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Variations on the power domination problem: Hypergraphs and robustness

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Variations on the power domination problem: Hypergraphs and robustness

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ABSTRACT

The power domination problem seeks to find the minimum number of sensors called phasor measurement units (PMUs) to monitor an electric power network. In this dissertation, we present two variations of the power domination problem.

The first variation is *infectious power domination*, which is a new way to generalize the power domination problem to hypergraphs using the infection rule from Bergen et al. (2018). We compare to the previous generalization by Chang and Roussel (2015). We examine general bounds; graph families such as complete k-partite hypergraphs, circular arc hypergraphs, and trees; and the impact of edge/vertex removal, linear sums, Cartesian products, and weak coronas.

The second variation considers how the minimum number of sensors and their placement changes when k sensors are allowed to fail. The *PMU-defect robust power domination number* is also a novel parameter, generalizing the work done by Pai, Chang, and Wang (2010) by allowing multiple sensors to be placed at the same location. We give general bounds, explicit values for some complete bipartite graphs, and computational results for small square grid graphs. We also give a new proof of the power domination number for trees and conjecture the PMU-defect robust power domination number for trees.

CHAPTER 1. GENERAL INTRODUCTION

During the Northeast Blackout in 2003, over 50 million people in North America lost power for up to two days [12]. This cascading power failure led to the Energy Policy of 2005, which resulted in an increase in the use of Phasor Measurement Units (PMUs). These sensors monitor the power grid and help indicate when conditions could lead to a blackout. While PMUs are useful, budget constraints mean that we must find an optimal placement using the fewest sensors possible. This became known as the *PMU placement problem*.

1.1 The power domination problem and graph theory

The PMU placement problem was redefined in graph theoretic terms in 2002 by Haynes et al. [10]. The process was simplified by Brueni and Heath in 2005 [6] and is now known as the *power domination problem*.

A *graph* G is a set of vertices, $V(G)$, and a set of edges, $E(G)$. Each edge consists of a set of two distinct unordered vertices. We say that vertices u and v are *neighbors* if $\{u, v\} \in E(G)$. The *closed neighborhood* of a vertex $v \in V(G)$ is denoted $N[v]$ and is the set of neighbors of v together with v. The *power domination process* on a graph G with initial set $S \subseteq V$ proceeds recursively:

1. $B = \bigcup_{v \in S} N[v]$
2. While there exists $v \in B$ such that exactly one neighbor, say u, of v is *not* in B, add u to B.

Step 1 is referred to as the *domination step* and each repetition of step 2 is called a *zero forcing step*. During the process, we say that a vertex in B is *observed* and a vertex not in B is *unobserved*. A *power dominating set* of a graph G is an initial set S such that $B = V(G)$ at the termination of the power domination process. For an example, see Figure 1.1. The *power domination number* of a graph G is the minimum cardinality of a power dominating set of G and is denoted by $\gamma_P(G)$.

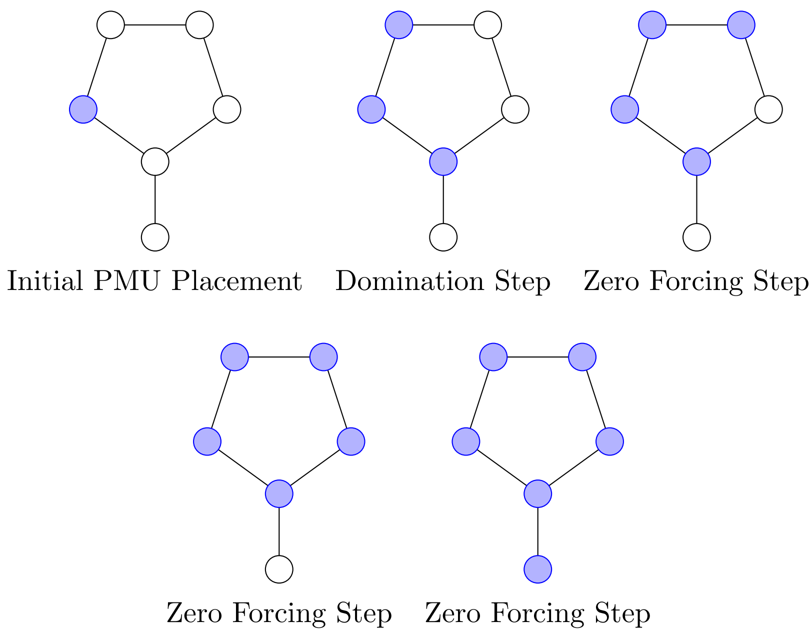

Figure 1.1 Power domination on a graph G

The power domination problem seeks to find a minimum power dominating set and the power domination number for a given graph.

The initial set of vertices, S, represents the placement of PMUs in the network. A PMU can directly take measurements for the vertex where it is placed and all neighboring vertices, which is modeled by the domination step. Then, Kirchhoff's Law and Ohm's Law can be used to solve a system of equations for the one missing value if there is an observed vertex for which only one neighbor is unobserved. This is precisely what occurs in the zero forcing step. Thus, the power domination number is the minimum number of PMUs needed to monitor the given network and the vertices in a minimum power dominating set give the PMU placement.

The PMU placement problem makes a connection between two other problems: domination and zero forcing. The connection with zero forcing was first used by Dean et al. in 2011 [9] and then described more generally in Benson et al. in 2018 [3]. *Domination* utilizes only the domination step to observe the entirety of a graph. For an overview of domination in graphs, see [11]. *Zero forcing* is a graph propagation process defined by a color change rule that began as a bound for the maximum nullity of a family of real symmetric matrices associated with a graph [2] and independently in control of quantum systems [7]. The zero forcing step in power domination is an application of this color change rule.

The PMU placement problem is NP-complete, as shown in [10]. Finding an optimal placement of the minimum number of PMUs is not solvable for most actual applications. However, theoretical results can yield bounds to determine how many PMUs are required and also give an idea of the structure of an optimal placement. Due to its connections to the related problems of domination and zero forcing, the power domination problem has also become interesting mathematically beyond the initial application.

In this dissertation, we examine two variations of the power domination problem. The first builds on an area of interest to graph theorists by defining a new extension to hypergraphs. The second is more application focused and explores how to find a placement of PMUs that is robust to sensor defects.

1.2 From graphs to hypergraphs

A *hypergraph* is a generalization of a graph in which the edges are allowed to be unordered sets of any number of distinct vertices. Power domination was first generalized to hypergraphs by Chang and Roussel in 2015 [8]. Chang and Roussel's generalization follows power domination on graphs closely; there is a domination step followed by a zero forcing step called the *observation step*. Specifically, their observation step uses one vertex v to observe a single edge that contains all unobserved neighbors of v. However, this is not the only way to generalize the zero forcing step to hypergraphs. Instead of only allowing one vertex to observe the other vertices in an edge, we use the *infection process* defined by Bergen et al. in 2018 [4] as the second step in what we call the *infectious power domination process*. That is, we use a domination step followed by an infection step in which a *set* of vertices A infects a single edge that contains both A and all uninfected vertices that are contained in edges with the set A.

In Chapter 2, we introduce the *infectious power domination number of a hypergraph*, which is the minimum size of an initial set of infected vertices so that the infectious power domination process terminates with every vertex infected. We then explore this novel parameter by way of general upper/lower bounds; an examination of specific hypergraph families; and bounds for how the infectious power domination number behaves under various hypergraph operations, including vertex/edge removal, linear sums, Cartesian products, and weak coronas. This chapter is modified from [5].

1.3 Maintaining power grid monitoring in the event of PMU failure

While PMUs are useful for monitoring the power grid to prevent cascading blackouts, they operate in real world conditions and are susceptible to mistakes. Errors can include things such as data corruption, time synchronization issues, or data loss [1]. As such, redundant PMUs are used in order to corroborate data and confirm if there is an issue with the power grid or a sensor [1]. How do we decide where to place PMUs if we know that some of them may be inaccurate or faulty?

In 2010, Pai, Chang, and Wang defined fault-tolerant power domination [13]. This problem asks for a set of vertices that describes a placement of PMUs that will still monitor the graph if some subset of the PMUs fail. Choose an initial set of vertices S, but before beginning the power domination process, remove any k vertices from S to create S'. If S' is a power dominating set regardless of which k vertices are removed to create S' from S, then we say that S is a *k-fault-tolerant power dominating set.*

We consider a similar question. Let the initial set S instead be a *multiset* of vertices, that is, multiple PMUs may be placed at a single vertex. Then k vertices are removed to create S' from S. If S' is still a power dominating set regardless of which k vertices are removed from S to create S', we say that S is a *k-robust power dominating set.* This power domination process is called *PMU-defect-robust power domination.*

PMU-defect-robust power domination focuses on issues with an individual PMU failing. Such failure could be in the form of data corruption or data loss. Fault-tolerant power domination focuses on failures based on location, so there is no advantage to placing multiple PMUs at a single site. For instance, if GPS signal is inconsistent at a site, placing multiple PMUs at the site may not correct the time synchronization error. Depending on the type of failure to be prevented, it may be better to have PMUs spread through the power network. On the other hand, it may prove to be more cost effective to place multiple PMUs at a single location than to place many PMUs at different sites.

In Chapter 3, we introduce the *k-robust power domination number* of a graph, which is the minimum size of an initial multiset of vertices such that every vertex is observed at the termination of the PMU-defect-robust power domination process. We investigate this new parameter by determining general upper/lower bounds and examining specific graph families such as complete bipartite graphs, grid graphs, and trees.

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CHAPTER 2. INFECTIOUS POWER DOMINATION OF HYPERGRAPHS

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Abstract

The power domination problem seeks to find the placement of the minimum number of sensors needed to monitor an electric power network. We generalize the power domination problem to hypergraphs using the infection rule from Bergen et al (2018): given an initial set of observed vertices, S_0, a set $A \subseteq S_0$ may infect an edge e if $A \subseteq e$ and for any unobserved vertex v, if $A \cup \{v\}$ is contained in an edge, then $v \in e$. We combine a domination step with this infection rule to create *infectious power domination*. We compare this new parameter to the previous generalization by Chang and Roussel (2015). We provide general bounds and determine the impact of some hypergraph operations.

Keywords: Power domination, hypergraph, infection number

2.1 Overview

The power domination problem seeks to find the placement of the mimimum number of sensors (called Phase Measurement Units or PMUs) needed to monitor an electric power network. In [8], Haynes et al. defined the power domination problem in graph theoretic terms by placing PMUs at a set of initial vertices and then applying observation rules to the vertices and edges of the graph. These observation rules consist of an initial domination step followed by what is now called the zero forcing process [5], [2]. Zero forcing is a graph propagation process that has its roots in determining the maximum nullity of the family of real symmetric matrices associated with the graph [1] and independently in control of quantum systems [6].

As a graph theory problem, zero forcing and its variants (including power domination) have been well studied and become interesting mathematically beyond the motivating applications. In particular, zero forcing has been generalized to hypergraphs in several different ways. Bergen et al. defined the infection number of a hypergraph in [3] to generalize the zero forcing process to hypergraphs. In [10], Hogben defines the zero forcing number of a hypergraph based on the skew symmetric zero forcing number of a graph and the maximum nullity of a family of hypermatrices. Chang and Roussel define k-power domination for hypergraphs in [7], which is a power domination process when $k = 1$. From this rule in [7], Hogben also defines the power domination zero forcing number in [10].

Just as a power domination rule can be used to define a zero forcing process for hypergraphs, a zero forcing process can be used to define a power domination process for hypergraphs. Hogben's zero forcing number is less useful for power domination, as the zero forcing number of a hypergraph can be zero, which eliminates the real world connection to sensor placement.

Our premise is to use Bergen et. al's definition of infection [3] to define infectious power domination of hypergraphs and compare this to the definition of power domination of hypergraphs introduced by Chang and Roussel. Both generalizations of power domination to hypergraphs reduce to the power domination problem for graphs when $\mathcal{H}$ is 2-uniform, i.e. $\mathcal{H}$ is a graph (Prop. 1.1 in [3] and page 1097 in [7]). However, Chang and Roussel's definition focuses on allowing one vertex to observe others whereas infectious power domination utilizes the fact that there may be multiple observed vertices in an edge which can be used to observe the edge. Using multiple vertices to observe an edge may be more natural as a model for physical problems, as this represents using measurements from multiple sensors.

In Section 2.2 we will review preliminary definitions from past work. Then in Section 2.3 we formally define infectious power domination and compare it to both the infection number and the power domination number. In Section 2.4 we determine general bounds for the power domination number and by extension the infectious power domination number. Section 2.5 consists of results for the infectious power domination number including hypergraphs which have infectious power domination number one and hypertrees. Section 2.6 determines bounds for the infectious power domination number for various hypergraph operations such as edge/vertex removal, linear sum, Cartesian products, and weak coronas. We make concluding remarks in Section 2.7.

2.2 Preliminaries

We use Bretto's *Hypergraph Theory* [4] as a reference for hypergraph notation and definitions.

A *hypergraph*, $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, is a set of vertices $V(\mathcal{H})$ along with a set of edges $E(\mathcal{H})$ so that $E(\mathcal{H})$ is a subset of the power set of $V(\mathcal{H})$. In the case that there is a constant k such that $|e| = k$ for all $e \in E(\mathcal{H})$, we say that $\mathcal{H}$ is *k-uniform* and denote such a hypergraph by $\mathcal{H}^{(k)}$.

A *path* in a hypergraph $\mathcal{H}$ is a sequence of vertices and edges

$$v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1}$$

so that the v_i are distinct vertices, the e_i are distinct edges, and $v_i, v_{i+1} \in e_i$ for $1 \leq i \leq \ell$. We say that the path $v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1}$ is a *path from* v_1 *to* $v_{\ell+1}$ and has *length* ℓ. A hypergraph is said to be *connected* if there is a path from any vertex to any other vertex. We say that a hypergraph $\mathcal{H}$ is *reduced* if for all distinct edges $e, e' \in E$, $e \not\subseteq e'$ and $e' \not\subseteq e$; that is, no edge is contained in another edge. We may reduce a given hypergraph by removing every edge that is a proper subset of another edge. Throughout what follows, we will consider only hypergraphs with at least one edge that are reduced.

The *closed neighborhood* of a vertex $a \in V$ is $N[a] = \bigcup_{a \in e \in E} e$. The *(open) neighborhood* of $a \in V$ is $N(a) = N[a] \setminus \{a\}$ and an element of $N(a)$ is called a *neighbor* of a. The *degree* of a vertex $a \in V$, denoted $\deg(a)$, is the number of edges that contain a. If $\deg(a) = 0$, that is, a is not contained in any edge, we say that a is an *isolated vertex*. An edge consisting of exactly one vertex is called a *loop*. If vertices a and b are neighbors, we say that a is *adjacent* to b. When vertex a is contained in edge e we say that e is *incident* to a.

