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Regularity theory for nonlocal space-time master equations

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DEDICATION

I would like to dedicate this thesis to my son Arnesh, my wife Amrita, my parents Sabita and Anathbandhu and my brother Aniruddha, without whose support I would not have been able to complete this work. I would also like to thank my friends and their families in the United States and in India for their loving support.

TABLE OF CONTENTS

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ABSTRACT

We analyze regularity estimates for solutions to nonlocal space time equations driven by fractional powers of parabolic operators in divergence form. These equations are fundamental in semipermeable membrane problems, biological invasion models and they also appear as generalized Master equations. We develop a parabolic method of semigroups that allows us to prove a local extension problem characterization for these nonlocal problems. As a consequence, we obtain interior and boundary Harnack inequalities and sharp interior and global parabolic Schauder estimates for solutions. For the latter, we also prove a characterization of the correct intermediate parabolic Hölder spaces in the spirit of Sergio Campanato.

CHAPTER 1. INTRODUCTION

In this dissertation we study regularity estimates of solutions to equations involving fractional powers of parabolic operators of the form

$$H^s u(t,x) \equiv (\partial_t + L)^s u(t,x) = f(t,x), \quad 0 < s < 1 \tag{1.0.1}$$

for $t \in \mathbb{R}$ and $x \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$, that may be unbounded, and L is an elliptic operator, i.e.

$$L = -\operatorname{div}(A(x)\nabla) \tag{1.0.2}$$

Here $A(x) = (A^{ij}(x))$ is a bounded, measurable, symmetric matrix defined in Ω, satisfying the uniform ellipticity condition, that is, for some $\Lambda \geq 1$,

$$\Lambda^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$. The operator L is subject to appropriate boundary conditions. Here we consider the boundary conditions to be either homogeneous Dirichlet or Neumann, that is

$$u = 0 \quad \text{or} \quad \partial_A u = A(x)\nabla_x u \cdot \nu = 0 \quad \text{on } \mathbb{R} \times \partial\Omega,$$

where ν is the exterior unit normal to $\partial\Omega$.

Fractional powers of parabolic operators, which occur in nonlocal equations in both space and time, appear in many different physical applications. For example, (1.0.1) appears in the Signorini problem, in the semipermeable membrane problem in biology, in the phenomenon of osmosis and in diffusion models for biological invasions, see [1, 4, 8, 15, 25, 42]. Besides, equations involving fractional powers of parabolic operators are examples of the so-called Master equations. Master equations are integro-differential equations that take the form

$$\int_0^\infty \int_{\mathbb{R}^n} (u(t-\tau,z) - u(t,x))K(t,x,\tau,z)\,dz\,d\tau = f(t,x) \tag{1.0.3}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $n \geq 1$ and K is some kernel.

In this work, we first define the fractional powers of parabolic operators by using the spectrum of L and Fourier transform in the time variable. Our definition is in the weak sense since L is a divergence form operator. It turns out that equations involving fractional powers of parabolic operators are nonlocal problems. Thus, we introduce a localization technique to change the nonlocal problem to a local degenerate parabolic partial differential equation (PDE) problem. We call this the *extension method*, which is in the same spirit as what Caffarelli and Silvestre did for the fractional Laplacian in [14]. The extension method is a crucial tool in proving different regularity estimates for solutions u to (1.0.1).

In terms of regularity estimates, we prove interior and boundary Harnack inequalities for nonnegative solutions to the homogeneous equation, $(\partial_t + L)^s u = 0$, under the assumption that the coefficients $A(x)$ are bounded and measurable. For the interior Harnack inequality, precisely, we show that for a compact set $K \subset \Omega$ and a bounded interval $I \subset \mathbb{R}$, there exists a constant $c > 0$ which depends on n, s, Λ, K, I, such that if u is a nonnegative solution then

$$\sup_{I\times K} u(t,x) \leq c \inf_{I\times K} u(t,x).$$

The interior Harnack inequality is important since, due to this inequality and our extension method, under minimal regularity assumptions on the coefficients $A(x)$, we prove that u is parabolically Hölder continuous inside $I \times K$, i.e. $u \in C^{\alpha/2,\alpha}_{t,x}(I \times K)$ for some $0 < \alpha < 1$, where $C^{\alpha/2,\alpha}_{t,x}(I \times K)$ is defined as the set of all continuous functions $u(t,x)$ such that

$$\|u\|_{C^{\alpha/2,\alpha}_{t,x}(\overline{I\times K})} = \|u\|_{L^\infty(I\times K)} + \sup_{t,\tau\in I,\, x,z\in K} \frac{|u(t,x)-u(\tau,z)|}{\max(|t-\tau|^{1/2}, |x-z|)^\alpha} < \infty.$$

On the other hand, for the boundary Harnack inequality, we consider a domain $\Omega_0 \subset \Omega$, $x_0 \in \partial\Omega_0$ and $B_r(x_0) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$, for some $r > 0$, such that $B_r(x_0) \subset \Omega$, and $B_r(x_0) \cap \partial\Omega_0$ can be represented as a graph of a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ in the $e_n = (0, 0, \ldots, 0, 1)$-direction. Next, let us assume that a nonnegative solution u vanishes continuously on $(\Omega/\Omega_0) \cap B_r(x_0)$ in some time interval $[-T, T] \in \mathbb{R}$. Now, if we fix $t_1 > T/2$ and $x_1 \in \Omega_0$, then we prove that there exists a constant $C > 0$, which depends on $n, s, \Lambda, r, T, t_1, g$, such that

$$\sup_{(-T/2,T/2)\times(\partial\Omega_0\cap B_{r/2}(x_0))} u(t,x) \leq Cu(t_1,x_1).$$

Our next regularity results are in terms of Schauder estimates. In the Schauder setting, we consider the nonlocal equation with a right hand side $(\partial_t + L)^s u = f$. But now, we assume some regularity on the coefficients $A(x)$ and on the right hand side $f(t,x)$. We prove both interior and global Schauder estimates for different regularity assumptions on $A(x), f(t,x)$, and also on the boundary $\partial\Omega$ (for the global case). We observe that the global regularity, under different boundary conditions, is consistent with the interior regularity, except for a special case when $u = 0$ on the boundary $\partial\Omega$ but f is not identically zero on $\partial\Omega$. The reason for this inconsistency can be explained by analyzing the behavior of particular one dimensional solutions to $(\partial_t - D^+_{xx})^s = f$ in $\mathbb{R} \times \mathbb{R}_+$, given that $u(t,0) = 0$ and $f(t,0) \neq 0$. Here $\mathbb{R}_+$ denotes the positive half line and D^+_{xx} is the second derivative operator on $\mathbb{R}_+$ subject to homogeneous Dirichlet boundary condition at $x = 0$.

The study of higher order Schauder estimates for our equations opens up a very different problem, namely, the Campanato characterization of higher order parabolic Hölder spaces, which in our opinion is of independent interest. We address such a problem in this dissertation. In the case of the space $C^{\alpha/2,\alpha}_{t,x}(I \times K)$, Campanato's characterization says that a function u will be in $C^{\alpha/2,\alpha}_{t,x}$ as soon as the following inequality holds

$$\inf_{c\in\mathbb{R}} \frac{1}{r^2|B_r(x)|} \int_{(t-r^2,t+r^2)\times B_r(x)} |u(\tau,z) - c|^2 \, d\tau dz \leq Cr^{2\alpha}$$

for every sufficiently small $r > 0$ and for all (t,x). The integral characterization above is actually measuring the mean square distance between $u(t,x)$ and any constant c, over the parabolic cylinder $(t - r^2, t + r^2) \times B_r(x)$. In this work, we extend this characterization for the higher order Hölder space $C^{(1+\alpha)/2,1+\alpha}_{t,x}(I \times K)$, where we have to compute the mean square distance between u and linear polynomial $P(x)$ in the x-variable.

1.1 Some Concrete Examples of Physical Applications Involving Fractional Powers of Parabolic Operators

In this section we will describe some physical processes involving fractional powers of parabolic operators.

1. **Semipermeable Membrane Problems:** For simplicity, we first consider the case of isotropic diffusion. More specifically, let us consider the heat equation in a region whose boundary is a semipermeable membrane. Let Γ be the membrane, which is assumed to be very thin, as in [25]. Again for simplicity, we consider Γ to be the hyperplane $y = 0$ in $\mathbb{R}^{n+1}$ and the diffusion process is occurring in the region $y > 0$. If $U(t, x, y)$ is the pressure of the fluid inside the membrane, i.e. for $y > 0$, and $\phi(t, x)$ is external pressure at the membrane, then $U(t, x, y)$ satisfies

$$\begin{cases} \partial_t U - \Delta_x U - \partial_{yy} U = 0 & \text{in } \mathbb{R}^{n+1} \times (0, \infty) \\ U(t, x, 0) \geq \phi(t, x) & \text{on } \mathbb{R}^{n+1} \\ -\partial_y U(t, x, 0) \geq 0 & \text{on } \mathbb{R}^{n+1} \\ -\partial_y U(t, x, 0) = 0 & \text{whenever } U(t, x, 0) > \phi(t, x) \end{cases} \tag{1.1.1}$$

   As an example, we can consider the phenomenon of osmosis across a cell membrane. The equations above indicate that when the external pressure $\phi(t, x)$ is smaller than the internal pressure $U(t, x, 0)$ (at the membrane Γ) then the cell is full and there is no fluid flow across the membrane, hence, $-\partial_y U(t, x, 0) = 0$. On the other hand, if the external pressure is greater than or equal to the internal pressure, then the cell is not full and the fluid enters inside the membrane, i.e. $-\partial_y U(t, x, 0) \geq 0$. The last three equations of (1.1.1) are known as the *Signorini complementary conditions.*

   Next, we want to understand more about the quantity $-\partial_y U(t, x, 0)$, i.e. the flow at the boundary $y = 0$. We apply the Fourier transform in t and x so that ρ and ξ are the corresponding Fourier variables for t and x, respectively. Then, from the diffusion equation,

$\widehat{U}(\rho,\xi,y) = e^{-y\sqrt{i\rho+|\xi|^2}}\widehat{U}(\rho,\xi,0)$. Therefore,

$$-\partial_y \widehat{U}(\rho,\xi,0) = \sqrt{i\rho+|\xi|^2}\widehat{U}(\rho,\xi,0)$$

and, by taking the inverse Fourier transform,

$$-\partial_y U(t,x,y)|_{y=0} = -\partial_y U(t,x,0) = \mathcal{F}^{-1}_{t,x}\left(\sqrt{i\rho+|\xi|^2}\widehat{U}(\rho,\xi,0)\right).$$

Now, $(i\rho+|\xi|^2)$ is the Fourier multiplier corresponding to the space-time operator $(\partial_t - \Delta)$. Then naturally, by writing $U(t,x,0) = u(t,x)$, we can define

$$\mathcal{F}^{-1}_{t,x}\left(\sqrt{i\rho+|\xi|^2}\widehat{U}(\rho,\xi,0)\right) = (\partial_t - \Delta_x)^{1/2}U(t,x,0) = (\partial_t - \Delta)^{1/2}u(t,x).$$

Let us next consider the case of anisotropic diffusion in the x-variable inside the membrane. We also assume that the domain in the x-variable is not $\mathbb{R}^n$ as before, but rather a bounded domain Ω, with appropriate boundary conditions on $\partial\Omega$. We still consider that the membrane is flat as before, i.e. the $y = 0$ hyperplane is the membrane. Then the diffusion process can be described by the following set of equations

$$\begin{cases} \partial_t U - \operatorname{div}_x(A(x)\nabla_x U) - \partial_{yy}U = 0 & \text{in } \mathbb{R}\times\Omega\times(0,\infty) \\ U(t,x,0) \geq \phi(t,x) & \text{on } \mathbb{R}\times\Omega \\ -\partial_y U(t,x,0) \geq 0 & \text{on } \mathbb{R}\times\Omega \\ -\partial_y U(t,x,0) = 0 & \text{whenever } U(t,x,0) > \phi(t,x) \\ U(t,x,y) = 0 \quad \text{or} \quad \partial_\nu U(t,x,y) = 0 & \text{on } \mathbb{R}\times\partial\Omega\times(0,\infty) \end{cases} \tag{1.1.2}$$

where ∂_ν denotes the exterior normal derivative at the boundary $\partial\Omega$. If we follow a similar procedure as in the case of isotropic diffusion, but now without the Fourier transform in the x-variable, rather with the spectrum of the operator $-\operatorname{div}_x(A(x)\nabla_x)$, then we can show that

$$-\partial_y U(t,x,y)|_{y=0} = (\partial_t - \operatorname{div}(A(x)\nabla))^{1/2}U(t,x,0).$$

To see this, we notice that $L = -\operatorname{div}(A(x)\nabla)$ with domain defined by suitable boundary conditions, has a discrete spectrum, say a set $\{\psi_k,\lambda_k\}_{k\geq 1}$ of eigenfunctions and eigenvalues.

Then we can write the following series expansions using the $L^2(\Omega)$-orthonormal basis $\{\psi_k\}_{k\geq 1}$:

$$U(t,x,y) = \sum_k U_k(t,y)\psi_k(x)$$

$$\widehat{U}(\rho,x,y) = \sum_k \widehat{U}_k(\rho,y)\psi_k(x)$$

$$-\operatorname{div}_x(A(x)\nabla_x U) = \sum_k \lambda_k U_k(t,y)\psi_k(x)$$

where $U_k(t,y) = \int_\Omega U(t,x,y)\psi_k(x)\,dx$, and $\widehat{U}_k(\rho,y)$ is the Fourier transform of $U_k(t,y)$ in the time variable t. Next, applying Fourier transform in the time variable to the equation $\partial_t U - \operatorname{div}_x(A(x)\nabla_x U) - \partial_{yy}U = 0$, we get

$$\sum_k (i\rho + \lambda_k + \partial_{yy})\widehat{U}_k(\rho,y)\psi_k(x) = 0.$$

Because of the orthogonality, we obtain, for all k and a.e. ρ,

$$(i\rho + \lambda_k + \partial_{yy})\widehat{U}_k(\rho,y) = 0.$$

Therefore,

$$\widehat{U}_k(\rho,y) = e^{-y\sqrt{i\rho+\lambda_k}}\widehat{U}_k(\rho,0) \quad \text{and} \quad -\partial_y\widehat{U}_k(\rho,0) = \sqrt{i\rho+\lambda_k}\widehat{U}_k(\rho,0)$$

and hence we can write,

$$-\partial_y U(t,x,y)|_{y=0} = \sum_k \mathcal{F}_t^{-1}\{\sqrt{i\rho+\lambda_k}\widehat{U}_k(\rho,0)\}\psi_k(x) = (\partial_t - \operatorname{div}(A(x)\nabla))^{1/2}U(t,x,0).$$

So we see that fractional powers of the parabolic operators appear very naturally in semipermeable membrane problems.

2. **Obstacle Problems:** Obstacle problems occur in many different physical applications. In [4], motivated by optimal control and finance, the authors studied the following obstacle problem: for $0 < s < 1$,

$$\begin{cases} u(t,x) \geq \phi(t,x) \\ (\partial_t - \Delta)^s u(t,x) \geq 0 \\ (\partial_t - \Delta)^s u(t,x) = 0 \quad \text{if } u(t,x) > \phi(t,x) \end{cases} \tag{1.1.3}$$

where $\phi(t,x)$ is the given obstacle and the $u(t,x)$ is the solution to the problem. This problem can be deduced from a corresponding Signorini problem or semipermeable membrane problem. Indeed, if we consider the semipermeable membrane problem as given in (1.1.1) then we observe that on the membrane, (1.1.1) reduces to the following form

$$\begin{cases} U(t,x,0) \geq \phi(t,x) \\ (\partial_t - \Delta)^{1/2} U(t,x,0) \geq 0 \\ (\partial_t - \Delta)^{1/2} U(t,x,0) = 0 \quad \text{if } U(t,x,0) > \phi(t,x). \end{cases} \tag{1.1.4}$$

If we denote $u(t,x) = U(t,x,0)$ then it is evident that (1.1.4) is a special case of (1.1.3) when $s = 1/2$.

If $L = -\operatorname{div}(A(x)\nabla)$ in a bounded domain with appropriate boundary conditions, the obstacle problem can be written as

$$\begin{cases} u(t,x) \geq \phi(t,x) \\ (\partial_t - \operatorname{div}(A(x)\nabla))^s u(t,x) \geq 0 \\ (\partial_t - \operatorname{div}(A(x)\nabla))^s u(t,x) = 0 \quad \text{if } u(t,x) > \phi(t,x) \\ u(t,x) = 0 \quad \text{or} \quad \partial_\nu u(t,x) = 0 \quad \text{on } \mathbb{R} \times \partial\Omega. \end{cases}$$

One can see that the obstacle problem above can be deduced from a semipermeable membrane problem similar to the problem described in (1.1.2), but for any $0 < s < 1$. We also want to mention that the regularity of solutions and free boundaries for this obstacle problem has not been analyzed yet.

3. **Biological Invasion Models:** Another interesting examples where fractional powers of parabolic operators appear are biological invasion models. In [8], the authors model many different biological phenomena with a single mathematical model. As examples, they mention phenomena like the black death in Europe, the movement of invasive species such as Processionary caterpillar of the pine tree in Europe, the invasion of Aedes albopictus mosquitoes in Europe and the movement of wolves in the Western Canadian forest. All of these examples

raise a question whether the inclusion of a line with fast diffusion affects the overall invasion speed or not.

To address such a question, the authors propose a model in the two-dimensional plane $\mathbb{R}^2$. According to their model, we assume that $\Gamma = \{(x,0) : x \in \mathbb{R}\}$ is the line, which they refer to as a 'road', for fast diffusion. Outside the line, which they call 'field', usual diffusion process takes place. Next, at time t, let $U(t,x,y)$ denote the density of the population at any point (x,y) in the field and let $\phi(t,x)$ be the density on the road Γ. Exchanges of population take place between the road and the field. Namely, a fraction $\nu \geq 0$ of the population $U(t,x,0)$ goes to $\phi(t,x)$ and a fraction $\mu \geq 0$ of the population $\phi(t,x)$ goes to $U(t,x,0)$. We denote the diffusion coefficient on the road by $d \geq 0$. Due to symmetry, it is enough to consider the equation only in the half plane $\mathbb{R} \times \mathbb{R}_+ = \{(x,y) : x \in \mathbb{R}, y > 0\}$ and for $t \in \mathbb{R}$. Then the model is described by

$$\begin{cases} \partial_t U(t,x,y) - \Delta U(t,x,y) = 0 & \text{if } (x,y) \in \mathbb{R} \times \mathbb{R}_+, t \in \mathbb{R} \\ \partial_t \phi(t,x) - d\partial_{xx}\phi(t,x) = \nu U(t,x,0) - \mu\phi(t,x) & \text{if } x \in \mathbb{R}, t \in \mathbb{R} \\ -\partial_y U(t,x,0) = \mu\phi(t,x) - \nu U(t,x,0) & \text{if } x \in \mathbb{R}, t \in \mathbb{R} \end{cases}$$

By applying the Fourier transform in the variables x and t, we see that the boundary expression $-\partial_y U(t,x,y)|_{y=0}$ equals $(\partial_t - \partial_{xx})^{1/2} U(t,x,0)$. Then, on the road we have the following local-fractional system of coupled equations

$$\begin{cases} \partial_t \phi(t,x) - d\partial_{xx}\phi(t,x) = \nu u(t,x) - \mu\phi(t,x) & \text{if } x \in \mathbb{R}, t \in \mathbb{R} \\ (\partial_t - \partial_{xx})^{1/2} u(t,x) = \mu\phi(t,x) - \nu u(t,x) & \text{if } x \in \mathbb{R}, t \in \mathbb{R} \end{cases}$$

where $u(t,x) = U(t,x,0)$. Notice that this local-fractional system is equivalent to the original local parabolic system mentioned above.

Let us consider the problem when the road is a bounded domain $\Omega \subset \mathbb{R}^n$ and the diffusion for U and ϕ in the x-variable are anisotropic with coefficient matrices A and A' respectively.

Then the system will be described by the following set of equations

$$\begin{cases} \partial_t \phi(t,x) - \operatorname{div}(A'(x)\nabla\phi) = \nu u(t,x) - \mu\phi(t,x) & \text{if } x \in \mathbb{R}, t \in \mathbb{R} \\ (\partial_t - \operatorname{div}(A(x)\nabla))^{1/2} u(t,x) = \mu\phi(t,x) - \nu u(t,x) & \text{if } x \in \mathbb{R}, t \in \mathbb{R} \end{cases}$$

where again we have denoted $u(t,x) = U(t,x,0)$.

1.2 Motivation

In this section we start our discussion with a very brief history of the nonlocal PDEs that are relevant to our work. In the last twenty years there has been a surge of interest in studying elliptic and parabolic equations in the nonlocal setting. Although fractional powers of operators on Banach spaces (which, in general, are nonlocal operators) have been studied for quite some time, for example, in [7, 13, 28], the recent resurgence in the study of this field was due to the works of L. Caffarelli and L. Silvestre, [14, 16, 39]. Let us give the definition of the fractional Laplacian. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a Schwartz's class function. We know that the Fourier transform of $(-\Delta)u$ is

$$\widehat{-\Delta u}(\xi) = |\xi|^2 \hat{u}(\xi).$$

Then, naturally, we can define, for $0 < s < 1$,

$$\widehat{(-\Delta)^s} u(\xi) = |\xi|^{2s} \hat{u}(\xi).$$

We can see that the above definition coincides with the following integral representation

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy$$

where PV denotes principal value integration and $c_{n,s} > 0$ is a constant which depends only on n and s. From this integral representation it is evident that the $(-\Delta)^s$ is an nonlocal operator. Hence if we want to study the regularity of solutions to $(-\Delta)^s u = f$, we cannot use the regularity theory of elliptic equations directly. This difficulty was overcome by the so-called extension method, first developed in the PDE setting by Caffarelli and Silvestre, see [14]. The extension method helps us to use well-established regularity results and techniques of local elliptic PDEs to obtain properties of solutions to $(-\Delta)^s u = f$.

In [41] P. R. Stinga and J. L. Torrea introduced the method of semigroups in connection with fractional powers of elliptic operators. They showed that, for any second order elliptic operator L on a Hilbert space, there is a solution U to a corresponding degenerate extension problem which characterizes solutions to $L^s u = f$. In addition, such solution U to the extension problem can be constructed explicitly using the semigroup $\{e^{-\tau L}\}_{\tau\geq 0}$ generated by the operator L. From [41] we can see that the methodology of semigroups helps us to define the fractional powers of any elliptic operator, not just the Laplacian, and to prove an extension problem characterization. In [17] the authors used the semigroup method and the extension characterization to prove Schauder regularity estimates for solutions u to the fractional nonlocal equation $L^s u(x) = f(x)$, where L is a divergence form elliptic operator in a bounded domain.

On the other hand, it is known that the most basic Master equation is given by the fractional powers of the heat operator $(\partial_t - \Delta)^s u = f$, $0 < s < 1$. This operator was analyzed in great detail in [42]. In that paper, Stinga and Torrea first provided a pointwise formula for $(\partial_t - \Delta)^s u(t, x)$, for $(t, x) \in \mathbb{R}^{n+1}$, by using the method of semigroups. Then they proved Harnack inequalities and Schauder estimates for solutions u to $(\partial_t - \Delta)^s u = f$ in $\mathbb{R}^{n+1}$.

In our work, we study different regularity estimates for solutions of the Master equation (1.0.1), which involves a fractional power of a parabolic operator and hence it is a nonlocal problem. One of the principal challenges in studying this problem is the limitation of the Fourier transform in the space variable. As the elliptic part L in the operator $\partial_t + L$ is *any* general elliptic operator and the domain Ω can be bounded, it turns out that the Fourier transform is not the most appropriate tool for our purposes. Instead, we use the spectrum of the operator L. Still, the precise notion of $(\partial_t + L)^s$ is a delicate point. Indeed, a natural definition in terms of Fourier transform in time and the spectral resolution of L leads to considering the multi-valued complex function $z \to z^s$. To overcome this difficulty we have to introduce a method which will help us select and effectively use the principal branch of this multi-valued function. Another challenge also lies in defining the operator $(\partial_t + L)^s$ in an appropriate pointwise sense. Notice that L is a divergence form operator. Hence the definition of the fractional power operator should be in the weak sense.

Studying regularity of solutions of these particular nonlocal equations opens up many interesting problems. First of all, as the problems are nonlocal, we cannot use the well-established local elliptic and parabolic regularity results directly. We introduce an extension method to overcome that difficulty. But then, developing the Schauder theory for this nonlocal equation still creates many complications. One of the reasons is the degeneracy in the corresponding parabolic extension equation. In principle, for the Schauder theory, it is not clear how to define *intermediate* Hölder spaces. When $0 < \alpha < 1$, the space $C_{t,x}^{\alpha/2,\alpha}$ is well-defined. Similarly, it is clear how to define the parabolic space $C_{t,x}^{1+\alpha/2,2+\alpha}$ where, as suggested by the heat equation $\partial_t u = \Delta u$, one derivative in time corresponds to two derivatives in space. But there is no universal definition of the space $C_{t,x}^{(1+\alpha)/2,1+\alpha}$, that contains functions $u(t,x)$ having only one derivative in space well-defined. In [29], N. V. Krylov used interpolation results to suggest a definition for the latter space. On the other hand, we need to characterize that space in an appropriate way to fit all the machinery of the compactness method that we use to obtain Schauder estimates. In fact, that characterization must be a Campanato-type characterization. Finally, establishing boundary Schauder regularity estimates for our nonlocal equations poses the following challenges. First of all, how does the given datum at the boundary affect the regularity? Secondly, is this regularity consistent with interior regularity estimates for different boundary conditions? All of these problems will be addressed and solved in this dissertation.

1.3 Notation

Before we give a short summary of our results, we would like to introduce some notation that will be used in the rest of our discussion.

Notation. Throughout the rest of this dissertation we will use the following notation. For $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and $r > 0$, we define

$$\begin{aligned} B_r(x) &= \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n : |x - z| < r\} \subset \mathbb{R}^n \\ Q_r(t,x) &= \{(\tau, z) \in \mathbb{R} \times \mathbb{R}^n : |t - \tau| < r^2,\ |x - z| < r\} \\ &= (t - r^2, t + r^2) \times B_r(x) \subset \mathbb{R} \times \mathbb{R}^n \\ B_r(x)^* &= \{(z, y) \in \mathbb{R}^n \times (0, \infty) : z \in B_r(x),\ 0 < y < r\} \\ &= B_r(x) \times (0, r) \subset \mathbb{R}^{n+1}_+ \\ Q_r(t,x)^* &= \{(\tau, z, y) \in \mathbb{R} \times \mathbb{R}^n \times (0, \infty) : |t - \tau| < r^2,\ z \in B_r(x),\ 0 < y < r\} \\ &= Q_r(t,x) \times (0, r) \subset \mathbb{R} \times \mathbb{R}^{n+1}_+. \end{aligned}$$

We write B_r, Q_r, etc, when $(t,x) = (0,0)$. If we let

$$B_r^+ = B_r \cap \{x_n > 0\} \subset \mathbb{R}^n_+$$

then we can also define Q_r^+, $(B_r^+)^*$ and $(Q_r^+)^*$ analogously. The fractional power is $s \in (0,1)$ and we will always denote $a = 1 - 2s \in (-1,1)$. Finally, we let $X = (x,y) \in \Omega \times (0,\infty)$ for $x \in \Omega \subset \mathbb{R}^n$, $y > 0$.

1.4 Description of Results

For the definition of fractional powers of parabolic operators we will use both the Fourier transform in time and the spectrum of the positive linear operator L. As mentioned before, L is an elliptic operator of the form

$$L = -\operatorname{div}(A(x)\nabla)$$

where the coefficient matrix $A(x)$ is bounded, measurable and satisfies the uniform ellipticity condition. In the following we will briefly discuss the main results that are contained in the next chapters of this dissertation.

1.4.1 Chapter 2 : Master Equation and Extension Method

The results in this chapter correspond to most of the paper [10] and some parts of [11].

In this chapter, we define the nonlocal operator $(\partial_t + L)^s$ for $0 < s < 1$ using the Fourier transform in time and the spectrum of L. As we mentioned before, this particular definition involves the multi-valued function $z \to z^s$. To capture the principal branch of that function in a way that can be explicitly used, we start by considering the numerical formula

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\tau\lambda} - 1) \frac{d\tau}{\tau^{1+s}},$$

where $\lambda > 0$ is a real number. We get the integral equation above from a corresponding integral representation of the Gamma function Γ evaluated at $-s \in (-1, 0)$. We prove that this representation holds true when λ is a complex number with positive real part. Furthermore, by using the Gamma function and the heat semigroup generated by the operator $H = (\partial_t + L)$, we are able to prove a pointwise integro-differential formula for $(\partial_t + L)^s$ in weak form. This approach, that we call "semigroup method", is described in Lemma 2.1.1. The pointwise formula is an integral representation of the operator $(\partial_t + L)^s$ which, on one hand, resembles the Master equation. On the other hand, it also shows that $(\partial_t + L)^s$ is a nonlocal operator.

We want to mention that the semigroup methodology given by Lemma 2.1.1 is very general and has wide applicability. Indeed, we can also consider other Master equations $(\partial_t + L)^s u = f$ in different settings. For example, the elliptic operator L can be replaced by the Laplace–Beltrami operator or the conformal Laplacian on a manifold, or by a subelliptic operator on a Lie group (like a Carnot or Heisenberg group), or by the Laplacian in infinite dimensions (Wiener space), or by nondivergence form elliptic operators, or by the Laplacian on a lattice.

Next we prove a parabolic extension problem characterization for our particular nonlocal operator, in the spirit of Caffarelli and Silvestre. Following [26], we show that if $U(t, x, y)$ is the unique weak solution to

$$\begin{cases} (\partial_t + L)U(t,x,y) - \frac{1-2s}{y}\partial_y U(t,x,y) - \partial_{yy}U(t,x,y) = 0 \text{ in } \mathbb{R} \times \Omega \times (0,\infty) \\ U(t,x,0) = u(t,x) \text{ on } \mathbb{R} \times \Omega \end{cases}$$

then

$$-\lim_{y\to 0} y^{1-2s}\partial_y U(t,x,y) = c_s(\partial_t + L)^s u(t,x) = c_s' \lim_{y\to 0} \frac{U(t,x,y)-U(t,x,0)}{y^{2s}}.$$

Here also we construct the solution U explicitly (see also [41]). To prove the extension theorem first we construct the following function

$$I_s(y,\lambda) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-\tau\lambda}\lambda^s e^{-y^2/4\tau}\frac{d\tau}{\tau^{1-s}}.$$

Then using the properties of modified Bessel function we prove different estimates for I_s and also show that I_s satisfies the following ordinary differential equation

$$\lambda I_s - \frac{1-2s}{y}\partial_y I_s - \partial_{yy} I_s = 0.$$

In Theorem 2.2.1 we prove the extension property for $(\partial_t + L)^s$ not only when L is any elliptic operator, but also when L is any nonnegative normal operator in a Hilbert space. Extension problems in such generality have proven to be very useful for several applications. For example, the extension problem allows us to find a monotonicity formula and prove regularity estimates for free boundary problems for fractional powers of the heat operator [4]. Also, they are central tools for the numerical analysis of fractional equations using finite elements methods [35].

Finally, in this chapter we prove several estimates of the fundamental solution of the nonlocal equation (1.0.1). Those estimates are obtained by two different methods. In one of them, we use the semigroup and heat kernel for the operator L and directly prove the estimates. In the other method, we use the extension theorem and then we apply estimates for the fundamental solution to such extension problem and we get the estimates for the fundamental solution of the nonlocal problem.

1.4.2 Chapter 3 : Harnack Inequalities

This chapter contains results from the paper [10].

The Harnack inequality is a very important regularity result in the realm of elliptic and parabolic PDEs. This inequality gives a bound on the oscillation of solutions, which, in turn, is necessary to prove Hölder regularity estimates. For Master equations as in (1.0.1) we prove parabolic interior and boundary Harnack inequalities in Theorems 3.1.1 and 3.1.3, respectively, and local boundedness and parabolic Hölder regularity of solutions.

For the interior Harnack inequality, let L be a uniformly elliptic operator in divergence form and let B_{2r} be a ball of radius $2r > 0$, such that $B_{2r} \subset\subset \Omega$. We prove that there exists a constant $C_H > 0$ depending only on n, s, Λ and r, such that if $u(t,x)$ is a solution to

$$\begin{cases} (\partial_t + L)^s u = 0 & \text{for } (t,x) \in R := (0,1) \times B_{2r} \\ u \geq 0 & \text{for } (t,x) \in (-\infty, 1) \times \Omega, \end{cases}$$

then

$$\sup_{R^-} u \leq C_H \inf_{R^+} u$$

where $R^- := (1/4, 1/2) \times B_r$ and $R^+ := (3/4, 1) \times B_r$. Moreover, we show that solutions u in R are locally bounded and locally parabolically α-Hölder continuous in R, for some exponent $0 < \alpha < 1$ depending on n, Λ and s.

On the other hand, for the boundary Harnack inequality, as we have mentioned earlier, we consider $\Omega_0 \subset \Omega, x_0 \in \partial\Omega_0$ and $B_r(x_0)$, the ball of radius r centered at x_0 such that $B_r(x_0) \subset \Omega$ and assume that $B_r(x_0) \cap \partial\Omega_0$ can be represented as a graph of Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ in the $e_n = (0, \ldots, 0, 1)$ direction. Let $(t_1, x_1) \in (-T, T) \times \Omega_0$ be a point such that $t_1 > T/2$. Then there exists a constant $C_{BH} > 0$ depending on n, s, Λ, t_1, r, g, such that if $u(t,x)$ is a solution to

$$\begin{cases} (\partial_t + L)^s u = 0 & \text{for } (t,x) \in R := (-T,T) \times (\Omega_0 \cap B_r(x_0)) \\ u \geq 0 & \text{for } (t,x) \in (-\infty, T) \times \Omega, \end{cases}$$

and u vanishes continuously on $(-T,T) \times ((\Omega/\Omega_0) \cap B_r(x_0))$ then

$$\sup_{(-T/2,T/2)\times(\Omega_0 \cap B_{r/2}(x_0))} u(t,x) \leq C_{BH} u(t_1, x_1).$$

The most important tool to prove the results above is the parabolic extension method that we introduced in Chapter 2. Once the extension changes the nonlocal problem to a degenerate parabolic problem, we are able to use the interior and boundary Harnack inequalities due to [27] to prove our results.

Finally, in this chapter we also develop a *transference method* for fractional powers of parabolic operators that allows us to transfer the Harnack inequalities and Hölder estimates for $(\partial_t+L)^s u = f$ from Theorems 3.1.1 and 3.1.3 to other Master equations of the form $(\partial_t+\bar{L})^s\bar{u} = \bar{f}$. Here, formally, $\bar{L} = (U \circ W)^{-1} \circ L \circ (U \circ W)$, where U is a multiplication operator by a smooth positive function and W is a smooth change of variables operator. This method is particularly useful when $\bar{L}$ is one of the elliptic operators listed in $(A) - (D)$ in Chapter 3. Notice that all of these new operators have a gradient term.

1.4.3 Chapter 4 : Parabolic Hölder Spaces

This chapter contains part of results of the paper [11].

As we mentioned before, to prove Schauder estimates we need to use an appropriate characterization of parabolic Hölder spaces. More specifically, due to our methodology, that relies on the compactness principle, we need to characterize parabolic Hölder spaces in terms of integrals involving the "mean distance" between solutions and constants or linear polynomials.

When we talk about the parabolic Hölder space $C^{\delta/2,\delta}_{t,x}$, we observe that it is clear how to define it in the case when $0 < \delta < 1$, namely, when there are no derivatives in time and space. It is also clear how to define the space $C^{1+\delta/2,2+\delta}_{t,x}$, that is, when we have one derivative in time and two derivatives in space. But it is not immediate how to define the appropriate *intermediate* Hölder space $C^{(1+\delta)/2,1+\delta}_{t,x}$, that is, the one that corresponds to one derivative in space. In [29], N. V. Krylov used interpolation results to suggest a definition. Indeed, in [29, Remark 8.8.7] he claims that "*with respect to the parabolic metric, one derivative in t is worth two derivatives in x. This suggests that $C^{(1+\delta)/2,1+\delta}(\mathbb{R}^{d+1})$ should be defined as the space of all functions with finite norm $\|u\|_0 + \|u_x\|_{\delta/2,\delta} + \sup_{s\neq t,x} \frac{|u(t,x)-u(s,x)|}{|t-s|^{(1+\delta)/2}}$.*"

Stinga and Torrea showed that Krylov's definition for the intermediate parabolic Hölder space $C^{(1+\delta)/2,1+\delta}_{t,x}(\mathbb{R}^{n+1})$ is correct in terms of the Poisson semigroup generated by the heat operator, see [42, Theorem 7.2]. They used such a semigroup characterization to prove Schauder estimates for solutions to $(\partial_t-\Delta)^{\pm s}u = f$.

In turn, here we show in Theorem 4.1.1(2) that Krylov's definition of intermediate parabolic Hölder space is also the correct one for bounded domains in terms of approximations by linear polynomials that depend only on space. Precisely, we show that u is in $C^{(1+\alpha)/2,1+\alpha}_{t,x}(\overline{I\times\Omega})$, where I is a time interval and Ω is a bounded domain, if and only if for every $(t,x)\in\overline{I\times\Omega}$ the following holds

$$\inf_{P(z)\in\mathcal{P}_1}\frac{1}{r^2|B_r(x)|}\int_{(t-r^2,t+r^2)\times B_r(x)}|u(\tau,z)-P(z)|^2\,d\tau dz\le Cr^{2(1+\alpha)}$$

for every sufficiently small $r>0$ and where $\mathcal{P}_1=\{P(z)=A_0+A_1\cdot z: A_0\in\mathbb{R}, A_1\in\mathbb{R}^n\}$. This is a Campanato-type characterization that, up to the best of our knowledge, has not been proved in the literature. In Theorem 4.1.1(1) we also prove the Campanato-type characterization for $C^{\alpha/2,\alpha}_{t,x}$. We remark that Campanato-type characterizations for $C^{\alpha/2,\alpha}_{t,x}$ are well-known results, see, for instance, [32, 38].

The proof of Theorem 4.1.1 depends mainly on the Lebesgue Differentiation Theorem. Let us describe briefly the ideas of proving Theorem 4.1.1(2). Let us assume that for each (t,x) and $r>0$ there is a polynomial $P(z)=P(z,(t,x),r,u)$ for which u satisfies the integral estimate,

$$\frac{1}{r^2|B_r(x)|}\int_{(t-r^2,t+r^2)\times B_r(x)}|u(\tau,z)-P(z)|^2\,d\tau dz\le Cr^{2(1+\alpha)}. \tag{1.4.1}$$

Let $a_0((t,x),r,u)=P(z,(t,x),r,u)\big|_{z=x}$ and $a_i((t,x,),r,u)=\partial_{z_i}P(z,(t,x),r,u)\big|_{z=x}$ for $i=1,\dots,n$. By using the Lebesgue Differentiation Theorem, we can prove that the following limits exist:

$$\lim_{r\to 0}a_i(z,(t,x),r,u)=v_i(t,x),\quad \text{for } i=0,1,\dots,n.$$

Then we show that $v_i(t,x)\in C^{\alpha/2,\alpha}_{t,x}$, $\frac{\partial v_0(t,x)}{\partial x_i}=v_i(t,x)$ for $i=1,\dots,n$ and $v_0(t,x)\in C^{(1+\alpha)/2,1+\alpha}_{t,x}$. We conclude by proving that $v_0=u$ almost everywhere in the domain $\overline{I\times\Omega}$.

1.4.4 Chapter 5 : Schauder Estimates

This chapter collects results from the paper [11].

In this chapter we continue the development of the regularity theory for (1.0.1). We obtain interior and boundary parabolic Schauder estimates for solutions u to (1.0.1) in the cases when f is Hölder continuous, see Theorems 5.1.1, 5.1.3, 5.1.4 and 5.1.5, and also when f is just L^p

integrable, for p large depending on s and n, see Theorems 5.1.2 and 5.1.6. For these results, the coefficients $A(x)$ are assumed to be at least continuous. In particular, when we prove the interior Schauder estimates in Theorem 5.1.1, we assume that the datum $f \in C_{t,x}^{\alpha/2,\alpha}$ for $0 < \alpha < 1$. Then, if $0 < \alpha + 2s < 1$ and the coefficients $A(x)$ are continuous, we prove that the solution $u \in C_{t,x}^{(\alpha+2s)/2,\alpha+2s}$. On the other hand, if $1 < \alpha + 2s < 2$ and the coefficients $A(x)$ are Hölder continuous, i.e. $A(x) \in C_x^{\alpha+2s-1}$ then we prove that solution $u \in C_{t,x}^{(\alpha+2s)/2,1+(\alpha+2s-1)}$. Here we want to stress the fact that we use the Campanato characterization of $C_{t,x}^{(\alpha+2s)/2,1+(\alpha+2s-1)}$ which we proved in Chapter 4. From these results we see that the regularity of solutions is higher than the regularity of the data by a factor of $2s$ in space and s in time, which are the orders of equation (1.0.1) in space and time, respectively.

Interior Hölder regularity for the solution u, when the datum is in L^p, is expected due to Calderón–Zygmund L^p estimates and Sobolev embeddings. For simplicity, first consider the elliptic nonlocal equation $H^s u = f$, where $H = L = -\Delta$ and $f \in L_x^p$. Using Calderón–Zygmund analysis for $(-\Delta)u = f$, see [12], the solution u is in the Sobolev space $W^{2,p}$. For the fractional equation $(-\Delta)^s u = f$ we in turn expect u to belong to some sort of fractional Sobolev space $W^{2s,p}$. Now, from Sobolev inequality, if $p > \frac{n}{2s}$, then $u \in C^{k+\alpha}$, where k, α depends on p, s. When we do a similar analysis for the parabolic case, i.e. when $H = \partial_t + L$, then the dimension will be $n + 2$ (parabolic dimension). Hence, to get Hölder regularity in the case of $f \in L_{t,x}^p$, we will need a restriction of the form $p > \frac{n+2}{2s}$. Indeed, in Theorem 5.1.2 we prove the following two results.

- If $p < \frac{n+2}{(2s-1)_+}$ then we have $k = 0, \alpha = 2s - (n+2)/p \in (0,1)$, hence $u \in C_{t,x}^{\alpha/2,\alpha}$. To get this regularity we need $A(x)$ to be continuous.

- If $p > \frac{n+2}{(2s-1)_+}$, $s > 1/2$, we have $k = 1, \alpha = 2s - (n+2)/p - 1 \in (0,1)$, hence $u \in C_{t,x}^{(1+\alpha)/2,1+\alpha}$. In this case we need $A(x)$ to be Hölder continuous with exponent α.

  Here $(2s-1)_+$ denotes the largest integer I such that $0 \leq I \leq 2s - 1$.

The main technique to prove interior Schauder estimates is the parabolic extension problem as described in Theorem 2.4.2. The extension result turns the nonlocal equation (1.0.1) into a local degenerate parabolic problem with Neumann boundary condition. As the extension (2.4.2) localizes the equation, we can prove energy estimates with appropriate test functions and then apply compactness arguments in the local parabolic setting. Indeed, we first prove a counterpart of the parabolic Caccioppoli inequality in Lemma 5.2.2. For this the Steklov averages are an essential tool. Second, the compactness provided by the Aubin–Lions lemma, see [5], together with the energy estimate, provide the existence of a solution W to a degenerate heat equation (5.2.8) that is 'close' to our solution U in the L^2-sense, see Corollary 5.2.3. This approximation is applied at any scale to finally transfer the regularity from W to U.

There are some intricate issues in the proof of global regularity, in particular, in Theorem 5.1.3. In this theorem, we deal with the scenario when the solution u is zero on the boundary but the data f is not identically zero on the boundary. In contrast to this situation, the regularity of u is improved when f is zero on the boundary, see Theorem 5.1.4. As mentioned before, this fact is better explained by computing particular one dimensional pointwise solutions to $(\partial_t - D_{xx}^+)^s u = f$ in $\mathbb{R} \times \mathbb{R}_+$, given that $u(t,0) = 0$ in $\mathbb{R}$ and f is nonzero on the boundary $x = 0$. On one hand, one dimensional particular solutions have a regularity estimate involving the distance function from the boundary. We show that in our solution u to (1.0.1) there is one component u_1 which can be constructed from that particular one dimensional solution. Hence, that component u_1 will have regularity in terms of $\operatorname{dist}(x, \partial\Omega)$. On the other hand, due to this solution, for $s \leq 1/2$, we need a little bit more regularity on the boundary $\partial\Omega$ and on the coefficients $A(x)$ to get $C_{t,x}^{(\alpha+2s)/2,1+(\alpha+2s-1)}(\overline{I \times \Omega})$ regularity for $u - u_1$. Therefore, to prove the above boundary regularity results, we need to develop sharp boundary estimates for half-space solutions.

