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Convergence and stability of variable-stepsize variable-formula multistep multiderivative methods

Gary Dale Buls
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CONVERGENCE AND STABILITY OF VARIABLE-STEP SIZE VARIABLE-FORMULA MULTISTEP MULTIDERIVATIVE METHODS

Iowa State University

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Convergence and stability of variable-stepsize variable-formula multistep multiderivative methods

by

Gary Dale Buls

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

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I. INTRODUCTION AND OVERVIEW

The general problem to be solved is the Initial Value Problem (IVP) in Ordinary Differential Equations (ODEs):

\[ y'(x) = f(x,y(x)) \]
\[ y(a) = y_0 \]

where \( x \in [a,b], \ f, \ y, \ y_0 \in \mathbb{R}^s \) and \( f(x,y) \) satisfies the following conditions:

(A) \( f(x,y) \) is defined and continuous on \([a,b] \times \mathbb{R}^s\).  
(B) \( f(x,y) \) is Lipschitz with respect to the second argument, i.e., for any vector norm \( ||\cdot|| \), there exists a constant \( L \) such that for every \( x \in [a,b] \) and any two vectors \( y \) and \( y^* \) in \( \mathbb{R}^s \), we have

\[ ||f(x,y) - f(x,y^*)|| \leq L||y - y^*||. \]

It is well-known that conditions (1.2) on \( f(x,y) \) ensure a unique solution \( y(x) \) on \([a,b]\) to the IVP (1.1); for a proof, see Henrici (1962, p 112).

To simplify the present discussion, only the case where the IVP (1.1) contains a single equation will be solved, i.e., \( s = 1 \). Generalizations to systems are straightforward but the notation becomes more complex. This reduces the IVP (1.1) to

\[ y'(x) = f(x,y) \]
\[ y(a) = y_0 \]

where \( x \in [a,b], \ f, \ y \) and \( y_0 \) are real numbers, and \( f(x,y) \)
satisfies the following conditions:

(A) \( f(x,y) \) is defined and continuous on \([a,b] \times \mathbb{R} \). 

(B) \( f(x,y) \) is Lipschitz with respect to the second argument, i.e., there exists a constant \( L \) such that for every \( x \in [a,b] \) and any two real numbers \( y \) and \( y^* \), we have

\[
|f(x,y) - f(x,y^*)| \leq L|y - y^*|.
\]

A. The General Multistep Method

There are several classes of methods to solve the IVP (1.1), but the present discussion will be limited to multistep methods. A basic understanding of multistep methods is assumed on the part of the reader. Henrici (1962) and Gear (1971) are good texts for the ideas being reviewed in this introduction.

To find a numerical solution to the IVP (1.3), the exact solution is approximated at discrete grid points which partition the interval \([a,b] \). Let \( a = x_0 < x_1 < ... < x_N = b \) be a finite partition of \([a,b] \). The classical derivation of multistep methods is based on a fixed-stepsizes \( k \)-step formula (FSF\([k]\)) of the form

\[
Y_n = \sum_{i=1}^{k} \alpha_{k-i} Y_{n-i} \\
+ h \sum_{i=0}^{k} \beta_{k-i} f(x_{n-i}, Y_{n-i}) \quad n \geq k
\]
where

\[ k \text{ is a fixed positive integer} \]

\[ h = x_m - x_{m-1}, \quad (m = 1, 2, \ldots, N) \]

(i.e., \( h = (b - a)/N \) is fixed)

\[ y_m = \text{approximation for the true solution } y(x_m), \quad (m = 1, 2, \ldots, N) \]

and \( \alpha_m \) (m = 0, 1, \ldots, k-1) and \( \beta_m \) (m = 0, 1, \ldots, k) are real constants of the formula not depending on n. Further, it is assumed that \( |\alpha_0| + |\beta_0| > 0 \).

Remark When \( \beta_k = 0 \), the \((\text{FSF}[k]) \) (1.5) is an explicit formula or a predictor formula. Otherwise, the \( \text{FSF}[k] \) (1.5) is an implicit formula or a corrector formula, and it is necessary to use an iterative technique to solve for \( y_n \). Although infinitely many iterations are usually needed to find \( y_n \), unless stated otherwise, it will be assumed that the corrector formula has been solved for exactly.

The general fixed-stepsize fixed-formula k-step method \((\text{FSFFM}[k])\), which is based on the \( \text{FSF}[k] \) (1.5), consists of finding \( y_n \) \((n = k, k+1, \ldots, N)\) by repeated use of the \( \text{FSF}[k] \) (1.5), assuming that \( y_0, y_1, \ldots, y_{k-1} \) have been obtained by other means.

B. Background on the \( \text{FSFFM}[k] \)

Define the polynomials \( \rho(y) \) and \( \sigma(y) \) associated with the \( \text{FSF}[k] \) (1.5) by
\[ \rho(f) = \sum_{i=0}^{k-1} \alpha_i f^i \quad \text{and} \quad \sigma(f) = \sum_{i=0}^{k} \beta_i f^i \] (1.6)

It will be assumed that \( \rho(f) \) and \( \sigma(f) \) have no factors in common.

To analyze the error committed by using the FSF[k] (1.5), define the difference operator \( L \), associated with the FSF[k] (1.5) by

\[
L[y(x_n); h] = y(x_n) - \sum_{i=1}^{k} \alpha_{k-i} y(x_n - ih) \\
- h \sum_{i=0}^{k} \beta_{k-i} y'(x_n - ih)
\] (1.7)

\( L \) may be applied to any differentiable function \( y(x) \). By Taylor's Theorem, it follows that

\[
L[y(x_n); h] = C_0 y(x_n) + C_1 y'(x_n) + \ldots \\
+ C_q h^q y^{(q)}(x_n) + \ldots
\]

where

\[
C_0 = 1 - \sum_{i=0}^{k} \alpha_{k-i}
\]

\[
C_1 = \sum_{i=1}^{k} i \alpha_{k-i} - \sum_{i=0}^{k} \beta_{k-i}
\]

\[
C_q = (-1)^{q+1} \frac{1}{q!} \sum_{i=1}^{k} i^q \alpha_{k-i} - \frac{1}{(q-1)!} \sum_{i=1}^{k} i^{q-1} \beta_{k-i}
\]

\((q = 2, 3, \ldots)\)

provided \( y(x) \) is sufficiently differentiable.
Remark L measures the discrepancy between the two sides of the FSF[k] (1.5) when $y_m$ is replaced by $y(x_m)$, where $y(x)$ is the solution of the IVP (1.3); i.e., L measures the amount by which the solution $y(x)$ fails to satisfy the FSF[k] (1.5).

Definition 1.1 The operator L (1.6) and the FSF[k] (1.5) are said to be of order $p$ if $C_0 = C_1 = \cdots = C_p = 0$, and $C_{p+1} \neq 0$.

Remark If a FSF[k] (1.5) is order $p$, then

(i) The FSFFM[k] solves the IVP (1.3) exactly when $y(x)$ is a polynomial of degree $p$ or less. This assumes that $y_0, y_1, \ldots, y_{k-1}$ are exactly known and that infinite precision is used.

(ii) $L[y(x_n); h] = C_{p+1} h^{p+1} y^{(p+1)}(x_n) + O(h^{p+2})$.

Remark One can see the order of a formula is a crude measure of accuracy, and the higher order formulas would tend to promote less error, assuming $h < 1$ and the higher order derivatives of $y(x)$ have comparable magnitude.

Definition 1.2 The FSFFM[k] is said to be consistent if the associated FSF[k] (1.5) is at least order 1.

Definition 1.3 The polynomial $p(x)$ is said to satisfy the root condition if all the roots of $p(x)$ are modulus less than or equal to 1, and all roots of modulus one are simple.

Definition 1.4 The polynomial $p(x)$ is said to satisfy the strong-root condition if all the roots of $p(x)$ are inside
the complex unit circle except for the root \( x = 1 \).

**Definition 1.5** The FSFFM\([k]\) is said to be **stable** if 
\( \rho(f) \) satisfies the root-condition.

**Remark** Intuitively, stability means that for small enough \( h \), small perturbations in the calculated values at \( x_n \) only cause small perturbations in later calculated values.

To be worth using, a FSFFM\([k]\) must be **convergent** in some sense. Intuitively, convergence means that the calculated solution approaches the true solution as the stepsize \( h \) approaches zero. More formally we have:

**Definition 1.6** A FSFFM\([k]\) is **convergent** if for all functions \( f(x,y) \) satisfying conditions (1.4), the following holds: If \( y(x) \) denotes the solution to the IVP (1.3) then the calculated solution \( y_n \) converges to \( y(x) \) for any \( x \in [a,b] \) as \( h \to 0 \), \( x_n \to x \) and the starting errors tend to zero.

A proof of the following fundamental theorem may be found in Henrici (1962).

**Theorem 1.1** A FSFFM\([k]\) is convergent if and only if it is consistent and stable.

**C. Changes to the FSFFM\([k]\)**

As have been seen, the results for the FSFFM\([k]\) depend on the use of only one multistep formula, the underlying FSF\([k]\) (1.5), and on the use of a stepsize \( h \) which is kept constant throughout the integration. In any practical application of
the FSFFM[k], keeping the stepsize constant is not sufficient. For most algorithms, the user usually supplies an error tolerance which represents the amount of error that is acceptable at each step of the integration. During the integration, one might find that in order to satisfy the user-supplied error tolerance, it is necessary to use a smaller stepsize. At the same time, for efficiency, one would like to perform the integration with as few steps as possible. Thus, it would be advantageous to keep the stepsize as large as possible while continuing to satisfy the error tolerance. So to use the FSFFM[k] as it has been defined, and still be both efficient and able to satisfy the error tolerance, this "optimal" stepsize must be known in advance, but this is not generally possible. Thus, some modifications to the FSFFM[k] need to be made.

Instead of finding one "optimal" stepsize to use throughout the integration on [a,b], the method can be modified so that an "optimal" stepsize is found before each step is taken, and thus, the resulting algorithm should be even more efficient. By changing the stepsize, the grid points are no longer equally spaced, and therefore, the FSF[k] (1.5) cannot be directly applied.

There are two common techniques of handling variable stepsizes for multistep methods. One technique is based on interpolation and consists of interpolating through the back
point information to obtain adjusted backpoint information at grid points which are then equally spaced with respect to the new stepsize, and thus, the $FSF[k]$ (1.5) can again be applied. The second technique consists of deriving multistep formulas directly based on the spacing of the grid points. The coefficients for the $k$-step formula are calculated at each step according to the spacing of the grid points. These techniques will be called the interpolation and variable-stepsize techniques, respectively.

As was noted earlier, higher order formulas tend to promote less error, so a higher order formula tends to allow the use of larger stepsizes while still being able to meet the prescribed error tolerance. Thus, another modification that can increase the efficiency of the method is to allow for a choice of formulas that can be used at the next step. This allows for the choice of a formula which will use the largest stepsize and still meet the prescribed error tolerance. This technique is referred to as the variable-formula technique. Another advantage for using the variable-formula technique is that if the method allows for a choice of using a 1-step formula, then the method is self starting, i.e., only $y_0$ needs to be given initially. Whereas, when the $FSFFM[k]$ is used with $k > 1$, the values $y_1, y_2, \ldots, y_{k-1}$ must be found by other means before the associated $FSF[k]$ (1.5) can be applied.
Remark  In the variable-stepsize technique, if a k-step formula is always being used and the method of determining the coefficients of the formula is the same regardless of the change in the stepsize, this is referred to as a fixed-formula technique.

Two codes based on the interpolation technique together with the variable-formula technique are the GEAR package (Hindmarsh, 1974) and the ISU package (Buls, 1981). EPISODE (Byrne and Hindmarsh, 1975) is a code based on the variable-stepsize and the variable-formula techniques.

In developing a practical algorithm, it is almost a necessity to allow for the use of different stepsizes, but it must be decided whether to use the interpolation technique or the variable-stepsize technique to handle the changes in stepsize. Brayton, Gustavson and Hachtel (1972) and Gear (1971, pp. 144-146) have shown that the two techniques for changing stepsize are not equivalent. So when deciding which technique to use, three considerations that should be taken into account are theoretical results, empirical results and computational efficiency.

For two important classes of formulas, the Adams formulas and the backward differentiation formulas (BDFs), theoretical and empirical results tend to favor the variable-stepsize technique as being more stable than the interpolation technique. With a method using the Adams formulas, Nordsieck
(1962) showed that using the interpolation technique can cause instability if the stepsize is changed too often, but it was shown by Gear and Watanbe (1974) that the method based on the Nordsieck form remains stable if after switching to a k-step formula, the stepsize and order are fixed for k or k+1 steps, depending on how one interpolates the higher derivatives; for further results, see also Skeel and Jackson (1983). In addition, Jackson (1978) showed that the Adams formulas implemented in a variable-stepsize variable-formula method are stable if increases in stepsize are bounded.

**Remark** It is important to note that the condition of taking k or k+1 steps without changing stepsize or order can not always be met if one also wants to satisfy the user-supplied error tolerance.

For methods based on the BDFs, Brayton, Gustavson and Hachtel (1972) give test examples which show that both stepsize changing techniques can lead to instability, but Byrne and Hindmarsh (1975) and Jackson and Sacks-Davis (1980) show empirically that the BDFs are more stable with variable-stepsize technique than with interpolation technique.

Computationally, there is a difference between the two techniques when applied to the IVP (1.1). For the variable-stepsize technique, at each step it is generally necessary to calculate the coefficients of the k-step formula which is going to be used, but there is no need to adjust the
backpoint information. Therefore, the computational effort
needed for the variable-stepsize technique depends only on k.
For the interpolation technique, at each step it requires
little effort, if any, to obtain the coefficients of the
k-step formula which is going to be used, but if the stepsize
is to be changed, then the backpoint information must also be
changed. Since there are s equations in the IVP (1.1), and
the amount of backpoint information needed to apply a k-step
formula to each equation depends on k, the computational
effort needed for the interpolation technique depends both on
k and s. After finding the coefficients and/or adjusting the
backpoint information, the remaining computational effort
necessary to solve the IVP (1.1) at the present grid point is
the same for both stepsize changing techniques. Gear and Tu
(1974), after a more thorough analysis, suggest that the
variable-stepsize technique be used for large values of s, and
that the interpolation technique be used for small values of
s. It is also noted that if frequent changes in stepsize is
expected, then the variable-stepsize technique should still be
used.

One last change to the FSFF[k] will be considered. As
noted above, it is desirable to be able to use higher order
formulas. One way of obtaining higher order formulas, without
increasing the number of backpoints, is to use higher
derivative formulas. Higher derivative multistep formulas
are of special interest for use in solving Stiff ODEs. In the fixed-steps size fixed-formula use, the higher derivative formulas allow for higher order A-stable or nearly A-stable formulas, which is desirable in solving Stiff ODEs; see Genin (1974) and Brown (1977) for further discussion. Therefore, the option of using higher derivative formulas will also be included.

The present discussion will now be restricted to variable-steps size variable-formula multistep multiderivative methods which will be defined more formally in the next chapter.
II. VARIABLE-STEP SIZE VARIABLE-FORMULA MULTISTEP MULTIDERIVATIVE METHODS

Let \( a = x_0 < x_1 < \cdots < x_N = b \) be a finite partition of \([a,b]\). Since the use of different formulas and unevenly spaced partitions are now allowed, advancing to the next grid point can result in a change of stepsize, a change in the coefficients of the formula, a change in the amount of backpoint information needed and a change in the number of derivatives being used. As a result, the form of a variable-step size \( k_n \)-step \( m_n \)-th-derivative formula (VSF[\(k_n,m_n\)]) used to find \( y_n \) is given by

\[
y_n = \sum_{i=1}^{k_n} \alpha_{n,k_n-i} y_{n-i}^2
\]

\[
+ \sum_{j=1}^{m_n} (h_n)^j \sum_{i=0}^{k_n} \beta_{n,k_n-i} f^{(j-1)}(x_{n-i}, y_{n-i}) \quad n > n_0,
\]

where \( k_n \) and \( m_n \) are positive integers and \( n_0 \) is a nonnegative integer,

\[
h_m = x_m - x_{m-1}, \quad (m = 1, 2, \ldots, N)
\]

\( y_m \) = approximation for the true solution \( y(x_m) \),

\[
(m = 1, 2, \ldots, N)
\]

\( f^{(j-1)} \) = the \( j \)th derivative of \( y(x) \) with respect to \( x \)

and \( \alpha_{n,m} \) (\( m = 0, 1, \ldots, k_n-1 \)) and \( \beta_{n,m}^{(j)} \) (\( m = 0, 1, \ldots, k_n, j = 1, 2, \ldots, m_n \)) are dependent on the relative spacing of the grid
points \( x_n, x_{n-1}, \ldots, x_{n-k_n} \). It is further assumed that

\[
|\alpha_{n,0}| + \sum_{j=1}^{m_n} |\beta_{n,0}^{(j)}| > 0 \quad \text{and} \quad \sum_{i=0}^{k_n} |\beta_{n,i}^{(m_n)}| > 0,
\]

which guarantees that the VSF \([k_n, m_n]\) (2.1) is in fact a \( k_n \)-step \( m_n \)-th-derivative formula.

