Estimation for time series subject to the error of rotation sampling

Édina Shisue Miazaki

Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Statistics and Probability Commons

Recommended Citation

Miazaki, Édina Shisue, "Estimation for time series subject to the error of rotation sampling " (1985). Retrospective Theses and Dissertations. 7872.
https://lib.dr.iastate.edu/rtd/7872

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document photographed is “Missing Page(s)”. If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.

2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.

3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of “sectioning” the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.

4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.

5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.
Miazaki, Edina Shisue

ESTIMATION FOR TIME SERIES SUBJECT TO THE ERROR OF ROTATION SAMPLING

Iowa State University

University Microfilms International

300 N. Zeeb Road, Ann Arbor, MI 48106

Ph.D. 1985
Estimation for time series subject to the
error of rotation sampling

by

Edina Shisue Miazaki

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major: Statistics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa
1985
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. LITERATURE REVIEW OF ROTATION SAMPLING</td>
<td>3</td>
</tr>
<tr>
<td>A. Classical Approach</td>
<td>4</td>
</tr>
<tr>
<td>B. Time Series Approach</td>
<td>14</td>
</tr>
<tr>
<td>III. PREVIOUS WORK IN AUTOREGRESSIVE TIME SERIES SUBJECT TO MEASUREMENT ERROR</td>
<td>22</td>
</tr>
<tr>
<td>A. Estimation of the Current Value when the Parameters are Known</td>
<td>22</td>
</tr>
<tr>
<td>B. Estimation for Autoregressive Signal</td>
<td>28</td>
</tr>
<tr>
<td>IV. AUTOREGRESSIVE SIGNAL PLUS MOVING AVERAGE NOISE</td>
<td>38</td>
</tr>
<tr>
<td>A. Model and Estimators</td>
<td>38</td>
</tr>
<tr>
<td>B. Limiting Properties of the Estimators</td>
<td>51</td>
</tr>
<tr>
<td>C. Error Autocovariance Function Estimated</td>
<td>88</td>
</tr>
<tr>
<td>D. Estimation of the True Values with Estimated Parameters</td>
<td>92</td>
</tr>
<tr>
<td>E. Extension to Nonormal Distributions</td>
<td>95</td>
</tr>
<tr>
<td>V. A SIMULATION STUDY</td>
<td>98</td>
</tr>
<tr>
<td>VI. APPLICATION TO THE NATIONAL CRIME SURVEY</td>
<td>107</td>
</tr>
<tr>
<td>A. Sample Design</td>
<td>107</td>
</tr>
<tr>
<td>B. Rotation Pattern</td>
<td>108</td>
</tr>
<tr>
<td>C. Data Analysis</td>
<td>110</td>
</tr>
<tr>
<td>VII. SUMMARY</td>
<td>127</td>
</tr>
<tr>
<td>VIII. BIBLIOGRAPHY</td>
<td>130</td>
</tr>
<tr>
<td>IX. ACKNOWLEDGMENTS</td>
<td>135</td>
</tr>
<tr>
<td>X. APPENDIX</td>
<td>136</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

Often, in sociological and economic research, studies are concerned with estimating the manner in which a given population is changing and the current value of the population mean. For example, the National Crime Survey is concerned with estimating yearly changes in the crime rate. One way of carrying out such studies is by surveying the population at regular time intervals. If the sampling is designed such that samples drawn on different occasions partially overlap, we have a rotation sampling scheme. Under such a scheme, suppose that a survey at time $t$ provides an estimate $Y_t$ of the population parameter $X_t$. We can write

$$ Y_t = X_t + u_t, $$

where $u_t$ is the sampling error. If $Y_t$ is unbiased, we have that, for given $X_t$, $E\{u_t\} = 0$, and $\text{var}\{u_t\} = \sigma_t^2$, the variance of the sampling error. In rotation sampling schemes, the errors $u_t$ are correlated over time. Hence, improved estimators of $X_t$ can be obtained by properly combining estimators from several time intervals. Together, this fact and the assumption that $X_t$ is a fixed parameter form the basis of the classical rotation sampling theory.

Recent work on rotation sampling has extended the model assumptions to allow for stochastic variation in $X_t$. By representing $X_t$ as a realization of a stochastic process, the problem of estimating $X_t$ becomes one of estimating the current value of a time series subject to
Consider the stationary sequence of population means $X_t$ satisfying the stochastic difference equation

$$X_t + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} = e_t$$

where $\{e_t\}$ is a sequence of $\text{NI}(0, \sigma_{ee})$ random variables. Let $\{Y_t\}$ be a sequence of unbiased survey estimates of $X_t$. We write

$$Y_t = X_t + u_t,$$

where $\{u_t\}$ is the sequence of sampling errors whose covariance structure depends upon the sampling design. If the sampling units appear in the sample for at most $q$ occasions, then the survey errors $u_t$ and $u_{t+j}$ are uncorrelated for $j > q$. Therefore, we assume

$$u_t = v_t + b_1 v_{t-1} + \ldots + b_q v_{t-q},$$

where $v_t$ is a sequence of $\text{NI}(0, \sigma_{vv})$ random variables and $E(e_t v_r) = 0$ for all $t$ and $r$. In this study, we propose estimators of $\alpha_1, \ldots, \alpha_p$ and $\sigma_{ee}$, assuming that $b_1, \ldots, b_q$ and $\sigma_{vv}$ are estimable from the survey data. The large sample distribution of the estimators is derived. The accuracy of the approximate distributions is investigated by means of a Monte Carlo experiment. Finally, an example of a time series subject to the error of rotation sampling is presented.
II. LITERATURE REVIEW OF ROTATION SAMPLING

In studies concerned with estimating a population parameter which changes over time, it is common practice to sample the population at regular time intervals. This repeated sampling allows the sample design to be very flexible. For example, on the t-th occasion, one may have parts of the sample that are matched with the (t-1)-th occasion, parts that are matched with the (t-2)-th occasion, and so on. If there exists a relationship between the value of an element in the population at time t and the value of the same element at time t + Δt, then it is possible to use the information contained in earlier samples to improve the current estimate of the population parameter.

If an estimator of the parameter at the t-th occasion does not use data reported on occasions other than t, it is called a simple or elementary estimator. If the estimator involves data reported earlier, then it is called a composite estimator.

Rotation sampling schemes, also called sampling on successive occasions with partial replacement of units by Patterson (1950), and sampling for a time series by Hansen et al. (1955), are sampling procedures in which the samples reporting at times t and t + Δt have some elements in common.

Rotation samples can be classified according to the reporting pattern of the samples. In one-level rotation sampling, there is an
overlap between the samples at times \( t \) and \( t-1 \). Each sampling unit reports data only for the current time. In two-level rotation sampling, the samples drawn at two consecutive occasions do not overlap, and each sampling unit reports data for the current and the immediately preceding times. The period covered by the interview is called reference period, and the time between the reported date and the date of interview is called recall lag or recall period.

Estimation of parameters in rotation sample designs has been considered by several authors. There are two different approaches to the problem: the classical approach in which the population parameter of interest \( \theta_t \), usually the mean, is considered to be a fixed quantity; and the time series approach in which \( \{\theta_t\} \), the sequence of means, is considered to be a realization of a stochastic process.

### A. Classical Approach

Under the classical approach, i.e., \( \theta_t \) fixed, the individual values \( Y_{ti} \) attached to the \( i \)-th unit at time \( t \), \( i = 1, 2, \ldots, t = 1, 2, \ldots \), are the random quantities. These values are assumed to be related to the previous values \( Y_{t-1,i} \) of the same unit with some known correlation structure.

Patterson (1950) provided a general theory for the rotation sampling problem. He assumed that samples are drawn from an infinite population according to the following pattern: at each given time, some of the old elements are eliminated from the sample and new elements are
added to the sample.

Assume that for all $t$,

$$y_{ti} - \theta_t = \rho(y_{t-1,i} - \theta_{t-1}) + \eta_{ti} \quad (2.1)$$

where $\{\eta_{ti}\}$ is a sequence of uncorrelated random variables,

$$E(\eta_{ti}) = 0,$$

$$\text{var}(\eta_{ti}) = (1 - \rho^2)s^2,$$

$$S^2 = E(y_{ti} - \theta_t)^2.$$

Let $n_t$ be the sample size at time $t$,

$n'_t$ (the number of sampling units on occasion $t$, which are in common with the $(t-1)$-th occasion, $n''_t$ be the number of new sampling units on occasion $t$, 

$$n''_t = n_t - n'_t,$$

$y_{tj}, 1 < j < n_t$, denote the $j$-th sampled value at time $t$, 

$x'_{t-1}$ be the mean of the $n'_t$ matched units on the $(t-1)$-th occasion, 

$x''_{t-1}$ be the mean of $(n_{t-1} - n'_t)$ unmatched units.
on the (t-1)-th occasion,
\( \bar{y}_t \) be the mean of the \( n_t \) new units, and
\( \tilde{\theta}_t \) be a linear unbiased estimator of \( \theta_t \), the population mean at time \( t \).

By definition, we can write,
\[
\tilde{\theta}_t = \sum_{i=1}^{n_i} \sum_{j=1}^{t} w_{ij} y_{ij},
\]
where
\[
\sum_{i=1}^{n_i} w_{ij} = 1, \text{ for } j = t
\]
\[
= 0, \text{ for } j \neq t.
\]

Therefore, the minimum variance linear unbiased estimator of \( \theta_t \) is obtained by minimizing the Lagrangean
\[
\text{Var}\{\sum_{i=1}^{n_i} \sum_{j=1}^{t} w_{ij} y_{ij}\} - 2 \sum_{j=1}^{t} k_{mj} \sum_{i=1}^{n_i} w_{ij},
\]
where the \( k_{mj} \) are the Lagrangean multipliers. The minimization leads to a set of equations which may be written
\[
\text{Cov}(Y_{hj}, \tilde{\theta}_t) = h_{tj}, \text{ for all } h \text{ and } j. \quad (2.3)
\]

Using expression (2.3), Patterson showed that the minimum variance
linear unbiased estimator of the population mean under model (2.1) is of the form

$$\tilde{\theta}_t = (1 - \phi_t') \tilde{y}_t + \rho (\tilde{\theta}_{t-1} - \tilde{x}_{t-1}' \gamma_{t-1}) + \phi_t' y_t,$$

where

$$1 - \phi_t' = \frac{n_t' n''_{t-1}}{n_t n_{t-1} - \rho^2 n_t' (n''_{t-1} - \phi_t' n_t')}, \quad n''_{t-1} \neq 0,$$

and its variance is given by

$$\text{var}(\tilde{\theta}_t) = \frac{\phi_t' S^2}{n_t} \quad \text{if} \quad n'' \neq 0.$$

If $n'' = 0$,

$$\text{var}(\tilde{\theta}_t) = S^2 \left(\frac{1 - \rho^2}{n_t} + \frac{\rho^2 \phi_{t-1}'}{n''_{t-1}}\right), \quad n''_{t-1} \neq 0.$$

The minimum variance linear unbiased estimator of the change in the mean for the last two occasions is given by

$$\Delta \tilde{\theta}_t = (1 + \rho \phi_{t-1}') \tilde{\theta}_t - \rho \phi_{t-1} \tilde{y}_t - \tilde{\theta}_{t-1}$$

and

$$\text{var}(\Delta \tilde{\theta}_t) = \frac{\phi_t' S^2}{n_t} + \frac{\phi_{t-1}' S^2}{n_{t-1}} \left\{1 - \rho^2 \phi_{t-1}' (1 - \phi_t') - 2\rho (1 - \phi_t')\right\}.$$
These expressions simplify for the case in which the sample size and the number of new units added to the sample at each time are the same on each occasion.

The problem of finding the optimum number of matching units between two successive occasions was also considered by Patterson. Patterson's work is an extension of the earlier work by Jessen (1942) for sampling on two occasions.

Eckler (1955) extended Patterson's methods to two and three-level rotation samplings. For the two-level rotation sampling, with \( n_t = n \) for all \( t \), the proposed minimum variance linear unbiased estimator of \( \theta_t \) is of the form

\[
\tilde{\theta}_t = \bar{y}_{t,1} - a_t \bar{y}_{t-1,1} + a_t \tilde{\theta}_{t-1},
\]

where \( \bar{y}_{t,1} \) is the mean for the \((t-1)\)-th occasion of the sample drawn on \( t \)-th occasion,

\( \bar{y}_{t-1,1} \) is the mean for the \((t-1)\)-th occasion of the sample drawn on the \((t-1)\)-th occasion, and

\[
a_t = 0 \quad \text{for} \quad t = 1,
\]

\[
a_t = \frac{\rho}{2 - a_{t-1} \rho} \quad \text{for} \quad t = 2, 3, \ldots
\]

The coefficients \( a_t \) were determined to satisfy the conditions

\[
\text{cov}(\bar{y}_{t-1,1}, \tilde{\theta}_t) = \frac{S^2}{n} (\rho - a_t)
\]
and

\[ \text{cov}(\tilde{y}_{t-1,2}, \tilde{\theta}_t) = \frac{s^2}{n} a_t (1 - a_{t-1} \rho) \]

for all \( t \). These conditions play the same role as condition (2.3) in the one-level rotation sampling.

The variance of the estimator \( \tilde{\theta}_t \) is given by

\[ \text{var} \tilde{\theta}_t = \frac{s^2}{n} (1 - a_t \rho). \]

The one-level rotation sampling estimators can be obtained from the two-level rotation sampling estimators through the relationships

\[ \tilde{\theta}_t^{(1)} = \frac{1}{2 - a_t \rho} \tilde{\theta}_t^{(2)} + \frac{1 - a_t \rho}{2 - a_t \rho} \tilde{y}_{t,2}, \]

and

\[ \text{var} \tilde{\theta}_t^{(1)} = 2n^{-1} s^2 (2 - a_t \rho)^{-1} (1 - a_t \rho). \]

where \( \tilde{\theta}_t^{(j)} \) is the minimum variance linear unbiased estimator of \( \theta_t \) based on \( j \)-level rotation sampling, \( j = 1, 2 \). This result was obtained by using a generalization of the method used to find the minimum variance linear combination of two uncorrelated estimates of the same parameter.

To get estimators for the three-level rotation sampling problem one
has to solve a system of four equations in four unknowns. In this case, the minimum variance linear unbiased estimator of $\theta_t$ depends upon the sample averages and the minimum variance linear unbiased estimators of $\theta_{t-1}$ and $\theta_{t-2}$.

Estimation in rotation sampling involving concepts of finite population was considered by Rao and Graham (1964). They considered a rotation pattern that allows sample units, which had been eliminated in previous occasions, to come back into the sample. Let $N$ and $n$ be the population and sample sizes, respectively (both assumed to be the same for all occasions). Also, let $N$ and $n$ be multiples of $n'$. A group of $n'$ units remains in the sample for $r$ occasions ($n = r n'$), leaves the sample for $m$ occasions, comes back for another $r$ occasions, and so on. If a unit returns to the sample after having dropped out $(k - 1)$ previous times, the unit is said to be in the $k$-th cycle.

A theory for composite estimates of the current population mean and of changes in level between successive occasions was developed for the one-level rotation sample designs. The composite estimator of the current population mean is

$$\tilde{\theta}_t = Q(\tilde{\theta}_{t-1} + \bar{y}_t' - \bar{x}_{t-1}') + (1 - Q)\bar{y}_t,$$

where $0 < Q < 1$, $\bar{y}_t'$ is the sample mean at time $t$, and $\bar{y}_t'$ and $\bar{x}_{t-1}'$ are as defined in (2.2). The composite estimator of the population change $\theta_t - \theta_{t-1}$ is
\[ \Delta \hat{\theta}_t = \hat{\theta}_t - \hat{\theta}_{t-1}. \]

For the cases in which either

i) \( \text{Var}(Y_{t,i}) = S^2, \ t = 1, 2, \ldots, i = 1, 2, \ldots, N \) and

\[ \text{Cov}(Y_{t,i}, Y_{t+k,i}) = \rho |i| S^2 \]

or

ii) \( \text{Var}(Y_{t,i}) = S^2, \ t = 1, 2, \ldots, i = 1, 2, \ldots, N \) and

\[ \text{Cov}(Y_{t,i}, Y_{t+k,i}) = \rho \text{[|i| - 1)d]S^2}, \text{ for } (|i| - 1)d < \rho \]

= 0 otherwise,

explicit formulas for the variances of the estimators were given.

The gain in efficiency of the composite estimators relative to the use of simple estimators was numerically investigated for cases (i) and (ii). For each combination of \((r, \rho, m)\) and \(((d, \rho, r), m = \infty)\), the optimum \( Q \) was defined to be the value which gives the maximum percent gain in efficiency. The percent gain in efficiency and the optimum \( Q \) were tabulated for selected values of \((r, \rho, m)\) and \(((d, \rho, r), m = \infty)\). This investigation shows that for model (i), the composite estimator is virtually the same for moderate \( m \) as for \( m = \infty \), and that the optimum \( Q \) is not affected by \( m \). Also, for \( m = \infty \), the optimum \( Q \) for model (i) either is equal or differs by 0.1 from the
optimum $Q$ for model (ii), for different values of $d$.

Two-level rotation schemes were studied by Wolter (1979). Under a finite population model, he considered surveys in which the sample is divided into $s (s > 0)$ fixed panels and $l (l > 1)$ rotating panels. The rotating panels continually rotate in an $m$-occasion cycle. At each time, point data are reported by a different rotating panel and all the fixed panels.

Because of the computational complexity of the MVLU estimators, Wolter proposed to approximate them by composite estimators which are easier to compute since they are linear combinations of simple estimators and past composite estimators. He suggested estimators for the population "monthly" total, "month-to-month" change, and "year-to-year" change, where "month" and "year" mean regular time periods of the survey.

Let $y_{1,t}$ and $y_{2,t-1}$ be unbiased simple estimates of the population totals $Y_t$ and $Y_{t-1}$, obtained from the group reporting at time $t$. Wolter considered two composite estimators of the total: the preliminary composite estimator

$$\hat{y}_t = (1 - \beta)y_{1,t} + \beta(y_{t-1} + y_{1,t} - y_{2,t-1}),$$

and the final composite estimator

$$\tilde{y}_t = (1 - \alpha)y_{2,t-1} + \alpha \hat{y}_{t-1}.$$
where $0 < a < 1$, and $0 < \beta < 1$.

The composite estimators of month-to-month change and year-to-year change are, respectively, defined as

$$\tilde{\Delta}_t = \tilde{y}(p) - \tilde{y}(f)$$

and

$$\tilde{\tilde{\Delta}}_t = \tilde{\tilde{y}}(p) - \tilde{\tilde{y}}(f)$$

Expressions for the approximate variances of the estimators were derived under the assumptions:

1) simple estimators derived from different groups are uncorrelated;

2) $\text{var}(y_{1,t}) = \text{var}(y_{2,t-1}) = S^2$, and $\text{cov}(y_{1,t}, y_{2,t-1}) = \rho S^2$;

3) the simple estimators are covariance stationary in the sense that

$$\text{cov}(y_{1,t}, y_{1,t-r}) = \text{cov}(y_{2,t-1}, y_{2,t-r-1})$$

$$= \text{cov}(y_{1,t}, y_{2,t-r-1})$$

$$= \text{cov}(y_{2,t-1}, y_{1,t-r})$$

$$= \rho^r S^2$$,

where $r$ is an integer multiple of the number of rotating panels.
Numerical computation led to the optimum values of $a$ and $b$ for particular correlation structures. The investigation showed that the optimum coefficients of the estimator of the current population total may not be the optimum coefficients of the estimators of the "month-to-month" change or "year-to-year" change. This suggests that different coefficients should be used for each of these estimates whenever permitted by the survey conditions. The suggested estimators were shown to be computationally and statistically efficient.

Estimation problems for repeated surveys have received considerable attention under the classical point of view. Studies not mentioned here include Yates (1949), Gurney and Daly (1965), and others.

B. Time Series Approach

Some authors argue that it is more natural to consider the population parameters $\theta_t$ as random quantities rather than as fixed constants. Based on this argument, they suggest using a time series representation for $\theta_t$. Under the time series approach, there are at least two different methods of estimation.

Scott and Smith (1974) pointed out that under the time series approach, data from previous surveys can be employed in nonoverlapping surveys in order to get improved estimates of the current population mean $\theta_t$.

Following Patterson's line, Blight and Scott (1973) assumed that conditional on the population mean $\theta_t$, the observations $Y_{ti}$ follow
model (2.1). In addition, they assumed that \( \theta_t \) follows a first-order autoregressive model

\[ \theta_t - \mu = \lambda(\theta_{t-1} - \mu) + \epsilon_t, \quad t = 2, 3, \ldots, \]

where \( \{\epsilon_t\} \) is a sequence of uncorrelated random variables,

\[ E(\epsilon_t) = 0, \quad \text{all } t, \]

\[ \operatorname{var}(\epsilon_t) = \sigma^2, \quad \text{all } t, \]

and \( \{\epsilon_t\} \) and \( \{\eta_t\} \) are normally distributed and uncorrelated with each other.

Assuming that the parameters \( \mu, \rho, S^2, \lambda, \) and \( \sigma^2 \) are known, Blight and Scott derived recursive relationships for estimators of the current mean and for the variance of the estimated mean.

Let \( \bar{y}_t' \), \( \bar{x}_t' \), \( \bar{y}_t'' \), \( \bar{x}_t'' \), \( n_t' \), and \( n_t'' \) be as defined in (2.2). Let

\[ w_t' = \frac{(1 - \rho^2)S^2}{n_t'}, \quad w_t = \frac{S^2}{n_t''}, \quad n_t'' \neq 0, \quad n_t' \neq 0, \]

and let

\[ \Delta_t = \left( \frac{\rho^2 + \frac{1}{w_t'} + \frac{\lambda^2}{\sigma^2}}{w_t' + \frac{1}{v_{t-1} + \frac{\lambda^2}{\sigma^2}}} \right) \left( \frac{1}{w_t'} + \frac{1}{w_t''} + \frac{1}{\sigma^2} \right) - \left( \frac{\rho}{w_t'} + \frac{\lambda}{\sigma^2} \right)^2, \]
\[ v_t = \Delta_t^{-1} \left( \frac{\rho^2}{w_t^2} + \frac{1}{v_{t-1}} + \frac{\lambda^2}{\sigma^2} \right). \]

The minimum mean square error estimator of \( \theta_t \), derived from Bayes' formula, is

\[ \tilde{\theta}_t = \Delta_t^{-1} \left[ \left( \frac{\rho^2}{w_t^2} + \frac{1}{v_{t-1}} + \frac{\lambda^2}{\sigma^2} \right) \frac{y_t'}{w_t^2} + \left( \frac{\rho}{w_t^2} + \frac{\lambda}{\sigma^2} \right) \frac{\tilde{\theta}_{t-1}}{v_{t-1}} \right. \]
\[ + \left. \left( \frac{1}{v_{t-1}} + \frac{\lambda(\lambda - \rho)}{\sigma^2} \right) \left( \frac{y_t' - \rho x_t'}{w_t} \right) \vphantom{\frac{1}{v_{t-1}}} \right] \]

and

\[ \text{var}(\tilde{\theta}_t) = v_t. \]

As \( \sigma^2 \) approaches infinity, the estimators derived by Blight and Scott reduce to Patterson's estimators.

Scott and Smith (1974), and Scott, Smith, and Jones (1977) suggested an alternative time series approach. Let \( y_t \) be an unbiased sample survey estimator of \( \theta_t \) based on the sample at time \( t \) alone. One can write

\[ y_t = \theta_t + e_t, \]
where $e_t$ is the sampling error,

$$E(e_t) = 0,$$

$$\text{var}(e_t) = s^2_y$$

Also assume that $\theta_t$ and $e_t$ are stationary processes uncorrelated with each other, and that $y_t$ admits an infinite autoregressive representation, i.e.,

$$y_t = \sum_{j=1}^{\infty} a_j y_{t-j} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is a sequence of uncorrelated random variables, and

$$E(\varepsilon_t) = 0, \text{ for all } t,$$

$$\text{var}(\varepsilon_t) = \sigma^2, \text{ for all } t.$$

Scott and Smith suggest fitting a linear time series model to the available data $y_t, y_{t-1}, \ldots$, etc. and then estimating $\theta_t$ from the fitted model. The estimator of $\theta_t$ is given by

$$\tilde{\theta}_t = y_t - \tilde{e}_t,$$
where $\tilde{e}_t$ is the best linear predictor of $e_t$ based on $y_t, y_{t-1}, \ldots, y_1$.

$$\tilde{e}_t = \sum_{j=0}^{\infty} \phi_j y_{t-j},$$

$$= \sum_{j=0}^{\infty} \phi_j \varepsilon_{t-j},$$

where

$$\phi_j = \sum_{k=0}^{\infty} \alpha_j \gamma_e(j+k)/\sigma^2,$$

are the coefficients which minimize $E(\tilde{e}_t - e_t)^2$, and $\gamma_e(h)$ is the autocovariance function of the process $\{e_t\}$. The mean square error of $\tilde{e}_t$ is

$$E(\tilde{e}_t - e_t)^2 = \sum_{j=0}^{\infty} \phi_j^2.$$

Jones (1980) showed that the results of Patterson (1950), Blight and Scott (1973), and Scott and Smith (1974) can be unified using the least squares theory. Let $y_i(t)$ be the $i$-th elementary unbiased estimator of $\theta_t$, $i = 1, 2, \ldots, \ell$, $t = 1, 2, \ldots, h$, say. One can write

$$y_i(t) = \theta_t + e_t, \quad t = 1, 2, \ldots, h, \quad i = 1, 2, \ldots, \ell.$$
or, in matrix form

\[ \mathbf{y}_h = \mathbf{X} \mathbf{\theta}_h + \mathbf{e}_h, \]  

(2.4)

where \( \mathbf{y}_h = [y_\lambda(h), y_{\lambda-1}(h), \ldots, y_1(1)]' \) is the \( \lambda \)-dimensional vector of elementary estimates, 

\( \mathbf{\theta}_h = (\theta_\lambda, \ldots, \theta_1)' \) is the \( \lambda \)-dimensional vector of unknown parameters, 

\( \mathbf{X} \) is the matrix of zeroes and ones, which selects the appropriate elements of \( \mathbf{\theta}_h \), and 

\( \mathbf{e}_h = [e_\lambda(h), \ldots, e_1(1)] \) is the \( \lambda \)-dimensional vector of sampling errors.