1.4.5 Chapter 6 : General Conclusion

In this chapter, we provide a brief summary of our work.

CHAPTER 2. MASTER EQUATION AND EXTENSION METHOD

We start our analysis with a general setting. We consider the linear elliptic operator L in (1.0.1) as given by

$$L = -\operatorname{div}(A(x)\nabla) + c(x) \tag{2.0.1}$$

for $x \in \Omega$, where Ω is an domain of $\mathbb{R}^n$, $n \geq 1$, that may be unbounded and $c(x) \in L^\infty_{\text{loc}}(\Omega)$ is a real-valued function. Here $A(x) = (A^{ij}(x))$ is a bounded, measurable, symmetric matrix defined in Ω, satisfying the uniform ellipticity condition, that is, for some $\Lambda \geq 1$,

$$\Lambda^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$. The operator L is subject to an appropriate boundary condition. Here we consider the boundary condition to be either homogeneous Dirichlet or Neumann, that is,

$$u = 0 \quad \text{or} \quad \partial_A u = A(x)\nabla_x u \cdot \nu = 0 \quad \text{on } \mathbb{R} \times \partial\Omega,$$

where ν is the exterior unit normal to $\partial\Omega$. Some concrete operators of the (2.0.1) we will work with are the following:

(1) $L = -\operatorname{div}(A(x)\nabla) + c(x)$ in a bounded domain Ω with homogeneous Dirichlet or Neumann (conormal) boundary condition. The potential function $c(x) \geq 0$ and $c(x) \in L^\infty(\Omega)$. If $c(x) = 0$ and $A(x) = I$, then we get $-\Delta_D$ and $-\Delta_N$, the Dirichlet and Neumann Laplacians, respectively.

(2) The harmonic oscillators $L = -\Delta + |x|^2$ and $L = -\Delta + |x|^2 - n$ in $\Omega = \mathbb{R}^n$.

(3) The Laguerre differential operator $L = \frac{1}{4}(-\Delta + |x|^2 + \sum_{i=1}^n \frac{1}{x_i^2}\left(\alpha_i^2 - \frac{1}{4}\right))$, for $\alpha_i > -1$, in $\Omega = (0,\infty)^n$.

(4) The ultraspherical operator $L = -\frac{d^2}{dx^2} + \frac{\lambda(\lambda-1)}{\sin^2 x}$, for $\lambda > 0$, in $\Omega = (0,\pi)$.

(5) The Laplacian $-\Delta$ in $\Omega = \mathbb{R}^n$.

(6) The Bessel operator $L = -\frac{d^2}{dx^2} + \frac{\lambda(\lambda-1)}{x^2}$, for $\lambda > 0$, in $\Omega = (0,\infty)$.

Observe that in (2)–(6) the ellipticity constant is $\Lambda = 1$. The operators (2)–(4) arise in classical orthogonal expansions and the Bessel operator in (6) appears when considering radial-in-space solutions to $(\partial_t - \Delta)^s u = f$.

2.1 Definition and Integro-differential Formula

In this section we present the precise definition of $H^s u(t,x) = (\partial_t + L)^s u(t,x)$ and show that, in general, this is a Master operator. Let L be a nonnegative normal linear operator on a Hilbert space $L^2(\Omega)$ with some positive measure $d\eta$. For concreteness and simplicity of the presentation, we will assume that L has discrete spectrum and $d\eta$ is the Lebesgue measure (notice that the eigenvalues of L will be nonnegative). For other types of spectrums, we can always obtain the general result by using the Spectral Theorem, the Fourier transform, the Hankel transform, the corresponding orthogonal expansions with respect to $d\eta$, etc.

Therefore, assume that L has a countable sequence of eigenvalues and eigenfunctions $(\lambda_k, \phi_k)_{k\geq 0}$ such that $0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$ and so that $\{\phi_k\}_{k\geq 0}$ forms an orthonormal basis of $L^2(\Omega)$. For simplicity of the presentation, we will also assume that the eigenfunctions ϕ_k are real-valued. In the case in which $\lambda_0 = 0$ (for instance, for the Neumann Laplacian) we assume that all the functions involved have zero integral mean over Ω. With this, any function $u(t,x) \in L^2(\mathbb{R}\times\Omega)$ can be written as

$$u(t,x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \sum_{k=0}^{\infty} \widehat{u_k}(\rho)\phi_k(x)e^{it\rho}\,d\rho,$$

where

$$u_k(t) = \int_{\Omega} u(t,x)\phi_k(x)\,dx$$

and $\widehat{u_k}(\rho)$ is the Fourier transform of $u_k(t)$ with respect to the variable $t \in \mathbb{R}$:

$$\widehat{u_k}(\rho) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} u_k(t)e^{-i\rho t}\,dt.$$

The domain of the operator $H^s \equiv (\partial_t + L)^s$, $0 \le s \le 1$, is defined as

$$\mathrm{Dom}(H^s) = \left\{ u \in L^2(\mathbb{R}\times\Omega) : \|u\|_{H^s}^2 := \int_{\mathbb{R}} \sum_{k=0}^{\infty} |i\rho + \lambda_k|^s |\widehat{u_k}(\rho)|^2 \, d\rho < \infty \right\}.$$

This is a complex Hilbert space with norm $\|\cdot\|_{H^s}$, whose dual is denoted by $\mathrm{Dom}(H^s)^*$. Moreover, $\mathrm{Dom}(H^t) \subset \mathrm{Dom}(H^s)$ whenever $0 \le s \le t \le 1$. For $u \in \mathrm{Dom}(H^s)$, we define $H^s u \in \mathrm{Dom}(H^s)^*$, in a 'weak sense', as acting on any $v \in \mathrm{Dom}(H^s)$ by

$$\langle H^s u, v \rangle \equiv \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho + \lambda_k)^s \widehat{u_k}(\rho) \overline{\widehat{v_k}(\rho)} \, d\rho \tag{2.1.1}$$

where $\overline{\widehat{v_k}(\rho)}$ denotes the complex conjugate of $\widehat{v_k}(\rho)$. We have

$$\|u\|_{H^s}^2 = \langle H^{s/2} u, H^{s/2} u \rangle \quad \text{for any } 0 \le s \le 1.$$

Notice that in the expression (2.1.1) we need to appropriately decide which s-power of the complex number $(i\rho + \lambda_k)$ we are taking. We are able to clarify this by developing a semigroup technique, in which the Gamma function plays a crucial role. The method permits us to show that (2.1.1) is indeed a Master equation, or nonlocal in space and time integro-differential operator, in divergence form. Observe as well that $\mathrm{Dom}(H^s)$ encodes the boundary condition on L.

As the family of eigenfunctions $\{\phi_k\}_{k\ge 0}$ is an orthonormal basis of $L^2(\Omega)$, we can write the semigroup $\{e^{-\tau L}\}_{\tau \ge 0}$ generated by L as

$$\langle e^{-\tau L}\varphi, \psi \rangle_{L^2(\Omega)} = \sum_{k=0}^{\infty} e^{-\tau\lambda_k} \varphi_k \psi_k = \int_\Omega \int_\Omega W_\tau(x,z) \varphi(z) \psi(x) \, dz \, dx$$

for any $\varphi, \psi \in L^2(\Omega)$, where $\varphi_k = \int_\Omega \varphi \phi_k \, dx$ and $\psi_k = \int_\Omega \psi \phi_k \, dx$. As it happens for (2.0.1) and all the other cases (1)–(6), we will always assume that the heat kernel for L is symmetric and nonnegative:

$$W_\tau(x,z) = W_\tau(z,x) \ge 0.$$

Since ∂_t and L commute, we define, for any $u \in L^2(\mathbb{R}\times\Omega)$,

$$e^{-\tau H} u(t,x) = e^{-\tau L}(e^{-\tau\partial_t} u)(t,x) = e^{-\tau L}(u(t-\tau,\cdot))(x)$$

in the sense that, for any $v \in L^2(\mathbb{R}\times\Omega)$,

$$\begin{aligned}\langle e^{-\tau H}u, v\rangle_{L^2(\mathbb{R}\times\Omega)} &= \int_{\mathbb{R}} \sum_{k=0}^{\infty} e^{-\tau(i\rho+\lambda_k)} \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)}\, d\rho \\ &= \int_{\mathbb{R}} \sum_{k=0}^{\infty} e^{-\tau\lambda_k} u_k(t-\tau)v_k(t)\, dt \\ &= \int_{\mathbb{R}} \iint_{\Omega} W_\tau(x,z)u(t-\tau,z)v(t,x)\, dz\, dx\, dt.\end{aligned} \tag{2.1.2}$$

Lemma 2.1.1. *Let $0 < s < 1$. If $u \in \mathrm{Dom}(H^s)$ then*

$$H^s u = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{-\tau H}u - u\right) \frac{d\tau}{\tau^{1+s}}$$

in the sense that, for any $v \in \mathrm{Dom}(H^s)$,

$$\langle H^s u, v\rangle = \frac{1}{\Gamma(-s)} \int_0^\infty \left(\langle e^{-\tau H}u, v\rangle_{L^2(\mathbb{R}\times\Omega)} - \langle u, v\rangle_{L^2(\mathbb{R}\times\Omega)}\right) \frac{d\tau}{\tau^{1+s}}.$$

Proof. Let $u, v \in \mathrm{Dom}(H^s)$. We will use the following numerical formula with the Gamma function that comes from performing the analytic continuation to $\mathrm{Re}(z) > 0$ of the function that maps $t \in [0,\infty)$ to t^s, see [9, 42],

$$(i\rho + \lambda_k)^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\tau(i\rho+\lambda_k)} - 1)\frac{d\tau}{\tau^{1+s}}, \quad \rho \in \mathbb{R}. \tag{2.1.3}$$

The integral above is absolutely convergent. Then, in (2.1.1) we have

$$\langle H^s u, v\rangle = \int_{\mathbb{R}} \sum_{k=0}^{\infty} \left[\frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\tau(i\rho+\lambda_k)} - 1)\frac{d\tau}{\tau^{1+s}}\right] \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)}\, d\rho.$$

On one hand,

$$\int_0^{1/|i\rho+\lambda_k|} |e^{-\tau(i\rho+\lambda_k)} - 1| \frac{d\tau}{\tau^{1+s}} \le C|i\rho+\lambda_k| \int_0^{1/|i\rho+\lambda_k|} \tau^{-s}\, d\tau = C|i\rho+\lambda_k|^s.$$

On the other hand,

$$\int_{1/|i\rho+\lambda_k|}^\infty |e^{-\tau(i\rho+\lambda_k)} - 1| \frac{d\tau}{\tau^{1+s}} \le C \int_{1/|i\rho+\lambda_k|}^\infty \tau^{-1-s}\, d\tau = C|i\rho+\lambda_k|^s.$$

Since $u, v \in \mathrm{Dom}(H^s)$, Fubini's Theorem and (2.1.2) allow us to get the conclusion. □

We observe that in the above lemma we are able to write down the 'weak form formulation' of $H^s u$ using the heat semigroup $e^{-\tau H}$. This is what we call the *semigroup method*. Next we see that by using our semigroup method for the concrete cases (1)–(6) we are able to obtain an integro-differential formula for $H^s u$ which shows that (1.0.1) is indeed a Master equation as in (1.0.3), but in divergence form.

Theorem 2.1.2. *Let L be as in (1)–(6). If $u, v \in \mathrm{Dom}(H^s) \cap C_c^\infty(\mathbb{R} \times \Omega)$ then*

$$\begin{aligned}
\langle H^s u, v\rangle &= \langle (\partial_t + L)^s u, v\rangle \\
&= \int_0^\infty \int_{\mathbb{R}} \int_\Omega \int_\Omega K_s(\tau, x, z)(u(t-\tau, x) - u(t-\tau, z))(v(t,x) - v(t,z))\, dz\, dx\, dt\, d\tau \\
&\quad + \int_0^\infty \Bigg[\int_{\mathbb{R}} \int_\Omega \frac{\big(1 - e^{-\tau L}1(x)\big)}{|\Gamma(-s)|\tau^{1+s}} u(t,x) v(t,x)\, dx\, dt \\
&\qquad\qquad - \int_{\mathbb{R}} \int_\Omega e^{-\tau L}1(x) \frac{(u(t-\tau, x) - u(t,x))}{|\Gamma(-s)|\tau^{1+s}} v(t,x)\, dx\, dt \Bigg] d\tau
\end{aligned}$$

where

$$K_s(\tau, x, z) = \frac{W_\tau(x,z)}{2|\Gamma(-s)|\tau^{1+s}}$$

$W_\tau(x,z)$ is the heat kernel for L, and

$$e^{-\tau L}1(x) = \int_\Omega W_\tau(x,z)\, dz.$$

Remark 2.1.3. There are cases in which $e^{-\tau L}1(x) \equiv 1$. This occurs, for example, when L is as in (2.0.1) with $c(x) = 0$ and has either Neumann boundary condition or $\Omega = \mathbb{R}^n$, or when L is the Laplacian $-\Delta$ on $\mathbb{R}^n$. Then, in Theorem 2.1.2 we get

$$\begin{aligned}
\langle H^s u, v\rangle &= \int_0^\infty \int_{\mathbb{R}} \int_\Omega \int_\Omega K_s(\tau, x, z)(u(t-\tau, x) - u(t-\tau, z))(v(t,x) - v(t,z))\, dz\, dx\, dt\, d\tau \\
&\quad - \int_0^\infty \int_{\mathbb{R}} \int_\Omega \frac{(u(t-\tau, x) - u(t,x))}{|\Gamma(-s)|\tau^{1+s}} v(t,x)\, dx\, dt\, d\tau.
\end{aligned}$$

The second integral term above is equal to

$$-\int_{\mathbb{R}} \int_\Omega (D_{\mathrm{left}})^s u(t,x) v(t,x)\, dx\, dt$$

where $(D_{\mathrm{left}})^s$ denotes the fractional power of the derivative from the left, which coincides with the Marchaud fractional derivative, acting on the variable $t \in \mathbb{R}$, see [9].

Remark 2.1.4. If the heat kernel $W_\tau(x,z)$ has Gaussian estimates (see, for example, [3, 17]) then it can be checked that the kernel $K_s(\tau,x,z)$ in Theorem 2.1.2 satisfies the size estimates of the kernels K of the equations considered in [15, 42].

Proof of Theorem 2.1.2. For $u,v\in \mathrm{Dom}(H^s)\cap C_c^\infty(\mathbb{R}\times\Omega)$ we have, by Lemma 2.1.1, up to the multiplicative constant $1/\Gamma(-s)$,

$$\begin{aligned}\langle H^s u,v\rangle &= \int_0^\infty \big(\langle e^{-\tau L}u(\cdot-\tau,\cdot),v(\cdot,\cdot)\rangle_{L^2(\mathbb{R}\times\Omega)}-\langle u,v\rangle_{L^2(\mathbb{R}\times\Omega)}\big)\,\frac{d\tau}{\tau^{1+s}}\\ &= \int_0^\infty\left[\int_{\mathbb{R}}\int_\Omega\int_\Omega W_\tau(x,z)u(t-\tau,z)v(t,x)\,dz\,dx\,dt-\int_{\mathbb{R}}\int_\Omega u(t,x)v(t,x)\,dx\,dt\right]\frac{d\tau}{\tau^{1+s}}.\end{aligned}$$

The integral in brackets can be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega\int_\Omega W_\tau(x,z)(u(t-\tau,z)-u(t-\tau,x))v(t,x)\,dz\,dx\,dt\\ &\qquad+\int_{\mathbb{R}}\int_\Omega\big(e^{-\tau L}1(x)u(t-\tau,x)-u(t,x)\big)v(t,x)\,dx\,dt.\end{aligned}\tag{2.1.4}$$

By exchanging the roles of x and z and using that $W_\tau(z,x)=W_\tau(x,z)$, the integrals above are also equal to

$$\begin{aligned}&-\int_{\mathbb{R}}\int_\Omega\int_\Omega W_\tau(x,z)(u(t-\tau,z)-u(t-\tau,x))v(t,z)\,dx\,dz\,dt\\ &\qquad+\int_{\mathbb{R}}\int_\Omega\big(e^{-\tau L}1(x)u(t-\tau,x)-u(t,x)\big)v(t,x)\,dx\,dt.\end{aligned}\tag{2.1.5}$$

By adding (2.1.4) and (2.1.5), we get that, up to the multiplicative constant $1/|\Gamma(-s)|$,

$$\begin{aligned}2\langle H^s u,v\rangle = \int_0^\infty\Bigg[&\int_{\mathbb{R}}\int_\Omega\int_\Omega W_\tau(x,z)(u(t-\tau,x)-u(t-\tau,z))(v(t,x)-v(t,z))\,dz\,dx\,dt\\ &+2\int_{\mathbb{R}}\int_\Omega\big(u(t,x)-e^{-\tau L}1(x)u(t-\tau,x)\big)v(t,x)\,dx\,dt\Bigg]\frac{d\tau}{\tau^{1+s}}.\end{aligned}$$

For the operators L in (1)–(6) we always have the Gaussian estimate

$$|W_\tau(x,z)|\le C\frac{e^{-|x-z|^2/(c\tau)}}{\tau^{n/2}}$$

(see, for instance, [2, 3, 17, 23]). Observe that u, v can be extended by zero outside of $\mathbb{R} \times \Omega$ so we can regard them as functions in $C_c^\infty(\mathbb{R}^{n+1})$. Then

$$\begin{aligned}
&\Big| \int_0^\infty \int_\Omega \int_\Omega W_\tau(x,z) \int_\mathbb{R} (u(t-\tau,x) - u(t-\tau,z))(v(t,x) - v(t,z))\, dt\, dz\, dx\, \frac{d\tau}{\tau^{1+s}} \Big| \\
&= \Big| \int_0^\infty \int_\Omega \int_\Omega W_\tau(x,z) \int_\mathbb{R} e^{i\tau\rho} (\widehat{u}(\rho,x) - \widehat{u}(\rho,z)) \overline{(\widehat{v}(\rho,x) - \widehat{v}(\rho,z))}\, d\rho\, dz\, dx\, \frac{d\tau}{\tau^{1+s}} \Big| \\
&\leq \int_\mathbb{R} \int_\Omega \int_\Omega |\widehat{u}(\rho,x) - \widehat{u}(\rho,z)| |\widehat{v}(\rho,x) - \widehat{v}(\rho,z)| \Big[\int_0^\infty W_\tau(x,z)\, \frac{d\tau}{\tau^{1+s}} \Big]\, dz\, dx\, d\rho \\
&\leq C \int_\mathbb{R} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\widehat{u}(\rho,x) - \widehat{u}(\rho,z)|^2}{|x-z|^{n+2s}}\, dz\, dx\, d\rho + C \int_\mathbb{R} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\widehat{v}(\rho,x) - \widehat{v}(\rho,z)|^2}{|x-z|^{n+2s}}\, dz\, dx\, d\rho \\
&= C \int_\mathbb{R} \big(\|(-\Delta)^{s/2} \widehat{u}(\rho,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{s/2} \widehat{v}(\rho,\cdot)\|^2_{L^2(\mathbb{R}^n)} \big)\, d\rho \\
&= C \int_{\mathbb{R}^{n+1}} |\xi|^{2s} \big(|\mathcal{F}_{\mathbb{R}^{n+1}}(u)(\rho,\xi)|^2 + |\mathcal{F}_{\mathbb{R}^{n+1}}(v)(\rho,\xi)|^2 \big)\, d\xi\, d\rho
\end{aligned}$$

where in the last identity we use Plancherel's identity in $\mathbb{R}^n$ and $\mathcal{F}_{\mathbb{R}^{n+1}}$ denotes the Fourier transform in $(t,x) \in \mathbb{R}^{n+1}$. The last integral above is finite because $u, v \in C_c^\infty(\mathbb{R}^{n+1})$. Therefore, we can write $\langle H^s u, v \rangle$ as the sum of

$$\frac{1}{2|\Gamma(-s)|} \int_0^\infty \int_\mathbb{R} \int_\Omega \int_\Omega W_\tau(x,z)(u(t-\tau,x) - u(t-\tau,z))(v(t,x) - v(t,z))\, dz\, dx\, dt\, \frac{d\tau}{\tau^{1+s}}$$

and

$$\frac{1}{|\Gamma(-s)|} \int_0^\infty \int_\mathbb{R} \int_\Omega \big(u(t,x) - e^{-\tau L} 1(x) u(t-\tau,x) \big) v(t,x)\, dx\, dt\, \frac{d\tau}{\tau^{1+s}}.$$

The conclusion readily follows from here. □

Remark 2.1.5. In Theorem 2.1.2 we have assumed that u and v are smooth with compact support. We can relax this assumption as soon as we are able to show that for any $u, v \in \mathrm{Dom}(H^s)$ we have

$$\int_\mathbb{R} \int_\Omega \int_\Omega |\widehat{u}(\rho,x) - \widehat{u}(\rho,z)| |\widehat{v}(\rho,x) - \widehat{v}(\rho,z)| \Big[\int_0^\infty W_\tau(x,z)\, \frac{d\tau}{\tau^{1+s}} \Big]\, dz\, dx\, d\rho < \infty.$$

This is true, for instance, in the case when L is as in (1) with either Dirichlet or Neumann boundary conditions, and with $c(x) = 0$. Indeed, by the results in [17], if $u, v \in \mathrm{Dom}(H^s)$ then it follows that $u, v \in L^2(\mathbb{R}; \mathrm{Dom}(L^s))$.

2.2 Extension Theorem for $(\partial_t + L)^s$ when L is a Normal Operator

The main tool to prove different regularity estimates for (1.0.1) is an extension problem characterization for the fractional operators $(\partial_t + L)^s$. Observe that, in general, L as in (2.0.1) may have discrete or continuous spectrum in different Hilbert spaces. The extension problem we present here not only works for (2.0.1), but for any fractional operator of the form $(\partial_t + L)^s$, where L is a nonnegative normal linear operator in a Hilbert space $L^2(\Omega)$ with some positive measure $d\eta$. Extension theorem for the case of the fractional heat operator $H^s = (\partial_t - \Delta)^s$ was proved in [42], see also [26]. Here we will present a similar theorem for $(\partial_t + L)^s$.

For the sake of simplicity and concreteness of the presentation we next assume that L is a nonnegative, normal linear operator in $L^2(\Omega)$, with countable eigenvalues and real-valued eigenfunctions and with a nonnegative, symmetric heat kernel, as in Section 2.1. Recall that if the first eigenvalue is $\lambda_0 = 0$ (as in the Neumann Laplacian) then we assume that all the functions involved have zero spatial mean. The general case follows by using the Spectral Theorem or the particular spectral resolution of the corresponding elliptic operator (like the Fourier transform or the Hankel transform). Details in those cases will be left to the interested reader.

Theorem 2.2.1 (Extension problem)**.** *Let L be a normal nonnegative linear operator on $L^2(\Omega)$ and $H = \partial_t + L$. Let $u \in \mathrm{Dom}(H^s)$. For $(t,x) \in \mathbb{R} \times \Omega$ and $y > 0$ we define*

$$\begin{aligned} U(t,x,y) &= \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-\tau H} u(t,x) \, \frac{d\tau}{\tau^{1+s}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-r} e^{-\frac{y^2}{4r}H} u(t,x) \, \frac{dr}{r^{1-s}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4r)} e^{-rH} (H^s u)(t,x) \, \frac{dr}{r^{1-s}}. \end{aligned} \tag{2.2.1}$$

Then $U(\cdot,\cdot,y) \in \mathrm{Dom}(H)$ for each $y > 0$, $U \in C^\infty((0,\infty); L^2(\mathbb{R}\times\Omega)) \cap C([0,\infty); L^2(\mathbb{R}\times\Omega))$ and $U \in L^2((0,\infty); \mathrm{Dom}(H), y^{1-2s}dy)$. Moreover, U is a solution to

$$\begin{cases} \langle HU, v\rangle = \left\langle \frac{1-2s}{y}\partial_y U + \partial_{yy} U, v \right\rangle_{L^2(\mathbb{R}\times\Omega)} & \text{for each } v \in \mathrm{Dom}(H) \text{ and } y > 0 \\ \lim_{y\to 0^+} U(t,x,y) = u(t,x) & \text{in } L^2(\mathbb{R}\times\Omega) \end{cases} \tag{2.2.2}$$

such that

$$\lim_{y\to\infty}\langle U,v\rangle_{L^2(\mathbb{R}\times\Omega)}=0,\quad \text{for every } v\in L^2(\mathbb{R}\times\Omega)$$

and

$$\sup_{y>0}|\langle y^{1-2s}\partial_y U,v\rangle_{L^2(\mathbb{R}\times\Omega)}|\le C_s\|u\|_{H^s}\|v\|_{H^s},\quad \text{for every } v\in \mathrm{Dom}(H^s).$$

In addition, for every $v\in \mathrm{Dom}(H^s)$*,*

$$\begin{aligned}-\frac{1}{2s}\lim_{y\to 0^+}\langle y^{1-2s}\partial_y U,v\rangle_{L^2(\mathbb{R}\times\Omega)}&=\frac{|\Gamma(-s)|}{4^s\Gamma(s)}\langle H^s u,v\rangle\\&=-\lim_{y\to 0^+}\left\langle\frac{U(\cdot,\cdot,y)-U(\cdot,\cdot,0)}{y^{2s}},v\right\rangle_{L^2(\mathbb{R}\times\Omega)}.\end{aligned}$$

Theorem 2.2.1 shows that the solution u to the nonlocal problem $H^s u=f$ can be characterized by the solution U to the local problem (2.2.2). Another main novelty is the set of explicit formulas for the solution U we discovered in (2.2.1). Observe that U is given in terms of the semigroup generated by H acting either on u or on $f=H^s u$.

To prove Theorem 2.2.1 we begin with an important preliminary result.

Lemma 2.2.2. *Let* $0<s<1$*. Denote by* $K_\nu(z)$ *the modified Bessel function of the second kind and order* ν*. For* $y>0$ *and* $\lambda\in\mathbb{C}$ *with* $\mathrm{Re}(\lambda)>0$ *we define*

$$\begin{aligned}I_s(y,\lambda)&=\frac{2^{1-s}}{\Gamma(s)}(y\sqrt{\lambda})^s K_s(y\sqrt{\lambda})\\&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-t}e^{-\frac{y^2}{4t}\lambda}\,\frac{dt}{t^{1-s}}\\&=\frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-y^2/(4r)}e^{-r\lambda}\,\frac{dr}{r^{1+s}}\\&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-y^2/(4\tau)}e^{-\tau\lambda}\lambda^s\,\frac{d\tau}{\tau^{1-s}}.\end{aligned}\tag{2.2.3}$$

The integrals are absolutely convergent. Fix any s *and* λ *as above. Then*

(1) $I_s(y,\lambda)$ *is a smooth function of* $y\in(0,\infty)$*.*

(2) *For each* $y>0$*,* $I_s(y,\lambda)$ *satisfies the equation*

$$\lambda u-\frac{1-2s}{y}\partial_y u-\partial_{yy}u=0.\tag{2.2.4}$$

(3) $\lim_{y\to 0^+} I_s(y,\lambda) = 1.$

(4) $-y^{1-2s}\partial_y I_s(y,\lambda) = \dfrac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}\lambda^s I_{1-s}(y,\lambda).$

(5) *The following estimates hold:*

(5.*a*) $|I_s(y,\lambda)| \le 1.$

(5.*b*) *There is a constant* $C_s > 0$ *such that*

$$|I_s(y,\lambda)| \le C_s(y|\lambda|^{1/2})^{s-1/2}e^{-\cos(\arg(\lambda)/2)y|\lambda|^{1/2}} \quad \textit{as } y\to\infty.$$

(5.*c*) *There is a constant* $C_s > 0$ *such that*

$$|\lambda I_s(y,\lambda)| + \left|\tfrac{1}{y}\partial_y I_s(y,\lambda)\right| + |\partial_{yy} I_s(y,\lambda)| \le C_s\frac{|\lambda|^s}{y^{2-2s}} \quad \textit{for every } y>0.$$

(6) *The function* $I_s(\lambda,y)$ *is the unique* C^∞ *solution to* (2.2.4) *such that*

$$\lim_{y\to 0} I_s(y,\lambda) = 1, \quad \lim_{y\to\infty} I_s(y,\lambda) = 0, \quad \textit{and} \quad y^{1-2s}\partial_y I_s(y,\lambda) \in L^\infty_y([0,\infty)).$$

Proof. It is well known that for ν arbitrary (see [31, eq. (5.10.25)])

$$K_\nu(z) = \frac{1}{2}\Big(\frac{z}{2}\Big)^\nu \int_0^\infty e^{-t}e^{-z^2/4t}t^{-\nu-1}dt \quad \text{for } |\arg z| < \frac{\pi}{4}.$$

As $K_\nu = K_{-\nu}$ we get the second identity in (2.2.3). The third one follows from the change of variables $r = y^2/(4t)$. The last one for $\lambda > 0$ is obtained from the third one via the change of variables $\tau = y^2/(4r\lambda)$, and the general case of $\mathrm{Re}(\lambda) > 0$ follows from the case of $\lambda > 0$ by analytic continuation.

Now (1) is easy to check by differentiating under the integral sign. Indeed, since

$$|\partial_y(y^{2s}e^{-y^2/(4\tau)})| = \Big|\Big(2sy^{2s-1} - \frac{y^{2s+1}}{2\tau}\Big)e^{-y^2/(4\tau)}\Big| \le C_s y^{2s-1}e^{-y^2/(c\tau)}, \tag{2.2.5}$$

we get

$$\partial_y I_s(y,\lambda) = \int_0^\infty \partial_y\Big(\frac{y^{2s}}{4^s\Gamma(s)}e^{-y^2/(4r)}\Big)e^{-r\lambda}\,\frac{dr}{r^{1+s}}.$$

Similarly for higher order derivatives. For (2) we can use integration by parts to get

$$\begin{aligned}
\lambda I_s(y,\lambda) &= -\frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-y^2/(4r)}\partial_r e^{-r\lambda}\,\frac{dr}{r^{1+s}} \\
&= \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty \partial_r\Big(\frac{e^{-y^2/(4r)}}{r^{1+s}}\Big)e^{-r\lambda}\,dr \\
&= \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty \Big(\partial_{yy}+\frac{1-2s}{y}\partial_y\Big)\Big(\frac{e^{-y^2/(4r)}}{r^{1+s}}\Big)e^{-r\lambda}\,dr \\
&= \partial_{yy}I_s(y,\lambda)+\frac{1-2s}{y}\partial_y I_s(y,\lambda).
\end{aligned}$$

The proof of (3) follows readily from the second identity in (2.2.3) and dominated convergence. By using that the Bessel function K_ν satisfies

$$\frac{\partial}{\partial z}[z^\nu K_\nu(z)] = -z^\nu K_{\nu-1}(z) = -z^\nu K_{1-\nu}(z)$$

we immediately obtain (4). Observe that (5.a) is clear from the second identity in (2.2.3). The asymptotic estimate (see [31, eq. (5.11.9)])

$$K_\nu(z) = Cz^{-1/2}e^{-z}\big(1+O(|z|^{-1})\big) \quad \text{as } |z|\to\infty,\ |\arg z|<\pi-\delta,\ \delta>0,$$

implies (5.b). To prove (5.c), observe that the function $g(t)=e^{-\frac{y^2}{4t}\mathrm{Re}(\lambda)}t^{s-1}$ has a maximum at $t=\frac{y^2\mathrm{Re}(\lambda)}{4(1-s)}$ which is $g_{max}=C_s\frac{\mathrm{Re}(\lambda)^{s-1}}{y^{2-2s}}$. Hence,

$$|I_s(y,\lambda)| \le \frac{1}{\Gamma(s)}\int_0^\infty e^{-t}g(t)\,dt \le C_s\frac{\mathrm{Re}(\lambda)^{s-1}}{y^{2-2s}}.$$

The estimate for $\frac{1}{y}\partial_y I_s(y,\lambda)$ follows from (4) and (5.a). We can bound $\partial_{yy}I_s(y,\lambda)$ by using (2.2.4) and the previous two estimates. We see from (5.b) that $I_s(y,\lambda)\to 0$ as $y\to\infty$. To prove (6), let $J(y)$ be a smooth solution to (2.2.4) such that $\lim_{y\to 0^+}J(y)=0$, $\lim_{y\to\infty}J(y)=0$ and $|y^{1-2s}\partial_y J(y)|\le C$ for all $y\ge 0$. Multiply (2.2.4) by $y^{1-2s}\overline{J(y)}$ and integrate by parts to get

$$\int_0^\infty y^{1-2s}\mathrm{Re}(\lambda)|J(y)|^2\,dy + \int_0^\infty y^{1-2s}|\partial_y J(y)|^2\,dy = 0.$$

Since $\mathrm{Re}(\lambda)>0$, it follows that $J(y)\equiv 0$. □

Remark 2.2.3. The fact that Bessel functions can be used to treat extension problems was first observed in [41]. Here we have extended [41] to apply to the case when λ is complex-valued. See also

[42] for solutions to the extension problem in terms of integral representations of Bessel functions for the particular case of $(\partial_t - \Delta)^s$, in which $\lambda = i\rho + |\xi|^2$.

Proof of Theorem 2.2.1. Let us denote $U(y) = U(\cdot,\cdot,y)$, for $y > 0$, where U is given by (2.2.1). Since

$$\frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)}\, \frac{d\tau}{\tau^{1+s}} = 1 \tag{2.2.6}$$

we find that, for any $v = v(t,x) \in L^2(\mathbb{R}\times\Omega)$,

$$\begin{aligned}\big|\langle U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)}\big| &\leq \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} \|e^{-\tau H}u\|_{L^2(\mathbb{R}\times\Omega)}\, \|v\|_{L^2(\mathbb{R}\times\Omega)}\, \frac{d\tau}{\tau^{1+s}} \\ &\leq \|u\|_{L^2(\mathbb{R}\times\Omega)}\, \|v\|_{L^2(\mathbb{R}\times\Omega)}\end{aligned}$$

so that

$$\langle U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)} = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} \langle e^{-\tau H}u, v\rangle_{L^2(\mathbb{R}\times\Omega)}\, \frac{d\tau}{\tau^{1+s}} < \infty. \tag{2.2.7}$$

In particular, for each $y > 0$, $U(y) \in L^2(\mathbb{R}\times\Omega)$, with

$$\|U(y)\|_{L^2(\mathbb{R}\times\Omega)} \leq \|u\|_{L^2(\mathbb{R}\times\Omega)}.$$

In addition, by using (2.1.2) and (2.2.3) from Lemma 2.2.2,

$$\langle U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)} = \int_{\mathbb{R}} \sum_{k=0}^\infty \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)} I_s(y, i\rho+\lambda_k)\, d\rho$$

and

$$U(y) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \sum_{k=0}^\infty \widehat{u_k}(\rho) I_s(y, i\rho+\lambda_k)\phi_k(x) e^{i\rho t}\, d\rho.$$

Next, by using Lemma 2.2.2 parts (5.a) and (5.c),

$$\int_{\mathbb{R}} \sum_{k=0}^\infty |i\rho+\lambda_k||\widehat{u_k}(\rho)|^2 |I_s(y, i\rho+\lambda_k)|^2\, d\rho \leq \frac{C_s}{y^{2-2s}} \int_{\mathbb{R}} \sum_{k=0}^\infty |i\rho+\lambda_k|^s |\widehat{u_k}(\rho)|^2\, d\rho < \infty,$$

we get that $U(y) \in \mathrm{Dom}(H)$ for each $y > 0$. Then, for any $v \in \mathrm{Dom}(H)$, (see (2.1.1))

$$\langle HU(y), v\rangle = \int_{\mathbb{R}} \sum_{k=0}^\infty \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)} (i\rho+\lambda_k) I_s(y, i\rho+\lambda_k)\, d\rho.$$

Let us check that $U \in C^\infty((0,\infty); L^2(\mathbb{R}\times\Omega))$ and that, for any $k \geq 1$,

$$\partial_y^k \langle U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)} = \langle \partial_y^k U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)}.$$

Indeed, first notice that

$$|\langle e^{-\tau H}u, v\rangle_{L^2(\mathbb{R}\times\Omega)}| \le e^{-\tau\lambda_i}\,\|u\|_{L^2(\mathbb{R}\times\Omega)}\,\|v\|_{L^2(\mathbb{R}\times\Omega)} \tag{2.2.8}$$

where $i = 0$ if $\lambda_0 \neq 0$ and $i = 1$ if $\lambda_0 = 0$. Here we have used that

$$\|e^{-\tau H}u\|^2_{L^2(\mathbb{R}\times\Omega)} = \sum_{k=i}^{\infty} e^{-2\tau\lambda_k}\int_{\mathbb{R}} |u_k(t-\tau)|^2\,dt \le e^{-2\tau\lambda_i}\|u\|^2_{L^2(\mathbb{R}\times\Omega)}.$$

By using (2.2.5),

$$\begin{aligned}\int_0^\infty \left|\partial_y\left(\frac{y^{2s}}{4^s\Gamma(s)}e^{-y^2/(4\tau)}\right)\langle e^{-\tau H}u, v\rangle_{L^2(\mathbb{R}\times\Omega)}\right|\frac{d\tau}{\tau^{1+s}} \\ \le C_s y^{2s-1}\,\|u\|_{L^2(\mathbb{R}\times\Omega)}\,\|v\|_{L^2(\mathbb{R}\times\Omega)}\int_0^\infty e^{-\tau\lambda_i}e^{-y^2/(c\tau)}\,\frac{d\tau}{\tau^{1+s}}\end{aligned}$$

so we can differentiate under the integral sign in (2.2.7). Similarly it can be done for higher order derivatives and we get $U(y) \in C^\infty((0,\infty); L^2(\mathbb{R}\times\Omega))$.

Observe that, by the first equation in (2.2.3),

$$\begin{aligned}\int_0^\infty y^{1-2s}\|U\|^2_{H^1}\,dy &= \int_0^\infty y^{1-2s}\int_{\mathbb{R}}\sum_{k=0}^{\infty}|i\rho+\lambda_k||\widehat{u}_k(\rho)|^2|I_s(y, i\rho+\lambda_k)|^2\,d\rho\,dy \\ &= \int_{\mathbb{R}}\sum_{k=0}^{\infty}|i\rho+\lambda_k||\widehat{u}_k(\rho)|^2\int_0^\infty y^{1-2s}|I_s(y, i\rho+\lambda_k)|^2\,dy\,d\rho \\ &\le C_s\int_{\mathbb{R}}\sum_{k=0}^{\infty}|i\rho+\lambda_k|^{1+s}|\widehat{u}_k(\rho)|^2\int_0^\infty y|K_s(y\sqrt{i\rho+\lambda_k})|^2\,dy\,d\rho.\end{aligned}$$

To estimate the integral in dy, let $r = y|\sqrt{i\rho+\lambda_k}|$ and $\theta = \arg(\sqrt{i\rho+\lambda_k})$, hence

$$\int_0^\infty y|K_s(y\sqrt{i\rho+\lambda_k})|^2\,dy = |i\rho+\lambda_k|^{-1}\int_0^\infty r|K_s(re^{i\theta})|^2 dr \le C_s|i\rho+\lambda_k|^{-1},$$

In the last inequality we used the fact that

$$K_s(z) \sim C_s z^{-s} \quad \text{as } z \to 0, \quad \text{and} \quad K_s(z) \sim\ z^{-1/2}e^{-z} \quad \text{as } z \to \infty, \tag{2.2.9}$$

see [31]. Then,

$$\int_0^\infty y^{1-2s}\|U\|^2\,dy \le C_s\int_{\mathbb{R}}\sum_{k=0}^{\infty}|i\rho+\lambda_k|^s|\widehat{u}_k(\rho)|^2\,d\rho = C_s\,\|u\|^2_{H^s} < \infty$$

so $U \in L^2((0,\infty); \mathrm{Dom}(H), y^{1-2s}dy)$.

For $v \in \mathrm{Dom}(H)$, by Lemma 2.2.2, we have that

$$\begin{aligned}\langle HU(y), v\rangle &= \int_{\mathbb{R}} \sum_{k=0}^{\infty} \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)}(i\rho+\lambda_k) I_s(y, i\rho+\lambda_k)\, d\rho \\ &= \int_{\mathbb{R}} \sum_{k=0}^{\infty} \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)} \left(\tfrac{1-2s}{y}\partial_y + \partial_{yy}\right) I_s(y, i\rho+\lambda_k)\, d\rho \\ &= \left\langle \left(\tfrac{1-2s}{y}\partial_y + \partial_{yy}\right) U(y), v \right\rangle_{L^2(\mathbb{R}\times\Omega)}.\end{aligned}$$

By Lemma 2.2.2 and Dominated Convergence Theorem,

$$\lim_{y\to 0}\langle U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)} = \int_{\mathbb{R}} \sum_{k=0}^{\infty} \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)}\, d\rho = \langle u, v\rangle_{L^2(\mathbb{R}\times\Omega)}$$

and

$$\begin{aligned}\langle -y^{1-2s}\partial_y U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)} &= \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho+\lambda_k)^s \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)} I_{1-s}(y, i\rho+\lambda_k)\, d\rho \\ &\to \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}\langle H^s u, v\rangle, \quad \text{as } y\to 0^+.\end{aligned} \tag{2.2.10}$$

Now, for every $v \in \mathrm{Dom}(H^s)$, since $I_s(0, i\rho+\lambda_k) = 1$,

$$\frac{1}{y^{2s}}\langle U(y) - U(0), v\rangle_{L^2(\mathbb{R}\times\Omega)} = \int_{\mathbb{R}} \sum_{k=0}^{\infty} \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)} \frac{I_s(y, i\rho+\lambda_k) - 1}{y^{2s}}\, d\rho.$$

From the third equation in (2.2.3), (2.2.6) and (2.1.3) we get

$$\begin{aligned}\frac{I_s(y, i\rho+\lambda_k) - 1}{y^{2s}} &= \frac{1}{4^s\Gamma(s)} \int_0^{\infty} e^{-y^2/(4\tau)} \left(e^{-\tau(i\rho+\lambda_k)} - 1\right) \frac{d\tau}{\tau^{1+s}} \\ &\to \frac{\Gamma(-s)}{4^s\Gamma(s)}(i\rho+\lambda_k)^s, \quad \text{as } y \to 0^+.\end{aligned}$$

Moreover, by applying Lemma 2.2.2(4) and (5.a),

$$\begin{aligned}\frac{|I_s(y, i\rho+\lambda_k) - 1|}{y^{2s}} &\le \frac{1}{y^{2s}} \int_0^y |\partial_r I_s(r, i\rho+\lambda_k)|\, dr \\ &\le \frac{C_s}{y^{2s}} |i\rho+\lambda_k|^s \int_0^y r^{2s-1}\, dr = C_s |i\rho+\lambda_k|^s.\end{aligned}$$

Thus, as $u, v \in \mathrm{Dom}(H^s)$, by Dominated Convergence Theorem,

$$\begin{aligned}\lim_{y\to 0^+} \frac{1}{y^{2s}}\langle U(y) - U(0), v\rangle_{L^2(\mathbb{R}\times\Omega)} &= \frac{\Gamma(-s)}{4^s\Gamma(s)} \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho+\lambda_k)^s \widehat{u_k}(\rho)\overline{\widehat{v_k}(\rho)}\, d\rho \\ &= \frac{\Gamma(-s)}{4^s\Gamma(s)}\langle H^s u, v\rangle.\end{aligned}$$

For any $v \in L^2(\mathbb{R}\times\Omega)$, by (2.2.8) and Lemma 2.2.2, we have

$$\begin{aligned} |\langle U(y), v\rangle_{L^2(\mathbb{R}\times\Omega)}| &\le \|u\|_{L^2(\mathbb{R}\times\Omega)}\|v\|_{L^2(\mathbb{R}\times\Omega)}\frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-\tau\lambda_i}e^{-\frac{y^2}{4\tau}}\frac{d\tau}{\tau^{1+s}} \\ &= \|u\|_{L^2(\mathbb{R}\times\Omega)}\|v\|_{L^2(\mathbb{R}\times\Omega)}I_s(y,\lambda_i), \end{aligned} \tag{2.2.11}$$

where $i = 0$ if $\lambda_0 \neq 0$ and $i = 1$ if $\lambda_0 = 0$. Since $I_s(y,\lambda_i) \to 0$ as $y \to \infty$, we get that U weakly vanishes as $y \to \infty$.