Remark Note that \( k_n \leq n \) is necessary, otherwise, there would not be enough backpoint information available to use the formula.

Remark When \( \sum_{j=1}^{m_n} |\beta_{n,k_n}^{(j)}| = 0 \), the VSF \([k_n, m_n]\) (2.1) is an explicit formula or a predictor formula. Otherwise, the VSF \([k_n, m_n]\) (2.1) is an implicit formula or a corrector formula, and it is usually necessary to use an iteration technique to solve for \( y_n \). Although infinitely many iterations are generally necessary to find \( y_n \), unless stated otherwise, it will be assumed that the corrector formula has been solved for exactly.

The class of first order ODEs that are integrable by higher derivative formulas is smaller than the class that can be solved by 1st derivative methods. The reason being, to use the VSF \([k_n, m_n]\) (2.1), it is necessary for \( y(x) \) to be \( m_n \) times differentiable. It will also be necessary to extend the conditions in (1.4) that \( f(x,y) \) must satisfy. If the VSF \([k_n, m_n]\) (2.1) is to be applied to the IVP (1.3), it will be assumed that \( f(x,y) \) satisfies the following conditions:
(A) \( f^{(j-1)}(x,y) \) is defined and continuous on \([a,b] \times \mathbb{R}, \quad (j = 1,2,\ldots,m_n). \) \hfill (2.2)

(B) \( f^{(j-1)}(x,y) \) is Lipshitz with respect to the second argument, \((j = 1,2,\ldots,m_n), \) i.e., there exists a constant \( L^{(j)} \) such that for every \( x \in [a,b] \) and any two real numbers \( y \) and \( y^* \), we have

\[
|f^{(j-1)}(x,y) - f^{(j-1)}(x,y^*)| \leq L^{(j)} |y - y^*|, \quad (j = 1,2,\ldots,m_n).
\]

**Definition 2.1**  If \( h_n = h_{n-1} = \ldots = h_{n-k_n+1} \), then the resulting VSF\([k_n,m_n]\) is called the underlying fixed-steps\(e\) derivative formula or the underlying (FSF\([k_n,m_n]\)).

Define

\[
t_{n,m} = \frac{x_n - x_{n-m}}{h_n} \quad (n = 1,2,\ldots,N, \ m = 0,1,\ldots,k_n). \hfill (2.3)
\]

It then follows that

\[
x_{n-m} = x_n - h_n t_{n,m} \quad (n = 1,2,\ldots,N, \ m = 0,1,\ldots,k_n). \hfill (2.4)
\]

**Lemma 2.1** \( t_{n,m} \ (n = 1,2,\ldots,N, \ m = 0,1,\ldots,k_n) \) is independent of the actual stepsizes and depends only on the ratios of the stepsizes.
Proof

\[ t_{n,m} = \frac{x_n - x_{n-m}}{h_n} = \frac{(x_{n-m} + \sum_{i=0}^{m-1} h_{n-i}) - x_{n-m}}{h_n} \]

\[ = \sum_{i=0}^{m-1} h_{n-1} / h_n \]

Q.E.D.

As in chapter one, to analyze the error committed by the VSF\([k_n,m_n]\) (2.1), define the difference operator \( L_n \) associated with the VSF\([k_n,m_n]\) (2.1) by

\[ L_n[y(x_n);h_n] = y(x_n) - \sum_{i=1}^{k_n} \alpha_{n,k_n-i}y(x_{n-i}) \]

\[ - \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)} (x_{n-i}) \]

Using (2.4) gives

\[ L_n[y(x_n);h_n] = y(x_n) - \sum_{i=1}^{k_n} \alpha_{n,k_n-i}y(x_{n-t_n,i}) \]

\[ - \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)} (x_{n-t_n,i}) \]

\( L_n \) may be applied to any function \( y(x) \) which is \( m_n \) times differentiable. Assuming \( y(x) \) is sufficiently differentiable Taylor's Theorem gives

\[ (h_n)^j y^{(j)}(x_n - h_n t_{n,m}) = (h_n)^j y^{(j)}(x_n) \]

\[ - (h_n)^{j+1} (t_{n,m}) y^{(j+1)}(x_n) \]
Using (2.7) in (2.6) and combining like powers of $h_n$ gives

$$L_n[y(x_n); h_n] = c_{n,0} y(x_n) + c_{n,1} h_n y'(x_n) + \cdots + c_{n,q} h_n^q y^{(q)}(x_n) + \cdots$$

where

$$c_{n,0} = 1 - \sum_{i=1}^{k_n} \alpha_{n,k_n-i}$$

$$c_{n,q} = (-1)^{q+1} \left( \frac{1}{q!} \left( \sum_{i=1}^{k_n} \frac{(tn,i)^q \alpha_{n,k_n-i}}{q} \right) \right)$$

$$+ \sum_{j=1}^{q} \frac{(-1)^j}{(q-j)!} \sum_{i=0}^{k_n} (tn,i)^{q-j} \beta_{n,k_n-i}$$

$$c_{n,q} = (-1)^{q+1} \left( \frac{1}{q!} \left( \sum_{i=1}^{k_n} \frac{(tn,i)^q \alpha_{n,k_n-i}}{q} \right) \right)$$

$$+ \sum_{j=1}^{m_n} \frac{(-1)^j}{(q-j)!} \sum_{i=0}^{k_n} (tn,i)^{q-j} \beta_{n,k_n-i}$$

**Definition 2.2** The operator $L_n$ (2.5) and the VSF[$k_n,m_n$] (2.1) are said to be of order $p$ if $c_{n,0} = c_{n,1} = \cdots = c_{n,p} = 0$, and $c_{n,p+1} \neq 0$.

**Remark** If the VSF[$k_n,m_n$] (2.1) is order $p$, then
(i) The VSF\([k_n, m_n]\) will find \(y_n\) exactly when \(y(x)\) is a polynomial of degree \(p\) or less and \(y_{n-k_n}, y_{n-k_n+1}, \ldots, y_{n-1}\) are exactly known. This also assumes the use of infinite precision.

(ii) \(L_n[y(x_n); h_n] = C_n, p+1(h_n)^{p+1} y^{(p+1)}(x_n) + O((h_n)^{p+2}).\)

Define the polynomials \(\rho_n(f)\) and \(\sigma_n^{(j)}(f)\), \(j = 1, 2, \ldots, m_n\), associated with the VSF\([k_n, m_n]\) (2.1) by

\[
\rho_n(f) = \sum_{i=0}^{k_n} \alpha_n, i f^i \quad \text{and} \quad \sigma_n^{(j)}(f) = \sum_{i=0}^{k_n} \beta_n, i f^i, \quad (j = 1, 2, \ldots m_n)
\]

It will be assumed that \(\rho_n(f)\) and \(\sigma_n^{(j)}(f)\), \(j = 1, 2, \ldots, m_n\), have no factors in common.

A. The Method

Ideally, one would like a general multistep method to consist of a set of multistep formulas which can be applied to solve the IVP (1.3) with as few restrictions as possible. The following definition is an attempt to determine such a method.

**Definition 2.3** A variable-stepsize variable-formula multistep multiderivative method (VSVFM) consists of a set \(\mathcal{F}\) of variable-stepsize formulas of the form (2.1), where

1. \(\mathcal{F}\) contains at least one 1-step formula;
(2) there exist positive integers $K$ and $M$ such that $K$ is the maximum number of backpoints and $M$ is the highest order derivative that a formula in $\mathcal{F}$ can use;

(3) the only restrictions on the order in which the formulas in $\mathcal{F}$ can be applied to solve the IVP (1.3) are as follows: In order to use a $k_n$-step formula from $\mathcal{F}$ to find $y_n$, $n > n_0$, it is necessary that

(i) $k_n \leq n$ and
(ii) the formula is appropriate for the spacing of the grid points $x_n, x_{n-1}, \ldots, x_{n-k_n}$, i.e., the coefficients of the formula are determined for the spacing of the grid points;

and (4) There exists a fixed constant $\Delta$ depending on $\mathcal{F}$ such that if a $k_n$-step formula from $\mathcal{F}$ can be used to find $y_n$, it then follows that

$$\max\{h_n, h_{n-1}, \ldots, h_{n-k_n+1}\}/h_n \leq \Delta.$$ 

Remark Notice that $\Delta \geq 1$.

Condition (4) of Definition 2.3 does not restrict how the stepsizes can be changed, it only restricts which formulas can be in $\mathcal{F}$. If $k_n > 1$ and

$$\max\{h_n, h_{n-1}, \ldots, h_{n-k_n+1}\}/h_n > \Delta,$$

then there is no $k_n$-step formula in $\mathcal{F}$ that can be used, so $k_n$ must be reduced. $k_n$ may have to be reduced to 1, but this creates no problem since (1) in the definition guarantees the
existence of a 1-step formula in $\mathcal{F}$. More restrictive
conditions than in (4) have been imposed. For example, Gear
and Tu (1974) and Gear and Watanbe (1974) require that
\[
\max(h_1,h_2,\ldots,h_N)/\min(h_1,h_2,\ldots,h_N) \leq \Delta',
\]
($\Delta'$ a fixed constant). This severely restricts the partitions
of $[a,b]$ that can be considered, and thereby puts further
limits on the formulas that are in $\mathcal{F}$.

Another reason for a VSVFM being required to have at
least one 1-step formula available is to allow the method to
be self-starting, i.e., $n_0 = 0$. On the other hand, if $y_0,y_1,$
\ldots,$y_{n_0}$ for $n_0 > 0$ are already available, then, if desired,
the VSVFM can be used just to find $y_n$ for $n > n_0$.

Remark It should be noted that in order to have a
convergent VSVFM, it may be necessary to put further
restrictions on the formulas that can be contained in $\mathcal{F}$.

Definition 2.4 The VSVFM is said to be of order $p$ if
all the formulas in $\mathcal{F}$ are at least order $p$.

The discussion will now concentrate on the stability,
consistency and convergence of the VSVFM. To help in the
discussion, it will be assumed that
\[
h = \max_{1 \leq n \leq N} h_n
\]
for any partition of $[a,b]$.

B. Stability of the VSVFM

In finding a numerical solution to the IVP (1.3), it is
important that the solution should not be too sensitive to small errors in the computations, for example, round-off errors. Although the individual errors may be small, their cumulative effect can grow very rapidly and completely invalidate the final result. So to be worth using, an algorithm should remain immune of the accumulation of these small errors, i.e., the algorithm should be (numerically) stable. The definition of stability will be similar to that of Gear and Tu (1974). Suppose that the VSVFM computes \( y_n \) from a VSF\([k_n,m_n]\) (2.1) and suppose \( \hat{y}_n \) is computed by

\[
\hat{y}_n = \sum_{i=1}^{k_n} \alpha_{n,k_n-i}[\hat{y}_{n-i} + r_{n,n-i}]
\]

\[
+ \sum_{j=1}^{m_n} (h_n)^j \sum_{i=0}^{k_n} \beta_{n,k_n-i}[f^{(j-1)}(x_{n-i},\hat{y}_{n-i}) + r_{n,n-i}]
\]

for \( n > n_0 \), with \( \hat{y}_{n_0} = y_{n_0} \), \( \hat{y}_{n_0-1} = y_{n_0-1} \), \( \cdots \), \( \hat{y}_0 = y_0 \).

Thus, \( \hat{y}_n \) is the solution to a problem which is perturbed at each step.

**Definition 2.5** A VSVFM is stable if for all functions \( f(x,y) \) satisfying conditions (2.2), there exists a constant \( h^* > 0 \), and, for any \( \varepsilon > 0 \), a \( \delta(\varepsilon) > 0 \) such that on any partition \( P \) with \( 0 < h < h^* \) it follows that

\[
\max_{1 \leq n \leq N} |\hat{y}_n - y_n| \leq \varepsilon
\]

whenever
Most definitions only require stability against perturbations in the calculated values $y_i$, $i < n$, whereas, this definition is more stringent in that it requires stability against perturbations in all the calculated values necessary to find $y_n$. This condition is the same as that of Gear and Tu (1974) except that it also protects against perturbations due to calculating the derivatives of $y$ at $x_n$. The main reason for considering such a stringent condition is that in the variable-stepsize methods, the coefficients are generally not fixed, and as a result, the coefficients can have a major effect on how a small perturbation in a calculated value effects the final result.

**Remark** In this representation, $\hat{y}_n$ is assumed to be the exact solution to (2.10) and $\hat{y}_n$ is not perturbed until it is used in another calculation.

**Theorem 2.1** If a VSVFM is stable then the coefficients of the formulas in $\mathcal{F}$ are uniformly bounded.

**Proof** Suppose the VSVFM is stable and let $h \leq \min\{h^*, 1\}$ be given. Let $\epsilon = \left(\frac{h}{\Delta}\right)^M$ and suppose the coefficients in $\mathcal{F}$ are not uniformly bounded. Then for any $\delta > 0$, there exists a $k_n$-step formula $F$ in $\mathcal{F}$ such that at least one of the coefficients of $F$ is larger than $1/\delta$ in magnitude. Let $P$ be a partition of $[a,b]$ such that $h = \sum_{n=n_0+1}^N \left( \sum_{i=1}^{k_n} |x_n, n-i| \right) + \sum_{j=1}^{m_n} \sum_{i=0}^{k_n} |x_n, n-i| \leq \delta(\epsilon)$.
max \( (h_n, h_{n-1}, \ldots, h_{n-k_n+1}) \) and the spacing of the grid points is appropriate to use \( F \) to find \( y_n \). Using (2.1) and (2.10) give that

\[
\hat{y}_n - y_n = \sum_{i=1}^{k_n} \alpha_{n,k_n-i} [\hat{y}_{n-i} - y_{n-i}]
\]

\[
+ \sum_{i=1}^{k_n} \alpha_{n,k_n-i} \beta_{n,k_n-i}^{(0)}
\]

\[
+ \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, \hat{y}_{n-i})
\]

\[
- \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, y_{n-i})
\]

\[
+ \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)} (j)
\]

\[
= \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, y_{n-i})
\]

\[
Y_n - Y_n = \sum_{i=1}^{k_n} \alpha_{n,k_n-i} \beta_{n,k_n-i}^{(0)}
\]

\[
+ \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, y_{n-i})
\]

\[
= \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, y_{n-i})
\]

If \( \hat{y}_{n-i} = y_{n-i}, (i = 1, 2, \ldots, k_n) \), it then follows that

\[
\hat{y}_n - y_n = \sum_{i=1}^{k_n} \alpha_{n,k_n-i} \beta_{n,k_n-i}^{(0)}
\]

\[
+ \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, y_{n-i})
\]

\[
+ \sum_{j=1}^{m_n} (h_n) \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)} (j)
\]

\[
= \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, y_{n-i})
\]

\[
= \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j-1)} (x_{n-i}, y_{n-i})
\]

If for some fixed \( i_0, 1 \leq i_0 \leq k_n, |\alpha_{n,k_n-i_0}| > 1/\delta \), then by letting \( r_{n,n-i_0}^{(0)} = \delta \) and all other \( r_{n,n-i}^{(j)} = 0 \), it then follows from (2.11) that

\[
|\hat{y}_n - y_n| = |\alpha_{n,k_n-i_0} r_{n,n-i_0}^{(0)}| > 1 > \epsilon.
\]
Similarly, if for some fixed $i_0$ and $j_0$, $0 \leq i_0 \leq k_n$ and $1 \leq j_0 \leq m_n$, $|\hat{\beta}_{n, k_n-i_0}| > 1/\delta$, then by letting $r_{n, n-i_0} = \delta$
and all other $r_{n, n-i-1} = 0$, it follows from (2.11) that

$$|\hat{y}_n - y_n| = |(h_n)^{j_0} \hat{\beta}_{n, k_n-i_0} r_{n, n-i_0}|$$

and thus,

$$|\hat{y}_n - y_n| > (h_n)^{j_0} \geq (h_n)^M \geq (h/\Delta)^M = \varepsilon.$$

Therefore, it is evident that if any of the coefficients
of formulas in $\mathcal{F}$ are allowed to become arbitrarily large in
magnitude, then one small perturbation by itself can cause
$|\hat{y}_n - y_n|$ to become arbitrarily large which contradicts the
stability of the VSVFM. Hence, the coefficients of the
formulas in $\mathcal{F}$ must be uniformly bounded. Q.E.D.

Before proceeding further, it will be convenient to
introduce some notation to help simplify the discussion.