It is assumed that

\[ \mathbf{E}(\mathbf{e}_h) = \mathbf{0}, \]

\[ \text{var}(\mathbf{e}_h) = \mathbf{K}_e, \]

where \( \mathbf{K}_e \) is known, and nonsingular. Under the classical approach, \( \mathbf{\theta}_h \) is fixed and the generalized least squares gives the best linear unbiased estimator (BLUE) of \( \mathbf{\theta}_h \) as

\[ \mathbf{\hat{\theta}}_h = (\mathbf{X}'\mathbf{K}_e^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{K}_e\mathbf{y}_h. \]
This is a more general result than the ones obtained by Patterson (1950) and Eckler (1955).

Under the time series approach, one has to add to (2.4) the assumption that $\theta_h$ is a random vector with $E(\theta_h) = \mu$, $\text{var}(\theta_h) = V$. The generalized least squares formulation gives the best linear unbiased estimator of $\theta_h$ as

$$
\tilde{\theta}_h = (X'K^{-1}_e X'K^{-1}_e V^{-1})^{-1} X'K^{-1}_e y_h
$$

with

$$
\text{var}(\tilde{\theta}_h) = (X'K^{-1}_e X'K^{-1}_e V^{-1})^{-1}.
$$

If the elements of $V$ are of the form

$$
v_{ij} = \frac{\lambda |i-j|}{1 - \lambda^2} \sigma^2,
$$

for some $\lambda$ and $\sigma^2$, $|\lambda| < 1$, $i, j = 1, \ldots, h$, (2.5) coincides with the estimates derived by Blight and Scott (1973).

Now, suppose $\lambda = 1$. Then, (2.4) becomes
\[ \xi_h = \beta_h + \epsilon_h \]

and (2.5) reduces to

\[ \hat{\xi}_h = (K_e^{-1} + \nu^{-1})^{-1} K_e^{-1} \xi_h , \]

with

\[ \text{var}(\hat{\xi}_h) = (K_e + \nu^{-1})^{-1} , \]

which are the results obtained by Scott and Smith (1974).
III. PREVIOUS WORK IN AUTOREGRESSIVE TIME SERIES

S objeto to Measurement Error

The methods presented in Section II.B approach estimation in rotation sampling designs as a signal detection problem. In such a formulation, the sequence of population means is the underlying time series that is observed with error. Several authors considered the signal measurement problem when the sequence of observational errors is white noise. For instance, Kendall (1944) studied the particular case of a time series generated by a second-order autoregressive process. Walker (1960) and Pagano (1974) proposed estimators for the parameters of a general p-th order autoregressive stationary time series. In this chapter, we review the usual signal detection model and some of the estimation methods that have been suggested.

A. Estimation of the Current Value When the Parameters are Known

Let \( \{X_t\} \) be a stationary autoregressive time series satisfying

\[
X_t + \alpha X_{t-1} = e_t, \quad t = 1, 2, \ldots
\]

(3.1)

where \( \{e_t\} \) is a sequence of uncorrelated random variables with mean zero and variance \( \sigma_e^2 \). Because of measurement error we do not observe \( X_t \) directly. Instead, we observe
\[ Y_t = X_t + u_t , \quad t = 1, 2, \ldots , \] (3.2)

where \( \{u_t\} \) is a sequence of uncorrelated random variables with zero mean and variance \( \sigma_{uu} \), and \( u_t \) is independent of \( e_j \) for all \( t \) and \( j \). A signal detection problem consists in building an estimator of \( X_t \) given a sample of a realization on \( Y_t \).

We consider a recursive estimation procedure proposed by Kalman (1960), the so-called Kalman filtering. This method was first developed using least squares theory. A development of Kalman filter from a Bayesian viewpoint is given by Meinhold and Singpurwalla (1983). We follow the original derivation.

To start the recursive procedure, we assume that

\[ \hat{X}_0 = X_0 + v_0 , \] (3.3)

where \( \hat{X}_0 \) is the initial estimate of \( X_0 \), and \( v_0 \) is a random variable with mean zero and variance \( \sigma_{v0} \), independent of \( (e_t, u_t) \) for all \( t \). Assume also that \( \sigma_{uu}, \sigma_{ee}, \sigma_{v0}, \alpha \) and \( \hat{X}_0 \) are known.

At time \( t = 1 \) we wish to estimate \( X_1 \) based on the observation \( Y_1 \) and the knowledge of \( \sigma_{uu}, \sigma_{ee}, \sigma_{v0}, \alpha \) and \( \hat{X}_0 \). From (3.1), (3.2) and (3.3), we have

\[ Y_1 = X_1 + u_1 , \]
\[ \alpha \hat{X}_0 = -X_1 + w_1 , \]
where \( w_1 = e_1 + \alpha v_0 \). Since \( u_1 \) and \( w_1 \) are uncorrelated, the best linear unbiased estimator of \( X_1 \), given \( Y_1 \), is

\[
\hat{X}_1 = (\sigma_{uu}^{-1} + \sigma_{ww11}^{-1})^{-1} (\sigma_{uu}^{-1} Y_1 - \sigma_{ww11}^{-1} \alpha \hat{X}_0),
\]

where \( \sigma_{ww11} = \sigma_{ee} + \alpha^2 \sigma_{vvo0} \). Let

\[
v_1 = \hat{X}_1 - X_1
\]

\[
= (\sigma_{uu}^{-1} + \sigma_{ww11}^{-1})^{-1} (\sigma_{uu}^{-1} u_t - \sigma_{ww11}^{-1} w_1).
\]

The variance of \( v_1 \) is

\[
\sigma_{vv11} = (\sigma_{uu}^{-1} + \sigma_{ww11}^{-1})^{-1}.
\]

At time \( t = 2 \) we use \( Y_2, Y_1 \) and \( \hat{X}_0 \) to estimate \( X_2 \). Now, the best linear unbiased estimator of \( X_2 \) based on \((Y_1, X_0)\) is

\[
\alpha \hat{X}_1 = \alpha X_1 - \alpha v_1. \tag{3.4}
\]

Therefore, the best linear unbiased estimator of \( X_2 \) is obtained by combining the information in \( \alpha \hat{X}_1 \) with that in \( Y_2 \). From (3.1) and (3.4),
\[ Y_2 = X_2 + u_2 \]

\[ \alpha \hat{X}_1 = -X_2 + \omega_2 , \quad (3.5) \]

where \( \omega_2 = e_2 + \alpha \nu_1 \).

Because \( \omega_2 \) is uncorrelated with \( u_2 \), the best linear unbiased estimator of \( \theta \) is

\[ \hat{X}_2 = (\sigma_{uu}^{-1} + \sigma_{ww22}^{-1})^{-1} (\sigma_{uu}^{-1} Y_2 - \sigma_{ww22}^{-1} \alpha \hat{X}_1) , \]

where \( \sigma_{ww22} = \sigma_{ee} + \alpha^2 \sigma_{vv11} \).

For a general \( t \),

\[ \hat{X}_t = (\sigma_{uu}^{-1} + \sigma_{wwtt}^{-1})^{-1} (\sigma_{uu}^{-1} Y_t - \sigma_{wwtt}^{-1} \alpha \hat{X}_{t-1}) , \]

where

\[ \sigma_{wwtt} = \sigma_{ee} + \alpha^2 \sigma_{vv,t-1,t-1} \]

\[ \sigma_{vvtt} = (\sigma_{uu}^{-1} + \sigma_{wwtt}^{-1})^{-1} . \]

The results for the model introduced by (3.1), (3.2) and (3.3) can be extended to the case where \( X_t \) is a \( p \)-th order autoregressive time series. Let
\[ Y_t = X_t + u_t, \quad t = 1, 2, \ldots, \]

\[ X_t + a_1 X_{t-1} + \ldots + a_p X_{t-p} = e_t, \quad t = 1, 2, \ldots, \quad (3.6) \]

\[ \hat{X}_0 = X_0 + e_0, \]

where \((u_t, e_t)\) is a sequence of uncorrelated vector random variables with mean zero and covariance matrix \(\Sigma = \text{diag}(\sigma_{uu}, \sigma_{ee})\),

\[ X_0 = (X_0, X_{-1}, \ldots, X_{-p+1})', \quad \hat{X}_0 \text{ is an initial estimate of } X_0 \]

and \(e_0\) is a random vector with mean zero and covariance matrix \(V_o\), independent of \((u_t, e_t)\).

Define the \(p\)-dimensional vectors

\[ \hat{X}_t = (X_t, X_{t-1}, \ldots, X_{t-p+1})', \]

\[ e_t = (e_t, 0, \ldots, 0)', \]

\[ F_o = (1, 0, \ldots, 0)', \]

and the \(p \times p\) matrix

\[
G = \begin{pmatrix}
-a_1 & \ldots & -a_p \\
\mathbf{I}_{p-1} & 0
\end{pmatrix},
\]
where denote the \((p - 1) \times (p - 1)\) identity matrix, and \(\mathbf{Q}\) is the \((p - 1) \times 1\) vector of zeroes. Then, (3.6) can be expressed as

\[
Y_t = \mathbf{F}' \mathbf{X}_t + u_t
\]

\[
\mathbf{X}_t = \mathbf{G} \mathbf{X}_{t-1} + e_t
\]

\[
\hat{X}_0 = X_0 + \nu_0
\]

The system of equations analogous to (3.5) is

\[
Y_t = \mathbf{F}' \mathbf{X}_t + u_t
\]

\[
\mathbf{G} \hat{X}_{t-1} = X_t + \omega_t
\]

where \(\omega_t = e_t + \mathbf{G} \nu_{t-1}\) and \(\nu_t = \hat{X}_t - X_t\). Hence, the best linear unbiased estimator of \(\mathbf{X}_t\) is the first component of

\[
\hat{X}_t = \left[\mathbf{G} \mathbf{G}' + \mathbf{V}_{tt}^{-1}\right]^{-1} \left[\mathbf{G} \mathbf{G}' \mathbf{Y}_t + \mathbf{V}_{tt}^{-1} \mathbf{G} \hat{X}_{t-1}\right],
\]

where

\[
\mathbf{V}_{tt} = \sigma_{ee} \mathbf{F}' \mathbf{F}' + \mathbf{G} \mathbf{V}_{t-1} \mathbf{G}'
\]

\[
\hat{X}_0 = X_0 + \nu_0
\]
B. Estimation for Autoregressive Signal

In the last section, we presented the signal detection problem when the parameters are known. We now study estimation methods for the parameters of the autoregressive signal.

Let \( \{X_t\} \) be a stationary \( p \)-th order autoregressive time series which cannot be observed directly. Assuming the observational error is additive, the observed value at time \( t \) is

\[
Y_t = X_t + u_t ,
\]

(3.7)

where

\[
X_t + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} = e_t ,
\]

(3.8)

\( \{u_t\} \) is the measurement error time series, and \((e_t, u_t)'\) is a sequence of independent normal \((0, \Sigma)\) random variables with

\[
\Sigma = \text{diag} (\sigma_{ee}, \sigma_{uu}) .
\]

The problem is to construct estimators of \( \alpha_1, \ldots, \alpha_p, \sigma_{ee} \) and \( \sigma_{uu} \) based on a sample of \( n \) observations from a realization on \( Y_t \).

We first consider the estimation procedure suggested by Walker (1960). Define

\[
Y_Y(h) = E\{Y_t Y_{t+h}\} , \quad h = 0, 1, \ldots
\]
Since \( \{Y_t\} \) is a normal process, its behavior is completely determined by the autocovariance function \( \gamma_Y(h) \). From (3.7),

\[
\gamma_Y(0) = \gamma_X(0) + \sigma_{uu}
\]

\[
\gamma_Y(h) = \gamma_X(h), \quad h = 1, 2, \ldots,
\]

so that the effect of the additional error term \( u_t \) is to reduce all the autocorrelations of the time series by the same proportion \( [1 + \sigma_{uu} \gamma_X(0)]^{-1} \). Thus,

\[
\frac{\gamma_Y(h)}{\gamma_Y(h')} = \frac{\rho_X(h, \alpha_1, \ldots, \alpha_p)}{\rho_X(h', \alpha_1, \ldots, \alpha_p)}, \quad h = 0, 1, \ldots,
\]

where \( \rho_X(h, \alpha_1, \ldots, \alpha_p) \) denotes the \( h \)-th autocorrelation of the process \( \{X_t\} \) expressed as a function of \( \alpha_1, \alpha_2, \ldots, \alpha_p \).

We create estimators of the unknown parameters by equating the first \( p + 2 \) sample autocovariances \( \hat{\gamma}_Y(h) \), \( h = 0, 1, \ldots, p + 1 \), to their expected values \( \gamma_Y(h) \). This gives the system of equations:

\[
\hat{\gamma}_Y(h) = \frac{\hat{\rho}_X(h, \hat{\alpha}_1, \ldots, \hat{\alpha}_p)}{\hat{\rho}_X(1, \hat{\alpha}_1, \ldots, \hat{\alpha}_p)}, \quad h = 2, 3, \ldots, p + 1
\]

\[
\hat{\gamma}_Y(1) = \hat{\sigma}_{uu} \hat{\rho}_X(1, \hat{\alpha}_1, \ldots, \hat{\alpha}_p)
\]

\[
\hat{\gamma}_Y(0) = \hat{\sigma}_{uu} + \hat{\sigma}_{ee}. \quad (3.9)
\]
Solving the first $p$ equations of (3.9) gives the estimator of 
$(a_1, \ldots, a_p)$. Substituting $(\hat{a}_1, \ldots, \hat{a}_p)$ in the last two equations 
gives $\hat{\sigma}_{ee}$ and $\hat{\sigma}_{uu}$.

Although the solution to the equations in (3.9) appears to be 
complicated, this system can be expressed in a simple form. Since 

$$
\rho(h) + a_1 \rho(h-1) + \ldots + a_p \rho(h-p) = 0, 
$$

$$
h = 2, 3, \ldots, p + 1,
$$

the first $p$ equation of (3.9) can be expressed as 

$$
\hat{\gamma}_X(p+1) + a_1 \hat{\gamma}_X(p) + \ldots + a_p \hat{\gamma}_X(1) = 0
$$

$$
\hat{\gamma}_X(h) + a_1 \hat{\gamma}_X(h-1) + \ldots + a_1 \frac{\hat{\gamma}_X(1)}{\hat{\rho}_X(1)} + \ldots + a_p \hat{\gamma}_X(h-p) = 0 , 
$$

$$
h = 2, 3, \ldots, p ,
$$

which determine $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p$ as ratios of two polynomials in 
$[\rho_X(1)]^{-1}$. Substitution of these polynomials in $\rho_X(1, \hat{a}_1, \ldots, \hat{a}_1)$ 
will then give an equation for $\hat{\rho}_u(1)$. Estimators of $\sigma_{ee}$ and $\sigma_{uu}$ 
are obtained as before from the last two equations in (3.9).

A further simplification can be achieved by using autocovariances 
for lags greater than $p + 1$. One can substitute $\hat{\gamma}_X(h)$ in any set 
of $p$ of the Yule-Walker equations, and solve the system
\[ \hat{\gamma}_X(h) + \hat{\alpha}_1 \gamma_X(h - 1) + \ldots + \hat{\alpha}_p \gamma_X(h - p) = 0, \]

\[ h = r, r + 1, \ldots, r + p, \]

for some \( r > p \). As \( E(\hat{\gamma}_X(h)) \to 0 \) as \( h \to \infty \), one would, in general, expect to obtain more accurate estimates from \( h = p + 1, \ldots, 2p \) than from any other set of \( p \) values of \( h \).

The estimates obtained from (3.9) are consistent and have asymptotic standard errors of \( O(n^{-1/2}) \); however, they are, in general, asymptotically inefficient. We define an estimator to be efficient if its covariance matrix is the same as that for the maximum likelihood estimator. An examination of the case \( p = 1 \) shows that, for a fixed \( \alpha_1 \neq 0 \), the efficiency of \( \hat{\alpha}_1 \) approaches one as the ratio \( \sigma_{uu}^{-1}(\sigma_{ee}) \) approaches zero.

Pagano (1974), using least squares theory, proposed an asymptotically efficient estimation procedure for the parameters of model (3.7). To introduce Pagano's procedure, define

\[ S_t = Y_t + \alpha_1 Y_{t-1} + \ldots + \alpha_p Y_{t-p} \]

\[ = e_t + u_t + \alpha_1 u_{t-1} + \ldots + \alpha_p u_{t-p}. \]  

(3.10)

The last equality follows directly from (3.8) and shows that \( \{S_t\} \) is a stationary time series. In fact, we can see that \( \gamma_S(h) = 0 \) for
Therefore, \( \{S_t\} \) is a \( p \)-th order normal moving average time series and hence, there exists a sequence of normal random variables \( \{z_t\} \) such that

\[
S_t = z_t + \beta_1 z_{t-1} + \cdots + \beta_p z_{t-p},
\]

where the roots of

\[
m^p + \beta_1 m^{p-1} + \cdots + \beta_p = 0,
\]

and of

\[
m^p + \alpha_1 m^{p-1} + \cdots + \alpha_p = 0
\]

are less than one in absolute value. To show that the polynomials (3.12) and (3.13) have no roots in common, we need the following definition. The autocovariance generating function of a stationary process \( \{Z_t\} \) is the complex function

\[
G_Z(m) = \sum_{h=-\infty}^{\infty} \gamma_Z(h) m^h,
\]

where \( \gamma_Z(h) \) is the autocovariance function of \( \{Z_t\} \).

Now, define the complex polynomials

\[
R_1(m) = \sum_{j=0}^{p} \alpha_j m^{p-j},
\]
and

\[ R_2(m) = \sum_{j=0}^{p} \beta_j m^{p-j}, \]

with \( \alpha_0 = \beta_0 = 1 \). From (3.10) and (3.11),

\[ G_S(m) = \sigma_{ee} + \sigma_{uu} R_1(m) R_1(m^{-1}) \]

\[ = \sigma_{zz} R_2(m) R_2(m^{-1}). \]

Therefore, if \( m_0 \) is a zero of \( R_1(m) \), then neither \( m_0 \) or \( m_0^{-1} \) is a zero of \( R_2(m) \). Thus, the polynomials (3.12) and (3.13) have no common roots.

Based on these results, the observable process \( \{Y_t\} \) can be represented as the following autoregressive moving average process

\[ Y_t + \alpha_1 Y_{t-1} + \ldots + \alpha_p Y_{t-p} = z_t + \beta_1 z_{t-1} + \ldots + \beta_p z_{t-p}. \]  

(3.14)

Model (3.14) has \( 2p + 1 \) unknown parameters, namely \( \alpha = (\alpha_1, \ldots, \alpha_p)' \), \( \beta = (\beta_1, \beta_2, \ldots, \beta_p)' \) and \( \sigma_{zz} \) or, equivalently, \( \alpha \) and \( \gamma_Y = [\gamma_Y(0), \ldots, \gamma_Y(p)]' \). By introducing \( p - 1 \) new parameters, model (3.7) is transformed into an autoregressive moving average model, for which the solution to the estimation problem is known.
To introduce Pagano's estimation method, let $\gamma_S = [\gamma_S(0), \ldots, 
\gamma_S(p)]'$, where $\gamma_S(h) = E[S_t S_{t+h}]$, and let $(\hat{\alpha}', \hat{\gamma}_S')'$ be a vector of efficient estimators $(\hat{\alpha}', \hat{\gamma}_S')$. Let $\hat{\alpha}$ be an estimator of $\alpha$, the information matrix of $(\hat{\alpha}', \hat{\gamma}_S')'$. Furthermore, assume that

$$\hat{\alpha} \rightarrow \alpha, \text{ a.s.}, \tag{3.15}$$

$$\hat{\gamma}_S \rightarrow \gamma_S, \text{ a.s.}, \tag{3.16}$$

and

$$\hat{H}_n \rightarrow H, \text{ a.s.}, \tag{3.17}$$

as $n \rightarrow \infty$. Using the sample spectral density, Parzen (1971) has developed estimators for the autoregressive moving average time series, which satisfy the conditions (3.15), (3.16), and (3.17).

The fact that $(\hat{\alpha}', \hat{\gamma}_S')'$ is not efficient for the parameters of the original model (3.7) becomes clear when one considers that $\gamma_S$ is a function of $\alpha$, and the information matrix $H$ is not block diagonal. In fact, from (3.10),

$$\gamma_S(h) = \sigma_{ee} \delta_{0h} + \sigma_{uu} \sum_{j=0}^{p} \alpha_j \alpha_{j+h}, \quad (h = 0, 1, \ldots, p), \tag{3.18}$$
where \( \delta_{0h} \) is the Kronecker delta. Denote the relationship in (3.18) by

\[
\gamma_S = \gamma_S(g', \sigma_{ee}, \sigma_{uu})',
\]

where \( \gamma_S = [\gamma_S(0), \ldots, \gamma_S(p)]' \).

The information matrix, \( \phi \), of \( (g', \sigma_{ee}, \sigma_{uu})' \) is obtained using the chain rule for vector derivatives. Let \( A \) be the matrix of derivatives of \( (g', \gamma_S') \) with respect to \( (g', \sigma_{ee}, \sigma_{uu})' \). The matrix \( A \) is a function of \( g \) and \( \sigma_{uu} \), i.e., \( A = A(g', \sigma_{uu}) \), and

\[
\phi = A' HA.
\]  

(3.19)

Pagano found an estimator of \( (g', \sigma_{ee}, \sigma_{uu})' \) with large sample covariance matrix \( (n \phi)^{-1} \).

Replacing \( \gamma_S(h), h = 0, 1, \ldots, p \), in (3.18) by their estimated values from the enlarged model (3.14) gives the equation

\[
\begin{pmatrix}
\dot{g}' \\
\dot{\gamma}_S
\end{pmatrix}
= \begin{pmatrix}
g' \\
\gamma_S(g', \sigma_{ee}, \sigma_{uu})'
\end{pmatrix} + \frac{s_n}{n},
\]

where the estimation error \( s_n \) is such that, as \( n \to \infty \),

\[
s_n \to 0 \quad \text{a.s.},
\]
and

\[ n^{1/2} \frac{L}{\sigma_n} \xrightarrow{\text{L}} N(Q, H^{-1}). \]

The generalized squares estimator of \( \hat{\theta} = (a', \sigma_{ee}, \sigma_{uu})' \) is obtained by minimizing the residual quadratic form

\[ Q_n(\theta) = ((\hat{a} - a)', [\hat{\gamma}_S - \gamma_S(\theta)]')H_n^{-1}((\hat{a} - a)', [\hat{\gamma}_S - \gamma_S(\theta)]'). \]

The generalized least squares estimator will be denoted by \( \hat{\theta} = (\hat{a}', \hat{\sigma}_{ee}, \hat{\sigma}_{uu})' \). Using a theory similar to that developed by Jennrich (1969), Pagano established that

\[ (\hat{a}', \hat{\sigma}_{ee}, \hat{\sigma}_{uu})' \rightarrow (a', \sigma_{ee}, \sigma_{uu})' \text{ a.s.}, \]

\[ n^{1/2} \begin{pmatrix} a' \\ \sigma_{ee} \\ \sigma_{uu} \end{pmatrix} - \begin{pmatrix} a \\ \sigma_{ee} \\ \sigma_{uu} \end{pmatrix} \xrightarrow{\text{L}} N(Q, \phi^{-1}), \]

and

\[ \tilde{A} H_n \tilde{A} \xrightarrow{\text{a.s.}} \phi, \]

where \( \phi \) is defined in (3.19) and \( \tilde{A} = \tilde{A}(\tilde{a}', \tilde{\sigma}_{ee}, \tilde{\sigma}_{uu})' \).
Other studies in autoregressive time series observed with error include estimation methods based on the asymptotic distribution of the periodogram ordinates. Dunsmuir (1979) derived a central limit theorem for the approximate maximum likelihood estimators obtained by maximizing a frequency domain approximation to the Gaussian likelihood. Because we will restrict our attention to estimation in the time domain, the frequency domain procedures will not be reviewed.
IV. AUTOREGRESSIVE SIGNAL PLUS MOVING AVERAGE NOISE

The estimation methods which have been presented are based upon the fact that the measurement errors form an independent stationary time series. However, in the case of repeated surveys, where the same units appear in the sample on more than one occasion, it is reasonable to assume that the errors form a sequence of correlated random variables. We consider a signal detection problem in which the signal is a p-th order autoregressive time series, and the noise is a q-th order moving average. This model applies to the situations in which the sampling units appear in the survey for a fixed finite number of occasions.