An *induced subhypergraph* $\mathcal{H}'$ of a hypergraph $\mathcal{H} = (V, E)$ has a vertex set $V' \subseteq V$ and the edge set is

$$E' = \{e_i \cap V' \neq \varnothing : e_i \in E, \text{ and either } e_i \text{ is a loop or } |e_i \cap V'| \geq 2\}.$$

In this case, we say that V' *induces* the subhypergraph $\mathcal{H}'$. Note that if $\mathcal{H}$ is a k-uniform hypergraph that $\mathcal{H}'$ need not be uniform.

A *dominating set* of a hypergraph $\mathcal{H}$ is a set of vertices $D \subseteq V(\mathcal{H})$ so that for every vertex $v \in V \setminus D$, there exists an edge $e \in E(\mathcal{H})$ for which $v \in e$ and $e \cap D \neq \varnothing$ [9]. That is, a dominating set is $D \subseteq V(\mathcal{H})$ so that $V = \cup_{d \in D} N[d]$. The size of a minimum dominating set of $\mathcal{H}$ is called the *domination number* of $\mathcal{H}$ and is denoted by $\gamma(\mathcal{H})$.

Definition 2.2.1. [3] The *infection rule* is defined so that a nonempty set A of infected vertices can infect the vertices in an edge e if

1. $A \subseteq e$, and
2. if v is an uninfected vertex such that $A \cup \{v\}$ is a subset of some edge in the hypergraph, then $v \in e$.

An initial set of infected vertices S_0 is called an *infection set* if after repeated application of the infection rule all vertices become infected. The size of a minimum infection set is called the *infection number* of the hypergraph $\mathcal{H}$ and is denoted by $I(\mathcal{H})$.

Definition 2.2.2. [7] The 1-*power domination process* consists of an initial subset of the vertices, S_0, called the *power dominating set* and the observation rules:

a. A vertex in the power dominating set observes itself and all of its neighbors.

b. If all unobserved neighbors of an observed vertex v are in one edge incident to v, then these unobserved vertices become observed as well.

We refer to step a as the *domination step* and each repetition of b as an *observation step*.

In this case, we say that a set of vertices A *observes* a set of vertices B if A causes B to become observed. A *power dominating set* of a hypergraph $\mathcal{H}$ is an initial set so that every vertex in $\mathcal{H}$ is observed at the termination of the 1-power domination process. The *power domination number* of a hypergraph $\mathcal{H}$, denoted $\gamma_P(\mathcal{H})$, is the minimum cardinality of a power dominating set of $\mathcal{H}$.

This is the same as a 1-power dominating set and the 1-power domination number as defined in [7] (originally denoted by $\gamma_P^1(\mathcal{H})$).

2.3 Infectious power domination

We can now generalize power domination to hypergraphs based on the definition of infection in [3].

Definition 2.3.1. Suppose $\mathcal{H} = (V, E)$ is a hypergraph. The *infectious power domination process* on $\mathcal{H}$ with initial set $S_0 \subseteq V$ proceeds by:

1. $S = \bigcup_{v \in S_0} N[v]$.
2. While there exists a nonempty $A \subseteq S$ so that A can infect the vertices in an edge e using Definition 2.2.1, add the vertices of e to S.

Definition 2.3.2. An *infectious power dominating set* of a hypergraph $\mathcal{H}$ is an initial set S_0 such that every vertex in $\mathcal{H}$ is in S after the termination of the infectious power domination process. The *infectious power domination number* of a hypergraph $\mathcal{H}$ is the minimum cardinality of an infectious power dominating set of $\mathcal{H}$, which we denote by $\gamma_{P_I}(\mathcal{H})$.

We say that a vertex in S is *infected* and that a set of vertices A *infects* a set of vertices B if A causes B to join S. We call step 1 the *domination step* and each repetition of step 2 an *infection step*.

As an infection set can infect a graph without needing the domination step, such a set is also an infectious power dominating set. Thus we have the following observation.

Observation 2.3.3. *For any hypergraph $\mathcal{H}$, $\gamma_{P_I}(\mathcal{H}) \leq I(\mathcal{H})$.*

Chang and Roussel's definition in [7], restated in Definition 2.2.2, is equivalent to Definition 2.3.1 with the restriction that A must always be a single vertex. Thus, the first inequality in proposition 2.3.4 is immediate and is also an easy consequence of Theorem 2.4 in [10]. Additionally, as power domination and infectious power domination consist of a domination step with the addition of the observation or infection step, the domination number gives the second inequality.

Proposition 2.3.4. *For any hypergraph $\mathcal{H}$, $\gamma_{P_I}(\mathcal{H}) \leq \gamma_P(\mathcal{H}) \leq \gamma(\mathcal{H})$.*

The first inequality in Proposition 2.3.4 is not an equality, as shown in Example 2.3.5.

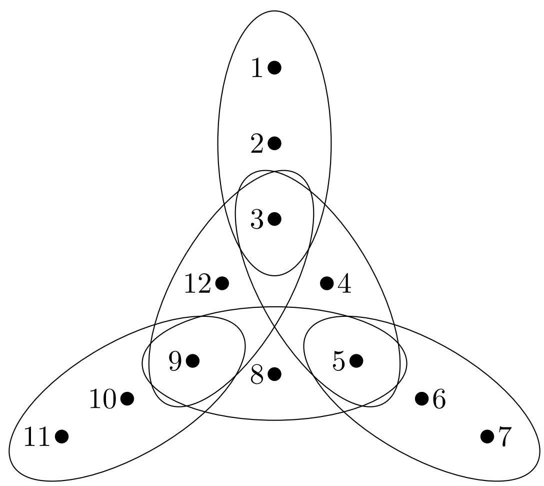

Figure 2.1 A hypergraph $\mathcal{H}^{(3)}$ with $\gamma_{P_I}(\mathcal{H}^{(3)}) < \gamma_P(\mathcal{H}^{(3)})$.

Example 2.3.5. For the hypergraph $\mathcal{H}^{(3)}$ in Figure 2.1, $\gamma_{P_I}(\mathcal{H}^{(3)}) = 1 < \gamma_P(\mathcal{H}^{(3)}) = 2$.

By symmetry, we need only check $\{2\}$, $\{3\}$, and $\{4\}$ as possible infectious power dominating sets or power dominating sets of size 1.

- $S_0 = \{2\}$: 2 observes 1 and 3. Then 1 and 2 have no unobserved neighbors and 3 has unobserved neighbors in $\{3, 12, 9\}$ and $\{3, 4, 5\}$, and no other subset of $\{1, 2, 3\}$ is contained edge with unobserved vertices so no observation (or infection) step can occur.

- $S_0 = \{4\}$: 4 observes $\{3, 4, 5\}$. Vertex 3 is in $\{1, 2, 3\}$ and $\{3, 12, 9\}$. Vertex 5 is in $\{5, 6, 7\}$ and $\{9, 8, 5\}$. Thus no more subsets of the observed vertices can observe an edge, nor can an infection step occur.

- $S_0 = \{3\}$: 3 observes 1, 2, 4, 5, 9, and 12. The only observed vertices adjacent to unobserved vertices are 5 and 9. Vertex 5 is contained in $\{9, 8, 5\}$ and $\{5, 6, 7\}$ so cannot observe an edge by itself. Similarly, $\{9\}$ cannot observe an edge. Thus no observation step can occur and so $\{3\}$ is not a power dominating set. However, $\{9,5\}$ can infect $\{9,8,5\}$. Then 5 infects $\{5,6,7\}$ and 9 infects $\{9,10,11\}$. Thus $\gamma_{P_I}(\mathcal{H}^{(3)}) = 1$.

For the power domination number, no one vertex is a power dominating set and so $\gamma_P(\mathcal{H}^{(3)}) > 1$. There is a power dominating set of size two: let $S_0 = \{3,5\}$. In the domination step, vertices 1, 2, 4, 6, 7, 8, 9, and 12 become observed. Then $\{9\}$ observes $\{9,10,11\}$. Therefore $\{3,5\}$ is a power dominating set of $\mathcal{H}^{(3)}$ and $\gamma_P(\mathcal{H}^{(3)}) = 2$.

Next we present examples showing that the infection number can be drastically different from the power domination number and infectious power domination number, particularly in the case of a k-uniform hypergraph when n is large and k is small. The complete k-uniform hypergraph, $\mathcal{K}_n^{(k)}$, is the k-uniform hypergraph with vertex set $\{1,2,\ldots,n\}$ and edge set all k-sets of the vertex set.

Proposition 2.3.6. [3, Lemma 3.1] $I\left(\mathcal{K}_n^{(k)}\right) = n-k+1$.

The next result is immediate as $N[v] = V(\mathcal{K}_n^{(k)})$ for any $v \in V(\mathcal{K}_n^{(k)})$.

Proposition 2.3.7. $\gamma_{P_I}\left(\mathcal{K}_n^{(k)}\right) = \gamma_P\left(\mathcal{K}_n^{(k)}\right) = \gamma\left(\mathcal{K}_n^{(k)}\right) = 1$.

For a less trivial example of the potential gap between the infection number and the infectious power domination number, we consider the *complete k-partite hypergraph*, $\mathcal{K}_{n_1,n_2,\ldots,n_k}^{(k)}$, which is the hypergraph that has its vertex set partitioned into k disjoint parts $V_1,\ldots,V_k$ where $|V_i| = n_i$. The edge set is the set of all k-sets with exactly one element from each V_i. Note that $\mathcal{K}_{n_1,n_2,\ldots,n_k}^{(k)}$ is k-uniform by definition.

Proposition 2.3.8. [3, Lemma 3.4] $I\left(\mathcal{K}_{n_1,\ldots,n_k}^{(k)}\right) = n_1+n_2+\cdots+n_k-k$.

Proposition 2.3.9. *For the complete k-partite hypergraph, we have the following:*

$$\gamma\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) = \begin{cases} 1 & \min\limits_{1\leq \ell\leq k}(n_\ell) = 1 \\ 2 & \text{otherwise} \end{cases}$$

$$\gamma_P\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) = \gamma_{P_I}\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) = \begin{cases} 1 & \min\limits_{1\leq \ell\leq k}(n_\ell) \leq 2 \\ 2 & \text{otherwise} \end{cases}.$$

Proof. If any $n_i = 1$, then the sole vertex in V_i is adjacent to every other vertex and so $1 = \gamma\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq \gamma_P\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq \gamma_{P_I}\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq 1$.

For the remainder of the proof, assume that $n_\ell \geq 2$ for all ℓ. To determine the domination number, consider any vertex $v_i \in V_i$. We see that v_i has at least one non-neighbor and so $\gamma\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq 2$. Consider $S_0 = \{v, u\}$ where v and u are adjacent (in different parts). Then v is adjacent to all non-neighbors of u and u is adjacent to all non-neighbors of v. Therefore, $\{u, v\}$ is a domination set and so $2 = \gamma\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq \gamma_P\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq \gamma_{P_I}\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right)$.

Next we consider the case in which $\min_{1\leq\ell\leq k}(n_\ell) \geq 3$. We will show that $\gamma_{P_I}\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq 2$. Choosing one vertex $v_i \in V_i$ infects all of the neighbors of v_i in the domination step and the uninfected vertices consist of the $n_i - 1 \geq 2$ non-neighbors of v_i. Let w_i, w_i' be two of these non-neighbors. Suppose A is a set of infected vertices such that $A \cup \{w_i\} \subseteq e$. Then $e' = (e \setminus \{w_i\}) \cup \{w_i'\}$ is also an edge. However, $A \cup \{w_i'\} \subseteq e'$ and $w_i' \notin e$. Thus the non-neighbors of v_i cannot be infected. Therefore, $2 \leq \gamma_{P_I}\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \leq \gamma_P\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \leq \gamma\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) = 2$.

For the remainder of the proof, we assume that, without loss of generality, $n_1 = 2$. Let $V_1 = \{v_1, v_1'\}$ and consider $S_0 = \{v_1\}$. We will show that $\gamma_P\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \leq 1$. For the power domination number, after the domination step only v_1' is unobserved. Let $v_j \in V_j$ with $j \neq 1$. Then there exists an edge e such that $v_1', v_j \in e$. Moreover, as v_1' is the only unobserved vertex, all unobserved neighbors of v_j are in one edge incident to v_j, namely, in e, and so v_1' becomes observed. Therefore, $1 \geq \gamma_P\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq \gamma_{P_I}\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) \geq 1$. □

Again we see that if the number of vertices is large and k is small, the discrepancy between the infection number and the infectious power domination number may be large.

2.4 General bounds

In this section, we give bounds for the power domination number and infectious power domination number in terms of the degrees of the vertices, the number of edges, the size of the edges, and the number of vertices.

Proposition 2.4.1. *Let $\mathcal{H}$ be a connected hypergraph. If $\mathcal{H}$ has at least one vertex of degree at least 3, then $\gamma_{P_I}(\mathcal{H}) \leq \gamma_P(\mathcal{H}) \leq |\{v \in V(\mathcal{H}) : \deg(v) \geq 3\}|$. If $\deg(v) \leq 2$ for all vertices v of $\mathcal{H}$, then $\gamma_{P_I}(\mathcal{H}) = \gamma_P(\mathcal{H}) = 1$.*

Proof. First, assume that $\mathcal{H}$ has at least one vertex of degree at least 3. Let S_0 be the set of vertices with degree at least 3. After the domination step, any remaining unobserved vertex has degree at most two. Moreover, each vertex in $S_1 = N(S_0) \setminus S_0$ has degree at most two. One of these two edges contains only observed vertices and so each vertex in S_1 can observe the precisely one edge containing observed vertices incident to it. Each of these newly observed vertices are now in at most one edge containing unobserved vertices and so can observe that remaining edge. This continues until the entire graph is observed.

On the other hand, if $\deg(v) \leq 2$ for all $v \in V(\mathcal{H})$, then select one vertex. After the domination step, the entire graph will become observed in the same way as in the previous case. □

Proposition 2.4.2. *For any connected hypergraph $\mathcal{H}$ with at least two edges, we have*

$$\gamma_{P_I}(\mathcal{H}) \leq \gamma_P(\mathcal{H}) \leq |E(\mathcal{H})| - 1.$$

This bound is tight.

Proof. If we select one vertex from each edge of $\mathcal{H}$ save one, then in the domination step we observe all but at most one edge. This edge is then the unique edge containing unobserved vertices and so can be observed via the observation step as $\mathcal{H}$ is connected.

It follows that any connected hypergraph $\mathcal{H}$ with exactly two edges has $\gamma_{P_I}(\mathcal{H}) = 1$ and so we see that the bound is tight. □

The next upper bound is similar to Proposition 1.2 in [3].

Proposition 2.4.3. *Let $\mathcal{H}$ be a nontrivial hypergraph on n vertices with at least two edges and let k be the size of the largest edge in $\mathcal{H}$. Then $\gamma_{P_I}(\mathcal{H}) \leq \gamma_P(\mathcal{H}) \leq n - k$. This bound is tight.*

Proof. Let e be the largest edge of $\mathcal{H}$. Let $S_0 = V(\mathcal{H}) \setminus e$. As $\mathcal{H}$ is connected with at least two edges, at least one vertex, say v, in e is adjacent to a vertex which is not in e, i.e. v is adjacent to a vertex in S_0. Thus in the domination step v becomes observed. Then all of the unobserved neighbors of v are contained in e and so they become observed in the observation step.

