38.

Figure 6.36 compares computed streamlines to experimental, oil-flow visualization. Qualitatively, all the formulations reproduce the streamline patterns near the nose of wing. Both experimental data and recent LES (Ryu et al. (2016)) show a weak recirculation zone at the trailing edge, which is well represented by Lag \( k - \omega \), Lag \( k - \varepsilon \) and \( k - \omega \) 2006, whereas \( k - \omega \) SST shows a bigger circulation zone. This can be attributed to the lower levels of eddy viscosity from the \( k - \omega \) SST model — as seen previously. Figure 6.37 shows the distribution of the coefficient of static pressure on the wall. All the models show reasonable agreement with the experimental data; however, the \( C_p \) lines from \( k - \omega \) SST show kinks that were not seen in the experiment. This is indicative of a
Figure 6.36: Streamlines on the plate for (a) Oil flow visualization by (b) Lag $k - \omega$, (c) Lag $k - \varepsilon$, (d) $k - \omega$ SST and (e) $k - \omega$ 2006.

pressure rise near the nose, that is predicted by $k - \omega$ SST to be stronger than the predictions of other models.
In the plane of symmetry, experimental data of Devenport et al. (1990) indicate a strong adverse pressure gradient, with the presence of a recirculation bubble. Comparisons for the streamwise and wall-normal velocities were carried at $x/T$ locations of -0.05, -0.1, -0.15, -0.2, -0.25, -0.3, -0.35, -0.4 and -0.45 and are presented in Figures 6.38. The location of the switch between attached and back-flow is around $x/T = -0.35$, as can be seen from the experiment. While the lag models and $k - \omega$ 2006 were able to reproduce the switch location accurately, the intensity of the flow reversal is best represented by $k - \omega$ 2006. $k - \omega$ SST tends to marginally over estimate the separation. Prediction of the wall normal component of the velocity showed similar trends and $k - \omega$ SST was seen to deviate more from the data compared to the other models.

It was interesting to see whether the lag approaches have similar accuracy to that seen for the swept bump, for the wing-body configuration. Streamwise and spanwise velocity profiles are
Figure 6.38: (a) $\frac{U}{U_{ref}}$, (b) $\frac{V}{U_{ref}}$ on the plane of symmetry at (a) $x/T=-0.05$, (b) $x/T=-0.1$, (c) $x/T=-0.15$, (d) $x/T=-0.2$, (e) $x/T=-0.25$, (f) $x/T=-0.3$, (g) $x/T=-0.35$, (h) $x/T=-0.4$, (i) $x/T=-0.45$ compared to experimental data of Olcmen et al. (1990) in Figure 6.39. In this case, the lag parameter has almost no effect.

Further comparisons, at streamwise planes $x/T = 0.76$ & $2.72$ — corresponding to maximum thickness and middle of the wing are provided in Figures 6.40 and 6.41. At these locations, $k - \omega$ SST predictions for both the streamwise and spanwise velocities are in good agreement with the experimental data.

Evolution of vortices downstream of the trailing edge can be critical. For instance, in compressor cascades, the performance of a stage is dependent upon the wake from the previous stage, incident upon it. Figures 6.42, 6.43, 6.44 and 6.45 represent a qualitative comparison of the streamwise mean flow and TKE levels at planes $x/T = 4.46$ and 6.38. At both locations, $k - \omega$ SST and $k - \omega$ 2006 were observed to produce higher levels of TKE than experiment. Overall, the flow field was predicted satisfactorily by all the models, although $k - \omega$ SST predicts a large circulation in
Figure 6.39: Comparison of U & W components of velocity at seven different locations around the nose of the wing with experimental data by.

Table 6.3: Comparison of vortex core coordinates from different models.