A. Model and Estimators

Consider the following model

\[ Y_t = X_t + u_t, \]
\[ X_t + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} = e_t, \]
\[ v_t + b_1 v_{t-1} + \ldots + b_q v_{t-q} = u_t, \]

where
\{Y_t\} is the observed time series,
\{X_t\} is the unobservable time series,
\{u_t\} is the time series of measurement errors,
\((e_t, v_t)'\) is a sequence of independent normal \((Q, \Sigma)\) random variables with \(\Sigma = \text{diag}(\sigma_{ee}, \sigma_{vv})\). It is assumed that the roots of
\[ m^p + a_1 m^{p-1} + \ldots + a_p = 0 \quad (4.2) \]
and of
\[ m^q + b_1 m^{q-1} + \ldots + b_q = 0 \]
are less than one in absolute value. We develop estimators of \(a = (a_1, \ldots, a_p)'\) and \(\sigma_{ee}\) under the assumption that \(b_1, b_2, \ldots, b_q\) and \(\sigma_{vv}\) are known. This is a reasonable approximation to reality for the repeated survey problem. This is because \(b_1, b_2, \ldots, b_q\) and \(\sigma_{vv}\) are functions of the sampling variance that can be estimated by standard sample survey methods from the individual observations. In a later section, we extend the model to recognize the estimation error of the within sample estimates.

Following Pagano (1974), we first enlarge the model (4.1) by finding the autoregressive moving average representation for the process \(\{Y_t\}\). Estimators of the parameters for the original model will be obtained by fitting an autoregressive moving average model with
constrained parameters to the available observations \( Y_1, Y_2, \ldots, Y_n \).

Define the time series \( \{ S_t \} \) by

\[
S_t = Y_t + \alpha_1 Y_{t-1} + \ldots + \alpha_p Y_{t-p} \tag{4.3}
\]

\[
= e_t + u_t + \alpha_1 u_{t-1} + \ldots + \alpha_p u_{t-p}.
\]

It has been shown, in Section III.B, that

\[
S_t = z_t + \beta_1 z_{t-1} + \ldots + \beta_{p+q} z_{t-p-q},
\]

where \( \{ z_t \} \) is a sequence of normal random variables, and

\( \beta_1, \beta_2, \ldots, \beta_{p+q} \) are such that the roots of

\[
m^{p+q} + \beta_1 m^{p+q-1} + \ldots + \beta_{p+q} = 0 \tag{4.4}
\]

are less than unity in modulus. In Section III.B, it has also been shown that (4.2) and (4.4) have no roots in common. Thus, the observed values \( (Y_1, Y_2, \ldots, Y_n) \) are a sample from the autoregressive moving average process

\[
Y_t + \alpha_1 Y_{t-1} + \ldots + \alpha_p Y_{t-p} = z_t + \beta_1 z_{t-1} + \ldots + \beta_{p+q} z_{t-p-q}, \tag{4.5}
\]

where \( \{ z_t \} \) is a sequence of independent normal \( (0, \sigma_{zz}) \) random variables. We call (4.5) the unrestricted model. This model has
2p + q + 1 parameters, whereas the original model has only p + 1 unknown parameters. The new parameters $\beta_1, \ldots, \beta_{p+q}$ and $\sigma_{zz}$ are functions of $\alpha_1, \ldots, \alpha_p$ and $\sigma_{ee}$. Let $\alpha_0 = \beta_0 = 1$. From (4.3),

$$\gamma_S(h) = \sum_{i=0}^{p} \sum_{j=0}^{p} \alpha_i \alpha_j \gamma(h + i - j)$$

$$= \sigma_{ee} \delta_{0h} + \sum_{i=0}^{p+q} \sum_{j=0}^{p} \beta_i \beta_j \gamma(h + i - j),$$

$$h = 0, 1, \ldots, p + q,$$

where $\delta_{0h}$ is the Kronecker delta and $\gamma_Z(h) = E\{Z_t Z_{t+h}\}$, $h = 0, \pm 1, \ldots$, for any stationary process $\{Z_t\}$. Equivalently,

$$\sigma_{ee} + \sum_{j=0}^{p} \sum_{k=0}^{p} \alpha_j \alpha_k \gamma(k - j) - \sigma_{zz} \sum_{j=0}^{p+q} \beta_j^2 = 0$$

$$+ \sum_{j=0}^{p+q-h} \sum_{k=0}^{p} \alpha_j \alpha_k \gamma(k - j + h) - \sigma_{zz} \sum_{j=0}^{p+q-h} \beta_j \beta_{j+h} = 0,$$

$$h = 1, \ldots, p + q .$$

Model (4.5) with parameters satisfying (4.6) will be called the restricted model. Under the assumption that $(b_1, \ldots, b_q)$ and $\sigma_{vv}$ are known, the $\gamma_v(h)$ are known for all $h$.

The problem is to construct estimators of $\varrho = (\alpha_1, \ldots, \alpha_p)'$ and $\sigma_{ee}$ assuming that $\gamma_v(h), h = 0, \ldots, q$ are known.
1. Preliminary estimators

We now define a procedure to build preliminary estimates of $\mathbf{a} = (a_1, \ldots, a_p)'$ and $\sigma_{ee}$. The procedure is similar to that suggested by Pagano. It differs in that least squares estimates are constructed at the first step and the set of statistics used at the second step differs from the set used by Pagano. The preliminary estimator of this section will be used as input to the estimation procedure of the next section. The steps of the procedure are:

(a) Obtain an asymptotically efficient estimator of

$$\hat{\mathbf{a}} = (\hat{a}', \hat{\beta}', \hat{\sigma}_{zz}')$$

for the unrestricted model. Denote this estimator by $\hat{\mathbf{a}}_n = (\hat{a}', \hat{\beta}', \hat{\sigma}_{zz}')$ and its estimated covariance matrix by $\hat{\Omega}_n$. See Fuller (1976) for a description of a Gauss-Newton estimation procedure.

(b) Find the generalized least squares estimator $(\bar{\mathbf{a}}', \bar{\sigma}_{ee})'$ for the model

$$\bar{\mathbf{a}}_n = \hat{\mathbf{a}} + \tilde{\mathbf{a}}$$

using $\tilde{\mathbf{a}}_n$ as the estimated covariance matrix of $\mathbf{a}$, where $f(\hat{\mathbf{a}}', \sigma_{ee})$ denotes the left-hand side of the set of restrictions in (4.6).
We suggest the Gauss-Newton procedure to obtain the least squares estimator of \((\hat{\delta}', \hat{\sigma}_{ee})'\) for the model

\[
\begin{pmatrix}
T' & 0 \\
0 & cI_{p+q+1}
\end{pmatrix}
\begin{pmatrix}
\hat{\delta}'_n \\
\hat{\varepsilon}'
\end{pmatrix}
= \begin{pmatrix}
T' & 0 \\
0 & cI_{p+q+1}
\end{pmatrix}
\begin{pmatrix}
\hat{\delta}' \\
\tilde{f}(\hat{\delta}', \hat{\sigma}_{ee})
\end{pmatrix}
+ \begin{pmatrix}
\tilde{a}_1 \\
\tilde{a}_2
\end{pmatrix}
\tag{4.7}
\]

where \(T\) is a matrix such that the \(T'T = G^{-1}\), \(I_d\) is the \(d \times d\) identity matrix, \(c\) is an arbitrary large number, and \((\tilde{a}_1', \tilde{a}_2')'\) is a random vector with zero mean and identity estimated covariance matrix.

In this formulation, the restrictions are included in the model as if they were actual observations. The error in the degree to which the set of restrictions are satisfied can be made very small by choosing the constant \(c\) sufficiently large.

Estimators for model (4.7) can be found using any standard nonlinear regression program. As initial estimators for the iterative process, one can use \(\tilde{\alpha} = \tilde{a}\) and

\[
\hat{\sigma}_{ee} = \tilde{\sigma}_{zz} \sum_{j=0}^{p+q} \tilde{\beta}_j^2 + \sum_{j=0}^p \sum_{k=0}^p \tilde{a}_j \tilde{a}_k \gamma_u (k - j).
\]

The estimator \((\hat{\delta}', \hat{\sigma}_{ee})'\) is consonant with the restricted model and will be used as the starting values of the nonlinear estimation procedure.
To illustrate this method consider a first-order autoregressive signal plus a first-order moving average noise. Let

\[ Y_t = X_t + u_t, \]

\[ X_t + \alpha_1 X_{t-1} = e_t, \]

\[ u_t = v_t + b_1 v_{t-1}, \]

where \( |\alpha_1| < 1, (e_t, v_t)' \) are vectors of \( \text{NI}(0, \Sigma) \) random variables, and \( \Sigma = \text{diag}(\sigma_{ee}, \sigma_{vv}) \). Assume \( b_1 \) and \( \sigma_{vv} \) are known. The observable process \( \{Y_t\} \) can be expressed as

\[ Y_t + \alpha_1 Y_{t-1} = z_t + \beta_1 z_{t-1} + \beta_2 z_{t-2}, \]

where the \( z_t \) are \( \text{NI}(0, \sigma_{zz}) \), and the parameters satisfy

\[ \sigma_{ee} + (1 + \alpha_1^2) \gamma_u(0) + 2\alpha_1 \gamma_u(1) - (1 + \beta_1^2 + \beta_2^2)\sigma_{zz} = 0 \]

\[ \alpha_1 \gamma_u(0) + (1 + \alpha_1^2) \gamma_u(1) - \beta_1 (1 + \beta_2) \sigma_{zz} = 0 \]

\[ \alpha_1 \gamma_u(1) - \beta_2 \sigma_{zz} = 0. \]

Let \( \hat{\Phi}_n \) be the least squares estimator of \( \Phi = (\alpha_1, \beta_1, \beta_2, \sigma_{zz})' \) for the unrestricted model, and let \( \hat{\Sigma}_n \) be the estimated covariance
matrix of \( \delta \). Let \( T \) be a 4 x 4 matrix such that \( T'T = G^{-1} \). To simplify the notation, we will assume that \( T = [t_{ij}] \) is the upper triangular matrix given by the Cholesky decomposition of the matrix \( G^{-1} \). Let

\[
\delta_T' = (t_{11} \sigma_1 + t_{12} \beta_1 + t_{13} \beta_2 + t_{14} \sigma_{zz}, t_{22} \beta_1 + t_{23} \beta_2 + t_{24} \sigma_{zz}, t_{33} \beta_2 + t_{34} \sigma_{zz}, t_{44} \sigma_{zz}, 0, 0, 0).
\]

Estimators of the parameters of interest, \( \alpha_1 \) and \( \sigma_{ee} \), are obtained by fitting the nonlinear model

\[
\tilde{\delta}_{iT} = D_{i1} \alpha_1 + D_{i2} \beta_1 + D_{i3} \beta_2 + D_{i4} \sigma_{zz}
\]

\[
+ D_{i5} \left[ \sigma_{ee} + (1 + \alpha_1^2) \gamma_u(0) + 2 \alpha_1 \gamma_u(1) - (1 + \beta_1^2 + \beta_2^2) \sigma_{zz} \right]
\]

\[
+ D_{i6} \left[ \alpha_1 \gamma_u(0) + (1 + \alpha_1^2) \gamma_u(1) - \beta_1 (1 + \beta_2) \sigma_{zz} \right]
\]

\[
+ D_{i7} [\alpha_1 \gamma_u(1) - \beta_2 \sigma_{zz}] + a_i, \ i = 1, \ldots, 7,
\]

where \( \tilde{\delta}_{iT} \) is the \( i \)-th component of \( \delta_T \),

\[
D_{ij} = t_{ij} \text{ for } i = 1, \ldots, j, j = 1, \ldots, 4,
\]

\[
= c \text{ for } ij = 55, 66, 77,
\]
c is an arbitrary large number, and \( a_i \) is the error component. The \( a_i \) have estimated mean zero and estimated variance one. The matrix of derivatives for the nonlinear model, before transformation, is given in Table 4.1.

By allowing an arbitrarily small error in the set of restrictions, this method releases us of the need of explicitly expressing \( \hat{\theta} \) and \( \sigma_{zz} \) as functions of the parameters \( \varphi \) and \( \sigma_{ee} \). This procedure gives estimators of \( \hat{\varphi} \) and \( \sigma_{zz} \) and an estimated covariance matrix for the vector of estimators.

Together, steps (a) and (b) are, essentially, Pagano's estimation method applied to model (4.1). Pagano parametrizes the unrestricted model in terms of \( \varphi \) and the autocovariance of \( \{ S_t \} \) at lags 0, ..., \( p + q \), whereas we take \( \varphi, \hat{\theta}, \) and \( \sigma_{zz} \) as parameters. Our parametrization has the advantage that, as we estimate \((\varphi', \hat{\theta}')\) in step (a), we obtain the least squares estimate of the covariance matrix of \((\hat{\varphi}', \hat{\theta}')\) as a direct result of the computations.

Moreover, in step (a), Pagano computes frequency domain estimators for the unrestricted model; we compute least squares estimators instead. Nerlove and Pinto (1984) conducted a Monte Carlo experiment to compare maximum likelihood in time domain with the approximate maximum likelihood in the frequency domain estimation as methods for autoregressive moving average time series. Their study suggests that the time domain estimators have both bias and mean square error
Table 4.1. Observations and derivatives used in construction of nonlinear estimators of (b)

<table>
<thead>
<tr>
<th>Observations</th>
<th>$\alpha$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\sigma_{zz}$</th>
<th>$\sigma_{ee}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\alpha}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{\beta}_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{\beta}_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\tilde{\sigma}_{zz}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$2[\alpha_1 \gamma_u(0) + \gamma_u(1)]$</td>
<td>$-2\beta_1 \sigma_{zz}$</td>
<td>$-2\beta_2 \sigma_{zz}$</td>
<td>$-(1+\beta_1^2+\beta_2^2)$</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>$\gamma_u(0) + 2\alpha_1 \gamma_u(1)$</td>
<td>$-(1+\beta_2) \sigma_{zz}$</td>
<td>$-\beta_1 \sigma_{zz}$</td>
<td>$-\beta_1 (1+\beta_2)$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$\gamma_u(1)$</td>
<td>0</td>
<td>$-\sigma_{zz}$</td>
<td>$-\beta_2$</td>
<td>0</td>
</tr>
</tbody>
</table>
that are smaller than those of the frequency domain estimators. Because least squares and maximum likelihood estimators are closely related, we feel that, at step (a), the least squares estimation method is preferred to the frequency domain estimation procedure.

2. An alternative generalized least squares estimator

The procedure of this section was developed on the basis of two properties of the preliminary estimators. First, the vector of covariances between $\tilde{\sigma}_{zz}$ and $(\tilde{\sigma}^t, \tilde{\sigma}^t)$ of step (a), Section IV.A.1, in the covariance matrix of the limiting distribution is zero. The limiting distribution of $\tilde{\sigma}_{zz}$ is approximated by a quadratic function of the $z_t$. Therefore, the information in $\tilde{\sigma}_{zz}$ can be combined with the information in the linear part of $Y_t$ by considering the vector of observations $(Y_1, Y_2, ..., Y_n, \tilde{\sigma}_{zz})$. This idea has been used by Jobson and Fuller (1980). Second, the preliminary estimator of the previous section and that developed by Pagano use an estimated covariance matrix constructed from the initial unrestricted estimates to compute the restricted estimates. It seems reasonable that a more efficient estimator of the covariance matrix would improve the efficiency of the estimators. The approximate covariance matrix of the error associated with the vector $(Y_1, Y_2, ..., Y_n, \tilde{\sigma}_{zz}^{1/2})$ is

$$\text{block diag} \{I_n, \sigma_{zz}, [2(n - 2p - q)]^{-1}\sigma_{zz}\}.$$
The use of the square root of $\tilde{\sigma}_{zz}$ in place of $\sigma_{zz}$ has two consequences. First, the distribution should be more nearly normal, and second, the approximate covariance matrix of the error in $(Y_1, Y_2, \ldots, Y_n, \tilde{\sigma}_{zz}^{1/2})$ is known up to a multiple.

The suggested estimation procedure applies simple nonlinear least squares to the system

\begin{equation}
Y_t = - \sum_{j=1}^{p} a_j Y_{t-j} + \sum_{j=1}^{p+q} \beta_j z_{t-j} + z_t, \quad t = 1, \ldots, n
\end{equation}

where $Y_0, Y_{-1}, \ldots, Y_{-p+1}$ are initial observed values, $\tilde{\sigma}_{zz}$ denotes the estimator of $\sigma_{zz}$ obtained at step (a), $d_f = n - 2p - q$, and $a_{n+1}$ is the estimation error of $[2d_f \tilde{\sigma}_{zz}]^{1/2}$. In expressions for estimators, we will often set $a_{n+1} = z_{n+1}$ to simplify the notation. This is not strictly proper because $a_{n+1}$ is not an observation from the stationary process $\{z_t\}$.

The method of adding the restrictions to the set of observations, as suggested for step (b) of Section IV.A.1, can be applied to solve for the estimators of (4.8). Let $\hat{\theta} = (\hat{\xi}', \sigma_{ee})'$. The equations (4.8) can be approximated by
\[ Y_t = -p \sum_{j=1}^{p} \alpha_j Y_{t-j} + p+q \sum_{j=1}^{p+q} \beta_j z_{t-j} + z_t, \quad t = 1, \ldots, n \]

\[ \left([2d_f \tilde{\sigma}_{zz}]\right)^{1/2} = \left([2d_f \sigma_{zz}]\right)^{1/2} + a_{n+1} \]

\[ 0 = c f_1(\theta) + a_{n+2} \]
\[ \vdots \]
\[ 0 = c f_{p+q+1}(\theta) + a_{n+p+q+1} \]

where \( c \) is an arbitrary large number, \( f_i(\theta) \) denotes the \( i \)-th component of \( f(\theta) \), and \((a_{n+2}, \ldots, a_{n+p+q+1})\) are introduced to signify that the equalities need not be exactly satisfied. Let

\[ z_t(Y; \theta) = Y_t - \sum_{j=1}^{p} \alpha_j Y_{t-j} - \sum_{j=1}^{p+q} \beta_j z_{t-j}, \quad t = 1, \ldots, n \]

\[ z_{n+1}(Y; \theta) = \left([2d_f \tilde{\sigma}_{zz}]\right)^{1/2} - \left([2d_f \sigma_{zz}]\right)^{1/2} \]

\[ z_{n+1+j}(Y; \theta) = -c f_j(\theta), \quad j = 1, \ldots, p + q + 1 \]

and let \( \tilde{W}_t(Y; \theta) \) be the \((2p + q + 2)\)-dimensional vector composed of the negative of the partial derivatives of \( z_t(Y; \theta) \) with respect to \( \theta \). The last \((2p + q)\) vectors of \( \tilde{W}_t(Y; \theta) \) are the last three rows of Table 4.1 for \( p = q = 1 \). The nonlinear regression estimator of \( \theta \) obtained at a step of the iterative procedure is
\hat{\theta} = \bar{\theta} + \Delta \hat{\theta},

where \( \bar{\theta} \) is the estimator of the preceding step, and

\[
\Delta \hat{\theta} = \left[ \sum_{t=1}^{n+p+q+2} W_t^t(\bar{\theta}; \theta) W_t(Y; \bar{\theta}) \right] \sum_{t=1}^{n+p+q+2} W_t^t(\bar{\theta}; \theta) z_t(Y; \bar{\theta}).
\]

The value of \( \theta \) for the first iteration is the value obtained in (b) of the previous section. Alternative start values could be used.

**B. Limiting Properties of the Estimators**

In this section, we derive some large sample properties of the estimators defined in Sections IV.A.1. and IV.A.2.

Consider the model

\[
Y_t + \sum_{j=1}^{p} \alpha_j Y_{t-j} = z_t + \sum_{j=1}^{q} \beta_j z_{t-j},
\]

where \( z_t \) are \( \text{NI}(0, \sigma_{zz}) \) and the roots of the characteristic polynomials associated with (4.9) are less than unity in modulus.

Let \( \theta = (\alpha_1, \ldots, \alpha_p)' \), \( \phi = (\beta_1, \ldots, \beta_q)' \), and write \( z_t \) as

\[
z_t(Y; \theta', \phi') = Y_t + \sum_{j=1}^{p} \alpha_j Y_{t-j} - \sum_{j=1}^{q} \beta_j z_{t-j}(Y; \theta', \phi'),
\]

where the notation \( z_t(Y; \theta', \phi') \) is to emphasize the fact that \( \theta \) and
$\beta$ are parameters of $z_t$. Define

$$U_{i1}(Y;\alpha', \beta') = -\frac{\partial z_t(Y;\alpha', \beta')}{\partial \alpha_i}, \quad i = 1, \ldots, p,$$

$$U_{j1}(Y;\alpha', \beta') = -\frac{\partial z_t(Y;\alpha', \beta')}{\partial \beta_j}, \quad j = 1, \ldots, q,$$

and

$$U_t(Y;\alpha', \beta') = [U_{i1}(Y;\alpha', \beta'), \ldots, U_{q1}(Y;\alpha', \beta')] .$$

(4.12)

Let $(\bar{\alpha}', \bar{\beta}')$ be an estimator of $(\alpha', \beta')$, such that

$$(\bar{\alpha}' - \alpha', \bar{\beta}' - \beta')' = o_p(n^{-1/4}).$$

Expanding $z_t(Y;\alpha', \beta')$ in a Taylor series about $(\bar{\alpha}', \bar{\beta}')$ gives

$$z_t(Y;\alpha', \beta') = z_t(Y;\bar{\alpha}', \bar{\beta}') - \sum_{i=1}^{p} U_{i1}(Y;\bar{\alpha}', \bar{\beta}') (\alpha_i - \bar{\alpha}_i)$$

$$- \sum_{j=1}^{q} U_{j1}(Y;\bar{\alpha}', \bar{\beta}') (\beta_j - \bar{\beta}_j) - 2^{-1} d_t(Y;\bar{\alpha}', \bar{\beta}') ,$$

(4.13)

where

$$d_t(Y;\bar{\alpha}', \bar{\beta}') = \sum_{j=1}^{p} \sum_{i=1}^{p} \frac{\partial U_{i1}(Y;\bar{\alpha}', \bar{\beta}')}{\partial \alpha_j} (\alpha_i - \bar{\alpha}_i) (\alpha_j - \bar{\alpha}_j)$$
and \((\alpha^*, \beta^*)'\) lies between \((\alpha', \beta')'\) and \((\alpha^*, \beta^*)'\).

Expression (4.13) is equivalent to

\[
z_t(Y;\alpha', \beta') = \sum_{i=1}^{q} \sum_{j=1}^{p} \beta_{ij} (Y;\alpha^*, \beta^*) (\beta_{ij} - \beta_{ij}) + 2 \sum_{i=1}^{q} \sum_{j=1}^{p} \alpha_{ij} (Y;\alpha^*, \beta^*) (\alpha_{ij} - \alpha_{ij}) (\beta_{ij} - \beta_{ij})
\]

\[
+ 2 \sum_{i=1}^{q} \sum_{j=1}^{p} \alpha_{ij} (Y;\alpha^*, \beta^*) (\beta_{ij} - \beta_{ij})
\]

\[
+ 2 \sum_{i=1}^{q} \sum_{j=1}^{p} \alpha_{ij} (Y;\alpha^*, \beta^*) (\beta_{ij} - \beta_{ij}) + z_t.
\]

Regressing \(z_t(Y;\alpha^*, \beta^*)\) on \(U_t(Y;\alpha, \beta)\) we obtain an estimator of \((\alpha' - \alpha^*, \beta' - \beta^*)'\). The one-step Gauss-Newton estimator of \((\alpha', \beta')'\) is then

\[
(\tilde{\alpha}', \tilde{\beta}') = (\alpha', \beta') + (\Delta \tilde{\alpha}', \Delta \tilde{\beta}'),
\]

where
\((\Delta \tilde{q}', \tilde{q}')' = \left( \sum_{t=1}^{n} \tilde{u}_t(Y; \tilde{q}', \tilde{q}') \tilde{u}_t(Y; \tilde{q}', \tilde{q}') \right)^{-1} \)

\[ \sum_{t=1}^{n} \tilde{u}_t(Y; \tilde{q}', \tilde{q}') z_t(Y; \tilde{q}', \tilde{q}') . \] (4.17)

The asymptotic properties of \((\tilde{q}', \tilde{q}')'\) are developed in Theorem 4.1 and Theorem 4.4.