Consider the hypergraph $\mathcal{H}$ consisting of n vertices with two edges: one edge containing $n-1$ vertices and the other containing one vertex from the first edge and the remaining vertex. The vertex in the intersection of the two edges infects all of the vertices in the domination step. Thus $\gamma_{P_I}(\mathcal{H}) = 1 = n - (n-1)$. □

For another upper bound, we may utilize the following domination bound from [9] with Proposition 2.3.4.

Proposition 2.4.4. [9, Theorem 2] *If $\mathcal{H}$ is a hypergraph with all edges of size at least three and no isolated vertex then $\gamma(\mathcal{H}) \leq \frac{|V(\mathcal{H})|}{3}$.*

Corollary 2.4.5. *If $\mathcal{H}$ is a hypergraph with all edges of size at least three and no isolated vertex then $\gamma_{P_I}(\mathcal{H}) \leq \gamma_P(\mathcal{H}) \leq \frac{|V(\mathcal{H})|}{3}$.*

Corollary 2.4.5 does not utilize the observation (or infection) step. For graphs, the well known domination upper bound of $\frac{|V(G)|}{2}$ was improved for power domination to $\frac{|V(G)|}{3}$ in [11]. We conjecture the following similar result.

Conjecture 2.4.6. *For any connected hypergraph $\mathcal{H}$ on at least 4 vertices with $|e| \geq 3$ for all $e \in E(\mathcal{H})$, then $\gamma_{P_I}(\mathcal{H}) \leq \gamma_P(\mathcal{H}) \leq \frac{|V(\mathcal{H})|}{4}$.*

The X-*private neighborhood* of a vertex $v \in X$ is the set

$$\mathrm{pn}(v, X) = N(v) \setminus \bigcup_{x \in X \setminus \{v\}} N[x],$$

a variant of the definition in [11]. The members of $\mathrm{pn}(v, X)$ are the X-*private neighbors of* v. Zhao, Kang, and Chang's proof from [11] is a counting argument in which they find two S_0-private neighbors for each vertex in the power dominating set, giving the bound via the inequality $|V(G)| \geq |S_0| + 2|S_0|$.

However, this strategy does not translate to hypergraphs. Consider the hypergraph $\mathscr{L}_1^{(3)}$ shown in Figure 2.2. We will use $\mathscr{L}_1^{(3)}$ to build a family of hypergraphs which achieves the bound in Conjecture 2.4.6, but in which any minimum power dominating set contains a vertex with at most two private neighbors.

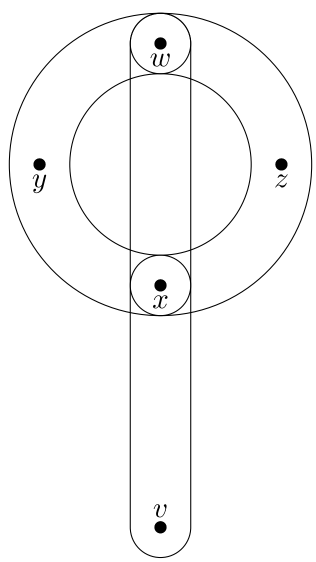

Figure 2.2 Shown is $\mathscr{L}_1^{(3)}$. The edges are $\{w, x, v\}$, $\{w, x, y\}$, and $\{w, x, z\}$.

The hypergraph $\mathscr{L}_q^{(3)}$ is constructed by taking q copies of $\mathscr{L}_1^{(3)}$, say $\mathscr{L}_{1,i}^{(3)}$, with vertex sets $\{v_i, x_i, y_i, z_i, w_i\}$ for each $1 \leq i \leq q$. Identify w_i with v_{i+1} for $1 \leq i \leq q-1$ and w_q with v_1. Then

$$\mathscr{L}_q^{(3)} = \left(\bigcup_{1 \leq i \leq q} \{x_i, y_i, z_i, w_i\}, \bigcup_{1 \leq i \leq q} E\left(\mathscr{L}_{1,i}^{(3)}\right) \right).$$

For an example, $\mathscr{L}_3^{(3)}$ is shown in Figure 2.3. We construct the family $\mathscr{L} = \{\mathscr{L}_q^{(3)} : q \geq 2\}$.

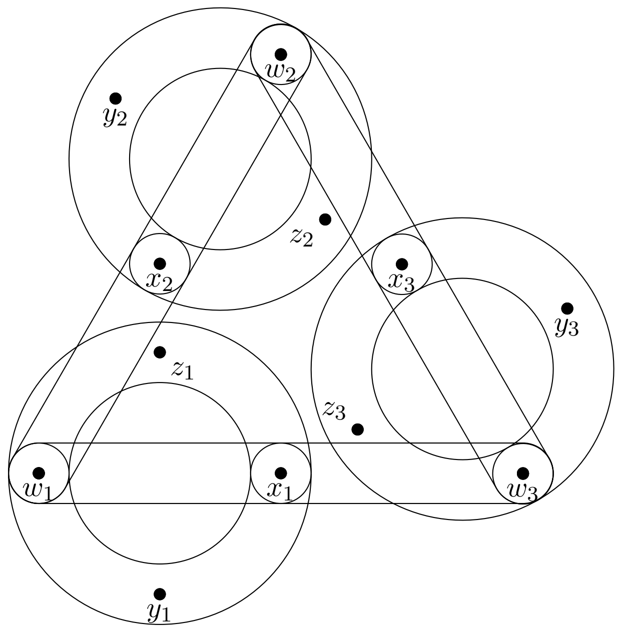

Figure 2.3 Shown is $\mathscr{L}_3^{(3)}$, with the identifications $w_1 = v_2$, $w_2 = v_3$, and $w_3 = v_1$.

Proposition 2.4.7. *The family of hypergraphs $\mathscr{L}$ satisfies*

$$\gamma_{P_I}(\mathscr{L}_q^{(3)}) = \gamma_P(\mathscr{L}_q^{(3)}) = \gamma(\mathscr{L}_q^{(3)}) = q = \frac{|V(\mathscr{L}_q^{(3)})|}{4}$$

for all $\mathscr{L}_q^{(3)} \in \mathscr{L}$. Any minimum power dominating set for a member of $\mathscr{L}$, S_0, contains a vertex with at most two S_0-private neighbors.

Proof. First we show $\gamma_{P_I}(\mathscr{L}_q^{(3)}) \geq q$. Suppose for contradiction that $\gamma_{P_I}(\mathscr{L}_q^{(3)}) < q$. Then there exists j so that $S_0 \cap \{x_j, y_j, z_j, w_j\} = \varnothing$. However, there is no way for y_j to become infected as its only neighbors, x_j and w_j, are also in an edge with the uninfected vertex z_j. This is a contradiction and so $\gamma_P(\mathscr{L}_q^{(3)}) \geq q$. This also implies that for all $1 \leq i \leq q$, at least one of $\{x_i, y_i, z_i, w_i\}$ must be in any infectious power dominating set or any power dominating set.

For equality, observe that $\{x_i : 1 \leq i \leq q\}$ is a dominating set of $\mathscr{L}_q^{(3)}$ of size q. Therefore, $q \leq \gamma_{P_I}(\mathscr{L}_q^{(3)}) \leq \gamma_P(\mathscr{L}_q^{(3)}) \leq \gamma(\mathscr{L}_q^{(3)}) \leq q$.

Now consider a minimum infectious power dominating set of $\mathscr{L}_q^{(3)}$, S_0. We will show that S_0 contains a vertex with at most two private neighbors.

If for some j, $y_j \in S_0$, then as y_j has only two neighbors, S_0 contains a vertex with at most two S_0-private neighbors. The same argument applies to z_j. Thus, we need only consider minimum power dominating sets consisting only of w and x type vertices and for all $1 \le i \le q$, either w_i or x_i must be in S_0.

Let $x_i \in S_0$. Without loss of generality, $N(x_i) = \{y_i, z_i, w_i, w_{i-1}\}$. Either w_{i+1} or x_{i+1} must be in S_0 and w_i is adjacent to both of these. Thus, $w_i \notin \mathrm{pn}(x_i, S_0)$. Similarly, either x_{i-1} or w_{i-1} must be in S_0, so $w_{i-1} \notin \mathrm{pn}(x_i, S_0)$. Hence $\mathrm{pn}(x_i, S_0) = \{y_i, z_i\}$. Finally, if $S_0 = \{w_i : 1 \le i \le p\}$, then $\mathrm{pn}(w_i, S_0) = \{y_i, z_i\}$ for all i, as x_i is a common neighbor with w_{i-1} and x_{i+1} is a common neighbor with w_{i+1}.

Therefore, $\mathscr{L}_q^{(3)}$ has at least one vertex in every minimum power dominating set which has at most two private neighbors. □

2.5 Infectious power domination number for particular hypergraphs

We have the following easy consequence of Observation 2.3.3.

Observation 2.5.1. *For any hypergraph $\mathcal{H}$, if $I(\mathcal{H}) = 1$, then $\gamma_{P_I}(\mathcal{H}) = 1$.*

Thus we obtain the several results directly from [3], after some definitions. A hypergraph is an *interval hypergraph* if there is a linear ordering of the vertices so that each edge consists of consecutive vertices. A *hypercycle* is a connected hypergraph with edge set $e_1, \dots, e_\ell$ with $\ell \ge 4$ so that $e_i \cap e_j \neq \varnothing$ if and only if $i - j \equiv \pm 1 \mod \ell$, [3].

Observation 2.5.2.

1. [3, Prop. 2.1] *If $E(\mathcal{H}) = \{V(\mathcal{H})\}$, then $\gamma_{P_I}(\mathcal{H}) = 1$.*

2. [3, Prop 3.2] *Let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph with $k \ge 3$. Then there exists a k-uniform hypergraph $\mathcal{H}'^{(k)}$ such that $V(\mathcal{H}^{(k)}) \subseteq V(\mathcal{H}'^{(k)})$ and $E(\mathcal{H}^{(k)}) \subseteq E(\mathcal{H}'^{(k)})$ with $\gamma_{P_I}(\mathcal{H}^{(k)}) = 1$.*

3. [3, Lem. 4.3] *If $\mathcal{H}$ is a connected interval hypergraph then $\gamma_{P_I}(\mathcal{H}) = 1$.*

4. [3, Prop. 4.5] *If $\mathcal{H}$ is a hypercycle with a vertex of degree 1, then $\gamma_{P_I}(\mathcal{H}) = 1$.*

In fact, Observation 2.5.2.4 is true without the restriction on vertex degrees, which follows from the next proposition.

A *circular arc interval hypergraph* is a hypergraph $\mathcal{H}$ with n vertices with a circular order of the vertices so that every edge is composed of consecutive vertices. Using this circular ordering of the vertices, the first end point of each edge is unique. If $\mathcal{H}$ has m edges, let the first end points be denoted by $v_1, v_2, \ldots, v_m$ with corresponding edges $e_1, \ldots, e_m$, in a similar way to [10]. When $\mathcal{H}$ is connected, we may choose an ordering so that $v_1, e_1, v_2, e_2, \ldots, e_{m-1}, v_m$ forms a path.

Proposition 2.5.3. *For any connected circular arc interval hypergraph $\mathcal{H}$,*

$$\gamma_{P_I}(\mathcal{H}) = \gamma_P(\mathcal{H}) = 1.$$

Proof. Let $S_0 = \{v_1\}$, the left endpoint of e_1. After the domination step, each vertex of e_1 is observed. As $\mathcal{H}$ is connected, this means that v_2 is observed. Since $\mathcal{H}$ is a circular arc hypergraph, any edge containing v_2 but not containing v_1 must have a left endpoint that is after v_1. The first left endpoint after v_1 is v_2. As v_2 is the left endpoint of e_2, the only edge containing v_2 and unobserved vertices is e_2, because any other edge containing v_2 would also have to contain v_1 and so would have become observed in the domination step. Thus v_2 observes e_2. In a similar way, v_3 is in e_2 and so v_3 observes e_3. We continue this process until all edges are observed. □

A *Berge cycle* in a hypergraph $\mathcal{H}$ is a sequence

$$C_m = (v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_1)$$

in which the v_i are distinct vertices, the e_i are distinct edges, $v_i, v_{i+1} \in e_i$ for $1 \leq i \leq m-1$, and $v_1, v_m \in e_m$. A *hypertree* is a connected hypergraph which contains no Berge cycle. A hypergraph is *linear* if distinct edges intersect in at most one vertex. Note that any hypertree is linear. To see this, consider if two edges e_1 and e_2 share vertices v_1 and v_2. Then $v_1e_1v_2e_2v_1$ is a Berge cycle. A *major vertex* is a vertex of degree at least 3 [7]. A *spider* is a nonempty hypertree with at most one major vertex. A *spider cover* of a hypertree $\mathcal{T}$ is a partition of $V(\mathcal{T})$, $V_1, \ldots, V_\ell$, such that each subset induces a spider. The *spider number* of a hypertree $\mathcal{T}$ is the minimum size of a spider cover of $\mathcal{T}$, denoted by $\mathrm{sp}(\mathcal{T})$.

Theorem 2.5.4. [7, Theorem 7] *For any hypertree $\mathcal{T}$, $\gamma_P(\mathcal{T}) = \mathrm{sp}(\mathcal{T})$.*

As a direct result of Theorem 2.5.4 and Proposition 2.3.4, we have the following proposition.

Corollary 2.5.5. *For any hypertree $\mathcal{T}$, $\gamma_{P_I}(\mathcal{T}) \leq \mathrm{sp}(\mathcal{T})$.*

For the infectious power domination number, this bound is not an equality as shown in Example 2.5.6.

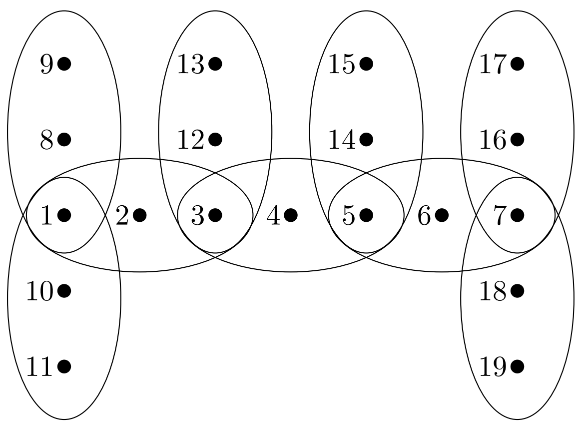

Figure 2.4 A hypertree $\mathcal{T}^{(3)}$ with $\gamma_{P_I}(\mathcal{T}^{(3)}) < \gamma_P(\mathcal{T}^{(3)})$.