<table>
<thead>
<tr>
<th>Plane</th>
<th>Model</th>
<th>y/T</th>
<th>z/T</th>
</tr>
</thead>
<tbody>
<tr>
<td>x/T = 6.38</td>
<td>Apsley et al. (2001)</td>
<td>0.285 ± 2%</td>
<td>0.800 ± 1%</td>
</tr>
<tr>
<td></td>
<td>Lag k − ω</td>
<td>0.278</td>
<td>0.571</td>
</tr>
<tr>
<td></td>
<td>Lag k − ε</td>
<td>0.279</td>
<td>0.626</td>
</tr>
<tr>
<td></td>
<td>k − ω 2006</td>
<td>0.278</td>
<td>0.657</td>
</tr>
<tr>
<td></td>
<td>k − ω SST</td>
<td>0.278</td>
<td>0.778</td>
</tr>
</tbody>
</table>

the corner region at x/T = 4.46, which is consistent with the streamline patterns discussed above.

Table 6.3 lists the coordinates of the vortex core at x/T = 6.38 from different models.
Figure 6.40: (a) $\frac{U}{U_{ref}}$, (b) $\frac{W}{W_{ref}}$ at streamwise plane location $x/T=0.76$, $z/T$ (from left to right) = 0.755, 0.85, 0.925, 1, 1.075, 1.175, 1.325, 1.525.

Figure 6.41: (a) $\frac{U}{U_{ref}}$, (b) $\frac{W}{W_{ref}}$ at streamwise plane location $x/T=2.72$, $z/T$ (from left to right) = 0.555, 0.655, 0.755, 0.855, 0.955, 1.055, 1.155, 1.355, 1.555, 1.997.
Figure 6.42: (a) Schematic of location of measurement plane \((x/T = 4.46)\), Contours of \(\frac{U}{U_{ref}}\) from (b) Experiment, (c) Lag \(k - \omega\), (d) Lag \(k - \varepsilon\), (e) \(k - \omega\) SST and (f) \(k - \omega\) 2006.

Figure 6.43: (a) Schematic of location of measurement plane \((x/T = 4.46)\), Contours of \(\frac{k}{U_{ref}^2}\) from (b) Experiment, (c) Lag \(k - \omega\), (d) Lag \(k - \varepsilon\), (e) \(k - \omega\) SST and (f) \(k - \omega\) 2006.
Figure 6.44: (a) Schematic of location of measurement plane \((x/T = 6.38)\), Contours of \(\frac{U}{U_{ref}}\) from (b) Experiment, (c) Lag \(k - \omega\), (d) Lag \(k - \varepsilon\), (e) \(k - \omega\) SST and (f) \(k - \omega\) 2006.

Figure 6.45: (a) Schematic of location of measurement plane \((x/T = 6.38)\), Contours of \(\frac{k}{U_{ref}^2}\) from (b) Experiment, (c) Lag \(k - \omega\), (d) Lag \(k - \varepsilon\), (e) \(k - \omega\) SST and (f) \(k - \omega\) 2006.
CHAPTER 7. CURRENT LIMITATIONS

While the Lag $k - \omega$ was mostly successful in providing superior prediction accuracy for separated flows, certain flow problems were identified where the lag parameter scaling did not prove to be effective. Such cases are presented in this chapter and can serve as motivation for future investigations. Predictions from the Lag $k - \omega$ model in comparison with $k - \omega$ SST formulation are presented.

7.1 Two dimensional asymmetric diffuser

This configuration is a wall bounded flow with one side being flat and the other expanding to create an adverse pressure gradient. The angle of expansion being $10^0$ and the expansion ratio being 4.7. The inflow is a fully developed channel flow with centerline velocity & channel height based Reynolds number of 20000. The adverse pressure gradient causes the flow to separate on the expanding wall. Experimental data for comparison are of Buice et al. (2000). The geometry is shown in Figure 7.1.