Let $e_m$ denoted the column vector in which all elements
but the $m^{th}$ element are zero, and the $m^{th}$ element is 1, i.e.,

$$e_m = (0, \ldots, 0, 1, 0, \ldots, 0)^t$$

To simplify the notation, the length of $e_m$ will depend on the
context in which it is used. Let $S$ be a $K \times K$ matrix given by

$$S = \begin{bmatrix}
0 & \cdots & 0 \\
1 & \ddots & \vdots \\
0 & \ddots & 1 \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix} \quad (2.12)$$
and let \( S' \) be the \( K(M+1) \times K(M+1) \) block diagonal matrix given by
\[
S' = \text{diag}(S, S, \ldots, S)
\]

Associated with the VSF\([k_n, m_n]\) (2.1) are the \( 1 \times K \) vectors \( a_n \) and \( b_n^{(j)} \), \( 1 \leq j \leq m_n \), given by
\[
a_n = (a_{n,K-1}, a_{n,K-2}, \ldots, a_{n,0})
\]
and
\[
b_n^{(j)} = (b_{n,K-1}, b_{n,K-2}, \ldots, b_{n,0})
\]

where
\[
a_{n,K-i} = \begin{cases} 
\alpha_{n,k_n-i} & 1 \leq i \leq k_n \\
0 & k_n < i \leq K 
\end{cases}
\]
and
\[
b_{n,K-i}^{(j)} = \begin{cases} 
\beta_{n,k_n-i}^{(j)} & 1 \leq i \leq k_n \text{ and } 1 \leq j \leq m_n \\
0 & k_n < i \leq K \text{ or } m_n < j \leq M 
\end{cases}
\]

Also associated with the VSF\([k_n, m_n]\) (2.1) are the \( K \times K \) matrices \( A_n \), \( A'_n \) and \( B'_n \) given by
\[
A_n = e_{1} a_{n} \\
A'_n = A_n + S \quad \text{and} \\
B_n^{(j)} = e_{1} b_{n}^{(j)}
\]

Remark \( A'_n \) is a companion matrix to the polynomial \( \rho_n(f)^{K-k_n} \).

During the computation, it is necessary to save some of the backpoint information, and since at most a \( K \)-step formula
using $M$ derivatives can be used, let

$$Y_n = (Y_n, Y_{n-1}, \ldots, Y_{n-K+1})^t$$

and

$$Y_n = (Y_n, Y_{n-1}, \ldots, Y_{n-K+1}; Y'_n, Y'_{n-1}, \ldots, Y'_{n-K+1}; \ldots; Y''_n, Y''_{n-1}, \ldots, Y''_{n-K+1})^t$$

where $Y_{n-i} = \frac{f^{(j-1)}(x_{n-i}, Y_{n-i})}{(M)^{j}}$. If $n < K - 1$, then some of the $y_i$'s and $y_i^{(j)}$'s are nonexistent and will be assumed to be zero. Similarly define $\hat{Y}_n$ and $\hat{y}_n$.

**Definition 2.6** If a finite sequence $(F_i)_{i=n_0+1}^N$ of formulas from $\mathcal{F}$ is such that there exists a partition of $[a,b]$ on which $F_i$ can be used to find $x_i$, $(i = n_0+1, n_0+2, \ldots, N)$, then $(F_i)_{i=n_0+1}^N$ is said to be a legitimate sequence from $\mathcal{F}$.

**Definition 2.7** An $n \times m$ matrix $A = (a_{ij})$ is said to be bounded by a constant $B$ if $|a_{ij}| \leq B$ for all $i$ and $j$, $(i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$.

Associated with a legitimate sequence $(F_i)_{i=n_0+1}^N$ from $\mathcal{F}$ are the products $A_n A_{n-1} \cdots A_q$, $(n_0 < q \leq n \leq N)$, where $A_i$ is determined by $F_i$, $(i = n_0+1, n_0+2, \ldots, N)$, as given in (2.14). Products of this form play a major role in the stability of the VSVFM.

**Theorem 2.2** If a VSVFM is stable then the products $A_n A_{n-1} \cdots A_q$ are uniformly bounded for all $n_0 < q \leq n \leq N$ and all legitimate sequences $(F_i)_{i=n_0+1}^N$ from $\mathcal{F}$. 
**Proof** Consider the IVP \( y'(x) = 0 \) with \( y(0) = 0 \).

On this problem, (2.1) gives

\[
Y_n = \sum_{i=1}^{k_n} \alpha_{n,k_i-1} Y_{n-i} = a_n Y_{n-1} \quad (2.16)
\]

To get \( Y_n \) from \( Y_{n-1} \), it is necessary to shift the old data and store the new \( y_n \) value. This can be represented by

\[
Y_n = s Y_{n-1} + Y_n a_1
\]

which, when used with (2.16), gives

\[
Y_n = s Y_{n-1} + (a_n Y_{n-1}) a_1 = (s + a_1 a_n) Y_{n-1} = a_n' Y_{n-1}
\]

Thus it follows that

\[
Y_n = a_n' Y_{n-1} = a_n' a_{n-1}' Y_{n-2} = \cdots = a_n' a_{n-1}' \cdots a_1' Y_{K-1}
\]

If the values \( y_0, y_1, \ldots, y_{K-1} \) are given, and \( \hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{K-1} \) are obtained by perturbing \( y_0, y_1, \ldots, y_{K-1} \) by \( r_0, r_1, \ldots, r_{K-1} \), respectively, and if no further perturbations are involved in calculating the solution \( \hat{y}_n \), then

\[
\hat{Y}_n = a_n' a_{n-1}' \cdots a_1' Y_{K-1}
\]

Thus

\[
\hat{Y}_n - Y_n = a_n' a_{n-1}' \cdots a_1' (\hat{Y}_{K-1} - Y_{K-1})
\]

\[
= a_n' a_{n-1}' \cdots a_1' r
\]

where

\[
r = (r_{K-1}, r_{K-2}, \ldots, r_0)^t.
\]

Suppose the products \( a_n' a_{n-1}' \cdots a_q' \) are not uniformly bounded for all \( n_0 < q \leq n \leq N \) and legitimate sequences
(F_{i})_{i=n_0+1}^{N} from \mathcal{F}. Let \varepsilon = 1 and \delta > 0 be given. Suppose 
A_1 A_{n-1} \cdots A_q is a product which is not bounded by 1/\delta.
Without loss of generality, assume q = K, for if not, there 
are other legitimate sequences which use the formulas F_q, F_{q+1}, 
\ldots, F_n to find Y_K, Y_{K+1}, \ldots, Y_{K+n-q}, respectively. Since 
A_1 A_{n-1} \cdots A_K = (a_{ij}) is not bounded by 1/\delta, there exist 
integers i_0 and j_0, 1 \leq i_0, j_0 \leq K, such that |a_{i_0 j_0}| > 1/\delta.
Thus, if r_{j_0-1} = \delta and the rest of the r_j's are zero, then it 
follows from (2.17) that
\[
\max_{1 \leq q \leq N} |\hat{Y}_q - Y_q| \geq ||\hat{Y}_n - Y_n||_{\infty} = \max_{1 \leq i \leq K} |a_{i j_0} r_{j_0-1}| \\
\geq |a_{i_0 j_0} r_{j_0-1}| > 1 = \varepsilon.
\]
Thus, if the products A_1 A_{n-1} \cdots A_q are not uniformly bounded, 
no matter how small \delta is, one perturbation of \delta by itself can 
cause an arbitrarily large change in later results. This 
contradicts the stability of the VSVFM, and therefore, the 
products A_1 A_{n-1} \cdots A_q associated with legitimate sequences 
from \mathcal{F} must be uniformly bounded. Q.E.D.

To be able to discuss stability of a VSVFM on a general 
problem, let T_n be the 1x(M+1)K vector given by
\[
T_n = (a_n, h_n b_n^{(1)}, (h_n)^2 b_n^{(2)}, \ldots, (h_n)^M b_n^{(M)})
\]
(2.18)
Then the VSF[k_n, m_n] (2.1) and equation (2.10) can be rewritten
in terms of \( T_n, Y_{n-1} \) and \( \hat{Y}_{n-1} \) by

\[
Y_n = T_n Y_{n-1} + \sum_{j=1}^{m_n} (h_n)^j \beta_{n,k_n}^{(j-1)} (x_n, Y_n)
\] (2.19)

and

\[
\hat{Y}_n = T_n \hat{Y}_{n-1} + \sum_{j=1}^{m_n} (h_n)^j \beta_{n,k_n}^{(j-1)} (x_n, \hat{Y}_n)
\]

\[
+ \sum_{i=1}^{k_n} \alpha_{n,k_n-i} \hat{Y}_{n-i} (0)
\]

\[
+ \sum_{j=1}^{m_n} (h_n)^j \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)} \hat{Y}_{n-i}
\] (2.20)

Therefore, (2.19) and (2.20) together give

\[
\hat{Y}_n - Y_n = T_n (\hat{Y}_{n-1} - Y_{n-1})
\]

\[
+ \sum_{j=1}^{m_n} (h_n)^j \beta_{n,k_n}^{(j-1)} [f^{(j-1)} (x_n, \hat{Y}_n) - f^{(j-1)} (x_n, Y_n)]
\]

\[
+ \sum_{i=1}^{k_n} \alpha_{n,k_n-i} (0)
\]

\[
+ \sum_{j=1}^{m_n} (h_n)^j \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)} \hat{Y}_{n-i}
\] (2.21)

Since \( f(x,y) \) satisfies the conditions (2.2), it follows for all \( j, (j = 1, 2, \ldots, M) \), that there exists constants \( \theta_n^{(j)} \) such that

\[
f^{(j-1)} (x_n, \hat{Y}_n) - f^{(j-1)} (x_n, Y_n) = \theta_n^{(j)} (Y_n - \hat{Y}_n)
\] (2.22)
where \(|\theta_n^{(j)}| \leq L^{(j)}\). Thus, using (2.22) in (2.21) gives

\[
\hat{Y}_n - Y_n = T_n(\hat{Y}_{n-1} - Y_{n-1})
\]

\[
+ \sum_{j=1}^{m_n} (h_n) j \beta_{n,k_n}^{(j)} (\hat{Y}_n - Y_n)
\]

\[
+ \sum_{i=1}^{k_n} \alpha_{n,k_n-i}^{(0)}
\]

\[
+ \sum_{j=1}^{m_n} (h_n) j \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)}
\]

or equivalently,

\[
(1 - \sum_{j=1}^{m_n} (h_n) j \beta_{n,k_n}^{(j)} \theta_n^{(j)}) (\hat{Y}_n - Y_n) = T_n(\hat{Y}_{n-1} - Y_{n-1})
\]

\[
+ \sum_{i=1}^{k_n} \alpha_{n,k_n-i}^{(0)}
\]

\[
+ \sum_{j=1}^{m_n} (h_n) j \sum_{i=0}^{k_n} \beta_{n,k_n-i}^{(j)}
\]

So if

\[
(1 - \sum_{j=1}^{m_n} (h_n) j \beta_{n,k_n}^{(j)} \theta_n^{(j)}) \neq 0
\]

then letting \(c_n\) be given by

\[
l/c_n = (1 - \sum_{j=1}^{m_n} (h_n) j \beta_{n,k_n}^{(j)} \theta_n^{(j)})
\]

transforms (2.23) into
\[ \hat{Y}_n - Y_n = c_n T_n(\hat{Y}_{n-1} - Y_{n-1}) \]

\[ + c_n \left( \sum_{i=1}^{k_n} d_{n, k_n-i} \gamma_n(0) \right) \]

\[ + c_n \left( \sum_{j=1}^{m_n} \sum_{i=0}^{k_n} \beta_{j}^{(j)}(j) \right) \]

At each step, the stored data must be updated. To get \( Y_n \) from \( Y_{n-1} \), it is necessary to shift the old data and store the new data which includes \( y_n \) and \( y_n^{(j)} = f^{(j-1)}(x_n, y_n) \), \( j = 1, 2, \ldots, M \). This can be represented by

\[ Y_n = s'(Y_n-1) + 0_{n}^{0} + \sum_{j=1}^{M} f^{(j-1)}(x_n, y_n) e_{jK+1} \]

\( \hat{Y}_n \) is similarly represented, giving

\[ \hat{Y}_n - Y_n = s'(\hat{Y}_{n-1} - Y_{n-1}) + (\hat{Y}_n - Y_n) e_1 \]

\[ + \sum_{j=1}^{M} [f^{(j-1)}(x_n, \hat{Y}_n) - f^{(j-1)}(x_n, Y_n)] e_{jK+1} \]

\[ = s'(\hat{Y}_{n-1} - Y_{n-1}) + (\hat{Y}_n - Y_n) e_1 \]

\[ + \sum_{j=1}^{M} \theta_{n}^{(j)} (\hat{Y}_n - Y_n) e_{jK+1} \]

\[ = s'(\hat{Y}_{n-1} - Y_{n-1}) + (\hat{Y}_n - Y_n) [e_1 + \sum_{j=1}^{M} \theta_{n}^{(j)} e_{jK+1}] \]

Using (2.26) to replace \( \hat{Y}_n - Y_n \) gives

\[ \hat{Y}_n - Y_n = s'(\hat{Y}_{n-1} - Y_{n-1}) \]
So
\[ \hat{y}_n - y_n = R_n(\hat{y}_{n-1} - y_{n-1}) + r_n \] (2.27)
where
\[ R_n = s' + c_n[\theta_1 + \sum_{j=1}^{M} \theta_n^{(j)} e^{jK+1}]T_n \]
and
(2.28)
\[ r_n = c_n(\sum_{i=1}^{k_n} N_{i-n}^{(0)} + \sum_{i=1}^{m_n} (h_n) \sum_{i=0}^{k_n} N_{i-n}^{(j)} + \sum_{i=1}^{m_n} (h_n) \sum_{i=0}^{k_n} N_{i-n}^{(j)}) \]
\[ * [\theta_1 + \sum_{j=1}^{M} \theta_n^{(j)} e^{jK+1}] \]
Thus
\[ \hat{y}_n - y_n = R_n(\hat{y}_{n-1} - y_{n-1}) + r_n \]
\[ = R_n[R_{n-1}(\hat{y}_{n-2} - y_{n-2}) + r_{n-1}] + r_n \]
\[ = R_nR_{n-1}(\hat{y}_{n-2} - y_{n-2}) + R_n r_{n-1} + r_n \]
\[ \vdots \]
\[ = R_nR_{n-1} \cdots R_{n-1}R_{n-0+1}(\hat{y}_{n-0} - y_{n-0}) + R_nR_{n-1} \cdots R_{n-0+2} r_{n-0+1} + \cdots + R_n r_{n-1} + r_n \]
By (2.10), $\hat{x}_{n_0} = x_{n_0}$, so for $n > n_0$,
\[
\hat{x}_n - x_n = R_n R_{n-1} \cdots R_{n_0+2} x_{n_0+1} + \cdots + R_n R_{n-1} \cdots R_q x_{q-1} + \cdots + R_n x_{n-1} + x_n
\]

(2.29)

For (2.29) to be valid, it was necessary to assume that
\[
(1 - \sum_{j=1}^{m} (h_n) j^{(j)} \beta_{n,j}^{(j)}) \neq 0
\]
for $n_0 < q < n$, see (2.24). The next lemma shows that this is the case, under the right circumstances.

**Lemma 2.1** For a VSVFM, if the coefficients of the formulas in $F$ are uniformly bounded, then there exists a constant $h^* > 0$ and a $C > 0$ such that $|c_n| \leq C$ uniformly for $n_0 < n \leq N$ and $0 < h \leq h^*$, where $c_n$ is defined by (2.25).

**Proof** By hypothesis, there exists a $B > 0$ such that $|\beta_{n,j}^{(j)}| \leq B$ and since $f(x,y)$ satisfies (2.2), $|\theta_{n,j}^{(j)}| \leq L$, where $L = \max_{1 \leq j \leq M} L^{(j)}$. Let $h^* = \min(1/2, 1/(2MBL))$ and assume $0 < h_n \leq h \leq h^* \leq 1/2$. Now,
\[
|\sum_{j=1}^{m} (h_n) j^{(j)} \beta_{n,j}^{(j)} \theta_{n,j}^{(j)}| = h_n \sum_{j=1}^{m} (h_n) j^{(j)} |\beta_{n,j}^{(j)}| |\theta_{n,j}^{(j)}|
\]
\[
\leq h^* \sum_{j=1}^{m} BL
\]
\[
\leq (\min(1/2, 1/(2MBL))) MBL
\]
\[
\leq 1/2
\]
Thus,

\[ 1/c_n = 1 - \sum_{j=1}^{m_n} (h_n)^j \beta(j) \theta(j) \]

\[ \geq 1 - \left| \sum_{j=1}^{m_n} (h_n)^j \beta(j) \theta(j) \right| \geq 1/2 \]

So let C = 2, then |c_n| \leq C uniformly for n_0 < n \leq N and 0 < h \leq h^*.

Q.E.D.

The following lemmas will be useful in proving that the products of the form R_nR_{n-1}...R_j in (2.29) are uniformly bounded with the appropriate conditions put on the VSVFM.

**Lemma 2.2** For any constant B \geq 0, there exists a constant B' such that if A is a nxn matrix which is bounded by B, then ||A||_1 \leq B'.