Example 2.5.6. For the hypertree $\mathcal{T}^{(3)}$ in Figure 2.4, $\gamma_{P_I}(\mathcal{T}^{(3)}) = 2 < 3 = \gamma_P(\mathcal{T}^{(3)})$.

We first show that there is no spider cover of $\mathcal{T}^{(3)}$ of size 2. The major vertices of $\mathcal{T}^{(3)}$ are $1, 3, 5, 7$. Suppose for contradiction that we have a spider cover V_1, V_2, and without loss of generality, let $1 \in V_1$.

First suppose that $1, 3, 5, 7 \in V_1$. However, this means that at least two of

$$\{8, 9\}, \{10, 11\}, \{12, 13\}, \{14, 15\}, \{16, 17\}, \{18, 19\}$$

are in V_2 and so we have a contradiction as V_2 must induce a connected hypergraph.

Next consider if $1, 3, 5 \in V_1$ and $7 \in V_2$. However, then at least one of

$$\{8, 9\}, \{10, 11\}, \{12, 13\}, \{14, 15\}$$

is not in V_1. However, this means that V_2 is disconnected as $7 \in V_2$. Hence we cannot have any three major vertices in either V_1 and similarly for V_2.

Thus we have at most two elements of $\{1, 3, 5, 7\}$ in V_1. Note that $1, 7 \in V_1$ or $1, 5 \in V_1$ would imply that $3 \in V_1$ as well because V_1 induces a connected hypergraph. Thus, we must have $1, 3 \in V_1$ and $5, 7 \in V_2$. If $4 \in V_1$, then 12 or one of $8, 10$ must be in V_2, but this is a contradiction to the connectedness of the subhypergraph induced by V_2. Similarly, $4 \notin V_2$. Therefore, $\text{sp}(\mathcal{T}^3) > 2$.

Observe that

$$\{\{1, 2, 8, 9, 10, 11\}, \{3, 4, 5, 12, 13, 14, 15\}, \{6, 7, 16, 17, 18, 19\}\}$$

is a spider cover of $\mathcal{T}^{(3)}$. Thus, $\text{sp}(\mathcal{T}^{(3)}) = 3$. By Theorem 2.5.4, $\text{sp}(\mathcal{T}^{(3)}) = \gamma_P(\mathcal{T}^{(3)}) = 3$.

We now consider possible infectious power dominating sets. By symmetry, for sets of size 1 we need only check $1, 2, 3, 4, 8$ and 12.

- $S_0 = \{1\}$: 1 infects $2, 3, 8, 9, 10, 11$. Then vertex 3 is the only infected vertex which has uninfected neighbors, however these neighbors occur in both edge $\{3, 12, 13\}$ and edge $\{3, 4, 5\}$, so no infection step can occur.

- $S_0 = \{2\}$: 2 infects $1, 3$. Then vertex 1 is adjacent to uninfected vertices in both $\{1, 8, 9\}$ and $\{1, 10, 11\}$. Vertex 3 is adjacent to uninfected vertices in both $\{3, 4, 5\}$ and $\{3, 12, 13\}$. No other subset of infected vertices is contained in an edge also containing uninfected vertices and so no infection step can occur.

- $S_0 = \{3\}$: 3 infects $1, 2, 4, 5, 12, 13$. However, 1 is in both $\{1, 8, 9\}$ and $\{1, 10, 11\}$. Vertex 5 is in both $\{5, 14, 15\}$ and $\{5, 6, 7\}$.

- $S_0 = \{4\}$ or $S_0 = \{12\}$: Since $\{3\}$ is not an infectious power dominating set, neither is $\{4\}$ or $\{12\}$ as $N(4), N(12) \subset N(3)$.

- $S_0 = \{8\}$: Since $\{1\}$ is not an infectious power dominating set, neither is $\{8\}$ as $N(8) \subset N(1)$.

Thus there is no infectious power dominating set of size 1. Next consider $S_0 = \{1, 7\}$. After the domination step, $1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 16, 17, 18,$ and 19 are infected. Then $\{3, 5\}$ infects 4. Finally, $\{3\}$ infects $\{12, 13\}$ and $\{5\}$ infects $\{14, 15\}$. Thus $\gamma_{P_I}(\mathcal{T}^{(3)}) = 2$.

Therefore, we have $\gamma_{P_I}(\mathcal{T}^{(3)}) = 2 < \mathrm{sp}(\mathcal{T}^{(3)}) = 3$.

2.6 Hypergraph operations

2.6.1 Edge/vertex removal

Let $\mathcal{H}$ be a hypergraph with $e \in E(\mathcal{H})$. The hypergraph $\mathcal{H} - e$ has vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H}) \setminus \{e\}$.

Theorem 2.6.1. *Let $\mathcal{H}$ be a hypergraph with an edge e. Then $\gamma_{P_I}(\mathcal{H}) - 1 \leq \gamma_{P_I}(\mathcal{H} - e) \leq \gamma_{P_I}(\mathcal{H}) + |e| - 1$. These bounds are tight.*

Proof. For the lower bound, if we have an infectious power dominating set $\hat{S}$ for $\mathcal{H} - e$ then adding an edge may ruin uniqueness for infection. By adding one vertex of e to $\hat{S}$, this new edge is infected in the domination step. Then $\hat{S}$ will infect the remainder of the graph as it did in $\mathcal{H} - e$. Thus we have $\gamma_{P_I}(\mathcal{H}) \leq \gamma_{P_I}(\mathcal{H} - e) + 1$ and so $\gamma_{P_I}(\mathcal{H}) - 1 \leq \gamma_{P_I}(\mathcal{H} - e)$.

For the upper bound, consider an infectious power dominating set S for $\mathcal{H}$. At some point in the infectious power domination process on $\mathcal{H}$, at least one vertex of e must become infected in order for the rest of e to become infected. Call this vertex v and let $e' = e \setminus \{v\}$. When we remove edge e, then the vertices in e' may no longer be infected. Consequently, $S \cup e'$ is an infectious power dominating set of $\mathcal{H} - e$ and has at most $\gamma_{P_I}(\mathcal{H}) + |e| - 1$ vertices.

For tightness of the lower bound, consider $\mathcal{K}^{(k)}_{n_1,\ldots,n_k}$ with $n_1 = 3$ and $n_i \geq 3$ for all $i \neq 1$. By Proposition 2.3.9, $\gamma_{P_I}\left(\mathcal{K}^{(k)}_{n_1,\ldots,n_k}\right) = 2$. Remove edge $e = \{v_1, \ldots, v_k\}$ with $v_i \in V_i$ and let v_1 have non-neighbors a and b. Let $S_0 = \{a\}$. In the domination step, every vertex except for v_1 and b have been infected. Let $A = \{v_2, \ldots, v_k\}$ and $e = A \cup \{b\}$. As $A \cup \{v_1\}$ is not an edge, A infects

b. Then v_1 is the one remaining infected vertex and so becomes infected by any vertex adjacent to v_1. Therefore $\gamma_{P_I}(\mathcal{K}^{(k)}_{n_1,\ldots,n_k} - e) = 1$.

To see the tightness of the upper bound, consider any connected linear interval hypergraph $\mathcal{H}$ with first edge e. By Observation 2.5.2 Part 3, $\gamma_{P_I}(\mathcal{H}) = 1$. The hypergraph $\mathcal{H} - e$ has $|e|-1$ isolated vertices and the remaining vertices form a connected interval hypergraph. Thus any infectious power dominating set must contain the $|e| - 1$ isolated vertices and the vertex that forms an infectious power dominating set for the remainder of the graph. Thus $\gamma_{P_I}(\mathcal{H} - e) = 1 + |e| - 1 = \gamma_{P_I}(\mathcal{H}) + |e| - 1$.

□

For the power domination number, we see that the proofs of the upper and lower bounds in Theorem 2.6.1 also apply. Thus we have the following.

Corollary 2.6.2. *Let $\mathcal{H}$ be a hypergraph with an edge e. Then $\gamma_P(\mathcal{H}) - 1 \leq \gamma_P(\mathcal{H} - e) \leq \gamma_P(\mathcal{H}) + |e| - 1$.*

We note that removing a vertex and its corresponding edges may drastically change the power domination number. Adding a dominating vertex (i.e., a vertex that is adjacent to every vertex in the graph) will lower the infectious power domination number to 1 regardless of the remaining graph structure. Similarly, removing such a vertex may drastically increase the infectious power domination number.

2.6.2 Linear sums

We will use the term *linear sum* of hypergraphs $\mathcal{H}_1, \mathcal{H}_2$ to describe the hypergraph defined by $\mathcal{H}_1 \star \mathcal{H}_2 = (V(\mathcal{H}_1) \cup V(\mathcal{H}_2), \{e_1 \cup e_2 : e_1 \in E(\mathcal{H}_1), e_2 \in E(\mathcal{H}_2)\})$. Note that the linear sum of two 2-uniform hypergraphs (i.e. two graphs) will yield a 4-uniform hypergraph. As the vertex set is the union of the vertex sets of the input hypergraphs, we call this operation a linear sum and use the notation $\mathcal{H}_1 \star \mathcal{H}_2$ instead of the term *direct product* and notation $\mathcal{H}_1 \times \mathcal{H}_2$ as used in [3].

Theorem 2.6.3. *For any connected hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$, $\gamma_{P_I}(\mathcal{H}_1 \star \mathcal{H}_2) \leq \gamma_P(\mathcal{H}_1 \star \mathcal{H}_2) \leq \gamma(\mathcal{H}_1 \star \mathcal{H}_2) \leq 2$. Furthermore, if $\gamma_{P_I}(\mathcal{H}_1) = 1$ or $\gamma_{P_I}(\mathcal{H}_2) = 1$ then $\gamma_{P_I}(\mathcal{H}_1 \star \mathcal{H}_2) = 1$.*

Proof. Take $v_1 \in V(\mathcal{H}_1)$ and $v_2 \in V(\mathcal{H}_2)$ and let $S_0 = \{v_1, v_2\}$. As each hypergraph is connected, $v_1 \in e_1 \in E(\mathcal{H}_1)$ and for any vertex $w_2 \in V(\mathcal{H}_2)$, there exists $f_2 \in E(\mathcal{H}_2)$ such that $w_2 \in f_2$. Then $v_1, w_2 \in e_1 \cup f_2 \in E(\mathcal{H}_1 \star \mathcal{H}_2)$ and so w_2 is adjacent to v_1. Similarly, we see that any vertex $w_1 \in V(\mathcal{H}_1)$ is adjacent to v_2. Therefore every vertex in $\mathcal{H}_1 \star \mathcal{H}_2$ is observed in the domination step and so $\gamma_{P_I}(\mathcal{H}_1 \star \mathcal{H}_2) \leq \gamma_P(\mathcal{H}_1 \star \mathcal{H}_2) \leq \gamma(\mathcal{H}_1 \star \mathcal{H}_2) \leq 2$.

Let $\gamma_{P_I}(\mathcal{H}_1) = 1$ with $\{v_1\}$ being an infectious power dominating set. We will show that $\{v_1\}$ is also an infectious power dominating set in $\mathcal{H}_1 \star \mathcal{H}_2$. In the domination step, all vertices of $\mathcal{H}_2$ are infected as are the original neighbors of v_1 from $\mathcal{H}_1$. Suppose that $A \subseteq V(\mathcal{H}_1)$ infected edge $e_1 \in E(\mathcal{H}_1)$ during the infectious power domination process for $\mathcal{H}_1$. This means that for any uninfected vertex $w_1 \in V(\mathcal{H}_1)$, $A \cup \{w_1\} \subseteq e_1' \in E(\mathcal{H}_1)$ implies that $w_1 \in e_1$. Let e_2 be any edge of $\mathcal{H}_2$. Consider $B = A \cup e_2$. Then for any uninfected w_1 in $\mathcal{H}_1 \star \mathcal{H}_2$, if $B \cup \{w_1\} \subseteq e' \cup e_2$ then $A \cup \{w_1\} \subseteq e'$ and so $w_1 \in e_1$. Thus $w_1 \in e_1 \cup e_2$ and so B infects $e_1 \cup e_2$. Continuing in this way, using the infection steps as in $\mathcal{H}_1$ but with the addition of the vertices in e_2, every vertex of $\mathcal{H}_1$ becomes infected. Therefore, $\gamma_{P_I}(\mathcal{H}_1 \star \mathcal{H}_2) = 1$. □

Corollary 2.6.4. *For any hypergraph $\mathcal{H}_1$ with $\gamma_{P_I}(\mathcal{H}_1) = 1$ and any hypergraphs $\mathcal{H}_2, \ldots, \mathcal{H}_\ell$, we have $\gamma_{P_I}(\mathcal{H}_1 \star \mathcal{H}_2 \cdots \star \mathcal{H}_\ell) = 1$.*

Theorem 2.6.3 shows that the infectious power domination number cannot increase for linear sums of hypergraphs. In particular, if $\mathcal{H}_1$ or $\mathcal{H}_2$ has infection power domination number 1, the linear sum must also have infection power domination number 1. This is a stark contrast with the following result for the infection number.

Proposition 2.6.5. [3, Proposition 6.5] *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are both hypergraphs with more than one edge and $I(\mathcal{H}_1) = I(\mathcal{H}_2) = 1$ then $I(\mathcal{H}_1 \star \mathcal{H}_2) = 2$.*

2.6.3 Cartesian products

For an edge $e = \{w_1, \ldots, w_k\}$ and a vertex v define

$$e \times v = \{(w_1, v), \ldots, (w_k, v)\} \text{ and } v \times e = \{(v, w_1), \ldots, (v, w_k)\}$$

as in [3]. The *Cartesian product* of two hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted $\mathcal{H}_1 \square \mathcal{H}_2$, has vertex set $V(\mathcal{H}_1) \times V(\mathcal{H}_2)$, and has edge set

$$\{e_1 \times v_2 : e_1 \in E(\mathcal{H}_1), v_2 \in V(\mathcal{H}_2)\} \cup \{v_1 \times e_2 : v_1 \in V(\mathcal{H}_1), e_2 \in E(\mathcal{H}_2)\}.$$

The infectious power domination number of the Cartesian product of two graphs may be greater than the infectious power domination number of either graph. In Proposition 2.6.6 we see that the infectious power domination number of a k-complete hypergraph with a ℓ-complete hypergraph is one such example. Recall from Proposition 2.3.7 that $\gamma_{P_I}\left(K_n^{(k)}\right) = 1$.