This is a common test case for assessing the accuracy of predictive models as capturing both the onset of separation and reattachment point correctly can be challenging. This is quantified through the plot of the skin friction coefficient on the expanding wall(Figure 7.2(a)). It can be seen that while $k - \omega$ SST accurately captures the reattachment point, prediction of the onset of separation is correctly captured by the Lag $k - \omega$ formulation, although it predicts a shorter separation extent compared to the experiment. Predictions on the flat wall are very accurate using the $k - \omega$ SST model. Numerous iterations of various combinations of coefficient values of the Lag $k - \omega$ model were tried in an attempt to capture both the critical points correctly, unfortunately it proved to be difficult. Streamwise velocity predictions were more accurate upon using $k - \omega$ SST than Lag $k - \omega$(Figure 7.3).
Dependence of predictions for this configuration on the realizability limiter (Durbin et al. (2010)) was looked at. Typically a limiter dampens the eddy-viscosity levels at regions of high strain. Most turbulence models employ the usage of such limiters, e.g., for the $k-\omega$ SST model eddy viscosity is defined by $\nu_t = \frac{0.3k}{\max(0.3\omega, |S|)}$, where $|S| = \sqrt{2S_{ij}S_{ij}}$. For the Lag $k-\omega$ model, $\nu_t = k \min\left(\frac{\varphi^*}{\omega}, \frac{\alpha_s}{|S|}\right)$, where $\alpha_s$ is calibrated to a value of 0.7. Figure 7.4 shows the effect of the limiter for the $k-\omega$ SST formulation on the coefficient of skin friction. It was observed that without the limiter, the onset of separation was predicted more accurately while the reattachment point and also the skin friction on the flat wall was poorly predicted.

Similar study was done for the Lag $k-\omega$ formulation, where two values of the $\alpha_s$ coefficient, being equal to 0.5 and 0.7 were chosen and was compared with the predictions where no limiter was used. Contrary to the $k-\omega$ SST formulation, the Lag $k-\omega$ predictions were less sensitive to the usage of the realizability limiter as can be seen from Figure 7.5.
Figure 7.2: 2D asymmetric diffuser, coefficient of skin friction on (a) expanding wall, (b) flat wall.
Figure 7.3: 2D asymmetric diffuser, U mean at different streamwise locations.

Figure 7.4: 2D asymmetric diffuser, dependence of coefficient of skin friction on the realizability limiter of $k-\omega$ SST formulation on (a) expanding wall, (b) flat wall.
Figure 7.5: 2D asymmetric diffuser, dependence of coefficient of skin friction on the realizability limiter of Lag $k - \omega$ formulation on (a) expanding wall, (b) flat wall.
Flow over a three dimensional flow geometry presents a complex flow problem often inaccurately predicted by linear eddy viscosity models. The geometry is of the Fundamental Aero Investigates the Hill (FAITH) configuration (Bell et al. (2012)). The hill height ($h$) relates to the radius ($r$) through,

$$h = 3\cos\left(\frac{\pi r}{9}\right) + 3$$

The hill height (H) at the centroid is 6 inches whereas the radius (R) is 9 inches. Reynolds number based on the hill height ($Re_H$) = 500,000 and the inflow velocity is 50.3 m/s. The domain used for the computation is $20H \times 5.3H \times 8H$ with the centroid of the hill being at $x/H = z/H = 0$. The configuration is shown in Figure 7.6. A plug flow type inflow was provided at the inlet and sufficient domain length was provided for the boundary layer to develop till it reaches the experimental value of momentum thickness before it encounters the hill.

![Figure 7.6: Faith Hill configuration.](image)

The flow separates on the leeward side of the hill creating a strong recirculation zone. While eddy-resolving simulations such a DES was very successful in predicting the extent and intensity of the separation, popular RANS models such as $k - \omega$ SST over predicts the separation zone.
Figure 7.7 shows profiles of $U$ mean immediately behind the hill on the constant $Z$ plane that splits the hill into half. The Lag $k-\omega$ model predicts a smaller separation bubble than $k-\omega$ SST, however the overall velocity prediction remains unsatisfactory. This is further illustrated through the contours of all the velocity components (Figure 7.8). Lower levels of eddy viscosity in close proximity of the wall immediately after the wall can be held accountable for the inferior predictions. One way to mitigate such an issue through the lag parameter scaling can be to sensitize the $\varphi^*$ equation to pressure gradients in spanwise direction and remains an open area for investigation.

Figure 7.7: Profiles of $U$ immediately after the hill at $Z=0$. 
Figure 7.8: Contours of velocity components on Z=0.
CHAPTER 8. DISCUSSION AND FUTURE WORK

This work presents the development of a novel linear eddy viscosity model by extending the elliptic blending lag parameter approach to an underlying $k-\omega$ model. The new formulation is termed as “Lag $k-\omega$” model. Predictive capabilities of the model is assessed by implementing the formulation in the open source code OpenFoam and testing it on two and three dimensional flows ranging from plane shear to separated flows.