Theorem 4.1. Let \(Y_t\) satisfy

\[ Y_t + \sum_{j=1}^{p} \alpha_j Y_{t-j} = \sum_{i=1}^{q} \beta_i z_{t-i} + z_t, \quad t = 1, 2, \ldots, \]

where \(Y_0, Y_{-1}, \ldots, Y_{-p+1}\) are initial observations and \(\{Y_t\}\) is a stationary process. Let the roots of

\[ m^p + \alpha_1 m^{p-1} + \ldots + \alpha_p = 0 \] (4.18)

and

\[ r^q + \beta_1 r^{q-1} + \ldots + \beta_q = 0 \] (4.19)

be less than one in absolute value, and let \(z_t\) be a sequence of \(N(0, \sigma_{zz})\) random variables. Let \((\tilde{q}', \tilde{q}')'\) be an initial estimator of \((\tilde{q}', \tilde{q}')'\) satisfying \((\tilde{q}' - \tilde{q}', \tilde{q}' - \tilde{q}')' = o(n^{-1/4})\) and such that the roots of \(m^p + \sum_{j=1}^{p} \alpha_j m^{p-j} = 0\) and of \(r^q + \sum_{j=1}^{q} \beta_j r^{q-j} = 0\) are less than unity in modulus. Let \(z_j, j = -q + 1, \ldots, 0\), be bounded
In probability. Then,

\[ [(\tilde{g} - g)', (\tilde{g} - g)')] = o_p(n^{-1/2}) , \]

where \((\tilde{g}', \tilde{g}')'\) is the estimator defined in (4.16).

**Proof:** From (4.15) and (4.17),

\[ [(\tilde{g} - g)', (\tilde{g} - g)')]' = \left[ \sum_{t=1}^{n} U_t'(Y; \tilde{g}', \tilde{g}') \sum_{t=1}^{n} U_t(Y; \tilde{g}', \tilde{g}') \right]^{-1} \]

\[ \left[ \sum_{t=1}^{n} U_t'(Y; \tilde{g}', \tilde{g}') R_t(Y; \tilde{g}', \tilde{g}') \right] , \]

(4.20)

where

\[ R_t(Y; \tilde{g}', \tilde{g}') = 2^{-1} \sum_{t=1}^{n} U_t'(Y; \tilde{g}', \tilde{g}') d_t(Y; \tilde{g}', \tilde{g}') \]

and \(d_t(Y; \tilde{g}', \tilde{g}')\) is defined in (4.14).

As a first step in this proof, we want to show that \(n^{-1} R_t(Y; \tilde{g}', \tilde{g}')\) is \(o_p(n^{-1/2})\). Differentiating both sides of (4.10) with respect to \(\alpha_i\), and using the invertibility property of \(\{Y_t\}\), we can establish the following difference equation relationship for \(U_{\alpha_i t}(Y; \tilde{g}', \tilde{g}')\),

\[ U_{\alpha_i t}(Y; \tilde{g}', \tilde{g}') + \sum_{j=1}^{p} \alpha_j \alpha_{i, t-j}(Y; \tilde{g}', \tilde{g}') = - z_{t-1} \]
From (4.21), we note that the difference equation defining $U_{a_2}(Y; a', b')$ is identical to that defining $U_{a_2}(Y; a', b')$, $i = 1, 2, \ldots, p - 1$, $s = 1, 2, \ldots, p - i$. Therefore, if the process started in the distant past, $U_{a_2}(Y; a', b') = U_{a_2}(Y; a', b')$, say.

The time series $U_{a_2}(Y; a', b')$ converges to a stationary autoregressive process with characteristic equation

$$m^p + \alpha_1 m^{p-1} + \ldots + \alpha_p = 0.$$
By the arguments used for $U_{\alpha_t}(Y;\varphi', \theta')$, it follows that

$$U_{\beta_{1,t}}(Y;\varphi', \theta') = U_{\beta_{2,t+1}}(Y;\varphi', \theta') = \ldots = U_{\beta_{q,t+q}}(Y;\varphi', \theta') = U_{\beta_t}(Y;\varphi', \theta') \text{, say.}$$

Furthermore, $U_{\beta_t}(Y;\varphi', \theta')$ converges to a stationary autoregressive process with characteristic equation

$$m^q + \beta_1 m^{q-1} + \ldots + \beta_q = 0. $$

Repeating the reasoning used for $U_{\alpha_t}(Y;\varphi', \theta')$ and $U_{\beta_t}(Y;\varphi', \theta')$, one can establish the convergence of the second partial derivatives of $z_t(Y;\varphi', \theta')$ to stationary processes.

Let $m^o = (m_1^o, \ldots, m_p^o)$ and $\tau^o = (\tau_1^o, \ldots, \tau_q^o)$ be the roots of (4.18) and (4.19), respectively. Let $F_1$ be a compact set within the unit circle of the $p$-dimensional Euclidean space, with $m^o$ as an interior point, such that

$$\sup_{m \in F_1} \max_{i=1, \ldots, p} |m_i| < \lambda_1 < 1.$$
Define \( S_1 = \{a : m^p \}
\) and \( S_2 \) be the analog be such that

\[
\sup_{x \in F, i=1,2} r
\]

Define \( \overline{S} = S_1 \cup S_2 \).

and

\[
U_{\alpha_c}(Y; a', y)
\]

and

\[
U_{\beta_c}(Y; a', y)
\]

where \( w_j \) and \( v_j \) s

\[
w_j + \alpha_1 w_j
\]

and

\[
v_j + \beta_1 v_j
\]

with initial condition.

Using a result given

Define \( S_1 \) and \( S_2 \) be such that

Define \( \overline{S} = S_1 \cup S_2 \).

and

where

and

with initial condition.

Using a result given
\[ |\omega_j| < M_1 \lambda_2^j \]

and

\[ |\nu_j| < M_2 \lambda_2^j \]

for some \( M_1 \) and \( M_2 \). Then,

\[ E\{U_{\omega_t}(Y;g', \delta')U_{\omega_t+h}(Y;g', \delta')\} < M_1^2 (1 - \lambda_2^2)^{-1} \lambda_1^h \sigma_{zz}, \]

and

\[ E\{U_{\beta_t}(Y;g', \delta')U_{\beta_t+h}(Y;g', \delta')\} < M_2^2 (1 - \lambda_2^2)^{-1} \lambda_2^h \sigma_{zz}, \]

and by theorem 6.2.1 in Fuller (1976), the variances of

\[ n^{-1/2} \sum_{t=1}^{n} U_{\omega_t}^2(Y;g', \delta') \quad \text{and} \quad n^{-1/2} \sum_{t=1}^{n} U_{\beta_t}^2(Y;g', \delta') \]

are bounded.

Since the second partial derivatives of \( z_t(Y;g', \delta') \) converge to stationary processes, it follows that, for \((g', \delta') \in \bar{S}\), the variances of

\[ n^{-1/2} \sum_{t=1}^{n} \frac{\partial U_{\omega_t}(Y;g', \delta')}{\partial g_j}, \quad n^{-1/2} \sum_{t=1}^{n} \frac{\partial U_{\omega_t}(Y;g', \delta')}{\partial \delta_j}, \]

are bounded.
and of

\[ n^{-1/2} \sum_{t=1}^{n} \frac{\partial U_t(Y; \mathbf{a}', \mathbf{b}')}{\partial \mathbf{b}_j} \]

are bounded. Therefore, for example,

\[ n^{-1} \sum_{t=1}^{n} U_{at}(Y; \mathbf{a}', \mathbf{b}')[\frac{\partial U_{at}(Y; \mathbf{a}', \mathbf{b})}{\partial \mathbf{b}_j}] = O_p(1) . \]

Because \((\mathbf{a}' - \mathbf{a}', \mathbf{b}' - \mathbf{b}'))' = o_p(n^{-1/4})\), given \(\varepsilon > 0\) there exists \(N\) such that, for all \(n > N\), \(P\{(\mathbf{a}', \mathbf{b}')' \in \tilde{S}\} > 1 - \varepsilon\). Hence,

\[ n^{-1} \mathbf{R}_t(Y; \mathbf{a}', \mathbf{b}') = O_p(n^{-1/2}) . \quad (4.22) \]

We can write

\[ n^{-1} \sum_{t=1}^{n} U'_t(Y; \mathbf{a}', \mathbf{b}') U_{-t}(Y; \mathbf{a}', \mathbf{b}') \]

\[ = n^{-1} \sum_{t=1}^{n} U'_t(Y; \mathbf{a}', \mathbf{b}') U_{-t}(Y; \mathbf{a}', \mathbf{b}') + O_p(n^{-1/2}) . \quad (4.23) \]

Let \(\{F_{a_t}\}\) and \(\{F_{b_t}\}\) be the limiting processes of \(U_{at}(Y; \mathbf{a}', \mathbf{b})\) and \(U_{bt}(Y; \mathbf{a}', \mathbf{b}')\), respectively. Thus, as \(n \to \infty\),


\[ n^{-1} \sum_{t=1}^{n} U_{\alpha t}(Y; a', \varepsilon') \alpha_{j t}(Y; a', \varepsilon') \xrightarrow{P} \gamma_{F_\alpha}(|i-j|), \]

\[ i, j = 1, \ldots, p, \]

\[ n^{-1} \sum_{t=1}^{n} U_{\beta j t}(Y; a', \varepsilon') \beta_{k t}(Y; a', \varepsilon') \xrightarrow{P} \gamma_{F_\beta}(|j-k|), \]

\[ j, k = 1, \ldots, q, \]

and

\[ n^{-1} \sum_{t=1}^{n} U_{\alpha t}(Y; a', \varepsilon') U_{\beta j t}(Y; a', \varepsilon') \xrightarrow{P} \gamma_{F_\alpha F_\beta}(|i-j|), \]

\[ i = 1, \ldots, p, \quad j = 1, \ldots, q, \]

where \( \gamma_{F_\alpha}(h) = E(F_{t} F_{t+h}), \) and \( \gamma_{F_\alpha F_\beta}(h) = E(F_{t} F_{t+h}). \) Hence,

\[ n^{-1} \sum_{t=1}^{n} U'_{t}(Y; a', \varepsilon') U_{t}(Y; a', \varepsilon') \xrightarrow{P} V, \text{ say.} \quad (4.24) \]

Now, expanding the \( r \)-th element of \( n^{-1} \sum_{t=1}^{n} U'_{t}(Y; a', \varepsilon') z_{t} \) in a Taylor series about \( (a', \varepsilon') \) gives
\[
\begin{align*}
\sum_{t=1}^{n-1} U_{rt}(Y; \bar{\alpha}', \bar{\beta}') z_t &= \sum_{t=1}^{n-1} [U_{rt}(Y; \bar{\alpha}', \bar{\beta}') \\
&\quad + \sum_{i=1}^{p} \frac{\partial U_{rt}(Y; \bar{\alpha}'', \bar{\beta}'', \bar{\alpha}', \bar{\beta}')} {\partial \alpha_i} (\alpha_i - \alpha_i) \\
&\quad + \sum_{i=1}^{q} \frac{\partial U_{rt}(Y; \bar{\alpha}'', \bar{\beta}'', \bar{\alpha}', \bar{\beta}')} {\partial \beta_i} (\beta_i - \beta_i)] z_t,
\end{align*}
\]
where \((\bar{\alpha}'', \bar{\beta}'')\) lies between \((\bar{\alpha}', \bar{\beta}')\) and \((\bar{\alpha}'', \bar{\beta}'')\).

Because
\[
\sum_{t=1}^{n-1} \frac{\partial U_{rt}(Y; \bar{\alpha}, \bar{\beta})} {\partial \alpha_i} z_t = o_p(n^{-1/2}), \quad i = 1, \ldots, p,
\]
and
\[
\sum_{t=1}^{n-1} \frac{\partial U_{rt}(Y; \bar{\alpha}', \bar{\beta}')} {\partial \beta_i} z_t = o_p(n^{-1/2}), \quad i = 1, \ldots, q,
\]
and
\[
[(\bar{\alpha} - \bar{\alpha}', \bar{\beta} - \bar{\beta}')] = o_p(n^{-1/4}),
\]
it follows that
From (4.20), (4.22), (4.23) and (4.25),

\[
\left[ (\tilde{\xi} - \alpha), (\tilde{\xi} - \beta) \right]' = [n^{-1} \sum_{t=1}^{n} \sum_{j=1}^{p} \tilde{\xi}'_{t} (Y;\tilde{\xi}', \tilde{\xi}')]^{-1} n^{-1} \sum_{t=1}^{n} \sum_{j=1}^{q} \xi_{t} z_{t-j} (Y;\tilde{\xi}', \tilde{\xi}^{'}) + O_{p}(n^{-1/2}) .
\]

The result follows from (4.25). \(\square\)

An estimator of \(\sigma_{zz}^{2}\) is

\[
\tilde{\sigma}_{zz} = n^{-1} \sum_{t=1}^{n} z_{t}^{2} (Y;\tilde{\xi}', \tilde{\xi}') ,
\] (4.26)

where

\[
 z_{t} (Y;\tilde{\xi}', \tilde{\xi}^{'}) = y_{t} + \sum_{j=1}^{p} \tilde{\alpha}_{j} y_{t-j} - \sum_{j=1}^{q} \tilde{\beta}_{j} z_{t-j} (Y;\tilde{\xi}', \tilde{\xi}^{'}) ,
\]

\[t = 1, \ldots, n\]

We show that \(\tilde{\sigma}_{zz}\) is a consistent estimator for \(\sigma_{zz}\).

Theorem 4.2. Let the conditions of Theorem 4.1 hold. Then,

\[
\tilde{\sigma}_{zz} = n^{-1} \sum_{t=1}^{n} z_{t}^{2} + O_{p}(n^{-1}) ,
\] (4.27)
\[ \sigma_{zz} = n^{-1} \sum_{t=1}^{n} z_t^2 + o_p(n^{-1}), \quad (4.27) \]

where \( \sigma_{zz} \) is defined in (4.26).

**Proof.** Because \( \{y_t\} \) is an invertible stationary time series, it can be represented as

\[ z_t = \sum_{j=0}^{\infty} d_j y_{t-j}, \]

where

\[ d_0 = 1 \]

\[ d_1 = \beta_1 + \alpha_1 \]

\[ d_2 = -\beta_1 d_1 - \beta_2 + \alpha_2 \]

\[ d_j = \begin{cases} \sum_{i=0}^{\min(j,q)} \beta_i d_{j-i} + \alpha_j, & j < p \\ \sum_{i=1}^{\min(j,q)} \beta_i d_{j-i}, & j > p \end{cases} \]

Hence,

\[ n^{-1} \left[ \sum_{t=1}^{n} z_t^2 (y_t; \beta', \theta') - \sum_{t=1}^{n} z_t^2 \right] = o_p(n^{-1}), \quad (4.28) \]

where
By the arguments used in Theorem 4.1 and (4.28),

\[ z_t(Y; \alpha', \beta') = \sum_{j=0}^{t-1} d_j Y_{t-j} \]

For the arguments used in Theorem 4.1 and (4.28),

\[ n^{-1} \sum_{t=1}^{n} z_t(Y; \alpha', \beta') = n^{-1} \sum_{t=1}^{n} z_t(Y; \alpha', \beta') \]

\[ + 2 \sum_{i=1}^{p} \sum_{t=1}^{n} z_t(Y; \alpha^{*'}, \beta^{*'}) U_{\alpha^{*'} t}(Y; \alpha^{*'}, \beta^{*'})(\tilde{\alpha} - \alpha) \]

\[ + 2 \sum_{j=1}^{q} \sum_{t=1}^{n} z_t(Y; \alpha^{*'}, \beta^{*'}) U_{\beta^{*'} t}(Y; \alpha^{*'}, \beta^{*'})(\tilde{\beta} - \beta) \]

\[ = n^{-1} \sum_{t=1}^{n} z_t^2 + o_p(n^{-1}), \]

where \((\alpha^{*'}, \beta^{*'})\) lies on the line segment joining \((\alpha', \beta')\) and \((\tilde{\alpha}', \tilde{\beta}')\). 

To derive the limiting distribution of \((\alpha', \beta', \tilde{\alpha})\), we need a central limit theorem for martingale differences. Theorem 4.3 is due to Scott (1973). Related results have been obtained by Brown (1971).

**Theorem 4.3.** Let \(\{Z_t: 1 < t < n, n > 1\}\) denote a triangular array of random variables defined on the probability space \((\Omega, B, \mathbb{P})\). Let

\[ S_{kn} = \sum_{t=1}^{n} Z_{tn} \]

for \(1 < k < n, n > 1\) with \(S_{on} = 0\) for \(n > 1\). Assume that for \(1 < t < n\),
\[ E[S_{kn} \mid B_{k-1,n}] = S_{k-1,n} \text{ a.s.}, \]

where \( B_{k-1,n} \) denotes the \( \sigma \)-algebra generated by \( S_{1n}, \ldots, S_{k-1,n} \).

Let

\[ \delta^2 = E[Z^2 \mid B_{n-1,n}], \]

\[ v^2 = \sum_{j=1}^{n} \delta^2_j, \]

\[ s^2 = E(v^2) = E(s^2). \]

Assume

1) \[ v^2 \frac{s^2}{n} \longrightarrow 1 \text{ in probability}, \]

and

2) \[ \lim_{n \to \infty} s^2 \sum_{j=1}^{n} \mathbb{E} \left[ \mathbf{1}_{\{|z_j| > \varepsilon \frac{s}{n}\}} z_j^2 \right] dP = 0 \text{ for all } \varepsilon > 0. \]

Then,

\[ s^{-1} \frac{S_n}{n} \xrightarrow{L} N(0, 1). \]

**Proof:** See Scott (1973).
Theorem 4.4. Assume that the model of Theorem 4.1 holds, and let \( \tilde{\theta} = (\tilde{a}', \tilde{b}', \tilde{\sigma}_{zz})' \). Then,

\[
n^{1/2} (\tilde{\theta} - \theta) \xrightarrow{L} N(0, G \sigma_{zz}),
\]

as \( n \to \infty \), where \( G = \text{block diag} \left( V^{-1}, 2\sigma_{zz} \right) \), \( V \) is defined in (4.24), \( \tilde{\theta} = (\tilde{a}', \tilde{b}', \tilde{\sigma}_{zz}) \), and \( (\tilde{a}', \tilde{b}') \) is defined in (4.16) and \( \tilde{\sigma}_{zz} \) is defined in (4.26).

Proof. From (4.26) and (4.27), the limiting distribution of \( n^{1/2} (\tilde{\theta} - \theta) \) is the same as the limiting distribution of

\[
n^{-1/2} \sum_{t=1}^{n} \left( \begin{array}{c}
V^{-1} U_t(Y; a', b') z_t \\
V^{-1} U_t(Y; a', b') z_t - \sigma_{zz}
\end{array} \right),
\]

or equivalently,

\[
\left( \begin{array}{cc}
V^{-1} & 0 \\
0 & 1
\end{array} \right) n^{-1/2} \sum_{t=1}^{n} \left( \begin{array}{c}
U_t(Y; a', b') z_t \\
U_t(Y; a', b') z_t - \sigma_{zz}
\end{array} \right), \quad (4.29)
\]

where \( U_t(Y; a', b') \) is defined in (4.12). Let \( \lambda = (\lambda_1, \lambda_2)' \) be a \((p + q + 1)\)-dimensional real vector with \( \lambda' \lambda \neq 0 \), and consider the distribution of

\[
n^{-1/2} \sum_{t=1}^{n} \lambda' \left( \begin{array}{c}
U_t(Y; a', b') z_t \\
U_t(Y; a', b') z_t - \sigma_{zz}
\end{array} \right),
\]
\[ \sum_{t=1}^{n} Z_{tn} = S_{nn}, \]

where

\[ Z_{tn} = n^{-1/2} \left[ \lambda'_1 \omega_t(Y; \alpha', \beta') z_t + \lambda_2 (z_{t}^2 - \sigma_{zz}) \right]. \]

Let \( \mathcal{B}_{t-1,n} \) be the sigma-algebra generated by \( \{z_j, j < t-1\} \). Then,

\[ E(Z_{tn} | \mathcal{B}_{t-1,n}) = 0 \quad \text{a.s.}, \]

\[ \delta_{tn}^2 = E(Z_{tn}^2 | \mathcal{B}_{t-1,n}) = n^{-1} \lambda'_1 \left[ \sum_{t=1}^{n} \omega_t(Y; \alpha', \beta') \omega_t(Y; \alpha', \beta') \right] \lambda_1 \sigma_{zz} + 2 n^{-1} \lambda_2^2 \sigma_{zz}^2 \]

\[ \nu_{nn}^2 = \sum_{j=1}^{n} \delta_{jn}^2 = n^{-1} \lambda'_1 \left[ \sum_{t=1}^{n} \omega_t(Y; \alpha', \beta') \omega_t(Y; \alpha', \beta') \right] \lambda_1 \sigma_{zz} + 2 \lambda_2^2 \sigma_{zz}^2. \]

From (4.24),

\[ \nu_{nn}^2 \longrightarrow \lambda'_1 \lambda_1 \sigma_{zz} + 2 \lambda_2^2 \sigma_{zz}^2 \quad \text{in probability}. \]

Now,

\[ E(S_{nn}^2) = \text{Var}(n^{-1/2} \sum_{t=1}^{n} \left[ \lambda'_1 \omega_t(Y; \alpha', \beta') z_t + \lambda_2 (z_{t}^2 - \sigma_{zz}) \right]) \]
\[ n^{-1} \sum_{t=1}^{n} E[\lambda'_t \lambda_t' (Y; a', b') \lambda_t \sigma_{zz} + 2 \lambda'_t \sigma_{zz}], \]

\[ = S_{nn}^2 \]

\[ \lim_{n \to \infty} E[S_{nn}^2] = \lambda'_1 \lambda_1 \sigma_{zz} + 2 \lambda'_1 \sigma_{zz} \]

\[ = \lim_{n \to \infty} S_{nn}^2 . \]

Hence,

\[ \frac{y_{nn}^2}{S_{nn}^2} \xrightarrow{n \to \infty} 1 , \text{ in probability}, \quad (4.30) \]

and condition (i) of Theorem 4.3 is satisfied.

Now, let \( \nu > 0 \) be such that \( E[z_t^{2+\nu}] < M \) for all \( t \), and let \( \epsilon > 0 \) be arbitrary. Then,

\[ \frac{S_{nn}^2}{S_{nn}^2} \sum_{t=1}^{n} E[Z_{tn}; |Z_{tn}| > \epsilon S_{nn}] \]

\[ \leq \frac{S_{nn}^2}{S_{nn}^2} \sum_{t=1}^{n} E[(\epsilon S_{nn})^{-\nu} |Z_{tn}|^{2+\nu}; |Z_{tn}| > \epsilon S_{nn}] \]

\[ \leq \frac{S_{nn}^2}{S_{nn}^2} \epsilon^{-\nu} \sum_{t=1}^{n} E[|Z_{tn}|^{2+\nu}] \]

\[ = \frac{S_{nn}^2}{S_{nn}^2} \epsilon^{-\nu} \sum_{t=1}^{n} \frac{2+\nu}{2} \lambda'_t \lambda_t' (Y; a', b') z_t \]

\[ + \lambda'_2 (z_t^2 - \sigma_{zz})^{2+\nu} . \]
By Minkowsky's inequality,

\[
\left[ \mathbb{E} \left( \left| \lambda_1 u_t^*(Y; g', \beta') z_t + \lambda_2 (z_t^2 - \sigma_{zz}) \right|^{2+\nu} \right) \right]^{\frac{1}{2+\nu}} < \left[ \mathbb{E} \left( \left| \lambda_1 u_t^*(Y; g', \beta') z_t \right|^{2+\nu} \right) \right]^{\frac{1}{2+\nu}} \\
+ \left[ \mathbb{E} \left( \left| \lambda_2 (z_t^2 - \sigma_{zz}) \right|^{2+\nu} \right) \right]^{\frac{1}{2+\nu}} \\
= \left[ \mathbb{E} \left( \left| \lambda_1 u_t^*(Y; g', \beta') \right|^{2+\nu} \mathbb{E} \left( \left| z_t \right|^{2+\nu} \right) \right) \right]^{\frac{1}{2+\nu}} \\
+ \left| \lambda_2 \right| \left[ \mathbb{E} \left( \left| z_t^2 - \sigma_{zz} \right|^{2+\nu} \right) \right]^{\frac{1}{2+\nu}} \\
< \mathbb{E} \left( \left| \lambda_1 u_t^*(Y; g', \beta') \right|^{2+\nu} \right)^{\frac{1}{2+\nu}} + C(\nu), \quad (4.31)
\]

where

\[
C(\nu) = \left| \lambda_2 \right| \left[ \mathbb{E} \left( \left| z_t^2 - \sigma_{zz} \right|^{2+\nu} \right) \right]^{\frac{1}{2+\nu}},
\]

which is finite.