Proposition 2.6.6. *Let $n \leq m$ with $3 \leq k \leq n-1$ and $3 \leq \ell \leq m-1$. Then $\gamma_{P_I}\left(\mathcal{K}_n^{(k)} \square \mathcal{K}_m^{(\ell)}\right) = n-1$.*

Proof. Denote the vertices of $\mathcal{K}_n^{(k)}$ by $v_1, \ldots, v_n$ and the vertices of $\mathcal{K}_m^{(\ell)}$ by $w_1, \ldots, w_m$. Then all vertices of $\mathcal{K}_n^{(k)} \square \mathcal{K}_m^{(\ell)}$ are of the form (v_i, w_j). For each $w_r \in V(\mathcal{K}_m^{(\ell)})$, define $K_n^{(k)} \times w_r = \{(v_i, w_r) : 1 \leq i \leq n\}$. Similarly, for each $v_s \in V(\mathcal{K}_n^{(k)})$ we define $v_s \times K_m^{(\ell)} = \{(v_s, w_j) : 1 \leq j \leq m\}$. Note that every edge of $\mathcal{K}_n^{(k)} \square \mathcal{K}_m^{(\ell)}$ is a subset of exactly one set of one of these forms.

Consider the set $S_0 = \{(v_1, w_1), (v_2, w_2), \ldots, (v_{n-1}, w_{n-1})\}$. In the domination step, as any vertex (v_i, w_j) is adjacent to each vertex in both $\mathcal{K}_n^{(k)} \times w_j$ and $v_i \times \mathcal{K}_m^{(\ell)}$, the only uninfected vertices are $\{(v_n, w_n), (v_n, w_{n+1}), \ldots, (v_n, w_m)\}$. However, this means that for each $n \leq x \leq m$, the set

$$A_x = \{(v_{n-k-1}, w_x), (v_{n-k}, w_x), \ldots, (v_{n-1}, w_x)\}$$

consists of infected vertices and is contained in the edge $e = A_x \cup \{(v_n, w_x)\}$, an edge made of vertices from $\mathcal{K}_n^{(k)} \times w_x$. Moreover, A_x is not contained in any other edge containing an uninfected vertex as every other vertex in $\mathcal{K}_n^{(k)} \times w_x$ is infected. Hence A_x infects (v_n, w_x) for $n \leq x \leq m$ and so all vertices become infected. Therefore, $\gamma_{P_I}\left(\mathcal{K}_n^{(k)} \square \mathcal{K}_m^{(\ell)}\right) \leq n-1$.

Assume for eventual contradiction that there exists some infectious power dominating set S_0 with $|S_0| \leq n-2$. By the Pigeonhole Principle, since $m \geq n \geq 4$, there are at least two i so that $v_i \times \mathcal{K}_m^{(\ell)}$ contains no vertex of S_0. Without loss of generality, let these be i and i'. In the same way, there exists j and j' so that $\mathcal{K}_n^{(k)} \times w_j$ and $\mathcal{K}_n^{(k)} \times w_{j'}$ contain no vertex of S_0. Now consider the vertices $(v_i, w_j), (v_i, w_{j'}), (v_{i'}, w_j)$, and $(v_{i'}, w_{j'})$.

We show that there is no set that can infect these vertices. Any set which could infect (v_i, w_j) must be of the form $A \subseteq \mathcal{K}_n^{(k)} \times w_j \setminus \{(v_i, w_j), (v_{i'}, w_j)\}$ with $|A| \leq k-1$. We see that by construction of the k-complete hypergraph, there is an edge e of $\mathcal{K}_n^{(k)} \times w_j$ containing $A \cup \{(v_i, w_j)\}$. There is also an edge e' so that $A \cup \{(v_{i'}, w_j)\} \subseteq (e \setminus \{(v_i, w_j)\}) \cup \{(v_{i'}, w_j)\} = e'$. Thus any such set A of infected vertices that are in an edge with (v_i, w_j) are also in a different edge with $(v_{i'}, w_j)$ and so no such A can infect (v_i, w_j) or $(v_{i'}, w_j)$. In a similar way, we can consider $A' \subseteq \mathcal{K}_n^{(k)} \times w_{j'} \setminus \{(v_i, w_{j'}), (v_{i'}, w_{j'})\}$ with $|A'| \leq k-1$, $A'' \subseteq v_i \times \mathcal{K}_m^{(\ell)} \setminus \{(v_i, w_j), (v_i, w_{j'})\}$ with $|A''| \leq \ell - 1$, and $A''' \subseteq v_{i'} \times \mathcal{K}_m^{(\ell)} \setminus \{(v_{i'}, w_j), (v_{i'}, w_{j'})\}$ with $|A'''| \leq \ell - 1$ to see that no such subsets can infect any of $(v_i, w_j), (v_i, w_{j'}), (v_{i'}, w_j)$, or $(v_{i'}, w_{j'})$. Therefore, there is no possible set of infected vertices that can infect these four vertices when $|S_0| \leq n-2$. Hence $\gamma_{P_I}\left(\mathcal{K}_n^{(k)} \square \mathcal{K}_m^{(\ell)}\right) \geq n-1$.

□

Bergen et al. [3] established a general upper bound on the infection number of Cartesian products which is greater than the infection number of either hypergraph.

Proposition 2.6.7. [3, Corollary 6.11]. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be hypergraphs, then $I(\mathcal{H}_1 \square \mathcal{H}_2) \leq I(\mathcal{H}_1)\left(I(\mathcal{H}_2) + |E(\mathcal{H}_2)|\right)$.*

2.6.4 Weak coronas

The *weak corona* of a k-uniform hypergraph $\mathcal{G}^{(k)}$ with a $(k-1)$-uniform hypergraph $\mathcal{H}^{(k-1)}$, as defined in [3], is the k-uniform hypergraph denoted by $\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}$, with vertex set

$$V\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) = V(\mathcal{G}^{(k)}) \cup \left(\bigcup_{v \in V(\mathcal{G}^{(k)})} V\left(\mathcal{H}_v^{(k-1)}\right) \right)$$

where $\mathcal{H}_v^{(k-1)}$ is a copy of $\mathcal{H}^{(k-1)}$ corresponding to vertex $v \in V\left(\mathcal{G}^{(k)}\right)$, and edge set

$$E\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) = E\left(\mathcal{G}^{(k)}\right) \cup \left\{ e_v \cup \{v\} : v \in V\left(\mathcal{G}^{(k)}\right), e_v \in E\left(\mathcal{H}_v^{(k-1)}\right) \right\}.$$

That is, the weak corona is formed by taking a copy of $\mathcal{H}^{(k-1)}$ for each vertex v of $\mathcal{G}^{(k-1)}$ and then adding v to each edge of its copy of $\mathcal{H}^{(k-1)}$.

In [3], it was shown that $I\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) \leq |V(\mathcal{G}^{(k)})| I\left(\mathcal{H}^{(k-1)}\right)$. We obtain a better result, reflecting the strength of the domination step.

Proposition 2.6.8. *For any hypergraphs $\mathcal{G}^{(k)}$ and $\mathcal{H}^{(k-1)}$,*

$$\gamma_{P_I}\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) \leq \gamma_P\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) \leq \gamma\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) \leq \left|V\left(\mathcal{G}^{(k)}\right)\right|.$$

This bound is tight whenever $\mathcal{H}^{(k-1)}$ has at least two edges.

Proof. Observe that $V(\mathcal{G}^{(k)})$ is a dominating set.

For tightness, let $\mathcal{G}^{(k)}$ be any hypergraph and let $\mathcal{H}^{(k-1)}$ be any $(k-1)$-uniform hypergraph with at least two edges. Suppose for contradiction that $\gamma_{P_I}\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) < |V(\mathcal{G}^{(k)})|$ and let S_0 be a minimum infectious power dominating set. Then there exists some $v \in V(\mathcal{G}^{(k)})$ so that $V(\mathcal{H}_v^{(k-1)}) \cup \{v\}$ contains no vertex of S_0. By the construction of the weak corona, the only vertex of $V(\mathcal{H}_v^{(k-1)}) \cup \{v\}$ adjacent to a vertex outside of $V(\mathcal{H}_v^{(k-1)}) \cup \{v\}$ is v. Thus the first vertex of $V(\mathcal{H}_v^{(k-1)}) \cup \{v\}$ to become infected is v. As $\mathcal{H}^{(k-1)}$ has at least two edges, we have $e, e' \in E(\mathcal{H}_v^{(k-1)})$. Then v is adjacent to uninfected vertices in the edges $e \cup \{v\}$ and $e' \cup \{v\}$. However, this implies that $V(\mathcal{H}_v^{(k-1)})$ cannot become infected, a contradiction. Thus $\gamma_{P_I}\left(\mathcal{G}^{(k)} \circ_w \mathcal{H}^{(k-1)}\right) \geq |V(\mathcal{G}^{(k)})|$.

□

Let $P_3^{(2)}$ denote the path graph on 3 vertices.

Corollary 2.6.9. *Let $\mathcal{G}^{(3)}$ be any 3-uniform hypergraph. Then*

$$\gamma_{P_I}\left(\mathcal{G}^{(3)} \circ_w P_3^{(2)}\right) = \gamma_P\left(\mathcal{G}^{(3)} \circ_w P_3^{(2)}\right) = \left|V\left(\mathcal{G}^{(3)}\right)\right|.$$

Corollary 2.6.9 demonstrates an infinite family of 3-uniform hypergraphs that achieves the bound in Conjecture 2.4.6 for both the power domination number and the infectious power domination number. Furthermore, $V(G)$ is an infectious power dominating set for which each vertex has exactly 3 private neighbors.

2.7 Concluding remarks

Several interesting questions remain for both infectious power domination and power domination for hypergraphs.

We have not yet found a generalized improvement of the bound in Corollary 2.4.5 or a counterexample to the bound in Conjecture 2.4.6 and so this remains an open question for further study. We have seen two examples of infinite families that achieve the conjectured bound, in Proposition 2.4.7 and Corollary 2.6.9. Each of these families achieves the bound with a dominating set. However, the family $\mathscr{L}$ indicates that a counting argument similar to that used in [11] will not work for hypergraphs.

While we know that the spider cover number is only an upper bound for the infectious power domination number for hypertrees as seen in Example 2.5.6, we do not know if there is a useful *lower* bound.

Cartesian products also present a large number of open questions. We showed in Proposition 2.6.6 that the infectious power domination number of the Cartesian product of two hypergraphs can be greater than either hypergraph but we do not have a general upper or lower bound.

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CHAPTER 3. PMU-DEFECT-ROBUST POWER DOMINATION

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Abstract

The power domination problem finds the placement of the minimum number of sensors called phasor measurement units (PMUs) needed to monitor an electric power network. We consider the minimum number of sensors and appropriate placement to ensure monitoring when k sensors are allowed to fail. That is, what is the minimum multiset of the vertices, S, such that for every $F \subseteq S$ with $|F| = k$, $S \setminus F$ is a power domination set. Such a set is called a *k-robust power domination set*. This generalizes the work done by Pai, Chang, and Wang in 2010 on vertex-fault-tolerant power domination, which did not allow for multiple sensors to be placed at the same vertex. We provide general bounds and determine the k-robust power domination number of some graph families.

3.1 Introduction

The power domination problem seeks to find the placement of the mimimum number of sensors called phasor measurement units (PMUs) needed to monitor an electric power network. In [7], Haynes et al. defined the power domination problem in graph theoretic terms by placing PMUs at a set of initial vertices and then applying observation rules to the vertices and edges of the graph. This process was simplified by Brueni and Heath in [3].

Pai, Chang, and Wang [10] generalized power domination to create *fault-tolerant power domination* in 2010. They ask for the minimum number of PMUs needed to monitor a power network given that k of the PMUs will fail. Pai, Chang, and Wang allow the placement of only one PMU per vertex, which they define as *vertex-fault-tolerant power domination*. We consider the related problem of the minimum number of PMUs needed to monitor a power network given that k PMUs will fail *but also allow for multiple PMUs to be placed at a given vertex.* We call this *PMU-defect-robust power domination*, as it is not the vertices (PMU locations) that cause a problem with monitoring the network, but the individual PMUs themselves.

In Section 3.2, we review definitions from past work and formally define PMU-defect-robust power domination. We also include some basic results in that section. Section 3.3 gives general bounds for k-robust power domination. Section 3.4 contains results for complete bipartite graphs and Section 3.5 has results for some square grid graphs. We examine trees in Section 3.6. Finally, in Section 3.7, we make suggestions for future work.

3.2 Preliminaries

In this section, we begin by giving relevant graph theory definitions. Then we will define power domination, vertex-fault-tolerant power domination, and PMU-defect-robust power domination. Finally, we include useful properties of the floor and ceiling functions.

3.2.1 Graph theory

A graph G is a set of vertices, $V(G)$, and a set of edges, $E(G)$. Each (unordered) edge consists of a set of two distinct vertices; the edge $\{u, v\}$ is often written as uv. When G is clear, we write $V = V(G)$ and $E = E(G)$. A *path* from v_1 to $v_{\ell+1}$ is a sequence of vertices and edges $v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1}$ so that the v_i are distinct vertices and $v_i \in e_i$ for all i and $v_i \in e_{i-1}$ for all $i \geq 2$. Such a path has *length* ℓ. The *distance* between vertices u and v is the minimum length of a path between u and v. A graph G is *connected* if there is a path from any vertex to any other vertex. *Throughout what follows, we consider only graphs that are connected.*

We say that vertices u and v are *neighbors* if $uv \in E$. The *neighborhood* of $u \in V$ is the set containing all neighbors of u and is denoted by $N(u)$. The *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$. The *degree* of a vertex $u \in V$ is the number of edges that contain u, that is, $\deg_G(u) = |N(u)|$. When G is clear, we omit the subscript. The *maximum degree* of a graph G is $\Delta(G) = \max_{v \in V} \deg(v)$.

A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* H of a graph G, denoted $H = G[V(H)]$, is a graph with vertex set $V(H) \subseteq V(G)$ and edge set $E(H) = \{uv : u, v \in V(H) \text{ and } uv \in E(G)\}$.

Graph Theory by Diestel [4] serves as a reference for graph definitions not given here.

3.2.2 Power domination, vertex-fault-tolerant power domination, and PMU-defect-robust power domination

What follows is an equivalent statement of the power domination process as defined in [7], as established by [3].

The *power domination process* on a graph G with initial set $S \subseteq V$ proceeds recursively by:

1. $B = \bigcup_{v \in S} N[v]$
2. While there exists $v \in B$ such that exactly one neighbor, say u, of v is *not* in B, add u to B.

Step 1 is referred to as the *domination step* and each repetition of step 2 is called a *zero forcing step*. During the process, we say that a vertex in B is *observed* and a vertex not in B is *unobserved*. If v causes u to join B, we say that v *forces* u and will sometimes write this as $v \to u$. A *power dominating set* of a graph G is an initial set S such that $B = V(G)$ at the termination of the power domination process. The *power domination number* of a graph G is the minimum cardinality of a power dominating set of G and is denoted by $\gamma_P(G)$. A minimum power dominating set gives an optimal placement of PMUs in the graph.

In [10], Pai, Chang, and Wang define the following variant of power domination. For a graph G and an integer k with $0 \leq k \leq |V|$, a set $S \subseteq V$ is called a *k-fault-tolerant power dominating set of G* if $S \setminus F$ is still a power dominating set of G for any subset $F \subseteq V$ with $|F| \leq k$. The *k-fault-tolerant power domination number*, denoted by $\gamma_P^k(G)$, is the minimum cardinality of a k-fault-tolerant power dominating set of G.