If $v \in \mathrm{Dom}(H^s)$ then we see from Lemma 2.2.2(5.a) and (2.2.10) that

$$|\langle y^{1-2s}\partial_y U, v\rangle_{L^2(\mathbb{R}\times\Omega)}| \le C_s\|u\|_{H^s}\|v\|_{H^s}, \quad \text{for all } y \ge 0.$$

□

2.3 Extension Problem for $(\partial_t + L)^s$ when L is a Divergence Form Operator

In this section we specialize the extension characterization for $(\partial_t + L)^s$ in Theorem 2.2.1 to the case when L is a divergence form elliptic operator. Then we can get an equivalent formulation of (2.2.2) in weak form.

Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain and

$$Lu = -\operatorname{div}(A(x)\nabla u) + c(x)u \quad \text{in } \Omega,$$

where $A(x) = (A^{ij}(x))$ is a bounded, measurable, symmetric matrix defined in Ω, satisfying the uniform ellipticity condition, that is, for some $\Lambda > 0$

$$\Lambda^{-1}|\xi|^2 \le A^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for a.e. $x \in \Omega$, for all $\xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n$, and $c(x) \in L^\infty_{\mathrm{loc}}(\Omega)$ is a real-valued function. For $u, f \in L^2(\Omega)$, the equation $Lu = f$ in Ω in the weak sense means that $\nabla u \in L^2(\Omega)$, $c^{1/2}u \in L^2(\Omega)$ and

$$\int_\Omega A(x)\nabla u\nabla v\,dx + \int_\Omega c(x)uv\,dx = \int_\Omega fv\,dx,$$

for every $v \in C_c^\infty(\Omega)$. We also assume appropriate boundary conditions on $\partial\Omega$ so that L has a countable family of nonnegative eigenvalues $(\lambda_k, \phi_k)_{k=0}^\infty$ such that the set of real-valued eigenfunctions $\{\phi_k\}_{k=0}^\infty$ forms an orthonormal basis for $L^2(\Omega)$. As before, if the first eigenvalue $\lambda_0 = 0$ then we assume that all the functions involved have zero spatial mean. In particular,

$$L\phi_k = \lambda_k \phi_k \quad \text{for all } k \geq 0 \text{ in the weak sense.}$$

Therefore, if we define

$$H_L^1(\Omega) \equiv \mathrm{Dom}(L) = \Big\{ u \in L^2(\Omega) : \sum_{k=0}^\infty \lambda_k |u_k|^2 < \infty \Big\}$$

where $u_k = \displaystyle\int_\Omega u\phi_k \, dx$, then, for any $u, v \in H_L^1(\Omega)$,

$$\int_\Omega A(x)\nabla u \nabla v \, dx + \int_\Omega c(x) uv \, dx = \sum_{k=0}^\infty \lambda_k u_k v_k.$$

The operators listed in (1)–(4) at the beginning of this chapter satisfy the conditions above.

Now, the extension equation takes the form

$$\partial_t U = y^{-(1-2s)} \operatorname{div}_{x,y}(y^{1-2s} B(x) \nabla_{x,y} U) - c(x)U,$$

where

$$B(x) = \begin{bmatrix} A(x) & 0 \\ 0 & 1 \end{bmatrix}$$

is also uniformly elliptic. Let us denote $D = \{(x, y) : x \in \Omega,\ y > 0\} \subset \mathbb{R}^{n+1}$. The $A_2(\mathbb{R}^N)$-class of Muckenhoupt weights is the set of all a.e. positive functions $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $N \geq 1$, for which there exists a constant $C_\omega > 0$ such that

$$\left(\frac{1}{|B|} \int_B \omega \right) \left(\frac{1}{|B|} \int_B \omega^{-1} \right) \leq C_\omega$$

for every ball $B \subset \mathbb{R}^N$, see [24]. It is straightforward to check that the weight $\omega(x, y) = |y|^{1-2s}$ belongs to the class $A_2(\mathbb{R}^{n+1})$. Define $H_{L,y}^1(D)$ as the set of functions $w = w(x, y) \in L^2(D, y^{1-2s} dxdy)$

such that

$$[w]^2_{H^1_{L,y}(D)} := \int_0^\infty \int_\Omega y^{1-2s}\big(A(x)\nabla w \nabla w + c(x)w^2\big)\,dx\,dy + \int_0^\infty \int_\Omega y^{1-2s}|\partial_y w|^2\,dx\,dy$$
$$= \int_0^\infty y^{1-2s}\sum_{k=0}^\infty \lambda_k |w_k(y)|^2\,dy + \int_0^\infty \int_\Omega y^{1-2s}|\partial_y w|^2\,dx\,dy < \infty,$$

where $w_k(y) = \int_\Omega w(x,y)\phi_k(x)\,dx$, under the norm

$$\|w\|^2_{H^1_{L,y}(D)} = \|w\|^2_{L^2(D,y^{1-2s}dxdy)} + [w]^2_{H^1_{L,y}(D)}.$$

Theorem 2.3.1. *Consider the extension problem in Theorem 2.2.1 with L is as above. Then U, defined in (2.2.1), belongs to $L^2(\mathbb{R}; H^1_{L,y}(D)) \cap C^\infty((0,\infty); L^2(\mathbb{R}\times\Omega)) \cap C([0,\infty); L^2(\mathbb{R}\times\Omega))$ and for any fixed $y > 0$ and $v \in C_c^\infty(\mathbb{R}\times\Omega)$,*

$$\langle HU, v\rangle = \int_\mathbb{R}\int_\Omega \Big(\tfrac{1-2s}{y}\partial_y + \partial_{yy}\Big)Uv\,dt\,dx = y^{2s-1}\int_\mathbb{R}\int_\Omega \partial_y(y^{1-2s}\partial_y U)v\,dt\,dx.$$

In particular, U is a weak solution to the parabolic extension problem

$$\begin{cases} \partial_t U = y^{-(1-2s)}\operatorname{div}_{x,y}(y^{1-2s}B(x)\nabla_{x,y}U) - c(x)U & \text{for } (t,x,y) \in \mathbb{R}\times\Omega\times(0,\infty) \\ -y^{1-2s}\partial_y U\Big|_{y=0^+} = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}H^s u & \text{for } (t,x) \in \mathbb{R}\times\Omega \end{cases}$$

in the following sense: for any $V(t,x,y) \in C_c^\infty(\mathbb{R}\times\Omega\times[0,\infty))$,

$$\begin{aligned}\int_\mathbb{R}\int_\Omega U\partial_t V\,dx\,dt = \int_\mathbb{R}\int_\Omega \big(A(x)\nabla_x U\nabla_x V + c(x)UV\big)\,dx\,dt \\ - \int_\mathbb{R}\int_\Omega \Big(\tfrac{1-2s}{y}\partial_y + \partial_{yy}\Big)UV\,dx\,dt\end{aligned} \tag{2.3.1}$$

and

$$\begin{aligned}\int_0^\infty \int_\mathbb{R}\int_\Omega y^{1-2s}U\partial_t V\,dx\,dt\,dy = \int_0^\infty \int_\mathbb{R}\int_\Omega y^{1-2s}\big(B(x)\nabla_{x,y}U\nabla_{x,y}V + c(x)UV\big)\,dx\,dt\,dy \\ - \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}\langle H^s u, V(t,x,0)\rangle.\end{aligned}$$

Proof. Let us first check that $U(t,x,y) \in L^2(\mathbb{R}; H^1_{L,y}(D))$. We found in (2.2.11) that

$$\|U(y)\|_{L^2(\mathbb{R}\times\Omega)} \le \|u\|_{L^2(\mathbb{R}\times\Omega)} I_s(y,\lambda_i)$$

where $i=0$ if $\lambda_0\neq 0$ and $i=1$ if $\lambda_0=0$. Then, from (2.2.3),

$$\begin{aligned}\int_0^\infty y^{1-2s}\|U(y)\|^2_{L^2(\mathbb{R}\times\Omega)}\,dy &\leq C_s\|u\|^2_{L^2(\mathbb{R}\times\Omega)}\int_0^\infty y^{1-2s}(y\sqrt{\lambda_i})^{2s}K_s^2(y\sqrt{\lambda_i})\,dy \\ &= C_s\|u\|^2_{L^2(\mathbb{R}\times\Omega)}\lambda_i^{s-1}\int_0^\infty rK_s^2(r)dr<\infty.\end{aligned} \tag{2.3.2}$$

In the last inequality we used (2.2.9). We are left to show that

$$\int_{\mathbb{R}}\int_0^\infty y^{1-2s}\sum_{k=0}^\infty \lambda_k|U_k(t,y)|^2\,dy\,dt+\int_{\mathbb{R}}\int_0^\infty\int_\Omega y^{1-2s}|\partial_yU(t,x,y)|^2\,dx\,dy\,dt<\infty,$$

where, for any $k\geq i$, for $i=0$ if $\lambda_0\neq 0$ and $i=1$ if $\lambda_0=0$,

$$\begin{aligned}U_k(t,y) &= \langle U(t,\cdot,y),\phi_k(\cdot)\rangle_{L^2(\Omega)} \\ &= \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-y^2/(4\tau)}\langle e^{-\tau L}u(t-\tau,\cdot),\phi_k(\cdot)\rangle_{L^2(\Omega)}\,\frac{d\tau}{\tau^{1+s}} \\ &= \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-y^2/(4\tau)}e^{-\tau\lambda_k}u_k(t-\tau)\,\frac{d\tau}{\tau^{1+s}}.\end{aligned}$$

From here and (2.2.3) we see that

$$\int_{\mathbb{R}}|U_k(t,y)|^2\,dt\leq\|u_k\|^2_{L^2(\mathbb{R})}|I_s(y,\lambda_k)|^2.$$

Therefore, as done in (2.3.2),

$$\begin{aligned}\int_{\mathbb{R}}\int_0^\infty y^{1-2s}\sum_{k=0}^\infty\lambda_k|U_k(t,y)|^2\,dy\,dt &\leq \sum_{k=0}^\infty\lambda_k\|u_k\|^2_{L^2(\mathbb{R})}\int_0^\infty y^{1-2s}(y\sqrt{\lambda_k})^{2s}K_s^2(y\sqrt{\lambda_k})\,dy \\ &\leq C_s\sum_{k=0}^\infty\lambda_k^s\|u_k\|^2_{L^2(\mathbb{R})}<\infty.\end{aligned}$$

Next, observe that

$$\partial_yU(t,x,y)=C_sy^{2s-1}\sum_{k=0}^\infty\left[\int_{\mathbb{R}}\widehat{u_k}(\rho)(i\rho+\lambda_k)^sI_{1-s}(y,i\rho+\lambda_k)e^{i\rho t}d\rho\right]\phi_k(x)$$

and then

$$\|\partial_yU\|^2_{L^2(\mathbb{R}\times\Omega)}=C_sy^{2s}\sum_{k=0}^\infty\int_{\mathbb{R}}|\widehat{u_k}(\rho)|^2|i\rho+\lambda_k|^{1+s}|K_{1-s}(y\sqrt{i\rho+\lambda_k})|^2\,d\rho.$$

Hence we have,

$$\begin{aligned}&\int_0^\infty y^{1-2s}\|\partial_yU\|^2_{L^2(\mathbb{R}\times\Omega)}\,dy \\ &\qquad= C_s\sum_{k=0}^\infty\int_{\mathbb{R}}|\widehat{u_k}(\rho)|^2|i\rho+\lambda_k|^{1+s}\int_0^\infty y|K_{1-s}(y\sqrt{i\rho+\lambda_k})|^2\,dy\,d\rho.\end{aligned}$$

To estimate the integral in dy, we write $r = y|\sqrt{i\rho+\lambda_k}|$ and $\theta = \arg\left(\sqrt{i\rho+\lambda_k}\right)$ to get

$$\int_0^\infty y|K_{1-s}(y\sqrt{i\rho+\lambda_k})|^2\,dy = \frac{1}{|i\rho+\lambda_k|}\int_0^\infty r|K_{1-s}(re^{i\theta})|^2\,dr \le \frac{C_s}{|i\rho+\lambda_k|},$$

because of (2.2.9). That gives us

$$\int_0^\infty y^{1-2s}\|\partial_y U\|^2_{L^2(\mathbb{R}\times\Omega)}\,dy \le C_s\sum_{k=0}^\infty\int_{\mathbb{R}}|\widehat{u_k}(\rho)|^2|i\rho+\lambda_k|^s d\rho < \infty.$$

Thus we show that $U(t,x,y) \in L^2(\mathbb{R}; H^1_{L,y}(D))$, as desired. Next let us assume that $V \in C_c^\infty(\mathbb{R}\times\Omega\times[0,\infty))$. The action of $\partial_t U$ on V is given by

$$\partial_t U(V) = -\int_{\mathbb{R}} U\partial_t V\,dt$$

for a.e. $(x,y)\in\Omega\times[0,\infty)$. For a fixed y, we already know that

$$\langle HU, V\rangle = \int_{\mathbb{R}}\int_\Omega\left(\tfrac{1-2s}{y}\partial_y + \partial_{yy}\right)UV\,dt\,dx = y^{2s-1}\int_{\mathbb{R}}\int_\Omega \partial_y(y^{1-2s}\partial_y U)V\,dt\,dx.$$

But now we notice that,

$$\begin{aligned}\langle HU, V\rangle &= -\int_{\mathbb{R}}\sum_{k=0}^\infty \widehat{u_k}(\rho)I_s(y, i\rho+\lambda_k)\overline{i\rho\widehat{V_k}(\rho,y)}\,d\rho\\ &\quad+\int_{\mathbb{R}}\sum_{k=0}^\infty \lambda_k\widehat{u_k}(\rho)I_s(y, i\rho+\lambda_k)\overline{\widehat{V_k}(\rho,y)}\,d\rho\\ &= -\int_{\mathbb{R}}\sum_{k=0}^\infty \widehat{u_k}(\rho)I_s(y, i\rho+\lambda_k)\overline{\widehat{\partial_t V_k}(\rho,y)}\,d\rho\\ &\quad+\int_{\mathbb{R}}\sum_{k=0}^\infty \lambda_k\widehat{u_k}(\rho)I_s(y, i\rho+\lambda_k)\overline{\widehat{V_k}(\rho,y)}\,d\rho\\ &= -\int_{\mathbb{R}}\int_\Omega U\partial_t V\,dx\,dt + \int_{\mathbb{R}}\int_\Omega\left(A(x)\nabla_x U\nabla_x V + c(x)UV\right)dx\,dt.\end{aligned}$$

Thus, (2.3.1) follows. Let us multiply (2.3.1) by y^{1-2s} and integrate in dy to obtain

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}}\int_\Omega y^{1-2s}\partial_t U(V)\,dx\,dt\,dy &= -\int_0^\infty\int_{\mathbb{R}}\int_\Omega y^{1-2s}\left(A(x)\nabla_x U\nabla_x V + c(x)UV\right)dx\,dt\,dy\\ &\quad+\int_0^\infty\int_{\mathbb{R}}\int_\Omega y^{1-2s}\left(\tfrac{1-2s}{y}\partial_y+\partial_{yy}\right)UV\,dx\,dt\,dy.\end{aligned}$$

Let us assume that $0 < a < b < \infty$. Since $U \in C^\infty((0,\infty); L^2(\mathbb{R}\times\Omega))$, we can apply Fubini's Theorem and integration by parts to get

$$\begin{aligned}
&\int_a^b \int_{\mathbb{R}} \int_\Omega y^{1-2s}\Big(\tfrac{1-2s}{y}\partial_y + \partial_{yy}\Big) UV \, dx\, dt\, dy \\
&\qquad = \int_{\mathbb{R}} \int_\Omega \int_a^b \partial_y (y^{1-2s}\partial_y U) V \, dy\, dt\, dx \\
&\qquad = -\int_a^b \int_{\mathbb{R}} \int_\Omega y^{1-2s}\partial_y U \partial_y V \, dy\, dx\, dt + \int_{\mathbb{R}} \int_\Omega y^{1-2s}\partial_y U V \, dx\, dt\Big|_{y=a}^{y=b}.
\end{aligned}$$

By letting $a \to 0$ and $b \to \infty$, we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \int_\Omega y^{1-2s}\Big(\tfrac{1-2s}{y}\partial_y + \partial_{yy}\Big) UV \, dx\, dt\, dy \\
&\qquad = -\int_0^\infty \int_{\mathbb{R}} \int_\Omega y^{1-2s}\partial_y U \partial_y V \, dy\, dx\, dt - \lim_{y\to 0^+} \int_{\mathbb{R}} \int_\Omega y^{1-2s}\partial_y U V \, dx\, dt.
\end{aligned}$$

To conclude,

$$\begin{aligned}
\lim_{y\to 0} \int_{\mathbb{R}} \int_\Omega \big(y^{1-2s}\partial_y U V\big) \, dx\, dt &= \lim_{y\to 0} \int_{\mathbb{R}} \int_\Omega y^{1-2s}\partial_y U \big(V(t,x,y) - V(t,x,0)\big) \, dx\, dt \\
&\quad + \lim_{y\to 0} \int_{\mathbb{R}} \int_\Omega y^{1-2s}\partial_y U V(t,x,0) \, dx\, dt \\
&= 0 - \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} \langle H^s u, V(\cdot,\cdot,0)\rangle,
\end{aligned}$$

where for the last identity we have used (2.2.10), the fact that $V \in C_c^\infty(\mathbb{R}\times\Omega\times[0,\infty))$ and Dominated Convergence Theorem. Indeed, we can prove that

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \int_\Omega y^{1-2s}\partial_y U \big(V(t,x,y) - V(t,x,0)\big) \, dx\, dt \right|^2 \le C_s \|u\|_{H^s}^2 \|V(\cdot,\cdot,y) - V(\cdot,\cdot,0)\|_{H^s}^2 \\
&\le C_s \|u\|_{H^s}^2 \|V(\cdot,\cdot,y) - V(\cdot,\cdot,0)\|_{H^1}^2 \\
&\le C_{s,\Lambda} \|u\|_{H^s}^2 \Big\{ \|V(\cdot,\cdot,y) - V(\cdot,\cdot,0)\|_{L^2(\mathbb{R}\times\Omega)}^2 + \int_{\mathbb{R}} \int_\Omega |\partial_t (V(t,x,y) - V(t,x,0))|^2 \, dx\, dt \\
&+ \int_{\mathbb{R}} \int_\Omega |\nabla_x (V(t,x,y) - V(t,x,0))|^2 \, dx\, dt + \int_{\mathbb{R}} \int_\Omega |c(x)||V(t,x,y) - V(t,x,0)|^2 \, dx\, dt \Big\} \\
&\to 0 \quad \text{as } y \to 0.
\end{aligned}$$

$\square$

Remark 2.3.2. If the elliptic operator L has continuous spectrum, then all the previous results are still valid.

Consider, for example, $L = -\Delta$ in $\Omega = \mathbb{R}^n$. We can use Fourier transform $\mathcal{F}$ in the variables t and x to define the operator $(\partial_t + L)^s$ as

$$\langle (\partial_t - \Delta)^s u, v \rangle_{L^2(\mathbb{R}^{n+1})} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} (i\rho + |\xi|^2)^s \mathcal{F}u(\rho, \xi) \overline{\mathcal{F}v(\rho, \xi)} \, d\xi \, d\rho.$$

The analogous to the expression

$$u(t, x) = \sum_{k=0}^{\infty} u_k(t) \phi_k(x)$$

in this case is just

$$u(t, x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{u}(t, \xi) e^{i\xi \cdot x} \, d\xi$$

where the Fourier transform is taken in the x variable by leaving t fixed. The eigenvalues and eigenfunctions $(\lambda_k, \phi_k)_{k=0}^{\infty}$ are replaced by $(|\xi|^2, e^{ix\cdot\xi})_{\xi \in \mathbb{R}^n}$. Consider another one, the Bessel operator $L = -\frac{d^2}{dx^2} + \frac{\lambda(\lambda-1)}{x^2}$, for $\lambda > 0$, in $\Omega = (0, \infty)$. In this case we can use Hankel transform in x and Fourier transform in t. Let $\phi_y(x) = (yx)^{1/2} J_{\lambda - 1/2}(yx)$, $x, y > 0$, where J_ν denotes the Bessel function of the first kind with order ν. Then $L\phi_y(x) = y^2 \phi_y(x)$ and the eigenvalues and eigenfunctions $(\lambda_k, \phi_k)_{k=0}^{\infty}$ are replaced by $(y^2, \phi_y(x))_{y>0}$. The Hankel transform in the variable x is defined as

$$\mathcal{H}u(t, y) = \int_0^{\infty} u(t, x) \phi_y(x) \, dx$$

and, since $\mathcal{H}^{-1} = \mathcal{H}$, we can write

$$u(t, x) = \int_0^{\infty} \mathcal{H}u(t, y) \phi_y(x) \, dy.$$

With this, we can let

$$\langle (\partial_t + L)^s u, v \rangle = \int_{\mathbb{R}} \int_0^{\infty} (i\rho + y^2)^s \mathcal{H}\widehat{u}(\rho, y) \overline{\mathcal{H}\widehat{v}(\rho, y)} \, dy \, d\rho.$$

We conclude this section with an important lemma which is useful for the next section and also when we prove Harnack inequalities in the next chapter. In this lemma, we will prove that the reflection extension of U, with respect to the variable y, also satisfies equations like in Theorem 2.3.1.

Lemma 2.3.3 (Reflection extension)**.** *Let L and U be as in Theorem 2.3.1. Let $\Omega_0 \subset \Omega$ be a bounded domain and $(T_0, T_1) \subset \mathbb{R}$. Suppose that*

$$\lim_{y \to 0^+} \langle y^{1-2s} \partial_y U, V \rangle_{L^2(\mathbb{R} \times \Omega)} = 0$$

for all $V \in C_c^\infty((T_0, T_1) \times \Omega_0 \times [0, \infty))$. Fix $Y_0 > 0$. Then, the even extension $\widetilde{U}$ of U in the variable y, defined by

$$\widetilde{U}(t,x,y) = \begin{cases} U(t,x,y) & \text{for } 0 \le y < Y_0 \\ U(t,x,-y) & \text{for } -Y_0 < y < 0 \end{cases} \tag{2.3.3}$$

is a weak solution to the degenerate parabolic equation

$$\partial_t \widetilde{U} = |y|^{-(1-2s)} \operatorname{div}_{x,y}(|y|^{1-2s} B(x) \nabla_{x,y} \widetilde{U}) - c(x) \widetilde{U} \tag{2.3.4}$$

in $(T_0, T_1) \times \Omega_0 \times (-Y_0, Y_0)$.

Proof. Let us assume that $V \in C_c^\infty((T_1, T_2) \times \Omega_0 \times (-Y_0, Y_0))$. We shall prove that

$$\begin{aligned} \int_{T_0}^{T_1} \int_{-Y_0}^{Y_0} \int_{\Omega_0} & |y|^{1-2s} \widetilde{U} \partial_t V \, dx \, dy \, dt \\ &= \int_{T_0}^{T_1} \int_{-Y_0}^{Y_0} \int_{\Omega_0} |y|^{1-2s} \big(B(x) \nabla_{x,y} \widetilde{U} \nabla_{x,y} V + c(x) \widetilde{U} V \big) \, dx \, dy \, dt. \end{aligned}$$

Suppose $\delta > 0$. From (2.3.1), for any $y > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}} \int_{\Omega} U \partial_t V \, dx \, dt = & \int_{\mathbb{R}} \int_{\Omega} \big(A(x) \nabla_x U \nabla_x V + c(x) U V \big) \, dx \, dt \\ & - \int_{\mathbb{R}} \int_{\Omega} |y|^{2s-1} \partial_y (|y|^{1-2s} \partial_y U) V \, dx \, dt. \end{aligned}$$

By multiplying this equation by $|y|^{1-2s}$, integrating in $y \in (\delta, Y_0)$, and using integration by parts we get

$$\begin{aligned} \int_{T_0}^{T_1} \int_{\delta}^{Y_0} \int_{\Omega_0} & |y|^{1-2s} \widetilde{U} \partial_t V \, dx \, dy \, dt \\ &= \int_{T_0}^{T_1} \int_{\delta}^{Y_0} \int_{\Omega_0} |y|^{1-2s} \big(B(x) \nabla_{x,y} \widetilde{U} \nabla_{x,y} V + c(x) \widetilde{U} V \big) \, dx \, dy \, dt \\ &\qquad + \int_{T_0}^{T_1} \int_{\Omega_0} \delta^{1-2s} \partial_y U(t,x,\delta) V(t,x,\delta) \, dx \, dt. \end{aligned}$$

From here we readily get

$$\int_{T_0}^{T_1}\int_{\delta<|y|<Y_0}\int_{\Omega_0}|y|^{1-2s}\widetilde{U}\partial_t V\,dx\,dy\,dt$$
$$=\int_{T_0}^{T_1}\int_{\delta<|y|<Y_0}\int_{\Omega_0}|y|^{1-2s}\big(B(x)\nabla_{x,y}\widetilde{U}\nabla_{x,y}V+c(x)\widetilde{U}V\big)\,dx\,dy\,dt$$
$$+\int_{T_0}^{T_1}\int_{\Omega_0}\delta^{1-2s}\partial_y U(t,x,y)|_{y=\delta}V(t,x,-\delta)\,dx\,dt$$
$$+\int_{T_0}^{T_1}\int_{\Omega_0}\delta^{1-2s}\partial_y U(t,x,\delta)V(t,x,\delta)\,dx\,dt.$$

The conclusion follows by taking $\delta\to 0$ in this last identity. □

We mention that the lemma above is also true for all the cases listed in Remark 2.3.2.

2.4 Fundamental Solution

2.4.1 Fundamental Solution Using Spectrum and Heat Kernel of L

Given $f\in L^2(\mathbb{R}\times\Omega)$, the solution $u\in \mathrm{Dom}(H^s)$ to $H^s u=f$ is given by

$$u(t,x)=H^{-s}f(t,x)=\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}}\sum_{k=1}^{\infty}(i\rho+\lambda_k)^{-s}\widehat{f}_k(\rho)\varphi_k(x)e^{i\rho t}\,d\rho.$$

Using the Gamma function identity

$$(i\rho+\lambda_k)^{-s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\tau(i\rho+\lambda_k)}\,\frac{d\tau}{\tau^{1-s}}$$

and the heat kernel $W_\tau(x,z)$ for L, we readily find that

$$\begin{aligned}u(t,x)=H^{-s}f(t,x)&=\frac{1}{\Gamma(s)}\int_0^\infty e^{-\tau L}f(t-\tau,x)\,\frac{d\tau}{\tau^{1-s}}\\&=\int_{-\infty}^{\infty}\int_\Omega K_{-s}(\tau,x,z)f(t-\tau,z)\,dz\,d\tau\end{aligned}$$

Definition 2.4.1 (Fundamental solution)**.** The function

$$K_{-s}(\tau,x,z)=\chi_{\tau>0}\frac{W_\tau(x,z)}{\Gamma(s)\tau^{1-s}}=\frac{\chi_{\tau>0}}{\Gamma(s)\tau^{1-s}}\sum_{k=0}^{\infty}e^{-\tau\lambda_k}\phi_k(x)\phi_k(z)$$

is the fundamental solution for the nonlocal equation $H^s u=f$.

We can estimate this kernel by applying known estimates for the heat kernel of L.

(a) If the coefficients $A(x)$ are bounded and measurable then, by [23], we find that

$$K_{-s}(\tau,x,z) \le \frac{C}{\tau^{n/2+1-s}} e^{-|x-z|^2/(c\tau)} \qquad x,z\in\Omega,\ \tau>0$$

for some constants $C, c > 0$.

(b) If the coefficients $A(x)$ are bounded and measurable in $\Omega = \mathbb{R}^n$ then, by Aronson's estimates [3],

$$\frac{C_1}{\tau^{n/2+1-s}} e^{-|x-z|^2/(c_1\tau)} \le K_{-s}(\tau,x,z) \le \frac{C_2}{\tau^{n/2+1-s}} e^{-|x-z|^2/(c_2\tau)} \qquad x,z\in\mathbb{R}^n,\ \tau>0$$

for some constants $C_1, c_1, C_2, c_2 > 0$.

(c) If the coefficients $A(x)$ are Hölder continuous with exponent $\alpha \in (0,1)$ and L is endowed with homogeneous Dirichlet boundary conditions then, from [36, Theorem 2.2], there exist positive constants c, c_1, c_2 and $\eta \le 1 \le \nu$ depending only on n, α, Ω and ellipticity, with c depending also on s, such that

$$\begin{aligned} c^{-1}\tau^{s-1} \min\left(1, \frac{\phi_0(x)\phi_0(z)}{\max(1,\tau^{\eta})}\right) e^{-\lambda_0\tau} \frac{e^{-c_1|x-z|^2/(\tau)}}{\max(1,\tau^{n/2})} &\le K_{-s}(\tau,x,z) \\ &\le c\tau^{s-1} \min\left(1, \frac{\phi_0(x)\phi_0(z)}{\max(1,\tau^{\nu})}\right) e^{-\lambda_0\tau} \frac{e^{-c_2|x-z|^2/(\tau)}}{\max(1,\tau^{n/2})} \end{aligned}$$

for all $x, z \in \Omega$, $t > 0$.

(d) Under the hypotheses of (c), if in addition we assume that Ω is a $C^{1,\gamma}$ domain for some $0 < \gamma < 1$, then the estimate above is true for $\eta = \nu = 1$ and the constant c depending also on γ. In particular, the estimate holds when $(\partial_t + L)^s = (\partial_t - \Delta_D)^s$, the fractional power of the heat operator with Dirichlet Laplacian in a $C^{1,\gamma}$ domain.

(e) For the case of Neumann boundary conditions, if Ω is an inner uniform domain then two-sided Gaussian estimates for the Neumann heat kernel hold and we obtain

$$\frac{C_1}{\tau^{n/2+1-s}} e^{-d(x,z)^2/(c_1\tau)} \le K_{-s}(\tau,x,z) \le \frac{C_2}{\tau^{n/2+1-s}} e^{-d(x,z)^2/(c_2\tau)} \qquad x,z\in\Omega,\ \tau>0$$

where $d(x,z)$ denotes the geodesic distance between x and z in Ω. In particular, if Ω is bounded and convex, or if it is the region above the graph of a globally Lipschitz function, then the geodesic distance $d(x,z)$ can be replaced by the Euclidean distance $|x-z|$. For details about inner uniform domains, see [37].

2.4.2 Fundamental Solution Using Extension Problem

Now we will estimate the fundamental solution of $(\partial_t + L)^s u = f$ when $L = -\operatorname{div}(A(x)\nabla)$ using the extension theorem. For the convenience of the reader, we write down the extension theorem, Theorem 2.3.1, for the case when $c(x) = 0$. Let $D = \{(x,y) : x \in \Omega,\ y > 0\} \subset \mathbb{R}^{n+1}$, as in Section 2.3 . We recall that the weight $\omega(x,y) = |y|^a$ belongs to the Muckenhoupt class $A_2(\mathbb{R}^{n+1})$. Again we define $H^1_{L,y}(D)$ as the set of functions $w = w(x,y) \in L^2(D, y^a dxdy)$ such that

$$
\begin{aligned}
[w]^2_{H^1_{L,y}(D)} &:= \int_0^\infty \int_\Omega y^a A(x)\nabla_x w \nabla_x w \, dx\, dy + \int_0^\infty \int_\Omega y^a |\partial_y w|^2 \, dx\, dy \\
&= \int_0^\infty y^a \sum_{k=0}^\infty \lambda_k |w_k(y)|^2 \, dy + \int_0^\infty \int_\Omega y^a |\partial_y w|^2 \, dx\, dy < \infty,
\end{aligned}
$$

where $w_k(y) = \displaystyle\int_\Omega w(x,y)\phi_k(x)\, dx$, under the norm

$$
\|w\|^2_{H^1_{L,y}(D)} = \|w\|^2_{L^2(D, y^a dxdy)} + [w]^2_{H^1_{L,y}(D)}.
$$

Recall that $\{e^{-\tau H}\}_{\tau \geq 0}$ denotes the semigroup generated by $H = \partial_t - \operatorname{div}(A(x)\nabla_x)$.

Theorem 2.4.2 (Extension Theorem)**.** *Let us assume that $u \in \operatorname{Dom}(H^s)$. For $(t,x) \in \mathbb{R} \times \Omega$ and $y > 0$, we define*

$$
\begin{aligned}
U(t,x,y) &= \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{-\tau H} u(t,x) \, \frac{d\tau}{\tau^{1+s}} \\
&= \frac{1}{\Gamma(s)} \int_0^\infty e^{-r} e^{-\frac{y^2}{4r} H} u(t,x) \, \frac{dr}{r^{1-s}} \\
&= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4r)} e^{-rH} (H^s u)(t,x) \, \frac{dr}{r^{1-s}}.
\end{aligned}
\tag{2.4.1}
$$

Then U belongs to $L^2(\mathbb{R}; H^1_{L,y}(D)) \cap C^\infty((0,\infty); L^2(\mathbb{R}\times\Omega)) \cap C([0,\infty); L^2(\mathbb{R}\times\Omega))$ and is a weak solution to the parabolic extension problem

$$\begin{cases} \partial_t U = y^{-a}\operatorname{div}(y^a B(x)\nabla U) & \text{for } (t,x,y) \in \mathbb{R}\times\Omega\times(0,\infty) \\ -y^a \partial_y U\Big|_{y=0^+} = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} H^s u & \text{for } (t,x) \in \mathbb{R}\times\Omega \\ U(t,x,0) = u(t,x) & \text{for } (t,x) \in \mathbb{R}\times\Omega \end{cases}$$

with boundary condition $U = 0$ or $\partial_A U = 0$ on $\mathbb{R}\times\partial\Omega\times(0,\infty)$, depending whether L is endowed with homogeneous Dirichlet or Neumann boundary conditions, respectively. Namely, for any $V(t,x,y) \in C_c^\infty(\mathbb{R}\times\Omega\times[0,\infty))$, in case of Dirichlet; or for any $V(t,x,y) \in C^\infty(\mathbb{R}\times\Omega\times[0,\infty))$ with compact support in t and y, in case of Neumann,

$$\begin{aligned} \int_{\mathbb{R}}\int_\Omega U\partial_t V\,dx\,dt = &\int_{\mathbb{R}}\int_\Omega A(x)\nabla_x U\nabla_x V\,dx\,dt \\ &- \int_{\mathbb{R}}\int_\Omega \Big(\tfrac{a}{y}\partial_y + \partial_{yy}\Big)UV\,dx\,dt \end{aligned}$$

$\lim_{y\to0^+} U(t,x,y) = u(t,x)$ *in $L^2(\mathbb{R}\times\Omega)$ and*

$$\begin{aligned} \int_0^\infty\int_{\mathbb{R}}\int_\Omega y^a U\partial_t V\,dx\,dt\,dy = &\int_0^\infty\int_{\mathbb{R}}\int_\Omega y^a B(x)\nabla U\nabla V\,dx\,dt\,dy \\ &- \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}\langle H^s u, V(t,x,0)\rangle. \end{aligned}$$

By density, these identities hold for test functions V in $L^2(\mathbb{R}; H^1_{L,a}(D))$. In addition, we have the estimate

$$\|U\|_{L^2(\mathbb{R};H^1_{L,a}(D))} \le C\|u\|_{\mathrm{Dom}(H^s)} \tag{2.4.2}$$

where $C > 0$ depends only on s.

Now let $K_{-s}(\tau,x,z)$ be the fundamental solution of H^s with pole at $\tau = 0$ and $z = x$. For fixed x, let $U^x = U^x(\tau,z,y)$ be the solution to the following extension problem

$$\begin{cases} y^a\partial_\tau U^x - \operatorname{div}(y^a B(z)\nabla U^x) = 0 & \text{in } \mathbb{R}\times\Omega\times(0,\infty) \\ -\lim_{y\to0} y^a (U^x)_y(\tau,z,y) = c_s\delta_{(0,x,0)} & \text{on } \mathbb{R}\times\Omega \end{cases}$$

with the appropriate boundary condition on $\mathbb{R}\times\partial\Omega\times[0,\infty)$. Here $\delta_{(0,x,0)}$ denotes the Dirac delta at $\tau=0$, $x\in\Omega$ and $y=0$. Then we see

$$K_{-s}(\tau,x,z)=U^x(\tau,z,0).$$

Let $\tilde{U}^x$ to be the even reflection of U^x with respect to the variable y, that is, $\tilde{U}^x(\tau,z,y)=U^x(\tau,z,|y|)$. Then, exactly as in Lemma 2.3.3, we find that $\tilde{U}^x$ solves

$$|y|^a\partial_\tau\tilde{U}^x-\operatorname{div}(|y|^aB(z)\nabla\tilde{U}^x)=c_s\delta_{(0,x,0)}\qquad\text{in }\mathbb{R}\times\Omega\times(-\infty,\infty)$$

with the corresponding boundary conditions. To see that, as in Lemma 2.3.3, for $V\in C_c^\infty((-\infty,\infty)\times\Omega\times(-\infty,\infty))$ and for some $\epsilon>0$, we write

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\epsilon<|y|<\infty}\int_\Omega |y|^{1-2s}\tilde{U}^x\partial_tV\,dz\,dy\,d\tau\\ =\int_{-\infty}^{\infty}\int_{\epsilon<|y|<\infty}\int_\Omega |y|^{1-2s}B(z)\nabla_{z,y}\tilde{U}^x\nabla_{z,y}V\,dz\,dy\,d\tau\\ +\int_{-\infty}^{\infty}\int_\Omega \epsilon^{1-2s}\partial_yU^x(\tau,z,\epsilon)[V(\tau,z,\epsilon)+V(\tau,z,-\epsilon)]\,dz\,d\tau.\end{aligned}$$

But $-\lim_{y\to0}y^a(U^x)_y(\tau,z,y)=c_s\delta_{(0,x,0)}$ on $\mathbb{R}\times\Omega$, which means

$$\lim_{\epsilon\to0}\int_{-\infty}^{\infty}\int_\Omega \epsilon^{1-2s}\partial_yU^x(\tau,z,\epsilon)V(\tau,z,\epsilon)\,dz\,d\tau=c_sV(0,x,0),$$

and similarly

$$\lim_{\epsilon\to0}\int_{-\infty}^{\infty}\int_\Omega \epsilon^{1-2s}\partial_yU^x(\tau,z,\epsilon)V(\tau,z,-\epsilon)\,dz\,d\tau=c_sV(0,x,0).$$

Then we have, as $\epsilon\to0$,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_\Omega |y|^{1-2s}\tilde{U}^x\partial_tV\,dz\,dy\,d\tau\\ =\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_\Omega |y|^{1-2s}B(z)\nabla_{z,y}\tilde{U}^x\nabla_{z,y}V\,dz\,dy\,d\tau+c_sV(0,x,0),\end{aligned}$$

which proves our claim.

Clearly, for $\mathcal{U}^x(\tau,z,y)=\chi_{\tau\geq0}\tilde{U}^x(\tau,z,y)$, we have

$$\begin{cases}|y|^a\partial_\tau\mathcal{U}^x-\operatorname{div}(|y|^aB(z)\nabla\mathcal{U}^x)=0 & \text{in }(0,\infty)\times\Omega\times(-\infty,\infty)\\ \lim_{\tau\to0}\mathcal{U}^x(\tau,z,y)=c_s\delta_{(0,x,0)}.\end{cases}$$

Then $\mathcal{U}^x$ is the heat kernel associated with the elliptic operator $\mathrm{div}(|y|^a B(x)\nabla)$ with pole at $(\tau, z, y) = (0, x, 0)$. Thus, from known heat kernel estimates for degenerate parabolic operators, we can derive bounds for the fundamental solution $K_{-s}(\tau, x, z)$.

Suppose that $\Omega = \mathbb{R}^n$, denote $X = (x, x_{n+1}), Z = (z, y) \in \mathbb{R}^{n+1}$ and let $W_\tau(X, Z)$ be the heat kernel for $\mathrm{div}(|y|^a B(x)\nabla)$ with pole at $\tau = 0$ and $Z = X$. From [22], we have the Gaussian estimate

$$|W_\tau(X, Z)| \leq \frac{C}{\sqrt{w_\tau(X)}\sqrt{w_\tau(Z)}} e^{-c|X-Z|^2/\tau}$$

where $w(Z) = |y|^a$ is an A_2-Muckenhoupt weight, $w_\tau(Z)$ is the w-volume of the ball centered at Z with radius $\sqrt{\tau}$ in the usual metric in $\mathbb{R}^{n+1}$ and $C, c > 0$ depend on s, n and ellipticity. It is easy to check that $w_\tau((z, 0)) \sim \tau^{n/2+1+s}$. Therefore, the fundamental solution for H^s, in $\Omega = \mathbb{R}^n$, verifies

$$K_{-s}(\tau, x, z) = W_\tau((x, 0), (z, 0)) \leq \frac{C}{\tau^{n/2+1+s}} e^{-c|x-z|^2/\tau} \qquad \text{when } \tau > 0$$

for $C, c > 0$ depending only on s, n and ellipticity. Compare this estimate with those in Section 2.4.1.

CHAPTER 3. HARNACK INEQUALITIES

3.1 Interior and Boundary Harnack Inequalities

We begin this chapter with the following interior Harnack inequality.

Theorem 3.1.1 (Parabolic interior Harnack inequality)**.** *Let L be as in (2.0.1). Let B_{2r} be a ball of radius $2r$, $r > 0$, such that $B_{2r} \subset\subset \Omega$. There exists a constant $c > 0$ depending only on n, s, Λ and r such that if $u = u(t,x) \in \mathrm{Dom}(H^s)$ is a solution to*

$$\begin{cases} H^s u = 0 & \text{for } (t,x) \in R := (0,1) \times B_{2r} \\ u \geq 0 & \text{for } (t,x) \in (-\infty,1) \times \Omega, \end{cases}$$

then

$$\sup_{R^-} u \leq c \inf_{R^+} u$$

where $R^- := (1/4, 1/2) \times B_r$ and $R^+ := (3/4, 1) \times B_r$. Moreover, solutions $u \in \mathrm{Dom}(H^s)$ to $H^s u = 0$ in R are locally bounded and locally parabolically α-Hölder continuous in R, for some exponent $0 < \alpha < 1$ depending on n, Λ and s. More precisely, for any compact set $K \subset R$ there exists $C = C(c, K, R) > 0$ such that

$$\|u\|_{C^{\alpha/2,\alpha}_{t,x}(K)} \leq C \|u\|_{L^2(\mathbb{R} \times \Omega)}.$$

Proof of Theorem 3.1.1. Consider the extension U of u given by Theorems 2.2.1 and 2.3.1. If $u \geq 0$ in $(-\infty, 1) \times \Omega$ then, since the heat kernel for L is nonnegative, the first formula in (2.2.1) gives that $U \geq 0$ in $(0,1) \times B_{2r} \times [0,2)$. Lemma 2.3.3 with $Y_0 = 2$ implies that $\widetilde{U}$, as defined by (2.3.3), is a nonnegative weak solution to (2.3.4) in $(t,x,y) \in (0,1) \times B_{2r} \times (-2,2)$. The parabolic Harnack inequality due to Ishige [27] gives the existence of a constant $C_H > 0$ such that

$$\begin{aligned}\sup_{R^-} u(t,x) = \sup_{R^-} \widetilde{U}(t,x,0) &\le \sup_{R^-\times(-1,1)} \widetilde{U}(t,x,y)\\ &\le C_H \inf_{R^+\times(-1,1)} \widetilde{U}(t,x,y)\\ &\le C_H \inf_{R^+} \widetilde{U}(t,x,0) = C_H \inf_{R^+} u(t,x).\end{aligned}$$

Now we prove the local boundedness and Hölder estimates on u. By using the results in [27] we get that $\widetilde{U}$ is locally bounded and locally parabolically Holder continuous of order $0 < \alpha < 1$ in R. Let K be a compact subset of R. We have

$$\|\widetilde{U}\|_{L^\infty(K\times(-1,1))} \le C\|\widetilde{U}\|_{L^2(R\times(-2,2))} = 2C\|U\|_{L^2(R\times(0,2))}.$$

Since $\|U\|_{L^2(R\times(0,2))} \le C\|u\|_{L^2(\mathbb{R}\times\Omega)}$, we obtain

$$\|u\|_{L^\infty(K)} \le \|\widetilde{U}\|_{L^\infty(K\times(-1,1))} \le C\|u\|_{L^2(\mathbb{R}\times\Omega)}.$$

Next, from the local Hölder continuity of $\widetilde{U}$,

$$[u]_{C^{\alpha/2,\alpha}_{t,x}(K)} = [\widetilde{U}]_{C^{\alpha/2,\alpha}_{t,x}(K\cap\{y=0\})} \le C\|\widetilde{U}\|_{L^\infty(K\times(-1,1))} \le C\|u\|_{L^2(\mathbb{R}\times\Omega)}.$$

□

Remark 3.1.2. If in Theorem 3.1.1 we substitute B_{2r} by an open set and B_r by a compact set contained in the open set, the result remains valid and the constant c also depends on both sets.