From arguments used in the proof of Theorem 4.1,

\[
U_{it}(Y; g', \beta') = \sum_{j=0}^{t-1} w_j z_{t-1-j}, \quad \text{for } i = 1, \ldots, p,
\]

\[
= \sum_{j=0}^{t-1} v_{j} z_{t+p-1-j}, \quad \text{for } i = p + 1, \ldots, p + q,
\]
where the weights \( w_j \) and \( v_j \) satisfy \( w_j = v_j = 0 \) for \( j < 0 \), \( w_j = v_j = 1 \) for \( j = 0 \), and

\[
    w_j + \alpha_1 w_{j-1} + \ldots + \alpha_p w_{j-p} = 0, \quad j > 1,
\]

\[
    v_j + \beta_1 v_{j-1} + \ldots + \beta_q v_{j-q} = 0, \quad j > 1.
\]

By Holder's inequality,

\[
    \sum_{j=0}^{t-1} w_j |z_{t-1-j}|^{2+\nu} \leq \left( \sum_{j=0}^{t-1} |w_j| \right) \left( \sum_{j=0}^{t-1} |z_{t-1-j}|^{2+\nu} \right)^{1/(2+\nu)}
\]

\[
    = \left( \sum_{j=0}^{t-1} |w_j| \right) \frac{1}{2+\nu} \left( \sum_{j=0}^{t-1} |z_{t-1-j}|^{2+\nu} \right)^{1/(2+\nu)}
\]

\[
    \leq \left( \sum_{j=0}^{t-1} |w_j| \right) \left( \sum_{j=0}^{t-1} |z_{t-1-j}|^{2+\nu} \right) \left( \sum_{j=0}^{t-1} |w_j| \right)^{1/(2+\nu)}
\]

(4.32)

Therefore, from (4.31) and (4.32),

\[
    E\left( \left| \frac{\lambda_1' \eta_t' (Y; \alpha', \beta') z_t + \lambda_2 (z_t^2 - \sigma_{z,z}) \right|^{2+\nu} \right)
\]

is bounded, and

\[
    s_{nn}^{-2} \sum_{t=1}^{n} E\left( Z_{ct}^2 \mid |Z_{ct}| > s_{nn} \right) \rightarrow 0 \text{ in probability. (4.33)}
\]
From (4.30), (4.33) and Theorem 4.3,

\[ n^{-\frac{1}{2}} \sum_{t=1}^{n} \left[ \lambda_1^t \left( \mathbf{Y}_i; \mathbf{a}^t, \mathbf{b}^t \right) z_t + \lambda_2 \left( z_t^2 - \sigma_{zz} \right) \right] \xrightarrow{L} N(0, \lambda_1^t \lambda_1^t \sigma_{zz}^2 + 2 \lambda_2^2 \sigma_{zz}^2), \]

and hence, using (4.29),

\[ n^{\frac{1}{2}} \lambda^t \left( \bar{\delta} - \bar{\delta} \right) \xrightarrow{L} N(0, \lambda^t \lambda^t \sigma_{zz}). \]

Since \( \bar{\lambda} \) is an arbitrary real vector the result follows.

We use the results in Theorem 4.4 to show that the least squares estimators for the restricted model obtained at step (b) of the estimation procedure of Section IV.A.1 are weakly consistent. The model we are considering is

\[ \bar{\delta}_n = \bar{\delta} + a_n, \]

(4.34)

\[ f(\delta', \sigma_{ee}) = 0, \]

where \( a_n \) is a random vector whose covariance matrix is estimated by \( n^{-1} \mathbf{C}_n \), and

\[ f(\delta', \sigma_{ee}) = [f_1(\delta', \sigma_{ee}), \ldots, f_{p+q+1}(\delta', \sigma_{ee})]', \]
The least squares estimator of \((\hat{\theta}', \sigma_{ee})\) is the value 
\((\hat{\theta}', \sigma_{ee})\) that minimizes \(f_1(\hat{\theta}', \sigma_{ee})\) subject to \(f_1(\hat{\theta}', \sigma_{ee}) = 0\).

For mathematical convenience, we use an alternative formulation of the minimization problem above. Because \(\sigma_{ee}\) appears only in \(f_1(\hat{\theta}', \sigma_{ee})\), and since \(\sigma_{ee} > 0\), the restriction \(f_1(\hat{\theta}', \sigma_{ee}) = 0\) is equivalent to

\[
\sigma_{ee} = \sigma_{zz} \sum_{j=0}^{p+q} \beta_j^2 - \sum_{j=0}^{p} \sum_{k=0}^{p+q} \alpha_j \gamma_k (k - j) \quad (4.36)
\]
or

\[ \sigma_{zz} \sum_{j=0}^{p+q} \beta_j^2 - \sum_{j=0}^{p} \sum_{k=0}^{p} a_j a_k \gamma_{(k-j)} > 0. \]

Since \( \sigma_{ee} \) in (4.36) is a continuous function of \( \tilde{\alpha} \), the least squares estimator \((\tilde{\alpha}', \tilde{\sigma}_{ee}')\) is obtained by finding \( \tilde{\alpha} \) that minimizes

\[ (\tilde{\alpha}_n - \tilde{\alpha})'G_n^{-1}(\tilde{\alpha}'_n - \tilde{\alpha}) \]

subject to

\[ \sigma_{zz} \sum_{j=0}^{p+q} \beta_j^2 - \sum_{j=0}^{p} \sum_{k=0}^{p} a_j a_k \gamma_{(k-j)} > 0. \]

We use \( f_i(\tilde{\alpha}') \) for \( f_i(\tilde{\alpha}', \sigma_{ee}) \) to emphasize the fact that \( f_i \) does not depend upon \( \sigma_{ee} \) for \( i = 2, \ldots, p+q+1 \). In this formulation, the least squares estimator of \( \sigma_{ee} \) is

\[ \sigma_{ee} = \sigma_{zz} \sum_{j=0}^{p+q} \beta_j^2 - \sum_{j=0}^{p} \sum_{k=0}^{p} a_j a_k \gamma_{(k-j)}. \]  

(4.37)

To show the consistency of \( (\tilde{\alpha}', \tilde{\sigma}_{ee}') \) we need the following lemma.

**Lemma 4.1.** Let \( \{X_n\} \) be a sequence of \( p \times 1 \) random vectors, let \( u_o \) be a fixed \( p \times 1 \) vector, and let \( A \) be a \( p \times p \) symmetric positive definite matrix. Assume that as \( n \to \infty \),
Then, \( \hat{x}_n \xrightarrow{P} \hat{\mu}_0 \).

**Proof.** Let \( \lambda_1 < \lambda_2 < \ldots < \lambda_p \) be the characteristic roots of \( A \), and let \( Q \) be the orthogonal matrix whose columns are the characteristic vectors associated with \( \lambda_1, \ldots, \lambda_p \). For a fixed vector \( \hat{x} \), we can write

\[
\hat{x}'A\hat{x} = \hat{x}'Q'\Lambda Q\hat{x} = \hat{x}'Ay = \sum_{i=1}^{p} \lambda_i y_i^2
\]

\[
> \lambda_1 \hat{y}'\hat{y}' = \lambda_1 \hat{x}'Q'Q\hat{x} = \lambda_1 \hat{x}'\hat{x}
\]

(4.38)

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p) \), and \( \hat{x} = Q\hat{x} \).

For a given \( \varepsilon > 0 \), and using (4.38),

\[
P[|\hat{x}_n - \mu_0|^2 > \varepsilon^2] = P[|\lambda_1|\hat{x}_n - \mu_0|^2 > \lambda_1 \varepsilon^2]
\]

\[
< P[(\hat{x}_n - \mu_0)'A(\hat{x}_n - \mu_0) > \lambda_1 \varepsilon^2]
\]

< \varepsilon \text{ for } n \text{ sufficiently large.} \)

**Theorem 4.5.** Let \( \hat{\phi} = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_{p+q}, \sigma_{zz})' \), let \( \hat{\phi}^0 \) be the true value of \( \hat{\phi} \), and let \( \gamma_u(h), h = 0, \ldots, q \) be known. Let \( D \) be a subset of \((2p + q + 1)\)-dimensional Euclidean space such that if \( \hat{\phi} \in D \),

Assume that $D$ is bounded, i.e., if $\hat{\delta} \in D$, $\hat{\delta}'\hat{\delta} < B$ for some $B < \infty$. Let $\bar{\hat{\delta}}_n$ be the least squares estimator of $\delta^0$ for the unrestricted model (4.6), and let $n^{-1}G_n$ be its estimated covariance matrix. Let $\bar{\hat{\delta}}$ be the value of $\bar{\hat{\delta}}$ that minimizes

$$(\bar{\hat{\delta}}_n - \bar{\hat{\delta}})'G_n^{-1}(\bar{\hat{\delta}}_n - \bar{\hat{\delta}})$$

over $D$. Assume that

$$\bar{\hat{\delta}}_n \xrightarrow{P} \delta^0,$$

$$G_n \xrightarrow{P} G,$$

as $n \to \infty$, where $\delta^0 \in D$ and $G$ is positive definite. Then,

$$\bar{\hat{\delta}} \xrightarrow{P} \delta^0,$$

as $n \to \infty$.

Proof. Because
\[
\begin{align*}
(\hat{\xi} - \hat{\xi}_n)'G_n^{-1}(\hat{\xi} - \hat{\xi}_n) - (\hat{\xi} - \hat{\xi}^0)'G_n^{-1}(\hat{\xi} - \hat{\xi}^0) \\
= \hat{\xi}'(G_n^{-1} - G^{-1})\hat{\xi} - 2 \hat{\xi}'(G_n^{-1}\hat{\xi}_n - G^{-1}\hat{\xi}^0) \\
+ \hat{\xi}_n'G_n^{-1}\hat{\xi}_n - \hat{\xi}^0'G^{-1}\hat{\xi}^0,
\end{align*}
\]

and because \( \hat{\xi}' \hat{\xi} < B \), it follows that

\[
(\hat{\xi} - \hat{\xi}_n)'G_n^{-1}(\hat{\xi} - \hat{\xi}_n) - (\hat{\xi} - \hat{\xi}^0)'G_n^{-1}(\hat{\xi} - \hat{\xi}^0) \xrightarrow{P} 0.
\]

Since \( \hat{\xi} \) minimizes \( (\hat{\xi}_n - \hat{\xi})'G_n^{-1}(\hat{\xi}_n - \hat{\xi}) \) over \( D \), \( \hat{\xi}^0 \in D \), and \( G_n \) is positive definite,

\[
0 < (\hat{\xi}_n - \hat{\xi})'G_n^{-1}(\hat{\xi}_n - \hat{\xi}^0) < (\hat{\xi}_n - \hat{\xi}^0)'G_n^{-1}(\hat{\xi}_n - \hat{\xi}^0).
\]

By hypothesis, \( \hat{\xi}_n \xrightarrow{P} \hat{\xi}^0 \), and \( G_n \xrightarrow{P} G \). Therefore,

\[
(\hat{\xi}_n - \hat{\xi}^0)'G_n^{-1}(\hat{\xi}_n - \hat{\xi}^0) \xrightarrow{P} 0,
\]

\[
(\hat{\xi}_n - \hat{\xi})'G_n^{-1}(\hat{\xi}_n - \hat{\xi}) \xrightarrow{P} 0,
\]

and

\[
(\hat{\xi} - \hat{\xi}^0)'G^{-1}(\hat{\xi} - \hat{\xi}^0) \xrightarrow{P} 0.
\]
Thus, by lemma 4.1, the results follows. □

Corollary 4.1. Given the assumptions of Theorem 4.1,
\[ \sigma_{ee}^0 \overset{P}{\to} \sigma_{ee}^0, \]

as \( n \to \infty \), where
\[ \sigma_{ee}^0 = \sigma_{zz}^{p+q} \sum_{j=0}^p \beta_j^0 \sum_{j=0}^p \sum_{k=0}^q \alpha_j^0 \gamma_u(k-j), \]

and \( \sigma_{ee}^0 \) is defined in (4.37).

Proof. Using the facts that \( \hat{\sigma} \overset{P}{\to} \sigma^0 \), and that \( \sigma_{ee} \) can be expressed as a continuous function of \( \hat{\sigma} \), we have
\[ \sigma_{ee}^0 \overset{P}{\to} \sigma_{ee}^0. \]

We now consider the estimator of Section IV.A.2. Let the restricted model be
\[ Y_t = -\sum_{j=1}^{p+q} \alpha_j \gamma_{t-j} + \sum_{j=1}^{p+q} \beta_j \gamma_{t-j} + z_t, \quad t = 1, \ldots, n \]
\[ [2d_x \tilde{\sigma}_{zz}]^{1/2} = [2d_x \sigma_{zz}]^{1/2} + a_{n+1}, \]
\[ f(\alpha', \beta', \sigma_{zz}, \sigma_{ee}) = 0, \]

where \( z_t \) are NI(0, \( \sigma_{zz} \)), and the characteristic polynomials
associated with the first equation in (4.39) have roots that are less than unity in modulus and have no common roots.

Expressing $\beta$ and $\sigma_{zz}$ in terms of $a$ and $\sigma_{ee}$ gives an alternative expression for equations (4.39). The existence and uniqueness of such representations are guaranteed by the implicit function theorem. We verify that $f(a', \beta', \sigma_{zz}, \sigma_{ee})$ satisfies the conditions of the theorem.

**Lemma 4.2.** Let $(a_1, \ldots, a_p, \beta_1, \ldots, \beta_{p+q}, \sigma_{zz})$ be the parameters of the autoregressive moving average

$$y_t + a_1 y_{t-1} + \ldots + a_p y_{t-p} = z_t + \beta_1 z_{t-1} + \ldots + \beta_q z_{t-q},$$

where the roots of $m^p + a_1 m^{p-1} + \ldots + a_p = 0$ and of $m^q + \beta_1 m^{q-1} + \ldots + \beta_q = 0$ are less than one in absolute value. Let $\sigma_{ee} > 0$. The parameter space of $(a_1, \ldots, a_p, \beta_1, \ldots, \beta_{p+q}, \sigma_{zz}, \sigma_{ee})$ is an open subset of the $(2p + q + 2)$-dimensional Euclidean space.

**Proof.** Let $B_1$ and $B_2$ be subsets of the $p$-dimensional and $(p + q)$-dimensional Euclidean spaces respectively.

Write the parameter space as $B_1 \times B_2 \times (0, \infty) \times (0, \infty)$.

Let $m_i$, $i = 1, \ldots, p$ be the roots of

$$m^p + a_1 m^{p-1} + \ldots + a_p = 0.$$

By assumption, the parameter space for the $m_i$ is
which is an open set in the p-dimensional Euclidean space. Since the mapping from \( C_1 \) to \( B_1 \) is continuous, \( B_1 \) is open in the p-dimensional Euclidean space.

The same reasoning applies to \( B_2 \), and the result follows. □

The function \( f(g', \beta', \sigma_{zz}, \sigma_{ee}) \) is twice continuously differentiable in the parameter space. Moreover, because the model (4.5), (4.6) is identified, the matrix of the partial derivatives of \( f(g', \beta', \sigma_{zz}, \sigma_{ee}) \) with respect to \( \beta \) and \( \sigma_{zz} \) is of full rank. Hence, the conditions of the implicit function theorem are satisfied.

This ensures the existence of the open neighborhoods \( N_1 \) of \((g', \sigma_{ee})\) and \( N_2 \) of \((\beta', \sigma_{zz})\) and of the unique solution \( \lambda(g', \sigma_{ee}) \), lying in \( N_1 \times N_2 \), to the equations (4.6). Furthermore, \( \lambda(g', \sigma_{ee}) \) is twice continuously differentiable in \( N_1 \). Hence, the parameters of the restricted model can be expressed as locally continuously differentiable functions of the parameters for the unrestricted model. Denote the i-th component of \( \lambda(g', \sigma_{ee}) \) by \( \beta_i(g', \sigma_{ee}) \), if \( i = 1, \ldots, p + q \), and by \( \sigma_{zz}(g', \sigma_{ee}) \) if \( i = p + q + 1 \).

Let \( \lambda^0 = (g^0', \sigma_{ee}^0) \), and let \( \lambda^0 \in N_1 \), an open set such that for any \( \lambda \in N_1 \),

\[
\begin{align*}
z_t(Y; \lambda^0) &= Y_t + \sum_{j=1}^{p} \alpha_j^0 Y_{t-j} - \sum_{j=1}^{p+q} \beta_j^0 \lambda^0_{t-j} Y_{t-j} (Y; \lambda^0), \\
t &= 1, \ldots, n
\end{align*}
\]
For $\theta \in N_1$, define

$$W_{it}(Y;\theta) = \begin{cases} \frac{\partial z_i(Y;\theta)}{\partial \alpha_i} , & i = 1, \ldots, p, t = 1, \ldots, n, \\ - \frac{\partial z_i(Y;\theta)}{\partial \sigma_{ee}} , & i = p + 1, t = 1, \ldots, n, \\ \frac{\partial a_{n+1}(Y;\theta)}{\partial \alpha_i} , & i = 1, \ldots, p, t = n + 1, \\ - \frac{\partial a_{n+1}(Y;\theta)}{\partial \sigma_{ee}} , & i = p + 1, t = n + 1, \end{cases}$$

and

$$W_t(Y;\theta) = [W_{1t}(Y;\theta), \ldots, W_{pt}(Y;\theta)] .$$

(4.41)

Let $\bar{\theta}$ be the estimator of $\theta^o$ obtained at step (b) in Section A.2. Then, the first-order Taylor expansion of $z_t(Y;\theta^o)$ about the point $\bar{\theta}$ is

$$z_t(Y;\theta^o) = z_t(Y;\bar{\theta}) - W_t(Y;\bar{\theta})(\theta^o - \bar{\theta})$$

$$+ 2^{-1}(\theta^o - \bar{\theta})'H_t(Y;\theta^*)(\theta^o - \bar{\theta}) ,$$

or equivalently,
\[ z_t(Y; \hat{\theta}) = W_t(Y; \hat{\theta})(\theta - \hat{\theta}) - 2^{-1}(\theta^0 - \hat{\theta})' H_t(Y; \hat{\theta}^*)(\theta^0 - \hat{\theta}) + z_t \]

(4.42)

where \( H_t(Y; \hat{\theta}^*) \) is the matrix of the second partial derivatives of \( z_t(Y; \hat{\theta}) \) with respect to \( \theta \) evaluated at \( \hat{\theta}^* \), and \( \hat{\theta}^* \) lies between \( \hat{\theta} \) and \( \theta \).

The Gauss-Newton estimator of \( \hat{\theta} \) is

\[ \hat{\theta} = \tilde{\theta} + \Delta \hat{\theta}, \]

(4.43)

where

\[ \Delta \hat{\theta} = \sum_{t=1}^{n+1} \left[ W_t(Y; \tilde{\theta}) W_t(Y; \tilde{\theta})^{-1} \right] \left[ \sum_{t=1}^{n} W_t(Y; \tilde{\theta}) z_t(Y; \tilde{\theta}) + W^t_{n+1}(Y; \tilde{\theta}) a_{n+1}(Y; \tilde{\theta}) \right] \]

(4.44)

**Theorem 4.6.** Let

\[ Y_t + \sum_{j=1}^{p} \alpha_j Y_{t-j} = z_t + \sum_{j=1}^{p+q} \beta_j z_{t-j}, \quad t = 1, \ldots, n, \]

\[ [2d_{\tilde{t}}]^{1/2} = [2d_{t}]^{1/2} + a_{n+1}, \]

\[ \mathcal{E}(\alpha', \beta', \sigma_{zz}, \sigma_{ee}) = 0, \]

where \( Y_0, Y_1, \ldots, Y_{p+1} \) are initial conditions, and \( Y_t \) is a
stationary process. Assume the \( z_t \) are \( \text{NI}(0, \sigma_{zz}) \), and the roots of
\[
m^p + \alpha_1 m^{p-1} + \ldots + \alpha = 0 \quad \text{and of} \quad m^{p+q} + \beta_1 m^{p+q-1} + \ldots + \beta = 0
\]
are less than unity in modulus. Let \( d_f = n - 2p - q \). Assume that \( \hat{\sigma}_{zz} \) is an estimator of \( \sigma_{zz} \) satisfying
\[\lim n^{1/2} (\hat{\sigma}_{zz} - \sigma_{zz}) = 0,\]

Let \( f(\alpha', \beta', \sigma_{zz}, \sigma_{ee}) \) be a mapping from \( \Omega \), an open subset of the \( (2p + q + 2) \)-dimensional Euclidean space, such that
\[
f_1(\alpha', \beta', \sigma_{zz}, \sigma_{ee}) = \sigma_{ee} + \sum_{j=0}^{p} \alpha_j \gamma_k (k-j) - \sigma_{zz} \sum_{j=0}^{p+q} \beta_j^2,
\]
\[
f_2(\alpha', \beta', \sigma_{zz}, \sigma_{ee}) = \sum_{j=0}^{p} \sum_{k=0}^{p} \alpha_j \gamma_k (k+j+1-1) - \sigma_{zz} \sum_{j=0}^{p+q-1} \beta_j^2,
\]
where the \( \gamma_k(h) \), \( h = 0, 1, \ldots, q \) are known. Let \( \tilde{\theta}_0 = (\tilde{\alpha}', \tilde{\beta}' \tilde{\sigma}_{ee}) \)
be an initial estimator of \( \theta \) satisfying \( \tilde{\theta}_0 - \theta = O_p(n^{-1/2}) \), and such that the roots of \( m^p + \sum_{j=1}^{p} \tilde{\alpha}_j m^{p-j} = 0 \) are less than unity in modulus. Let \( z_i, i = -q + 1, \ldots, 0 \) be bounded in probability. Then,
\[n^{1/2} (\hat{\theta} - \theta) \overset{L}{\rightarrow} N(0, \Omega \sigma_{zz}).\]
where $\hat{\theta}$ is the one-step Gauss-Newton estimator defined in (4.43),

$$M^{-1} = \operatorname{plim} n^{-1} \sum_{t=1}^{n} W_t^r(Y; \hat{\theta}) W_t^q(Y; \hat{\theta}),$$

and $W_t^r(Y; \hat{\theta})$ is the vector of derivatives defined in (4.41).

**Proof:** From (4.42) and (4.44),

$$\hat{\theta} - \theta = n^{1 \ldots 1} \left[ \sum_{t=1}^{n} W_t'(Y; \hat{\theta}) W_t(Y; \hat{\theta}) \right]^{-1} \left[ \sum_{t=1}^{n} W_t'(Y; \hat{\theta}) z_t + \sum_{t=n+1}^{n+1} W_t'(Y; \hat{\theta}) \right] \sigma_{n+1}(\hat{\theta})$$

$$+ R(Y; \hat{\theta})$$

where $W_t(Y; \hat{\theta})$ is defined in (4.41), the $r$-th element of $R(Y; \hat{\theta})$ is given by

$$R(Y; \hat{\theta})_{r} = \sum_{i=1}^{p+1} \sum_{j=1}^{p+1} W_{rt}(Y; \hat{\theta}) H_{ij, t}(Y; \theta^*) \left( \theta_i - \theta^*_i \right) \left( \theta_j - \theta^*_j \right), \quad (4.45)$$

and $H_{ij, t}(Y; \theta^*)$ is the negative of the derivative of $W_{rt}(Y; \hat{\theta})$ with respect to $\theta_j$ evaluated at $\theta^*$, a point lying between $\theta$ and $\hat{\theta}$.

By defining

$$\delta(\theta) = [a_1, \ldots, a_p, b_1(\theta), \ldots, b_{p+q}(\theta), \sigma_{zz}(\theta)]',$$

we obtain the partial derivatives $W_{rt}(Y; \hat{\theta})$, $r = 1, \ldots, p+1$, by the chain rule, and

$$W_t(Y; \hat{\theta}) = U_t(Y; \hat{\theta}) D(\theta), \quad t = 1, \ldots, n.$$
\[
\frac{\partial a_{n+1}(Y; \hat{\theta})}{\partial \hat{\theta}} - D(\theta), \quad t = n + 1,
\]

where \( U_t(Y; \hat{\theta}) = \left[ U_{1t}(Y; \hat{\theta}), \ldots, U_{2p+q}(Y; \hat{\theta}), 0 \right] \), \( U_{1t}(Y; \hat{\theta}) \) is the negative of the partial derivative of \( z_t(Y; \hat{\theta}) \) with respect to the \( i \)-th element of \( \hat{\theta} \), and \( D(\theta) = [d_{ij}(\theta)] \) is the matrix of derivatives of \( \delta(\theta) \) with respect to \( \theta \).

From (4.45), one can write the \( r \)-th element of \( R(Y; \theta) \) as

\[
\sum_{i=1}^{p+1} \sum_{j=1}^{p+1} \sum_{t=1}^{n+1} \left( \frac{\partial}{\partial \theta_i} \left[ \frac{\partial}{\partial \theta_j} U_t(Y; \hat{\theta}) \right] \right) (\theta_i - \hat{\theta}_i)(\theta_j - \hat{\theta}_j)
\]

where \( d_i(\theta) \) is the \( i \)-th column of \( D(\theta) \), and

\[
U_{n+1}(Y; \hat{\theta}) = [0, \ldots, (d_{i\theta})^{1/2} (2\sigma_{zz})^{-1/2}] .
\]

It has been shown in the proof of Theorem 4.4 that

\[
n^{-1} \sum_{t=1}^{n} U_t(Y; \hat{\theta}) \frac{\partial U_t(Y; \hat{\theta})}{\partial \hat{\theta}} = o_p(1) .
\]

Moreover, \( D(\theta) \) is twice continuously differentiable at \( \theta \), and

\[
\theta - \hat{\theta} = o_p(n^{-1/2}) . \quad \text{Thus,}
\]

\[
n^{-1} R(Y; \theta) = o_p(n^{-1}) . \quad (4.46)
\]
Now,

\[ n^{-1} \sum_{t=1}^{n+1} W_t^i(Y; \theta) W_t(Y; \theta) \]

\[ = n^{-1} \sum_{t=1}^{n+1} W_t^i(Y; \theta) W_t(Y; \theta) + o_p(n^{-1/2}), \]

and as \( n \to \infty \),

\[
\operatorname{plim} n^{-1} \sum_{t=1}^{n+1} W_t^i(Y; \theta) W_t(Y; \theta) = D'(\theta) \approx D(\theta) (2 \sigma_{zz})^{-1} \quad (4.47)
\]

where \( V \) is defined in (4.24), and \( G \) is defined in Theorem 4.4.