**Proof** Let A = (a_{ij}) be any nxn matrix bounded by B, then

\[ ||A||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} B = nB, \] so let B' = nB.

Q.E.D.

**Lemma 2.3** If a nxn matrix A is such that ||A||_1 \leq B, then A is also bounded by B.

**Proof** For any element a_{i0j0} of A = (a_{ij}),

\[ |a_{i0j0}| \leq \sum_{i=1}^{n} |a_{ij}| \leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| = ||A||_1 \leq B. \]

Q.E.D.

**Remark** Lemmas 2.2 and 2.3 show that the concept of "uniform boundedness" of matrices can be determined by element...
(Definition 2.7) or by norm, since they are equivalent.

Lemma 2.4 If $D_i$, $E_i$ and $F_i$ are $m \times m$ matrices such that
$D_i = E_i + F_i$, $(i = j, j+1, \ldots, n)$, then

$$D_n D_{n-1} \cdots D_j = E_n E_{n-1} \cdots E_j$$

(2.30)

where $E_n \cdots E_{i+1} = I$ for $i = n$ and $D_{i-1} \cdots D_j = I$ for $i = j$.

Proof The proof is by induction on $n$. (2.30) clearly holds for $n = j$. Suppose (2.30) holds for all $n$ such that $j \leq n \leq q$, then for $n = q + 1$,

$$D_{q+1} D_{q} \cdots D_j = (E_{q+1} + F_{q+1}) D_q D_{q-1} \cdots D_j$$

$$= E_{q+1} D_q D_{q-1} \cdots D_j + F_{q+1} D_q D_{q-1} \cdots D_j$$

Using the inductive assumption gives

$$D_{q+1} D_{q} \cdots D_j = E_{q+1} (E_q E_{q-1} \cdots E_j + \sum_{i=j}^{q} E_q \cdots E_{i+1} F_{i} D_{i-1} \cdots D_j)$$

$$+ F_{q+1} D_q D_{q-1} \cdots D_j$$

$$= E_{q+1} E_q \cdots E_j + \sum_{i=j}^{q} E_q \cdots E_{i+1} F_{i} D_{i-1} \cdots D_j$$

$$+ F_{q+1} D_q D_{q-1} \cdots D_j$$

Combining the last two terms gives

$$D_{q+1} D_{q} \cdots D_j = E_{q+1} E_q \cdots E_j + \sum_{i=j}^{q+1} E_q \cdots E_{i+1} F_{i} D_{i-1} \cdots D_j$$

Q.E.D.
Except for some slight modifications, the statement of the following lemma is due to Gear and Tu (1974). A proof is included for completeness.

Lemma 2.5 If $D_1$, $E_1$ and $F_1$ are $m \times m$ matrices such that $D_i = E_i + h_i F_i$, $||F_i||_1 \leq D_0$ and $||EnE_{n-1}\cdots E_j||_1 \leq D_1$, with $D_1 \geq 1$, $(i = j, j+1, \ldots, n)$, then

$$||D_n D_{n-1}\cdots D_j||_1 \leq D_1 e^{D_0 D_1 H_{j,n}}$$

where $H_{j,n} = \sum_{i=j}^{n} h_i$.

Proof The proof is by induction on $n$.

For $n = j$, $||D_n||_1 = ||E_n + h_n F_n||_1$

$$\leq ||E_n||_1 + h_n ||F_n||_1$$

$$\leq D_1 + h_n D_0$$

$$\leq D_1 (1 + D_0 D_1 h_n)$$

$$\leq D_1 e^{D_0 D_1 h_n}$$

Assume the conclusion holds for $j \leq n < q$. Using Lemma 2.4,

$$||D_{q+1} D_q \cdots D_j||_1$$

$$= ||E_{q+1} E_{q} \cdots E_j + \sum_{i=j}^{q+1} h_i E_{q+1} \cdots E_i F_{i+1} D_{i+1} \cdots D_j||_1$$

$$\leq ||E_{q+1} E_{q} \cdots E_j||_1$$

$$+ \sum_{i=j}^{q+1} h_i ||E_{q+1} \cdots E_i F_{i+1} D_{i+1} \cdots D_j||_1$$

$$\leq D_1 e^{D_0 D_1 H_{j,n}}$$
\[ \leq D_1 + D_0 \sum_{i=j}^{q+1} h_i |D_{i-1} \cdots D_{j}|_1 \]

Using the inductive assumption gives
\[ ||D_{q+1}D_q \cdots D_j||_1 \leq D_1 + D_0 \sum_{i=j}^{q+1} h_i D_1 e^{D_0 D_1 h_i, i-1} \]
\[ = D_1 + D_0 \sum_{i=j}^{q+1} h_i D_0 D_1 e^{D_0 D_1 h_i, i-1} \]

Now, \( 1 + D_0 D_1 h_i \leq e \) implies \( D_0 D_1 h_i \leq (e - 1) \), giving
\[ ||D_{q+1}D_q \cdots D_j||_1 \leq D_1 \left( 1 + \sum_{i=j}^{q+1} (e - 1) e^{D_0 D_1 h_i, i-1} \right) \]
\[ = D_1 \left( 1 + \sum_{i=j}^{q+1} (e - e^{D_0 D_1 h_i, i-1}) \right) \]

Cancellation of terms in the sum gives
\[ ||D_{q+1}D_q \cdots D_j||_1 \leq D_1 e^{D_0 D_1 H_j, q+1} \quad Q.E.D. \]

R_n in (2.28) can be rewritten in the form
\[ R_n = [I + (C_n - 1)D] R_n' = R_n' + (C_n - 1) D R_n' \] (2.31)

where
\[ I = \text{the (M+1)Kx(M+1)K identity matrix} \]
\[ D = \text{the (M+1)Kx(M+1)K diagonal matrix} \]
\[ = \text{diag}(1,0,\ldots,0|1,0,\ldots,0|\cdots|1,0,\ldots,0) \]

and
\[ R_n' = s + [e_1 + \sum_{j=1}^{M} \theta_n^{(j)} e_{jK+1}] T_n. \]
Now,
\[
R_n' = \begin{bmatrix}
\theta_n^{(1)} a_n' & 0 & \cdots & 0 \\
\theta_n^{(2)} a_n' & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\theta_n^{(M)} a_n' & 0 & \cdots & 0 \\
\end{bmatrix}
\] (2.32)

So
\[
R_n' = U_n + h_n U_n'
\] (2.33)

where \(U_n\) and \(U_n'\) are the obvious choices.

Also,
\[
c_n - 1 = \frac{\sum_{j=1}^{m_n} (h_n)^j \beta_n, k_n \theta_n^{(j)}}{1 - \sum_{j=1}^{m_n} (h_n)^j \beta_n, k_n \theta_n^{(j)}}
\]
thus,
\[ c_n - 1 = h_n c_n \left( \sum_{j=1}^{m_n} (h_n)^{j-1} \theta_{n,j} \right) \]  \hspace{1cm} (2.34)

Using (2.33) and (2.34) in (2.31) gives
\[ R_n = U_n + h_n (U_n + c_n \left( \sum_{j=1}^{m_n} (h_n)^{j-1} \theta_{n,j} \right) ) D_R' \]

So
\[ R_n = U_n + h_n V_n \]  \hspace{1cm} (2.35)

where
\[ V_n = U_n + c_n \left( \sum_{j=1}^{m_n} (h_n)^{j-1} \theta_{n,j} \right) D_R' \]

and as a result, using (2.35) with Lemma 2.4 gives
\[ R_n R_{n-1} \cdots R_j = U_n U_{n-1} \cdots U_j \]
\[ + \sum_{i=j}^{n} h_i U_n \cdots U_{i+1} V_i R_{i-1} \cdots R_j \]

where \( U_n \cdots U_{i+1} = I \) for \( i = n \) and \( R_{i-1} \cdots R_j = I \) for \( i = j \).

The next lemma will show that the products \( U_n U_{n-1} \cdots U_j \) are uniformly bounded when the products \( A_p A_{p-1} \cdots A_j \), \( p = j, j+1, \ldots, n \), are uniformly bounded.

**Lemma 2.6**

For a VSVFM, if the coefficients of the formulas in \( \mathcal{G} \) are uniformly bounded and if the products \( A_n A_{n-1} \cdots A_q \) are uniformly bounded for all \( n_0 < q \leq n \leq N \) and legitimate sequences \( (F_i)_{i=n_0+1}^N \) from \( \mathcal{G} \), then the products \( U_n U_{n-1} \cdots U_q \) are also uniformly bounded.

**Proof**

\( U_n \) is given by (2.33) from which it follows that \( U_n U_{n-1} \cdots U_q \) is equal to
where $A_i^{-1} \cdots A_q^i = I$ when $i = q$. Thus,

\[
\begin{align*}
|u_n u_{n-1} \cdots u_q|_1 & \leq |A_n A_{n-1}^i \cdots A_q^i|_1 + M|s^{n-q+1}|_1 \\
& + \sum_{j=1}^{M} \sum_{i=q}^n |s^{n-i} \theta_n^{(j)} A_i A_{i-1}^i \cdots A_q^i|_1 \\
& \leq |A_n A_{n-1}^i \cdots A_q^i|_1 + M|s^{n-q+1}|_1 \\
& + \sum_{j=1}^{M} \sum_{i=q}^n |\theta_n^{(j)}| |s^{n-i}|_1|A_i|_1|A_{i-1}^i \cdots A_q^i|_1
\end{align*}
\]

(2.37)

By hypothesis, the products $A_p A_{p-1}^i \cdots A_q^i$ are uniformly bounded for $n_0 < p \leq q \leq N$, so there exists a constant $D \geq 1$ such that $|A_i^{-1} A_{i-2}^i \cdots A_q^i|_1 \leq D$, $(i = q, q+1, \ldots, n+1)$. The requirement that $D \geq 1$ is made to take care of the case when
i = q, which results in $A_{i-1}A_{i-2}\cdots A_q = I$ and $||I||_1 = 1$. The coefficients of the formulas are uniformly bounded, so there exists a $B \geq 0$ such that $||A_i||_1 \leq B$ and by conditions (2.2), there exists a $L \geq 0$ such that $|\theta_n^{(j)}| \leq L$. Therefore, from (2.37) it follows that

$$||U_nU_{n-1}\cdots U_q||_1 \leq D + M||S^{n-q+1}||_1 + LBD \sum_{j=1}^{M} \sum_{i=q}^{n} ||S^{n-i}||_1$$

(2.38)

Now, $||S^p||_1 \leq 1$ for all $p > 0$, and in addition, $S^p = 0$ for $p \geq K$, so $\sum_{i=q}^{n} ||S^{n-i}||_1$ contains at most $K$ nonzero terms, as a result, (2.38) gives

$$||U_nU_{n-1}\cdots U_q||_1 \leq D + M + LBDMK = D_1$$

Thus, the products $U_nU_{n-1}\cdots U_q$ are uniformly bounded since the bound $D_1$ is independent of the legitimate sequence of formulas which is used. Q.E.D.

Lemma 2.7 For a VSVFM, if the coefficients of the formulas in $\mathcal{F}$ are uniformly bounded and if the products $A_nA_{n-1}\cdots A_q$ are uniformly bounded for all $n_0 < q < n \leq N$ and legitimate sequences $(F_i)_{i=n_0+1}^N$ from $\mathcal{F}$, then there exists a $h^* \leq 1$ such that for $0 < h \leq h^*$, the products $R_nR_{n-1}\cdots R_q$ are also uniformly bounded.

Proof Let $B \geq 0$ be a uniform bound of the coefficients of the formulas in $\mathcal{F}$, and let $D_1 \geq 1$ be a uniform bound of $||U_nU_{n-1}\cdots U_p||_1$, ($p = q, q+1, \ldots, n$),
guaranteed by Lemma 2.6. Lemma 2.1 guarantees the existence of constants $h^* \leq 1$ and $C \geq 0$ such that if $0 < h < h^*$ then $|c_p| \leq C$, $(p = q, q+1, \ldots, n)$, where $c_p$ is given by (2.25).

Assume $h$ satisfies $0 < h \leq h^*$. For $U_p^i$ defined in (2.32) and (2.33),

$$||U_p^i||_1 \leq \max_{1 \leq j \leq M} ||B_p^{(j)}||_1 (h_p)^{j-1} (1 + \sum_{i=1}^M |\theta_p^{(i)}|)$$

$$\leq (1 + ML) \max_{1 \leq j \leq M} ||B_p^{(j)}||_1$$

(2.39)

where $L = \max_{1 \leq j \leq M} L^{(j)}$. By the form of $B_p^{(j)}$ in (2.14), it follows that $||B_p^{(j)}||_1 \leq B$, so (2.39) gives

$$||U_p^i||_1 \leq B(1 + ML)$$

(2.40)

By (2.33), $R_p^i = U_p^i + h_p U_p^i$, so

$$||R_p^i||_1 \leq ||U_p^i||_1 + h_p ||U_p^i||_1 \leq D_1 + B(1 + ML)$$

(2.41)

For $V_p$ in (2.35), it follows that

$$||V_p||_1 \leq ||U_p^i||_1$$

$$+ |c_p| \sum_{j=1}^{mp} (h_p)^{j-1} (h_p)^{(j)} \theta_p^{(j)} ||D||_1 ||R_p^i||_1$$

$$\leq B(1 + ML) + \text{CMBL}(D_1 + B(1 + ML)) = D_0$$

From (2.35), $R_p = U_p + h_p V_p$, and since $||U_n U_{n-1} \cdots U_p||_1 \leq D_1$ with $D_1 \geq 1$, and $||V_p||_1 \leq D_0$, by Lemma 2.5,

$$||R_n R_{n-1} \cdots R_q||_1 \leq D_0 D_1 H_q, n$$

$D_0 D_1 e$.
and since \( H_q, n = \sum_{i=q}^{n} h_i \leq (b-a) \), it follows that for all \( n_0 < q \leq n \leq N \),

\[
\| R_n R_{n-1} \ldots R_q \|_1 \leq D_1 e^{D_0 (b-a)} = D_2
\]

Since \( D_2 \) is independent of \( n, q \) and the sequence of formulas used, the products \( R_n R_{n-1} \ldots R_q \) are uniformly bounded.

Q.E.D.

With the use of the previous lemmas, the following theorem on stability can be proven quite easily.

**Theorem 2.3**  For a VSVFM, if the coefficients of the formulas in \( \mathcal{F} \) are uniformly bounded and if the products \( \lambda_n \lambda_{n-1} \ldots \lambda_q \) are uniformly bounded for all \( n_0 < q < n < N \) and legitimate sequences \( \{ F_i \}_{i=n_0+1}^N \) from \( \mathcal{F} \), then the VSFVM is stable.

**Proof**  By Lemma 2.7 and Lemma 2.1, there exists a \( h^* < 1 \) such that for \( h \) satisfying \( 0 < h < h^* \), constants \( D_2 \geq 0 \) and \( C > 0 \) exist with \( \| R_n R_{n-1} \ldots R_q \|_1 \leq D_2 \) and \( |c_q| \leq C \) uniformly for all \( n_0 < q < n \leq N \) and legitimate sequence \( \{ F_i \}_{i=n_0+1}^N \) from \( \mathcal{F} \). Using (2.29) gives

\[
|\hat{y}_n - y_n| \leq |\hat{y}_n - y_n|_1 \leq D_2 \sum_{p=n_0+1}^{N} \| x_p \|_1
\]

(2.42)

Now by (2.28),

\[
\| x_p \|_1 = \| e_1 + \sum_{j=1}^{M} \beta_p^{(j)} e_{jK+1} \|_1
\]
Using this in (2.42) gives

\[ |\hat{Y}_n - Y_n| \leq CB(1 + ML) \]

for \( n_0 < n < N \).

Therefore, for every \( \varepsilon > 0 \), let \( \delta(\varepsilon) = \varepsilon/[CB(1 + ML)] \).

Then if

\[ \sum_{p=n_0+1}^{N} (\sum_{i=1}^{k_p} |r_{p,i}| + \sum_{j=1}^{m_p} k_p |r_{j}|) \leq \delta(\varepsilon) \]

it follows from (2.43) that \( \max_{1 \leq n \leq N} |\hat{Y}_n - Y_n| \leq \varepsilon \).  Q.E.D.

Theorems 2.1, 2.2 and 2.3 together result in the following stability theorem which gives necessary and sufficient conditions for the stability of a VSVFM.

**Theorem 2.4** A VSVFM is stable if and only if the coefficients of the formulas in \( \mathcal{G} \) are uniformly bounded and the products \( A_n A_{n-1} \cdots A_q \) are uniformly bounded for all \( n_0 < q \leq n < N \) and legitimate sequences \((F_i)_{i=n_0+1}^{N} \) from \( \mathcal{G} \).
Proof Immediate from Theorems 2.1, 2.2 and 2.3.
Q.E.D.

Remark The advantage of these necessary and sufficient conditions over those in Definition 2.3 is that the stability of a VSVMF is entirely in terms of the coefficients of the formulas. Although, it should be noted that satisfying these conditions in general is no simple task.