While k-fault-tolerant power domination allows us to examine what occurs when a previously chosen PMU location is no longer usable (yet the vertex remains in the graph), it is also interesting to study when an individual PMU fails. That is, allow for multiple PMUs to be placed at the same location and consider if a subset of the PMUs fail. This also avoids issues with poorly connected graphs, such as in Figure 3.1, where $\gamma_P^1(G)$ may be close to the number of vertices of G. Thus we define *PMU-defect-robust power domination* as follows.

Definition 3.2.1. For a given graph G and integer $k \geq 0$, we say that a multiset S, each of whose elements is in V, is a *k-robust power dominating set* of G if $S \setminus F$ is a power dominating set of G for any submultiset F of S with $|F| = k$. We shorten k-robust power dominating set of G to k-rPDS of G. The size of a minimum k-rPDS is denoted by $\ddot{\gamma}_P^k(G)$ and such a multiset is also referred to as a $\ddot{\gamma}_P^k$-set of G. The *number of PMUs* at a vertex $v \in S$ is its multiplicity in S, which we denote by $\#\text{PMU}(v)$.

To demonstrate the difference between vertex-fault-tolerant power domination and PMU-defect-robust power domination, and the effect of low connectivity on vertex-fault-tolerant power domination, consider a star on 16 vertices with $k = 1$. This is shown in Figure 3.1. Notice that in vertex-fault-tolerant power domination, if one PMU was placed in the center of the star and F is taken to be the center, then all but one of the leaves must be in S in order to still form a power dominating set. On the other hand, for PMU-defect-robust power domination, we may simply place 2 PMUs on the center vertex. When one of these PMUs fails, the other will complete the power domination process on its own. Futhermore, S_{16} does not have a vertex-fault-tolerant power dominating set for $k \geq 3$, as removing PMUs from the center vertex and 2 leaves prevents the observation of the two leaves.

There are several observations that one can quickly make.

Observation 3.2.2. *For any graph G, $\ddot{\gamma}_P^0(G) = \gamma_P^0(G) = \gamma_P(G)$.*

Observation 3.2.3. *For any graph G, $\ddot{\gamma}_P^k(G) \leq \gamma_P^k(G)$.*

Observation 3.2.4. *For any graph G, $\gamma_P(G) = 1$ if and only if $\ddot{\gamma}_P^k(G) = k + 1$ for all $k \geq 0$.*

For any minimum k-rPDS, having more than $k + 1$ PMUs at a single vertex is redundant.

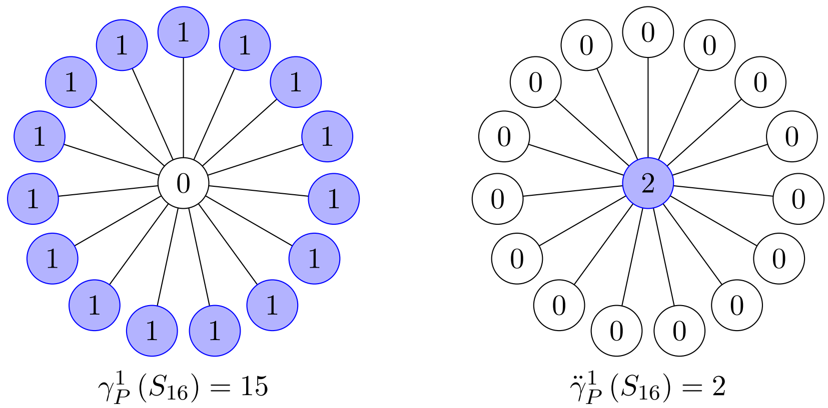

Figure 3.1 The difference between vertex-fault-tolerant power domination and PMU-defect-robust power domination shown for S_{16} when $k = 1$.

Observation 3.2.5. *Let G be a graph and $k \geq 0$. If S is a $\ddot{\gamma}_P^k$-set of G, then for all $v \in S$ we have* $\#\mathrm{PMU}(v) \leq k + 1$.

3.2.3 Floor and ceiling functions

Throughout what follows, recall the following rules for the floor and ceiling functions. Most can be found in Chapter 3 in [9] and we provide proofs for the rest.

Proposition 3.2.6. [9, Equation 3.11] *If m is an integer, n is a positive integer, and x is any real number, then*

$$\left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil = \left\lceil \frac{x + m}{n} \right\rceil.$$

Proposition 3.2.7. [9, Ch. 3 Problem 12] *If n is a positive integer and m is a real number, then*

$$\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m-1}{n} \right\rfloor + 1.$$

Proposition 3.2.8. [9, Equation 3.4] *For any real number x, $\lceil -x \rceil = -\lfloor x \rfloor$.*

Proposition 3.2.9. *If x and y are real numbers then*

$$\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil.$$

Proof. Observe that

$$\lceil x \rceil - 1 + \lceil y \rceil - 1 < x + y$$

and so

$$\lceil x \rceil - 1 + \lceil y \rceil - 1 < \lceil x + y \rceil$$
$$\lceil x \rceil + \lceil y \rceil - 2 < \lceil x + y \rceil,$$

which is a strict inequality of integers, so

$$\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil. \qquad \square$$

We can repeatedly apply the inequality in Proposition 3.2.9 to obtain

Corollary 3.2.10. *If x is a real number and a is a positive integer then*

$$a\lceil x \rceil \leq \lceil ax \rceil + a - 1.$$

3.3 General bounds

First we will show that $\ddot{\gamma}_P^k(G)$ is strictly increasing in k.

Proposition 3.3.1. *Let $k \geq 0$. For any graph G, $\ddot{\gamma}_P^k(G) + 1 \leq \ddot{\gamma}_P^{k+1}(G)$.*

Proof. Consider a $\ddot{\gamma}_P^{k+1}$-set of G, S. Let $v \in S$. Create $S' = S \setminus \{v\}$, that is, S' is S with one fewer PMU at v. Observe that for any $F' \subseteq S'$ with $|F'| = k$, we have $F' \cup \{v\} \subseteq S$ and $|F' \cup \{v\}| = k + 1$. Hence $S \setminus (F' \cup \{v\})$ is a power dominating set of G. Thus, for any such F', we have $(S \setminus \{v\}) \setminus F' = S' \setminus F'$ is a power dominating set of G. Therefore, S' is a k-robust power dominating set of G of size $|S| - 1$. $\square$

Proposition 3.3.1 can be applied repeatedly to obtain the next result.

Corollary 3.3.2. *Let $k \geq 0$ and $j \geq 1$. For any graph G,*

$$\ddot{\gamma}_P^k(G) + j \leq \ddot{\gamma}_P^{k+j}(G).$$

Corollary 3.3.2 implies the lower bound in the next proposition. The upper bound follows from taking $k+1$ copies of any minimum power dominating set for G to form a k-rPDS.

Proposition 3.3.3. *Let $k \geq 0$. For any graph G,*

$$\gamma_P(G) + k \leq \ddot{\gamma}_P^k(G) \leq (k+1)\gamma_P(G).$$

Notice that for any graph G with $\gamma_P(G) = 1$, Proposition 3.3.3 is an equality with $k+1 = \ddot{\gamma}_P^k(G) = k+1$, in agreement with Observation 3.2.4.

The next explanation is analogous to the argument for [7, Observation 4]. That is, consider a graph G with $\Delta(G) \geq 3$ and a $\ddot{\gamma}_P^k$-set S containing a vertex v with $\deg(v) \leq 2$. Let u be the vertex of G of degree 3 at minimum distance from v. Move all PMUs from v to u to create S'. Then any vertex that would have been observed by a result of a force performed by v will instead be observed via u. If $\#\text{PMU}(v) < k+1$ and this movement results in less than $k+1$ PMUs at u, then if u is removed from S' the power domination process will continue as if v had been removed from S and so S' is a k-rPDS.

Observation 3.3.4. *Let $k \geq 0$. If G is a graph with $\Delta(G) \geq 3$, then G contains a $\ddot{\gamma}_P^k$-set in which every vertex has degree at least 3.*

A *terminal path* from a vertex v in G is a path from v to a vertex u such that $\deg(u) = 1$ and every internal vertex on the path has degree 2. A *terminal cycle* from a vertex v in G is a cycle $v, u_1, u_2, \ldots, u_\ell, v$ in which $\deg_G(u_i) = 2$ for $i = 1, \ldots, \ell$.

If we consider a connected graph G with $\Delta(G) \geq 3$, then any vertex v that has two terminal paths or a terminal cycle has degree at least 3. For any $\ddot{\gamma}_P^k$-set in which every vertex has degree at least 3, there are at least two neighbors of v which can only be observed via v. As v can only observe both of these neighbors via the domination step, it must be the case that $\#\text{PMU}(v) = k+1$. This is formalized in Proposition 3.3.5.

Proposition 3.3.5. *Let $k \geq 0$ and let G be a graph with $\Delta(G) \geq 3$. Let S be a $\ddot{\gamma}_P^k$-set in which every vertex has degree at least 3. Any vertex $v \in S$ that has at least two terminal paths from v*

must have $\#\text{PMU}(v) = k+1$. *Any vertex* $v \in S$ *that has at least one terminal cycle must have* $\#\text{PMU}(v) = k+1$.

Zhao, Kang, and Chang [11] defined the family of graphs $\mathcal{T}$ to be those graphs obtained by taking a connected graph H and for each vertex $v \in V(H)$ adding two vertices, v' and v''; and two edges vv' and vv'', with the edge $v'v''$ optional. The complete bipartite graph $K_{3,3}$ is the graph with vertex set $X \cup Y$ with $|X| = |Y| = 3$ and edge set $E = \{xy : x \in X, y \in y\}$.

Theorem 3.3.6. [11, Theorem 3.] *If* G *is a connected graph on* $n \geq 3$ *vertices then* $\gamma_P(G) \leq \frac{n}{3}$ *with equality if and only if* $G \in \mathcal{T} \cup \{K_{3,3}\}$.

This gives an upper bound for $\ddot{\gamma}_P^k(G)$ in terms of the size of the vertex set and equality conditions, as demonstrated in the next corollary.

Corollary 3.3.7. *If* G *is a graph with* $n \geq 3$ *vertices, then* $\ddot{\gamma}_P^k(G) \leq (k+1)\frac{n}{3}$ *for* $k \geq 0$. *When* $k = 0$, *this is an equality if and only if* $G \in \mathcal{T} \cup \{K_{3,3}\}$. *When* $k \geq 1$, *this is an equality if and only if* $G \in \mathcal{T}$.

Proof. The upper bound is given by Proposition 3.3.3 and Theorem 3.3.6. From these results, we need only consider $\mathcal{T} \cup \{K_{3,3}\}$ for equality. The $k = 0$ case follows directly from the power domination result. Let $k \geq 1$.

First consider $G \in \mathcal{T}$, constructed from H. Note that $\Delta(G) \geq 3$, so there exists a $\ddot{\gamma}_P^k$-set, say S, in which every vertex has degree at least 3, so every vertex in S is a vertex of H. For each $v \in V(H)$, $\deg_G(v) \geq 3$ and there are either two terminal paths (if $v'v'' \notin E(H)$) or a terminal cycle (if $v'v'' \in E(H)$). By Proposition 3.3.5, each $v \in V(H)$ must have at least $k+1$ PMUs.

Finally, consider $K_{3,3}$. Note that $\gamma_P(K_{3,3}) = 2$. We will see in Theorem 3.4.1 that $\ddot{\gamma}_P^k(K_{3,3}) = k + \left\lfloor \frac{k}{5} \right\rfloor + 2 < 2(k+1)$ for $k \geq 1$. □

For any graph G, define $s(G)$ to be the size of the largest set $A \subseteq V$ such that for any $B \subseteq A$ with $|B| = \gamma_P(G)$, B is a power dominating set of G. Observe that $\gamma_P(G) \leq s(G)$. For example, the star graph S_{16} shown in Figure 3.1 has $\gamma_P(S_{16}) = s(S_{16}) = 1$. The complete bipartite graph

$K_{3,3}$ has $\gamma_P(K_{3,3}) = 2$ and $s(K_{3,3}) = 6$ as any two vertices of $K_{3,3}$ form a power dominating set. In Section 3.5, we will determine $s(G)$ for small square grid graphs. From the definition of $s(G)$ and Proposition 3.3.3, we have the following proposition.

Proposition 3.3.8. *For any graph G and $k \geq 0$, if $s(G) \geq k + \gamma_P(G)$ then $\ddot{\gamma}_P^k(G) = k + \gamma_P(G)$.*

Proof. If $s(G) \geq k + \gamma_P(G)$, then there exists a set S of size at least $k + \gamma_P(G)$ so that any $\gamma_P(G)$ elements of S form a power dominating set of G. Thus, any $\gamma_P(G) + k$ elements of S form a k-rPDS of G of size $\gamma_P(G) + k$ and so $\ddot{\gamma}_P^k(G) \leq \gamma_P(G) + k$. By the lower bound in Proposition 3.3.3, $\ddot{\gamma}_P^k(G) \geq \gamma_P(G) + k$. □

When $s(G) > \gamma_P(G) \geq 2$, the following upper bound sometimes improves the upper bound from Proposition 3.3.3.