We can write

\[
\sigma_{zz}^{1/2} = \sigma_{zz}^{1/2} + (4 \sigma_{zz})^{-1/2} (\sigma_{zz} - \sigma_{zz}) + o_p(n^{-1})
\]

\[
= \sigma_{zz}^{1/2} + (4 \sigma_{zz})^{-1/2} n^{-1} \sum_{t=1}^{n} (z_t^2 - \sigma_{zz}) + o_p(n^{-1})
\]

and hence,

\[
a_{n+1} = (d_x) \frac{1}{2} n^{-1/2} (2 \sigma_{zz})^{-1/2} n \sum_{t=1}^{n} (z_t^2 - \sigma_{zz}) + o_p(n^{-1/2}),
\]

and

\[
U_{n+1}(Y; \theta) a_{n+1} = [0, \ldots, (2 \sigma_{zz})^{-1} n \sum_{t=1}^{n} (z_t^2 - \sigma_{zz})] + o_p(n^{-1/2}).
\]
Now,

\[ n^{-1/2} \left[ \sum_{t=1}^{n} W_t'(Y; \hat{\theta}) z_t + W_{n+1}'(Y; \hat{\theta}) a_{n+1} \right] \]

\[ = n^{-1/2} \left[ \sum_{t=1}^{n} W_t'(Y; \hat{\theta}) z_t + W_{n+1}'(Y; \hat{\theta}) a_{n+1} \right] + o_p(1) \]

\[ = D'(\theta) n^{-1/2} \left[ \sum_{t=1}^{n} U_t'(Y; \theta) z_t + U_{n+1}'(Y; \theta) a_{n+1} \right] + o_p(1) \]

\[ = D'(\theta) n^{-1/2} \sum_{t=1}^{n} U_t'(Y; \theta', \hat{\theta}') z_t (2\sigma_{zz}^{-1}(z_t^2 - \sigma_{zz})) + o_p(1) \]

(4.48)

where \( U_t(Y; \theta', \hat{\theta}') \) is defined in (4.12).

Therefore, from (4.46), (4.47) and (4.48), the limiting distribution of \( n^{1/2} (\hat{\theta} - \theta) \) is the same as the limiting distribution of

\[ [D'(\theta) G^{-1} D(\theta)]^{-1} D'(\theta) n^{-1/2} \sum_{t=1}^{n} U_t'(Y; \theta', \hat{\theta}') z_t (2\sigma_{zz}^{-1}(z_t^2 - \sigma_{zz})) \]

The result follows from Theorem 4.4. \qed
C. Error Autocovariance Function Estimated

The estimation procedure described in Section IV.A assumed the error covariance structure to be known. We now investigate the effect of using a survey estimator of the autocovariance function on the estimation of the remaining parameters.

Let \( \hat{\chi} \) be an estimator of \( \chi = [Y_u(0), \ldots, Y_u(q)]' \), and let \( d^{-1} \Sigma_{YY} \) be an estimator of the covariance matrix of \( (\chi - \chi) \) satisfying \( \Sigma_{YY} \xrightarrow{P} \hat{\Sigma}_{YY} \). Given a sample of \( n \) observations, we construct an estimator of \( \tilde{\eta} = (a', \sigma_{ee}, \chi') \) by solving the expanded system

\[
Y_t + \sum_{j=1}^{p+q} a_j Y_{t-j} = z_t + \sum_{j=1}^{p+q} b_j z_{t-j}, \quad t = 1, \ldots, n
\]

\[
[2d_f \tilde{\sigma}_{zz}]^{1/2} = [2d_f \sigma_{zz}]^{1/2} + a \tilde{r}_{n+1} \tag{4.49}
\]

\[
\mathcal{L}(a', b', \sigma_{zz}, \sigma_{ee}, \chi') = 0
\]

\[
\tilde{\chi} = \chi + \tilde{r}_{n+2},
\]

where \( \tilde{r}_{n+2} = (r_{n+2}, \ldots, r_{n+2+q})' \) is the estimation error of \( \tilde{\chi} \), and \( \mathcal{L}(a', b', \sigma_{zz}, \sigma_{ee}, \chi') = 0 \) denotes the set of restrictions in (4.6).

It is worth noting that, in this model, \( \chi \) is an argument of \( \mathcal{L}(a', b', \sigma_{ee}, \sigma_{zz}, \chi') \).
In our notation, model (4.49) becomes

\[ z_t(Y; \eta) = y_t + \sum_{j=1}^{p} a_j y_{t-j} - \sum_{j=1}^{p+q} \beta_j(\eta) z_{t-j}, \quad t = 1, \ldots, n \]

\[ a_{n+1}(Y; \eta) = [2d_x \sigma_Z^2]\frac{1}{2} - [2d_x \sigma_Z(\eta)]\frac{1}{2} \]

\[ a_{n+2}(Y; \eta) = d \frac{1}{2} \sum_{t=1}^{n} y_t^2 - d \frac{1}{2} \sum_{t=n+1}^{n+j} y_t^2 \]

Define

\[ \psi_t(Y; \eta) = \frac{\partial z_t(Y; \eta)}{\partial \eta}, \quad t = 1, \ldots, n \]

\[ = \frac{\partial a_{n+j}(Y; \eta)}{\partial \eta}, \quad t = n + j, \quad j = 1, \ldots, q + 2. \]

Let \( \bar{\eta} \) be the initial estimator of \( \eta \). Thus, the one-step Gauss–Newton estimator is defined by

\[ \tilde{\eta} = \bar{\eta} + \Delta \tilde{\eta}, \quad \text{(4.51)} \]

where

\[ \Delta \tilde{\eta} = \left[ \sum_{t=1}^{m} \psi_t(Y; \tilde{\eta}) \psi_t(Y; \tilde{\eta}) \right]^{-1} \left[ \sum_{t=1}^{n} \psi_t(Y; \tilde{\eta}) z_t(Y; \tilde{\eta}) \right] \]

\[ + \sum_{t=n+1}^{m} \psi_t(Y; \tilde{\eta}) a_t(Y; \tilde{\eta}) ] \]
and \( m = n + q + 2 \).

**Theorem 4.7.** Let

\[
Y_t + \sum_{j=1}^{p} a_j Y_{t-j} = z_t + \sum_{j=1}^{p+q} \beta_j z_{t-j}, \quad t = 1, \ldots, n
\]

\[
[2d_f \tilde{\sigma}_{zz}]^{1/2} = [2d_f \sigma_{zz}]^{1/2} + \alpha_{n+1}
\]

\[
\chi = \chi + \Sigma_{n+2}
\]

\[
\sigma_{ee} + \sum_{j=0}^{p} \sum_{k=0}^{p} a_j a_k \gamma_u(k-j) = \sigma_{zz} \sum_{j=0}^{p+q} \beta_j^2
\]

\[
\sum_{j=0}^{p} \sum_{k=0}^{p} a_j a_k \gamma_u(k-j+i-1) = \sigma_{zz} \sum_{j=0}^{p+q-i+1} \beta_j^2 \beta_{j+i-1}, \quad i = 2, \ldots, p + q + 1,
\]

where \( z_t \) are \( N(0, \sigma_{zz}) \), \( Y_0, Y_{-1}, \ldots, Y_{-p+1} \) are initial conditions, and \( Y_t \) is a stationary process. Assume the roots of the characteristic polynomial associated with \( Y_t \) are less than unity in absolute value. Let \( \tilde{\sigma}_{zz} \) be an estimator of \( \sigma_{zz} \) satisfying

\[
\tilde{\sigma}_{zz} - n^{-1} \sum_{t=1}^{n} z_t^2 = O_p(n^{-1}).
\]

Let \( \tilde{\gamma} \) be an estimator of \( \gamma = [\gamma_u(0), \ldots, \gamma_u(q)]' \), \( \gamma_u(k) = E\{u_t u_{t+k}\} \), such that
\[
d \frac{1}{2} \left( \hat{\gamma} - \gamma \right) \overset{L}{\rightarrow} N(0, \sigma_{zz} \Sigma_{\gamma \gamma}) ,
\]

where \( d \) satisfies \( \lim_{n \to \infty} n d^{-1} = c, \ 0 < c < \infty \). Assume there exists a sequence of matrices \( \hat{\Sigma}_{\gamma \gamma} \) converging to \( \Sigma_{\gamma \gamma} \) in probability and that the distribution of \( \hat{\gamma} \) is independent of the distribution of \( Y_t \).

Let \( \hat{\eta} \) be an initial estimator of \( \eta \) satisfying \( \hat{\eta} - \eta = o_p(n^{-1/2}) \), and let \( z^*_t, i = p - q + 1, \ldots, 0 \) be bounded in probability. Then,

\[
n \frac{1}{2} (\hat{\eta} - \eta) \overset{L}{\rightarrow} N(0, \Omega \sigma_{zz}) ,
\]

where \( \hat{\eta} \) is defined in (4.51), \( V_t(Y;\hat{\eta}) \) is defined in (4.50) and

\[
\hat{\Omega}^{-1} = \text{plim} \ n^{-1} \sum_{t=1}^{m} V_t(Y;\hat{\eta}) V_t(Y;\hat{\eta}) .
\]

**Proof.** The proof parallels that of Theorem 4.4. Let \( \hat{\xi} = (\hat{\alpha}', \hat{\beta}', \sigma_{zz})' \). From the discussion in Section IV.B, we can write

\[
V_t(Y;\hat{\eta}) = V_t(Y;\hat{\xi}) [D_1(\eta) \ D_2(\eta)] , \ t = 1, \ldots, n + 1 ,
\]

where

\[
V_t(Y;\hat{\xi}) = \begin{bmatrix} \frac{\partial z_t(Y;\hat{\xi})}{\partial \alpha_1} , & \ldots , & \frac{\partial z_t(Y;\hat{\xi})}{\partial \beta_{p+q}} , & 0 \end{bmatrix} , \ t = 1, \ldots, n
\]

\[
V_{n+1}(Y;\hat{\xi}) = [0, \ldots, 0, (d_{\eta})^{1/2} (2\sigma_{zz})^{-1/2}]
\]

and \( D_1(\eta) \) and \( D_2(\eta) \) are the matrices of derivatives of \( \hat{\xi}(\eta) \) with
respect to \( \alpha_1, \ldots, \alpha_p, \sigma_{ee} \) and \( \gamma_u(0), \ldots, \gamma_u(q) \), respectively.

Therefore,

\[
\begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix} = n^{-1} \sum_{t=1}^{m} Y'_t(Y; \theta)Y_t(Y; \theta),
\]

where

\[
V_{11} = D_1^*(\eta) \left[ \sum_{t=1}^{n+1} U'_t(Y; \xi)U_t(Y; \xi) \right] D_1^*(\eta),
\]

\[
V_{12} = D_1^*(\eta) \left[ \sum_{t=1}^{n+1} U'_t(Y; \xi)U_t(Y; \xi) \right] D_2^*(\eta),
\]

\[
V_{22} = D_2^*(\eta) \left[ \sum_{t=1}^{n+1} U'_t(Y; \xi)U_t(Y; \xi) \right] D_2^*(\eta) + d \Sigma_{YY}^{-1},
\]

\[m = n + q + 2.\]

Because the elements of \( U'_t(Y; \xi) \) converge to stationary autoregressive time series and because \( 0 < \lim_{n \to \infty} n d^{-1} < \infty \), the matrix \( \Omega \) is well defined. The proof of normality follows the proof of Theorem 4.4.

\[\square\]

**D. Estimation of the True Values with Estimated Parameters**

Constructing predictions of \( \mathbf{X}_T^* = (X_T, X_{T-1}, \ldots, X_1)' \) for the model (4.1) is of particular interest when one is dealing with rotation.
sample designs. Given knowledge of the parameters, one can construct a predictor of $X_T$ superior to $X_T$. However, in general, the parameters of the time series are unknown and they are replaced by their estimators in the prediction formulas. In this section, we investigate the use of estimated parameters in the estimation of the true process $X_t$.

Consider the model

$$Y_t = X_t + u_t, \; t = 1, 2, \ldots,$$

$$X_t + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} = e_t, \; t = 1, 2, \ldots, \quad (4.52)$$

$$u_t = v_t + b_1 v_{t-1} + \ldots + b_q v_{t-q}, \; t = 1, 2, \ldots,$$

where $(v_t, e_t)'$ is a sequence of independent normal random vectors with mean zero and covariance matrix $\Sigma = \text{diag}(\sigma_{vv}, \sigma_{ee})$, and $X_t$ is a stationary process.

Under this model, the covariance matrix of $Y_T = (Y_1, \ldots, Y_T)'$ is $\Sigma_{XX} + \Sigma_{uu}$, where $\Sigma_{XX}$ and $\Sigma_{uu}$ are the covariance matrices of $X_T$ and $u_T = (u_T, \ldots, u_1)'$, respectively. When the parameters are known, the stochastic least squares theory gives the best linear unbiased predictor of $X_T$ as

$$\hat{X}_T = \Sigma_{XX}(\Sigma_{XX} + \Sigma_{uu})^{-1}Y_T$$

$$= [I - \Sigma_{uu}(\Sigma_{XX} + \Sigma_{uu})^{-1}]Y_T \quad (4.53)$$
and

$$\text{Var}(\hat{X}_T - X_T) = \Sigma_{XX} - \Sigma_{XX}(\Sigma_{XX} + \Sigma_{uu})^{-1}\Sigma_{XX}.$$ 

Expression (4.53) confirms one's expectation that the estimator of $X_T$ is constructed by subtracting the predictor of $u_T$ from the observed values $Y_T$.

The covariance matrices $\Sigma_{XX}$ and $\Sigma_{uu}$ depend on the parameter values. Substituting the estimators for the respective parameters in (4.47) gives

$$\hat{X}_T = \hat{\Sigma}_{XX}(\hat{\Sigma}_{XX} + \hat{\Sigma}_{uu})^{-1}Y_T.$$  \hspace{1cm} (4.54)

**Theorem 4.8.** Let the model (4.52) hold. Define

$\gamma_u(k) = E[u_{t+k}^tu_t]$ and $\gamma_u = [\gamma_u(0), \ldots, \gamma_u(q)]'$. Let $(\hat{\alpha}', \hat{\gamma}')$ be an estimator of $(\alpha', \gamma)$ such that $(\hat{\alpha}' - \alpha', \hat{\gamma}' - \gamma') = O_p(T^{-1/2})$. Then,

$$\hat{X}_T - X_T = O_p(T^{-1/2}),$$

where $\hat{X}_T$ and $X_T$ are defined in (4.53) and (4.54), respectively.

**Proof.** From (4.54),

$$\hat{X}_T = \gamma_T - \Sigma_{uu}(\Sigma_{XX} + \Sigma_{uu})^{-1}Y_T$$

$$- [\Sigma_{uu}(\Sigma_{XX} + \Sigma_{uu})^{-1} - \Sigma_{uu}(\Sigma_{XX} + \Sigma_{uu})^{-1}]Y_T.$$
For all $t$, the elements of $\Sigma_{XX}$ and $\Sigma_{uu}$ are continuously differentiable functions at the point $(q', \chi')$. Because 

$$(\hat{q}' - q', \hat{\chi}' - \chi') = O_p(T^{-1/2})$$

it follows that

$$\hat{\Sigma}_{uu}(\hat{\Sigma}_{XX} + \hat{\Sigma}_{uu})^{-1} = \Sigma_{uu}(\Sigma_{XX} + \Sigma_{uu})^{-1} + O_p(T^{-1/2})$$

Since $Y_t$ is a stationary process, the result follows.

Kalman filtering, introduced in Section III.A, is an efficient computational method for obtaining the estimated $X_{zt}$.

E. Extension to Nonnormal Distributions

The derivations of the preceding sections were based on the assumption that the time series were normal. A number of results can be obtained without the normality assumption. First, the asymptotic covariance matrix of $(q', \hat{\theta}')$ is the same for $\{e_t\}$ and $\{u_t\}$ that are independent sequences, each a sequence of independent and identically distributed random variables with finite $(2 + \nu)$-th moments, $\nu > 0$. The initial estimator of $\sigma_{z^2}$ obtained at step (a) in Section IV.A.1 is asymptotically normally distributed for $z_t$ with finite fourth moment. However, the variance of the limiting distribution of $n^{1/2}(\tilde{\sigma}_{z^2} - \sigma_{z^2})$ is no longer $2 \sigma_{z^2}^2$. Also, $n^{1/2}(\tilde{\sigma}_{z^2} - \sigma_{z^2})$ is asymptotically independent of $n^{1/2}[(\tilde{q} - q)', (\tilde{\theta} - \theta)']$ if and only if the distribution of $z_t$ is symmetric.
Assume that \( \{z_t\} \) is a sequence of independent and identically distributed random variables with finite fourth moments. Assume that the estimators are constructed by the procedure in Section IV.A.1. Then these estimators have errors that are \( O_p(n^{-1/2}) \), but the estimated covariance matrix is not a consistent estimator of the covariance matrix of the estimator.

Let

\[
\hat{z}_t = z_t(Y; \theta)
\]

be the estimated residual computed with the estimates obtained at step (b) in Section IV.A.1. Then, for \( z_t \) with finite eighth moments,

\[
\hat{\kappa}_3 = (n - p)^{-1} \sum_{t=1}^{n} (z_t^2 - \hat{\sigma}_{zz}) z_t
\]

\[
\hat{\kappa}_4 = (n - p)^{-1} \sum_{t=1}^{n} (z_t^2 - \hat{\sigma}_{zz})^2,
\]

where

\[
\hat{\sigma}_{zz} = n^{-1} \sum_{t=1}^{n} z_t^2,
\]

are consistent estimators of the covariance of \( z_t \) and \( z_t^2 \) and of the variance of \( z_t^2 \), respectively.

If one is willing to postulate a distribution for the \( z_t \), alternative consistent estimators of the third and fourth moments of
$z_t$ can be calculated. Under the general moment assumptions the approximate covariance matrix of the error in $(Y_1, Y_2, \ldots, Y_n, \tilde{\sigma}_{zz})$ is

$$
\begin{pmatrix}
I & \sigma_{zz} \\
\sigma_{zz} & n^{-1}J\kappa_3 \\
J\kappa_3 & n^{-1}J\kappa_4 \\
\end{pmatrix},
$$

where $J$ is an $n$-dimensional column vector of ones, $\kappa_3$ is the third moment of $z_t$ and $\kappa_4$ is the second moment about the mean of $z_t^2$. Using a consistent estimator of this covariance matrix, one can compute the generalized nonlinear least squares estimators defined in Section IV.A.2. The estimated nonlinear least squares estimator of the covariance matrix of the estimators is a consistent estimator of the covariance matrix of the limiting distribution.
V. A SIMULATION STUDY

Asymptotic properties of the least squares estimators were derived in the previous chapter. In this chapter, a Monte Carlo study is presented. The Monte Carlo study had two aims:

(i) to assess the accuracy of the approximate distributions of the estimators;

(ii) to compare the estimators of Sections A.1 and A.2 of Chapter IV.

The cases of a first-order autoregressive signal plus white noise and a first-order autoregressive signal plus a first-order moving average noise were investigated in the Monte Carlo experiment. All computations were performed using the SAS package. Normal random variables were generated using the function NORMAL. Step (a) in Section IV.A.1 and the procedure of Section IV.A.2 were computed using the procedure MATRIX, whereas step (b) of Section IV.A.1 used the procedure NLIN.

The time series used in the study has the form

\[ Y_t = X_t + u_t \]

where

\[ X_t + a X_{t-1} = e_t , \]
\[ u_t = v_t + b v_{t-1}, \]

\((e_t, v_t)\) are \(NI(0, \Sigma)\), and \(\Sigma = \text{diag}(\sigma_{ee}, \sigma_{vv})\).

The initial observation for the process \(\{X_t\}\) was generated by

\[ X_0 = (1 - \alpha^2)^{-\frac{1}{2}} e_0, \]

and the remaining observations as

\[ X_t + \alpha X_{t-1} = e_t, \quad t = 1, 2, \ldots, n. \]

Series of length 30 and 100 observations were considered. In both cases, the results are based on 100 samples. The constant \(c\) used in the computation was 400 and 600 for the preliminary and final estimates, respectively. For each replication, we recorded the estimates obtained at the three stages of the estimation procedure and the values of the auxiliary statistics

\[ P_1 = \sum_{t=1}^{n} Y_t^2, \quad P_2 = \sum_{t=1}^{n} (e_t + v_t)^2, \]

and of

\[ P_3 = \sum_{t=2}^{n} (e_t + v_t)(e_{t-1} + v_{t-1}). \]

The estimators \((\hat{\alpha}, \hat{\sigma}_{ee})\), \((\tilde{\alpha}, \tilde{\sigma}_{ee})\) and \((\ddot{\alpha}, \ddot{\sigma}_{ee})\) are the estimators obtained at steps (a) and (b) in Section IV.A.1, and the estimator of
Section IV.A.2., respectively. The initial values $z_{-1}$ and $z_0$ used in the estimation procedure were set equal to zero. In the computations, the estimates were constrained to lie within the parameter space. The three sets of parameter values used in this experiment are given in Table 5.1.

Table 5.1. Parameter values in the Monte Carlo study

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\sigma_{\epsilon\epsilon}$</th>
<th>$b$</th>
<th>$\sigma_{v\nu}$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.8</td>
<td>1.00</td>
<td>0.0</td>
<td>1.00</td>
<td>15</td>
</tr>
<tr>
<td>-0.8</td>
<td>1.00</td>
<td>0.0</td>
<td>1.00</td>
<td>30</td>
</tr>
<tr>
<td>-0.8</td>
<td>1.00</td>
<td>0.0</td>
<td>1.00</td>
<td>100</td>
</tr>
<tr>
<td>-0.95</td>
<td>1.00</td>
<td>0.5</td>
<td>1.00</td>
<td>15</td>
</tr>
<tr>
<td>-0.95</td>
<td>1.00</td>
<td>0.5</td>
<td>1.00</td>
<td>100</td>
</tr>
</tbody>
</table>

We first investigate the correlation of the estimators and the variables $P_1$, $P_2$, and $P_3$. Table 5.2 displays the sample multiple correlation coefficient of the different estimators with $(P_1, P_2, P_3)$ for the three sets of parameter values. An inspection of Table 5.2 suggests the use of the regression estimators to obtain estimates of the mean of the empirical distribution of $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$. The regression estimators are of the form $\sum_{i=1}^{100} c_i \tilde{\alpha}_i$, $\sum_{i=1}^{100} c_i \tilde{\beta}_i$, and $\sum_{i=1}^{100} c_i \tilde{\gamma}_i$, where the $c_i$ are functions of $P_1$, $P_2$, and $P_3$. We computed the weights in the regression estimators using the program MWEIGHTS.
developed by Huang (1978). The Huang procedure is a refinement of regression estimator as described in Cochran (1977). This program has

Table 5.2. Sample multiple correlation coefficients of \((P_1, P_2, P_3)\) with the estimators of Section A.1 and A.2

<table>
<thead>
<tr>
<th>((n, a, b))</th>
<th>(\hat{\alpha})</th>
<th>(\ddot{\alpha})</th>
<th>(\hat{\alpha})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((15, -0.8, 0.0))</td>
<td>0.39</td>
<td>0.55</td>
<td>0.58</td>
</tr>
<tr>
<td>((30, -0.8, 0.0))</td>
<td>0.42</td>
<td>0.38</td>
<td>0.46</td>
</tr>
<tr>
<td>((100, -0.8, 0.0))</td>
<td>0.34</td>
<td>0.61</td>
<td>0.62</td>
</tr>
<tr>
<td>((15, -0.95, 0.5))</td>
<td>0.38</td>
<td>0.59</td>
<td>0.60</td>
</tr>
<tr>
<td>((100, -0.95, 0.5))</td>
<td>0.36</td>
<td>0.48</td>
<td>0.50</td>
</tr>
</tbody>
</table>

the advantage of calculating positive weights. Moreover, Huang showed that under simple random sampling, the regression estimator with nonnegative weights has the same large sample properties as the usual regression estimator. All estimates of bias and mean square error are the regression estimators.