8.1 Summary of the chapters

In chapter 3, issues related to near wall turbulence has been discussed. Approaches to model the wall effects on the turbulence have been reviewed. The elliptic relaxation approach proposed by Durbin (1991) has a sound physical rationale to predict the near wall asymptotic behaviour of the wall normal fluctuations. Based on this, an elliptic blending Reynolds stress model was developed by Manceau et al. (2002). To reduce the complexity of an RSM, the elliptic relaxation was introduced to a linear eddy viscosity modelling framework by Durbin (1991). This was further revised by several researcher such as Billard (2011) etc. Billard’s model $BL - \overline{\nu^2}/k$ uses an elliptic blending parameter $\alpha$ to blending between the near-wall and far from wall terms in the additional transport equation, thereby ensuring correct asymptotic behaviour without impairing numerical robustness. This is an important concept that was used in the development of the lag models.

Chapter 4, describes the development of the Lag $k-\varepsilon$ model, where the stress-strain misalignment (‘lag”) parameter, originally defined by Revell et al. (2011) was combined with an underlying $k-\varepsilon$ formulation by Lardeau et al. (2016). The Lag $k-\varepsilon$ model was successful in improving predictions for separated flows which motivated the development of a the Lag $k-\omega$ version. Derivation of the same is explained in detail in this chapter and the calibration of the model coefficients is discussed.
Application of the lag models to canonical separated flows are shown in chapter 6. It was observed that one of the most important feature of the lag approaches was the improved scaling of the near wall eddy viscosity, thereby predicting a better turbulent mixing in close proximity of viscous walls. This is a key reason for the observed improvements in separated flows. The formulations were also tested on 3D flows where improvements were observed when compared to the existing RANS models such as $k - \omega$ SST etc.

8.2 Future Work

Several avenues regarding the lag modelling approach explained in this study are still left unexplored. The key ones are,

- The closure of the $a_{ij}$ term is quadratic in terms of $S_{ij}$ & $W_{ij}$ in the proposed formulation. Involving a third order term and studying the effect of that particularly for three dimensional geometries can be beneficial. Also, sensitivity of the $\beta_2$ needs to be looked at for more complex flow cases.

- Rotation and curvature corrections have not been considered for the present proposal. Basic cases of such nature, e.g., rotating channel flow etc., should be evaluated using the Lag $k - \omega$ formulation.

- The focus of this formulation was restricted to incompressible flow only for the current study. Compressible effects need to be taken in to account to develop a more robust version of the model. It will be particularly interesting to see the behaviour of the lag parameter transport equation in compressible flow regimes.
REFERENCES


APPENDIX A. LAG $k - \omega$ TRANSPORT EQUATIONS

\begin{align}
\frac{Dk}{Dt} &= P - \beta^* k \omega + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] \\
\frac{D\omega}{Dt} &= \gamma_{new} \frac{P}{k} \omega - \beta \omega^2 + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right] \\
\frac{D\varphi^*}{Dt} &= -(1 - \alpha^3) C_w \varphi^* \omega - \alpha^3 \left( \tilde{C}_1 + C_{1^*} \frac{P}{\beta^* k \omega} \right) \varphi^* k \omega - C_{p1} \frac{P}{k} \varphi^* \\
&\quad + \alpha^3 C_{p2} \varphi^* S + \alpha^3 C_{p3} \varphi^* \omega + \alpha^3 \beta^* \omega \varphi_h \left( C_{4^*} a_{ik} S_{kj} - C_{5^*} a_{ik} W_{kj} \right) S_{ij} \\
&\quad + \frac{\partial}{\partial x_j} \left[ \left( \nu + \frac{\nu_t}{\sigma_\varphi} \right) \frac{\partial \varphi^*}{\partial x_j} \right].
\end{align}

\(\alpha - L^2 \nabla^2 \alpha = 1\)

The eddy viscosity is defined as

\[ \nu_t = k \min \left( \varphi^* \frac{\alpha_s}{\omega S} \right) \]

Turbulent length scale is defined by,

\[ L^2 = C_L^2 \left( \frac{k^3}{(\beta^* k \omega)^2} + C_\eta^2 \sqrt{\frac{\nu^3}{\beta^* k \omega}} \right) \]