Theorem 3.3.9. *If $s(G) > \gamma_P(G) \geq 2$, then $\ddot{\gamma}_P^k(G) \leq \left\lceil \frac{s(G)(k+\gamma_P(G)-1)}{s(G)-\gamma_P(G)+1} \right\rceil$ for $k \geq 1$.*

Proof. Let $A = \{v_1, v_2, \ldots, v_{s(G)}\} \subseteq V$ be a maximum set such that any subset of size $\gamma_P(G)$ is a power dominating set of G. For what follows, let $p = \left\lceil \frac{s(G)(k+\gamma_P(G)-1)}{s(G)-\gamma_P(G)+1} \right\rceil$. Construct $S = \left\{v_1^{m_1}, v_2^{m_2}, \ldots, v_{s(G)}^{m_{s(G)}}\right\}$ where

$$m_1 = \left\lceil \frac{p}{s(G)} \right\rceil$$
$$m_i = \min\left\{ \left\lceil \frac{p}{s(G)} \right\rceil, p - \sum_{j=1}^{i-1} m_j \right\}, i \geq 2.$$

In order to show that S is a k-rPDS of G, we will show that $p - k \geq (\gamma_P(G) - 1)\left\lceil \frac{p}{s(G)} \right\rceil + 1$. Assume this is true. Then whenever we have p PMUs and k fail, there are at least $(\gamma_P(G) - 1)\left\lceil \frac{p}{s(G)} \right\rceil + 1$ working PMUs. As each vertex has at most $\left\lceil \frac{p}{s(G)} \right\rceil$ PMUs, there are at least $\gamma_P(G)$ vertices of A that must have at least one PMU remaining and so form a power dominating set. We prove the equivalent statement

$$p - k - (\gamma_P(G) - 1)\left\lceil \frac{p}{s(G)} \right\rceil \geq 1.$$

Observe that by Proposition 3.2.6,

$$p - k - (\gamma_P(G) - 1)\left\lceil \frac{p}{s(G)} \right\rceil = p - k - (\gamma_P(G) - 1)\left\lceil \frac{k + \gamma_P(G) - 1}{s(G) - \gamma_P(G) + 1} \right\rceil.$$

Then by Corollary 3.2.10 and simplifying, we see that

$$\begin{aligned}
p - k - (\gamma_P(G) - 1)\left\lceil \frac{p}{s(G)} \right\rceil &\geq p - k - \left(\left\lceil \frac{(k + \gamma_P(G) - 1)(\gamma_P(G) - 1)}{s(G) - \gamma_P(G) + 1} \right\rceil + \gamma_P(G) - 2 \right) \\
&= p - \left(\left\lceil \frac{(k + \gamma_P(G) - 1)(\gamma_P(G) - 1)}{s(G) - \gamma_P(G) + 1} \right\rceil + \gamma_P(G) - 2 + k \right) \\
&= p - \left\lceil \frac{(k + \gamma_P(G) - 1)(\gamma_P(G) - 1) + (s(G) - \gamma_P(G) + 1)(\gamma_P(G) - 2 + k)}{s(G) - \gamma_P(G) + 1} \right\rceil \\
&= p - \left\lceil \frac{s(G)(k + \gamma_P(G) - 1) - s(G) + \gamma_P(G) - 1}{s(G) - \gamma_P(G) + 1} \right\rceil \\
&= p - \left\lceil \frac{s(G)(k + \gamma_P(G) - 1)}{s(G) - \gamma_P(G) + 1} \right\rceil + 1 \\
&= 1.
\end{aligned}$$

□

To see the difference between the upper bound in Theorem 3.3.9 and Proposition 3.3.3, let $s(G) = \gamma_P(G) + r$. Then Theorem 3.3.9 becomes

$$\begin{aligned}
\ddot{\gamma}_P^k(G) &\leq \left\lceil \frac{(\gamma_P(G) + r)(k + \gamma_P(G) - 1)}{r + 1} \right\rceil \\
&= \left\lceil \frac{\gamma_P(G)(k + 1) + \gamma_P(G)(\gamma_P(G) - 2) + r(k + \gamma_P(G) - 1)}{r + 1} \right\rceil \\
&= \gamma_P(G)(k + 1) + \left\lceil \frac{-r\gamma_P(G)(k + 1) + \gamma_P(G)(\gamma_P(G) - 2) + r(k + \gamma_P(G) - 1)}{r + 1} \right\rceil \\
&= \gamma_P(G)(k + 1) + \left\lceil \frac{-kr(\gamma_P(G) - 1) + \gamma_P(G)(\gamma_P(G) - 2) - r}{r + 1} \right\rceil.
\end{aligned}$$

This means that the bound from Theorem 3.3.9 improves the upper bound in Proposition 3.3.3 when the second term above is negative. Since $r \geq 1$ and $\gamma_P(G) \geq 2$, the second term is negative as k approaches infinity. Thus for $s(G) > \gamma_P(G) \geq 2$, we see that there exists some k' such that for every $k \geq k' \geq 1$, the bound from Theorem 3.3.9 is better than the upper bound from Proposition 3.3.3. For Section 3.4, it will be useful to look specifically at Theorem 3.3.9 when $\gamma_P(G) = 2$.

Corollary 3.3.10. *If $s(G) > \gamma_P(G) = 2$, then $\ddot{\gamma}_P^k(G) \leq \left\lceil \frac{s(G)(k+1)}{s(G)-1} \right\rceil$ for $k \geq 1$.*

3.4 Complete bipartite graphs

The *complete bipartite graph*, $K_{a,b}$ is the graph with vertex set $V = X \cup Y$ such that $|X| = a$, $|Y| = b$, and edge set is $E = \{xy : x \in X, y \in Y\}$.

Theorem 3.4.1. *Let $k \geq 0$. Let $K_{3,3}$ be the complete bipartite graph with parts $X = \{x_1, x_2, x_3\}$ and $Y = \{x_4, x_5, x_6\}$. Then*

$$\ddot{\gamma}_P^k(K_{3,3}) = k + \left\lfloor \frac{k}{5} \right\rfloor + 2.$$

Proof. We begin by observing that any two vertices of $K_{3,3}$ form a power dominating set, that is, $s(K_{3,3}) = 6$. First we prove the lower bound $k + \left\lfloor \frac{k}{5} \right\rfloor + 2 \leq \ddot{\gamma}_P^k(K_{3,3})$ where $k = 5m$. Assume for contradiction that there exists a $\ddot{\gamma}_P^{5m}$-set S of size $5m + \left\lfloor \frac{5m}{5} \right\rfloor + 1 = 6m + 1$. By the pigeonhole principle, some x_i contains at least $\left\lceil \frac{6m+1}{6} \right\rceil = m + 1$ of the PMUs. Observe that $|S| - 5m = m + 1$. Thus, we can remove $5m$ PMUs so that some vertex x_i contains all remaining PMUs. This is a contradiction, as $\gamma_P(K_{3,3}) = 2$. Thus, $\ddot{\gamma}_P^{5(m+1)}(K_{3,3}) \geq 6m + 2 = 5m + \left\lfloor \frac{5m}{5} \right\rfloor + 2$, as desired. The lower bounds when k is not a multiple of 5 then follow by Corollary 3.3.2.

For the upper bound, observe that by Corollary 3.3.10,

$$\begin{aligned} \ddot{\gamma}_P^k(K_{3,3}) &\leq \left\lceil \frac{6(k+1)}{5} \right\rceil \\ &= k + 1 + \left\lceil \frac{k+1}{5} \right\rceil. \end{aligned}$$

Then by Proposition 3.2.7, we see that

$$\ddot{\gamma}_P^k(K_{3,3}) \le k+1+\left\lfloor \frac{k+1-1}{5} \right\rfloor + 1$$
$$= k + \left\lfloor \frac{k}{5} \right\rfloor + 2. \qquad \square$$

Theorem 3.4.1 gives an example of a graph for which Theorem 3.3.9 is tight and the structure of the $\ddot{\gamma}_P^k$-set suggested by the proof of Theorem 3.3.9 is shown in Figure 3.2. A larger family of complete bipartite graphs follows the same pattern, as shown in Theorem 3.4.2.

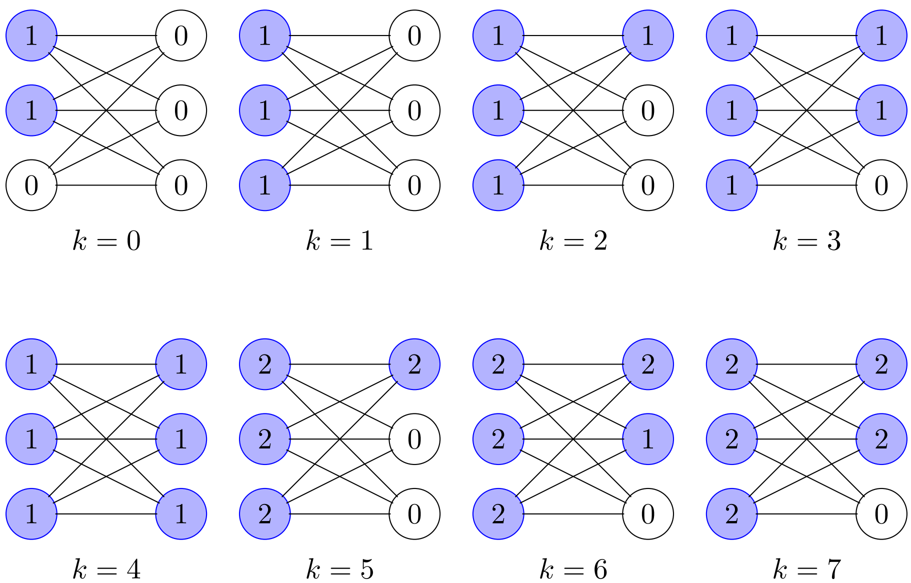

Figure 3.2 Minimum k-rPDS for $K_{3,3}$ for $k = 0, 1, \ldots, 7$.

Theorem 3.4.2. *Let $k \ge 0$. Let $K_{3,b}$ be the complete bipartite graph with parts $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$. For $b \ge \lfloor \frac{k}{3} \rfloor + 3$,*

$$\ddot{\gamma}_P^k(K_{3,b}) = k + \left\lfloor \frac{k}{3} \right\rfloor + 2.$$

Proof. First we prove the lower bound when $k = 3m$. Assume for eventual contradiction that there exists a $\ddot{\gamma}_P^{3m}$-set, S, of size $3m + \lfloor \frac{3m}{3} \rfloor + 1 = 4m + 1$. Let $y = \sum_{i=1}^{b} \#\mathrm{PMU}(y_i)$. Then

$$\#\mathrm{PMU}(x_1) + \#\mathrm{PMU}(x_2) + \#\mathrm{PMU}(x_3) + y = 4m + 1.$$

By the pigeonhole principle, we see that one of x_1, x_2, x_3 or y must represent at least

$$\left\lceil \frac{4m+1}{4} \right\rceil = m+1$$

of the PMUs. Observe that $|S| - 3m = m + 1$. Thus, we can remove $3m$ PMUs such that either:

1. All $m+1$ remaining PMUs are on a single x_i, which is a contradiction as this is only one vertex and $\gamma_P(K_{3,b}) = 2$; or

2. All $m+1$ remaining PMUs are on the y_i vertices. In order for the PMUs on the y_i's to form a power dominating set of $K_{3,b}$, $b-1$ of the y_i's must have a PMU. However, we also have that

$$\begin{aligned} b-1 &\geq \left\lfloor \frac{3m}{3} \right\rfloor + 3 - 1 \\ &= m+2. \end{aligned}$$

   This means that at least $m+2$ PMUs are needed but after $3m$ PMUs are removed only $m+1$ PMUs remain, a contradiction.

Therefore, $\ddot{\gamma}_P^{3m}(K_{3,b}) > 4m+1$. Hence, $\ddot{\gamma}_P^{3m}(K_{3,b}) \geq 4m+2 = 3m + \left\lfloor \frac{3m}{3} \right\rfloor + 2$, as desired. The lower bounds for the remaining cases then follow by Corollary 3.3.2.

For the upper bound, the case of $k = 0$ is given by the power domination number. If $b = 3$ we need only consider when $k = 0, 1, 2$; this is covered by Theorem 3.4.1. If $b \geq 4$ and $k \geq 1$, we have $s(K_{3,b}) = 4$. Then by Corollary 3.3.10,

$$\begin{aligned} \ddot{\gamma}_P^{k}(K_{3,b}) &\leq \left\lceil \frac{4(k+1)}{3} \right\rceil \\ &= k+1+\left\lceil \frac{k+1}{3} \right\rceil \end{aligned}$$

and by Proposition 3.2.7,

$$\begin{aligned} \ddot{\gamma}_P^{k}(K_{3,b}) &\leq k+1+\left\lfloor \frac{k+1-1}{3} \right\rfloor + 1 \\ &= k + \left\lfloor \frac{k}{3} \right\rfloor + 2. \end{aligned}$$

□

3.5 Square grid graphs

The *Cartesian product* of graphs G and H, denoted $G\Box H$, has vertex set $V(G\Box H) = V(G) \times V(H)$, and edge set $E(G\Box H) = \{(x,y)(x',y') : x = x' \text{ and } yy' \in E(H) \text{ or } y = y' \text{ and } xx' \in E(G)\}$. The n *by* m *grid graph* for $n \le m$ is $G_{n,m} = P_n \Box P_m$. In [5], Dorfling and Henning found the power domination number of grid graphs.

Theorem 3.5.1. [5, Theorem 1] *If $G_{n,m}$ is an $n \times m$ grid graph where $m \ge n \ge 1$, then*

$$\gamma_P\left(G_{n,m}\right) = \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil & \text{if } n \equiv 4 \mod 8 \\ \left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}.$$

For what follows, we will consider only the case $n = m$, i.e., square grid graphs, and let the vertices of $P_n = \{1, 2, \ldots, n\}$. As an example, Figure 3.3 shows $G_{4,4}$.

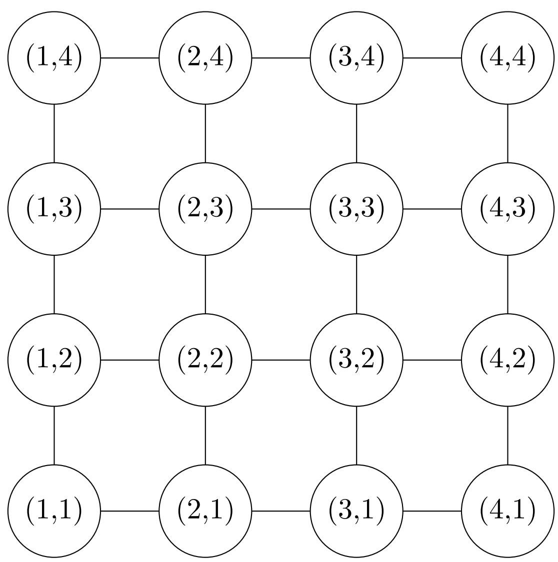

Figure 3.3 $G_{4,4}$

Using Sage (see [1]) and modifying the power domination code from [8], we were able to use brute force in order to find the k-robust power domination number for $k = 1, \ldots, 7$ for $G_{4,4}$ and for $k = 1, 2, 3$ for $G_{5,5}$ and $G_{6,6}$. This data is displayed in the following tables, which demonstrate example minimum $\ddot{\gamma}_P^k$-sets for each case.