To present the parabolic boundary Harnack inequality, let $\Omega_0 \subset \Omega$ and $\tilde{x} \in \partial\Omega_0$ such that $B_{2r}(\tilde{x}) \subset \Omega$, for some $r > 0$ fixed. Suppose that, up to a rotation and translation, $B_{2r}(\tilde{x}) \cap \partial\Omega_0$ can be represented as the graph of a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ in the $e_n = (0,\ldots,0,1)$-direction, such that g has Lipschitz constant $M > 0$. Thus, we denote

$$\Omega_0 \cap B_{2r}(\tilde{x}) = \{(x', x_n) : \ x_n > g(x')\} \cap B_{2r}(\tilde{x})$$

$$\partial\Omega_0 \cap B_{2r}(\tilde{x}) = \{(x', x_n) : \ x_n = g(x')\} \cap B_{2r}(\tilde{x}).$$

Fix a point $(t_0, x_0) \in (-2,2) \times \Omega_0$ such that $t_0 > 1$.

Theorem 3.1.3 (Parabolic boundary Harnack inequality)**.** *Let L be as in (2.0.1). Assume the geometric conditions on Ω_0 and Ω described above. Then there exists a constant $C > 0$ depending on n, Λ, r, M, s, $t_0 - 1$ and g, such that if $u(t,x) \in \mathrm{Dom}(H^s)$ is a solution to*

$$\begin{cases} H^s u = 0 & \textit{for } (t,x) \in (-2,2) \times (\Omega_0 \cap B_{2r}(\tilde{x})) \\ u \geq 0 & \textit{for } (t,x) \in (-\infty, 2) \times \Omega \end{cases}$$

such that u vanishes continuously on $(-2,2) \times ((\Omega \setminus \Omega_0) \cap B_{2r}(\tilde{x}))$ then

$$\sup_{(-1,1)\times(\Omega_0 \cap B_r(\tilde{x}))} u(t,x) \leq C u(t_0, x_0).$$

Proof. For simplicity, and without loss of generality, we will assume that $\tilde{x} = 0$. Let $\widetilde{U}$ be the reflection in y of the extension U of u. By Lemma 2.3.3, $\widetilde{U}$ is a nonnegative weak solution to (2.3.4) in $(t,x,y) \in (-2,2) \times (B_{2r}(0) \cap \Omega_0) \times (-2r, 2r)$ that vanishes continuously in $(t,x,y) \in (-2,2) \times ((\Omega \setminus \Omega_0) \cap B_{2r}(0)) \times \{0\}$.

As a first step we flatten the boundary of Ω_0 inside $B_{2r}(0)$. We use a bi-Lipschitz transformation Ψ such that $\Psi(0) = 0$ and $\Psi(\Omega_0 \cap B_{2r}(0)) = \Omega_1$, where Ω_1 is a new domain with flat boundary at $x_n = 0$, which can be extended as constant in t and y. Without loss of generality we can assume that the flat part of $B_{2r}(0) \cap \mathbb{R}^n_+$ is the flat part of the new domain Ω_1. Then the transformed function $\widetilde{U}_1 := \widetilde{U} \circ \Psi^{-1}$ satisfies the same type of degenerate parabolic equation with bounded measurable coefficients in the domain $(-2,2) \times (\mathbb{R}^n_+ \cap B_{2r}(0)) \times (-2r, 2r)$ and vanishes continuously on $(-2,2) \times ((\mathbb{R}^n \setminus \mathbb{R}^n_+) \cap B_{2r}(0)) \times \{0\}$.

As a second step, we define a transformation which maps $\mathbb{R}^{n+1} \setminus \{x_n \leq 0, y = 0\}$ into $\mathbb{R}^{n+1} \cap \{x_n > 0\}$ and is extended to be constant in t. This construction is standard, see [42]. After this transformation is performed, we obtain a function $\widetilde{U}_2$ that solves again a degenerate parabolic equation with bounded measurable coefficients in the domain $(-2,2) \times (\mathbb{R}^n_+ \cap B_{2r}(0)) \times (-2r, 2r)$ and that vanishes continuously for $(t,x,y) \in (-2,2) \times \{(x', 0, y) : (x')^2 + y^2 < (2r)^2\}$.

Now we can apply the boundary Harnack inequality of Ishige [27] to $\widetilde{U}_2$ to get

$$\sup_{(-1,1)\times(\Omega \cap B_r(0))} u(t,x) = \sup_{(-1,1)\times(\mathbb{R}^n_+ \cap B_r(0))} \widetilde{U}_2(t,x,0) \leq C \widetilde{U}_2(t_0, \widetilde{x}_0, 0) = u(t_0, x_0),$$

where $\widetilde{x}_0$ is the point obtained from x_0 via the two transformations. $\square$

Remark 3.1.4. If in Theorem 3.1.3 we substitute $B_{2r}(\tilde{x})$ by an open set and $B_r(\tilde{x})$ by another open subset of the first one, the result remains still valid and the constant C also depends on both open sets.

3.2 Transference Method

In this section, we develop a *transference method* for the fractional powers of parabolic operators that allows us to transfer the Harnack inequalities and Hölder estimates of the solutions to $(\partial_t + L)^s u = f$ in Theorems 3.1.1 and 3.1.3 to the solutions of other Master equations of the form $(\partial_t + \bar{L})^s \bar{u} = \bar{f}$. Here, formally, $\bar{L} = (U \circ W)^{-1} \circ L \circ (U \circ W)$, where U is a multiplication operator by a smooth positive function and W is a smooth change of variables operator. This method is particularly useful when $\bar{L}$ is one of the following elliptic operators having gradient term.

(A) The Ornstein–Uhlenbeck operator $\bar{L} = -\Delta + 2x \cdot \nabla$ in $\Omega = \mathbb{R}^n$ with the Gaussian measure.

(B) The Laguerre operators

- $\bar{L} = \sum_{i=1}^n \left(-x_i \frac{\partial^2}{\partial x_i^2} - (\alpha_i + 1)\frac{\partial}{\partial x_i} + \frac{x_i}{4}\right)$,
- $\bar{L} = \frac{1}{4}(-\Delta + |x|^2 - \sum_{i=1}^n \frac{2\alpha_i+1}{x_i}\frac{\partial}{\partial_{x_i}})$,
- $\bar{L} = \sum_{i=1}^n \left(-x_i \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial x_i} + \frac{x_i}{4} + \frac{\alpha_i^2}{4x_i}\right)$,
- $\bar{L} = \sum_{i=1}^n \left(-x_i \frac{\partial^2}{\partial x_i^2} - (\alpha_i + 1 - x_i)\frac{\partial}{\partial x_i}\right)$,

for $\alpha_i > -1$ in $\Omega = (0, \infty)^n$, with their corresponding Laguerre measures.

(C) The ultraspherical operator $\bar{L} = -\frac{d^2}{dx^2} - 2\lambda \cot x \frac{d}{dx} + \lambda^2$, for $\lambda > 0$ in $\Omega = (0, \pi)$ with the measure $d\eta(x) = \sin^{2\lambda} x\, dx$.

(D) The Bessel operator $\bar{L} = -\frac{d^2}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx}$, for $\lambda > 0$ in $\Omega = (0, \infty)$ with the measure $d\eta(x) = x^{2\lambda} dx$.

These are related to classical orthogonal expansions and can be obtained by transference from the operators L listed in (2)–(6) in Chapter 2. Transference techniques in the elliptic case have been widely used in harmonic analysis, see [2], and also for fractional elliptic PDEs, see [43]. We also point out that pointwise formulas for the nonlocal operators $(\partial_t+\bar{L})^s\bar{u}(t,x)$ when $\bar{L}$ is as in (A)–(D) can be deduced exactly as in Theorem 2.1.2 by using the corresponding heat kernels. Details are left to the interested reader.

Theorem 3.2.1 (Transference method)**.** *If Theorems 3.1.1 and 3.1.3 hold true for solutions* $u \in \mathrm{Dom}(H^s)$ *to* $(\partial_t + L)^s u = 0$*, where* L *is as in* (2.0.1)*, then they also hold true for solutions* $\bar{u} \in \mathrm{Dom}(\bar{H}^s)$ *to* $(\partial_t + \bar{L})^s\bar{u} = 0$*.*

In this section we assume that

$$Lu = -\operatorname{div}(a(x)\nabla u) + c(x)u \quad \text{in } \Omega$$

is an operator as in Section 2.3.

3.2.1 Change of Variables

Let $\tilde{\Omega} \subset \mathbb{R}^n$ be a domain and $h : \Omega \to \widetilde{\Omega}$ be a smooth change of variables from $x \in \Omega$ into $\widetilde{x} = h(x) \in \widetilde{\Omega}$, that is, h is one-to-one, onto and differentiable with inverse $h^{-1} : \widetilde{\Omega} \to \Omega$ differentiable as well. We denote by $J_h(x) = |\det \nabla h(x)|$, for $x \in \Omega$, and $J_{h^{-1}}(\widetilde{x}) = |\det \nabla h^{-1}(\widetilde{x})|$, for $\widetilde{x} \in \widetilde{\Omega}$. Let us define the change of variables application

$$W : L^2(\widetilde{\Omega}, J_{h^{-1}} d\widetilde{x}) \to L^2(\Omega, dx)$$

as

$$W(\widetilde{f})(x) = \widetilde{f}(h(x)) \quad \text{for } x \in \Omega.$$

Then W is one-to-one, onto and, for any $f \in L^2(\Omega, dx)$,

$$W^{-1}(f)(\widetilde{x}) = f(h^{-1}(\widetilde{x})),\ \widetilde{x} \in \widetilde{\Omega}.$$

It is readily seen that

$$\|W\widetilde{f}\|_{L^2(\Omega,dx)} = \|\widetilde{f}\|_{L^2(\widetilde{\Omega},J_{h^{-1}}d\widetilde{x})}.$$

Let $\{\phi_k\}_{k=0}^{\infty}$ be the orthonormal basis of $L^2(\Omega, dx)$ consisting of eigenfunctions of L. We claim that $\{\widetilde{\phi}_k := W^{-1}\phi_k\}_{k=0}^{\infty}$ is an orthonormal basis of $L^2(\widetilde{\Omega}, J_{h^{-1}} d\widetilde{x})$. Indeed, by changing variables,

$$\int_{\widetilde{\Omega}} \widetilde{\phi}_k(\widetilde{x})\widetilde{\phi}_\ell(\widetilde{x}) J_{h^{-1}}(\widetilde{x})\, d\widetilde{x} = \int_\Omega \phi_k(x)\phi_\ell(x)\, dx = \delta_{k\ell}.$$

Also, if $\widetilde{f} \in L^2(\widetilde{\Omega}, J_{h^{-1}} d\widetilde{x})$ is orthogonal to each $\widetilde{\phi}_k$ then

$$0 = \int_{\widetilde{\Omega}} \widetilde{f}(\widetilde{x})\widetilde{\phi}_k(\widetilde{x}) J_{h^{-1}}(\widetilde{x})\, d\widetilde{x} = \int_\Omega W(\widetilde{f})(x)\phi_k(x)\, dx$$

for all $k \geq 0$, which gives $\widetilde{f} = 0$, and the orthonormal set $\{\widetilde{\phi}_k\}_{k=0}^{\infty}$ is complete in $L^2(\widetilde{\Omega}, J_{h^{-1}} d\widetilde{x})$.

If $u \in \mathrm{Dom}(L)$ and we define $\widetilde{u} = W^{-1}u = u \circ h^{-1}$ then we can write $u = W\widetilde{u} = \widetilde{u} \circ h$ and the change rule gives

$$u_{x_i}(x) = \sum_{k=1}^{n} \widetilde{u}_{\widetilde{x}_k}(h(x))(\nabla h(x))_{ki}$$

where $(\nabla h(x))_{ki} = \left(\frac{\partial h_k(x)}{\partial x_i}\right)_{ki}$ denotes the ki-th entry of the matrix $\nabla h(x)$. From the definition of the action of L on u we have, for any $v \in \mathrm{Dom}(L)$,

$$\begin{aligned}
\langle Lu, v\rangle &= \int_\Omega \Big(\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i}(x) v_{x_j}(x) + c(x)u(x)v(x) \Big)\, dx \\
&= \int_\Omega \Big[\sum_{k,\ell=1}^{n} \Big(\sum_{i,j=1}^{n} a^{ij}(x)(\nabla h(x))_{ki}(\nabla h(x))_{\ell j} \Big) \widetilde{u}_{\widetilde{x}_k}(h(x))\widetilde{v}_{\widetilde{x}_\ell}(h(x)) + c(x)u(x)v(x) \Big]\, dx \\
&= \int_{\widetilde{\Omega}} \big(\widetilde{a}(\widetilde{x})\nabla\widetilde{u}\nabla\widetilde{v} + \widetilde{c}(\widetilde{x})\widetilde{u}\widetilde{v} \big) J_{h^{-1}}(\widetilde{x})\, d\widetilde{x}
\end{aligned}$$

where

$$\widetilde{a}^{kl}(\widetilde{x}) = \sum_{i,j=1}^{n} a^{ij}(h^{-1}(\widetilde{x}))(\nabla h(h^{-1}(\widetilde{x})))_{ki}(\nabla h(h^{-1}(\widetilde{x})))_{\ell j}$$

and

$$\widetilde{c}(\widetilde{x}) = c(h^{-1}(\widetilde{x})).$$

With this identity we define a new operator $\widetilde{L}$ in the following way. Let $\widetilde{u}, \widetilde{v} \in L^2(\widetilde{\Omega}, J_{h^{-1}} d\widetilde{x})$ such that $u = W\widetilde{u}$ and $v = W\widetilde{v}$ belong to $\mathrm{Dom}(L)$. We define

$$\langle \widetilde{L}\widetilde{u}, \widetilde{v}\rangle := \langle Lu, v\rangle.$$

With this, $(\lambda_k, \widetilde{\phi}_k)_{k=0}^{\infty}$ are the eigenvalues and eigenfunctions of $\widetilde{L}$, where λ_k are the eigenvalues of L. Moreover,

$$\mathrm{Dom}(\widetilde{L}) = \Big\{ \widetilde{u} \in L^2(\widetilde{\Omega}, J_{h^{-1}} d\widetilde{x}) : \sum_{k=0}^{\infty} \lambda_k \widetilde{u}_k^2 < \infty \Big\},$$

where $\widetilde{u}_k = \int_{\widetilde{\Omega}} \widetilde{u}\widetilde{\phi}_k J_{h^{-1}}(\widetilde{x})\, d\widetilde{x}$. We also notice that, if $\widetilde{u} \in L^2(\widetilde{\Omega}, J_{h^{-1}} d\widetilde{x})$ and $v \in L^2(\Omega, dx)$ then

$$\int_{\Omega} (W\widetilde{u})(x) v(x)\, dx = \int_{\widetilde{\Omega}} \widetilde{u}(\widetilde{x})(W^{-1}v)(\widetilde{x}) J_{h^{-1}}(\widetilde{x})\, d\widetilde{x}.$$

Then we can formally write

$$\langle \widetilde{L}\widetilde{u}, \widetilde{v} \rangle = \langle L(W\widetilde{u}), (W\widetilde{v}) \rangle = \langle W^{-1} L W \widetilde{u}, \widetilde{v} \rangle,$$

or

$$\widetilde{L} = W^{-1} \circ L \circ W.$$

3.2.2 Multiplication Operator

Let $M = M(x) \in C^{\infty}(\Omega)$ be a positive function. We define the multiplication operator

$$U : L^2(\Omega, M(x)^2 dx) \to L^2(\Omega, dx)$$

as

$$U(\breve{u})(x) = M(x)\breve{u}(x),$$

for $\breve{u} \in L^2(\Omega, M(x)^2 dx)$. If $\{\phi_k\}_{k=0}^{\infty}$ is the orthonormal basis of $L^2(\Omega, dx)$ consisting of eigenfunctions of L then $\{\breve{\phi}_k = U^{-1}\phi_k\}_{k=0}^{\infty}$ is an orthonormal basis of $L^2(\Omega, M(x)^2 dx)$.

Now given $u \in \mathrm{Dom}(L)$ we define $\breve{u}(x) = U^{-1}u(x) = M(x)^{-1}u(x)$, so that

$$u_{x_i}(x) = M(x)\breve{u}_{x_i}(x) + M_{x_i}(x)\breve{u}(x).$$

Therefore, for any $v \in \mathrm{Dom}(L)$, we have

$$\begin{aligned}
\langle Lu, v \rangle &= \int_{\Omega} \big(a^{ij}(x) u_{x_i} v_{x_j} + c(x) u v \big)\, dx \\
&= \int_{\Omega} \Big[a^{ij}(x) \Big(\breve{u}_{x_i} + \frac{M_{x_i}(x)}{M(x)} \breve{u} \Big) \Big(\breve{v}_{x_j} + \frac{M_{x_j}(x)}{M(x)} \breve{v} \Big) + c(x) \breve{u}\breve{v} \Big] M(x)^2\, dx.
\end{aligned}$$

This allows us to define the operator $\breve{L}$ in the following way. For $\breve{u}, \breve{v} \in L^2(\Omega, M(x)^2 dx)$ such that $u = U(\breve{u}) = M \cdot \breve{u}$ and $v = U(\breve{v}) = M \cdot \breve{v}$ are in $\mathrm{Dom}(L)$, we define

$$\langle \breve{L}\breve{u}, \breve{v} \rangle := \langle Lu, v \rangle.$$

With this, $(\lambda_k, \breve{\phi}_k)_{k=0}^{\infty}$ are the eigenvalues and eigenfunctions of $\breve{L}$, where λ_k are the eigenvalues of L. Hence we define,

$$\mathrm{Dom}(\breve{L}) = \Big\{ \breve{u} \in L^2(\Omega, M(x)^2 dx) : \sum_{k=0}^{\infty} \lambda_k \breve{u}_k^2 < \infty \Big\},$$

where $\breve{u}_k = \displaystyle\int_\Omega \breve{u}\breve{\phi}_k M(x)^2 \, dx = \int_\Omega u\phi_k \, dx = u_k$. Observe that

$$\int_\Omega U(\breve{u})(x)v(x) \, dx = \int_\Omega \breve{u}(x)U^{-1}v(x)M(x)^2 \, dx.$$

Then we can formally write

$$\langle \breve{L}\breve{u}, \breve{v} \rangle = \langle L(U\breve{u}), (U\breve{v}) \rangle = \langle U^{-1}LU\breve{u}, \breve{v} \rangle,$$

or

$$\breve{L} = U^{-1} \circ L \circ U.$$

3.2.3 Composition of Multiplication and Change of Variables

We consider the following composition of the multiplication operator U with the change of variables operator W:

$$U \circ W : L^2(\widetilde{\Omega}, M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}} d\widetilde{x}) \to L^2(\Omega, dx).$$

Notice that if $\bar{f} \in L^2(\widetilde{\Omega}, M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}} d\widetilde{x})$ then

$$\int_\Omega |[(U \circ W)\bar{f}](x)|^2 \, dx = \int_{\widetilde{\Omega}} |\bar{f}(\widetilde{x})|^2 M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}}(\tilde{x}) \, d\widetilde{x}.$$

By using a similar technique as we used in cases of W and U separately, we can define a new operator $\bar{L}$ in the following way. For $\bar{u}, \bar{v} \in L^2(\widetilde{\Omega}, M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}} d\widetilde{x})$ such that $u := (U \circ W)\bar{u}$ and $v := (U \circ W)\bar{v}$ are in $\mathrm{Dom}(L)$, we let

$$\langle \bar{L}\bar{u}, \bar{v} \rangle = \langle Lu, v \rangle.$$

By proceeding as in the previous cases we can formally write

$$\bar{L} = (U \circ W)^{-1} \circ L \circ (U \circ W).$$

3.2.4 Transference Method from $(\partial_t + L)^s$ to $(\partial_t + \bar{L})^s$

Now we consider the parabolic operators $H = \partial_t + L$ and $\bar{H} = \partial_t + \bar{L}$, where L and $\bar{L}$ are as above. If $\bar{u} = \bar{u}(t, \widetilde{x})$ is a function of $t \in \mathbb{R}$ and $\widetilde{x} \in \widetilde{\Omega}$ then the composition operator will act on $\bar{u}$ by leaving the variable t fixed:

$$(U \circ W)\bar{u}(t, x) = M(x)\bar{u}(t, h(x)), \quad \text{for } x \in \Omega,$$

so that

$$U \circ W : L^2(\mathbb{R}, dt; L^2(\widetilde{\Omega}, M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}} d\widetilde{x})) \to L^2(\mathbb{R}, dt; L^2(\Omega, dx)) = L^2(\mathbb{R} \times \Omega).$$

Recall that

$$\mathrm{Dom}(H) = \Big\{ u \in L^2(\mathbb{R} \times \Omega) : \int_{\mathbb{R}} \sum_{k=0}^{\infty} |(i\rho + \lambda_k)| |\widehat{u}_k(\rho)|^2 \, d\rho < \infty \Big\}$$

and, for $u \in \mathrm{Dom}(H)$ any $v \in C_c^\infty(\mathbb{R} \times \Omega)$,

$$\langle Hu, v \rangle_{L^2(\mathbb{R} \times \Omega)} = \int_{\mathbb{R}} \int_{\Omega} \Big(-uv_t + \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i}(t, x) v_{x_j}(t, x) + c(x) u(t, x) v(t, x) \Big) \, dx \, dt.$$

Now, for $\bar{u} \in L^2(\mathbb{R}, dt; L^2(\widetilde{\Omega}, M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}} d\widetilde{x}))$ such that $u := (U \circ W)\bar{u} \in \mathrm{Dom}(H)$, and $v := (U \circ W)\bar{v}$, we define the parabolic operator

$$\langle \bar{H}\bar{u}, \bar{v} \rangle := \langle Hu, v \rangle.$$

As a matter of fact, we can write,

$$\begin{aligned}
\langle Hu, v\rangle_{L^2(\mathbb{R}\times\Omega)} = \int_{\mathbb{R}}\int_{\Omega}\Big[&- M(x)\bar{u}(t,h(x))M(x)\bar{v}_t(t,h(x)) \\
&+ \sum_{i,j=1}^{n} a^{ij}(x)\Big(M_{x_i}(x)\bar{u}(t,h(x)) + \sum_{k=1}^{n} M(x)\bar{u}_{\widetilde{x}_k}(t,h(x))(\nabla h(x))_{ki}\Big) \\
&\times \Big(M_{x_j}(x)\bar{v}(t,h(x)) + \sum_{\ell=1}^{n} M(x)\bar{v}_{\widetilde{x}_\ell}(t,h(x))(\nabla h(x))_{\ell j}\Big) \\
&+ c(x)M(x)\bar{u}(t,h(x))M(x)\bar{v}(t,h(x))\Big]\, dx\, dt \\
= \int_{\mathbb{R}}\int_{\widetilde{\Omega}}\Big[&- \bar{u}\bar{v}_t \\
&+ \sum_{i,j=1}^{n} a^{ij}(h^{-1}(\widetilde{x}))\Big(\frac{M_{x_i}(h^{-1}(\widetilde{x}))}{M(h^{-1}(\widetilde{x}))}\bar{u} + \sum_{k=1}^{n} \bar{u}_{\widetilde{x}_k}(\nabla h(h^{-1}(\widetilde{x})))_{ki}\Big) \\
&\times \Big(\frac{M_{x_j}(h^{-1}(\widetilde{x}))}{M(h^{-1}(\widetilde{x}))}\bar{v}(t,\widetilde{x}) + \sum_{\ell=1}^{n} \bar{v}_{\widetilde{x}_\ell}(\nabla h(h^{-1}(\widetilde{x})))_{\ell j}\Big) \\
&+ c(h^{-1}(\widetilde{x}))\bar{u}\bar{v}\Big] M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}}\, d\widetilde{x}\, dt \\
= \langle \bar{H}\bar{u}, \bar{v}\rangle&
\end{aligned}$$

By using a similar argument as before we can formally write

$$\bar{H} = (U\circ W)^{-1}\circ H\circ (U\circ W).$$

Next, for $u \in \mathrm{Dom}(H)$ set $u_k(t) = \int_{\Omega} u\phi_k\, dx$, and write

$$u(t,x) = \sum_{k=0}^{\infty} u_k(t)\phi_k(x).$$

We know from the previous discussion that $(\lambda_k, \bar{\phi}_k)_{k=0}^{\infty}$ is the family of eigenvalues and eigenfunctions of $\bar{L}$, where

$$\bar{\phi}_k(\widetilde{x}) = \frac{1}{M(h^{-1}(\widetilde{x}))}\phi_k(h^{-1}(\widetilde{x})) \quad \text{for } x \in \widetilde{\Omega}.$$

So if we have $u(t,x) \in L^2(\mathbb{R}, dt; L^2(\widetilde{\Omega}, M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}} d\widetilde{x}))$, then

$$\bar{u}(t,\widetilde{x}) = \sum_{k=0}^{\infty} \bar{u}_k(t)\frac{1}{M(h^{-1}(\widetilde{x}))}\phi_k(h^{-1}(\widetilde{x})).$$

But as we know

$$\bar{u}_k(t) = \int_{\widetilde{\Omega}} \bar{u}(t,\widetilde{x})\bar{\phi}_k(\widetilde{x})M^2(h^{-1}(\widetilde{x}))J_{h^{-1}}\, d\widetilde{x} = \int_{\Omega} u(t,x)\phi_k(x)\, dx = u_k(t),$$

hence,

$$\langle \bar{H}\bar{u},\bar{v}\rangle = \langle Hu, v\rangle = \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho+\lambda_k)\widehat{u}_k(\rho)\overline{\widehat{v}_k(\rho)}\, d\rho = \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho+\lambda_k)\widehat{\bar{u}}_k(\rho)\overline{\widehat{\bar{v}}_k(\rho)}\, d\rho.$$

Therefore, for any $0 \leq s \leq 1$, we prove

$$\langle \bar{H}^s\bar{u},\bar{v}\rangle = \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho+\lambda_k)^s\widehat{\bar{u}}_k(\rho)\overline{\widehat{\bar{v}}_k(\rho)}\, d\rho = \langle H^s u, v\rangle.$$

Whence, we can formally write

$$\bar{H}^s = (U\circ W)^{-1}\circ H^s \circ (U\circ W).$$

Proof of Theorem 3.2.1. Let us first show how to transfer Theorem 3.1.1. Let $\bar{u} \in \mathrm{Dom}(\bar{H}^s)$ be a solution to

$$\begin{cases} \bar{H}^s\bar{u} = 0 & \text{in } (0,1)\times\widetilde{O} \\ \bar{u} \geq 0 & \text{in } (-\infty,1)\times\widetilde{\Omega}, \end{cases}$$

for some open set $\widetilde{O} \subset \widetilde{\Omega}$. From the definition, $\langle \bar{H}^s\bar{u},\bar{v}\rangle = \langle H^s u, v\rangle$, where $u = (U\circ W)\bar{u}$ and $v = (U\circ W)\bar{v}$. Then, by taking any $v \in C_c^\infty((0,1)\times O)$, where $O = h^{-1}(\tilde{O})$, we can let $\bar{v} = (U\circ W)^{-1}v \in C_c^\infty((0,1)\times\widetilde{O})$ and thus conclude that $H^s u = 0$ in $(0,1)\times h^{-1}(\widetilde{O}) = (0,1)\times O$. Also we notice that $u \geq 0$ in $(-\infty,1)\times h^{-1}(\widetilde{\Omega}) = (-\infty,1)\times\Omega$. Let $\widetilde{J}$ be a compact subset of $\widetilde{O}$. Then $h^{-1}(\widetilde{J})$ is a compact subset of O and, by Harnack inequality for H^s, (see Remark 3.1.2),

$$\sup_{(\frac{1}{4},\frac{1}{2})\times h^{-1}(\widetilde{J})} u \leq C \inf_{(\frac{3}{4},1)\times h^{-1}(\widetilde{J})} u.$$

Since $M(x)$ is strictly positive, continuous and bounded in $h^{-1}(\widetilde{J})$,

$$\sup_{(\frac{1}{4},\frac{1}{2})\times h^{-1}(\widetilde{J})} W\bar{u} \leq C' \inf_{(\frac{3}{4},1)\times h^{-1}(\widetilde{J})} W\bar{u}.$$

The change of variable h is a smooth diffeomorphism, so that

$$\sup_{(\frac{1}{4},\frac{1}{2})\times \widetilde{J}} \bar{u} \leq C' \inf_{(\frac{3}{4},1)\times \widetilde{J}} \bar{u}.$$

Thus Harnack inequality holds for $\bar{H}^s$. Let $\widetilde{K}$ be a compact subset of $(0,1)\times\widetilde{O}$. Then $K = h^{-1}(\widetilde{K})$ is a compact subset of $(0,1)\times O$ and u is parabolically Hölder continuous in K with

$$\|u\|_{C^{\alpha/2,\alpha}_{t,x}(K)} \le C\|u\|_{L^2(\mathbb{R}\times\Omega)} = C\|\bar{u}\|_{L^2(\mathbb{R},dt;L^2(\widetilde{\Omega},M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}}d\widetilde{x}))}.$$

Notice that $\bar{u}(t,\widetilde{x}) = [(U\circ W)^{-1}u](t,\widetilde{x}) = \frac{1}{M(h^{-1}(\widetilde{x}))}u(t,h^{-1}(\widetilde{x}))$, which, for any $(t_i,x_i) = (t_i,h^{-1}(\widetilde{x}_i)) \in K$, $i = 1,2$, gives

$$\begin{aligned}
|\bar{u}(t_1,\widetilde{x}_1) - \bar{u}(t_2,\widetilde{x}_2)| &= \left|\frac{u(t_1,x_1)}{M(x_1)} - \frac{u(t_2,x_2)}{M(x_2)}\right| \\
&\le \left|\frac{u(t_1,x_1)}{M(x_1)} - \frac{u(t_1,x_1)}{M(x_2)}\right| + \left|\frac{u(t_1,x_1)}{M(x_2)} - \frac{u(t_2,x_2)}{M(x_2)}\right| \\
&\le C\|M^{-1}\|_{C^{\alpha}_{x}(K)}\|u\|_{C^{\alpha/2,\alpha}_{t,x}(K)} d((t_1,x_1),(t_2,x_2))^{\alpha} \\
&\le C'\|\bar{u}\|_{L^2(\mathbb{R},dt;L^2(\widetilde{\Omega},M(h^{-1}(\widetilde{x}))^2 J_{h^{-1}}d\widetilde{x}))} d((t_1,\widetilde{x}_1),(t_2,\widetilde{x}_2))^{\alpha}
\end{aligned}$$

where d denotes the parabolic distance. In the last identity we used the fact that h^{-1} is a smooth diffeomorphism.

Let us next transfer the boundary Harnack inequality of Theorem 3.1.3. Again, for simplicity and without loss of generality, we consider $\tilde{x} = 0$. Let $\bar{u} \in \mathrm{Dom}(\bar{H}^s)$ be a solution to

$$\begin{cases} \bar{H}^s\bar{u} = 0 & \text{in } (-2,2)\times(\widetilde{\Omega}_0 \cap \widetilde{B}_{2r}(0)) \\ \bar{u} \ge 0 & \text{in } (-\infty,2)\times\widetilde{\Omega}, \end{cases}$$

such that $\bar{u}$ vanishes continuously on $(-2,2)\times((\widetilde{\Omega}\setminus\widetilde{\Omega}_0)\cap\widetilde{B}_{2r}(0))$. Let $(t_0,\widetilde{x}_0)$ be a fixed point in $(-2,2)\times\widetilde{\Omega}_0$ such that $t_0 > 1$. Then $H^s u = 0$ in $(-2,2)\times(\Omega_0\cap h^{-1}(\widetilde{B}_{2r}(0)))$, where $\Omega_0 = h^{-1}(\widetilde{\Omega}_0)$, $u \ge 0$ in $(-\infty,2)\times\Omega$ and, as h is a smooth diffeomorphism, we can also see that $u = (U\circ W)\bar{u}$ vanishes continuously in $(-2,2)\times((\Omega\setminus\Omega_0)\cap h^{-1}(\widetilde{B}_{2r}(0)))$. We assume, again for simplicity, that $h(0) = 0$ and let $K = h^{-1}(\widetilde{B}_r(0))$. Then $0 \in K$ and K is compactly contained in $h^{-1}(\widetilde{B}_{2r}(0)))$. We know that (see Remark 3.1.4)

$$\sup_{(-1,1)\times(\Omega_0\cap K)} u(t,x) \le Cu(t_0,x_0),$$

for $C > 0$. Since $M > 0$ is bounded and continuous, and h is a smooth diffeomorphism,

$$\sup_{(-1,1)\times(\widetilde{\Omega}_0\cap\widetilde{B}_r(0))} \bar{u}(t,\widetilde{x}) \le C'\bar{u}(t_0,\widetilde{x}_0).$$

□

Remark 3.2.2. As it was explained in Remark 2.3.2, one can check that if the differential operator L has continuous spectrum, then all the previous transference results are still valid.

Now we use the transference method of Theorem 3.2.1 to prove the following result.

Theorem 3.2.3. *Theorems 3.1.1 and 3.1.3 hold true for solutions u to $(\partial_t + L)^s u = f$, where L is any of the elliptic operators in (A)–(D).*

Proof. We have already proven Theorems 3.1.1 and 3.1.3 for the elliptic operators L in (2)–(6) in Chapter 2. Now we will transfer these theorems to the operators with L in (A)–D.

Transference from (2) *to* (A). In this case, $H^s = (\partial_t - \Delta + |x|^2 - n)^s$ in $\mathbb{R} \times \Omega = \mathbb{R} \times \mathbb{R}^n$ with Lebesgue measure and with zero boundary condition at infinity whereas $\bar{H}^s = (\partial_t - \Delta + 2x \cdot \nabla)^s$ in $\mathbb{R} \times \widetilde{\Omega} = \mathbb{R} \times \mathbb{R}^n$ with Gaussian measure $\pi^{-n/4} e^{-|x|^2/2} dx$. For the transference we use $h(x) = x$ and $M(x) = \pi^{-n/4} e^{-|x|^2/2}$.

Transference from (3) *to* (B). In all these examples we have $\widetilde{\Omega} = \Omega$. In the first three cases we start with $H^s = (\partial_t - \frac{1}{4}(\Delta + |x|^2 + \sum_{i=1}^n \frac{1}{x_i^2}\left(\alpha_i^2 - \frac{1}{4}\right)))^s$, for $\alpha_i > -1$, in $\mathbb{R} \times \Omega = \mathbb{R} \times (0, \infty)^n$. By using the transference method we can obtain the result for the other Laguerre systems.

- For $\bar{H}^s = (\partial_t + \sum_{i=1}^n (-x_i \frac{\partial^2}{\partial x_i^2} - (\alpha_i + 1)\frac{\partial}{\partial x_i} + \frac{x_i}{4}))^s$ with measure $x_1^{\alpha_1} \cdots x_n^{\alpha_n} dx$, which is related to the Laguerre system l_k^α, we choose $h(x) = (x_1^2, x_2^2, \ldots, x_n^2)$ and $M(x) = 2^{n/2} x_1^{\alpha_1 + 1/2} \cdots x_n^{\alpha_n + 1/2}$.

- For $\bar{H}^s = (\partial_t + \frac{1}{4}(-\Delta + |x|^2) - \sum_i^n \frac{2\alpha_i + 1}{4x_i} \frac{\partial}{\partial x_i})^s$ with measure $x_1^{2\alpha_1 + 1} \cdots x_n^{2\alpha_n + 1} dx$, which is related to the Laguerre system ψ_k^α, we choose $h(x) = x$ and $M(x) = x_1^{\alpha_1 + 1/2} \cdots x_n^{\alpha_n + 1/2}$.

- For $\bar{H}^s = (\partial_t + \sum_{i=1}^n (-x_i \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial x_i} + \frac{x_i}{4} + \frac{\alpha_i^2}{4x_i}))^s$ with Lebesgue measure, which is related to the Laguerre system $\mathcal{L}_k^\alpha$, we choose $h(x) = (x_1^2, x_2^2, \ldots, x_n^2)$ and $M(x) = 2^{n/2} x_1^{1/2} \cdots x_n^{1/2}$.

In the last case, we start with $H^s = (\partial_t - \frac{1}{4}(\Delta + |x|^2 + \sum_{i=1}^n \frac{1}{x_i^2}\left(\alpha_i^2 - \frac{1}{4}\right)) - \frac{\alpha+1}{2})^s$. Thus, we apply the transference method for $\bar{H}^s = (\partial_t + \sum_{i=1}^n (-x_i \frac{\partial^2}{\partial x_i^2} - (\alpha_i + 1 - x_i)\frac{\partial}{\partial x_i}))^s$ with measure $x_1^{\alpha_1} e^{-x_1} \cdots x_n e^{-x_n} dx$, which is related to the Laguerre polynomials system L_k^α, by choosing $h(x) = (x_1^2, x_2^2, \ldots, x_n^2)$ and $M(x) = 2^{n/2} e^{-|x|^2/2} x_1^{\alpha_1 + 1/2} \cdots x_n^{\alpha_n + 1/2}$.

Transference from (4) *to* (*C*). In this case, $H^s = (\partial_t - \frac{d^2}{dx^2} + \frac{\lambda(\lambda-1)}{\sin^2 x})^s$ in $\mathbb{R} \times \Omega = \mathbb{R} \times (0, \pi)$ with Lebesgue measure, and $\bar{H}^s = (\partial_t - \frac{d^2}{dx^2} - 2\lambda \cot x \frac{d}{dx} + \lambda^2)^s$ in $\mathbb{R} \times \widetilde{\Omega} = \mathbb{R} \times (0, \pi)$ with measure $\sin^{2\lambda} x dx$. For the transference method we use $h(x) = x$ and $M(x) = (\sin x)^\lambda$.

Transference from (6) *to* (*D*). Here $\Omega = \widetilde{\Omega} = (0, \infty)$, $H^s = (\partial_t - \frac{d^2}{dx^2} + \frac{\lambda^2-\lambda}{x^2})^s$ in $\mathbb{R} \times (0, \infty)$ with Lebesgue measure and $\bar{H}^s = (\partial_t - \frac{d^2}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx})^s$ in $\mathbb{R} \times (0, \infty)$ with measure $x^{2\lambda} dx$. For the transference method we use $h(x) = x$ and $M(x) = x^\lambda$. $\square$

CHAPTER 4. PARABOLIC HÖLDER SPACES

In this chapter, we define and prove Campanato-type characterizations of parabolic Hölder spaces.

4.1 Definition of Parabolic Hölder Spaces

Parabolic Hölder spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with Lipschitz constant $M > 0$, and let $I \subset \mathbb{R}$ be a bounded interval. Fix any $0 < \beta \leq 1$.

The classical parabolic Hölder space $C^{\beta/2,\beta}_{t,x}(\overline{I \times \Omega})$ is the set of continuous functions $u = u(t,x) : \overline{I \times \Omega} \to \mathbb{R}$ such that

$$\|u\|_{C^{\beta/2,\beta}_{t,x}(\overline{I\times\Omega})} = \|u\|_{L^\infty(I\times\Omega)} + [u]_{C^{\beta/2,\beta}_{t,x}(I\times\Omega)} < \infty$$

where

$$[u]_{C^{\beta/2,\beta}_{t,x}(I\times\Omega)} = \sup_{t,\tau\in I,\, x,z\in\Omega} \frac{|u(t,x) - u(\tau,z)|}{\max(|t-\tau|^{1/2}, |x-z|)^\beta}.$$

It is also customary to define the space $C^{(2+\beta)/2,2+\beta}_{t,x}(\overline{I \times \Omega}) = C^{1+\beta/2,2+\beta}_{t,x}(\overline{I \times \Omega})$ by requiring that $u_t, D^2u \in C^{\beta/2,\beta}_{t,x}(\overline{I \times \Omega})$. For these two definitions see [29, Chapter 8].

We define the space $C^{(1+\beta)/2,1+\beta}_{t,x}(\overline{I \times \Omega})$, as the set of continuous functions $u = u(t,x) : \overline{I \times \Omega} \to \mathbb{R}$ such that

- u is $(1+\beta)/2$-Hölder continuous in t uniformly in x, that is,

$$[u]_{L^\infty_x(\Omega; C^{(1+\beta)/2}_t(I))} = \sup_{x\in\Omega} [u(\cdot,x)]_{C^{(1+\beta)/2}_t(I)} = \sup_{x\in\Omega}\sup_{t,\tau\in I} \frac{|u(t,x) - u(\tau,x)|}{|t-\tau|^{(1+\beta)/2}} < \infty.$$

- $\nabla_x u \in C(\overline{I \times \Omega})$ and

$$[\nabla_x u]_{C^{\beta/2,\beta}_{t,x}(I\times\Omega)} = \sup_{t,\tau\in I,\, x,z\in\Omega} \frac{|\nabla_x u(t,x) - \nabla_x u(\tau,z)|}{\max(|t-\tau|^{1/2}, |x-z|)^\beta} < \infty.$$

The norm in $C_{t,x}^{(1+\beta)/2,1+\beta}(\overline{I\times\Omega})$ is given by

$$\begin{aligned}\|u\|_{C_{t,x}^{(1+\beta)/2,1+\beta}(\overline{I\times\Omega})} &= \|u\|_{L^\infty(I\times\Omega)} + \|\nabla_x u\|_{L^\infty(I\times\Omega)} \\ &\quad + [u]_{L_x^\infty(\Omega;C_t^{(1+\beta)/2}(I))} + [\nabla_x u]_{C_{t,x}^{\beta/2,\beta}(I\times\Omega)}.\end{aligned}$$

For a point $(t,x)\in\mathbb{R}^{n+1}$ and $r>0$ recall that $Q_r(t,x)=(t-r^2,t+r^2)\times B_r(x)$. Notice that $|Q_r(t,x)|=C_n r^{n+2}$, for some universal constant $C_n>0$. For the rest of this section we let

$$r_0=\min\{|I|^{1/2},\operatorname{diam}(\Omega)\}>0.$$

Observe that there exists a constant $C>0$ depending on n and M such that for any $(t,x)\in\overline{I\times\Omega}$ and $0<r\le r_0$ we have (see, for instance, [19, eq. (1.1)])

$$|Q_r(t,x)\cap(I\times\Omega)|=|(t-r^2,t+r^2)\cap I||B_r(x)\cap\Omega|\ge C_n r^{n+2}.$$

Let $\mathcal{P}_1$ be the set of polynomials of degree 1 in x, that is,

$$\mathcal{P}_1=\big\{P(z)=A_0+A_1\cdot z : A_0\in\mathbb{R},\ A_1\in\mathbb{R}^n\big\}.$$

Theorem 4.1.1 (Campanato-type characterizations)**.** *Let $0<\beta\le 1$. Suppose that $u=u(t,x)\in L^2(I\times\Omega)$. Then:*

(1) *$u\in C_{t,x}^{\beta/2,\beta}(\overline{I\times\Omega})$ if and only if there is a constant $C>0$ such that*

$$\inf_{c\in\mathbb{R}}\frac{1}{|Q_r(t,x)\cap(I\times\Omega)|}\int_{Q_r(t,x)\cap(I\times\Omega)}|u(\tau,z)-c|^2\,d\tau\,dz\le Cr^{2\alpha} \tag{4.1.1}$$

for all $(t,x)\in\overline{I\times\Omega}$ and $0<r\le r_0$ small. In this case, if we denote by $C_*>0$ the least constant for which the inequality above holds, then $\|u\|^2_{L^2(I\times\Omega)}+C_*$ is equivalent to $\|u\|^2_{C_{t,x}^{\beta/2,\beta}(\overline{I\times\Omega})}$.

(2) *$u\in C_{t,x}^{(1+\beta)/2,1+\beta}(\overline{I\times\Omega})$ if and only if there is a constant $C>0$ such that*

$$\inf_{P\in\mathcal{P}_1}\frac{1}{|Q_r(t,x)\cap(I\times\Omega)|}\int_{Q_r(t,x)\cap(I\times\Omega)}|u(\tau,z)-P(z)|^2\,d\tau\,dz\le Cr^{2(1+\beta)} \tag{4.1.2}$$

for all $(t,x)\in\overline{I\times\Omega}$ and $0<r\le r_0$ small. In this case, if we denote by $C_{**}>0$ the least constant for which the inequality above holds, then $\|u\|^2_{L^2(I\times\Omega)}+C_{**}$ is equivalent to $\|u\|^2_{C_{t,x}^{(1+\beta)/2,1+\beta}(\overline{I\times\Omega})}$.