C. Consistency of the VSVMF

Let \( \tilde{y}_n \) be computed using the same formula that is used to find \( y_n \), but assume that all the information used is exact instead of using the computed approximations, i.e.,

\[
\tilde{y}_n = \sum_{i=1}^{k_n} \alpha_n, k_n-i \gamma(x_{n-i})
\]

(2.44)

\[
+ \sum_{j=1}^{m_n} (h_n) \sum_{j=0}^{k_n} (j) \beta_n, k_n-i \gamma^j (x_{n-i})
\]

Let \( d_n = y(x_n) - \tilde{y}_n \), which is the local truncation error at \( x_n \).

**Definition 2.8** A VSVMF is consistent if for all functions \( f(x,y) \) satisfying conditions (2.2),

\[
\lim_{h \to 0} \sum_{n=n_0+1}^{N} |d_n| = 0.
\]

**Remark** It should be noted that \( d_n \) is just the difference operator \( L_n[y(x_n);h_n] \) from (2.5).
It is well known that fixed-stepsize fixed-formula first derivative methods of order one are consistent. For VSVFMs, adding an additional condition that the coefficients of the formulas in \( \mathcal{F} \) be uniformly bounded will be sufficient.

**Remark** It should be noted that the fixed-stepsize fixed-formula first derivative methods automatically satisfy this additional condition.

**Theorem 2.5** If a VSVFM is order 1 and the coefficients of the formulas in \( \mathcal{F} \) are uniformly bounded, then the VSVFM is consistent.

**Proof** For \( \epsilon > 0 \), define

\[
Z(\epsilon) = \max \left| y'(x^*) - y'(x) \right|
\]

where the maximum is over \( x^*, x \in [a, b] \) with \( |x^* - x| \leq \epsilon \).

Then for any \( f_q \) such that \( x_{n-q} \leq f_q \leq x_n \), since \( h_t, x_{n-q} = x_n - x_{n-q} \), it follows that

\[
y'(f_q) = y'(x_n) + G_{n,q}Z(h_t, x_{n-q}) \tag{2.46}
\]

for some \( G_{n,q} \) satisfying \( |G_{n,q}| \leq 1 \). Furthermore, since

\[
y(x_{n-q}) = y(x_n) - h_t, x_{n-q}y'(f_q)
\]

for some \( f_q \) satisfying \( x_{n-q} \leq f_q \leq x_n \), (2.46) gives that

\[
y(x_{n-q}) = y(x_n) - h_t, x_{n-q}[y'(x_n) + G_{n,q}Z(h_t, x_{n-q})] \tag{2.47}
\]

with \( |G_{n,q}| \leq 1 \). Let \( d'_{n} \) be given by

\[
d'_{n} = y(x_n) - \sum_{i=1}^{k_n} \alpha_{n, k_n-i} y(x_{n-i}) - h_t \sum_{i=0}^{k_n} \beta_{n, k_n-i} y'(x_{n-i})
\]
Since (2.45) and (2.47) hold for \( n_0 < q \leq n \leq N \),
\[
d_n' = y(x_n)
\]
\[
= \sum_{i=1}^{k_n} \alpha_{n,k_n-i}y(x_n) - h_{n-t_n,i}[y'(x_n) + G_{n,i}Z(h_{n-t_n,i})]
\]
\[
- \beta^{(1)}_{n,k_n}y'(x_n)
\]
\[
- h_n \sum_{i=1}^{k_n} \beta^{(1)}_{n,k_n-i}[y'(x_n) + G_{n,i}Z(h_{n-t_n,i})]
\]
\[
= [1 - \sum_{i=1}^{k_n} \alpha_{n,k_n-i}]y(x_n)
\]
\[
+ h_n \left( \sum_{i=1}^{k_n} \alpha_{n,k_n-i} + \sum_{i=0}^{k_n} \beta^{(1)}_{n,k_n-i}y'(x_n) \right)
\]
\[
+ h_n \sum_{i=1}^{k_n} (\alpha_{n,k_n-i}G_{n,i} + \beta^{(1)}_{n,k_n-i}G_{n,i})Z(h_{n-t_n,i})
\]
The VSVFM is assumed to be order 1, so by (2.8) it follows that
\[
d_n' = \sum_{i=1}^{k_n} h_{n-t_n,i} \alpha_{n,k_n-i}G_{n,i}Z(h_{n-t_n,i})
\]
\[
+ \sum_{i=1}^{k_n} h_n \beta^{(1)}_{n,k_n-i}G_{n,i}Z(h_{n-t_n,i})
\]
(2.48)
Let \( B \geq 0 \) be a uniform bound of the coefficients of the formulas in \( \mathcal{F} \). Since \( h_n \leq h_{n-t_n,i} \leq h_{n-t_n,k} \), \( |G_{n,i}| \leq 1 \), \( |G'_{n,i}| \leq 1 \) and \( Z(h_{n-t_n,i}) \leq Z(Kh) \), \( (i = 1, 2, \ldots, k_n) \), (2.48) gives
\[ |d_n'| \leq h_n t_n, k_n^2 K BZ(Kh) \] (2.49)

Since \( d_n = y(x_n) - \hat{y}_n \), (2.45) gives

\[ d_n = d_n' - \sum_{j=2}^{m_n} (h_n)^j \sum_{i=0}^{k_n} \beta_n, k_n - i y^{(j)}(x_{n-1}) \]

so

\[ |d_n| \leq |d_n'| + h_n \sum_{j=2}^{m_n} (h_n)^{j-1} \sum_{i=0}^{k_n} B_1 \] (2.50)

where \( B_1 = \max \{ y'', y''', \ldots, y^{(M)} \} \) over \([a, b]\).

Since \( h \to 0 \), without loss of generality, assume \( h < 1 \), then (2.49) and (2.50) give

\[ |d_n| \leq h_n t_n, k_n^2 K BZ(Kh) + h_n h M K B B_1 \] (2.51)

Now

\[ \sum_{n=n_0+1}^{N} h_n t_n, k_n = \sum_{n=n_0+1}^{N} (h_n + h_{n-1} + \cdots + h_{n-k_n+1}) \]

\[ \leq K \left( \sum_{n=n_0+1}^{N} h_n \right) \]

\[ \leq K(b - a) \]

Thus (2.51) gives

\[ \sum_{n=n_0+1}^{N} |d_n| \leq 2K^2 (b - a) B Z(Kh) + h(b - a) M K B B_1 \]

and since \( Z(Kh) \to 0 \) as \( h \to 0 \), it follows that

\[ \lim_{h \to 0} \sum_{n=n_0+1}^{N} |d_n| = 0. \quad Q.E.D. \]
D. Convergence of the VSVFM

Intuitively, convergence of a VSVFM means that the calculated solution approaches the true solution as the partitions of \([a,b]\) become finer. The following definition formalizes this concept.

**Definition 2.9** A VSVFM is **convergent** if for all functions \(f(x,y)\) satisfying conditions (2.2), the following holds: If \(y(x)\) denotes the solution to the IVP (1.3), then

\[
\lim_{n \to \infty} \max_{n_0 \leq n \leq N} |y(x_n) - y_n| = 0
\]

as \(h \to 0\) and the starting errors tend to zero.

The following theorem gives sufficient conditions on a VSVFM to guarantee convergence. It should be noted that these conditions do correspond to the classical ones.

**Theorem 2.6** If a VSVFM is consistent and stable then the it is convergent.

**Proof** Let \(Y(x_n)\) be the \(1 \times (M+1)K\) column vector defined similar to \(Y_n\) in (2.15), except the entries are the exact values coming from the solution \(y(x)\) to the IVP (1.3). Now \(d_n = y(x_n) - \tilde{y}_n\) so

\[
y(x_n) - y_n = \tilde{y}_n - y_n + d_n
\]

Using (2.19) and (2.44) gives

\[
y(x_n) - y_n = T_n(Y(x_{n-1}) - y_{n-1})
\]

(2.52)
\[ + \sum_{j=1}^{m_n} (h_n)^j \beta_n^{(j)} [Y_n(j)(x_n) - f^{(j-1)}(x_n, y_n)] + d_n \]

Since \( y^{(j)}(x_n) = f^{(j-1)}(x_n, y(x_n)) \) and \( f(x, y) \) satisfies conditions (2.2), there exist \( \theta_n^{(j)} \), \( (j = 1, 2, \ldots, M) \) such that \( f^{(j-1)}(x_n, y(x_n)) - f^{(j-1)}(x_n, y_n) \)

\[ = \theta_n^{(j)} (y(x_n) - y_n) \]

Thus, using (2.53) in (2.52) gives

\[ y(x_n) - y_n = T_n(y(x_{n-1}) - y_{n-1}) + \sum_{j=1}^{m_n} (h_n)^j \beta_n^{(j)} [y_n(x_n) - y_n] + d_n \]

which reduces to

\[ (1 - \sum_{j=1}^{m_n} (h_n)^j \beta_n^{(j)} [y_n(x_n) - y_n]) (y(x_n) - y_n) \]

\[ = T_n(y(x_{n-1}) - y_{n-1}) + d_n \]

Since the method is stable, as a result of Theorem 2.4 and Lemma 2.1, there exists a \( h^* > 0 \) such that if \( 0 < h < h^* \), defining \( c_n \) as in (2.25) is valid and there exists a \( C > 0 \) such that \( |c_n| \leq C \), \( (n = n_0+1, n_0+2, \ldots, N) \). Thus, (2.54) can be rewritten as

\[ y(x_n) - y_n = c_n T_n(y(x_{n-1}) - y_{n-1}) + c_n d_n \]

Now
Thus, using (2.53) gives

\[ Y(x_n) - Y_n = s'(Y(x_{n-1}) - Y_{n-1}) + (y(x_n) - y_n) \left[ e_1 + \sum_{j=1}^{M} \theta_n^{(j)} e_{jK+1} \right] \]

Replacing \( y(x_n) - Y_n \) by (2.55) gives

\[ Y(x_n) - Y_n = R_n(Y(x_{n-1}) - Y_{n-1}) + d_n \]

where \( R_n \) is defined as in (2.28) and \( d_n \) is given by

\[ d_n = c_n[d_n e_1 + \sum_{j=1}^{M} \theta_n^{(j)} e_{jK+1}] \quad (2.56) \]

Therefore,

\[ Y(x_n) - Y_n = R_n(Y(x_{n-1}) - Y_{n-1}) + d_n \]

\[ = R_n[R_n-1(Y(x_{n-2}) - Y_{n-2}) + d_{n-1}] + d_n \]

\[ = R_nR_n-1(Y(x_{n-2}) - Y_{n-2}) + R_n d_{n-1} + d_n \]

\[ = R_nR_n-1 \cdots R_0(Y(x_0) - Y_0) + R_n d_{n-1} + d_n \]

Since the method is stable, Theorem 2.4 and Lemma 2.7 give
that the products $R_n R_{n-1} \cdots R_j$, $(j = n_0+1, n_0+2, \ldots, n)$, are uniformly bounded so there exists a $D_2 \geq 1$ such that $||R_n R_{n-1} \cdots R_j||_1 \leq D_2$, $(j = n_0+1, n_0+2, \ldots, n)$, and since $|y(x_n) - y_n| \leq ||y(x_n) - y_n||_1$, (2.57) leads to

$$|y(x_n) - y_n| \leq D_2 ||y(x_{n_0}) - y_{n_0}||_1$$

(2.58)

$$+ D_2 \sum_{i=n_0+1}^{n} ||d_i||_1$$

Now

$$||d_i||_1 \leq c_i d_i (||e_1 + \sum_{j=1}^{n} e_{n} e_{j} x_{1+i}||_1) \leq C (1 + ML) |d_i|$$

Using this in (2.58) gives

$$|y(x_n) - y_n| \leq D_2 ||y(x_{n_0}) - y_{n_0}||_1$$

(2.59)

$$+ D_2 C (1 + ML) \sum_{i=n_0+1}^{N} |d_i|$$

Since the starting errors tend to zero, $||y(x_{n_0}) - y_{n_0}||_1 \to 0$ as $h \to 0$, and since the method is consistent, $\sum_{i=n_0+1}^{N} |d_i| \to 0$ as $h \to 0$. Therefore, since (2.59) holds for all $n$ such that $n_0 < n \leq N$, it is immediate that

$$\max_{n_0 \leq n \leq N} |y(x_n) - y_n| \to 0$$

as $h \to 0$ and the starting errors tend to zero. Hence, the VSVFM is convergent. Q.E.D.

The discussion will now show that the conditions of Theorem 2.6, in addition to being sufficient, are in fact necessary for a VSVFM to be convergent.
Lemma 2.8 A convergent VSVFM must be at least order zero.

Proof Consider any formula from \( \mathcal{F} \). Since only one formula is going to be considered, for simplicity, the extra subscripts in (2.1) will be suppressed. Also, since this \( k \)-step formula will only be used to find \( y_k \), it follows that

\[
Y_k = \sum_{i=1}^{k} \alpha_{k-i} Y_{k-i} + \sum_{j=1}^{m} (h_k)^j \sum_{i=0}^{k} \beta_{k-i}^{(j)} (j-1) (x_{k-i}, y_{k-i})
\]

Consider the IVP \( y'(x) = 0 \) and \( y(0) = 1 \). When the formula is applied to this problem, (2.60) gives

\[
Y_k = \sum_{i=1}^{k} \alpha_{k-i} Y_{k-i}
\]

Suppose the value \( y_0 = 1 \) is given exactly and the values \( y_1, y_2, \ldots, y_{k-1} \) are found by the VSVFM. For fixed \( j \), \( x_j \to 0 \) as \( h \to 0 \), so it follows that if the VSVFM is to be convergent, then for \( 0 < j \leq k \), \( y_j \to y(0) = 1 \) as \( h \to 0 \). Therefore, (2.61) gives

\[
1 = \lim_{h \to 0} Y_k = \sum_{i=1}^{k} \alpha_{k-i} (\lim_{h \to 0} Y_{k-i}) = \sum_{i=1}^{k} \alpha_{k-i}
\]

Thus, \( C_0 = 1 - \sum_{i=1}^{k} \alpha_{k-i} = 0 \) and the formula must be at least order 0. Since this is true for all \( k \)-step formulas, \( (k = 1, 2, \ldots, K) \), the VSVFM must be at least order 0. Q.E.D.
Lemma 2.9  If a VSVFM is convergent, then all 1-step formulas in $\mathcal{F}$ are at least order 1.

Proof  Consider any 1-step formula from $\mathcal{F}$. Since only one formula will be used, the extra subscripts will be suppressed. Lemma 2.8 shows that a 1-step formula is at least order 0, so $\alpha_0 = 1$. Consider the IVP $y'(x) = 1$ and $y(0) = 0$ on $[0,1]$. When applied to this problem, the 1-step formula gives

$$Y_n = Y_{n-1} + h_n(\beta_0 + \beta_1) \quad \text{for } 0 < n \leq N$$

Then for any partition $P$ of $[a,b]$, and if $y_0 = 0$, it follows that

$$Y_1 = y_0 + h_1(\beta_0 + \beta_1) = h_1(\beta_0 + \beta_1)$$
$$Y_2 = Y_1 + h_2(\beta_0 + \beta_1) = (h_2 + h_1)(\beta_0 + \beta_1)$$
$$\vdots$$
$$Y_N = \left( \sum_{i=1}^{N} h_i \right)(\beta_0 + \beta_1) = \beta_0 + \beta_1$$

Since $y(x) = x$ is the solution to the IVP, it follows that

$$\lim_{h \to 0} y_N = y(1) = 1 \text{ must hold, but } 1 = \lim_{h \to 0} y_N = \lim_{h \to 0} (\beta_0 + \beta_1) = (\beta_0 + \beta_1) = 1.$$  
Therefore, by Lemma 2.8, all 1-step formulas in $\mathcal{F}$ must be at least order 1. Q.E.D.

Lemma 2.10  If a VSVFM is convergent, then it must be at least order 1.