First, consider the estimation of \(\alpha\). Table 5.3 displays the empirical bias of the different estimators. All three estimators are positively biased. The biases of \(\ddot{\alpha}\) and \(\hat{\alpha}\) are very similar. The bias for \(n = 30\) is about three times larger than that for \(n = 100\). For \(n = 15\), the biases are at least five times larger than that for \(n = 100\).
Table 5.3. Empirical bias of estimators of \( \alpha \) obtained for a sample of 100 samples

<table>
<thead>
<tr>
<th>((n, \alpha, b))</th>
<th>(\tilde{\alpha})</th>
<th>(\hat{\alpha})</th>
<th>(\hat{\alpha})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((15, -0.8, 0.0))</td>
<td>0.133</td>
<td>0.113</td>
<td>0.129</td>
</tr>
<tr>
<td></td>
<td>(0.042)</td>
<td>(0.027)</td>
<td>(0.027)</td>
</tr>
<tr>
<td>((30, -0.8, 0.0))</td>
<td>0.059</td>
<td>0.034</td>
<td>0.036</td>
</tr>
<tr>
<td></td>
<td>(0.017)</td>
<td>(0.014)</td>
<td>(0.014)</td>
</tr>
<tr>
<td>((100, -0.8, 0.0))</td>
<td>0.022</td>
<td>0.0110</td>
<td>0.0107</td>
</tr>
<tr>
<td></td>
<td>(0.0094)</td>
<td>(0.0069)</td>
<td>(0.0067)</td>
</tr>
<tr>
<td>((15, -0.95, 0.5))</td>
<td>0.171</td>
<td>0.139</td>
<td>0.129</td>
</tr>
<tr>
<td></td>
<td>(0.033)</td>
<td>(0.027)</td>
<td>(0.027)</td>
</tr>
<tr>
<td>((100, -0.95, 0.5))</td>
<td>0.0245</td>
<td>0.0216</td>
<td>0.0218</td>
</tr>
<tr>
<td></td>
<td>(0.0056)</td>
<td>(0.0051)</td>
<td>(0.0050)</td>
</tr>
</tbody>
</table>

The empirical mean square errors multiplied by 100 are shown in Table 5.4. The last column lists the corresponding values of the asymptotic distribution of \( \tilde{\alpha} \) and \( \hat{\alpha} \). Both \( \tilde{\alpha} \) and \( \hat{\alpha} \) are much superior to the preliminary estimator. The estimator \( \hat{\alpha} \) has a marginally smaller mean square error.

For each of the 100 samples, we also computed the "t-statistics"

\[
\tilde{\tau} = \frac{\tilde{\alpha} - \alpha}{\text{s.e.}(\tilde{\alpha})}, \quad \hat{\tau} = \frac{\hat{\alpha} - \alpha}{\text{s.e.}(\hat{\alpha})} \quad \text{and} \quad \hat{\tau} = \frac{\hat{\alpha} - \alpha}{\text{s.e.}(\hat{\alpha})},
\]

where the estimated standard errors are estimators of the standard errors of the limiting distributions. Table 5.5 shows that for the sample sizes used in this study, the empirical distributions of
Table 5.4. Empirical mean square error of $\tilde{a}$, $\bar{a}$ and $\hat{a}$ multiplied by 100

<table>
<thead>
<tr>
<th>$(n, a, b)$</th>
<th>$\tilde{a}$</th>
<th>$\bar{a}$</th>
<th>$\hat{a}$</th>
<th>$a_{asy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(15, -0.8, 0.0)$</td>
<td>7.21</td>
<td>5.30</td>
<td>5.27</td>
<td>3.97</td>
</tr>
<tr>
<td></td>
<td>(1.02)</td>
<td>(0.60)</td>
<td>(0.56)</td>
<td></td>
</tr>
<tr>
<td>$(30, -0.8, 0.0)$</td>
<td>3.47</td>
<td>2.04</td>
<td>2.04</td>
<td>1.99</td>
</tr>
<tr>
<td></td>
<td>(0.56)</td>
<td>(0.27)</td>
<td>(0.23)</td>
<td></td>
</tr>
<tr>
<td>$(100, -0.8, 0.0)$</td>
<td>1.27</td>
<td>0.752</td>
<td>0.74</td>
<td>0.596</td>
</tr>
<tr>
<td></td>
<td>(0.32)</td>
<td>(0.083)</td>
<td>(0.10)</td>
<td></td>
</tr>
<tr>
<td>$(15, -0.95, 0.5)$</td>
<td>3.51</td>
<td>2.16</td>
<td>1.99</td>
<td>1.76</td>
</tr>
<tr>
<td></td>
<td>(1.10)</td>
<td>(0.59)</td>
<td>(0.55)</td>
<td></td>
</tr>
<tr>
<td>$(100, -0.95, 0.5)$</td>
<td>0.54</td>
<td>0.298</td>
<td>0.288</td>
<td>0.264</td>
</tr>
<tr>
<td></td>
<td>(0.14)</td>
<td>(0.049)</td>
<td>(0.048)</td>
<td></td>
</tr>
</tbody>
</table>

$\tilde{r}$, $\bar{r}$ and $\hat{r}$ are slightly skewed. The empirical probabilities were computed by adding up the weights assigned by MWEIGHTS to the observations lying on the tails of the distributions. Note that the values of the empirical probabilities that $\tilde{r} \in [-a, a]$, for $a = 1.64$, 1.78, 1.96 and 2.24, can be approximated by the corresponding probabilities computed from the standard normal, namely, 0.90, 0.925, 0.95 and 0.975. Hence, despite the fact that the distribution of $\hat{r}$ is skewed, the use of the standard normal approximation in two-sided tests will result in modest deviations from nominal error rates. Furthermore, the approximation error becomes smaller as the sample size increases.
Table 5.5. Empirical tail probabilities for $\tilde{\tau}$, $\bar{\tau}$ and $\hat{\tau}$ obtained for a sample of 100 samples

<table>
<thead>
<tr>
<th>Region</th>
<th>$\tilde{\tau}$</th>
<th>$\bar{\tau}$</th>
<th>$\hat{\tau}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(15, -0.8, 0.0)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt;-1.64, &gt; 1.64$</td>
<td>0.030, 0.040</td>
<td>0.020, 0.050</td>
<td>0.040, 0.040</td>
</tr>
<tr>
<td>$&lt;-1.78, &gt; 1.78$</td>
<td>0.020, 0.040</td>
<td>0.010, 0.040</td>
<td>0.030, 0.040</td>
</tr>
<tr>
<td>$&lt;-1.96, &gt; 1.96$</td>
<td>0.010, 0.040</td>
<td>0.000, 0.040</td>
<td>0.020, 0.030</td>
</tr>
<tr>
<td>$&lt;-2.24, &gt; 2.24$</td>
<td>0.000, 0.020</td>
<td>0.000, 0.040</td>
<td>0.010, 0.020</td>
</tr>
<tr>
<td>$(30, -0.8, 0.0)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt;-1.64, &gt; 1.64$</td>
<td>0.048, 0.011</td>
<td>0.060, 0.017</td>
<td>0.059, 0.027</td>
</tr>
<tr>
<td>$&lt;-1.78, &gt; 1.78$</td>
<td>0.048, 0.000</td>
<td>0.041, 0.008</td>
<td>0.049, 0.018</td>
</tr>
<tr>
<td>$&lt;-1.96, &gt; 1.96$</td>
<td>0.038, 0.000</td>
<td>0.039, 0.000</td>
<td>0.039, 0.000</td>
</tr>
<tr>
<td>$&lt;-2.24, &gt; 2.24$</td>
<td>0.030, 0.000</td>
<td>0.030, 0.000</td>
<td>0.020, 0.000</td>
</tr>
<tr>
<td>$(100, -0.8, 0.0)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt;-1.64, &gt; 1.64$</td>
<td>0.045, 0.021</td>
<td>0.046, 0.012</td>
<td>0.068, 0.021</td>
</tr>
<tr>
<td>$&lt;-1.78, &gt; 1.78$</td>
<td>0.045, 0.021</td>
<td>0.046, 0.000</td>
<td>0.060, 0.009</td>
</tr>
<tr>
<td>$&lt;-1.96, &gt; 1.96$</td>
<td>0.020, 0.013</td>
<td>0.034, 0.000</td>
<td>0.046, 0.000</td>
</tr>
<tr>
<td>$&lt;-2.24, &gt; 2.24$</td>
<td>0.020, 0.000</td>
<td>0.000, 0.000</td>
<td>0.012, 0.000</td>
</tr>
</tbody>
</table>
Now, consider the estimation of $\sigma_{ee}$. Recall that in the preliminary stage of estimation, $\sigma_{ee}$ is estimated only at step (b). Therefore, there are only two estimators to be considered: $\hat{\sigma}_{ee}$ and $\hat{\sigma}_{ee}$. The empirical biases are given in Table 5.6, with all of them being less than one standard error. The estimator $\hat{\sigma}_{ee}$ has the smallest empirical bias, but the differences are very small.

Table 5.7 contains one hundred times the mean square error of the estimators of $\sigma_{ee}$ and the corresponding values of the approximate distribution. Both estimators have mean square error larger than the
Table 5.6. Empirical bias of $\hat{\sigma}_{ee}$ and $\hat{\sigma}_{ee}$ obtained for a sample of 100 samples

<table>
<thead>
<tr>
<th>$(n, a, b)$</th>
<th>$\hat{\sigma}_{ee}$</th>
<th>$\hat{\sigma}_{ee}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(15, -0.8, 0.0)</td>
<td>0.012 (0.078)</td>
<td>0.010 (0.078)</td>
</tr>
<tr>
<td>(30, -0.8, 0.0)</td>
<td>0.009 (0.054)</td>
<td>0.006 (0.051)</td>
</tr>
<tr>
<td>(100, -0.8, 0.0)</td>
<td>0.004 (0.032)</td>
<td>0.002 (0.032)</td>
</tr>
<tr>
<td>(15, -0.95, 0.5)</td>
<td>0.010 (0.061)</td>
<td>0.009 (0.060)</td>
</tr>
<tr>
<td>(100, -0.95, 0.5)</td>
<td>0.002 (0.032)</td>
<td>0.002 (0.032)</td>
</tr>
</tbody>
</table>

Table 5.7. Empirical mean square error of $\hat{\sigma}_{ee}$ and $\hat{\sigma}_{ee}$

<table>
<thead>
<tr>
<th>$(n, a, b)$</th>
<th>$\hat{\sigma}_{ee}$</th>
<th>$\hat{\sigma}_{ee}$</th>
<th>$\hat{\sigma}_{ee}$ asy</th>
</tr>
</thead>
<tbody>
<tr>
<td>(15, -0.8, 0.0)</td>
<td>67.0 (12.0)</td>
<td>63.7 (9.9)</td>
<td>61.3</td>
</tr>
<tr>
<td>(30, -0.8, 0.0)</td>
<td>32.8 (4.3)</td>
<td>32.7 (4.5)</td>
<td>30.8</td>
</tr>
<tr>
<td>(100, -0.8, 0.0)</td>
<td>9.7 (1.4)</td>
<td>9.8 (1.3)</td>
<td>9.2</td>
</tr>
<tr>
<td>(15, -0.95, 0.5)</td>
<td>69.8 (9.9)</td>
<td>65.1 (9.0)</td>
<td>64.2</td>
</tr>
<tr>
<td>(100, -0.95, 0.5)</td>
<td>9.88 (0.80)</td>
<td>9.84 (0.81)</td>
<td>9.63</td>
</tr>
</tbody>
</table>
mean square error of the limiting distribution. As expected, this characteristic is more evident in small samples. For samples of size $n = 15$, the estimator $\hat{\sigma}_{ee}$ has smaller mean square error. For the larger samples in this study, the estimator $\hat{\sigma}_{ee}$ has smaller mean square error in two out of the three sets of parameters, but the differences are small.

For the sample sizes of this experiment, we conclude that there are only modest differences between the estimator $({\bar{\alpha}}, \hat{\sigma}_{ee})$ and the estimator $({\bar{\alpha}}, \hat{\sigma}_{ee})$. 
VI. APPLICATION TO THE NATIONAL CRIME SURVEY

The survey under study is the National Crime Survey (N.C.S.), a nationwide general population survey conducted monthly since July of 1972, by the U.S. Bureau of the Census for the Law Enforcement and Administration of the U.S. Department of Justice. The purpose of the N.C.S. is to estimate the occurrence of certain types of crime and to investigate the nature of criminal incidents and identify the types of persons that are victims of crimes.

A. Sample Design

The survey is based on a stratified multistage cluster design whose objective is to obtain a self-weighting probability sample of approximately 75,000 households. The stages of sampling are as follows. The primary sampling units (PSUs) were formed from the counties or groups of contiguous counties in the entire United States. These PSUs were then combined into 367 strata: 156 of them consisting of only one PSU, and the remaining 220 being the combination of PSUs with similar demographic characteristics, such as geographic regions, population density, etc. The strata were designed to have their 1970 population sizes approximately equal. From each stratum, one PSU was selected with probability proportional to size. If a PSU is selected with probability one, then it is called self-representing; otherwise, it is called nonself-representing. So, the sample of primary units consists
of 376 units: 220 nonself representing and 156 self-representing PSUs.

The next stage in sample selection consisted of selecting enumeration districts, geographic areas used for the 1970 census that usually have well-defined boundaries and contain, on the average, about three hundred households. The enumeration districts were arranged in a predetermined geographic manner and then selected systematically with probability proportionate to their 1970 population size.

At the final step of the selection procedure, the selected enumeration districts were subdivided into segments whose expected size is four housing units. A sample of these segments was then taken. In the urban areas, segmentation was accomplished by the list of addresses compiled during the 1970 census. Area sampling was used whenever the address list was incomplete or inaccurate.

An independent sampling operation was used to take into account those housing units built after the 1970 census had been conducted and that were not included in the above sampling process. Units were selected from a list of new construction building permits issued from permit issuing offices in the sampled area. The areas that are not permit issuing were sampled for new construction by means of a sample of area segments.

B. Rotation Pattern

Because of the concern for respondent fatigue, which might occur if a respondent is interviewed indefinitely, a six-level rotation scheme
within PSU is used: respondents are asked to supply information about criminal victimization they may have experienced during each of the preceding six months. This six-month span is called recall or reference period. Table 6.1 illustrates the interviewing scheme. The "X"s denote the months in the six-month recall period. Note that sample data for eighteen months of interviewing are required to produce an annual estimate.

Table 6.1. Recall period scheme in the National Crime Survey

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>February</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>March</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>April</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>May</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>June</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>July</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>August</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>September</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>October</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
</tbody>
</table>

The rotation pattern is as follows. A sample of 75,000 housing units is systematically divided into six rotating groups and each of
them is further divided into six panels. Every month one panel of each group is interviewed so that households in each group are interviewed once in a period of six months. Panels are interviewed at six-month intervals for a period of three and one-half years; each group of units which completes its seven-cycle tenure is retired from the N.C.S. In any one month, one panel is in its first month of enumeration, another panel is in its second month of enumeration, etc., with the last in its sixth time. Thus, sample segments that are $6k$, $k = 1, \ldots, 5$ months apart have $(6 - k)/6$ of their elements in common. Additional samples of 75,000 households selected in the above manner are similarly assigned to rotating groups and panels to replace the retired ones.

Table 6.2 describes the rotation pattern. The entry $(IJ)$, $I = 1, 2, \ldots, 6$, represents rotating panel $I$ and group $J$. The first interview is to establish a time frame to avoid recording duplicative reports on subsequent visits, i.e., for purposes of bounding, and its data are not used in the succeeding analysis. A new rotating group enters the sample every six months and the corresponding rotating group from the previous sample is phased out.

C. Data Analysis

The data set to be analyzed is the data on crime described in Section VI.B. The data set contains information on households victimizations from 1973 to 1982 by month and year of occurrence of victimization and month and year of interview. The observations are the
victimization level, i.e., the number of persons victimized by a crime, averaged across the rotating panels. The survey started in July of 1972.

Table 6.2 Rotation pattern in the National Crime Survey

<table>
<thead>
<tr>
<th>Months</th>
<th>Sample 1</th>
<th>Sample 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>13 14 15 16</td>
<td>11 12 13</td>
</tr>
<tr>
<td>t+1</td>
<td>23 24 25 26</td>
<td>21 22 23</td>
</tr>
<tr>
<td>t+2</td>
<td>33 34 35 36</td>
<td>31 32 33</td>
</tr>
<tr>
<td>t+3</td>
<td>43 44 45 46</td>
<td>41 42 43</td>
</tr>
<tr>
<td>t+4</td>
<td>53 54 55 56</td>
<td>51 52 53</td>
</tr>
<tr>
<td>t+5</td>
<td>63 64 65 66</td>
<td>61 62 63</td>
</tr>
<tr>
<td>t+6</td>
<td>14 15 16</td>
<td>11 12 13 14</td>
</tr>
<tr>
<td>t+7</td>
<td>24 25 26</td>
<td>21 22 23 24</td>
</tr>
<tr>
<td>t+8</td>
<td>34 35 36</td>
<td>31 32 33 34</td>
</tr>
<tr>
<td>t+9</td>
<td>44 45 46</td>
<td>41 42 43 44</td>
</tr>
<tr>
<td>t+10</td>
<td>54 55 56</td>
<td>51 52 53 54</td>
</tr>
<tr>
<td>t+11</td>
<td>64 65 66</td>
<td>61 62 63 64</td>
</tr>
</tbody>
</table>

and the rotation sampling scheme was not fully operative until July of 1974, hence data for 1972 and 1973 were omitted from the analysis.

As the first step in our analysis, we construct a model for the fixed effects associated with the crime data. In the analysis, we use the concept of a group. Two observations are in the same group if their months of interview coincide. For example, group 1 consists of all the
households reporting in January and July; group 2 consists of all the households reporting in February and August; and so on. We fit the model

$$ Y_{ijl} = \mu + \alpha_i + \beta_j + \gamma_k + \delta_{ij} + \eta_{l} + \beta_j \times \eta_{l} + \nu_{ijl} , \quad (6.1) $$

where $ Y_{ijl} $ is the mean across rotation panels of the number of persons victimized by a crime for month $ i $, year $ j $, reported by group $ l $,

$ \alpha_i $ denotes the month effect, $ i = 1, \ldots, 12 $,

$ \beta_j $ denotes the year effect, $ j = 74, \ldots, 82 $,

$ \gamma_k $ denotes the recall period effect, $ k = 1, \ldots, 6 $,

$ \delta_{ij} $ denotes the year linear by month effect, $ i = 1, \ldots, 12 $,

$ \eta_l $ denotes the group effect, $ l = 1, \ldots, 6 $, and

$ \beta_j \times \eta_l $ denotes the interaction of year by group,

$$ j = 74, \ldots, 82, \quad l = 1, \ldots, 6 . $$

No subscript for recall period is required for $ Y $ because the recall period is determined by group and month. The sums of squares associated with the different effects obtained from an ordinary least squares fit are summarized in Table 6.3.

To investigate the across-time correlation in the data, we analyze the residuals from the ordinary least squares fit of model (6.1). Our aim is to develop a model for $ \nu_{ijl} $. To help us in this task, we partition the time component into years, semesters within years, pairs of months within semesters, and months within pairs of months.
Table 6.3. An analysis of variance for victimization level in the household sector, 1974 to 1982

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>S.S.</th>
<th>F-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>11</td>
<td>72,793</td>
<td>62.10</td>
</tr>
<tr>
<td>$\beta</td>
<td>\alpha$</td>
<td>8</td>
<td>6,971</td>
</tr>
<tr>
<td>$\gamma</td>
<td>\alpha,\beta$</td>
<td>5</td>
<td>901,925</td>
</tr>
<tr>
<td>$\delta</td>
<td>\alpha,\beta,\gamma$</td>
<td>11</td>
<td>649</td>
</tr>
<tr>
<td>$\eta</td>
<td>\alpha,\beta,\gamma,\delta$</td>
<td>5</td>
<td>797</td>
</tr>
<tr>
<td>$\eta \times \beta</td>
<td>\alpha,\beta,\gamma,\delta,\eta$</td>
<td>40</td>
<td>3012</td>
</tr>
<tr>
<td>Error</td>
<td>567</td>
<td>60,417</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>647</td>
<td>1,046,564</td>
<td></td>
</tr>
</tbody>
</table>

(bimester). We write

$$v_{jsbmj} = \beta_j + S_{js} + B_{jsb} + M_{jsbm} + \eta_j \delta$$

$$+ \eta_j \delta \times S_{js} + \eta_j \delta \times B_{jsb} + u_{jsbmj} \quad (6.2)$$

where $v_{jsbmj}$ denotes the error of model (6.1), for year $j$, semester $s$, bimester $b$, month $m$ and group $\ell$,

$\beta_j$ denotes the year effect, $j = 74, ..., 82$,

$S_{js}$ denote the effect of semester $s$ nested in year $j$, $s = 1, 2$, $j = 74, ..., 82$,

$B_{jsb}$ denotes the effect of bimester $b$ within semester $s$, $b = 1, 2, 3$, $s = 1, 2$, $j = 74, ..., 82$, ...
\( \eta_{j\ell} \) denotes the effect of group \( \ell \) within year \( j \)
\( i = 1, \ldots, 6, \quad j = 74, \ldots, 82, \)

\( \eta_{j\ell} \times S_{js} \) denotes the group by semester interaction
\( j = 74, \ldots, 82, \quad \ell = 1, \ldots, 6, \quad s = 1, 2, \)

\( \eta_{j\ell} \times B_{jsb} \) denotes the group by bimester interaction \( j = 74, \)
\( \ldots, 82, \quad \ell = 1, \ldots, 6, \quad s = 1, 2, \quad b = 1, 2, 3, \) and

\( u_{jsbml} \) is the deviation of group \( \ell \) for month \( m \) within
bimester \( b \), semester \( s \) and year \( j \).

Table 6.4 gives an analysis of the variance of the least squares
residuals \( v_{jsbml} \) calculated from the least squares fit of model
(6.1). The last three rows are consistent with the hypothesis of no
time correlation in the \( u_{jsbml} \) after removing the year by group
effect. Tables 6.3 and 6.4 show that the variation among groups is
smaller than the variation within groups, suggesting that the groups are
negatively correlated. This result is consistent with the fact that the
groups are formed within each PSU. In addition, recall that our
analysis is for averages over PSUs. To incorporate these facts and to
permit a time correlation, we specify the model

\[
 v_{t\ell} = X_t + u_{t\ell},
\]

(6.3)

where \( t \) is a simple time subscript replacing the double subscripts
\( ij \) of model (6.1), i.e., \( t = 12(j - 74) + i \), and \( ij \) denoted year
and month. Hence, t denotes the monthly observations in chronological order, t = 1, 2, ... Furthermore,

Table 6.4. An analysis of variance for the deviations from the regression (6.1)

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>S.S.</th>
<th>M.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td>β</td>
<td>8</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>S</td>
<td>β</td>
<td>9</td>
<td>1334.71</td>
</tr>
<tr>
<td>B</td>
<td>S</td>
<td>β</td>
<td>36</td>
</tr>
<tr>
<td>M</td>
<td>B</td>
<td>S</td>
<td>β</td>
</tr>
<tr>
<td>η</td>
<td>β</td>
<td>45</td>
<td>0.00</td>
</tr>
<tr>
<td>(n x S)</td>
<td>β</td>
<td>45</td>
<td>5265.02</td>
</tr>
<tr>
<td>(n x B</td>
<td>S)</td>
<td>β</td>
<td>180</td>
</tr>
<tr>
<td>Error</td>
<td>270</td>
<td>31,476.45</td>
<td>116.58</td>
</tr>
</tbody>
</table>

\[ X_t = (367)^{-1} \sum_{r=1}^{367} X_{rt} , \]

\[ u_{t \ell} = (75,000)^{-1} \sum_{t} u_{\ell rt} , \]

where \( X_{rt} \) is the effect of the r-th PSU at time t, \( u_{\ell rt} \) is the observation \( o \) within group \( \ell \) and PSU r at time t, and
\[ \Sigma_{t} \] denotes the sum over all observations in the sample at time \( t \).

We assume \( X_t \) is a stationary time series,

\[ X_t = \alpha X_{t-1} + e_t, \quad (6.4) \]

with \( e_t \) independent of \( u_{v\ell} \), for all \( t, v \) and \( \ell \). Furthermore, let \( E[u_{t\ell}] = 0 \) and

\[ E[u_{t\ell} v_f] = \sigma_{uu} \quad t = v, \quad \ell = f \]

\[ = \lambda \sigma_{uu} \quad t = v, \quad \ell \neq f \quad (6.5) \]

\[ = 0 \quad \text{otherwise}. \]

Expression (6.5) is a model for the between group correlation. Our objective is to express the expected values of the mean squares of Table 6.4 in terms of the parameters of model (6.3), (6.4), (6.5). To this end, let

\[ C_1 = 6^{-1} \begin{pmatrix} 2 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & 2 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & 2 \end{pmatrix}, \]
\[ c_2 = 12^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \]

A be a 6 x 6 matrix with (i,j)-th element given by
\[ a_{ij} = \alpha |i-j| \]

(6.6)

B be a 12 x 12 matrix with (i,j)-th element given by
\[ b_{ij} = \alpha |i-j| \], and

\[ \sigma_{XX} = \text{Var}(X_t) \]

The expected values of the mean squares in Table 6.4 are given in Table 6.5.