4.2 Proof of Theorem 4.1.1(1)

The integral quantity in (4.1.1) is a quadratic polynomial in c. Hence, if u, (t,x) and r are fixed then the infimum is achieved at a unique constant $c = c((t,x),r,u)$. Therefore, we can restate condition (4.1.1) in the following equivalent way: *for any* $(t,x) \in \overline{I \times \Omega}$ *and* $0 < r \le r_0$ *there is a constant* $c((x,t),r,u)$ *such that*

$$\frac{1}{|Q_r(t,x) \cap (I \times \Omega)|} \int_{Q_r(t,x)\cap(I\times\Omega)} |u(\tau,z) - c((t,x),r,u)|^2 \, d\tau \, dz \le C r^{2\alpha}.$$

We will use the notation for such constant c repeatedly along this section.

Lemma 4.2.1. *Let* $u \in L^2(I \times \Omega)$ *satisfy* (4.1.1)*. There exists* $C > 0$ *such that, for any* $k \ge 0$ *and* $0 < r \le r_0$*,*

$$|c((t,x),r,u) - c((t,x),r/2^k)| \le C(C_*)^{1/2} r^{\alpha}.$$

Proof. We have

$$\begin{aligned}
&r^{n+2}|c((t,x),r,u) - c((t,x),r/2,u)|^2 \\
&\quad\le C \int_{Q_{r/2}(t,x)\cap(I\times\Omega)} \left(|c((t,x),r,u) - u(\tau,z)|^2 + |u(\tau,z) - c((t,x),r/2,u)|^2\right) d\tau \, dz \\
&\quad\le C \int_{Q_r(t,x)\cap(I\times\Omega)} |c((t,x),r,u) - u(\tau,z)|^2 \, d\tau \, dz + CC_*(r/2)^{2\alpha} \\
&\quad\le CC_* r^{2\alpha} r^{n+2}.
\end{aligned} \tag{4.2.1}$$

From here, it follows that

$$\sum_{k=0}^{\infty} |c((t,x),r/2^k,u) - c((t,x),r/2^{k+1},u)| \le \sum_{k=0}^{\infty} \frac{C(C_*)^{1/2}}{2^{\alpha k}} r^{\alpha} = C(C_*)^{1/2} r^{\alpha}.$$

The conclusion follows by noticing that for any k, we have

$$|c((t,x),r,u) - c((t,x),r/2^k)| \le \sum_{k=0}^{\infty} |c((t,x),r/2^k,u) - c((t,x),r/2^{k+1},u)|.$$

$\square$

Lemma 4.2.2. *Let* $u \in L^2(I \times \Omega)$ *satisfy* (4.1.1)*. Then*

$$\sup_{(t,x)\in\overline{I\times\Omega}} |c((t,x),r_0,u)| \le C((C_*)^{1/2} + \|u\|_{L^2(I\times\Omega)}) < \infty.$$

Proof. We have

$$\begin{aligned}|c((t,x),r_0,u)|^2 &\le \frac{C}{r_0^{n+2}}\int_{Q_{r_0}(t,x)\cap(I\times\Omega)}|u(\tau,z)-c((t,x),r_0,u)|^2\,d\tau\,dz\\ &\quad+\frac{C}{r_0^{n+2}}\int_{Q_{r_0}(t,x)\cap(I\times\Omega)}|u(\tau,z)|^2\,d\tau\,dz\\ &\le CC_*r_0^{2\alpha}+\frac{C}{r_0^{n+2}}\|u\|^2_{L^2(I\times\Omega)}.\end{aligned}$$

□

Lemma 4.2.3. *Let $u \in L^2(I\times\Omega)$ satisfy (4.1.1). Then*

$$\widetilde{u}(t,x)=\lim_{r\to 0}c((t,x),r,u)$$

exists and is finite, for all $(t,x)\in\overline{I\times\Omega}$. Moreover, for any $0<r\le r_0$,

$$|c((t,x),r,u)-\widetilde{u}(t,x)|\le C(C_*)^{1/2}r^{\alpha}.$$

Proof. Let $0<r\le r_0$ and $k\ge j$ integers. We have

$$|c((t,x),r/2^j,u)-c((t,x),r/2^k,u)|\le\sum_{i=j}^{k-1}|c((t,x),r/2^i,u)-c((t,x),r/2^{i+1},u)|.$$

We saw in the proof of Lemma 4.2.1 that the series

$$\sum_{i=0}^{\infty}|c((t,x),r/2^i,u)-c((t,x),r/2^{i+1},u)|$$

converges. Whence,

$$\lim_{k\to\infty}c((t,x),r/2^k,u)$$

exists and is finite. Moreover, we claim that the limit does not depend on the choice of r. For, if $0<r_1<r_2\le r_0$, then, by a parallel computation to the one we did in (4.2.1), we find that

$$|c((t,x),r_1/2^k,u)-c((t,x),r_2/2^k,u)|^2\le\frac{CC_*}{(2^k)^{2\alpha}}\left(\frac{r_1^{n+2+2\alpha}+r_2^{n+2+2\alpha}}{r_1^{n+2}}\right).$$

Hence,

$$\lim_{k\to\infty}|c((t,x),r_1/2^k,u)-c((t,x),r_2/2^k,u)|=0$$

and the claim follows. For any fixed $0 < r \le r_0$, let us define

$$\widetilde{u}(t,x) = \lim_{k\to\infty} c((t,x), r/2^k, u).$$

By taking $k \to \infty$ in Lemma 4.2.1, it follows that

$$|c((t,x), r, u) - \widetilde{u}(t,x)| \le C(C_*)^{1/2} r^\alpha.$$

□

Theorem 4.2.4. *Let $u \in L^2(I \times \Omega)$ satisfy (4.1.1) and let $\widetilde{u}$ be as in Lemma 4.2.3. Then $\widetilde{u} \in L^\infty(\overline{I \times \Omega})$ with*

$$\|\widetilde{u}\|_{L^\infty(\overline{I\times\Omega})} \le C((C_*)^{1/2} + \|u\|_{L^2(I\times\Omega)})$$

and

$$\widetilde{u} = u \qquad \text{for a.e. } (t,x) \in I \times \Omega.$$

Proof. For the boundedness of $\widetilde{u}$, by Lemmas 4.2.3 (with $r = r_0$) and 4.2.2,

$$|\widetilde{u}(t,x)| \le |\widetilde{u}(t,x) - c((t,x), r_0, u)| + |c((t,x), r_0, u)| \le CC_* r_0^\alpha + C((C_*)^{1/2} + \|u\|_{L^2(I\times\Omega)})$$

for all $(t,x) \in \overline{I \times \Omega}$. To verify that $\widetilde{u} = u$ a.e. in $I \times \Omega$, in view of Lemma 4.2.3, we need to show that

$$\lim_{r\to 0} c((t,x), r, u) = u(t,x) \qquad \text{a.e.}$$

Indeed,

$$\begin{aligned}
|c((t,x), r, u) - u(t,x)|^2 &\le \frac{C}{r^{n+2}} \int_{Q_r(t,x)\cap(I\times\Omega)} |c((t,x), r, u) - u(\tau, z)|^2 \, d\tau \, dz \\
&\quad + \frac{C}{r^{n+2}} \int_{Q_r(t,x)\cap(I\times\Omega)} |u(\tau, z) - u(t,x)|^2 \, d\tau \, dz \\
&\le CC_* r^{2\alpha} + \frac{C}{r^{n+2}} \int_{Q_r(t,x)\cap(I\times\Omega)} |u(\tau, z) - u(t,x)|^2 \, d\tau \, dz.
\end{aligned}$$

As $r \to 0$, the last term above converges to zero for a.e. $(t,x) \in I \times \Omega$ because a.e. point in $I \times \Omega$ is a Lebesgue point for u with respect to the parabolic distance, see [18]. □

Lemma 4.2.5. *Let $u \in L^2(I \times \Omega)$ satisfy (4.1.1). For any $(t_0, x_0), (s_0, y_0) \in \overline{I \times \Omega}$ such that $d_0 = \max(|t_0 - s_0|^{1/2}, |x_0 - y_0|) < r_0$ we have*

$$|c((t_0, x_0), 2d_0, u) - c((s_0, y_0), 2d_0, u)| \leq C(C_*)^{1/2} d_0^\alpha.$$

Proof. Let $K = Q_{2d_0}(t_0, x_0) \cap Q_{2d_0}(s_0, y_0) \cap (I \times \Omega)$. Then, by noticing that

$$|K| \geq |Q_{d_0}(t_0, x_0) \cap (I \times \Omega)| \geq C d_0^{n+2}$$

we find that

$$\begin{aligned}
&|c((t_0, x_0), 2d_0, u) - c((s_0, y_0), 2d_0, u)|^2 \\
&\qquad \leq \frac{C}{d_0^{n+2}} \int_K |c((t_0, x_0), 2d_0, u) - u(\tau, z)|^2 \, d\tau \, dz \\
&\qquad \quad + \frac{C}{d_0^{n+2}} \int_K |u(\tau, z) - c((s_0, y_0), 2d_0, u)|^2 \, d\tau \, dz \\
&\qquad \leq \frac{C}{d_0^{n+2}} \int_{Q_{2d_0}(t_0, x_0) \cap (I \times \Omega)} |c((t_0, x_0), 2d_0, u) - u(\tau, z)|^2 \, d\tau \, dz \\
&\qquad \quad + \frac{C}{d_0^{n+2}} \int_{Q_{2d_0}(s_0, y_0) \cap (I \times \Omega)} |u(\tau, z) - c((s_0, y_0), 2d_0, u)|^2 \, d\tau \, dz \\
&\qquad \leq C C_* d_0^{2\alpha}
\end{aligned}$$

as desired. ∎

Theorem 4.2.6. *Let $u \in L^2(I \times \Omega)$ satisfy (4.1.1) and define $\widetilde{u}$ as in Lemma 4.2.3. Then $\widetilde{u}$ is in $C^{\alpha/2, \alpha}_{t,x}(\overline{I \times \Omega})$ and for every $(t, x), (s, y) \in \overline{I \times \Omega}$ we have*

$$|\widetilde{u}(t, x) - \widetilde{u}(s, y)| \leq C(C_*)^{1/2} \max(|t - s|^{1/2}, |x - y|)^\alpha.$$

Proof. Let $(t, x), (s, y) \in \overline{I \times \Omega}$ such that $d = \max(|t - s|^{1/2}, |x - y|) < r_0/2$. Then, by Lemmas 4.2.3 and 4.2.5,

$$\begin{aligned}
|\widetilde{u}(t, x) - \widetilde{u}(s, y)| &\leq |\widetilde{u}(t, x) - c((t, x), 2d, u)| \\
&\quad + |\widetilde{u}(s, y) - c((s, y), 2d, u)| + |c((t, x), 2d, u) - c((s, y), 2d, u)| \\
&\leq C(C_*)^{1/2} d^\alpha = C(C_*)^{1/2} \max(|t - s|^{1/2}, |x - y|)^\alpha.
\end{aligned}$$

In the case when $d = \max(|t-s|^{1/2}, |x-y|) \geq r_0/2$, we can connect (t,x) and (s,y) with a polygonal contained in $\overline{I \times \Omega}$, whose segments have length less than $r_0/2$ and apply the inequality above to each pair of consecutive vertices. Notice that the number of segments needed for any pair of points (t,x) and (s,y) can be universally bounded in terms of the size of $I \times \Omega$, see [19, p. 184]. $\square$

Corollary 4.2.7. *Let $u \in L^2(I \times \Omega)$ satisfy (4.1.1). Then $u \in C^{\alpha/2,\alpha}_{t,x}(\overline{I \times \Omega})$ with*

$$\|u\|_{L^\infty(I\times\Omega)} \leq C((C_*)^{1/2} + \|u\|_{L^2(I\times\Omega)})$$

and

$$[u]_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)} \leq C(C_*)^{1/2}.$$

Conversely, if $u \in C^{\alpha/2,\alpha}_{t,x}(\overline{I \times \Omega})$ then u satisfies (4.1.1)

Proof. The first part of the result follows from Theorems 4.2.4 and 4.2.6. For the converse, suppose that $u \in C^{\alpha/2,\alpha}_{t,x}(\overline{I \times \Omega})$. For any $(t,x) \in \overline{I \times \Omega}$ and $0 < r \leq r_0$ we let $c = c(t,x) = u(t,x)$. Then,

$$\begin{aligned}
&\frac{1}{|Q_r(t,x) \cap (I \times \Omega)|} \int_{Q_r(t,x)\cap(I\times\Omega)} |u(\tau,z) - c|^2 \, d\tau \, dz \\
&\qquad \leq \frac{C_n}{r^{n+2}} \int_{Q_r(t,x)\cap(I\times\Omega)} |u(\tau,z) - u(t,x)|^2 \, d\tau \, dz \\
&\qquad \leq \frac{C_n}{r^{n+2}} [u]^2_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)} \int_{Q_r(t,x)\cap(I\times\Omega)} \left(|\tau - t|^\alpha + |z-x|^{2\alpha}\right) d\tau \, dz \\
&\qquad \leq \frac{C_n}{r^{n+2}} [u]^2_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)} r^{2\alpha} |Q_r(t,x)| \leq C_n [u]^2_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)} r^{2\alpha}.
\end{aligned}$$

In addition, $C_* \leq C_n [u]^2_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)}$ and $\|u\|^2_{L^2(I\times\Omega)} \leq |I \times \Omega| \|u\|^2_{L^\infty(I\times\Omega)}$, whence

$$\|u\|^2_{L^2(I\times\Omega)} + C_* \leq C_{n,I,\Omega} \|u\|^2_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)}.$$

$\square$

One can also see the these works, [32, 38] for the alternative proofs of Theorem 4.1.1(1).

4.3 Proof of Theorem 4.1.1(2)

We have the following preliminary result.

Lemma 4.3.1. *There exists a constant $c = c_{n,\Omega} > 0$ such that for any $P(z) \in \mathcal{P}_1$, $(t_0, x_0) \in \overline{I \times \Omega}$ and $0 < r \leq r_0$,*

$$|P(x_0)|^2 \leq \frac{c}{r^{n+2}} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |P(z)|^2 \, dz \, d\tau.$$

and, for any $i = 1, \ldots, n$,

$$|\partial_{z_i} P(x_0)|^2 \leq \frac{c}{r^{n+2+2}} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |P(z)|^2 \, dz \, d\tau.$$

Proof. Observe that if β is a multi-index with $|\beta| \leq 1$ then

$$\begin{aligned}\frac{1}{r^{n+2+2|\beta|}} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |P(z)|^2 \, dz \, d\tau &= \frac{|(t_0 - r^2, t_0 + r^2) \cap I|}{r^{n+2+2|\beta|}} \int_{B_r(x_0)\cap\Omega} |P(z)|^2 \, dz \\ &\geq \frac{1}{2r^{n+2|\beta|}} \int_{B_r(x_0)\cap\Omega} |P(z)|^2 \, dz.\end{aligned}$$

Notice that there is a constant $A = A_\Omega > 0$ such that $|E| = |B_r(x_0) \cap \Omega| \geq Ar^n$. Then, by [19, Lemma 2.I], there is a constant $c > 0$, depending only on n and A such that

$$\frac{c}{r^{n+2|\beta|}} \int_{B_r(x_0)\cap\Omega} |P(z)|^2 \, dz \geq |D^\beta P(x_0)|^2.$$

□

It is easy to see that the infimum for the integral quantity in (4.1.2) is achieved at a unique polynomial (see [19]). Therefore, (4.1.2) is restated as follows: *for any $(t, x) \in \overline{I \times \Omega}$ and $0 < r \leq r_0$ there is a unique polynomial $P(z, (t, x), r, u) \in \mathcal{P}_1$ such that*

$$\frac{1}{|Q_r(t,x) \cap (I \times \Omega)|} \int_{Q_r(t,x)\cap(I\times\Omega)} |u(\tau, z) - P(z, (t, x), r, u)|^2 \, d\tau \, dz \leq Cr^{2(1+\beta)}.$$

A generic polynomial $P \in \mathcal{P}_1$ is written as

$$P(z) = a_0 + \sum_{j=1}^{n} a_j (z_j - x_j).$$

For the unique polynomial $P(z, (t, x), r) \equiv P(z, (t, x), r, u)$ above we have

$$a_0((t, x), r) = P(z, (t, x), r)\big|_{z=x}$$

and

$$a_i((t, x), r) = \left[\partial_{z_i} P(z, (t, x), r)\right]\Big|_{z=x} \qquad \text{for } i = 1, \ldots, n.$$

Lemma 4.3.2. *Let u satisfy (4.1.2). There exists $c = c(n,\beta) > 0$ such that for any $(t_0, x_0) \in \overline{I \times \Omega}$, $0 < r \le r_0$ and $k \ge 0$, we have*

$$\int_{Q_{r/2^{k+1}}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),r/2^k) - P(z,(t_0,x_0),r/2^{k+1})|^2 \, d\tau \, dz \le C_{**}c(r/2^k)^{n+2+2(1+\beta)}.$$

Proof. We have

$$\begin{aligned}
&\int_{Q_{r/2^{k+1}}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),r/2^k) - P(z,(t_0,x_0),r/2^{k+1})|^2 \, d\tau \, dz \\
&\quad \le 2\int_{Q_{r/2^k}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),r/2^k) - u(\tau,z)|^2 \, d\tau \, dz \\
&\qquad + 2\int_{Q_{r/2^{k+1}}(t_0,x_0)\cap(I\times\Omega)} |u(\tau,z) - P(z,(t_0,x_0),r/2^{k+1})|^2 \, d\tau \, dz \\
&\quad \le C_{**}c(r/2^k)^{n+2+2(1+\beta)}.
\end{aligned}$$

□

Lemma 4.3.3. *Let u satisfy (4.1.2). There exists $c = c(n,\beta) > 0$ such that for any $(t_0, x_0), (s_0, y_0) \in \overline{I \times \Omega}$, if we denote by $d_0 = \max(|t_0 - s_0|^{1/2}, |x_0 - y_0|) \le r_0$, then*

$$|a_0((t_0,x_0),2d_0) - a_0((s_0,x_0),2d_0)|^2 \le cC_{**}|t_0 - s_0|^{1+\beta}$$

and, for $i = 1, \ldots, n$,

$$|a_i((t_0,x_0),2d_0) - a_i((s_0,y_0),2d_0)|^2 \le cC_{**}d_0^{2\beta}.$$

Proof. Consider first the case $i = 0$ and the polynomial

$$P(z) \equiv P(z,(t_0,x_0),2d_0) - P(z,(s_0,x_0),2d_0).$$

By Lemma 4.3.1 with $r = d_0$,

$$
\begin{aligned}
|a_0((t_0,x_0),2d_0) - a_0((s_0,x_0),2d_0)|^2 &= |P(x_0,(t_0,x_0),2d_0) - P(x_0,(s_0,x_0),2d_0)|^2 \\
&\le \frac{c}{d_0^{n+2}} \int_{Q_{d_0}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),2d_0) - P(z,(s_0,x_0),2d_0)|^2 \, d\tau \, dz \\
&\le \frac{2c}{d_0^{n+2}} \int_{Q_{2d_0}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),2d_0) - u(\tau,z)|^2 \, d\tau \, dz \\
&\quad + \frac{2c}{d_0^{n+2}} \int_{Q_{2d_0}(s_0,x_0)\cap(I\times\Omega)} |u(\tau,z) - P(z,(s_0,x_0),2d_0)|^2 \, d\tau \, dz \\
&\le cC_{**} d_0^{2(1+\beta)} = cC_{**}|t_0 - s_0|^{1+\beta}.
\end{aligned}
$$

For $i = 1, \dots, n$, the proof is similar using Lemma 4.3.1. ☐

Lemma 4.3.4. *Let u satisfy (4.1.2). There exists $c = c(n,\beta,\Omega) > 0$ such that for any $(t_0,x_0) \in \overline{I\times\Omega}$, $0 < r \le r_0$ and $k \ge 0$,*

$$|a_0((t_0,x_0),r) - a_0((t_0,x_0),r/2^k)| \le c(C_{**})^{1/2} \sum_{j=0}^{k-1} (r/2^j)^{1+\beta}$$

and, for $i = 1, \dots, n$,

$$|a_i((t_0,x_0),r) - a_i((t_0,x_0),r/2^k)| \le c(C_{**})^{1/2} \sum_{j=0}^{k-1} (r/2^j)^{\beta}.$$

Proof. By applying Lemmas 4.3.1 and 4.3.2, for $i = 1, \dots, n$,

$$
\begin{aligned}
&|a_i((t_0,x_0),r) - a_i((t_0,x_0),r/2^k)| \le \sum_{j=0}^{k-1} |a_i((t_0,x_0),r/2^j) - a_i((t_0,x_0),r/2^{j+1})| \\
&= \sum_{j=0}^{k-1} |\partial_{z_i} P(x_0,(t_0,x_0),r/2^j) - \partial_{z_i} P(x_0,(t_0,x_0),r/2^{j+1})| \\
&\le c \sum_{j=0}^{k-1} \left[\frac{c}{(r/2^{j+1})^{n+2+2}} \int_{Q_{r/2^{j+1}}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),r/2^j) - P(z,(t_0,x_0),r/2^{j+1})|^2 \, dz \, d\tau \right]^{1/2} \\
&\le c(C_{**})^{1/2} \sum_{j=0}^{k-1} (r/2^j)^{\beta}.
\end{aligned}
$$

The case $i = 0$ follows the same lines. ☐

Lemma 4.3.5. *Let u satisfy (4.1.2). Then there exists a family of functions $\{v_i(t,x)\}_{i=0}^n$ defined in $\overline{I \times \Omega}$ such that for all $0 < r \le r_0$,*

$$|a_0((t_0,x_0),r) - v_0(t_0,x_0)| \le C(C_{**})^{1/2} r^{1+\beta}$$

and, for all $i = 1, 2, \ldots, n$,

$$|a_i((t_0,x_0),r) - v_i(t_0,x_0)| \le C(C_{**})^{1/2} r^{\beta}.$$

Moreover, for all $i = 0, 1, \ldots, n$,

$$\lim_{r \to 0} a_i((t_0,x_0),r) = v_i(t_0,x_0)$$

uniformly with respect to (t_0,x_0).

Proof. Using Lemma 4.3.4, for $i = 1, 2, \ldots, n$, if $j < k$ then we find that

$$|a_i((t_0,x_0),r/2^j) - a_i((t_0,x_0),r/2^k)| \le c(C_{**})^{1/2} \sum_{m=j}^{k-1} (r/2^m)^{\beta}.$$

If j, k are large then the sum above can be made very small. Hence the limit

$$\lim_{k \to \infty} a_i((t_0,x_0),r/2^k) = v_i(t_0,x_0) \tag{4.3.1}$$

exists. We claim that the limit does not depend on r. Indeed, let $0 < r_1 < r_2 < r_0$. Then we have,

$$\begin{aligned}
&|a_i((t_0,x_0),r_1/2^k) - a_i((t_0,x_0),r_2/2^k)|^2 \\
&\quad = |\partial_{z_i} P(x_0,(t_0,x_0),r_1/2^k) - \partial_{z_i} P(x_0,(t_0,x_0),r_2/2^k))|^2 \\
&\quad \le \frac{c2^{k(n+4)}}{r_1^{n+4}} \int_{Q_{r_1/2^k}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),r_1/2^k) - P(z,(t_0,x_0),r_2/2^k)|^2 \, d\tau \, dz \\
&\quad \le \frac{c2^{k(n+4)+2}}{r_1^{n+4}} \int_{Q_{r_1/2^k}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),r_1/2^k) - u(\tau,z)|^2 \, d\tau \, dz \\
&\qquad + \frac{c2^{k(n+4)+2}}{r_1^{n+4}} \int_{Q_{r_1/2^k}(t_0,x_0)\cap(I\times\Omega)} |u(\tau,z) - P(z,(t_0,x_0),r_2/2^k)|^2 \, d\tau \, dz \\
&\quad \le \frac{c2^{2k}}{r_1^2} \int_{Q_{r_1/2^k}(t_0,x_0)\cap(I\times\Omega)} |P(z,(t_0,x_0),r_1/2^k) - u(\tau,z)|^2 \, d\tau \, dz \\
&\quad + \frac{c2^{2k} r_2^{n+2}}{r_1^{n+4}} \int_{Q_{r_2/2^k}(t_0,x_0)\cap(I\times\Omega)} |u(\tau,z) - P(z,(t_0,x_0),r_2/2^k)|^2 \, d\tau \, dz \\
&\quad \le cC_{**} \left(\frac{r_1}{2^k}\right)^{2\beta} + cC_{**} \left(\frac{1}{2^k}\right)^{2\beta} \frac{r_2^{n+2+2(1+\beta)}}{r_1^{n+4}} = \frac{cC_{**}}{2^{2k\beta}} \frac{r_1^{n+4+2\beta} + r_2^{n+4+2\beta}}{r_1^{n+4}}
\end{aligned}$$

Hence,

$$\lim_{k\to\infty} |a_i((t,x), r_1/2^k) - a_i((t,x), r_2/2^k)| = 0$$

and the limit (4.3.1) does not depend on r. Now, recall that we have

$$|a_i((t_0,x_0), r) - a_i((t_0,x_0), r/2^k)| \le c(C_{**})^{1/2} \sum_{j=0}^{k-1} (r/2^j)^\beta$$

Then taking the limit $k \to \infty$, $|a_i((t_0,x_0), r) - v_i(t_0,x_0)| \le c(C_{**})^{1/2} r^\beta$. For $i = 0$, the proof is the same. □

Theorem 4.3.6. *Let u satisfy (4.1.2) and define v_i as in Lemma 4.3.5 for $i = 1, 2, \ldots, n$. Then v_i is in $C^{\beta/2,\beta}_{t,x}(\overline{I \times \Omega})$ and for every $(t,x), (s,y) \in \overline{I \times \Omega}$ we have*

$$|v_i(t,x) - v_i(s,y)| \le C(C_{**})^{1/2} \max(|t-s|^{1/2}, |x-y|)^\beta.$$

Proof. Let $(t,x), (s,y) \in \overline{I \times \Omega}$ such that $d = \max(|t-s|^{1/2}, |x-y|) < r_0/2$. Then, by Lemmas 4.3.3 and 4.3.5,

$$\begin{aligned} |v_i(t,x) - v_i(s,y)| &\le |v_i(t,x) - a_i((t,x), 2d)| \\ &\quad + |v_i(s,y) - a_i((s,y), 2d)| + |a_i((t,x), 2d) - a_i((s,y), 2d)| \\ &\le C(C_{**})^{1/2} d^\beta = C(C_{**})^{1/2} \max(|t-s|^{1/2}, |x-y|)^\beta. \end{aligned}$$

In the case when $d = \max(|t-s|^{1/2}, |x-y|) \ge r_0/2$, then we can construct a polygonal connecting (t,x) and (s,y), contained in $\overline{I \times \Omega}$, whose segments have length less than $r_0/2$. After that we can apply the inequality above to each pair of consecutive vertices. Again notice that the number of segments needed for any pair of points (t,x) and (s,y) can be universally bounded in terms of the size of $I \times \Omega$, see [20, p. 149]. □

Theorem 4.3.7. *Let u satisfy (4.1.2) and define v_i as in Lemma 4.3.5 for $i = 0, 1, \ldots, n$. Then, for every $(t,x) \in I \times \Omega$*

$$\frac{\partial v_0(t,x)}{\partial x_i} = v_i(t,x) \quad \text{for } i = 1, \ldots, n.$$

Proof. Let $(t,x) \in I \times \Omega$ be any point and $r > 0$ sufficiently small such that $Q_r(t,x) \subset I \times \Omega$. Now we see that

$$a_0((t,x+re_i),2r) = P(z,(t,x+re_i),2r)\big|_{z=x+re_i}.$$

Using Taylor series expansion we can write,

$$\begin{aligned} P(z,(t,x+re_i),2r)\big|_{z=x} &= P(z,(t,x+re_i),2r)\big|_{z=x+re_i} - \partial_{z_i} P(z,(t,x+re_i),2r)\big|_{z=x+re_i} r \\ &= a_0((t,x+re_i),2r) - r a_i((t,x+re_i),2r). \end{aligned}$$

Then,

$$\begin{aligned} &\frac{a_0((t,x+re_i),2r) - a_0((t,x),2r)}{r} \\ &\qquad = \frac{P(z,(t,x+re_i),2r)\big|_{z=x} - P(z,(t,x),2r)\big|_{z=x}}{r} + a_i((t,x+re_i),2r) \end{aligned} \tag{4.3.2}$$

Now using Lemma 4.3.1 we see that

$$\begin{aligned} &\left| \frac{P(z,(t,x+re_i),2r)\big|_{z=x} - P(z,(t,x),2r)\big|_{z=x}}{r} \right|^2 \\ &= \frac{1}{r^2} \left| P(z,(t,x+re_i),2r)\big|_{z=x} - P(z,(t,x),2r)\big|_{z=x} \right|^2 \\ &\le \frac{c}{r^{2+n+2}} \int_{Q_r(t,x)\cap(I\times\Omega)} |P(z,(t,x+re_i),2r) - P(z,(t,x),2r)|^2 \, d\tau \, dz \\ &\le \frac{c}{r^{2+n+2}} \int_{Q_{2r}(t,x+re_i)\cap(I\times\Omega)} |P(z,(t,x+re_i),2r) - u(\tau,z)|^2 \, d\tau \, dz \\ &\quad + \frac{c}{r^{2+n+2}} \int_{Q_r(t,x)\cap(I\times\Omega)} |u(\tau,z) - P(z,(t,x),2r)|^2 \, d\tau \, dz \\ &\le \frac{cC_{**}}{r^{n+4}} r^{n+2+2(1+\beta)} = cC_{**} r^{2\beta} \to 0 \end{aligned}$$

as $r \to 0$. Next we see that, by Lemma 4.3.5 and since v_i are continuous functions (see Theorem 4.3.6),

$$\begin{aligned} |a_i((t,x+re_i),2r) - v_i(t,x)| &\le |a_i((t,x+re_i),2r) - v_i(t,x+re_i)| \\ &\quad + |v_i(t,x+re_i) - v_i(t,x)| \\ &\le c(C_{**})^{1/2} r^\beta + |v_i(t,x+re_i) - v_i(t,x)| \to 0 \end{aligned}$$

as $r \to 0$. Thus, it follows in (4.3.2) that

$$\lim_{r\to 0} \frac{a_0((t,x+re_i),2r) - a_0((t,x),2r)}{r} = v_i(t,x).$$

But now observe that

$$\lim_{r\to 0} \frac{a_0((t,x+re_i),2r) - a_0((t,x),2r)}{r} = \lim_{r\to 0} \frac{v_0(t,x+re_i) - v_0(t,x)}{r} = \partial_{x_i} v_0(t,x)$$

because, by Lemma 4.3.5,

$$\left| \frac{v_0(t,x+re_i) - a_0((t,x+re_i),2r)}{r} \right| \le c(C_{**})^{1/2} r^{\beta}$$

and

$$\left| \frac{v_0(t,x) - a_0((t,x),2r)}{r} \right| \le c(C_{**})^{1/2} r^{\beta}.$$

□

The following result is a direct consequence of Theorems 4.3.6 and 4.3.7.

Corollary 4.3.8. *Let u satisfy (4.1.2). If v_0 is as in Lemma 4.3.5 then $v_0 \in C^{(1+\beta)/2,1+\beta}_{t,x}(\overline{I\times\Omega})$ with the estimate*

$$[v_0]_{L^\infty_x(C^{(1+\beta)/2}_t)} + [\nabla v_0]_{C^{\beta/2,\beta}_{t,x}} \le c(C_{**})^{1/2}.$$

Proof. Let $(t,x),(s,x) \in \overline{I\times\Omega}$ such that $d = |t-s|^{1/2} < r_0/2$. Then, by Lemmas 4.3.3 and 4.3.5,

$$\begin{aligned} |v_0(t,x) - v_0(s,x)| &\le |v_0(t,x) - a_0((t,x),2d)| \\ &\quad + |v_0(s,x) - a_0((s,t),2d)| + |a_0((t,x),2d) - a_0((s,x),2d)| \\ &\le C(C_{**})^{1/2} d^{1+\beta} = C(C_{**})^{1/2}|t-s|^{(1+\beta)/2}. \end{aligned}$$

In the case when $d > r_0/2$ we can apply a polygonal argument as in [19, p. 149]. Also we have already shown that, $v_i = \frac{\partial v_0}{\partial z_i}$ is in $C^{\beta/2,\beta}_{t,x}$ for each $i = 1,2,\ldots,n$ and

$$|v_i(t,x) - v_i(s,y)| \le C(C_{**})^{1/2} \max(|t-s|^{1/2}, |x-y|)^{\beta}.$$

See Theorems 4.3.7 and 4.3.6. Hence by definition of $C^{(1+\beta)/2,1+\beta}_{t,x}$ we have

$$v_0 \in C^{(1+\beta)/2,1+\beta}_{t,x}(\overline{I\times\Omega})$$

with the corresponding estimate. □

Theorem 4.3.9. *Let u satisfy (4.1.2). Then $u \in C^{(1+\beta)/2,1+\beta}_{t,x}(\overline{I \times \Omega})$ with the estimates*

$$[u]_{L^\infty_x(C^{(1+\beta)/2}_t)} + [\nabla u]_{C^{\beta/2,\beta}_{t,x}} \le c(C_{**})^{1/2}$$

and

$$\|u\|_{L^\infty(I\times\Omega)} + \|\nabla u\|_{L^\infty(I\times\Omega)} \le c\big((C_{**})^{1/2} + \|u\|_{L^2(I\times\Omega)}\big).$$

Proof. For any $(t_0, x_0) \in I \times \Omega$, we have, by Lebesgue differentiation theorem,

$$\lim_{r\to 0} \frac{1}{|Q_r(t_0,x_0) \cap (I\times\Omega)|} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |u(t,x) - u(t_0,x_0)|^2 \, dt\, dx = 0$$

see [18]. Then, for any $0 < r \le r_0$,

$$\begin{aligned} |a_0((t_0,x_0),r) - u(t_0,x_0)|^2 \le{}& \frac{C}{r^{n+2}} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |P(x,(t_0,x_0),r) - a_0((t_0,x_0),r)|^2 \, dt\, dx \\ &+ \frac{C}{r^{n+2}} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |P(x,(t_0,x_0),r) - u(t,x)|^2 \, dt\, dx \\ &+ \frac{C}{r^{n+2}} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |u(t,x) - u(t_0,x_0)|^2 \, dt\, dx. \end{aligned}$$

Now, using (4.1.2) and the following equation,

$$P(x,(t_0,x_0),r) = a_0((t_0,x_0),r) + \sum_{j=1}^n a_j((t_0,x_0),r)(x_j - (x_0)_j)$$

We get

$$\frac{1}{r^{n+2}} \int_{Q_r(t_0,x_0)\cap(I\times\Omega)} |P(x,(t_0,x_0),r) - a_0((t_0,x_0),r)|^2 \, dt\, dx \le C \sum_{j=1}^n |a_j((t_0,x_0),r)|^2 r^2.$$

For a fixed (t_0, x_0), $|a_j((t_0,x_0),r)|^2$ converges as $r \to 0$, see Lemma 4.3.5. Hence, as $r \to 0$, using all the previous results and estimates we see that

$$v_0(t_0,x_0) = \lim_{r\to 0} a_0((t_0,x_0),r) = u(t_0,x_0).$$

Therefore, u can be modified on a set of measure zero so that $u = v_0$. In particular, by Theorem 4.3.7, u is differentiable in $I \times \Omega$ and, by using Corollary 4.3.8, seminorm estimates follow. For the boundedness of u and ∇u, we use Lemmas 4.3.1 and 4.3.5 to bound in the following way. On one hand,

$$\begin{aligned}
|u(t,x)|^2 &\le C|u(t,x) - a_0((t,x),r_0)|^2 + C|a_0((t,x),r_0)|^2 \\
&= C|v_0(t,x) - a_0((t,x),r_0)|^2 + C|P(x,(t,x),r_0)|^2 \\
&\le cC_{**}r_0^{2(1+\beta)} + \frac{C}{r_0^{n+2}} \int_{Q((t,x),r_0)\cap(I\times\Omega)} |P(z,(t,x),r_0)|^2 \, dz \, d\tau \\
&\le cC_{**}r_0^{2(1+\beta)} + \frac{C}{r_0^{n+2}} \int_{Q((t,x),r_0)\cap(I\times\Omega)} |P(z,(t,x),r_0) - u(\tau,z)|^2 \, dz \, d\tau \\
&\quad + \frac{C}{r_0^{n+2}} \int_{I\times\Omega} |u(\tau,z)|^2 \, dz \, d\tau \\
&\le cC_{**}r_0^{2(1+\beta)} + \frac{C}{r_0^{n+2}} \|u\|^2_{L^2(I\times\Omega)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|u_{x_i}(t,x)|^2 &\le C|u_{x_i}(t,x) - a_i((t,x),r_0)|^2 + C|a_i((t,x),r_0)|^2 \\
&\le cC_{**}r_0^{2\beta} + \frac{C}{r_0^{n+4}} \|u\|^2_{L^2(I\times\Omega)}.
\end{aligned}$$

$\square$

CHAPTER 5. SCHAUDER ESTIMATES

In this chapter, we present our results on Schauder estimates for solutions to the nonlocal equation (1.0.1):

$$H^s u(t,x) \equiv (\partial_t + L)^s u(t,x) = f(t,x), \quad 0 < s < 1$$

for $t \in \mathbb{R}$ and $x \in \Omega$, where Ω is a bounded Lipschitz domain of $\mathbb{R}^n$, $n \geq 1$, and L is an elliptic operator, i.e.

$$L = -\operatorname{div}(A(x)\nabla)$$

Here $A(x) = (A^{ij}(x))$ is a bounded, measurable, symmetric matrix defined in Ω, satisfying the uniform ellipticity condition, that is, for some $\Lambda \geq 1$,

$$\Lambda^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$. The operator L is subject to an appropriate boundary condition. Here we consider the boundary condition to be either Dirichlet or Neumann, that is

$$u = 0 \quad \text{or} \quad \partial_A u = A(x)\nabla_x u \cdot \nu = 0 \quad \text{on } \mathbb{R} \times \partial\Omega,$$

where ν is the exterior unit normal to $\partial\Omega$. From now on, we fix $T_1 < 0 < T_2$ and we call $I = (T_1, T_2)$. For other notation, let us denote $D = \{(x,y) : x \in \Omega,\ y > 0\} \subset \mathbb{R}^{n+1}$, as in Section 2.4.2 . We recall from the definition of Muckenhoupt weight that $|y|^a$ belongs to the Muckenhoupt class $A_2(\mathbb{R}^{n+1})$. Again we define $H^1_{L,y}(D)$ as the set of functions $w = w(x,y) \in L^2(D, y^a dxdy)$ such that

$$\begin{aligned}[w]^2_{H^1_{L,y}(D)} &:= \int_0^\infty \int_\Omega y^a A(x)\nabla_x w \nabla_x w \, dx\, dy + \int_0^\infty \int_\Omega y^a |\partial_y w|^2 \, dx\, dy \\ &= \int_0^\infty y^a \sum_{k=0}^\infty \lambda_k |w_k(y)|^2 \, dy + \int_0^\infty \int_\Omega y^a |\partial_y w|^2 \, dx\, dy < \infty,\end{aligned}$$

where $w_k(y) = \displaystyle\int_\Omega w(x,y)\phi_k(x)\, dx$, under the norm

$$\|w\|^2_{H^1_{L,y}(D)} = \|w\|^2_{L^2(D, y^a dxdy)} + [w]^2_{H^1_{L,y}(D)}.$$

Recall that $\{e^{-\tau H}\}_{\tau\geq 0}$ denotes the semigroup generated by $H = \partial_t - \operatorname{div}(A(x)\nabla_x)$. We refer the reader to Section 4.1 for the definition of parabolic Hölder spaces. In the first two statements, we present the interior regularity when f is parabolically Hölder continuous in $I \times \Omega$ and when f is in $L^p(I \times \Omega)$, respectively, under precise continuity assumptions on $A(x)$. Interior regularity in both cases does not depend on the prescribed boundary conditions nor on the regularity of the boundary.

5.1 Main Theorems

Theorem 5.1.1 (Interior regularity for f Hölder)**.** *Let $0 < \alpha < 1$ and suppose that $f \in C^{\alpha/2,\alpha}_{t,x}(I \times \Omega)$. Let $u \in \operatorname{Dom}(H^s)$ be a weak solution to (1.0.1) such that $u = 0$ or $\partial_A u = 0$ on $\mathbb{R} \times \partial\Omega$.*

(i) Assume that $0 < \alpha + 2s < 1$ and that $A(x)$ is continuous in Ω. Then

$$u \in C^{(\alpha+2s)/2,\alpha+2s}_{t,x,\mathrm{loc}}(I \times \Omega)$$

and for any open subset $K \subset\subset I \times \Omega$ we have the estimate

$$\|u\|_{C^{(\alpha+2s)/2,\alpha+2s}_{t,x}(K)} \leq C\big(\|u\|_{\operatorname{Dom}(H^s)} + \|f\|_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)}\big).$$

(ii) Assume that $1 < \alpha + 2s < 2$ and that $A(x) \in C^{0,\alpha+2s-1}(\Omega)$. Then

$$u \in C^{(\alpha+2s)/2,1+(\alpha+2s-1)}_{t,x,\mathrm{loc}}(I \times \Omega)$$

and for any open subset $K \subset\subset I \times \Omega$ we have the estimate

$$\|u\|_{C^{(\alpha+2s)/2,1+(\alpha+2s-1)}_{t,x}(K)} \leq C\big(\|u\|_{\operatorname{Dom}(H^s)} + \|f\|_{C^{\alpha/2,\alpha}_{t,x}(I\times\Omega)}\big).$$

The constants $C > 0$ above depend only on $s, \alpha, K, I \times \Omega$ and the modulus of continuity of $A(x)$.

Theorem 5.1.2 (Interior regularity for f in L^p)**.** *Suppose that $f \in L^p(I \times \Omega)$ for some $2 \leq p < \infty$. Let $u \in \operatorname{Dom}(H^s)$ be a weak solution to (1.0.1) such that $u = 0$ or $\partial_A u = 0$ on $\mathbb{R} \times \partial\Omega$.*

(i) Assume that $(n + 2)/(2s) < p < (n + 2)/(2s - 1)_+$ and that $A(x)$ is continuous in Ω. Then

$$u \in C^{\alpha/2,\alpha}_{t,x,\mathrm{loc}}(I \times \Omega)$$

where $\alpha = 2s-(n+2)/p \in (0,1)$ *and, for any open subset* $K \subset\subset I\times\Omega$*, we have the estimate*

$$\|u\|_{C^{\alpha/2,\alpha}_{t,x}(K)} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{L^p(I\times\Omega)}\big).$$

(*ii*) *Assume that* $s>1/2$, $p>(n+2)/(2s-1)$ *and that* $A(x)\in C^{0,\alpha}(\Omega)$ *for* $\alpha = 2s-(n+2)/p-1 \in (0,1)$*. Then*

$$u \in C^{(1+\alpha)/2,1+\alpha}_{t,x,\mathrm{loc}}(I\times\Omega)$$

and for any open subset $K\subset\subset I\times\Omega$ *we have the estimate*

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}_{t,x}(K)} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{L^p(I\times\Omega)}\big).$$

The constants $C>0$ *above depend only on* $s,p,K,I\times\Omega$ *and the modulus of continuity of* $A(x)$*.*

Next we state our results on global regularity. The first one, Theorem 5.1.3, deals with solutions satisfying the Dirichlet boundary condition $u=0$ on $\mathbb{R}\times\partial\Omega$ when f is Hölder continuous in $\overline{I\times\Omega}$ and, in addition, is allowed to be nonzero on the boundary $I\times\partial\Omega$. The fact that f is nonzero on the boundary will affect the global regularity of the solution. Instead, when f is identically zero on the boundary, we get better global regularity which is consistent with the interior estimates of Theorem 5.1.1, see Theorem 5.1.4. This is in high contrast with the local case of parabolic equations, namely, when $s=1$, see [32]. Such feature had already been observed in the case of fractional elliptic equations in divergence form in [17]. Our statements are also precise in terms of the sharp regularity of the coefficients and the boundary $\partial\Omega$.