Also,

\[ C_{p3} = 1.7, \text{ if } \frac{\alpha}{\eta} < 10 \]

\[ = 0.64, \text{ OTHERWISE} \]

And,

\[ \gamma_{new} = (1 - \alpha^3) \gamma_1 + \alpha^3 \gamma_2 \]
Model coefficients are,

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<th>$\beta^*$</th>
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<th>$\gamma_2$</th>
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<table>
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<th>$\varphi_h$</th>
<th>$C_4^\ast$</th>
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In both the models, strain and vorticity rate tensors are defined as

$$S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad W_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right),$$

respectively. Also the production term $P$, is given by $P = \nu_t |S|^2$, with $|S|^2 = 2S_{ij}S_{ij}$.

$a_{ij}$ is closed using the relation,

$$a_{ij} = -2\frac{\nu_t}{k} \left[ S_{ij} + \beta_2 \frac{2(S_{ik}W_{kj} - W_{ik}S_{kj})}{(|S_{kl} + W_{kl})(S_{kl} + W_{kl})|} \right] \quad (A.7)$$

with, $\beta_2 = \frac{2 - 2C_5}{C_1 + C_1^\ast + 1}$.
APPENDIX B. LAG $k - \varepsilon$ TRANSPORT EQUATIONS

\[
\frac{Dk}{Dt} = P - \varepsilon + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu}{2} + \nu_t \right) \frac{\partial k}{\partial x_j} \right] \quad \text{(B.1)}
\]

\[
\frac{D\varepsilon}{Dt} = \frac{1}{\tau}(C_{\varepsilon 1}P - C_{\varepsilon 2}\varepsilon) + E + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu}{2} + \nu_t \right) \frac{\partial \varepsilon}{\partial x_j} \right] \quad \text{(B.2)}
\]

WITH,

\[
E = C_k \nu \nu_t (1 - \alpha^3) \left( \frac{\partial |2S_{ij}n_j| n_k}{\partial x_k} \right)^2 \quad \text{(B.3)}
\]

\[
\frac{D\varphi}{Dt} = -(1 - \alpha^3)C_w \varphi \frac{\varphi}{\tau} - \alpha^3 \left( \hat{C}_1 + C_{\varphi 1} \frac{P}{\varepsilon} \right) \frac{\varphi}{\tau} + \frac{\alpha^3}{S^2 \tau} (C_{\varphi 2} a_{ik} S_{kj} - C_{\varphi 4} a_{ik} W_{kj}) S_{ij} - C_{\varphi 5} \varphi \frac{P}{k} + \alpha^3 C_{\varphi 2} \varphi S + \alpha^3 C_{\varphi 5} \frac{\varphi}{\tau} + \frac{\partial}{\partial x_j} \left[ \left( \frac{\nu}{2} + \nu_t \frac{\varphi}{\sigma_{\varphi}} \right) \frac{\partial \varphi}{\partial x_j} \right]
\]

\[
\alpha - L^2 \nabla^2 \alpha = 1 \quad \text{(B.5)}
\]

**Eddy viscosity is defined by,**

\[
\nu_t = C_{\mu} \varphi k \min \left( \tau_{lim}, \frac{C_T}{C_{\mu} \sqrt{3 \varphi S}} \right) \quad \text{(B.6)}
\]

**Turbulent length and time scales are defined by,**

\[
L = C_L \sqrt{\frac{k^3}{\varepsilon^2} + C_\eta^2 \sqrt{\frac{P}{\varepsilon}}} \quad \text{(B.7)}
\]

\[
\tau = \frac{k}{\varepsilon} \quad \text{(B.8)}
\]
\[ \tau_{\text{lim}} = \sqrt{\tau^2 + C_t^2 \frac{\nu}{\varepsilon}} \]  
\hspace{1cm} (B.9)

\[ C_{p3} = \frac{S\tau + \alpha^3}{C_{\mu \max}(\eta, 1.87)} \left( \frac{2}{3} - \frac{C_3}{2} \right) \]  
\hspace{1cm} (B.10)

Model coefficients are given by,

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