<table>
<thead>
<tr>
<th>Source</th>
<th>EMS</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>β</td>
</tr>
<tr>
<td>B</td>
<td>S</td>
</tr>
<tr>
<td>M</td>
<td>B</td>
</tr>
<tr>
<td>(η×S)</td>
<td>β</td>
</tr>
<tr>
<td>(η×B</td>
<td>S)</td>
</tr>
<tr>
<td>Error</td>
<td>( \sigma_{uu} )</td>
</tr>
</tbody>
</table>
Table 6.5 shows that $\sigma_{uu}$ can be estimated by pooling the last three sums of squares in Table 6.4, and the pooled estimator of $\sigma_{uu}$ is

$$
\hat{\sigma}_{uu} = 110.0 \quad (6.7)
$$

Earlier studies by the Bureau of the Census showed that 20% of the total sampling variance can be attributed to the variation among PSUs and 80% to variation among households within PSUs. Assume that $X_{rt}$ follows an autoregressive process with coefficient $\alpha$, and that $X_{rt}$ is independent of $X_{r't'}$, for all $r$, $r'$, $t$ and $t'$. With this assumption, and because the sample consists of 75,000 households in 367 PSUs we can write the sampling variance as

$$
(367)^{-1} \text{Var}(X_{rt}) + (75,000)^{-1} \text{Var}(u_{rotl})
$$

$$
= \text{Var}(X_t) + 6^{-1} \text{Var}(u_{xt})
$$

$$
= \sigma_{XX} + 6^{-1} \sigma_{uu}
$$

where $X_{rt}$ is the between PSU component and $u_{rotl}$ is the within PSU component, where both are defined in (6.3). Because 20% of the sampling variance is due to variation among PSUs, we have

$$
6 \sigma_{XX}^{-1} \sigma_{uu}^{-1} = 0.25
$$
and hence, from (6.7), an estimator of \( \sigma_{XX} \) is

\[
\hat{\sigma}_{XX} = 4.58^{+30}. 
\]

Estimates of \((\lambda, \alpha, \sigma_{ee})\) can be obtained using the first three mean square errors in Table 6.4 and the estimator \( \hat{\sigma}_{XX} \). The expected values of the mean squares under the model (6.3), (6.4), (6.5) are nonlinear functions of \((\lambda, \alpha, \sigma_{ee})\). Therefore, we apply generalized least squares to the model:

\[
34.04 = \sigma_{uu}(1 + 5\lambda) + 6[\sigma_{XX}(1 - \alpha)] + g_1
\]
\[
77.49 = \sigma_{uu}(1 + 5\lambda) + 12 \sigma_{XX} \text{tr}[C_1A_1] + g_2
\]
\[
148.30 = \sigma_{uu}(1 + 5\lambda) + 36 \sigma_{XX} \text{tr}[C_2B_2] + g_3
\]
\[
4.58 = \sigma_{XX} + g_4,
\]

where \((g_1, g_2, g_3, g_4)'\) is the vector of estimation errors, and \(C_1, A, B, \) and \(C_2\) are defined in (6.6). The estimation errors are independent and under normality,

\[
\text{var}(g_1) = 2(54)^{-1}\{\sigma_{uu}(1 + 5\lambda) + 6[\sigma_{XX}(1 - \alpha)]\}^2,
\]
\[
\text{var}(g_2) = 2(36)^{-1}\{\sigma_{uu}(1 + 5\lambda) + 12 \sigma_{XX} \text{tr}[C_1A_1]\}^2,
\]
\[ \text{var}(g_3) = 2(9)^{-1}(\sigma_{uu} (1 + 5\lambda) + 36 \sigma_{XX} \text{tr}[C_2', B C_2])^2, \]

\[ \text{var}(g_4) = 2(495)^{-1} \sigma_{XX}^2. \]

The generalized least squares estimates of the parameters are

\[ \hat{\lambda} = -0.149, \quad (0.023) \]

\[ \hat{\alpha} = 0.68, \quad (0.27) \quad (6.8) \]

and

\[ \hat{\sigma}_{XX} = 4.58, \quad (0.20) \]

Given estimates of \( \alpha_t, \sigma_{uu}, \lambda \) and \( \sigma_{XX} \), and a set of observations, it is possible to construct estimates of annual victimization level. Let \( Y \) denote the vector of observations, \( Y \) denote the vector of errors defined in \( (6.1) \), and let \( X_T = (X_T, X_{T-1}, \ldots)' \). If all available data for nine years were used in the analysis, \( X_T \) would be of dimension \( T = 108 \) and \( Y \) would be of dimension \( 6T = 648 \). In matrix notation, model \( (6.1) \) becomes

\[ Y = Z_1 Y + \gamma, \]

\[ = Z_1 Y + Z_2 X_T + \gamma, \quad (6.9) \]
where \( \mu \) denotes the vector of the fixed effects in (6.1), and
\( Z_1 \) and \( Z_2 \) are the incidence matrices of \( \mu \) and \( \mathbf{x}_T \) respectively.
The second equality in (6.9) is obtained by substituting the expression of (6.3) for \( \gamma \). Moreover,

\[
\Sigma_{Y Y} = \text{Var}(Y) = Z_2 \Sigma_{XX} Z_2' + \Sigma_{uu},
\]

and

\[
\Sigma_{X_T Y} = \text{Cov}(X_T, Y) = \Sigma_{XX} Z_2',
\]

where \( \Sigma_{uu} \) is constructed from (6.5), and \( \Sigma_{XX} \) is the \( T \times T \) matrix
whose \((i, j)\)-th element is \( \alpha_{XX}^{i-j} \). Table 6.6 gives some elements of the covariance matrix of \( Y \). Harville (1976) showed that the best linear unbiased estimator of \( X_T \) is

\[
\hat{X}_T = \Sigma_{XX} Z_2' \Sigma_{YY}^{-1} (\hat{\gamma} - Z_1 \hat{\mu}),
\]

where \( \hat{\mu} \) is any solution to

\[
Z_1' \Sigma_{YY}^{-1} Z_1 \hat{\mu} = Z_1' \Sigma_{YY}^{-1} \gamma.
\] (6.10)

For any estimable linear function \( \lambda_1' \mu \),
Table 6.6. Selected elements from the covariance matrix $\Sigma_{YY}$

<table>
<thead>
<tr>
<th>time</th>
<th>group</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>group 1</td>
<td>group 2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\sigma_{XX} + \sigma_{uu}$</td>
<td>$\sigma_{XX} + \lambda \sigma_{uu}$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\sigma_{XX} + \lambda \sigma_{uu}$</td>
<td>$\sigma_{XX} + \sigma_{uu}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$\alpha \sigma_{XX}$</td>
<td>$\alpha \sigma_{XX}$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$\alpha \sigma_{XX}$</td>
<td>$\alpha \sigma_{XX}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$\alpha^2 \sigma_{XX}$</td>
<td>$\alpha^2 \sigma_{XX}$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>$\alpha^2 \sigma_{XX}$</td>
<td>$\alpha^2 \sigma_{XX}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>$\alpha^2 \sigma_{XX}$</td>
<td>$\alpha^2 \sigma_{XX}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>
\[
\text{Var} \left( \begin{pmatrix} \lambda_1' \hat{y} \\ \hat{X}_T - X_T \end{pmatrix} \right) = \begin{pmatrix} C & D \\ D' & F \end{pmatrix} \tag{6.11}
\]

where \( C = \lambda_1' (Z_1' \Sigma_{YY}^{-1} Z_1)^{-1} \lambda_1 \)

\[D = - \lambda_1' (Z_1' \Sigma_{YY}^{-1} Z_1)^{-1} Z_1' \Sigma_{YY}^{-1} Z_2 \Sigma_{XX}\]

\[F = \Sigma_{XX} - \Sigma_{XX} \Sigma_{YY}^{-1} \Sigma_{XX} + \Sigma_{XX} Z_2' \Sigma_{YY}^{-1} Z_1 (Z_1' \Sigma_{YY}^{-1} Z_1)^{-1} Z_1' \Sigma_{YY}^{-1} Z_2 \Sigma_{XX},\]

and \( G^- \) denotes the Moore-Penrose generalized inverse of \( G \).

Hence, the best linear unbiased estimator of \( \lambda_1' \hat{y} + \lambda_2' \hat{X}_T \), where \( \lambda_1' \hat{y} \) is estimable, is

\[\lambda_1' \hat{y} + \lambda_2' \hat{X}_T \tag{6.12}\]

with variance

\[
\text{var}(\lambda_1' \hat{y}) + \lambda_2' \text{var}(\hat{X}_T - X_T) \lambda_2 + 2 \lambda_2' \text{cov}(\hat{X}_T - X_T, \lambda_1' \hat{y}),
\]

\[
\text{var}(\hat{X}_T - X_T), \text{ var}(\lambda_1' \hat{y}), \text{ and } \text{cov}(\hat{X}_T - X_T, \lambda_1' \hat{y}) \text{ are defined in (6.11).}
\]

In this survey, all the data for a particular year will only be available in June of the following year. However, in practice, it is
important to have estimates of annual rates before June. Using data for one year, we compute the estimated variances of the prediction error of the estimators based on data available in January, February, ..., June of the following year. The computations were performed ignoring the recall period effect. Data for a particular year consist of month-group observations, and hence the only fixed effect associated with the observations is the yearly mean. For this simplified model and sample size, $Z_1$ is a column of ones and $\mu$ is the mean for year $j$, denoted by $\mu_j$. For data available in January, the matrix $Z_2$ is a $57 \times 12$ matrix with $I_6 \otimes (1, 1, 1, 1, 1, 1)$ for the first 42 rows, where $\otimes$ denotes the Kronecker product. The quantity to be estimated is the monthly average of victimization level for a year, denoted by

$$\mu_j + (12)^{-1} \sum_{t(j)} X_t$$

where $\sum_{t(j)}$ denotes the sum across months within year $j$. The variance of the prediction error can be computed from (6.13), with $\lambda_1 = 1$ and $\lambda_2$ equal to the 12-dimensional vector of ones divided by twelve. Estimates of the mean square error of the prediction error were computed substituting the estimates in (6.8) for their true values in (6.13).

Under the classical sampling approach, the vector $X_T$ is considered to be fixed. In this case, the best linear unbiased estimator of the monthly victimization level for a year is the sample mean of the monthly means because the month-to-month correlation in
reported crimes for a group is zero. In January, the data for the previous year consist of reports from all the six groups for January, February, ..., July, and reports from five, four, three, two and one groups for, respectively, August, September, October, November and December. Hence, using data available in January, the variance of the estimation error is

\[
(144)^{-1} \sigma^2 + \frac{7(1 + 5\lambda)}{6} + \frac{1 + 4\lambda}{5} + \frac{1 + 3\lambda}{4} + \frac{1 + 2\lambda}{3} + \frac{1 + \lambda}{2} + 1.
\]

The remaining variances can be obtained similarly. We replaced \( \sigma \) and \( \lambda \) by their estimated values in (6.8) to obtain estimates of the variance. Table 6.7 gives the estimated efficiency of the

<table>
<thead>
<tr>
<th>Interview date</th>
<th>Time Series Approach</th>
<th>Classical Sampling Approach</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>0.7587</td>
<td>1.6559</td>
<td>2.18</td>
</tr>
<tr>
<td>February</td>
<td>0.6119</td>
<td>0.9237</td>
<td>1.51</td>
</tr>
<tr>
<td>March</td>
<td>0.5105</td>
<td>0.6309</td>
<td>1.24</td>
</tr>
<tr>
<td>April</td>
<td>0.4430</td>
<td>0.4845</td>
<td>1.09</td>
</tr>
<tr>
<td>May</td>
<td>0.4018</td>
<td>0.4113</td>
<td>1.02</td>
</tr>
<tr>
<td>June</td>
<td>0.3820</td>
<td>0.3820</td>
<td>1.00</td>
</tr>
</tbody>
</table>
estimators (6.12) relative to the classical sampling estimators. The first three rows in Table 6.7 show a considerable gain in using the time series estimator. When all the data for a year have been obtained, the sample mean and the time series estimator are equivalent. The covariance matrices used in the computation of the variance of the prediction error, using data available in January, are listed in the Appendix.
VII. SUMMARY

Rotation sampling and the related subject of estimation of a population mean which changes over time have been considered by several authors. A unified approach to the problem is given by Jones (1980) using least squares theory. Under the assumption that the sequence of population means is a realization of a stochastic process, we consider the application of time series methods to the analysis of repeated surveys.

Let \( X_t \) be the mean of the population at time \( t \), and let \( Y_t \) be a survey estimate of \( X_t \). Let \( u_t \) be the sampling error. The model under study is

\[
Y_t = X_t + u_t,
\]

\[
X_t + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} = e_t,
\]

\[
u_t = v_t + b_1 v_{t-1} + \ldots + b_q v_{t-q},
\]

where \((e_t, v_t)\) is a sequence of normal independent random vectors with mean 0 and covariance matrix \( \Sigma = \text{diag}(\sigma_{ee}, \sigma_{vv}) \), and the roots of

\[
m^p + \alpha_1 m^{p-1} + \ldots + \alpha_p = 0
\]
are less than unity in absolute value. The moving average representation for \( u_t \) comes from the fact that the sampling units stay in the survey only for a fixed finite number of occasions.

Assuming that survey estimates of \( b_1, ..., b_q \) and \( \sigma_{vv} \) are available, we wish to estimate \( \alpha, ..., \alpha_p \) and \( \sigma_{ee} \). To solve this problem, we represent \( y_t \) as an autoregressive moving average time series whose parameters satisfy a set of nonlinear restrictions. Based upon the properties of the least squares estimators of an autoregressive moving average time series an efficient estimation procedure is developed. The limiting distribution is derived as an application of a version of the central limit theorem for martingale differences.

Given estimators of \((\alpha_1, ..., \alpha_p, \sigma_{ee})\) and the vector of observations \( \tilde{y} = (y_n, y_{n-1}, ..., y_1) \), an estimate of the population mean at a particular time \( T \) is

\[
\hat{X}_T = \hat{\Sigma}_{XY}^{-1} \hat{\Sigma}_{YY} \tilde{y} ,
\]

where \( \hat{\Sigma}_{XY} \) is the estimated covariance matrix between \( X_T \) and \( \tilde{y} \), \( \hat{\Sigma}_{YY} \) is the estimated covariance matrix of \( \tilde{y} \), and \( \hat{\Sigma}_{XY} \) and \( \hat{\Sigma}_{YY} \) are obtained by replacing \((\alpha_1, ..., \alpha_p, \sigma_{ee})\) with estimated values in the expressions for the covariance matrices. It is shown that the use of \( \hat{\Sigma}_{XY} \) and \( \hat{\Sigma}_{YY} \) in the prediction of \( X_T \) increases the prediction error by a quantity of order in probability \( n^{-1/2} \).

A Monte Carlo study is conducted. The distributional properties of the estimators showed reasonable agreement with the asymptotic theory for samples of thirty observations from the first order autoregressive
process with autoregressive parameter less than 0.95.

Application of time series techniques to the National Crime Survey is considered. One set of estimates suggests that the use of time series procedures produces sizable gains for estimates of yearly victimization level constructed in the first three months of the following year.
VIII. BIBLIOGRAPHY


IX. ACKNOWLEDGMENTS

I wish to express my gratitude to Professor Wayne A. Fuller for his expert guidance and commitment. His enthusiasm and insight contributed a great deal to making my graduate work an enjoyable and stimulating experience.

Thanks are due to Professor Yasuo Amemiya who was a constant source of help during the years I spent in the Survey Section.

Coordenação de Aperfeiçoamento de Pessoal de Nível Superior and Universidade Federal de São Carlos supported my studies at Iowa State University.

I thank Jo Ann Ilershey and Marlene Sposito for the repeated typings required to bring the manuscript to final form.

Special thanks are given to my husband, Ed, for his patience, assistance and love during the course of this work.

This research was supported by joint statistical agreement with the U.S. Bureau of the Census.
X. APPENDIX

In this section, we give the matrices needed in the computation of the variance of the prediction error in Section VI.C. using data available in January. Let \( Y_{ij} \) denote the mean across rotating panels of the number of persons victimized by a crime for month \( i \) within year \( j \), reported by group \( \lambda \). Let \( \gamma \) be the 57-dimensional vector of data for one year that is available in January. Write

\[
\gamma = (Y_{1j1}, Y_{1j2}, \ldots, Y_{1j6}, Y_{2j1}, \ldots, Y_{12j1})'.
\]

Let \( \chi = (X_1, X_2, \ldots, X_{12}) \). Let \( \Sigma_{YY} \) be the covariance matrix of \( \gamma \) and \( \Sigma_{XY} \) be the covariance matrix of \( \chi \) and \( \gamma \). Define the matrix of the regression coefficients of \( \gamma \) or \( \chi \) by \( R \), i.e.,

\[
R = \Sigma_{XY} \Sigma_{YY}^{-1}.
\]

We present estimates of \( \Sigma_{YY} \), \( \Sigma_{XY} \) and \( R \). For the sake of this presentation, we partition the matrices as

\[
\Sigma_{YY} = \begin{pmatrix}
Q_{11} & Q_{21}^t & Q_{31}^t \\
Q_{21} & Q_{22} & Q_{32}^t \\
Q_{31} & Q_{32} & Q_{33}
\end{pmatrix},
\]

\[
\Sigma_{XY} = \begin{pmatrix}
Q_{11} & Q_{21}^t & Q_{31}^t \\
Q_{21} & Q_{22} & Q_{32}^t \\
Q_{31} & Q_{32} & Q_{33}
\end{pmatrix},
\]

\[
R = \Sigma_{XY} \Sigma_{YY}^{-1}.
\]
$$\Sigma_{XY} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \end{pmatrix},$$

and

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \end{pmatrix},$$

where $Q_{ij}$ are 19-dimensional square matrices, and $R_{ij}$ and $S_{ij}$ are $12 \times 19$ matrices.

Tables 10.1 - 10.6 give the estimated $Q_{11}$, $Q_{21}$, $Q_{31}$, $Q_{22}$, $Q_{32}$ and $Q_{33}$, respectively. The estimate of $\Sigma_{XY}$ is presented in Tables 10.7 - 10.9. Finally, the estimate of the regression matrix is given in Tables 10.10 - 10.12.
Table 10.1. Matrix Q11 in the covariance matrix of Y, using data available in January

<p>| | | | | | | | | | | | | | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
</tr>
<tr>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
</tbody>
</table>
Table 10.2. Matrix Q21 in the covariance matrix of Y, using data available in January

<table>
<thead>
<tr>
<th></th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>-11.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
</tr>
<tr>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
</tr>
<tr>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
</tr>
<tr>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>
Table 10.3. Matrix Q31 in the covariance matrix of Y, using data available in January

|     | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 1.06 |
|-----|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 0.25| 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 1.06 |
| 0.25| 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 1.06 |
| 0.25| 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 1.06 |
| 0.25| 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 | 0.65 | 0.65 | 0.65 | 0.65 | 1.06 |
| 0.15| 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 |
| 0.15| 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 |
| 0.15| 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 |
| 0.15| 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.65 |
| 0.09| 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 |
| 0.09| 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 |
| 0.09| 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 |
| 0.09| 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 | 0.40 |
| 0.06| 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 |
| 0.06| 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.09 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 |
| 0.03| 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 |
| 0.03| 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.06 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 |
| 0.02| 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.25 |
Table 10.4. Matrix $Q_{22}$ in the covariance matrix of $Y$, using data available in January

<table>
<thead>
<tr>
<th></th>
<th>114.6</th>
<th>-11.9</th>
<th>-11.9</th>
<th>-11.9</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.06</th>
<th>1.06</th>
</tr>
</thead>
<tbody>
<tr>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
</tr>
</tbody>
</table>
Table 10.5. Matrix $Q32$ in the covariance matrix of $Y$, using data available in January

<table>
<thead>
<tr>
<th></th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>2.82</th>
<th>-11.9</th>
<th>-11.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
</tr>
<tr>
<td></td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
</tr>
<tr>
<td></td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
</tr>
<tr>
<td></td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td></td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td></td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td></td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
</tr>
</tbody>
</table>

142
Table 10.6. Matrix Q33 in the covariance matrix of Y, using data available in January

<table>
<thead>
<tr>
<th></th>
<th>1.73</th>
<th>1.73</th>
<th>1.73</th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>0.65</th>
<th>0.65</th>
<th>0.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td></td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td></td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>-11.9</td>
<td>2.82</td>
<td></td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>2.82</td>
<td></td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>2.82</td>
<td></td>
</tr>
<tr>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>-11.9</td>
<td>114.6</td>
<td>114.6</td>
<td>114.6</td>
<td>114.6</td>
<td>114.6</td>
<td>114.6</td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>1.06</td>
</tr>
</tbody>
</table>
Table 10.7. Matrix $S_{11}$ in the covariance matrix of $X$ and $Y$, using data available in January

\[
\begin{array}{cccccccccc}
4.60 & 4.60 & 4.60 & 4.60 & 4.60 & 2.82 & 2.82 & 2.82 & 2.82 & 2.82 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.06 \\
2.82 & 2.82 & 2.82 & 2.82 & 2.82 & 4.60 & 4.60 & 4.60 & 4.60 & 4.60 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.06 \\
1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 2.82 & 2.82 & 2.82 & 2.82 & 2.82 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.06 \\
1.06 & 1.06 & 1.06 & 1.06 & 1.06 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.73 & 1.06 \\
0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 0.65 & 1.06 \\
0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.40 & 0.65 \\
0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.25 & 0.40 \\
0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.15 & 0.40 \\
0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.09 & 0.65 \\
0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.06 & 0.40 \\
0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\
0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 \\
\end{array}
\]
<table>
<thead>
<tr>
<th></th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>1.06</th>
<th>0.65</th>
<th>0.65</th>
<th>0.65</th>
<th>0.65</th>
<th>0.65</th>
<th>0.40</th>
<th>0.40</th>
<th>0.40</th>
<th>0.40</th>
<th>0.40</th>
<th>0.40</th>
<th>0.25</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.06</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>0.65</td>
</tr>
<tr>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>0.65</td>
</tr>
<tr>
<td>2.82</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
</tr>
<tr>
<td>4.60</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.06</td>
</tr>
<tr>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>2.82</td>
<td>2.82</td>
<td></td>
</tr>
<tr>
<td>2.82</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>1.73</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td>4.60</td>
<td></td>
</tr>
<tr>
<td>1.73</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td>1.06</td>
<td></td>
</tr>
<tr>
<td>1.06</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td>0.65</td>
<td></td>
</tr>
<tr>
<td>0.65</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>0.40</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td></td>
</tr>
<tr>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td></td>
</tr>
<tr>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
<td></td>
</tr>
</tbody>
</table>
Table 10.9. Matrix S13 in the covariance matrix of X and Y, using data available in January

\[
\begin{array}{cccccccccccc}
0.25 & 0.25 & 0.25 & 0.25 & 0.15 & 0.15 & 0.15 & 0.15 & 0.09 & 0.09 & 0.09 & 0.09 & 0.03 & 0.03 & 0.02 \\
0.40 & 0.40 & 0.40 & 0.40 & 0.25 & 0.25 & 0.25 & 0.25 & 0.15 & 0.15 & 0.15 & 0.15 & 0.09 & 0.09 & 0.09 \\
0.65 & 0.65 & 0.65 & 0.65 & 0.40 & 0.40 & 0.40 & 0.40 & 0.25 & 0.25 & 0.25 & 0.25 & 0.15 & 0.15 & 0.09 \\
1.06 & 1.06 & 1.06 & 1.06 & 0.65 & 0.65 & 0.65 & 0.65 & 0.40 & 0.40 & 0.40 & 0.40 & 0.25 & 0.25 & 0.15 \\
1.73 & 1.73 & 1.73 & 1.73 & 1.06 & 1.06 & 1.06 & 1.06 & 0.65 & 0.65 & 0.65 & 0.65 & 0.40 & 0.40 & 0.40 \\
2.82 & 2.82 & 2.82 & 2.82 & 1.73 & 1.73 & 1.73 & 1.73 & 1.06 & 1.06 & 1.06 & 0.65 & 0.65 & 0.65 & 0.40 \\
4.60 & 4.60 & 4.60 & 4.60 & 2.82 & 2.82 & 2.82 & 2.82 & 1.73 & 1.73 & 1.73 & 1.06 & 1.06 & 0.65 & 0.65 \\
2.82 & 2.82 & 2.82 & 2.82 & 4.60 & 4.60 & 4.60 & 4.60 & 2.82 & 2.82 & 2.82 & 1.73 & 1.73 & 1.73 & 1.06 \\
1.73 & 1.73 & 1.73 & 1.73 & 2.82 & 2.82 & 2.82 & 2.82 & 4.60 & 4.60 & 4.60 & 4.60 & 2.82 & 2.82 & 1.73 \\
1.06 & 1.06 & 1.06 & 1.06 & 1.73 & 1.73 & 1.73 & 1.73 & 2.82 & 2.82 & 2.82 & 2.82 & 4.60 & 4.60 & 2.82 \\
0.65 & 0.65 & 0.65 & 0.65 & 1.06 & 1.06 & 1.06 & 1.06 & 1.73 & 1.73 & 1.73 & 1.73 & 2.82 & 2.82 & 4.60 \\
0.40 & 0.40 & 0.40 & 0.40 & 0.65 & 0.65 & 0.65 & 0.65 & 1.06 & 1.06 & 1.06 & 1.06 & 2.82 & 2.82 & 4.60 \\
\end{array}
\]
Table 10.10. Matrix R11 of regression coefficients of Y on X, using data available in January

<table>
<thead>
<tr>
<th></th>
<th>0.01</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
<th>0.07</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 10.11. Matrix $R_{12}$ of regression coefficients of $Y$ on $X$, using data available in January.
Table 10.12. Matrix R13 of regression coefficients of $Y$ on $X$, using data available in January

\[
\begin{array}{cccccccccccccccc}
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.01 & 0.01 & 0.01 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.02 & 0.02 & 0.02 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \\
0.07 & 0.07 & 0.07 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 & 0.02 \\
0.03 & 0.03 & 0.03 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\
0.01 & 0.01 & 0.01 & 0.01 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \\
0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \\
0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
\end{array}
\]