Proof  Consider the IVP $y'(x) = 1$ and $y(0) = 0$ on $[0,1]$. The proof is by induction. By Lemma 2.9, all 1-step
formulas in $\mathcal{F}$ are at least order 1. Suppose all $j$-step formulas in $\mathcal{F}$, $(j = 1, 2, \ldots, k-1)$, are at least order 1. Thus, a $j$-step formula from $\mathcal{F}$, $1 \leq j < k$, is exact on this IVP, if the backpoint information it uses is exact. Suppose $F_k$ is a $k$-step formula from $\mathcal{F}$. Use $F_k$ to find $y_k$ and any formulas $F_1, F_2, \ldots, F_{k-1}$ from $\mathcal{F}$ to find $y_1, y_2, \ldots, y_{k-1}$, respectively. Clearly, $F_1, F_2, \ldots, F_{k-1}$ are at most $(k-1)$-step formulas, so it follows that $y_1, y_2, \ldots, y_{k-1}$ are exact. Thus, after suppressing extra subscripts, it follows that

$$Y_k = \sum_{i=1}^{k} a_{k-i} (x_{k-i} + y_0) + h_k \sum_{i=0}^{k} \beta_{k-i}$$

Using $x_{k-i} = x_k - h_k t_{k,i}$ gives

$$Y_k = \sum_{i=1}^{k} a_{k-i} (x_k - h_k t_{k,i} + y_0) + h_k \sum_{i=0}^{k} \beta_{k-i}$$

$$= (x_k + y_0) \left( \sum_{i=1}^{k} a_{k-i} \right) - h_k \left( \sum_{i=1}^{k} a_{k-i} t_{k,i} - \sum_{i=0}^{k} \beta_{k-i} \right)$$

Using the definitions of $C_0$ and $C_1$ from (2.8) and the fact that Lemma 2.8 guarantees a $k$-step formula from $\mathcal{F}$ must be at least order 0, (2.62) gives

$$Y_k = (x_k + y_0) - h_k C_1$$

Let $x_n$ be such that $N x_n = 1$. Integrate the problem by repeating the sequence $(F_1, F_2, \ldots, F_k)$ of formulas $N$ times, i.e., use formula $F_q$ to find $y_q, y_{k+q}, \ldots, y_{(N-1)k+q}$, $(q = 1, 2, \ldots, k)$. Each time the sequence of formulas is repeated, it is like starting over with the new initial value, i.e.,
\[ Y_{nk} = (x_k + y(n-1)k) - h_kc_1 \]

In particular, for \( n = N \),
\[ Y_{Nk} = (x_k + y(N-1)k) - h_kc_1 \]
\[ = x_k + [(x_k + y(N-2)k) - h_kc_1] - h_kc_1 \]
\[ = 2x_k + y(N-2)k - 2h_kc_1 \]
\[ = Nx_k + y_0 - Nh_kc_1 \]

Using \( N x_k = 1 \) and the fact that \( y_0 = 0 \) gives
\[ Y_{Nk} = 1 - Nh_kc_1 \]

For the VSVFM to be convergent, it is necessary that \( \lim_{h \to 0} y(l) = 1 \). Therefore, it is necessary for \( \lim_{h \to 0} Nh_kc_1 \) to be zero, but since \( Nh_k = h_k/x_k \) is a nonzero constant, this is only possible if \( c_1 = 0 \). Hence, \( C_0 = C_1 = 0 \) for all k-step formulas in \( G \), \( 1 \leq k \leq K \), giving the VSVFM must be at least order 1. Q.E.D.

**Theorem 2.7** If a VSVFM is convergent, then the coefficients of the formulas in \( G \) are uniformly bounded.

**Proof** Consider the IVP \( y'(x) = 0 \) and \( y(0) = 0 \) to be solved on \([0,1]\) and let \( k \) be a fixed integer such that \( 1 \leq k \leq K \). Suppose the set of k-step formulas in \( G \) do not have uniformly bounded coefficients. Then there exists a sequence \( (F_q)_{q=1}^\infty \) of k-step formulas in \( G \) such that for any \( q \geq 1 \), there is at least one coefficient of \( F_q \) of magnitude greater than \( 2^q \).
For each $q \geq 1$, define a partition $P_q$ of $[0,1]$, denoted by $0 = x_{q,0} < x_{q,1} < \cdots < x_{q,N_q} = 1$, as follows:

1. the spacing of the grid points $x_{q,0}, x_{q,1}, \ldots, x_{q,k}$ is appropriate for the use of formula $F_q$ to find the approximation $y_{q,k}$ to $y(x_{q,k})$,
2. the maximum of the stepsizes $h_{q,i} = x_{q,i} - x_{q,i-1}$, $i = 1, 2, \ldots, k$, is $1/(kq)$, and
3. the remaining stepsizes $h_{q,i}$, $i > k$, are less than or equal to $1/(kq)$.

$h = 1/(kq)$ for the partition $P_q$, therefore, as $q \to \infty$, $h \to 0$, and as a result, since $k$ is fixed, $x_{q,k} \to x_{q,0} = 0$. So if the VSVFM is to be convergent, it follows that as $q \to \infty$, $y_{q,k}$ must approach $y(0) = 0$, independent of the formulas which are used to integrate on the partitions $P_q$. Therefore, if $F_q$ is used to find $y_{q,k}$, it is necessary that $y_{q,k} \to 0$ as $q \to \infty$, provided the starting errors tend to zero.

The only formulas of concern are the $F_q$'s, so with straightforward changes in notation, formula $F_q$ which is applied to partition $P_q$ to find $y_{q,k}$ can be given by

$$y_{q,k} = \sum_{i=1}^{k} a_{q,i} y_{q,k-i} + \sum_{j=1}^{m_q} (h_{q,k})^j \sum_{i=0}^{k} b_{q,i} y_{q,k-i}$$

where $y_{q,k-i}$ is the approximation to $y^{(i)}(x_{q,k-i})$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, m_q$.

The solution to the IVP is $y(x) = 0$, so the values $y_{q,i}$
(i = 0, 1, ..., k - 1) and $y_{q,i}^{(j)}$ ($i = 0, 1, ..., k$, $j = 1, 2, ..., m_q$) are the actual starting errors that $F_q$ will be using. In the following, if more than one coefficient of $F_q$ tie for having the largest magnitude, arbitrarily designate one of them as having the largest magnitude.

For $1 \leq i \leq k$, let

$$y_{q,i} = \begin{cases} \frac{1}{2^q} & \text{if } \alpha_{q,i} \text{ is the coefficient of } F_q \\ \text{with the largest magnitude} \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq i \leq k$ and $1 \leq j \leq m_q$, let

$$y_{q,i}^{(j)} = \begin{cases} \frac{j}{h^j 2^q} & \text{if } \beta_{q,i}^{(j)} \text{ is the coefficient of } F_q \\ \text{with the largest magnitude} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, for each $q \geq 1$, there is only one nonzero starting error, and since $h = 1/(kq)$ and $\Delta$ is constant, it follows that the starting errors go to zero as $q \to \infty$.

If one of the $\alpha$'s of $F_q$ has the largest magnitude, then for some $i$, $1 \leq i \leq k$,

$$y_{q,k}^{(j)} = \frac{\beta_{q,k-i}^{(j)}}{2^q}$$

and since $|\alpha_{q,k-i}| > 2^q$, it follows that $|y_{q,k}| > 1$. Now if one of the $\beta$'s of $F_q$ has the largest magnitude, then for some
i, 0 \leq i \leq k, \text{ and some } j, 1 \leq j \leq m_q,

\begin{align*}
y_{q,k} &= \frac{(h_q,k)^{j \cdot j(p(j))}}{h_{q,k}^2 q} = \frac{(h_q,k)^{j \cdot j(p(j))}}{h_{q,k}^2 q},
\end{align*}

and since \( |\beta_{q,k-1}^{(j)}| > 2^q \) and \( h_q,k \Delta \geq h \), it follows that
\( |y_{q,k}| > 1 \). Therefore, for all \( q \geq 1 \), \( |y_{q,k}| > 1 \), and so as
\( q \to \infty \), \( y_{q,k} \) does not go to zero. This contradicts the method
being convergent, thus the coefficients of the k-step formulas
in \( \mathcal{E} \) must be uniformly bounded.

Lastly, since this holds for all \( k \) such that \( 1 \leq k \leq K \),
the coefficients of the formulas in \( \mathcal{E} \) are uniformly bounded.

Q.E.D.

Theorem 2.8 If a VSVFM is convergent then the
products \( A_{n}^i A_{n-1}^i \cdots A_{m}^i \) are uniformly bounded for all \( n_0 < m < n \)
\( \leq N \) and all legitimate sequences \( (F_i)_{i=n_0+1}^N \) from \( \mathcal{F} \).

Proof Suppose the products are not uniformly bounded.
Then for every integer \( q > 0 \), there exists a product
\( A_{q,n_q}^i A_{q,n_q-1}^i \cdots A_{q,k}^i \) which is not bounded by \( q \), where
\( A_{q,n_q}^i A_{q,n_q-1}^i \cdots A_{q,k}^i \) is associated with a legitimate sequence
\( (F_q,i)_{i=K}^{n_q} \) from \( \mathcal{F} \).

Consider the IVP \( y'(x) = 0 \) and \( y(0) = 0 \) on \([0,1]\). For
each \( q > 0 \), let \( P_q \) be a partition of \([0,1]\), denoted by
\( 0 = x_q,0 < x_q,1 < \cdots < x_q,N_q = 1 \), such that the largest
stepsize is at most \( 1/q \) and such that \( y_{q,k},y_{q,k+1},\ldots,y_{q,n_q} \)
can be found by using formulas $F_{q,K}, F_{q,K+1}, \ldots, \dot{F}_{q,n_q}$ respectively. This implies $n_q < N_q$ and that $(\dot{F}_{q,i})_{i=K}^{n_q}$ might only be a subsequence of the formulas used to integrate the problem on $P_q$. Now it follows that

$$y_{q,n} = A_{q,n}^{n_q}A_{q,n_q-1}^{n_q}A_{q,K}^{v_q,K}$$

(2.63)

where

$$y_{q,m} = (y_{q,m}, y_{q,m-1}, \ldots, y_{q,m-K+1})^t \quad K-1 \leq m \leq n_q$$

In the following, if more than one element of $A_{q,n}^{n_q}A_{q,n_q-1}^{n_q}A_{q,K}^{v_q,K}$ tie for having largest magnitude, arbitrarily designate one of them as having the largest magnitude. For $0 < i < K - 1$, let

$$y_{q,i} = \begin{cases} \frac{1}{q} & \text{if the } i^{th} \text{ column of } A_{q,n}^{n_q}A_{q,n_q-1}^{n_q}A_{q,K}^{v_q,K} \\
0 & \text{otherwise} \end{cases}$$

Clearly, for each $i$, $0 \leq i \leq K - 1$, $y_{q,i} \to 0$ as $q \to \infty$, so the starting errors are going to zero as $q \to \infty$. Then, since $A_{q,n}^{n_q}A_{q,n_q-1}^{n_q}A_{q,K}^{v_q,K}$ is not bounded by $q$, it follows from (2.63) that for all $q > 0$, $||y_{q,n}||_\infty > 1$, which gives

$$\max_{K \leq n \leq N_q} |y(x_{q,n}) - y_{q,n}| = \max_{K \leq n \leq N_q} |y_{q,n}| \geq ||y_{q,n}||_\infty > 1$$

for all $q > 0$. But, since $q \to \infty$ implies $h \to 0$, this contradicts the method being convergent.

Therefore, the products $A_{n}^{v_n}A_{n-1}^{v_{n-1}} \cdots A_{m}^{v_m}$ must be uniformly
bounded for $n_0 < m \leq n \leq N$ and all legitimate sequences

$(F_i)_{i=n_0+1}^N$ from $\mathcal{F}$. Q.E.D.

Combining these results on convergence together with those on stability and consistency yield the following theorem.

**Theorem 2.9** A VSVFM is convergent if and only if it is stable and consistent.

**Proof** Immediate from the following,

$$(\text{convergent}) \implies \begin{cases} \text{order 1} \implies \text{consistent} \\ \text{stable} \implies \text{stable} \end{cases} \implies \text{convergent}$$

where (1) is shown by Lemma 2.10 and Theorems 2.7 and 2.8

(2) is shown by Theorem 2.5

(3) is shown by Theorem 2.6. Q.E.D.

**Remark** Theorem 2.9 can restated in the equivalent form: A VSVFM is convergent if and only if it is stable and at least order 1. In this form the theorem is more usable since the necessary and sufficient conditions are completely in terms of the coefficients of the formulas in $\mathcal{F}$.

**E. Reduction of the VSVFM to the FSFFM[k,m]**

In this section, it will be shown that if a VSVFM is restricted to the use of a single fixed-stepsize $k$-step $m^{th}$-derivative formula ($FSF[k,m]$), then the stability of the VSVFM reduces to the condition of stability for a FSFFM[k,m].

**Remark** It is assumed that the stability of a
FSFFM[k,m] is defined as in Definition 1.5.

It will be useful to introduce some notation which will be used here as well as in the next chapter. Let $A$ be an $n \times n$ matrix and $J$ its Jordan Normal Form. Then there exists a nonsingular matrix $Q$ such that $J = Q^{-1}AQ$ where

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_q \end{bmatrix}$$

(2.64)

and each $J_i$ is an $n_i \times n_i$ matrix of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

with $\sum_{i=1}^{q} n_i = n$, and $\lambda_1, \lambda_2, \ldots, \lambda_q$ the distinct eigenvalues of $A$. Now (2.64) gives

$$J^m = \begin{bmatrix} J^m_1 & 0 & \cdots & 0 \\ 0 & J^m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J^m_q \end{bmatrix}$$

(2.65)

for $m > 0$ and if $J^m_i = (d_{jk}^{(m)}(i))$, $1 \leq j, k \leq n_i$, then
\[ d_{jk}^{(m)}(i) = \begin{cases} 
0 & k < j \\
(m-j)\lambda_i^{m-k+j} & j \leq k \leq \min(n_i, m+j) \\
0 & m+j < k \leq n_i 
\end{cases} \tag{2.66} \]

For a more detailed description, see Varga (1962, pp. 13-14).

**Theorem 2.10** If \( A \) is an \( nxn \) matrix, then the matrices \( A^m \) are uniformly bounded for all \( m > 0 \) if and only if the eigenvalues of \( A \) are modulus less than or equal to one and those of modulus one are simple.

**Proof** Let \( \lambda_1, \lambda_2, \ldots, \lambda_q \) be the distinct eigenvalues of \( A \) and \( J \) its Jordan Form. Since \( J = Q^{-1}AQ \), \( J^m = Q^{-1}A^mQ \), and thus, it follows that

\[
\|J^m\|_1 \leq \|A^m\|_1 \leq \|Q^{-1}\|_1\|J^m\|_1\|Q\|_1 \]

Therefore, the matrices \( A^m \) are uniformly bounded for \( m > 0 \) if and only if the matrices \( J^m \) are uniformly bounded for \( m > 0 \).

Now,

\[
\|J^m\|_1 = \max(\|J_{11}^m\|_1, \|J_{22}^m\|_1, \ldots, \|J_{qq}^m\|_1),
\]

so it suffices to show that the matrices \( J_i^m \), \( (i = 1, 2, \ldots, q) \), are uniformly bounded for all \( m > 0 \) if and only if the eigenvalues of \( A \) are modulus less than or equal to one and those of modulus one are simple.

Suppose the matrices \( J_i^m \) are uniformly bounded by \( B \geq 0 \). Then (2.66) gives \( |\lambda_i^m| \leq \|J_i^m\|_1 \leq B \) for all \( m > 0 \), which
implies that $|\lambda_i| \leq 1$. Suppose that $\lambda_i$ is an eigenvalue of modulus 1 which is not simple. Then $J_i$ is an $n_i \times n_i$ matrix with $n_i > 1$. By (2.66), $d_{12}^{(m)}(i) = m \lambda_i^{m-1}$ so $|d_{12}^{(m)}(i)| = m$ and hence, $|d_{12}^{(m)}(i)| \to \infty$ as $m \to \infty$, contradicting the uniform boundedness of $J_i^m$. Thus, the eigenvalues of $A$ must be modulus less than or equal to one, and those of modulus one are simple.

Suppose the eigenvalues of $A$ are modulus less than or equal to one and those of modulus one are simple. If $|\lambda_i| = 1$, then $J_i^m$ is a $1 \times 1$ matrix whose element is always bounded by 1. By (2.66),

$$d_{jk}^{(m)}(i) = \begin{cases} (k-j)\lambda_i^{m-k+j} & \text{if } j \leq k \leq n_i \\ 0 & \text{otherwise} \end{cases}$$

for $m > n_i$

Therefore, if $|\lambda_i| < 1$, it follows that $J_i^m \to 0$ as $m \to \infty$, and thus the matrices $J_i^m$ are uniformly bounded for $m > 0$. Q.E.D.

With Theorem 2.10, the following result is now simple to show.

**Theorem 2.11** The stability of a VSVFM reduces to the stability of a FSFFM[k,m] when $I$ is restricted to only one FSF[k,m].

**Proof** Since only one formula is being used, the uniform boundedness of the coefficients is immediate, and uniform boundedness of the products $A_n A_{n-1} \cdots A_j$ reduces to the
matrices \((A')^m\) being uniformly bounded for \(m > 0\), where \(A'\) is the matrix associated with \(\rho(\xi)\) and given by (2.14). Since \(A'\) is the companion matrix for the polynomial \(\rho(\xi)\), it follows immediately from Theorem 2.10 that \(\rho(\xi)\) must satisfy the root condition, which is the condition of stability for the \(\text{FSFFM}[k,m]\). Q.E.D.

Theorem 2.11, together with Theorem 2.9, gives the following extension of Theorem 1.1 to \(m^{\text{th}}\)-derivative methods.