Theorem 5.1.3 (Global regularity for Dirichlet and f Hölder)**.** *Let* $0<\alpha<1$ *and suppose that* $f\in C^{\alpha/2,\alpha}_{t,x}(\overline{I\times\Omega})$*. Let* $u\in\mathrm{Dom}(H^s)$ *be a weak solution to* (1.0.1) *such that* $u=0$ *on* $\mathbb{R}\times\partial\Omega$*.*

(*i*) *Assume that* $0<\alpha+2s<1$, $\partial\Omega$ *is* $C^{1,\alpha}$ *and that* $A(x)\in C^{0,\alpha}(\overline{\Omega})$*. Then*

$$u(t,x) \sim \mathrm{dist}(x,\partial\Omega)^{2s} + v(t,x) \qquad \textit{for all } t\in I$$

where

$$v\in C^{(\alpha+2s)/2,\alpha+2s}_{t,x}(\overline{I\times\Omega})$$

and we have the estimate

$$\|v\|_{C_{t,x}^{(\alpha+2s)/2,\alpha+2s}(\overline{I\times\Omega})} \leq C\big(1+\|u\|_{\mathrm{Dom}(H^s)}+\|f\|_{C_{t,x}^{\alpha/2,\alpha}(\overline{I\times\Omega})}\big).$$

(*ii*) *Assume that* $s=1/2$, $\partial\Omega$ *is* $C^{1,\alpha+\varepsilon}$ *and that* $A(x)\in C^{0,\alpha+\varepsilon}(\overline{\Omega})$, *for some* $\varepsilon>0$ *such that* $0<\alpha+\varepsilon<1$. *Then*

$$u(t,x)\sim \mathrm{dist}(x,\partial\Omega)|\log \mathrm{dist}(x,\partial\Omega)|+v(t,x) \qquad \textit{for all } t\in I$$

where

$$v\in C_{t,x}^{(1+\alpha)/2,1+\alpha}(\overline{I\times\Omega})$$

and we have the estimate

$$\|v\|_{C_{t,x}^{(1+\alpha)/2,1+\alpha}(\overline{I\times\Omega})} \leq C\big(1+\|u\|_{\mathrm{Dom}(H^s)}+\|f\|_{C_{t,x}^{\alpha/2,\alpha}(\overline{I\times\Omega})}\big).$$

(*iii*) *Assume that* $s>1/2$, $1<\alpha+2s<2$, $\partial\Omega$ *is* $C^{1,\alpha+2s-1}$ *and that* $A(x)\in C^{0,\alpha+2s-1}(\overline{\Omega})$. *Then*

$$u(t,x)\sim \mathrm{dist}(x,\partial\Omega)+v(t,x) \qquad \textit{for all } t\in I$$

where

$$v\in C_{t,x}^{(\alpha+2s)/2,1+(\alpha+2s-1)}(\overline{I\times\Omega})$$

and we have the estimate

$$\|v\|_{C_{t,x}^{(\alpha+2s)/2,1+(\alpha+2s-1)}(\overline{I\times\Omega})} \leq C\big(1+\|u\|_{\mathrm{Dom}(H^s)}+\|f\|_{C_{t,x}^{\alpha/2,\alpha}(\overline{I\times\Omega})}\big).$$

The constants $C>0$ *above depend only on* n,s,α *and the modulus of continuity of* $\partial\Omega$ *and* $A(x)$.

Theorem 5.1.4 (Global regularity for Dirichlet and f Hölder, $f\equiv 0$ on the boundary)**.** *Let* $0<\alpha<1$ *and suppose that* $f\in C_{t,x}^{\alpha/2,\alpha}(\overline{I\times\Omega})$ *is such that* $f=0$ *on* $I\times\partial\Omega$. *Let* $u\in\mathrm{Dom}(H^s)$ *be a weak solution to* (1.0.1) *such that* $u=0$ *on* $\mathbb{R}\times\partial\Omega$.

(*i*) *Assume that* $0<\alpha+2s<1$, $\partial\Omega$ *is* C^1 *and that* $A(x)$ *is continuous in* $\overline{\Omega}$. *Then*

$$u\in C_{t,x}^{(\alpha+2s)/2,\alpha+2s}(\overline{I\times\Omega})$$

and we have the estimate

$$\|u\|_{C^{(\alpha+2s)/2,\alpha+2s}_{t,x}(\overline{I\times\Omega})} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{C^{\alpha/2,\alpha}_{t,x}(\overline{I\times\Omega})}\big).$$

(*ii*) *Assume that* $1<\alpha+2s<2$, $\partial\Omega$ *is* $C^{1,\alpha+2s-1}$ *and that* $A(x)\in C^{0,\alpha+2s-1}(\overline{\Omega})$. *Then*

$$u\in C^{(\alpha+2s)/2,1+(\alpha+2s-1)}_{t,x}(\overline{I\times\Omega})$$

and we have the estimate

$$\|u\|_{C^{(\alpha+2s)/2,1+(\alpha+2s-1)}_{t,x}(\overline{I\times\Omega})} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{C^{\alpha/2,\alpha}_{t,x}(\overline{I\times\Omega})}\big).$$

The constants $C>0$ *above depend only on* n,s,α *and the modulus of continuity of* $\partial\Omega$ *and* $A(x)$.

In the following, we turn our attention to global regularity results for the case of the Neumann boundary condition $\partial_A u = 0$ on $\mathbb{R}\times\partial\Omega$, when f is Hölder continuous. In contrast with the case of Dirichlet boundary condition, here the global estimates do not depend on the values of f on the boundary and, therefore, are consistent with the interior regularity obtained in Theorem 5.1.1.

Theorem 5.1.5 (Global regularity for Neumann and f Hölder)**.** *Let* $0<\alpha<1$ *and suppose that* $f\in C^{\alpha/2,\alpha}_{t,x}(\overline{I\times\Omega})$. *Let* $u\in\mathrm{Dom}(H^s)$ *be a weak solution to* (1.0.1) *such that* $\partial_A u = 0$ *on* $\mathbb{R}\times\partial\Omega$.

(*i*) *Assume that* $0<\alpha+2s<1$, $\partial\Omega\in C^1$ *and that* $A(x)$ *is continuous in* $\overline{\Omega}$. *Then*

$$u\in C^{(\alpha+2s)/2,\alpha+2s}_{t,x}(\overline{I\times\Omega})$$

and we have the estimate

$$\|u\|_{C^{(\alpha+2s)/2,\alpha+2s}_{t,x}(\overline{I\times\Omega})} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{C^{\alpha/2,\alpha}_{t,x}(\overline{I\times\Omega})}\big).$$

(*ii*) *Assume that* $1<\alpha+2s<2$, $\partial\Omega\in C^{1,\alpha+2s-1}$ *and that* $A(x)\in C^{0,\alpha+2s-1}(\overline{\Omega})$. *Then*

$$u\in C^{(\alpha+2s)/2,1+(\alpha+2s-1)}_{t,x}(\overline{I\times\Omega})$$

and we have the estimate

$$\|u\|_{C^{(\alpha+2s)/2,1+(\alpha+2s-1)}_{t,x}(\overline{I\times\Omega})} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{C^{\alpha/2,\alpha}_{t,x}(\overline{I\times\Omega})}\big).$$

The constants $C > 0$ above depend only on n, s, α and the modulus of continuity of $\partial\Omega$ and $A(x)$.

Finally, we state our global Schauder estimates for the case of L^p right hand side, which are in accordance with the interior estimates of Theorem 5.1.2.

Theorem 5.1.6 (Global regularity for f in L^p)**.** *Suppose that $f \in L^p(I \times \Omega)$ for some $2 \leq p < \infty$. Let $u \in \mathrm{Dom}(H^s)$ be a weak solution to (1.0.1) such that $u = 0$ or $\partial_A u = 0$ on $\mathbb{R} \times \partial\Omega$.*

(i) Assume that $(n+2)/(2s) < p < (n+2)/(2s-1)_+$, $\partial\Omega \in C^1$ and that $A(x)$ is continuous in $\overline{\Omega}$. Then

$$u \in C_{t,x}^{\alpha/2,\alpha}(\overline{I \times \Omega})$$

where $\alpha = 2s - (n+2)/p \in (0,1)$ and we have the estimate

$$\|u\|_{C_{t,x}^{\alpha/2,\alpha}(\overline{I\times\Omega})} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{L^p(I\times\Omega)}\big).$$

(ii) Assume that $s > 1/2$, $p > (n+2)/(2s-1)$ and that $A(x) \in C^{0,\alpha}(\overline{\Omega})$ for $\alpha = 2s-(n+2)/p-1 \in (0,1)$. Then

$$u \in C_{t,x}^{(1+\alpha)/2,1+\alpha}(\overline{I \times \Omega})$$

and we have the estimate

$$\|u\|_{C_{t,x}^{(1+\alpha)/2,1+\alpha}(\overline{I\times\Omega})} \leq C\big(\|u\|_{\mathrm{Dom}(H^s)} + \|f\|_{L^p(I\times\Omega)}\big).$$

The constants $C > 0$ above depend only on n, s, p and the modulus of continuity of $\partial\Omega$ and $A(x)$.

5.2 Caccioppoli Estimate and Approximation

Before we state the Caccippoli energy inequality lemma we want to state the following proposition.

Proposition 5.2.1. *Let U be as in (2.4.1) and assume that $f = H^s u \in L^2(\mathbb{R} \times \Omega)$. Then $U_t \in L^2(\mathbb{R}; (H^1_{L,a}(D))^*)$ and, in particular, $U \in C(\mathbb{R}; L^2(D, y^a dX))$. Furthermore, for every*

$\phi \in H^1([-1,1];L^2(B_1^*, y^a dX)) \cap L^2([-1,1];H^1_{L,a}(B_1^*))$ such that $\phi = 0$ on $\partial Q_1^* \setminus (Q_1 \times \{0\})$ and a.e. $t_1, t_2 \in [-1,1]$, we have

$$\int_{B_1^*} y^a \big[U\phi\big]_{t=t_1}^{t=t_2} dX - \int_{t_1}^{t_2}\int_{B_1^*} y^a U \partial_t \phi \, dX dt + \int_{t_1}^{t_2}\int_{B_1^*} y^a B(x)\nabla U \nabla \phi \, dX dt$$
$$= \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} \int_{t_1}^{t_2}\int_{B_1} f(t,x)\phi(t,x,0)\,dx\,dt.$$

Proof. We claim that

$$U_t = y^{-a}\operatorname{div}(y^a B(x)\nabla U) \in (H^1_{L,a}(D))^* \tag{5.2.1}$$

in the weak sense, namely, that for any $\psi(t) \in C_c^\infty(\mathbb{R})$ and any $\phi(x,y) \in H^1_{L,a}(D)$,

$$\int_0^\infty y^a \int_\Omega \left(\int_\mathbb{R} U\psi_t\,dt\right)\phi\,dx\,dy$$
$$= \int_\mathbb{R}\int_0^\infty\int_\Omega y^a B(x)\nabla U\nabla\phi\,dX\psi\,dt + c_s\int_\mathbb{R}\int_\Omega f(t,x)\phi(x,0)\,dx\,\psi\,dt.$$

Indeed, notice that, by Theorem 2.4.2, $U \in L^2(\mathbb{R}; H^1_{L,a}(D))$, so that

$$U_t(\psi) = -\int_\mathbb{R} U\psi_t\,dt \in H^1_{L,a}(D).$$

Therefore,

$$[U_t(\psi)](\phi) = -\int_0^\infty y^a\int_\Omega\left(\int_\mathbb{R} U\psi_t\,dt\right)\phi\,dx\,dy$$

is well defined. On the other hand, for a.e. $t \in \mathbb{R}$,

$$[y^{-a}\operatorname{div}(y^a B(x)\nabla U)](\phi) = -\int_0^\infty\int_\Omega y^a B(x)\nabla U\nabla\phi\,dx\,dy + c_s\int_\Omega f(t,x)\phi(x,0)\,dx$$

is a well defined bounded linear functional on $H^1_{L,a}(D)$, because $U \in L^2(\mathbb{R}; H^1_{L,a}(D))$, $f \in L^2(\mathbb{R}\times\Omega)$ and the trace inequality $\|\phi\|_{L^2(\Omega)} \le C_s\|\phi\|_{H^1_{L,a}(D)}$ holds true. On the other hand, from Theorem 2.4.2, we see that

$$-[U_t(\psi)](\phi) = \int_\mathbb{R}\left[\int_0^\infty\int_\Omega y^a B(x)\nabla U\nabla\phi\,dx\,dy - c_s\int_\Omega f(t,x)\phi(x,0)\,dx\right]\psi\,dt.$$

Thus, (5.2.1) holds.

Moreover, it is clear that

$$\left\| \int_0^\infty \int_\Omega y^a B(x)\nabla U \nabla \phi \, dx\, dy - c_s \int_\Omega f(t,x)\phi(x,0)\, dx \right\|_{L^2(\mathbb{R})} \leq C\big(\|U\|_{L^2(\mathbb{R},H^1_{L,a}(D))} + \|f\|_{L^2(\mathbb{R}\times\Omega)}\big)\|\phi\|_{H^1_{L,a}(D)}.$$

This gives that $U_t \in L^2(\mathbb{R};(H^1_{L,a}(D))^*)$ and (5.2.1) holds a.e., namely,

$$\int_0^\infty y^a \int_\Omega U_t \phi \, dx\, dy = \int_0^\infty \int_\Omega y^a B(x)\nabla U \nabla\phi \, dx\, dy - c_s \int_\Omega f(t,x)\phi(x,0)\, dx \tag{5.2.2}$$

for a.e. $t \in \mathbb{R}$.

For the second claim, notice that $U \in H^1([-1,1];(H^1_{L,a}(D))^*)$. Then, for any $\psi \in C_c^\infty(-1,1)$ and a.e. $t_1, t_2 \in (-1,1)$, by using a standard mollifier argument, we have

$$\big[U\psi\big]_{t=t_1}^{t=t_2} = (U\psi)(t_2) - (U\psi)(t_1) = \int_{t_1}^{t_2} U_t \psi \, dt + \int_{t_1}^{t_2} U\psi_t \, dt.$$

Whence, multiplying by ψ and integrating from t_1 to t_2 in (5.2.2), we find that

$$\int_0^\infty y^a \int_\Omega \big[U\psi\big]_{t=t_1}^{t=t_2} \phi\, dx\, dy - \int_0^\infty y^a \int_\Omega \int_{t_1}^{t_2} U\psi_t \phi \, dt\, dx\, dy = \int_{t_1}^{t_2}\int_0^\infty \int_\Omega y^a B(x)\nabla U\nabla\phi\, dx\, dy\, \psi\, dt - c_s \int_{t_1}^{t_2}\int_\Omega f(t,x)\phi(x,0)\, dx\, \psi\, dt.$$

The conclusion is true by approximation. □

In view of Proposition 5.2.1, we define weak solutions to the extension problem in Q_1^* in the following way. Consider the problem

$$\begin{cases} y^a \partial_t U - \operatorname{div}(y^a B(x)\nabla U) = -\operatorname{div}(y^a F) & \text{in } Q_1^* \\ -y^a U_y\big|_{y=0} = f & \text{on } Q_1. \end{cases} \tag{5.2.3}$$

Here $F = F(t,x) = (F_1, \ldots, F_n, F_{n+1})$ is an $\mathbb{R}^{n+1}$-valued vector field on Q_1^* such that $F_{n+1} = 0$ and $|F| \in L^2(Q_1^*)$, and $f = f(t,x) \in L^2(Q_1)$. We say that $U \in C([-1,1];L^2(B_1^*, y^a dX)) \cap L^2([-1,1];H^1_{L,a}(B_1^*))$ is a weak solution to (5.2.3) if for every $-1 < t_1 < t_2 < 1$

$$\int_{B_1^*} y^a U\phi\big|_{t=t_1}^{t=t_2} dX - \int_{t_1}^{t_2}\int_{B_1^*} y^a U \partial_t\phi\, dX\, dt + \int_{t_1}^{t_2}\int_{B_1^*} y^a B(x)\nabla U\nabla\phi\, dX\, dt = \int_{t_1}^{t_2}\int_{B_1} f(t,x)\phi(t,x,0)\, dx\, dt + \int_{t_1}^{t_2}\int_{B_1^*} y^a F \nabla\phi\, dX\, dt \tag{5.2.4}$$

holds for every $\phi \in H^1([-1,1]; L^2(B_1^*, y^a dX)) \cap L^2([-1,1]; H^1_{L,a}(B_1^*))$ such that $\phi = 0$ on $\partial Q_1^* \backslash (Q_1 \times \{0\})$. Any such ϕ will be called a test function. In the following let us state the lemma about the Caccippoli energy inequality.

Lemma 5.2.2. *Suppose that U is a weak solution to* (5.2.3) *in the sense of* (5.2.4) *with F as described above. Then, for any $\eta \in C_c^\infty(Q_1 \times [0,1))$ and for any $-1 < t_1 < t_2 < 1$,*

$$\begin{aligned} &\sup_{t_1<t<t_2} \int_{B_1^*} y^a U^2 \eta^2 \, dX + \int_{t_1}^{t_2} \int_{B_1^*} y^a \eta^2 |\nabla U|^2 \, dX dt \\ &\leq C \bigg[\int_{t_1}^{t_2} \int_{B_1^*} y^a \left((|\partial_t(\eta^2)| + |\nabla \eta|^2) U^2 + |F|^2 \eta^2 \right) \, dX dt \\ &\quad + \int_{t_1}^{t_2} \int_{B_1} (\eta(t,x,0))^2 |U(t,x,0)||f(t,x)| \, dx dt \bigg] + \int_{B_1^*} y^a U^2(t_1, X) \eta^2(t_1, X) \, dX \end{aligned}$$

where $C > 0$ depends only on ellipticity, n and s.

Proof of Lemma 5.2.2. First we will define the Steklov averages of U and state some of their properties (see, for example, [30]). Let $-1 < t < 1$ and $h > 0$ such that $t + h < 1$. We define

$$U_h(t,x,y) = \frac{1}{h} \int_t^{t+h} U(\tau, x, y) \, d\tau \qquad \text{for } t \in (-1, 1-h]$$

and $U_h(t,x,y) = 0$ for $t > 1 - h$, for all $(x,y) \in B_1^*$. Since $U(\cdot, x, y) \in L^2([-1,1])$ for almost every $(x,y) \in B_1^*$, it follows that U_h is differentiable almost everywhere in $(-1,1)$, for almost every $(x,y) \in B_1^*$, and

$$\partial_t U_h(t,x,y) = \frac{U(t+h,x,y) - U(t,x,y)}{h} \in L^2([-1,1]).$$

Moreover, since $U \in C([-1,1]; L^2(B_1^*; y^a dX))$, we have that

$$\lim_{h \to 0} U_h = U \qquad \text{in } L^2(B_1^*; y^a dX), \text{ for every } t \in (-1, 1-\delta)$$

for any $\delta \in (0,2)$. Additionally, for any $\delta \in (0,2)$,

$$\lim_{h \to 0} U_h = U \qquad \text{in } L^2([-1, 1-\delta]; L^2(B_1^*; y^a dX)).$$

Now we see that U_h satisfies

$$\begin{aligned} &\int_{B_1^*} \left[y^a (U_h)_t \varphi + y^a B(x) \nabla U_h \nabla \varphi \right] dX \\ &\qquad = \int_{B_1} f_h(t,x) \varphi(x,0) \, dx + \int_{B_1^*} y^a F_h \nabla \varphi \, dX \end{aligned} \tag{5.2.5}$$

for almost every $-1 < t < 1-h$ and for every $\varphi = \varphi(x,y) \in H^1(B_1^*)$ such that $\varphi = 0$ on $\partial B_1^* \setminus (B_1 \times \{0\})$, where F_h, f_h are defined in the similar fashion. This follows by choosing $t_1 = t$ and $t_2 = t+h$ such that $[t_1, t_2] \subset [-1,1]$ and $\phi = \varphi$ (which is independent of the time variable) in the weak formulation (5.2.4).

Next, fix a subinterval $[t_1, t_2] \subseteq [-1,1]$ such that $t_2 + h < 1$. In (5.2.5) we take $\varphi = \phi$, where $\phi = \phi(t,x,y)$ is a test function as in the definition of the weak formulation (5.2.4). Then (5.2.5) holds for almost every $t \in (-1, 1-h)$ and, if we integrate in the t-variable over $[t_1, t_2]$ and use integration by parts in t, we finally get

$$\begin{aligned}\int_{B_1^*} [y^a U_h \phi]_{t_1}^{t_2}\, dX + &\int_{t_1}^{t_2}\int_{B_1^*} [-y^a U_h \phi_t + y^a B(x) \nabla U_h \nabla \phi]\, dX dt \\ &= \int_{t_1}^{t_2}\int_{B_1} f_h(t,x)\phi(t,x,0)\, dxdt + \int_{t_1}^{t_2}\int_{B_1^*} y^a F_h \nabla \phi\, dX dt.\end{aligned} \tag{5.2.6}$$

Observe that, from the earlier properties of Steklov average, by taking $h \to 0$ in (5.2.6) one arrives to (5.2.4).

For the proof of the Caccioppoli inequality, let $\phi = \eta^2 U_h$ in (5.2.6). Since

$$\begin{aligned}\int_{t_1}^{t_2}\int_{B_1^*} y^a U_h \partial_t(\eta^2 U_h)\, dX dt &= \int_{t_1}^{t_2}\int_{B_1^*} y^a U_h^2 \partial_t(\eta^2)\, dX dt + \frac{1}{2}\int_{t_1}^{t_2}\int_{B_1^*} y^a \eta^2 \partial_t(U_h^2)\, dX dt \\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_{B_1^*} y^a U_h^2 \partial_t(\eta^2)\, dX dt + \frac{1}{2}\int_{B_1^*} [y^a \eta^2 U_h^2]_{t_1}^{t_2} dX\end{aligned}$$

it follows that

$$\begin{aligned}\frac{1}{2}\int_{B_1^*} [y^a \eta^2 U_h^2]_{t_1}^{t_2}\, dX &+ \int_{t_1}^{t_2}\int_{B_1^*} y^a B(x) \eta^2 \nabla U_h \nabla U_h\, dX dt \\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_{B_1^*} y^a U_h^2 \partial_t(\eta^2)\, dX dt - 2\int_{t_1}^{t_2}\int_{B_1^*} y^a B(x) \eta U_h \nabla U_h \nabla \eta\, dX dt \\ &\quad + \int_{t_1}^{t_2}\int_{B_1^*} y^a \eta^2 F_h \nabla U_h\, dX dt + 2\int_{t_1}^{t_2}\int_{B_1^*} y^a F_h U_h \eta \nabla \eta\, dX dt \\ &\quad + \int_{t_1}^{t_2}\int_{B_1} (\eta(t,x,0))^2 U_h(t,x,0) f_h(t,x)\, dxdt.\end{aligned}$$

By the properties of Steklov averages, we can take the limit as $h \to 0$ above to deduce that the same identity holds for U, F and f in place of U_h, F_h and f_h, respectively. Then, by ellipticity and the Cauchy inequality with $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{2}\int_{B_1^*}[y^a\eta^2U^2]_{t_1}^{t_2}\,dX + \lambda\int_{t_1}^{t_2}\int_{B_1^*} y^a\eta^2|\nabla U|^2\,dX\,dt \\
&\quad\le \frac{1}{2}\int_{t_1}^{t_2}\int_{B_1^*} y^aU^2|\partial_t(\eta^2)|\,dX\,dt \\
&\qquad + \frac{C}{\varepsilon}\int_{t_1}^{t_2}\int_{B_1^*} y^aU^2|\nabla\eta|^2\,dX\,dt + \varepsilon\int_{t_1}^{t_2}\int_{B_1^*} y^a\eta^2|\nabla U|^2\,dX\,dt \\
&\qquad + \frac{C}{\varepsilon}\int_{t_1}^{t_2}\int_{B_1^*} y^a\eta^2|F|^2\,dX\,dt + \varepsilon\int_{t_1}^{t_2}\int_{B_1^*} y^a\eta^2|\nabla U|^2\,dX\,dt \\
&\qquad + C\int_{t_1}^{t_2}\int_{B_1^*} y^a\eta^2|F|^2\,dX\,dt + C\int_{t_1}^{t_2}\int_{B_1^*} y^aU^2|\nabla\eta|^2\,dX\,dt \\
&\qquad + \int_{t_1}^{t_2}\int_{B_1^*} \eta^2|U||f|\,dxdt.
\end{aligned}$$

The conclusion follows in a standard way by choosing $\varepsilon > 0$ sufficiently small. □

Let us consider a test function $\phi \in H^1([-1,1];L^2(B_1^*, y^a dX)) \cap L^2([-1,1];H^1_{L,a}(B_1^*))$ with $\phi = 0$ on $\partial Q_1^* \setminus (Q_1 \times \{0\})$. Let $U \in C([-1,1];L^2(B_1^*, y^a dX)) \cap L^2([-1,1];H^1_{L,a}(B_1^*))$. If U is a weak solution to (5.2.3) in the sense of (5.2.4) then, by letting $t_2 \to 1$ and $t_1 \to -1$, we find that

$$\begin{aligned}
-\int_{Q_1^*} y^aU\partial_t\phi\,dX\,dt + \int_{Q_1^*} y^aB(x)\nabla U\nabla\phi\,dX\,dt \\
= \int_{Q_1} f(t,x)\phi(t,x,0)\,dx\,dt + \int_{Q_1^*} y^aF\nabla\phi\,dX\,dt.
\end{aligned} \tag{5.2.7}$$

Conversely, if U satisfies (5.2.7) for all such ϕ then, by using arguments similar to Proposition 5.2.1 we get that (5.2.4) holds. Therefore, when referring to weak solutions to (5.2.3), we will mean that (5.2.4) or, equivalently, (5.2.7), hold for the corresponding test functions.

Corollary 5.2.3. *Let U be a weak solution to (5.2.3). Suppose that*

$$\int_{Q_1} U(t,x,0)^2\,dxdt + \int_{Q_1^*} y^aU^2\,dX\,dt \le 1.$$

Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$\int_{Q_1} f^2\,dxdt + \int_{Q_1^*} y^a|F|^2\,dX\,dt + \int_{B_1} |A(x) - I|^2\,dx \le \delta^2$$

where I denotes the identity matrix, then there exists a weak solution W to

$$\begin{cases} y^a \partial_t W - \operatorname{div}(y^a \nabla W) = 0 & \text{in } Q_{3/4}^* \\ -y^a W_y\big|_{y=0} = 0 & \text{on } Q_{3/4} \end{cases} \tag{5.2.8}$$

such that

$$\int_{Q_{3/4}^*} y^a |U - W|^2 \, dX dt < \varepsilon^2.$$

Proof. We will prove this by contradiction. Let us assume that there exists $\varepsilon > 0$, coefficients $A_k(x)$, data f_k, vector fields F^k and solutions U_k in Q_1^*, $k \geq 1$, such that

$$\int_{Q_1} U_k^2 \, dx dt + \int_{Q_1^*} y^a U_k^2 \, dX dt \leq 1,$$

$$\int_{Q_1} f_k^2 \, dx dt + \int_{Q_1^*} y^a |F^k|^2 \, dX dt + \int_{B_1} |A_k(x) - I|^2 \, dx < \frac{1}{k^2}$$

and such that, for any weak solution W to (5.2.8),

$$\int_{Q_{3/4}^*} y^a |U_k - W|^2 \, dX dt \geq \varepsilon^2. \tag{5.2.9}$$

If in Lemma 5.2.2 we choose η such that $\eta \equiv 1$ in $Q_{3/4}^*$, $0 \leq \eta \leq 1$ in Q_1^*, and we let $t_1 \to -1$ and $t_2 \to 1$, then we find that

$$\int_{Q_{3/4}^*} y^a |\nabla U_k|^2 \, dX dt \leq C$$

for all $k \geq 1$. Let us define $T_1 = -9/16$, $T_2 = 9/16$. The previous estimate says that the sequence $\{U_k\}_{k=1}^\infty$ is bounded in $L^2([T_1, T_2]; H^1(B_{3/4}^*, y^a dX))$. By the Aubin–Lions Lemma, this space is compactly embedded in $L^2([T_1, T_2]; L^2(B_{3/4}^*, y^a dX))$, so that there exists a subsequence, again denoted by $\{U_k\}_{k=1}^\infty$, and a function U_∞ such that

$$U_k \to U_\infty \text{ strongly in } L^2([T_1, T_2]; L^2(B_{3/4}^*, y^a dX))$$

$$U_k \to U_\infty \text{ weakly in } L^2([T_1, T_2]; H^1(B_{3/4}^*, y^a dX)).$$

We show next that U_∞ is a solution to (5.2.8) and this will give a contradiction to (5.2.9). Indeed, for any $k \geq 1$ and any test function ϕ,

$$- \int_{Q^*_{3/4}} y^a U_k \partial_t \phi \, dX dt + \int_{Q^*_{3/4}} y^a B_k(x) \nabla U_k \nabla \phi \, dX dt$$
$$= \int_{Q_{3/4}} f_k(t,x)\phi(t,x,0) \, dx dt + \int_{Q^*_{3/4}} y^a F^k \nabla \phi \, dX dt.$$

By letting $k \to \infty$, the equation above reduces to

$$- \int_{Q^*_{3/4}} y^a U_\infty \partial_t \phi \, dX dt + \int_{Q^*_{3/4}} y^a \nabla U_\infty \nabla \phi \, dX dt = 0$$

as desired. □

Similarly as with (5.2.3)–(5.2.4), we can define the notion of weak solutions to

$$\begin{cases} y^a \partial_t U - \operatorname{div}(y^a B(x) \nabla U) = -\operatorname{div}(y^a F) & \text{in } (Q_1^+)^* \\ -y^a U_y\big|_{y=0} = f & \text{on } Q_1^+ \\ U = 0 \quad \text{or} \quad \partial_A U = 0 & \text{on } Q_1^* \cap \{x_n = 0\} \end{cases} \tag{5.2.10}$$

with test functions ϕ such that $\phi = 0$ on $\partial(Q_1^+)^* \backslash (Q_1^+ \times \{0\})$ (for Dirichlet boundary condition), or $\phi = 0$ on $\partial(Q_1^+)^* \backslash [(Q_1^+ \times \{0\}) \cup (Q_1^* \cap \{x_n = 0\})]$ (for Neumann boundary condition). Then, exactly as with Corollary 5.2.3, we can prove the following approximation result up to the boundary.

Corollary 5.2.4. *Let U be a weak solution to (5.2.10). Suppose that*

$$\int_{Q_1^+} U(t,x,0)^2 \, dx dt + \int_{(Q_1^+)^*} y^a U^2 \, dX dt \leq 1.$$

Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$\int_{Q_1^+} f^2 \, dx dt + \int_{(Q_1^+)^*} y^a |F|^2 \, dX dt + \int_{B_1^+} |A(x) - I|^2 \, dx \leq \delta^2$$

where I denotes the identity matrix, then there exists a weak solution W to

$$\begin{cases} y^a \partial_t W - \operatorname{div}(y^a \nabla W) = 0 & \text{in } (Q_{3/4}^+)^* \\ -y^a W_y\big|_{y=0} = 0 & \text{on } Q_{3/4}^+ \\ W = 0 \quad \text{or} \quad \partial_{x_n} W = 0 & \text{on } Q_{3/4}^* \cap \{x_n = 0\} \end{cases}$$

such that

$$\int_{(Q^+_{3/4})^*} y^a |U - W|^2 \, dXdt < \varepsilon^2.$$

Next, we present the regularity of W.

Proposition 5.2.5. *Let W be a weak solution to*

$$\begin{cases} y^a \partial_t W - \operatorname{div}(y^a \nabla W) = 0 & \text{in } Q_1^* \\ -y^a W_y\big|_{y=0} = 0 & \text{on } Q_1. \end{cases} \tag{5.2.11}$$

Then following estimates hold.

(1) *For every integer $k \geq 0$, multi-index $\beta \in \mathbb{N}_0^n$ and each $Q_r(t_0, x_0) \subset Q_1$, we have*

$$\sup_{Q_{r/2}(t_0,x_0)\times[0,r/2)} |\partial_t^k D_x^\beta W| \leq \frac{C(n,s)}{r^{k+|\beta|}} \operatorname*{osc}_{Q_r(t_0,x_0)\times[0,r)} W.$$

(2) *For each $Q_r(t_0, x_0) \subset Q_1$,*

$$\max_{Q_{r/2}(t_0,x_0)\times[0,r/2)} |W| \leq C(r,n,s) \|W\|_{L^2(Q_r(t_0,x_0)\times[0,r), y^a dXdt)}.$$

(3) *We have*

$$\sup_{(t,x)\in Q_{1/2}} |W_y(t,x,y)| \leq C(n,s) \|W\|_{L^2(Q_1^*, y^a dXdt)} y \qquad \text{for all } 0 \leq y < 1/2.$$

Proof. The proof of (1) follows as in the proof of Corollary 1.13 of [42].

To prove (2), we see from [42] and [10] that $\tilde{W}(t,x,y) = W(t,x,|y|)$ is a weak solution to $|y|^a \partial_t \tilde{W} - \operatorname{div}(|y|^a \nabla \tilde{W}) = 0$ in $Q_1 \times (-1, 1)$. Then, by [21], $\tilde{W}$ is locally bounded and controlled by its L^2-norm.

To prove (3), we see that, since the coefficients of the equation in (5.2.11) are smooth in Q_1^*, we can differentiate through to get

$$y^a \partial_t W - y^a \left(\Delta_x W + \frac{a}{y} W_y + W_{yy} \right) = 0 \qquad \text{in } Q_1^*.$$

It is easy to check that $V = y^a W_y$ is a weak solution to

$$\begin{cases} y^{-a}\partial_t V - \operatorname{div}(y^{-a}\nabla V) = 0 & \text{in } Q_1^* \\ V\big|_{y=0} = 0 & \text{on } Q_1 \end{cases}$$

(the test functions for this equation vanish on ∂Q_1^*). Let

$$\tilde{V}(t,x,y) = \begin{cases} V(t,x,y) & \text{for } y > 0 \\ -V(t,x,-y) & \text{for } y \le 0. \end{cases}$$

Then $\tilde{V}$ is a weak solution to the degenerate parabolic equation

$$|y|^a \partial_t \tilde{V} - \operatorname{div}(|y|^a \nabla \tilde{V}) = 0 \qquad \text{for } (t,x,y) \in Q_1 \times (-1,1).$$

Since $|y|^a$ is a Muckenhoupt A_2-weight, it follows that $\tilde{V}$ is locally Hölder continuous [21]. Therefore, $y^a W_y \to 0$ locally uniformly as $y \to 0^+$. Now, by substituting $z = \left(\frac{y}{1-a}\right)^{1-a}$ in the equation for W above, we find that

$$\partial_t W - (\Delta_x W + z^\alpha W_{zz}) = 0$$

for $z > 0$ small, where $\alpha = -\frac{2a}{1-a}$. Additionally, $y^a W_y = W_z$, so that W is differentiable with respect to z up to the boundary $z = 0$, with $W_z\big|_{z=0} = 0$. Next, for $z > 0$ small, by (1) and (2),

$$|W_{zz}| \le \frac{|-\partial_t W + \Delta_x W|}{|z^\alpha|} \le \frac{C}{|z|^\alpha} \|W\|_{L^2(Q_1^*, y^a dX dt)}$$

which in turn implies that, for $z_0 > 0$ small,

$$|W_z(t,x,z_0)| = \left| \int_0^{z_0} W_{zz}(t,x,z)\, dz \right| \le C \|W\|_{L^2(Q_1^*, y^a dX dt)} z_0^{1-\alpha}$$

for all $(t,x) \in Q_{1/2}$. After transforming back to y we get the final result. □

Corollary 5.2.6. *Let W be a weak solution to*

$$\begin{cases} y^a \partial_t W - \operatorname{div}(y^a \nabla W) = 0 & \textit{in } (Q_1^+)^* \\ -y^a W_y\big|_{y=0} = 0 & \textit{on } Q_1^+ \\ W = 0 \quad \textit{or} \quad \partial_{x_n} W = 0 & \textit{on } Q_1^* \cap \{x_n = 0\}. \end{cases}$$

Then Proposition 5.2.5 holds for this W if we replace the cubes Q by half-cubes Q^+ in all the estimates there.

Proof. This is an immediate consequence of Proposition 5.2.5. Indeed, the odd reflection of W with respect to x_n (for Dirichlet boundary condition) and the even reflection of W with respect to x_n (for Neumann boundary condition) are weak solutions to (5.2.11). □

Lemma 5.2.7 (Trace inequality)**.** *There exists a constant $C > 0$, depending only on n and s, such that, for any $U \in L^2((-1,1); H^1_{L,a}(B_1^*))$,*

$$\int_{-r^2}^{r^2} r^{2-2s} \|U(t,\cdot,0)\|^2_{L^2(B_r)}\,dt \leq C \int_{-r^2}^{r^2} y^a \big(\|U(t,\cdot,\cdot)\|^2_{L^2(B_r^*, y^a dX)} + r^2 \|\nabla U(t,\cdot,\cdot)\|^2_{L^2(B_r^*, y^a dX)} \big)\,dt$$

for all $0 < r < 1$. The same is true if we replace B_r by B_r^+.

Proof. The general estimate follows by scaling from the case $r = 1$. From [34], we have that, for a.e $t \in (-1,1)$, $\|U(t,\cdot,0)\|^2_{L^2(B_1)} \leq C \|U(t,\cdot,\cdot)\|^2_{H^1(B_1^*, y^a dX)}$. Then we just integrate in time. □

5.3 Proofs of Interior Regularity Theorems

In this section, we present the proofs of the interior regularity results contained in Theorems 5.1.1 and 5.1.2. We say that a function $f \in L^2(Q_1)$ is in $L^{\alpha/2,\alpha}(0,0)$, for $0 < \alpha \leq 1$, whenever

$$[f]^2_{L^{\alpha/2,\alpha}(0,0)} = \sup_{0<r\leq 1} \frac{1}{r^{n+2+2\alpha}} \int_{Q_r} |f - f(0,0)|^2\,dt\,dx < \infty$$

where $f(0,0) = \lim_{r\to 0} \frac{1}{|Q_r|} \int_{Q_r} f(t,x)\,dt\,dx$. In view of Theorem 4.1.1, we see that if f satisfies this property uniformly in balls centered at points close to the origin then f is parabolically α-Hölder continuous at the origin. Futhermore, Theorem 5.1.1 will follow directly from the following statement after rescaling and translation, and by using estimate (2.4.2).

Theorem 5.3.1. *Let $u \in \mathrm{Dom}(H^s)$ be as in Theorem 5.1.1, with $f \in L^2(\mathbb{R} \times \Omega)$. Suppose that $B_1 \subset \Omega$ and that $f \in L^{\alpha/2,\alpha}(0,0)$, for some $0 < \alpha < 1$.*

(1) *Assume that* $0 < \alpha + 2s < 1$. *There exist* $0 < \delta < 1$, *depending only on* n, *ellipticity,* α *and* s, *and a constant* $C_1 > 0$ *such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r} |A(x) - A(0)|^2 \, dx < \delta^2$$

then there exists a constant c *such that*

$$\frac{1}{r^{n+2}} \int_{Q_r} |u(t,x) - c|^2 \, dt \, dx \leq C_1 r^{2(\alpha+2s)}$$

for all $r > 0$ *small. Moreover,*

$$|c| + C_1^{1/2} \leq C_0(\|u\|_{\mathrm{Dom}(H^s)} + |f(0,0)| + [f]_{L^{\alpha/2,\alpha}(0,0)})$$

where $C_0 > 0$ *depends on* $A(x)$, n, s, α *and ellipticity.*

(2) *Assume that* $1 < \alpha + 2s < 2$. *There exists* $0 < \delta < 1$, *depending only on* n, *ellipticity,* α *and* s, *and a constant* $C_1 > 0$ *such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^{n+2(\alpha+2s-1)}} \int_{B_r} |A(x) - A(0)|^2 \, dx < \delta^2$$

then there exists a linear function $\ell(x) = \mathcal{A} + \mathcal{B} \cdot x$ *such that*

$$\frac{1}{r^{n+2}} \int_{Q_r} |u(t,x) - \ell(x)|^2 \, dt \, dx \leq C_1 r^{2(\alpha+2s)}$$

for all $r > 0$ *small. Moreover,*

$$|\mathcal{A}| + |\mathcal{B}| + C_1^{1/2} \leq C_0(\|u\|_{\mathrm{Dom}(H^s)} + |f(0,0)| + [f]_{L^{\alpha/2,\alpha}(0,0)})$$

where $C_0 > 0$ *depends on* $A(x)$, n, s, α *and ellipticity.*

We say that a function $f \in L^2(Q_1)$ is in $L^{-s+\alpha/2,-2s+\alpha}(0,0)$, for $0 < \alpha < 1$, whenever

$$[f]^2_{L^{-s+\alpha/2,-2s+\alpha}(0,0)} = \sup_{0<r\leq 1} \frac{1}{r^{n+2+2(-2s+\alpha)}} \int_{Q_r} |f(t,x)|^2 \, dt \, dx < \infty$$

and that is in $L^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)$ whenever

$$[f]^2_{L^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)} = \sup_{0<r\leq 1} \frac{1}{r^{n+2+2(-2s+\alpha+1)}} \int_{Q_r} |f(t,x)|^2 \, dt \, dx < \infty.$$

Then we have the following consequences

- If $f \in L^2(Q_1)$ is also in $L^p(Q_1)$, for $(n+2)/(2s) < p < (n+2)/(2s-1)^+$, then we have the estimate $[f]_{L^{-s+\alpha/2,-2s+\alpha}(0,0)} \leq C_n \|f\|_{L^p(Q_1)}$, for $\alpha = 2s - (n+2)/p$.
- If $s > 1/2$ and $f \in L^p(Q_1)$ for $p > (n+2)/(2s-1)$, then $[f]_{L^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)} \leq C_n \|f\|_{L^p(Q_1)}$, for $\alpha = 2s - (n+2)/p - 1$.

In view of these observations, Theorem 5.1.2 will follow immediately from the next result.

Theorem 5.3.2. *Let $u \in \mathrm{Dom}(H^s)$ be as in Theorem 5.1.2, with $f \in L^2(\mathbb{R}\times\Omega)$. Suppose that $B_1 \subset \Omega$ and let $0 < \alpha < 1$.*

(1) *Assume that $f \in L^{-s+\alpha/2,-2s+\alpha}(0,0)$. Then there exist $0 < \delta < 1$, depending only on n, ellipticity, α, s, and a constant $C_1 > 0$ such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r} |A(x) - A(0)|^2 dx < \delta^2$$

then there a exists constant c such that

$$\frac{1}{r^{n+2}} \int_{Q_r} |u(t,x) - c|^2 \, dt \, dx \leq C_1 r^{2\alpha}$$

for all $r > 0$ small. Moreover,

$$|c| + C_1^{1/2} \leq C_0\big(\|u\|_{\mathrm{Dom}(H^s)} + [f]_{L^{-s+\alpha/2,-2s+\alpha}(0,0)}\big)$$

where $C_0 > 0$ depends on $A(x)$, n, s, α and ellipticity.

(2) *Assume that $f \in L^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)$. Then there exist $0 < \delta < 1$, depending only on n, ellipticity, α, s, and a constant $C_1 > 0$ such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^{n+2\alpha}} \int_{B_r} |A(x) - A(0)|^2 \, dx < \delta^2$$

then there exists a linear function $\ell(x) = \mathcal{A} + \mathcal{B}\cdot x$ such that

$$\frac{1}{r^{n+2}} \int_{Q_r} |u(t,x) - \ell(x)|^2 \, dt \, dx \leq C_1 r^{2(1+\alpha)}$$

for all $r > 0$ small. Moreover,

$$|\mathcal{A}| + |\mathcal{B}| + C_1^{1/2} \leq C_0\big(\|u\|_{\mathrm{Dom}(H^s)} + [f]_{L^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)}\big)$$

where $C_0 > 0$ depends on $A(x)$, n, s, α and ellipticity.

Therefore, the rest of this section is devoted to the proofs of Theorems 5.3.1 and 5.3.2.