Table 3.1 Computational results for $G_{4,4}$.

k	$\ddot{\gamma}_P^k(G_{4,4})$	Example $\ddot{\gamma}_P^k$-set
0	2	$\{(1,2)^1,(2,3)^1\}$
1	3	$\{(1,2)^1,(2,3)^1,(4,2)^1\}$
2	4	$\{(1,2)^1,(2,3)^1,(4,2)^1,(3,3)^1\}$
3	6	$\{(1,2)^2,(2,3)^2,(4,2)^2\}$
4	7	$\{(1,2)^2,(2,3)^2,(4,2)^2,(3,3)^1\}$
5	8	$\{(1,2)^2,(2,3)^2,(4,2)^2,(3,3)^2\}$
6	10	$\{(1,2)^3,(2,3)^3,(4,2)^3,(3,3)^1\}$
7	11	$\{(1,2)^3,(2,3)^3,(4,2)^3,(3,3)^2\}$
⋮	⋮	⋮

Table 3.2 Computational results for $G_{5,5}$.

k	$\ddot{\gamma}_P^k(G_{5,5})$	Example $\ddot{\gamma}_P^k$-set
0	2	$\{(2,4)^1,(3,2)^1\}$
1	3	$\{(2,4)^1,(3,2)^1,(2,1)^1\}$
2	5	$\{(2,4)^2,(3,2)^2,(2,1)^1\}$
3	6	$\{(2,4)^2,(3,2)^2,(2,1)^2\}$
⋮	⋮	⋮

Table 3.3 Computational results for $G_{6,6}$.

k	$\ddot{\gamma}_P^k(G_{6,6})$	Example $\ddot{\gamma}_P^k$-set
0	2	$\{(2,4)^1,(3,2)^1\}$
1	4	$\{(2,4)^2,(3,2)^2\}$
2	5	$\{(2,4)^1,(3,2)^1,(6,5)^1,(4,4)^1,(2,6)^1\}$
3	7	$\{(2,4)^2,(3,2)^2,(6,5)^1,(4,4)^1,(2,6)^1\}$
⋮	⋮	⋮

We can use this data to determine $s(G)$ for these graphs. For example, Table 3.1 shows that there is no set of 5 vertices for which any subset of $2 = \gamma_P(G_{4,4})$ vertices forms a power dominating set of $G_{4,4}$, as this would be a $\ddot{\gamma}_P^3$-set of size 5. Thus, $s(G_{4,4}) \le 4$. The exhibited $\ddot{\gamma}_P^2$-set for $G_{4,4}$ shows $s(G_{4,4}) \ge 4$. Similarly, we can derive from Table 3.2 that $s(G_{5,5}) = 3$ and from Table 3.3 that $s(G_{6,6}) = \gamma_P(G_{6,6}) = 2$.

As $\gamma_P(G_{4,4}) = 2$, the upper bound from Corollary 3.3.10 is $\ddot{\gamma}_P^k(G_{4,4}) \le \left\lceil \frac{4(k+1)}{3} \right\rceil$. For the given k values, we see that $\ddot{\gamma}_P^k(G_{4,4})$ achieves this upper bound. Moreover, notice that the structure of the found $\ddot{\gamma}_P^k$-sets for $G_{4,4}$ follows the structure given by the proof of Theorem 3.3.9. Whether this pattern will continue for larger values of k is an open question.

We see that $\gamma_P(G_{5,5}) = 2$ and the upper bound from Corollary 3.3.10 is $\ddot{\gamma}_P^k(G_{5,5}) \le \left\lceil \frac{3(k+1)}{2} \right\rceil$. Here as well, for the computed k values, we see that $\ddot{\gamma}_P^k(G_{5,5})$ achieves this upper bound. Again we see that the structure of the found $\ddot{\gamma}_P^k$-sets for $G_{5,5}$ follows the structure given by the proof of Theorem 3.3.9. This case also requires further study to determine if this pattern continues for larger values of k.

For $G_{6,6}$, $s(G_{6,6}) = \gamma_P(G_{6,6}) = 2$, so Corollary 3.3.10 does not apply to this case. The relevant upper bound from Proposition 3.3.3 is $\ddot{\gamma}_P^k(G_{6,6}) \le 2(k+1)$. We see that $\ddot{\gamma}_P^2(G_{6,6}) = 5 < 2(2+1) = 6$, so the upper bound is not tight in this instance. The lower bound from Proposition 3.3.3 is $\ddot{\gamma}_P^k(G_{6,6}) \ge k+2$, but $\ddot{\gamma}_P^2(G_{6,6}) = 5 > 2+2 = 4$, so the lower bound is also not tight. Similarly, we find that neither bound is tight for $k = 3$ as well. It may be the case that $k + \gamma_P(G_{6,6}) < \ddot{\gamma}_P^k(G_{6,6}) < \gamma_P(G_{6,6})(k+1)$ for all $k \ge 2$; more investigation is needed.

3.6 Trees

A *tree* is a connected graph with no cycles. A *spider* is a tree with at most one vertex of degree 3 or more. A *spider cover* of a tree T is a partition of V, say $\{V_1, \ldots, V_\ell\}$ such that $G[V_i]$ is a spider for all i. The *spider number* of a tree T, denoted by $\operatorname{sp}(T)$, is the minimum number of partitions in a spider cover.

Theorem 3.6.1. [7, Theorem 12] *For any tree T, $\gamma_P(T) = \operatorname{sp}(T)$.*

In [7], Haynes et al. present a proof of Theorem 3.6.1. We will present an issue with the argument presented in [7] and then give an alternate proof.

First we repeat relevant notation used in [7]: "If T is a tree rooted at r and v is a vertex of T, then the *level number* of v, which we denote by $\ell(v)$, is the length of the unique rv path in T. If a vertex v of T is adjacent to u and $\ell(u) > \ell(v)$, then u is called a *child* of v, and v is the *parent* of u, written $v = parent(u)$. A vertex w is a *descendant* of v (and v is an *ancestor* of w) if the level numbers of the vertices on the vw path are monotonically increasing. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree of T rooted at v* is the subtree induced by $D[v]$ and is denoted by T_v." Also needed is the following observation.

Observation 3.6.2. [7, Observation 4] *If G is a graph with $\Delta(G) \geq 3$, then G contains a minimum power dominating set in which every vertex has degree at least 3.*

Next, we delve into the issue of the proof given in [7] of Theorem 3.6.1. Specifically, we have found a counterexample to a subcase of Case 2 in the proof of Lemma 3.6.3.

Lemma 3.6.3. [7, Lemma 10] *For any tree T, $\mathrm{sp}(T) \leq \gamma_P(T)$.*

What follows is the relevant portion of the proof of Lemma 3.6.3, reproduced directly from [7].

Proof. "We proceed by induction on $m = \gamma_P(T)$ ... Let T be a tree with $\gamma_P(T) = m + 1$. Let $S = \{v_1, v_2, \ldots, v_{m+1}\}$ be a [minimum power dominating set of T]. By [Observation 3.6.2], we may assume that each vertex of S has degree at least 3 in T.

Let T be rooted at the vertex v_{m+1}. Relabeling if necessary, we may assume that v_1 is the vertex of S at maximum distance from v_{m+1} in T. Then v_1 has at least two children, and each descendant of v_1 has degree at most 2 in T. Let u_1 be the ancestor of v_1 of degree at least 3 that is at a minimum distance from v_1 [1]. Then either u_1 is the parent of v_1 or every internal vertex on the u_1v_1 path has degree 2 in T. We consider two possibilities, depending on whether $u_1 \in S$.

...

Case 2. $u_1 \notin S$.

[1] $\gamma_P(T) \geq 2$ implies u_1 exists.

Let $w_1, w_2, \ldots, w_k$ be the children of u_1, where w_1 is the child on the u_1v_1 path (possibly $v_1 = w_1$). For $i = 1, 2, \ldots, k$, let $W_i = D[w_i]$. In particular, $v_1 \in W_1$. Since $u_1 \notin S$ and since S is a power dominating set of T, all except possibly one of the sets $W_1, W_2, \ldots, W_k$ contains a vertex of S.

...

Suppose, on the other hand, that each of the sets $W_2, \ldots, W_k$ contains a vertex of S. Then $|W_i \cap S| \geq 1$ for $i = 1, 2, \ldots, k$. Let T_1 be the tree induced by $W_1 \cup \{u_1\}$. Then T_1 is a spider with v_1 as the vertex of degree exceeding 2. For $i = 2, \ldots, k$, let T_i be the maximal subtree rooted at w_i. Let $T_{k+1} = T - D[u_1]$. For $i = 1, 2, \ldots, k+1$, let $S_i = S \cap V(T_i)$. Then ... each S_i is a [minimum power dominating set of T_i] ..." □

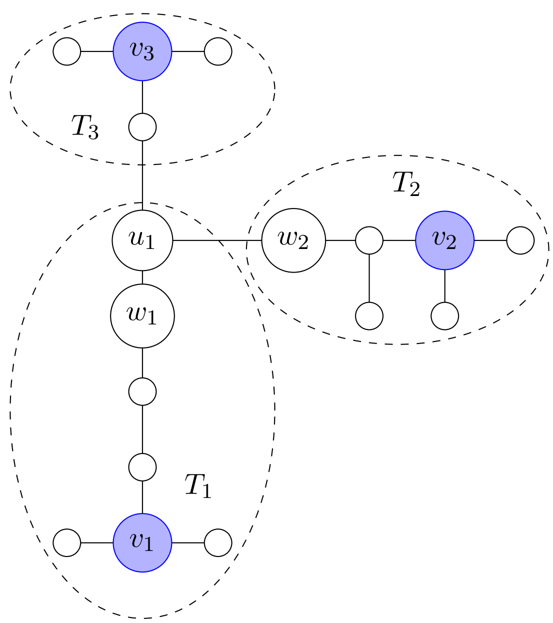

Figure 3.4 A counterexample to the proof technique used in [7] for Lemma 3.6.3.

In Figure 3.4, we demonstrate a tree T with minimum power dominating set $S = \{v_1, v_2, v_3\}$ satisfying the following:

1. every vertex of S has degree at least 3,

2. v_1 is the element of S of maximum distance from v_3,

3. $u_1 \notin S$, and

4. each $D[w_i]$ contains an element of S.

However, $S_2 = \{v_2\}$ is *not* a power dominating set of T_2.

We now present an alternative proof, which is a generalization to power domination of the argument used by Ekstrand et al. in [6]. Let G be a graph and let $S = \{v_1, v_2, \ldots, v_{\gamma_P(G)}\}$ be a minimum power dominating set of G. Construct the *chronological list of forces*, $\mathcal{F}$ in the following manner.

1. Add the forces $v_i \to u$ one at a time for all $u \in N(v_i) \cap \left(V - \left(S \cup \bigcup_{j<i} N(v_j)\right)\right)$ for $1 \leq i \leq \gamma_P(G)$ to $\mathcal{F}$.

2. List the forces from the zero forcing step in the order in which they occur.

Definition 3.6.4. Given a tree T, power dominating set S, chronological list of forces, $\mathcal{F}$, and vertex $v \in S$, define S_v to be the set of vertices w such that there is a sequence of vertices in $\mathcal{F}$, $v = v_1 \to v_2 \to \cdots \to v_r = w$, including the case of no forces, i.e., $v = w$. The *forcing spider* S_v is the induced subgraph $S_v = T[S_v]$. The *forcing spider cover* is $\mathscr{S} = \{S_v : v \in S\}$.

Theorem 3.6.5. *If T is a tree, S is a power dominating set of T, and $\mathcal{F}$ is a chronological list of forces of S then*

1. *$T[S_v]$ is a spider for all $v \in S$*

2. *$\mathscr{S} = \{S_v : v \in S\}$ is a spider cover of T*

3. *$\operatorname{sp}(T) \leq \gamma_P(T)$*

Proof. By construction, every vertex $u \in V$ is forced exactly one way in $\mathcal{F}$, that is, there is a unique force which ends in u. Moreover, the only vertices which can possibly perform multiple forces are the vertices of S. This means that each S_v is disjoint. Moreover, each S_v contains at most one vertex that performed multiple forces, and so $T[S_v]$ contains at most one vertex of degree 3. As T is a tree, $T[S_v]$ is a tree. Therefore, $T[S_v]$ is a spider.

Each $T[S_v]$ forms a spider and S_v are disjoint by construction. As S is a power dominating set of T, every vertex is forced at some point and so every vertex is in some S_v, Thus we see that $\mathscr{S} = \{S_v : v \in S\}$ is a spider cover of T.

This holds for *any* initial power dominating set S of T. Thus, any power dominating set S yields a forcing spider cover. As a forcing spider cover forms a spider cover of T, we see that $\operatorname{sp}(T) \leq \gamma_P(T)$. □

Therefore, Theorem 3.6.1 does hold.

As in Section 3.5, we have used Sage (see [2]) to find the 1-robust power domination number for all trees on 19 or fewer vertices. In every case, $\ddot{\gamma}_P^1(T) = 2\operatorname{sp}(T)$. We also tested all trees on 6 to 13 vertices for $2 \leq k \leq 10$ and found $\ddot{\gamma}_P^k(T) = (k+1)\operatorname{sp}(T)$. This supports the following conjecture.

Conjecture 3.6.6. *For any tree T, $\ddot{\gamma}_P^k(T) = (k+1)\operatorname{sp}(T)$.*

3.7 Concluding remarks

PMU-defect-robust power domination allows us to place multiple PMUs at the same location and consider the consequences if some of these PMUs fail. There are many questions left to examine in future work.

Is there an improvement to the lower bound given in Proposition 3.3.3 for $\gamma_P(G) > 1$? As $K_{3,3}$ demonstrates in Theorem 3.4.1, it seems likely that there is a better lower bound based on the number of vertices and the power domination number that utilizes the pigeonhole principle to show that the lower bound must increase at certain values of k.

We have begun the study of k-robust power domination for certain families of graphs but work remains to be done. For complete bipartite graphs, we still have the case of $\ddot{\gamma}_P^k(K_{3,b})$ for $4 \leq b < \left\lfloor \frac{k}{3} \right\rfloor + 3$. The question of $\ddot{\gamma}_P^k(K_{a,b})$ for $a, b \geq 4$ is also open. We have some computational results for small grid graphs, which indicate that both $G_{4,4}$ and $G_{5,5}$ may attain the upper bound from Corollary 3.3.10, but this still needs to be shown. On the other hand, computational results show that $G_{6,6}$ can have a k-robust power domination number strictly between the upper and lower bounds given by Proposition 3.3.3 and so would be interesting to study. Finally, while we have found an alternate proof of the power domination number for trees and have computational results supporting the generalization to k-robust power domination in Conjecture 3.6.6, this remains to be proven.

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CHAPTER 4. GENERAL CONCLUSION

The power domination problem is interesting both as an application to the real world power grid and also as a graph theory problem. In Chapter 1, we introduced the history of the problem and explained how graph theory can be used to solve it.

Chapter 2 presented our first variation of the power domination problem: what happens if we change the structure we want to monitor from a graph to a hypergraph? Infectious power domination is a new way to generalize the power domination problem to hypergraphs. We examined general bounds; graph families such as complete k-partite hypergraphs, circular arc hypergraphs, and trees; and the impact of hypergraph operations including edge/vertex removal, linear sums, Cartesian products, and weak coronas. There are a plethora of questions that remain. In particular, the general upper bound in Conjecture 2.4.6 would be an interesting place to start, as we have found several families of graphs that achieve this bound. There are many families of hypergraphs that have yet to be studied; this includes finding a lower bound for hypertrees. The infectious power domination number of a Cartesian product can be greater than that of either input hypergraph, as shown in Theorem 2.6.6, but general upper and lower bounds are still needed.

In Chapter 3, we examined how to handle sensor failure: how do we find a power dominating set that works if any k of the sensors fail? The PMU-defect robust power domination number is also a novel parameter. We presented general bounds, gave explicit values for some complete bipartite graphs, and found computational results for small square grid graphs. We also gave a new proof of the power domination number for trees and conjectured the PMU-defect robust power domination number for trees. There are many questions left to explore. One avenue of future study is to find an improvement to the lower bound in Proposition 3.3.3. There are also many interesting families to be studied, including grid graphs.