**Theorem 2.12** A \(\text{FSFFM}[k,m]\) is convergent if and only if it is consistent and stable.

**Proof** Immediate from Theorem 2.11 and Theorem 2.9. Q.E.D.
III. CONVERGENT VSVFMS

In the previous chapter, necessary and sufficient conditions were obtained for a VSVFM, as defined in Definition 2.3, to be convergent. The present discussion will now focus on showing that certain VSVFMs are convergent, but first, some comments and results which will be useful.

For a FSFFM\([k]\), stability is ensured if the polynomial \(\rho(\bar{y})\) associated with the method satisfies the root condition. Even if a VSVFM consists of formulas which have uniformly bounded coefficients, adding the condition that all the formulas in \(\mathcal{F}\) have their associated \(\rho\)-polynomials satisfy the root condition, or even the strong-root condition, is not enough to ensure stability of the method. As an example, consider the 3-step formulas \(F_1\) and \(F_2\) which have associated polynomials \(\rho_1(\bar{y}) = (\bar{y} - 1)(\bar{y} - 1/2)^2 = \bar{y}^3 - 2\bar{y}^2 + 5/4\bar{y} - 1/4\) and \(\rho_2(\bar{y}) = (\bar{y} - 1)(\bar{y} + 1/2)^2 = \bar{y}^3 - 3/4\bar{y} - 1/4\), respectively. \(\rho_1(\bar{y})\) and \(\rho_2(\bar{y})\) clearly satisfy the strong-root condition.

The associated matrices \(A_1^i\) and \(A_2^i\) are given by

\[
A_1^i = \begin{bmatrix} 2 & -5/4 & 1/4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A_2^i = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]

and thus,
\[ A_1 A_2^t = \begin{bmatrix} -5/4 & 7/4 & 1/2 \\ 0 & 3/4 & 1/4 \\ 1 & 0 & 0 \end{bmatrix} \]

\( A_1 A_2^t \) has eigenvalues 1 and \(-3/4 \pm \sqrt{2}/2\), thus, by Theorem 2.10 it follows that the products \((A_1 A_2^t)^n\) are not uniformly bounded, and hence, the sequence of formulas \(F_1, F_2, F_1, F_2, \ldots\) is unstable.

**Remark** Even though this example shows that having the \(\rho\)-polynomials satisfy the root condition is not sufficient for stability, there are some cases where stability is guaranteed.

The following lemmas will be useful in showing that a VSVFM is stable.

**Lemma 3.1** The products \(A_n A_{n-1} \cdots A_j\) associated with legitimate sequences \((F_i)_{i=n_0+1}^N\) from \(\mathfrak{G}\) are uniformly bounded if there exists an invertible matrix \(H\) such that for any formula \(F\) from \(\mathfrak{G}\), \(\|H^{-1} A' H\|_1 \leq 1\), where \(A'\) is determined by \(F\) and given by (2.14).

**Proof**

\[ \|A_n A_{n-1} \cdots A_j\|_1 = \|H \left( \prod_{i=j}^N H^{-1} A_i^t H\right) H^{-1}\|_1 \]

\[ \leq \|H\|_1 \|H^{-1}\|_1 \prod_{i=j}^N \|H^{-1} A_i^t H\|_1 \]

\[ \leq \|H\|_1 \|H^{-1}\|_1 \]

Q.E.D.
Lemma 3.2 If the spectral radius of an nxn matrix A is less than 1, then there exists an invertible matrix H such that \(|\|H^{-1}AH\|_1 < 1\).

Proof Let J be the Jordan Form of A. Then there exists an invertible matrix Q such that \(J = Q^{-1}AQ\), where J has the form given in (2.64). Let \(\varepsilon\) be the spectral radius of A, then \(|\lambda_i| \leq \varepsilon\), \((i = 1, 2, \ldots, q)\), where the \(\lambda_i\)'s are the q distinct eigenvalues of A.

If the \(n_i\times n_i\) Jordan block \(J_i\) is such that \(n_i > 1\), let \(H_i = \text{diag}(d_1, d_2, \ldots, d_{n_i})\) where \(d_j > 0\), \((j = 1, 2, \ldots, n_i)\). Now,

\[
H_i^{-1}J_iH_i = \begin{bmatrix}
    \lambda_1 & d_2/d_1 & 0 & \cdots & 0 \\
    0 & \lambda_2 & d_3/d_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & \cdots & \cdots & \lambda_q \frac{d_{n_i}}{d_{n_i-1}} \\
    0 & \cdots & \cdots & \cdots & \lambda_q
\end{bmatrix}
\]

Letting \(d_1 = 1\) and choosing the remaining \(d_j\)'s such that \(d_{j+1}/d_j \leq (1 - \varepsilon)/2\), \((j = 1, 2, \ldots, n_i - 1)\), then \(\|H_i^{-1}J_iH_i\|_1 \leq |\lambda_i| + \max_{1 \leq j \leq n_i - 1} d_{j+1}/d_j \leq \varepsilon + (1 - \varepsilon)/2 = (1 + \varepsilon)/2 < 1\).

If the \(n_i\times n_i\) Jordan block \(J_i\) is such that \(n_i = 1\), let \(H_i\) be the 1x1 matrix \([1]\). Then \(\|H_i^{-1}J_iH_i\|_1 = \|J_i\|_1 = |\lambda_i| < 1\).

Therefore, if \(H = Q\text{diag}(H_1, H_2, \ldots, H_q)\), then

\(\|H^{-1}AH\|_1 = \max_{1 \leq i \leq q} \|H_i^{-1}J_iH_i\|_1 < 1. \) Q.E.D.
Lemma 3.3  If the polynomial $p(f)$ associated with formula $F$ satisfies the strong-root condition, then there exists an invertible matrix $H$ such that $\|H^{-1}A'H\|_1 = 1$, where $A'$ given by (2.14).

Proof  Suppose that $F$ is a $k$-step formula and let

$$Q_1 = \begin{bmatrix} 1 & 0 & & 0 \\ 1 & \epsilon & & 0 \\ 1 & & & 0 \\ 1 & & & \epsilon \end{bmatrix} \quad (3.1)$$

where $\epsilon > 0$. Then,

$$Q_1^{-1} = \begin{bmatrix} 1 & 0 & & 0 \\ -1 & 1/\epsilon & & 0 \\ 0 & -1/\epsilon & 1/\epsilon & 0 \\ 0 & 0 & -1/\epsilon & 1/\epsilon \end{bmatrix} \quad (3.2)$$

Since $p(f)$ has a root of 1, it follows that

$$Q_1^{-1}A'Q_1 = \begin{bmatrix} 1 & \epsilon X \\ 0 & M \end{bmatrix} \quad (3.3)$$

where

$$X = (x_2, x_3, \ldots, x_K) \text{ with } x_j = \sum_{i=0}^{K-j} a_i, \quad (j = 2,3,\ldots,K)$$

and
Since \( \rho(f) \) satisfies the strong-root condition, the spectral radius of \( M \) is less than 1 so by Lemma 3.2, there exists an invertible matrix \( Q_2 \) such that \( \|Q_2^{-1}MQ_2\|_1 < 1 \). Thus, if

\[
H = Q_1 \text{diag}(1, Q_2),
\]

it follows that

\[
H^{-1}A'H = \begin{bmatrix}
1 & \varepsilon XQ_2 \\
0 & Q_2^{-1}MQ_2
\end{bmatrix}
\]

and thus, \( \|H^{-1}A'H\|_1 \leq \max(1, \varepsilon \|XQ_2\|_1 + \|Q_2^{-1}MQ_2\|_1) \).

If \( \|XQ_2\|_1 = 0 \), then let \( \varepsilon = 1 \), otherwise, let \( \varepsilon = (1 - \|Q_2^{-1}MQ_2\|_1)/\|XQ_2\|_1 \). It then follows that \( \|H^{-1}A'H\|_1 = 1 \). Q.E.D.

Remark In the previous theorem, it should be noted that since \( \rho(f) \) satisfies the strong-root condition, \( \|H^{-1}A'H\|_1 \) cannot be made any smaller than 1, for any choice of \( H \).

Before showing any particular VSVFMs are convergent, it will be convenient to replace condition (4) of Definition 2.3 with a more restrictive condition. This condition will be
referred to as condition (4') and is given by

(4') There exists fixed constants $\delta$ and $\Delta$ ($0 < \delta < 1 < \Delta$) depending on $\mathcal{F}$ such that if a $k_n$-step formula from $\mathcal{F}$ can be used to find $y_n$, where $k_n > 1$, it then follows that

$$\delta \leq h_i/h_{i-1} \leq \Delta, \; (i = n-k_n+2, n-k_n+3, \ldots, n).$$

Remark Condition (4'), as with condition (4) of Definition 2.3, does not restrict the partitions of $[a,b]$ which can be considered. But, condition (4') does further restrict which VSF$[k_n,m_n]$ can be in $\mathcal{F}$.

The following lemma will be useful in showing the coefficients of the formulas in $\mathcal{F}$ are uniformly bounded for various VSVFMs.

Lemma 3.4 Condition (4') is equivalent to the existence of fixed constants $\delta'$ and $\Delta'$ ($0 < \delta' < 1 < \Delta'$) depending on $\mathcal{F}$ such that if a $k_n$-step formula from $\mathcal{F}$ can be used to find $y_n$, where $k_n > 1$, it then follows that

$$\delta' \leq h_i/h_n \leq \Delta', \; (i = n-k_n+1, n-k_n+2, \ldots, n).$$

Proof Suppose $\delta \leq h_i/h_{i-1} \leq \Delta, \; (i = n-k_n+2, n-k_n+3, \ldots, n)$. Using the fact that

$$h_i/h_n = (h_i/h_{i+1})(h_{i+1}/h_{i+2}) \cdots (h_{n-1}/h_n)$$
gives

$$(1/\delta)^K \leq (1/\delta)^{n-1} \leq h_i/h_n \leq (1/\delta)^{n-1} \leq (1/\delta)^K$$

Thus, if $\delta' = (1/\delta)^K$ and $\Delta' = (1/\delta')^K$, it follows that
\( \delta' \leq \frac{h_i}{h_n} \leq \Delta', \ (i = n-k_n+1, n-k_n+2, \ldots, n) \).

Suppose \( \delta' \leq \frac{h_i}{h_n} \leq \Delta', \ (i = n-k_n+1, n-k_n+2, \ldots, n) \).

Using the fact that \( \frac{h_i}{h_{i-1}} = \frac{h_i}{h_n} \left( \frac{h_n}{h_{i-1}} \right) \) gives

\[
\delta'/\Delta' \leq \frac{h_i}{h_{i-1}} \leq \Delta'/\delta'
\]

Thus, if \( \delta = \delta'/\Delta' \) and \( \Delta = \Delta'/\delta' \), it follows that

\( \delta \leq \frac{h_i}{h_{i-1}} \leq \Delta, \ (i = n-k_n+2, n-k_n+3, \ldots, n) \). Q.E.D.

Remark As a result of Lemma 3.4, it follows that condition (4') implies condition (4).

Now for the formal definitions of the VSVFMs which will be considered.

**Definition 3.1** An Adams-Bashforth VSF[k] \( (AB-VSF[k]) \) has the form

\[
Y_n = Y_{n-1} + h_n \sum_{i=1}^{k} \beta_{n,k} f(x_{n-i}, Y_{n-i})
\] (3.6)

where the coefficients are determined to make the formula order \( k \). An Adams-Bashforth VSVFM \( (AB-VSVFM) \) consists of the \( AB-VSF[k] \)s for \( 1 \leq k \leq K \).

**Definition 3.2** An Adams-Moulton VSF[k] \( (AM-VSF[k]) \) has the form

\[
Y_n = Y_{n-1} + h_n \sum_{i=0}^{k} \beta_{n,k} f(x_{n-i}, Y_{n-i})
\] (3.7)

where the coefficients are determined to make the formula order \( k+1 \). An Adams-Moulton VSVFM \( (AM-VSVFM) \) consists of the \( AM-VSF[k] \)s for \( 1 \leq k \leq K \).
Definition 3.3  
A Backward-Differentiation VSF[k] (BD-VSF[k]) has the form

\[ Y_n = \sum_{i=1}^{k} \alpha_{n,i} Y_{n-i} + h_n \beta_{n,k} f(x_n, x_{n}) \]  

(3.8)

where the coefficients are determined to make the formula order \( k \). A Backward-Differentiation VSVFM (BD-VSVFM) consists of the BD-VSF[k]'s for \( 1 \leq k \leq K \).

Definition 3.4  
A Second-Derivative VSF[k] (2D-VSF[k]) has the form

\[ Y_n = \sum_{i=1}^{k} \alpha_{n,i} Y_{n-i} + h_n \beta_{n,k} f(x_n, x_{n}) \]

\[ + (h_n)^2 \gamma_{n,k}^{(3)} (x_n, Y_n) \]  

(3.9)

where the coefficients are determined to make the formula order \( k + 1 \). A Second-Derivative VSVFM (2D-VSVFM) consists of the 2D-VSF[k]'s for \( 1 \leq k \leq K \).

A. Convergence of AB-VSVFMs and AM-VSVFMs

Definition 3.5  
A VSVFM is said to be a \( \rho \)-constant VSVFM if there exists an integer \( k_0 \geq 0 \) such that the associated \( \rho \)-polynomials of all formulas in \( \mathcal{F} \) which are allowed to be used to find \( y_n \), \( (k_0 < n \leq N) \), have identical nonzero roots, including multiplicities.

Remark  
If two formulas from a VSVFM have \( \rho \)-polynomials with identical nonzero roots, then the two
formulas have the identical matrix $A^i$ which is given by (2.14).

**Theorem 3.1** Assume a $\rho$-constant VSVFM satisfies the following.

1. The VSVFM is at least order 1.
2. The $\rho$-polynomials associated with the formulas in $\mathcal{F}$ satisfy the strict-root condition.
3. The coefficients of the formulas in $\mathcal{F}$ are uniformly bounded.

Then the VSVFM is convergent.

**Proof** Hypothesis (3) gives that the products $A_{k_0}A_{k_0-1}\cdots A_j, (n_0 < j \leq k_0)$, are uniformly bounded. Also, the previous remark together with hypothesis (2) and Theorem 2.10 gives that the products $A_nA_{n-1}\cdots A_j, (k_0 < j \leq n \leq N)$, are uniformly bounded. Thus, the products $A_nA_{n-1}\cdots A_j, (n_0 < j \leq n \leq N)$, associated with legitimate sequences $(F_i)_{i=n_0+1}^N$ from $\mathcal{F}$ are uniformly bounded. Using this result together with hypotheses (1) and (2), Theorems 2.4 and 2.5 give that the VSVFM is stable and consistent, and thus, Theorem 2.9 implies that the VSVFM is convergent. Q.E.D.

**Theorem 3.2** The coefficients of the formulas in the AB-VSVFM are uniformly bounded if the AB-VSVFM satisfies condition (4').

**Proof** Consider a AB-VSF[k] (3.6). The coefficients of the AB-VSF[1] are constant and independent of any stepsize
changes. Now, for the AB-VSF[k] where $k > 1$. If the AB-VSF[k] is to be order $k$, it follows that the coefficients $\beta_n,i, \ (i = 0,1,\ldots,k-1)$, must satisfy the following equations:

\[
1 - \sum_{i=1}^{k} \beta_n,k-i = 0
\]

\[
1/2 - \sum_{i=1}^{k} t_n,i \beta_n,k-i = 0
\]

\[
\vdots
\]

\[
1/k! - 1/(k-1)! \sum_{i=1}^{k} (t_n,i)^{k-1} \beta_n,k-i = 0
\]

Multiplying the $j$th equation by $(j-1)!$ gives

\[
\sum_{i=1}^{k} \beta_n,k-i = 1
\]

\[
\sum_{i=1}^{k} t_n,i \beta_n,k-i = 1/2
\]

\[
\vdots
\]

\[
\sum_{i=1}^{k} (t_n,i)^{k-1} \beta_n,k-i = 1/k
\]

This system can be written in the form

\[
BZ = C
\]

where
The determinant of matrix $B$ is a Vandermonde determinant and therefore,

$$z = (\beta_{n,k-1}, \beta_{n,k-2}, \ldots, \beta_n, 0)^t$$

and

$$c = (1, 1/2, 1/3, \ldots, 1/k)^t$$

The determinant of matrix $B$ is a Vandermonde determinant and therefore,

$$\det(B) = \prod_{1 \leq j < i \leq k} (t_{n,i} - t_{n,j})$$

For details, see Isaacson and Keller (1966, p.188). Since the AB-VSVFM satisfies condition (4'), Lemma 3.4 ensures the existence of constants $\delta'$ and $\Delta'$ such that $0 < \delta' < h_i/h_n < \Delta'$ ($i = n-k+1, n-k+2, \ldots, n$). Now for $j < i$,

$$t_{n,j} - t_{n,i} = [(x_n - x_{n-j}) - (x_n - x_{n-i})]/h_n$$

$$= (x_{n-i} - x_{n-j})/h_n$$

$$\geq h_{n-i}/h_n$$

$$\geq \delta'$$

Thus, for $j < i$, $t_{n,i} - t_{n,j}$ is uniformly bounded away from 0, and as a result, $\det(B)$ is uniformly bounded away from zero.
Also, $1 \leq t_n, j = \sum_{i=n^{2k+1}}^{n} \frac{h_i}{h_n} \leq K$, $(j = 1, 2, \ldots, k)$, so the entries of $B$ are uniformly bounded.