5.3.1 Proof of Theorem 5.3.1(1)

In view of the extension problem characterization in Theorem 2.4.2, we only need to prove the theorem for $u(t,x) = U(t,x,0)$, where U is a solution to (5.2.3) in Q_1^* with $F \equiv 0$. We will consider normalized solutions U as defined next. Without loss of generality, we can assume that $A(0) = I$ and $f(0,0) = 0$ (otherwise, one needs to take $U - \frac{y^{1-a}}{1-a} f(0,0)$). Given $\delta > 0$, we say that U is a δ-normalized solution if the following conditions hold:

1. $\displaystyle\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r} |A(x) - I|^2 dx < \delta^2$;
2. $\displaystyle[f]^2_{L^{\alpha/2,\alpha}(0,0)} = \sup_{0<r\leq 1} \frac{1}{r^{n+2+2\alpha}} \int_{Q_r} |f|^2 \, dt \, dx < \delta^2$;
3. $\displaystyle\int_{Q_1} U(t,x,0)^2 \, dt \, dx + \int_{Q_1^*} y^a U^2 \, dt \, dX \leq 1.$

Notice that (1) can always be assumed by scaling, while (2) and(3) hold after normalizing

$$U(x,y)\bigg(\int_{Q_1} U(t,x,0)^2 \, dt \, dx + \int_{Q_1^*} y^a U^2 \, dt \, dX + \frac{1}{\delta}[f]^2_{L^{\alpha/2,\alpha}(0,0)} \bigg)^{-1}. \tag{5.3.1}$$

Lemma 5.3.3. *Given $0 < \alpha + 2s < 1$, there exist $0 < \delta, \lambda < 1$ depending on n, s and ellipticity, a constant c and a universal constant $D > 0$ such that, for any δ-normalized solution U to (5.2.3),*

$$\frac{1}{\lambda^{n+2}} \int_{Q_\lambda} |U(t,x,0) - c|^2 \, dt \, dx + \frac{1}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |U - c|^2 \, dt \, dX < \lambda^{2(\alpha+2s)}$$

and $|c| \leq D$.

Proof. Let $0 < \varepsilon < 1$ be fixed. We use Corollary 5.2.3 to get a function W which satisfies (5.2.8). Then, since U is a normalized solution,

$$\int_{Q_{1/2}^*} y^a |W|^2 \, dt \, dX \leq 2 \int_{Q_{1/2}^*} y^a |U - W|^2 \, dt \, dX + 2 \int_{Q_{1/2}^*} y^a U^2 \, dt \, dX \leq 2\varepsilon^2 + 2 \leq 4.$$

Define $c = W(0,0,0)$. Hence, by Proposition 5.2.5(2), we get that $|c| \leq D$, for some universal constant D. Now, for any $(t,X) \in Q_{1/4}^*$, by Proposition 5.2.5,

$$\begin{aligned}|W(t,X) - c| &\leq |W(t,x,y) - W(t,x,0)| + |W(t,x,0) - W(t,0,0)| + |W(t,0,0) - c| \\ &\leq N(y^2 + |x| + |t|) \leq N(|X| + |t|^{1/2})\end{aligned}$$

for some universal constant $N > 0$. Then for any $0 < \lambda < 1/4$,

$$\begin{aligned}
&\frac{1}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |U - c|^2 \, dt \, dX \\
&\quad \le \frac{2}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |U - W|^2 \, dt \, dX + \frac{2}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |W - c|^2 \, dt \, dX \\
&\quad \le \frac{2\varepsilon^2}{\lambda^{n+3+a}} + \frac{2N^2}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a (|X|^2 + |t|) \, dt \, dX \\
&\quad \le \frac{2\varepsilon^2}{\lambda^{n+3+a}} + c_{n,a}\lambda^2.
\end{aligned}$$

Next we apply the trace inequality of Lemma 5.2.7 to $(U - c)$ to get

$$\begin{aligned}
\lambda^{1+a} \int_{Q_\lambda} |U(t,x,0) - c|^2 \, dt \, dx &\le C \int_{Q_\lambda^*} y^a |U - c|^2 \, dt \, dX + C\lambda^2 \int_{Q_\lambda^*} y^a |\nabla U|^2 \, dt \, dX \\
&\le 2C\varepsilon^2 + C c_{n,a} \lambda^{n+5+a} + C\lambda^2 \int_{Q_\lambda^*} y^a |\nabla U|^2 \, dt \, dX.
\end{aligned}$$

Now we estimate the last integral by applying Lemma 5.2.2 to $(U - c)$. For this purpose, take η such that $\eta = 1$ in Q_λ^*, $\eta = 0$ outside $Q_{2\lambda}^*$, and $|\partial_t \eta| + |\nabla \eta| \le \frac{2}{\lambda}$ in $Q_{2\lambda}^*$. Then

$$\begin{aligned}
&\lambda^2 \int_{Q_\lambda^*} y^a |\nabla U|^2 \, dt \, dX \\
&\quad \le C\lambda^2 \left(\int_{Q_{2\lambda}^*} y^a \frac{1}{\lambda^2} |U - c|^2 \, dt \, dX + \int_{Q_{2\lambda}} |U(t,x,0) - c| |f(t,x)| \, dt \, dx \right) \\
&\quad \le C \int_{Q_{2\lambda}^*} y^a |U - c|^2 dt \, dX + C\big(\|U(\cdot,\cdot,0)\|_{L^2(Q_{2\lambda})} + |c| |Q_{2\lambda}|^{1/2} \big) \|f\|_{L^2(Q_{2\lambda})} \\
&\quad \le 2C\varepsilon^2 + C c_{n,a} \lambda^{n+5+a} + C(1 + |c|)\delta.
\end{aligned}$$

Thus, for any $0 < \lambda < 1/8$,

$$\begin{aligned}
&\frac{1}{\lambda^{n+2}} \int_{Q_\lambda} |U(t,x,0) - c|^2 dt \, dx + \frac{1}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |U - c|^2 \, dt \, dX \\
&\qquad\qquad < \frac{C\varepsilon^2}{\lambda^{n+3+a}} + c_{n,a}\lambda^2 + \frac{C\delta}{\lambda^{n+3+a}}.
\end{aligned}$$

Next if we make λ sufficiently small we have $c_{n,a}\lambda^2 \le \frac{1}{3}\lambda^{2(\alpha+2s)}$. Then we can choose ε small such that $\frac{C\varepsilon^2}{\lambda^{n+3+a}} \le \frac{1}{3}\lambda^{2(\alpha+2s)}$. Finally, with this ε in Corollary 5.2.3, we can let δ small enough such that $C(1 + |c|)\delta \le \frac{1}{3}\lambda^{2(\alpha+2s)}$. □

Lemma 5.3.4. *Assume the conditions on Lemma 5.3.3. Then there exist a sequence of constants c_k, $k \geq 0$, and a universal constant $D > 0$ such that*

$$|c_k - c_{k+1}| \leq D\lambda^{k(\alpha+2s)}$$

and

$$\frac{1}{\lambda^{k(n+2)}}\int_{Q_{\lambda^k}} |U(t,x,0) - c_k|^2 dt\, dx + \frac{1}{\lambda^{k(n+3+a)}}\int_{Q^*_{\lambda^k}} y^a|U - c_k|^2 dt\, dX < \lambda^{2k(\alpha+2s)}$$

for all $k \geq 0$.

Proof. We prove this lemma by induction. First we consider the base $k = 0$. We let $c_0 = 0$ and notice that the estimates on U hold because U is a normalized solution. Next, we let c_1 be the constant c from Lemma 5.3.3, so clearly the conclusion holds in this case. Now we assume that the lemma is true for some $k \geq 1$. We define

$$\tilde{U}(t,X) = \frac{U(\lambda^{2k}t, \lambda^k X) - c_k}{\lambda^{k(\alpha+2s)}} \qquad \text{for } (t,X) \in Q^*_1.$$

Recall that, in particular, U satisfies

$$-\int_{Q^*_{\lambda^k}} y^a U \partial_t \phi\, dX\, dt + \int_{Q^*_{\lambda^k}} y^a B(x)\nabla U \nabla \phi\, dX\, dt = \int_{Q_{\lambda^k}} f(t,x)\phi(t,x,0)\, dx\, dt$$

for suitable test functions ϕ. Therefore, by changing variables here, it is easy to see that $\tilde{U}$ satisfies

$$-\int_{Q^*_1} y^a \tilde{U} \partial_t \tilde{\phi}\, dX\, dt + \int_{Q^*_1} y^a \tilde{B}(x)\nabla \tilde{U} \nabla \tilde{\phi}\, dX\, dt = \int_{Q_1} \tilde{f}(t,x)\tilde{\phi}(t,x,0)\, dx\, dt$$

where $\tilde{\phi}(t,X) = \phi(\lambda^{2k}t, \lambda^k X)$, $\tilde{B}(x) = B(\lambda^k x)$, $\tilde{f}(t,x) = \lambda^{-k\alpha} f(\lambda^{2k}t, \lambda^k x)$. Furthermore, $\tilde{A}(0) = I$, $\tilde{f}(0,0) = 0$ and, by changing variables and using the induction hypotheses,

$$\frac{1}{r^n}\int_{B_r} (\tilde{A}(x) - I)^2\, dx + \frac{1}{r^{n+2+2\alpha}}\int_{Q_r} |\tilde{f}(t,x)|^2\, dt\, dx < \delta^2$$

and

$$\int_{Q_1} \tilde{U}(t,x,0)^2 dx\, dt + \int_{Q^*_1} y^a \tilde{U}^2 dX\, dt \leq 1.$$

In other words, $\tilde{U}$ is a δ-normalized weak solution to

$$\begin{cases} y^a \partial_t \tilde{U} - \operatorname{div}(y^a \tilde{B}(x)\nabla \tilde{U}) = 0 & \text{in } Q^*_1 \\ -y^a \tilde{U}_y|_{y=0} = \tilde{f} & \text{on } Q_1. \end{cases}$$

Thus we can apply Lemma 5.3.3 to $\tilde{U}$ to get the existence of a constant c such that

$$\frac{1}{\lambda^{n+2}}\int_{Q_\lambda}|\tilde{U}(t,x,0)-c|^2\,dt\,dx+\frac{1}{\lambda^{n+3+a}}\int_{Q_\lambda^*}y^a|\tilde{U}-c|^2\,dt\,dX<\lambda^{2(\alpha+2s)}.$$

If we change variables back we obtain

$$\frac{1}{\lambda^{(n+2)(k+1)}}\int_{Q_{\lambda^{k+1}}}|U(t,x,0)-c_k-c\lambda^{k(\alpha+2s)}|^2\,dt\,dx \\ +\frac{1}{\lambda^{(n+3+a)(k+1)}}\int_{Q_{\lambda^{k+1}}}y^a|U-c_k-c\lambda^{k(\alpha+2s)}|^2\,dt\,dX<\lambda^{2(k+1)(\alpha+2s)}.$$

Defining $c_{k+1}=c_k+\lambda^{k(\alpha+2s)}c$ we see that $|c_{k+1}-c_k|\le D\lambda^{k(\alpha+2s)}$. □

Proof of Theorem 5.3.1(1). If $\{c_k\}_{k\ge0}$ is the sequence of constants from Lemma 5.3.4 then we see that $c_\infty=\lim_{k\to\infty}c_k$ exists and is finite. Indeed, to show that $\{c_k\}_{k\ge0}$ is a Cauchy sequence of real numbers, let $m,k\ge0$ and suppose that $m=k+j$ for some $j\ge1$. Then

$$\begin{aligned}|c_k-c_m|=|c_k-c_{k+j}|&\le\sum_{\ell=0}^{j-1}|c_{k+\ell}-c_{k+\ell+1}|\\ &\le D\sum_{\ell=0}^{j-1}\lambda^{(k+\ell)(\alpha+2s)}\le D\lambda^{k(\alpha+2s)}\sum_{\ell=0}^{\infty}\lambda^{\ell(\alpha+2s)}\\ &\le C(D,\lambda,\alpha,s)\lambda^{k(\alpha+2s)}\to0\qquad\text{as }k\to\infty.\end{aligned}$$

Given any $0<r<1/8$, let $k\ge0$ such that $\lambda^{k+1}<r\le\lambda^k$. Then, by Lemma 5.3.4,

$$\begin{aligned}&\frac{1}{r^{n+2}}\int_{Q_r}|U(t,x,0)-c_\infty|^2\,dt\,dx\\ &\quad\le\frac{2}{r^{n+2}}\int_{Q_r}|U(t,x,0)-c_k|^2\,dt\,dx+2C_n|c_k-c_\infty|^2\\ &\quad\le\frac{2}{\lambda^{n+2}}\frac{1}{\lambda^{k(n+2)}}\int_{Q_{\lambda^k}}|U(t,x,0)-c_k|^2\,dt\,dx+\frac{C_n}{(1-\lambda^{\alpha+2s})^2}D^2\lambda^{2k(\alpha+2s)}\le C_1r^{2(\alpha+2s)}\end{aligned}$$

where $C_1=C_1(n,\lambda,D,\alpha,s)>0$. □

5.3.2 Proof of Theorem 5.3.1(2)

As before, we will prove Theorem 5.3.1(2) for $u(t,x)=U(t,x,0)$, where U is a solution to (5.2.3) in Q_1^*. We will consider normalized solutions U as defined next. Again, without loss of generality, we can assume that $A(0) = I$ and $f(0,0) = 0$. Given $\delta > 0$, we say that U is a δ-normalized solution (with F not identically 0) if the following conditions hold:

1. $\displaystyle\sup_{0<r\leq 1} \frac{1}{r^{n+2(\alpha+2s-1)}} \int_{B_r} |A(x) - I|^2 dx < \delta^2$;

2. $\displaystyle[f]^2_{L^{\alpha/2,\alpha}(0,0)} = \sup_{0<r\leq 1} \frac{1}{r^{n+2+2\alpha}} \int_{Q_r} |f|^2 \, dt \, dx < \delta^2$;

3. $\displaystyle\sup_{0<r\leq 1} \frac{1}{r^{n+3+a+2(\alpha+2s-1)}} \int_{Q_r^*} y^a |F|^2 \, dt \, dX < \delta^2$;

4. $\displaystyle\int_{Q_1} U(t,x,0)^2 \, dt \, dx + \int_{Q_1^*} y^a U^2 \, dt \, dX \leq 1$.

Notice that (1) can always be assumed by scaling, and (2), (3) and (4) hold after an appropriate normalization, see (5.3.1).

Lemma 5.3.5. *Given $1 < \alpha + 2s < 2$, there exist $0 < \delta, \lambda < 1$ depending on n, s and ellipticity, a linear function $\ell(x) = \mathcal{A} + \mathcal{B} \cdot x$ and a universal constant $D > 0$ such that for any δ-normalized solution U to (5.2.3),*

$$\frac{1}{\lambda^{n+2}} \int_{Q_\lambda} |U(t,x,0) - \ell(x)|^2 \, dt \, dx + \frac{1}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |U - \ell(x)|^2 \, dt \, dX < \lambda^{2(\alpha+2s)}$$

and $|\mathcal{A}| + |\mathcal{B}| \leq D$.

Proof. Let $0 < \varepsilon < 1$. Then, as in Lemma 5.3.3, there exists a function W which satisfies Corollary 5.2.3, the smoothness estimates of Proposition 5.2.5 and also

$$\int_{Q_{1/2}^*} y^a |W|^2 \, dt \, dX \leq 4.$$

Now define

$$\ell(x) = W(0,0,0) + \nabla_x W(0,0,0) \cdot x = \mathcal{A} + \mathcal{B} \cdot x.$$

By Proposition 5.2.5, there exists a universal constant D such that $|\mathcal{A}| + |\mathcal{B}| \leq D$. Next, for any $(t, X) \in Q^*_{1/4}$ we have, for some universal constant $N > 0$,

$$\begin{aligned} |W(t,x,y) - \ell(x)| &\leq |W(t,x,y) - W(t,x,0)| + |W(t,x,0) - W(0,x,0)| \\ &\quad + |W(0,x,0) - W(0,0,0) - \nabla_x W(0,0,0) \cdot x| \\ &\leq C|W_y(t,x,\xi)|y + Ct + C|x|^2 \\ &\leq C\xi y + Ct + C|x|^2 \leq N(|X|^2 + t) \end{aligned}$$

where we used the mean value theorem for some $0 \leq \xi \leq y$ and Proposition 5.2.5(3). Then, for any $0 < \lambda < 1/4$,

$$\begin{aligned} &\frac{1}{\lambda^{n+3+a}} \int_{Q^*_\lambda} y^a |U - \ell(x)|^2 \, dt \, dX \\ &\quad \leq \frac{2}{\lambda^{n+3+a}} \int_{Q^*_\lambda} y^a |U - W|^2 \, dt \, dX + \frac{2}{\lambda^{n+3+a}} \int_{Q^*_\lambda} y^a |W - \ell(x)|^2 \, dt \, dX \\ &\quad \leq \frac{2\varepsilon^2}{\lambda^{n+3+a}} + \frac{2N^2}{\lambda^{n+3+a}} \int_{Q^*_\lambda} y^a (|X|^4 + |t|^2) \, dt \, dX \\ &\quad \leq \frac{2\varepsilon^2}{\lambda^{n+3+a}} + c_{n,a} \lambda^4. \end{aligned}$$

In the next step, we apply the trace inequality (Lemma 5.2.7) to $U - \ell$. Hence, for $0 < \lambda < 1/8$,

$$\begin{aligned} &\lambda^{1+a} \int_{Q_\lambda} |U(t,x,0) - \ell(x)|^2 \, dt \, dx \\ &\qquad \leq C \int_{Q^*_\lambda} y^a |U - \ell(x)|^2 \, dt \, dX + C\lambda^2 \int_{Q^*_\lambda} y^a |\nabla(U - \ell)|^2 \, dt \, dX. \end{aligned}$$

Observe that $V = U - \ell$ is a weak solution to

$$\begin{cases} y^a \partial_t V - \operatorname{div}(y^a B(x) \nabla V) = -\operatorname{div}(y^a (F + G)) & \text{in } Q^*_1 \\ -y^a V_y|_{y=0} = f & \text{on } Q_1 \end{cases}$$

where the vector field G is given by $G = ((I - A(x))\nabla_x \ell, 0)$ and $G(0) = 0$. Thus, by Lemma 5.2.2,

$$\begin{aligned}
\int_{Q_\lambda^*} |\nabla(U-\ell)|^2 y^a \, dt \, dX &\leq C \int_{Q_{2\lambda}^*} y^a \left(\frac{1}{\lambda^2}|U-\ell|^2 + |F+G|^2 \right) dt \, dX \\
&\quad + C \int_{Q_{2\lambda}} |U(t,x,0) - \ell(x)||f(t,x)| \, dt \, dx \\
&\leq \frac{C}{\lambda^2} \int_{Q_{2\lambda}^*} y^a |U-\ell|^2 \, dt \, dX + C \, \|F+G\|^2_{L^2(Q_{2\lambda}^*, y^a dt dX)} \\
&\quad + C\big(\|U(\cdot,\cdot,0)\|_{L^2(Q_{2\lambda})} + \|\ell\|_{L^2(Q_{2\lambda})} \big) \, \|f\|_{L^2(Q_{2\lambda})} \\
&\leq \frac{C}{\lambda^2} \int_{Q_{2\lambda}^*} y^a |U-\ell|^2 \, dt \, dX + C\delta^2 \lambda^{n+3+a+2(\alpha+2s-1)} \\
&\quad + C(1+D)\delta \\
&\leq \frac{C}{\lambda^2} \int_{Q_{2\lambda}^*} y^a |U-\ell|^2 \, dt \, dX + C\delta.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{\lambda^{n+2}} \int_{Q_\lambda} |U(t,x,0) - \ell(x)|^2 \, dt \, dx &+ \frac{1}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |U - \ell(x)|^2 \, dt \, dX \\
&\leq \frac{C\varepsilon^2}{\lambda^{n+3+a}} + c_{n,a}\lambda^4 + \frac{C\delta}{\lambda^{n+3+a}} < \lambda^{2(\alpha+2s)}
\end{aligned}$$

where the last inequality follows by first choosing λ small, then ε sufficiently small and, for this $\varepsilon > 0$, a $0 < \delta < 1$ in Lemma 5.2.3 small enough. □

Lemma 5.3.6. *Assume the conditions on Lemma 5.3.5. Then there exist a sequence of linear functions $\ell_k(x) = \mathcal{A}_k + \mathcal{B}_k \cdot x$, $k \geq 0$, and a universal constant $D > 0$ such that*

$$|\mathcal{A}_k - \mathcal{A}_{k+1}| + \lambda^k |\mathcal{B}_k - \mathcal{B}_{k+1}| \leq D\lambda^{k(\alpha+2s)}$$

and

$$\frac{1}{\lambda^{k(n+2)}} \int_{Q_{\lambda^k}} |U(t,x,0) - \ell_k|^2 \, dt \, dx + \frac{1}{\lambda^{k(n+3+a)}} \int_{Q_{\lambda^k}^*} y^a |U - \ell_k|^2 \, dt \, dX < \lambda^{2k(\alpha+2s)}$$

for all $k \geq 0$.

Proof. The proof is by induction. For the base step $k = 0$, we set $\ell_0(x) = 0$ and hence the estimates on U are true because U is a δ-normalized solution. For $k = 1$ we choose $\ell_1(x) = \ell(x)$ from Lemma

5.3.5 and obviously the conclusion holds. Suppose the result is true for some $k \geq 1$. Define

$$\tilde{U}(t,X) = \frac{U(\lambda^{2k}t, \lambda^k X) - \ell_k(\lambda^k x)}{\lambda^{k(\alpha+2s)}} \qquad \text{for } (t,X) \in Q_1^*.$$

Recall that U satisfies

$$\int_{Q^*_{\lambda^k}} y^a U \partial_t \phi \, dt \, dX + \int_{Q^*_{\lambda^k}} y^a B(x) \nabla U \nabla \phi \, dt \, dX$$
$$= \int_{Q^*_{\lambda^k}} y^a F \nabla \phi \, dt \, dX + \int_{Q_{\lambda^k}} f(t,x)\phi(t,x,0) \, dt \, dx$$

for suitable test functions ϕ. Now, by the change of variables $X = \lambda^k X, t = \lambda^{2k} t$, we find that $\tilde{U}$ is a weak solution to

$$\begin{cases} y^a \partial_t \tilde{U} - \operatorname{div}(y^a \tilde{B}(x) \nabla \tilde{U}) = -\operatorname{div}(y^a(\tilde{F} + \tilde{G})) & \text{in } Q_1^* \\ -y^a \tilde{U}_y|_{y=0} = \tilde{f} & \text{on } Q_1 \end{cases}$$

where $\tilde{B}(x) = B(\lambda^k x)$, $\tilde{F}(t,X) = \lambda^{-k(\alpha+2s-1)} F(\lambda^{2k}t, \lambda^k X)$, $\tilde{f}(t,x) = \lambda^{-k\alpha} f(\lambda^{2k}t, \lambda^k x)$ and

$$\tilde{G} = \left(\frac{I - \tilde{B}(x)}{\lambda^{k(\alpha+2s-1)}} \nabla_x \ell_k(\lambda^k x), 0 \right) \qquad \text{with } \tilde{G}(0) = 0.$$

Moreover, by the hyptheses on f, $A(x)$ and F,

$$\frac{1}{r^{n+2+2\alpha}} \int_{Q_r} |\tilde{f}|^2 \, dt \, dx < \delta^2$$

and

$$\frac{1}{r^{n+3+a+2(\alpha+2s-1)}} \int_{Q_r^*} y^a |\tilde{F} + \tilde{G}|^2 \, dt \, dX$$
$$\leq \frac{2}{(\lambda^k r)^{n+3+a+2(\alpha+2s-1)}} \int_{Q^*_{\lambda^k r}} y^a (|F|^2 + |I - B(x)|^2 |\mathcal{B}_k|^2) \, dt \, dX$$
$$\leq 2(1 + D^2 C^2)\delta^2$$

where we used that

$$|\mathcal{B}_k| \leq \sum_{j=1}^{k} |\mathcal{B}_j - \mathcal{B}_{j-1}| \leq D \sum_{j=0}^{\infty} \lambda^{j(\alpha+2s-1)} \leq DC$$

Additionally, by changing variables and the induction hypothesis,

$$\int_{Q_1} \tilde{U}(t,x,0)\,dt\,dx + \int_{Q_1^*} y^a \tilde{U}^2\,dt\,dX \le 1$$

so that $\tilde{U}$ is a δ-normalized solution. Whence, by Lemma 5.3.5, there exists a linear function $\ell(x)$ such that

$$\frac{1}{\lambda^{n+2}} \int_{Q_\lambda} |\tilde{U}(t,x,0) - \ell(x)|^2\,dt\,dx + \frac{1}{\lambda^{n+3+a}} \int_{Q_\lambda^*} y^a |\tilde{U} - \ell|^2\,dt\,dX < \lambda^{2(\alpha+2s)}.$$

By changing variables back,

$$\frac{1}{\lambda^{(k+1)(n+2)}} \int_{Q_{\lambda^{k+1}}} |U(t,x,0) - \ell_{k+1}(x)|^2\,dt\,dx \\ + \frac{1}{\lambda^{(k+1)(n+3+a)}} \int_{Q^*_{\lambda^{k+1}}} y^a |U - \ell_{k+1}|^2\,dt\,dX < \lambda^{2(k+1)(\alpha+2s)}$$

where $\ell_{k+1}(x) = \ell_k(x) + \lambda^{k(\alpha+2s)}\ell(\lambda^{-k}x)$. Then

$$|\ell_{k+1}(x) - \ell_k(x)| = \lambda^{k(\alpha+2s)} |\ell(\lambda^{-k}x)| \le D\lambda^{k(\alpha+2s)}(1 + \lambda^{-k}|x|)$$

so that $|\mathcal{A}_{k+1} - \mathcal{A}_k| = |\ell_{k+1}(0) - \ell_k(0)| \le D\lambda^{k(\alpha+2s)}$ and, by construction, $|\mathcal{B}_{k+1} - \mathcal{B}_k| \le \lambda^{k(\alpha+2s-1)}|\mathcal{B}| \le D\lambda^{k(\alpha+2s-1)}$ □

Proof of Theorem 5.3.1(2). It follows the same procedure as the proof of Theorem 5.3.1(1), but instead we need to use now Lemmas 5.3.5 and 5.3.6. □

5.3.3 Proof of Theorem 5.3.2

The proof follows very similar lines to those for Theorem 5.3.1 with minor changes.

Proof. Indeed, in the proof of Theorem 5.3.1(1) we need to replace the exponent α by $-2s + \alpha$, while in the proof of Theorem 5.3.1(2) we substitute the exponent α by $-2s + \alpha + 1$. Notice also that we do not need the normalization $f(0,0) = 0$. □

In the next section we will discuss about the boundary regularity for fractional heat equations. The analysis on the boundary regularity for fractional heat will provide us important tools which will be useful later discussing the boundary regularity for fractional parabolic equations.

5.4 Boundary Regularity for Fractional Heat Equations

In this section we perform a detailed analysis of boundary regularity and asymptotic behavior of half space solutions for Master equations driven by fractional powers of heat operators. First we state known estimates for the fractional heat operator from [42]. In the following we let $\Lambda^{1/2,1}(\mathbb{R}^{n+1})$ be the Hölder–Zygmund space of continuous functions $u = u(t,x)$ such that the norm

$$\|u\|_{\Lambda^{1/2,1}(\mathbb{R}^{n+1})} = \|u\|_{L^\infty(\mathbb{R}^{n+1})} + \sup_{(t,x),(\tau,z)\in\mathbb{R}^{n+1}} \frac{|u(\tau,x-z)+u(\tau,x+z)-2u(t,x)|}{|t-\tau|^{1/2}+|z|}$$

is finite.

Proposition 5.4.1. *Let $u, f \in L^\infty(\mathbb{R}^{n+1})$ be such that*

$$(\partial_t - \Delta)^s u = f \qquad \text{in } \mathbb{R}^{n+1}.$$

(1) *Suppose that $f \in C^{\alpha/2,\alpha}(\mathbb{R}^{n+1})$ for $0 < \alpha \leq 1$.*

(a) *If $\alpha + 2s$ is not an integer then $u \in C^{\alpha/2+s,\alpha+2s}(\mathbb{R}^{n+1})$, with the estimate*

$$\|u\|_{C^{\alpha/2+s,\alpha+2s}(\mathbb{R}^{n+1})} \leq C\big(\|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}^{n+1})} + \|u\|_{L^\infty(\mathbb{R}^{n+1})}\big).$$

(b) *If $\alpha + 2s = 1$ then $u(t,x)$ is in the Hölder–Zygmund space $\Lambda^{1/2,1}(\mathbb{R}^{n+1})$, with the estimate*

$$\|u\|_{\Lambda^{1/2,1}(\mathbb{R}^{n+1})} \leq C\big(\|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}^{n+1})} + \|u\|_{L^\infty(\mathbb{R}^{n+1})}\big).$$

The constants $C > 0$ above depend only on n, s and α.

(2) *Suppose that $f \in L^\infty(\mathbb{R}^{n+1})$.*

(a) *If $s \neq 1/2$ then $u \in C^{s,2s}(\mathbb{R}^{n+1})$, with the estimate*

$$\|u\|_{C^{s,2s}(\mathbb{R}^{n+1})} \leq C\big(\|f\|_{L^\infty(\mathbb{R}^{n+1})} + \|u\|_{L^\infty(\mathbb{R}^{n+1})}\big).$$

(b) *If $s = 1/2$ then u is in the Hölder-Zygmund space $\Lambda^{1/2,1}(\mathbb{R}^{n+1})$, with the estimate*

$$\|u\|_{\Lambda^{1/2,1}(\mathbb{R}^{n+1})} \leq C\big(\|f\|_{L^\infty(\mathbb{R}^{n+1})} + \|u\|_{L^\infty(\mathbb{R}^{n+1})}\big).$$

The constants $C > 0$ above depend only on n and s.

5.4.1 Boundary Regularity in Half Space – Dirichlet

In the half space $\mathbb{R} \times \mathbb{R}^n_+$ we consider the heat operator $\partial_t - \Delta^+_D$, where Δ^+_D is the Dirichlet Laplacian in $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. For a function $u(t,x)$ defined on $\mathbb{R} \times \overline{\mathbb{R}^n_+}$ with $u(t,x',0) = 0$ and $0 < s < 1$ we define

$$(\partial_t - \Delta^+_D)^s u(t,x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{\tau\Delta^+_D} u(t-\tau,x) - u(t,x)\right) \frac{d\tau}{\tau^{1+s}}$$

where $\{e^{\tau\Delta^+_D}\}_{\tau\geq 0}$ is the semigroup generated by Δ^+_D. Let $x^* = (x', -x_n)$ for $x \in \mathbb{R}^n$ and $u_0(t,x)$ be the odd extension of $u(t,x)$ about the x_n axis given by

$$u_0(t,x) = \begin{cases} u(t,x) & \text{if } x_n \geq 0 \\ -u(t,x^*) = -u(t,x',-x_n) & \text{if } x_n < 0. \end{cases}$$

Now

$$\begin{aligned} e^{\tau\Delta^+_D} u(t-\tau,x) &= e^{\tau\Delta} u_0(t-\tau,x) \\ &= \frac{1}{(4\pi\tau)^{n/2}} \int_{\mathbb{R}^n_+} \left(e^{-|x-z|^2/(4\tau)} - e^{-|x-z^*|^2/(4\tau)}\right) u(t-\tau,z)\, dz \end{aligned}$$

for any $\tau > 0$, $x \in \mathbb{R}^n_+$. Hence, for $x \in \mathbb{R}^n_+$,

$$(\partial_t - \Delta^+_D)^s u(t,x) = \frac{1}{(4\pi)^{n/2}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n_+} \left(\frac{e^{-|x-z|^2/(4\tau)} - e^{-|x-z^*|^2/(4\tau)}}{\tau^{n/2+1+s}}\right) (u(t-\tau,z) - u(t,x))\, dz\, d\tau$$

and

$$\begin{aligned} (\partial_t - \Delta^+_D)^{-s} f(t,x) &= \frac{1}{\Gamma(s)} \int_0^\infty e^{\tau\Delta^+_D} f(t-\tau,x) \frac{d\tau}{\tau^{1-s}} \\ &= \frac{1}{(4\pi)^{n/2}\Gamma(s)} \int_0^\infty \int_{\mathbb{R}^n_+} \left(\frac{e^{-|x-z|^2/(4\tau)} - e^{-|x-z^*|^2/(4\tau)}}{\tau^{n/2+1-s}}\right) f(t-\tau,z)\, dz\, d\tau. \end{aligned}$$

Theorem 5.4.2 (Boundary regularity in half space – Dirichlet)**.** *Let $u, f \in L^\infty(\mathbb{R} \times \mathbb{R}^n_+)$ be such that*

$$\begin{cases} (\partial_t - \Delta^+_D)^s u = f & \text{in } \mathbb{R} \times \mathbb{R}^n_+ \\ u = 0 & \text{on } \mathbb{R} \times \partial\mathbb{R}^n_+ = \mathbb{R} \times \{x \in \mathbb{R}^n : x_n = 0\}. \end{cases}$$

(1) *Suppose that* $f \in C^{\alpha/2,\alpha}(\mathbb{R}\times\overline{\mathbb{R}^n_+})$ *for some* $0<\alpha\le 1$*. In addition, assume that* $f(t,x',0)=0$*, for all* $t\in\mathbb{R}$*,* $x'\in\mathbb{R}^{n-1}$*.*

(a) *If* $\alpha+2s$ *is not an integer then* $u \in C^{\alpha/2+s,\alpha+2s}(\mathbb{R}\times\overline{\mathbb{R}^n_+})$*, with the estimate*

$$\|u\|_{C^{\alpha/2+s,\alpha+2s}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \le C\big(\|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})}\big).$$

(b) *If* $\alpha+2s=1$ *then* $u(t,x)$ *is in the Hölder–Zygmund space* $\Lambda^{1/2,1}(\mathbb{R}\times\overline{\mathbb{R}^n_+})$*, with the estimate*

$$\|u\|_{\Lambda^{1/2,1}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \le C\big(\|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})}\big).$$

The constants $C>0$ *above depend only on* n*,* s *and* α*.*

(2) *Let* $f\in L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})$*.*

(a) *If* $s\neq 1/2$ *then* $u\in C^{s,2s}(\mathbb{R}\times\overline{\mathbb{R}^n_+})$ *with the estimate*

$$\|u\|_{C^{s,2s}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \le C\left(\|f\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})}\right)$$

(b) *If* $s=1/2$ *then* u *is in parabolic Hölder-Zygmund space* $\Lambda^{1/2,1}(\mathbb{R}\times\overline{\mathbb{R}^n_+})$ *with the estimate*

$$\|u\|_{\Lambda^{1/2,1}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \le C\left(\|f\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})}\right)$$

The constants $C>0$ *above depend only on* n *and* s*.*

Proof. This result follows by observing that if f_0 and u_0 are the odd reflections of f and u with respect to the variable x_n, respectively, then $(\partial_t-\Delta)^s u_0 = f_0$ in $\mathbb{R}^{n+1}$. Thus we can invoke Proposition 5.4.1. From the pointwise formula we see that $(\partial_t-\Delta)^s u_0(t,x) = (\partial_t-\Delta_D^+)^s u(t,x) = f(t,x) = f_0(t,x)$ when $x\in\mathbb{R}^n_+$. Now, for some (t,x) such that $x_n<0$ we have

$$\begin{aligned}(\partial_t-\Delta)^s u_0(t,x) &= \frac{1}{\Gamma(-s)}\int_0^\infty \big(e^{\tau\Delta}u_0(t,x)-u_0(t,x)\big)\frac{d\tau}{\tau^{1+s}}\\ &= \frac{1}{\Gamma(-s)}\int_0^\infty \Big(u(t,x^*)-e^{\tau\Delta_D^+}u(t,x^*)\Big)\frac{d\tau}{\tau^{1+s}}\\ &= -(\partial_t-\Delta_D^+)^s u(t,x^*) = -f(t,x^*) = f_0(t,x).\end{aligned}$$

Also we can see that if (t,x) is such that $x_n=0$ then $u_0(t,x)=0$ and

$$(\partial_t-\Delta)^s u_0(t,x) = \frac{1}{(4\pi)^{n/2}|\Gamma(-s)|}\int_0^\infty\int_{\mathbb{R}^n}\frac{e^{-|z|^2/(4\tau)}}{\tau^{n/2+1+s}}u_0(t-\tau,x-z)\,dz\,d\tau = 0$$

because $u_0(t-\tau,x-z)$ is an odd function in the variable z_n. ☐

5.4.2 Boundary Behavior in Half Space – Dirichlet

We collect some particular one dimensional pointwise solutions that will be useful later. Consider the problem

$$\begin{cases} (\partial_t - D^+_{xx})^s u = f & \text{in } \mathbb{R}\times\mathbb{R}_+ \\ u(t,0) = 0 & \text{in } \mathbb{R} \end{cases}$$

where D^+_{xx} denotes the Dirichlet Laplacian in the half line $[0,\infty)$ and

$$f(t,x) = \begin{cases} 1 & \text{when } 0 < s < 1/2 \\ \chi_{[0,1]}(x) & \text{when } 1/2 \leq s < 1. \end{cases}$$

Since f is independent of t, we have that u is also independent of t and solves

$$\begin{cases} (-D^+_{xx})^s u = f & \text{in } \mathbb{R}_+ \\ u(0) = 0. \end{cases}$$

Then we have the following results (see also [17]).

Case 1: $0 < s < 1/2$. There exists a constant $c_s > 0$ such that

$$u(t,x) = c_s x^{2s} \qquad \text{for } (t,x) \in \mathbb{R}\times\mathbb{R}^+.$$

Case 2: $s = 1/2$. We have

$$\begin{aligned} u(t,x) &= c\int_0^1 \big(\log|x+z| - \log|x-z|\big)\,dz \\ &= cx\int_0^{1/x} \big(\log|1+\omega| - \log|1-\omega|\big)\,d\omega. \end{aligned}$$

For $0 < x < 1$,

$$\begin{aligned} u(t,x) &= cx\int_0^1 \big(\log(1+\omega) - \log(1-\omega)\big)\,d\omega + cx\int_1^{1/x} \big(\log(1+\omega) - \log(\omega-1)\big)\,d\omega \\ &= c\big((1+x)\log(1+x) - (1-x)\log(1-x) - 2x\log x\big). \end{aligned}$$

Hence, there exists $C > 0$ such that, for any $0 < x < 1$,

$$u(t,x) = -Cx\log x + \eta_1(x)$$

where $\eta_1(x) \sim x$ as $x \to 0$. Therefore,

$$u(t,x) \sim -x \log x \qquad \text{as } x \to 0, \text{ uniformly in } t \in \mathbb{R}.$$

On the other hand, if $x \geq 1$ then,

$$\begin{aligned} u(t,x) &= cx \int_0^{1/x} \big(\log(1+\omega) - \log(1-\omega)\big)\, d\omega \\ &= cx\big[(1/x+1)\log(1/x+1) + (1-1/x)\log(1-1/x)\big]. \end{aligned}$$

Hence, for any $x \geq 1$,

$$u(t,x) = x\eta_2\Big(\frac{1}{x}\Big)$$

where

$$\eta_2(x) = c\big[(1+x)\log(1+x) + (1-x)\log(1-x)\big].$$

To study the behavior of $u(t,x)$ near infinity we need to study the behavior of $\eta_2(x)$ near 0. Using the series expansion for $\log(1 \pm x)$ we see that $\eta_2(x) \sim x^2$ as $x \to 0$. Therefore,

$$u(t,x) \sim \frac{1}{x} \qquad \text{as } x \to \infty.$$

Case 3: $1/2 < s < 1$**.** We have

$$\begin{aligned} u(t,x) &= c \int_0^1 \Big(|x-z|^{2s-1} - |x+z|^{2s-1}\Big)\, dz \\ &= cx^{2s} \int_0^{1/x} \Big(|1-\omega|^{2s-1} - (1+\omega)^{2s-1}\Big)\, d\omega. \end{aligned}$$

Let us consider $0 < x < 1$. Then

$$\begin{aligned} u(t,x) &= cx^{2s}\Big[\int_0^1 (1-\omega)^{2s-1}\, d\omega + \int_1^{1/x} (\omega-1)^{2s-1}\, d\omega - \int_0^{1/x} (1+\omega)^{2s-1}\, d\omega\Big] \\ &= c_s\big[2x^{2s} + (1-x)^{2s} - (1+x)^{2s}\big]. \end{aligned}$$

On the other hand, if $x \geq 1$, then

$$\begin{aligned} u(t,x) &= cx^{2s} \int_0^{1/x} \Big((1-\omega)^{2s-1} - (1+\omega)^{2s-1}\Big)\, d\omega \\ &= c_s x^{2s}\big[2 - (1-1/x)^{2s} - (1+1/x)^{2s}\big]. \end{aligned}$$

Whence, there exists $c_s > 0$ such that

$$u(t,x) = \begin{cases} 2c_s x^{2s} + \eta_{s1}(x) & \text{if } 0 < x < 1, \\ c_s x^{2s}\Big(2 - \eta_{s2}\Big(\frac{1}{x}\Big)\Big) & \text{if } x \geq 1, \end{cases}$$

where η_{s1} and η_{s2} are smooth up to $x = 0$. Using the series expansions of $(1 \pm x)^{2s}$, we get

$$\eta_{s1}(x) \sim -4sx \quad \text{and} \quad \eta_{s2}(x) \sim 2 + 2s(2s-1)x^2 \qquad \text{as } x \to 0. \tag{5.4.1}$$

Using these estimates we conclude that

$$u(t,x) \sim x \qquad \text{as } x \to 0, \text{ uniformly in } t \in \mathbb{R},$$

and

$$u(t,x) \sim x^{2s-2} \qquad \text{as } x \to \infty, \text{ uniformly in } t \in \mathbb{R}.$$

Consider next the problem in a higher dimensional half space

$$\begin{cases} (\partial_t - \Delta_D^+)^s w = g & \text{in } \mathbb{R} \times \mathbb{R}^n_+ \\ w(t,x',0) = 0 & \text{on } \mathbb{R} \times \partial\mathbb{R}^n_+ \end{cases} \tag{5.4.2}$$

where

$$g(t,x) = \begin{cases} 1 & \text{when } 0 < s < 1/2 \\ \chi_{[0,1]}(x_n) & \text{when } 1/2 \leq s < 1. \end{cases} \tag{5.4.3}$$

The study of these solutions relies on the following observation. Suppose that $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is a function depending only on the x_n variable, that is, $g(t,x) = \phi(x_n)$ for some function $\phi : \mathbb{R} \to \mathbb{R}$, for all $(t,x) \in \mathbb{R}^{n+1}$. Let w satisfy

$$(\partial_t - \Delta)^s w = g \qquad \text{in } \mathbb{R}^{n+1}.$$

Then w is a function that depends only on x_n. More precisely, $w(t,x) = \psi(x_n)$ for all $(t,x) \in \mathbb{R}^{n+1}$, where $\psi : \mathbb{R} \to \mathbb{R}$ solves the one dimensional problem

$$(-D_{xx})^s \psi = \phi \qquad \text{in } \mathbb{R}.$$

Indeed, that w does not depend on t is clear because g does not depend on t. Then w will satisfy $(-\Delta)^s w = g$ and therefore the conclusion follows as in [17].

Thus, the pointwise solution $w(t,x)$ to (5.4.2) with g as in (5.4.3) will be

$$w(t,x) = \begin{cases} c_s x_n^{2s} & \text{if } 0 < s < 1/2, \\ -Cx_n \log x_n + \eta_1(x_n) & \text{for } 0 < x_n < 1, \text{ if } s = 1/2, \\ x_n \eta_2\left(\frac{1}{x_n}\right) & \text{for } x_n \geq 1, \text{ if } s = 1/2, \\ 2c_s x_n^{2s} + \eta_{s1}(x_n) & \text{for } 0 < x_n < 1, \text{ if } 1/2 < s < 1 \\ c_s x_n^{2s}\left(2 - \eta_{s2}\left(\frac{1}{x_n}\right)\right) & \text{for } x_n \geq 1, \text{ if } 1/2 < s < 1 \end{cases} \tag{5.4.4}$$

for some constants $c_s, C > 0$.

Now, if we consider the following extension problem

$$\begin{cases} y^a \partial_t W - \text{div}(y^a \nabla W) = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n_+ \times (0,\infty) \\ -y^a W_y\big|_{y=0} = \theta g & \text{on } \mathbb{R} \times \mathbb{R}^n_+ \\ W = 0 & \text{on } \mathbb{R} \times \partial\mathbb{R}^n_+ \times [0,\infty) \end{cases} \tag{5.4.5}$$

with g as in (5.4.3) and $\theta \in \mathbb{R}$, then the pointwise solution $W(t,x,y)$ will satisfy

$$W(t,x,0) = \theta w(t,x) \qquad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^n_+$$

where $w(t,x)$ is as in (5.4.4). Though these solutions W can be computed explicitly, we will only need bounds for them and their derivatives in the x_n-direction (see the proof of the following Lemmas).

Lemma 5.4.3. *The solution $W(t,x,y)$ to (5.4.5) satisfies the following estimates.*

(1) *If $s < 1/2$ then $|W(t,x,y)| \leq C|\theta| x_n^{2s}$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R}^n_+ \times (0,\infty)$, where $C > 0$ depends only on s.*

(2) *If $s \geq 1/2$ then $\|W\|_{L^\infty(\mathbb{R}\times\mathbb{R}^n_+\times(0,\infty))} \leq C|\theta|$, where $C > 0$ depends only on s.*

Proof. After dividing by θ, we can assume that $\theta = 1$. Recall that the solution W to (5.4.5) is given by

$$W(t,x,y) = \frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-y^2/(4\tau)}e^{\tau\Delta}w_o(t-\tau,x)\,\frac{d\tau}{\tau^{1+s}}$$

where w_o denotes the odd reflection of w with respect to the x_n variable.

Consider first the case of $s < 1/2$. Then $w(t,x) = c_s x_n^{2s}$ and

$$\begin{aligned}
&e^{\tau\Delta}w_o(t-\tau,x)\\
&\quad= \frac{C}{\tau^{1/2}}\bigg[\int_{-\infty}^{x_n} e^{-z_n^2/(4\tau)}(x_n-z_n)^{2s}\,dz_n - \int_{x_n}^\infty e^{-z_n^2/(4\tau)}(z_n-x_n)^{2s}\,dz_n\bigg]\\
&\quad= \frac{Cx_n^{2s+1}}{\tau^{1/2}}\bigg[\int_{-\infty}^{1} e^{-x_n^2\omega^2/(4\tau)}(1-\omega)^{2s}\,d\omega - \int_{1}^\infty e^{-x_n^2\omega^2/(4\tau)}(\omega-1)^{2s}\,d\omega\bigg]\\
&\quad= \frac{Cx_n^{2s+1}}{\tau^{1/2}}\bigg[\int_{-1}^{\infty} e^{-x_n^2\omega^2/(4\tau)}(1+\omega)^{2s}\,d\omega - \int_{1}^\infty e^{-x_n^2\omega^2/(4\tau)}(\omega-1)^{2s}\,d\omega\bigg]\\
&\quad\leq \frac{Cx_n^{2s+1}}{\tau^{1/2}}\bigg[\int_{-1}^{2} e^{-x_n^2\omega^2/(4\tau)}\,d\omega + \int_{2}^\infty e^{-x_n^2\omega^2/(4\tau)}[(1+\omega)^{2s}-(\omega-1)^{2s}]\,d\omega\bigg].
\end{aligned}$$

The first integral above can be estimated by

$$\int_{-\infty}^\infty e^{-x_n^2\omega^2/(4\tau)}\,d\omega = C\frac{\tau^{1/2}}{x_n}. \tag{5.4.6}$$

For the second integral we use the mean value theorem to estimate $(1+\omega)^{2s} - (\omega-1)^{2s} \leq C$, whenever $2 < \omega < \infty$. Therefore, by applying again (5.4.6), we conclude that

$$e^{\tau\Delta}w_o(t-\tau,x) \leq C_s x_n^{2s}.$$

Hence, from the explicit formula for W we conclude (1).

For the case when $s \geq 1/2$, notice that w in (5.4.4) is bounded, so that there exists $C_s > 0$ such that $|e^{\tau\Delta_D^+}w(t-\tau,x)| \leq C_s$ for all $t \in \mathbb{R}$, $\tau > 0$ and $x \in \mathbb{R}^n_+$. Whence (2) follows from the explicit formula for W. □

Lemma 5.4.4. *The solution $W(t,x,y)$ to (5.4.5) satisfies the following estimates,*

(1) *If $s < 1/2$, then $|\partial_{x_n}W(t,x,y)| \leq Cy^{2s-1}$ for all $(t,x,y) \in (Q_1^+)^*$, where $C > 0$ depends only on s and θ.*

(2) *If* $s = 1/2$ *then* $|\partial_{x_n} W(t,x,y)| \leq C|\log(x_n^2 + y^2)|$ *for all* $(t,x,y) \in (Q^+_{1/2})^*$*, where* $C > 0$ *depends only on* s *and* θ*.*

(3) *If* $s > 1/2$ *then* $|\partial_{x_n} W(t,x,y)| \leq C$ *for all* $(t,x,y) \in (Q^+_1)^*$*, where* $C > 0$ *depends only on* s *and* θ*.*

Proof. The solution W to (5.4.5) for $\theta = 1$ is given by

$$\begin{aligned} W(t,x,y) &= \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{\tau\Delta_D^+} w(t-\tau,x)\,\frac{d\tau}{\tau^{1+s}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4\tau)} e^{\tau\Delta_D^+} g(t-\tau,x)\,\frac{d\tau}{\tau^{1-s}}. \end{aligned} \tag{5.4.7}$$

Consider first the case of $s < 1/2$. Using the second formula in (5.4.7) and the fact that g depends only on x_n, we get that

$$W(t,x,y) = C_s \int_0^\infty e^{-y^2/(4\tau)} \int_0^\infty \left(\frac{e^{-|x_n - z_n|^2/(4\tau)}}{\tau^{1/2}} - \frac{e^{-|x_n + z_n|^2/(4\tau)}}{\tau^{1/2}} \right) g(z_n)\,dz_n\,\frac{d\tau}{\tau^{1-s}}.$$

We would like to apply Fubini's Theorem above. Since g is bounded and $x_n, z_n > 0$, we only need to check that

$$0 \leq I := \int_0^\infty e^{-y^2/(4\tau)} \int_0^\infty \left(\frac{e^{-|x_n - z_n|^2/(4\tau)}}{\tau^{1/2}} - \frac{e^{-|x_n + z_n|^2/(4\tau)}}{\tau^{1/2}} \right) dz_n\,\frac{d\tau}{\tau^{1-s}} < \infty.$$

Indeed

$$\begin{aligned} I &= \int_0^\infty \frac{e^{-y^2/(4\tau)}}{\tau^{1/2}} \left[\int_0^\infty e^{-(x_n - z_n)^2/(4\tau)} dz_n - \int_{-\infty}^0 e^{-(x_n - z_n)^2/(4\tau)} dz_n \right] \frac{d\tau}{\tau^{1-s}} \\ &= \int_0^\infty e^{-y^2/(4\tau)} \left[\int_{-\infty}^{x_n/(2\sqrt{\tau})} e^{-\omega^2}\,d\omega - \int_{x_n/(2\sqrt{\tau})}^\infty e^{-\omega^2}\,d\omega \right] \frac{d\tau}{\tau^{1-s}} \\ &= \int_0^\infty e^{-y^2/(4\tau)} \operatorname{erf}(x_n/(2\sqrt{\tau})) \frac{d\tau}{\tau^{1-s}} \end{aligned}$$

where we have denoted $\operatorname{erf}(r) = \int_{-r}^r e^{-\omega^2}\,d\omega$. One one hand, if $0 < \tau < 1$ then $\operatorname{erf}(x_n/(2\sqrt{\tau})) < C$, so that

$$\int_0^1 e^{-y^2/(4\tau)} \operatorname{erf}(x_n/(2\sqrt{\tau}))\,\frac{d\tau}{\tau^{1-s}} \leq C \int_0^1 \frac{d\tau}{\tau^{1-s}} < \infty.$$

On the other hand, when τ is large, by using the Taylor expansion of $e^{-\omega^2}$, we can estimate $\operatorname{erf}(x_n/(2\sqrt{\tau})) \sim Cx_n/(2\sqrt{\tau})$ so we have

$$\int_1^\infty e^{-y^2/(4\tau)} \operatorname{erf}(x_n/(2\sqrt{\tau}))\,\frac{d\tau}{\tau^{1-s}} \leq Cx_n \int_1^\infty \tau^{s-1/2}\,\frac{d\tau}{\tau} < \infty.$$

Hence I is convergent. Thus, for each fixed (t,x,y), after Fubini's Theorem,

$$W(t,x,y) = C_s \int_0^\infty g(z_n) \int_0^\infty \left(\frac{e^{-(y^2+|x_n-z_n|^2)/(4\tau)}}{\tau^{1/2-s}} - \frac{e^{-(y^2+|x_n+z_n|^2)/(4\tau)}}{\tau^{1/2-s}} \right) \frac{d\tau}{\tau}\, dz_n$$
$$= C_s \int_0^\infty \left(\frac{1}{(y^2+(x_n-z_n)^2)^{(1-2s)/2}} - \frac{1}{(y^2+(x_n+z_n)^2)^{(1-2s)/2}} \right) dz_n.$$

Since $s<1/2$, it is easy to check that we can differentiate inside the integral to finally obtain

$$\partial_{x_n} W(t,x,y) = \frac{C_s}{(x_n^2+y^2)^{(1-2s)/2}}$$

from which the estimate in (1) follows.

For $s=1/2$, we use the second formula in (5.4.7) and a similar computation as in [40] to find that, since g_o is independent of t and has zero mean,

$$W(t,x,y) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{-(y^2+z_n^2)/(4\tau)} g_o(t-\tau, x_n-z_n)\, dz_n \,\frac{d\tau}{\tau}$$
$$= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{-r(y^2+z_n^2)} g_o(x_n-z_n)\, dz_n \,\frac{dr}{r}$$
$$= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \left(e^{-r(y^2+z_n^2)} - \chi_{(0,1)}(r)\right) g_o(x_n-z_n)\, dz_n \,\frac{dr}{r}$$
$$= -\frac{1}{2\pi} \int_{-\infty}^\infty \log\left((x_n-z_n)^2+y^2\right) g_o(z_n)\, dz_n$$
$$= \frac{1}{2\pi} \int_0^\infty \left[\log\left((x_n+z_n)^2+y^2\right) - \log\left((x_n-z_n)^2+y^2\right)\right] g(z_n)\, dz_n.$$

Next, since $g(z_n)=\chi_{[0,1]}(z_n)$, by using integration by parts,

$$W(t,x,y) = (1+x_n)\log((1+x_n)^2+y^2) - (1-x_n)\log((1-x_n)^2+y^2) - 2x_n \log(x_n^2+y^2)$$
$$+ 2y\arctan((1+x_n)/y) - 2y\arctan((1-x_n)/y) - 4y\arctan(x_n/y)$$

Therefore,

$$\partial_{x_n} W(t,x,y) = \log((1+x_n)^2+y^2) + \log((1-x_n)^2+y^2) - 2\log(x_n^2+y^2)$$

from which (2) follows.

To prove (3) for $s>1/2$, we notice that

$$w(t,x) = \begin{cases} c_s\left[2x_n^{2s} + (1-x_n)^{2s} - (1+x_n)^{2s}\right] & \text{for } 0<x_n<1, \\ c_s x_n^{2s}\left[2-(1-1/x_n)^{2s} - (1+1/x_n)^{2s}\right] & \text{for } x_n \ge 1 \end{cases}$$

Then, for $0 < x_n < 1$,

$$\partial_{x_n} w(t,x) = c_s\big[2x_n^{2s-1} - (1-x_n)^{2s-1} - (1+x_n)^{2s-1}\big]$$

and, for $x_n \geq 1$,

$$\partial_{x_n} w(t,x) = c_s x_n^{2s-1}\big[2 - (1-1/x_n)^{2s} - (1+1/x_n)^{2s}\big] \\ + c_s x_n^{2s-2}\big[(1-1/x_n)^{2s-1} + (1+1/x_n)^{2s-1}\big].$$

Now using the estimate for $\eta_{s2}(1/x_n)$ in (5.4.1), we conclude that $\partial_{x_n} w \sim C$ as $x_n \to 0$, and $\partial_{x_n} w \sim x_n^{2s-2}$ as $x_n \to \infty$. Then we see that $|\partial_{x_n} w|$ is bounded everywhere. From here and the first formula in (5.4.7) is it easy to check that $|\partial_{x_n}(e^{\tau\Delta_D^+} w(t-\tau,x)| \leq C$ for all $\tau > 0$ and $(t,x) \in \mathbb{R} \times \mathbb{R}^n_+$, which in turn establishes (3). □

5.4.3 Boundary Regularity in Half Space – Neumann

In the half space $\mathbb{R} \times \mathbb{R}^n_+$ we consider the heat operator $\partial_t - \Delta_N^+$, where Δ_N^+ is the Neumann Laplacian in $\mathbb{R}^n_+$. For a function $u(t,x)$ defined on $\mathbb{R} \times \overline{\mathbb{R}^n_+}$ with $u_{x_n}(t,x',0) = 0$ and $0 < s < 1$ we define

$$(\partial_t - \Delta_N^+)^s u(t,x) = \frac{1}{\Gamma(-s)} \int_0^\infty \big(e^{\tau\Delta_N^+} u(t-\tau,x) - u(t,x)\big) \frac{d\tau}{\tau^{1+s}}$$

where $\{e^{\tau\Delta_N^+}\}_{\tau\geq 0}$ is the semigroup generated by Δ_N^+. As before, let $x^* = (x', -x_n)$ for $x \in \mathbb{R}^n$. Denote by $u_e(t,x)$ the even extension of $u(t,x)$ about the x_n axis given by

$$u_e(t,x) = \begin{cases} u(t,x) & \text{if } x_n \geq 0 \\ u(t,x^*) = u(t,x',-x_n) & \text{if } x_n < 0. \end{cases}$$

Now

$$e^{\tau\Delta_N^+} u(t-\tau,x) = e^{\tau\Delta} u_e(t-\tau,x) \\ = \frac{1}{(4\pi\tau)^{n/2}} \int_{\mathbb{R}^n_+} \Big(e^{-|x-z|^2/(4\tau)} + e^{-|x-z^*|^2/(4\tau)}\Big)\, u(t-\tau,z)\, dz$$

for any $\tau > 0$, $x \in \mathbb{R}^n_+$. Hence, for $x \in \mathbb{R}^n_+$,

$$(\partial_t - \Delta_N^+)^s u(t,x) = \frac{1}{(4\pi)^{n/2}\Gamma(-s)} \int_0^\infty \int_{\mathbb{R}^n_+} \left(\frac{e^{-|x-z|^2/(4\tau)} + e^{-|x-z^*|^2/(4\tau)}}{\tau^{n/2+1+s}} \right) (u(t-\tau,z) - u(t,x))\, dz\, d\tau$$

and

$$\begin{aligned}(\partial_t - \Delta_N^+)^{-s} f(t,x) &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-\tau(\partial_t - \Delta_N^+)} f(t,x) \frac{d\tau}{\tau^{1-s}} \\ &= \frac{1}{(4\pi)^{n/2}\Gamma(s)} \int_0^\infty \int_{\mathbb{R}^n_+} \left(\frac{e^{-|x-z|^2/(4\tau)} + e^{-|x-z^*|^2/(4\tau)}}{\tau^{n/2+1-s}} \right) f(t-\tau,z)\, dz\, d\tau.\end{aligned}$$

Theorem 5.4.5 (Boundary regularity in half space – Neumann)**.** *Let $u, f \in L^\infty(\mathbb{R} \times \mathbb{R}^n_+)$ be such that $\int_{\mathbb{R}^n_+} f(t,x)\, dx = 0$ for all $t \in \mathbb{R}$ and*

$$\begin{cases} (\partial_t - \Delta_N^+)^s u = f & \text{in } \mathbb{R} \times \mathbb{R}^n_+ \\ -u_{x_n} = 0 & \text{on } \mathbb{R} \times \partial\mathbb{R}^n_+. \end{cases}$$

(1) *Suppose that $f \in C^{\alpha/2,\alpha}(\mathbb{R} \times \overline{\mathbb{R}^n_+})$ for $0 < \alpha \le 1$.*

(a) *If $\alpha + 2s$ is not an integer then $u \in C^{\alpha/2+s,\alpha+2s}(\mathbb{R} \times \overline{\mathbb{R}^n_+})$ with the estimate*

$$\|u\|_{C^{\alpha/2+s,\alpha+2s}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \le C\big(\|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \big).$$

(b) *If $\alpha + 2s = 1$ then $u(t,x)$ is in the Hölder–Zygmund space $\Lambda^{1/2,1}(\mathbb{R} \times \overline{\mathbb{R}^n_+})$ with the estimate*

$$\|u\|_{\Lambda^{1/2,1}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \le C\big(\|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \big).$$

The constants $C > 0$ depend only on n, s and α.

(2) *Let $f \in L^\infty(\mathbb{R} \times \overline{\mathbb{R}^n_+})$.*

(a) *If $s \neq 1/2$ then $u \in C^{s,2s}(\mathbb{R} \times \overline{\mathbb{R}^n_+})$ with the estimate*

$$\|u\|_{C^{s,2s}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \le C\big(\|f\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \big).$$

(b) If $s = 1/2$ then u is in the Hölder–Zygmund space $\Lambda^{1/2,1}(\mathbb{R} \times \overline{\mathbb{R}^n_+})$ with the estimate

$$\|u\|_{\Lambda^{1/2,1}(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \leq C\big(\|f\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} + \|u\|_{L^\infty(\mathbb{R}\times\overline{\mathbb{R}^n_+})} \big).$$

The constants $C > 0$ above depend only on n and s.

Proof. We show this result by noticing that if f_e and u_e are the even reflections of f and u with respect to the variable x_n, respectively, then $(\partial_t - \Delta)^s u_e = f_e$ in $\mathbb{R}^{n+1}$, so that Proposition 5.4.1 applies. From the pointwise formula we see that $(\partial_t - \Delta)^s u_e(t,x) = (\partial_t - \Delta_N^+)^s u(t,x) = f(t,x) = f_e(t,x)$ for $x \in \mathbb{R}^n_+$. Now, for $x \in \mathbb{R}^n$ is such that $x_n < 0$,

$$\begin{aligned}(\partial_t - \Delta)^s u_e(t,x) &= \frac{1}{\Gamma(-s)} \int_0^\infty \big(e^{\tau\Delta} u_e(t-\tau,x) - u_e(t,x)\big) \frac{d\tau}{\tau^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty \Big(e^{\tau\Delta_N^+} u(t-\tau,x^*) - u(t,x^*)\Big) \frac{d\tau}{\tau^{1+s}} \\ &= (\partial_t - \Delta_N^+)^s u(t,x^*) = f(t,x^*) = f_e(t,x).\end{aligned}$$

□

5.5 Proofs of Global regularity results

5.5.1 Global Regularity for Dirichlet Boundary Condition and f Hölder:

Here we present the proof of the Theorem 5.1.3. For that we assume that $\Omega \subset \mathbb{R}^n_+$ is a bounded domain such that its boundary contains a flat portion on $\{x_n = 0\}$ in such a way that $B_1^+ \subset \Omega$.

We say that $f \in L^2(Q_1^+)$ is in $L_+^{\alpha/2,\alpha}(0,0)$, $0 < \alpha \leq 1$, if

$$[f]^2_{L_+^{2,\alpha/2,\alpha}(0,0)} := \sup_{0<r\leq 1} \frac{1}{r^{n+2+2\alpha}} \int_{Q_r^+} |f - f(0,0)|^2 \, dt \, dx < \infty$$

where $f(0,0) = \lim_{r\to 0} \frac{1}{|Q_r^+|} \int_{Q_r^+} f(t,x) \, dt \, dx$.

Theorem 5.1.3 follows from the next result after flattening the boundary, translation and rescaling, and by taking into account estimate (2.4.2) and the properties of half space solutions, see subsection 5.4.2, and Theorem 4.1.1.

Theorem 5.5.1. *Let $u \in \mathrm{Dom}(H^s)$ be a solution to (1.0.1) with Dirichlet boundary condition and assume that $f \in L_+^{\alpha/2,\alpha}(0,0)$, for some $0<\alpha<1$. Let w be the half space solution to (5.4.2).*

(1) *Assume that $0<\alpha+2s<1$. There exist $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_1>0$ such that if*

$$\sup_{0<r\le 1} \frac{1}{r^{n+2\alpha}} \int_{B_r^+} |A(x)-A(0)|^2\,dx < \delta^2$$

then

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |u(t,x)-f(0,0)w(t,x)|^2\,dt\,dx \le C_1 r^{2(\alpha+2s)}$$

for all $r>0$ small. Moreover,

$$C_1^{1/2} \le C_0\big(1+\|u\|_{\mathrm{Dom}(H^s)}+|f(0,0)|+[f]_{L_+^{\alpha/2,\alpha}(0,0)}\big)$$

where $C_0>0$ depends on $A(x)$, n, s, α and ellipticity.

(2) *Assume that $s=1/2$ and $1<\alpha+2s<2$. Let $0<\varepsilon<1/2$ such that $0<\alpha+\varepsilon<1$. There exists $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_1>0$ such that if*

$$\sup_{0<r\le 1/2} \frac{1}{r^{n+2(\alpha+\varepsilon)}} \int_{B_r^+} |A(x)-A(0)|^2\,dx < \delta^2$$

then there exists a linear function $l(x)=\mathcal{B}\cdot x$ such that

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |u(t,x)-f(0,0)w(t,x)-l(x)|^2\,dt\,dx \le C_1 r^{2(\alpha+1)}$$

for all $r>0$ small. Moreover,

$$C_1^{1/2}+|\mathcal{B}| \le C_0\big(1+\|u\|_{\mathrm{Dom}(H^s)}+|f(0,0)|+[f]_{L_+^{\alpha/2,\alpha}(0,0)}\big)$$

where $C_0>0$ depends on $A(x)$, n, s, α and ellipticity.

(3) *Assume that $s>1/2$ and $1<\alpha+2s<2$. There exists $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_1>0$ such that if*

$$\sup_{0<r\le 1} \frac{1}{r^{n+2(\alpha+2s-1)}} \int_{B_r^+} |A(x)-A(0)|^2\,dx < \delta^2$$

then there exists a linear function $l(x) = \mathcal{B}\cdot x$ *such that*

$$\frac{1}{r^{n+2}}\int_{Q_r^+} |u(t,x) - f(0,0)w(t,x) - l(x)|^2\,dt\,dx \le C_1 r^{2(\alpha+2s)}$$

for all $r > 0$ *small. Moreover,*

$$C_1^{1/2} + |\mathcal{B}| \le C_0\big(1 + \|u\|_{\mathrm{Dom}(H^s)} + |f(0,0)| + [f]_{L_+^{\alpha/2,\alpha}(0,0)}\big)$$

where $C_0 > 0$ *depends on* $A(x)$*,* n*,* s*,* α *and ellipticity.*

Proof of Theorem 5.5.1. Here in the following we will give proofs of all three parts of the above Theorem.

- **Proof of Theorem 5.5.1(1)**

  Let U be the solution to the extension problem for u, so that U is a weak solution to

$$\begin{cases} y^a\partial_t U - \mathrm{div}(y^a B(x)\nabla U) = 0 & \text{in } (Q_1^+)^* \\ -y^a U_y\big|_{y=0} = f & \text{on } Q_1^+ \\ U = 0 & \text{on } Q_1 \cap \{x_n = 0\}. \end{cases}$$

  Without loss of generality, we can assume that $B(0) = I$. We need to compare U with the solution W to the extension problem for the half space solution w. Let W solve (5.4.5) with $\theta = f(0,0)$, so that it is a weak solution to

$$\begin{cases} y^a\partial_t W - \mathrm{div}(y^a \nabla W) = 0 & \text{in } (Q_1^+)^* \\ -y^a W_y\big|_{y=0} = f(0,0) & \text{on } Q_1^+ \\ W = 0 & \text{on } Q_1 \cap \{x_n = 0\}. \end{cases}$$

  Let $V = U - W$. Then V is a weak solution to

$$\begin{cases} y^a\partial_t V - \mathrm{div}(y^a B(x)\nabla V) = -\,\mathrm{div}(y^a F) & \text{in } (Q_1^+)^* \\ -y^a V_y\big|_{y=0} = h & \text{on } Q_1^+ \\ V = 0 & \text{on } Q_1 \cap \{x_n = 0\}. \end{cases} \tag{5.5.1}$$

where

$$F=(I-B(x))\nabla W,\ F_{n+1}=0 \qquad \text{and} \qquad h=f-f(0,0),\ h(0,0)=0.$$

We observe that F satisfies a certain Morrey-type integrability condition. Indeed, when $s<1/2$, by Lemma 5.4.4,

$$\begin{aligned}
[F]^2_{\alpha,s} &:= \sup_{0<r\le 1}\frac{1}{r^{n+3+a+2(\alpha+2s-1)}}\int_{(Q_r^+)^*} y^a|F|^2\,dt\,dX\\
&= \sup_{0<r\le 1}\frac{1}{r^{n+3+a+2(\alpha+2s-1)}}\int_{(Q_r^+)^*} y^a|(I-B(x))\nabla W|^2\,dt\,dX\\
&\le \sup_{0<r\le 1}\frac{1}{r^{n+3+a+2(\alpha+2s-1)}}\int_{B_r^+}\int_{-r^2}^{r^2}\int_0^r y^a|(I-A(x))|^2y^{4s-2}\,dy\,dt\,dx\\
&= \sup_{0<r\le 1}\frac{C_s}{r^{n+2+2\alpha}}\int_{B_r^+}\int_{-r^2}^{r^2}|(I-A(x))|^2\,dt\,dx\\
&= \sup_{0<r\le 1}\frac{C_s}{r^{n+2\alpha}}\int_{B_r^+}|(I-A(x))|^2\,dx < C_s\delta^2
\end{aligned}$$

We say that, given $\delta>0$, V is a δ-normalized solution to (5.5.1) if the following conditions hold:

1. $\displaystyle\sup_{0\le r\le 1}\frac{1}{r^{n+2\alpha}}\int_{B_r^+}|A(x)-I|^2dx<\delta^2$;
2. $\displaystyle[h]^2_{L_+^{\alpha/2,\alpha}(0,0)}:=\sup_{0<r\le 1}\frac{1}{r^{n+2+2\alpha}}\int_{Q_r^+}|h|^2\,dt\,dx<\delta^2$;
3. $\displaystyle[F]^2_{\alpha,s}=\sup_{0<r\le 1}\frac{1}{r^{n+3+a+2(\alpha+2s-1)}}\int_{(Q_r^+)^*}y^a|F|^2\,dt\,dX<\delta^2$;
4. $\displaystyle\int_{Q_1^+}V(t,x,0)^2\,dt\,dx+\int_{(Q_1^+)^*}y^aV^2\,dt\,dX\le 1.$

By scaling and by considering

$$V(t,x,y)\Bigg[\bigg(\int_{Q_1^+}V(t,x,0)^2\,dt\,dx+\int_{(Q_1^+)^*}y^aV^2\,dt\,dX\bigg)^{1/2}+\frac{1}{\delta}\big([F]_{\alpha,s}+[h]_{L_+^{\alpha/2,\alpha}(0,0)}\big)\Bigg]^{-1}$$

we can always assume that V is a δ-normalized solution.

Now we follow similar steps as in the proof of Lemma 5.3.3 with necessary changes. Namely, we replace balls by half-balls and use Corollaries 5.2.4 and 5.2.6 and Lemma 5.2.7. There is another change in the computation we need to consider because, unlike the proof of Theorem 5.3.1, here we have $F \neq 0$. Indeed, we perform the following estimate:

$$\begin{aligned}
&\lambda^2 \int_{(Q_\lambda^+)^*} y^a |\nabla V|^2 \, dt \, dX \\
&\leq C\lambda^2 \bigg(\int_{(Q_{2\lambda}^+)^*} y^a \frac{1}{\lambda^2} |V - c|^2 \, dt \, dX + \|F\|_{L^2((Q_{2\lambda}^+)^*)} + \int_{Q_{2\lambda}^+} |V(t,x,0) - c||h(t,x)| \, dt \, dx \bigg) \\
&\leq C \int_{(Q_{2\lambda}^+)^*} y^a |V - c|^2 dt \, dX + C\delta^2 + C \left(\|V(\cdot,\cdot,0)\|_{L^2(Q_{2\lambda}^+)} + |c||Q_{2\lambda}^+|^{1/2} \right) \|h\|_{L^2(Q_{2\lambda}^+)} \\
&\leq 2C\varepsilon^2 + Cc_{n,a}\lambda^{n+5+a} + C(1+|c|)\delta.
\end{aligned}$$

Therefore, we obtain the existence of $0 < \delta, \lambda < 1$ such that if V is a δ-normalized solution then

$$\frac{1}{\lambda^{n+2}} \int_{Q_\lambda^+} |V(t,x,0)|^2 dt \, dx + \frac{1}{\lambda^{n+3+a}} \int_{(Q_\lambda^+)^*} |V|^2 dt \, dX < \lambda^{2(\alpha+2s)}$$

Notice that here $c = V(0,0,0) = 0$. Using the above result, if we follow similar steps as in the proof of Lemma 5.3.4, with similar necessary changes as above, and setting $c_k = 0$ for all induction step k, and we can prove that

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |V(t,x,0)|^2 dt \, dx < C_1 r^{2(\alpha+2s)}$$

for all $r > 0$ sufficiently small. The constant C_1 satisfies the following bound

$$\begin{aligned}
C_1 \leq C_0^2 \bigg(&\int_{Q_1^+} U(t,x,0)^2 \, dt \, dx + \int_{(Q_1^+)^*} y^a U^2 \, dt \, dX + \int_{Q_1^+} W(t,x,0)^2 \, dt \, dx \\
&+ \int_{(Q_1^+)^*} y^a W^2 \, dt \, dX + \frac{1}{\delta^2} [F]_{\alpha,s}^2 + \frac{1}{\delta^2} [f]^2_{L_+^{\alpha/2,\alpha}(0,0)} \bigg).
\end{aligned}$$

Notice that, from Lemma 5.4.3,

$$\begin{aligned}
&\int_{Q_1^+} W(t,x,0)^2 \, dt \, dx + \int_{(Q_1^+)^*} y^a W^2 \, dt \, dX \\
&\qquad \leq C|f(0,0)|^2 \int_{Q_1^+} x_n^{4s} \, dt \, dx + C|f(0,0)|^2 \int_{(Q_1^+)^*} y^a x_n^{4s} \, dt \, dX = C|f(0,0)|^2,
\end{aligned}$$

so we conclude that the estimate for C_1 in the statement holds.

- **Proof of Theorem 5.5.1(2)**

Let U, V, F and h be as in the proof of Theorem 5.5.1(1). Observe that, by Lemma 5.4.4, F now satisfies the following Campanato-type integrability condition:

$$\begin{aligned}
[F]^2_{\alpha,1/2} &:= \sup_{0<r\le 1/2} \frac{1}{r^{n+3+2\alpha}} \int_{(Q_r^+)^*} |(I - A(x))\nabla_x W|^2 \, dt \, dX \\
&\le \sup_{0<r\le 1/2} \frac{C}{r^{n+3+2\alpha}} \int_{(Q_r^+)^*} |(I - A(x))|^2 |\log y|^2 \, dt \, dX \\
&\le \sup_{0<r\le 1/2} \frac{C}{r^{n+3+2\alpha}} \int_{(Q_r^+)^*} |(I - A(x))|^2 y^{-2\varepsilon} \, dt \, dX \\
&= \sup_{0<r\le 1/2} \frac{C}{r^{n+2(\alpha+\varepsilon)}} \int_{B_r^+} |(I - A(x))|^2 \, dx < C\delta^2.
\end{aligned}$$

By scaling and normalization, we can assume that V is a δ-normalized solution to (5.5.1) in the sense that

1. $\displaystyle \sup_{0<r\le 1/2} \frac{1}{r^{n+2(\alpha+\varepsilon)}} \int_{B_r^+} |A(x) - I|^2 \, dx < \delta^2$;
2. $\displaystyle [h]^2_{L_+^{\alpha/2,\alpha}(0,0)} := \sup_{0<r\le 1/2} \frac{1}{r^{n+2+2\alpha}} \int_{Q_r^+} |h|^2 \, dt \, dx < \delta^2$;
3. $\displaystyle [F]^2_{\alpha,1/2} = \sup_{0<r\le 1/2} \frac{1}{r^{n+3+2\alpha}} \int_{(Q_r^+)^*} |F|^2 \, dt \, dX < \delta^2$;
4. $\displaystyle \int_{Q_1^+} V(t,x,0)^2 \, dt \, dx + \int_{(Q_1^+)^*} V^2 \, dt \, dX \le 1$.

Then we follow the proof of Theorem 5.3.1(2). We have a linear polynomial $\ell(x)$ such that $V - \ell$ is a weak solution to

$$\begin{cases}
\partial_t V - \operatorname{div}(B(x)\nabla V) = -\operatorname{div}(F + G) \quad \text{in } (Q_{1/2}^+)^* \\
-(V - \ell)_y|_{y=0} = h \quad \text{on } (Q_{1/2}^+)
\end{cases}$$

where the vector field G is given by

$$G = ((I - A(x))\nabla_x \ell, 0) \quad \text{and} \quad G(0) = 0$$

Then we can see that G also satisfies the same Campanato-type condition as F. Indeed, as $|\nabla\ell| \leq C$,

$$[G]^2_{\alpha,1/2} = \sup_{0<r\leq 1/2} \frac{1}{r^{n+3+2\alpha}} \int_{(Q_r^+)^*} |(I - A(x))\nabla_x \ell|^2 \, dt \, dX$$
$$\leq \sup_{0<r\leq 1/2} \frac{C}{r^{n+2\alpha}} \int_{B_r^+} |(I - A(x))|^2 \, dx \leq C\delta^2.$$

With this we can continue as in the proof of Theorem 5.3.1(2) and get $l_\infty(x) = \mathcal{B}_\infty \cdot x$ such that

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |V(t,x,0) - l_\infty(x)|^2 dt \, dx \leq C_1 r^{2(\alpha+2s)}$$

for $r > 0$ sufficiently small. As in Theorem 5.5.1(1),

$$C_1^{1/2} + |\mathcal{B}_\infty| \leq C_0 \big(1 + \|u\|_{\mathrm{Dom}(H^s)} + |f(0,0)| + [f]_{L_+^{\alpha/2,\alpha}(0,0)}\big)$$

where C_0 depends on δ, n, s, α and ellipticity. In this particular case we observe that, the term $\mathcal{A}$ from Lemma 5.3.5 will be 0 because the our approximating function $W = 0$ at the origin and hence $\mathcal{A}_\infty$ will be 0.

- **Proof of Theorem 5.5.1(3)** Let U, V, F and h be as in the proof of Theorem 5.5.1(1). Observe that, by Lemma 5.4.4, F satisfies the following Campanato-type condition:

$$[F]^2_{\alpha,s} \leq \sup_{0<r\leq 1} \frac{C}{r^{n+3+a+2(\alpha+2s-1)}} \int_{(Q_r^+)^*} y^a |(I - A(x))|^2 \, dt \, dX$$
$$\leq \sup_{0<r\leq 1} \frac{C}{r^{n+2(\alpha+2s-1)}} \int_{B_r^+} |(I - A(x))|^2 \, dx \leq C\delta^2.$$

Then again we can normalize V and follow the proof of Theorem 5.3.1(2). Details are left to the interested reader.

□

As we have observed that Theorem 5.1.3 is a consequence of Theorem 5.5.1 similarly Theorem 5.1.4 is a direct consequence of the following result.

Theorem 5.5.2. *Let $u \in \mathrm{Dom}(H^s)$ be a solution to (1.0.1) with Dirichlet boundary condition and assume that $f \in L_+^{\alpha/2,\alpha}(0,0)$, for some $0 < \alpha < 1$, and that $f(t,0) = 0$ for all $t \in [-1,1]$.*

(1) *Assume that* $0 < \alpha + 2s < 1$. *There exist* $0 < \delta < 1$, *depending only on* n, *ellipticity,* α *and* s, *and a constant* $C_1 > 0$ *such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r^+} |A(x) - A(0)|^2 \, dx < \delta^2$$

then

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |u(t,x)|^2 \, dt \, dx \leq C_1 r^{2(\alpha+2s)}$$

for all $r > 0$ *small. Moreover,*

$$C_1^{1/2} \leq C_0 \big(\|u\|_{\mathrm{Dom}(H^s)} + [f]_{L_+^{\alpha/2,\alpha}(0,0)} \big)$$

where $C_0 > 0$ *depends on* $A(x)$, n, s, α *and ellipticity.*

(2) *Assume that* $1 < \alpha + 2s < 2$. *There exists* $0 < \delta < 1$, *depending only on* n, *ellipticity,* α *and* s, *and a constant* $C_1 > 0$ *such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^{n+2(\alpha+2s-1)}} \int_{B_r^+} |A(x) - A(0)|^2 \, dx < \delta^2$$

then there exists a linear function $l(x) = \mathcal{B} \cdot x$ *such that*

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |u(t,x) - l(x)|^2 \, dt \, dx \leq C_1 r^{2(\alpha+2s)}$$

for all $r > 0$ *small. Moreover,*

$$C_1^{1/2} + |\mathcal{B}| \leq C_0 \big(\|u\|_{\mathrm{Dom}(H^s)} + [f]_{L_+^{\alpha/2,\alpha}(0,0)} \big)$$

where $C_0 > 0$ *depends on* $A(x)$, n, s, α *and ellipticity.*

Proof of Theorem 5.5.2. The proof is very similar to the proof of Theorem 5.3.1 with minor changes. If we replace Q_r by Q_r^+ and follow the other steps then we get our result. □

5.5.2 Global Regularity for Neumann Boundary Condition and f Hölder:

Theorem 5.1.5 follows from the next statement.

Theorem 5.5.3. *Let u be a solution to (1.0.1) with Neumann boundary condition. Assume that $f \in L_+^{\alpha/2,\alpha}(0,0)$ for some $0<\alpha<1$.*

(1) *Assume that $0<\alpha+2s<1$. There exist $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_1>0$ such that if*

$$\sup_{0<r\leq 1}\frac{1}{r^n}\int_{B_r^+}|A(x)-A(0)|^2dx\ <\delta^2$$

then there exists a constant c such that

$$\frac{1}{r^{n+2}}\int_{Q_r^+}|u(t,x)-c|^2\,dt\,dx\leq C_1r^{2(\alpha+2s)}$$

for all $r>0$ small. Moreover,

$$C_1^{1/2}+|c|\leq C_0\big(\,\|u\|_{\mathrm{Dom}(H^s)}+|f(0,0)|+[f]_{L_+^{\alpha/2,\alpha}(0,0)}\big)$$

where $C_0>0$ depends on $A(x)$, n, s, α and ellipticity.

(2) *Assume that $1<\alpha+2s<2$. There exists $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_1>0$ such that if*

$$\sup_{0<r\leq 1}\frac{1}{r^{n+2(\alpha+2s-1)}}\int_{B_r^+}|A(x)-A(0)|^2\,dx<\delta^2$$

then there exists a linear function $l(x)=\mathcal{A}+\mathcal{B}\cdot x$ such that

$$\frac{1}{r^{n+2}}\int_{Q_r^+}|u(t,x)-l(x)|^2\,dt\,dx\leq C_1r^{2(\alpha+2s)}$$

for all $r>0$ small. Moreover,

$$C_1^{1/2}+|\mathcal{A}|+|\mathcal{B}|\leq C_0\big(\,\|u\|_{\mathrm{Dom}(H^s)}+|f(0,0)|+[f]_{L_+^{\alpha/2,\alpha}(0,0)}\big)$$

where $C_0>0$ depends on $A(x)$, n, s, α and ellipticity.

Proof of Theorem 5.5.3. We prove the regularity of the solution for the extension problem about the origin like we did in the case of Dirichlet boundary condition. The extension problem is

$$\begin{cases} y^a\partial_tU-\operatorname{div}(y^aB(x)\nabla U)=0 & \text{in } (Q_1^+)^* \\ -y^aU_y|_{y=0}=f & \text{on } Q_1^+ \\ \partial_AU=0 & \text{on } Q_1^*\cap\{x_n=0\}. \end{cases}$$

Then the proof follows the similar steps as in the proof of Theorem 5.3.1 except we need to replace the Q_r by Q_r^+. □

5.5.3 Global Regularity for f in L^p:

We say that a function $f \in L^2(Q_1^+)$ is in $L_+^{-s+\alpha/2,-2s+\alpha}(0,0)$, for $0<\alpha<1$, whenever

$$[f]^2_{L_+^{-s+\alpha/2,-2s+\alpha}(0,0)} = \sup_{0<r\leq 1} \frac{1}{r^{n+2+2(-2s+\alpha)}} \int_{Q_r^+} |f(t,x)|^2\,dt\,dx < \infty$$

and that is in $L_+^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)$ whenever

$$[f]^2_{L_+^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)} = \sup_{0<r\leq 1} \frac{1}{r^{n+2+2(-2s+\alpha+1)}} \int_{Q_r^+} |f(t,x)|^2\,dt\,dx < \infty.$$

By Hölder's inequality (see the remarks before Theorem 5.3.2), it is clear that Theorem 5.1.6 will follow from the next result.

Theorem 5.5.4. *Let $u \in \mathrm{Dom}(H^s)$ be a solution to (1.0.1) with either Dirichlet or Neumann boundary condition and let $0<\alpha<1$.*

(1) *Assume that $f \in L_+^{-s+\alpha/2,-2s+\alpha}(0,0)$. There exist $0<\delta<1$, depending only on n, ellipticity, α, s and a constant $C_1>0$ such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r^+} |A(x)-A(0)|^2\,dx < \delta^2$$

then there exists a constant c such that

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |u(t,x)-c|^2\,dt\,dx \leq C_1 r^{2\alpha}$$

for all $r>0$ small. Moreover,

$$C_1^{1/2} \leq C_0\big(\|u\|_{\mathrm{Dom}(H^s)} + [f]_{L_+^{-s+\alpha/2,-2s+\alpha}(0,0)}\big)$$

where $C_0>0$ depends on $A(x)$, n, s, α and ellipticity.

(2) *Assume that $f \in L_+^{-s+(1+\alpha)/2,-2s+\alpha+1}(0,0)$. There exist $0<\delta<1$, depending only on n, ellipticity, α, s, and a constant $C_1>0$ such that if*

$$\sup_{0<r\leq 1} \frac{1}{r^{n+2\alpha}} \int_{B_r^+} |A(x)-A(0)|^2\,dx < \delta^2$$

then there exists a linear function $l(x) = \mathcal{A} + \mathcal{B} \cdot x$ *such that*

$$\frac{1}{r^{n+2}} \int_{Q_r^+} |u(t,x) - l(x)|^2 \, dt \, dx \leq C_1 r^{2(1+\alpha)}$$

for all $r > 0$ *small. Moreover,*

$$C_1^{1/2} + |\mathcal{A}| + |\mathcal{B}| \leq C_0 \big(\|u\|_{\mathrm{Dom}(H^s)} + [f]_{L_+^{-s+(1+\alpha)/2, -2s+\alpha+1}(0,0)} \big)$$

where $C_0 > 0$ *depends on* $A(x)$, n, s, α *and ellipticity.*

In particular, for the case of Dirichlet boundary condition, $c = 0$ *and* $\mathcal{A} = 0$ *above.*

Proof of Theorem 5.5.4. The proof follows very similar lines to those for Theorem 5.3.2 with minor changes, by replacing Q_r by Q_r^+. □

CHAPTER 6. GENERAL CONCLUSION

In this dissertation, we studied different regularity estimates for solutions to the nonlocal space-time equation,

$$(\partial_t + L)^s u(t,x) = f(t,x), \quad \text{for } 0 < s < 1,$$

for $t \in \mathbb{R}$ and $x \in \Omega$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, $n \geq 1$, that may be unbounded, and L is an elliptic operator in divergence form, i.e.

$$L = -\operatorname{div}(A(x)\nabla)$$

Here $A(x) = (A^{ij}(x))$ is a bounded, measurable, symmetric matrix defined in Ω, satisfying the uniform ellipticity condition, that is, for some $\Lambda \geq 1$,

$$\Lambda^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^n$. The operator L is subject to homogeneous Dirichlet or Neumann boundary conditions, that is,

$$u = 0 \quad \text{or} \quad \partial_A u = A(x)\nabla_x u \cdot \nu = 0 \quad \text{on } \mathbb{R} \times \partial\Omega,$$

where ν is the exterior unit normal to $\partial\Omega$.

Our nonlocal equation appears in several different physical processes and it is also an example of Master equation. In our work, we first defined the nonlocal operator $(\partial_t + L)^s$ for $0 < s < 1$ using spectral analysis and analytic continuation of the Gamma function. From here, we developed a semigroup method that, in particular, provided a meaningful pointwise formula for the nonlocal operator. In terms of regularity, we established interior and boundary Harnack inequalities and interior and boundary Schauder estimates for the solution. To this end, we proved a local characterization of the nonlocal problem. We also proved interior and global Schauder estimates by means of the extension technique. Along the way, we obtained a characterization of the intermediate parabolic Hölder space $C^{(1+\alpha)/2,1+\alpha}_{t,x}$, where $0 < \alpha < 1$, in the spirit of Campanato.

The results of this dissertation are contained in the papers [10, 11].

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