Since $\det(B) \neq 0$, $Z = B^{-1}C$. Now, the elements of $B$ are bounded and $\det(B)$ is bounded away from 0, so the elements of $B^{-1} = (\text{Adjoint of } B)/\det(B)$ are also uniformly bounded. This, together with the elements of $C$ being bounded, give that the entries of $Z$ are uniformly bounded. Hence, the coefficients of the formulas in the AB-VSVFM are uniformly bounded. Hence, the coefficients of the formulas in the AB-VSVFM are uniformly bounded. Q.E.D.

Theorem 3.3 The coefficients of the formulas in the AM-VSVFM are uniformly bounded if the AM-VSVFM satisfies condition (4').

Proof The proof is similar to that of Theorem 3.2.

Theorem 3.4 If the AB-VSVFM satisfies condition (4'), then it is convergent.

Proof The AB-VSVFM is a $\rho$-constant VSVFM and the associated $\rho$-polynomials satisfy the strong-root condition. Thus, Theorems 3.2 together with Theorem 3.1 gives the desired result. Q.E.D.

Theorem 3.5 If the AM-VSVFM satisfies condition (4'), then it is convergent.

Proof The proof is similar to that of Theorem 3.4.

Remark If the AB-VSVFM and the AM-VSVFM satisfy condition (4'), convergence is guaranteed independent of the choice of $\delta$ and $\Delta$ ($0 < \delta < 1 < \Delta$). With the BD-VSVFM and the
2D-VSVFM, it will not turn out to be that simple.

B. Convergence of a BD-VSVFM and a 2D-VSVFM

Theorem 3.6 The coefficients of the formulas in the BD-VSVFM are uniformly bounded if the BD-VSVFM satisfies condition \(4'\).

Proof Consider a BD-VSF[\(k\)] (3.8). The coefficients of the BD-VSF[\(k\)] are constant and independent of any stepsize changes. Now, for the BD-VSF[\(k\)] where \(k > 1\). If the BD-VSF[\(k\)] is to be order \(k\), it follows that the coefficients \(\alpha_{n,i}, (i = 0,1,2,\ldots,k-1)\) and \(\beta_{n,k}\) must satisfy the following equations:

\[
1 - \sum_{i=1}^{k} \alpha_{n,k-i} = 0
\]

\[
\sum_{i=1}^{k} t_{n,i} \alpha_{n,k-i} - \beta_{n,k} = 0
\]

\[
(1/2) \sum_{i=1}^{k} (t_{n,i})^2 \alpha_{n,k-i} = 0
\]

\[
\vdots
\]

\[
(1/k!) \sum_{i=1}^{k} (t_{n,i})^{k-1} \alpha_{n,k-i} = 0
\]

Without loss of generality, assume \(\beta_{n,k} \neq 0\). Multiply the \(j\)th equation by \(j!/\beta_{n,k}\), \((j = 1,2,\ldots,k+1)\), then the system can be written in the form
\[ \frac{1}{\beta_{n,k}} - \sum_{i=1}^{k} \alpha_{n,k-i}/\beta_{n,k} = 0 \] 

(3.11)

and

\[ BZ = C \]

where

\[ B = \begin{bmatrix}
(t_n,1)^2 & (t_n,2)^2 & (t_n,3)^2 & \cdots & (t_n,k)^2 \\
(t_n,1)^3 & (t_n,2)^3 & (t_n,3)^3 & \cdots & (t_n,k)^3 \\
& & & \ddots & \\
(t_n,1)^k & (t_n,2)^k & (t_n,3)^k & \cdots & (t_n,k)^k
\end{bmatrix} \]

and

\[ C = (1,0,\ldots,0)^t \]

The determinant of matrix \( B \) is just a multiple of a Vandermonde determinant and

\[ \det(B) = (t_{n,1} t_{n,2} \cdots t_{n,k}) \prod_{1 \leq j < i \leq k} (t_{n,i} - t_{n,j}) \]

Using an argument similar to that in the proof of Theorem 3.2, it follows that \( Z = B^{-1}C \) and the elements of \( Z \) are uniformly bounded. That is, \( \alpha_{n,k-1}/\beta_{n,k}, \alpha_{n,k-2}/\beta_{n,k}, \ldots, \alpha_{n,0}/\beta_{n,k} \) are uniformly bounded. Thus, by equation (3.11) it follows that \( 1/\beta_{n,k} \) is bounded, which in turn implies that \( \alpha_{n,k-1}, \alpha_{n,k-2}, \ldots, \alpha_{n,0} \) are uniformly bounded. Lastly, the first equation of (3.10) gives that \( \beta_{n,k} \) is uniformly bounded. Q.E.D.
Theorem 3.7  The coefficients of the formulas in the 2D-VSVFM are uniformly bounded if the 2D-VSVFM satisfies condition (4').

Proof  The proof is similar to that of Theorem 3.6.

The stability of a VSVFM which is not a $\rho$-constant VSVFM is in general not a simple task to show. The following theorem will be helpful in showing that the BD-VSVFM and the 2D-VSVFM are stable under the appropriate conditions.

Definition 3.6  For $n \geq k > 1$, define $H_n(k)$ by

$$H_n(k) = (h_n/h_{n-1}, h_{n-1}/h_{n-2}, \ldots, h_{n-k+2}/h_{n-k+1}).$$

Assume the VSVFM is such that $G = \bigcup_{q=1}^{\lambda} G^{(q)}$ where the formulas in $G^{(q)}$ are $k(q)$-step $m(q)$-derivative formulas all of which have their coefficients determined in the same manner regardless of the changes in stepsize. That is, the formulas in $G^{(q)}$ can be written in the form

$$F^{(q)}: y_n = \sum_{i=1}^{k(q)} \alpha_{n,i(q)} y_{n-i} + \sum_{j=1}^{m(q)} (h_n)^j \sum_{i=0}^{k(q)} \beta_{n,i(q)} f^{(j-1)}(x_{n-i}, y_{n-i})$$

where the coefficients $\alpha_{n,i(q)}$ ($i = 1, 2, \ldots, k(q)$) and $\beta_{n,i(q)}$ ($i = 0, 1, \ldots, k(q)$, $j = 1, 2, \ldots, m(q)$) are functions of $H_n(k(q))$ if $k(q) > 1$ and constant if $k(q) = 1$. Let $M_q$ denote the matrix $M$ associated with $F^{(q)}$ as defined in (3.3), $(q = 1, 2, \ldots, \lambda)$. 
Remark The formulas in the BD-VSVFM and the formulas in the 2D-VSVFM satisfy these properties.

For this class of VSVFMs, the following stability result can be stated.

**Theorem 3.8** Let the VSVFM satisfy the following conditions.

1. It is order $p > 0$.
2. The coefficients $\alpha_{n,i}(q)$ and $\beta_{n,i}(q)$ of $F(q)$, for $k(q) > 1$, are continuous functions of $H_n(k(q))$ in a neighborhood of $H_n(k(q)) = (1,1,\ldots,1)$.
3. There exists an invertible matrix $Q_2$ such that
   
   (i) $k(q) = 1$ implies $||Q_2^{-1}H_nQ_2||_1 < 1$, and
   
   (ii) $k(q) > 1$ implies $||Q_2^{-1}H_nQ_2||_1 < 1$, for $H_n(k(q)) = (1,1,\ldots,1)$.

Then, for each $q$ such that $k(q) > 1$, there exists fixed constants $\delta_q$ and $\Lambda_q$ ($0 < \delta_q < 1 < \Lambda_q$) such that the VSVFM is stable if $\delta_q$ satisfies the following statement:

If a formula from $\mathcal{G}(q)$ can be used to find $y_n$,

where $k(q) > 1$, it then follows that

$$\delta_q \leq \frac{h_i}{h_{i-1}} \leq \Lambda_q, \quad (i = n-k(q)+2,n-k(q)+3,\ldots,n).$$

**Proof** Let $A_q$ and $X_q$ be associated with $F(q)$ as defined in (2.14) and (3.3) respectively. Also, let

$H = Q_1 \text{diag}(1,Q_2)$, where $Q_1$ is given in (3.1). Then, as in (3.5), it follows that
and

\[
H^{-1}A_q^t H = \begin{bmatrix} 1 & \varepsilon X_q Q_2 \\ 0 & Q_2^{-1} M_q Q_2 \end{bmatrix}
\]

and

\[
||H^{-1}A_q^t H||_1 \leq \max(1, \varepsilon ||X_q Q_2||_1 + ||Q_2^{-1} M_q Q_2||_1)
\]

Suppose \(q\) is such that \(k(q) = 1\). The formulas in \(\mathcal{F}(q)\) are 1-step formulas and thus, \(X_q = 0\) and \(M_q\) is constant for all formulas in \(\mathcal{F}(q)\). Thus, by (3.13), \(||H^{-1}A_q^t H||_1 = 1\) for all formulas in \(\mathcal{F}(q)\).

Suppose \(q\) is such that \(k(q) > 1\). By hypotheses (2) and (3), there exist fixed constants \(\delta_q\) and \(\Delta_q\) \((0 < \delta_q < 1 < \Delta_q)\) such that \(\delta_q \leq \frac{h_i}{h_{i-1}} \leq \Delta_q\) \((i = n-k(q)+2, n-k(q)+3, \ldots, n)\) implies that the coefficients of the formulas in \(\mathcal{F}(q)\) are uniformly bounded and \(||Q_2^{-1} M_q Q_2||_1 \leq c_q < 1\) for all formulas in \(\mathcal{F}(q)\) and some \(c_q\) \((0 < c_q < 1)\). Assume that \(\mathcal{F}(q)\) satisfies condition (3.12) with \(\delta_q\) and \(\Delta_q\). Since the coefficients of formulas in \(\mathcal{F}(q)\) are bounded, the elements of \(X_q\) are bounded. As a result, there exists an \(\varepsilon_q > 0\) such that \(\varepsilon_q ||X_q Q_2||_1 + ||Q_2^{-1} M_q Q_2||_1 < 1\) for all formulas in \(\mathcal{F}(q)\).

Let \(\xi = \min \varepsilon_q\), where the minimum is taken over \(q\) such that \(k(q) > 1\). If \(\mathcal{F}\) satisfies condition (3.12), then for any formula in \(\mathcal{F}(q)\), where \(k(q) > 1\), it follows that \(\varepsilon||X_q Q_2||_1 + ||Q_2^{-1} M_q Q_2||_1 < 1\), and thus, by (3.13), \(||H^{-1}A_q^t H||_1 = 1\).

Therefore, for all formulas \(F\) in \(\mathcal{F}\), \(||H^{-1}A_q^t H||_1 = 1\), and as a result, using Lemma 3.1 together with the fact that
the coefficients are uniformly bounded, Theorem 2.9 shows the VSVFM is stable. Q.E.D.

Remark If a VSVFM satisfies the hypotheses of Theorem 3.8 and during the integration a drastic change in stepsize is needed, Theorem 3.8 shows that the use of a 1-step formula will preserve the stability of the VSVFM.

To help simplify the code of many practical algorithms, single bounds on the stepsize ratios are preferred for the k-step formulas with \( k > 1 \). Single bounds are guaranteed by the following corollary to Theorem 3.8. This slightly weaker result is an extension to a similar result given by Crouziex and Lisbona (1984).

**Corollary 3.1** Let the VSVFM satisfy the hypotheses of Theorem 3.8. Then there exists fixed constants \( \delta \) and \( \Delta \) \((0 < \delta < 1 < \Delta)\) such that if \( \mathcal{F} \) satisfies condition \((4')\) with \( \delta \) and \( \Delta \), then the VSVFM is stable.

**Proof** It is immediate from the proof of Theorem 3.8 by letting \( \delta = \max \delta_q \) and \( \Delta = \min \Delta_q \), where the maximum and the minimum are taken over \( q \) with \( \kappa(q) > 1 \). Q.E.D.

To make use of Theorem 3.8 or Corollary 3.1, it is necessary to find an invertible matrix \( Q_2 \) which satisfies condition \((3)\) of the theorem. This in general is no simple task. One such matrix will be exhibited in the proofs of the following two convergence theorems.
Theorem 3.9  For $K = 3$, there exists fixed constants $\delta$ and $\Delta$ ($0 < \delta < 1 < \Delta$) such that if a BD-VSVFM satisfies condition (4') with $\delta$ and $\Delta$, then the BD-VSVFM is convergent.

Proof  It is immediate from Definition 3.3 that a BD-VSVFM satisfies hypotheses (1) and (2) of Theorem 3.8. Let $Q_2$ be given by

$$Q_2 = \begin{bmatrix} 1 & 1 \\ 27/32 - i & 27/32 + i \end{bmatrix}$$

The fixed stepsize, 1, 2 and 3-step BDFs are given by

$$Y_n = Y_{n-1} + h_nf(x_n, Y_n)$$

$$Y_n = 4/3Y_{n-1} - 1/3Y_{n-2} + 2/3h_nf(x_n, Y_n)$$

$$Y_n = 18/11Y_{n-1} - 9/11Y_{n-2} + 2/11Y_{n-3} + 6/11h_nf(x_n, Y_n)$$

If the respective matrices $M_1$, $M_2$ and $M_3$ are defined as in (3.3), then

$$M_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1/3 & 0 \\ 1 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 7/11 & -3/11 \\ 1 & 0 \end{bmatrix}$$

As a result,

$$||Q_2^{-1}M_1Q_2||_1 = 1,$$

$$||Q_2^{-1}M_2Q_2||_1 = \frac{5783}{9216} < 1$$

and
\[ \|Q_2^{-1}M_3Q_2\|_1 = \frac{\sqrt{3186337} + \sqrt{8794705}}{2\sqrt{31719424}} < 1 \]

Therefore, the BD-VSVFM satisfies hypothesis (3) of Theorem 3.8. Thus, by Corollary 3.1, there exists fixed constants \( \delta \) and \( \Lambda \) (\( 0 < \delta < 1 < \Lambda \)) such that if \( \mathcal{F} \) satisfies condition (4') with \( \delta \) and \( \Lambda \), the BD-VSVFM is stable. In addition, the BD-VSVFM is order 1, and thus by Theorem 2.9, it is convergent. Q.E.D.

**Theorem 3.10**

For \( K = 3 \), there exists fixed constants \( \delta \) and \( \Lambda \) (\( 0 < \delta < 1 < \Lambda \)) such that if a 2D-VSVFM satisfies condition (4') with \( \delta \) and \( \Lambda \), then the 2D-VSVFM is convergent.

**Proof**

The proof is similar to the proof of Theorem 3.9 and uses the same \( Q_2 \). Q.E.D.

**C. Empirical Results for a BD-VSVFM and a 2D-VSVFM**

Theorem 3.8 and Corollary 3.1 are useful in showing that a VSVFM is stable for "some" changes in the stepsize, but they only guarantee the existence of bounds on the stepsize ratios which will ensure stability. As a result, Theorems 3.9 and 3.10 also give no direct means of determining how much change is acceptable when using a \( k \)-step formula (\( k > 1 \)) in either the BD-VSVFM or the 2D-VSVFM. One way to determine bounds on the stepsize ratios is to try to find a matrix \( Q_2 \) which satisfies condition (3) of Theorem 3.8, and for \( k(q) > 1 \),
satisfies $||Q_2^{-1}M_2Q_2||_1 \leq c < 1$ for an acceptable range of stepsize ratios. Finding such a matrix $Q_2$ is not a simple task, so a numerical search can be employed.

The matrix $Q_2$, which was used in the proofs of Theorems 3.9 and 3.10, was found by using a numerical search. It was chosen so as to allow for a large range of stepsize ratios when used with the BD-VSVFM. Matrix $Q_2$ also turned out to be acceptable with the 2D-VSVFM, and it even allowed for a larger range of stepsize ratios with the 2D-VSVFM than it did with the BD-VSVFM. The results are summarized below.

Results for the BD-VSVFM with $K=3$

1. For the BD-VSF[1],
   $$||Q_2^{-1}M_1Q_2||_1 = 1$$
   for all stepsize changes.

2. For the BD-VSF[2],
   $$||Q_2^{-1}M_2Q_2||_1 \leq 0.999225$$
   for $h_n/h_{n-1} = j/32$
   ($j = 1, 2, \ldots, 76$).

3. For the BD-VSF[3],
   $$||Q_2^{-1}M_3Q_2||_1 \leq 0.999201$$
   for $h_i/h_{i-1} = j/32$
   ($i = n-1, n, j = 1, 2, \ldots, 47$).

Results for the 2D-VSVFM with $K=3$

1. For the 2D-VSF[1],
   $$||Q_2^{-1}M_1Q_2||_1 = 1$$
   for all stepsize changes.

2. For the 2D-VSF[2],
\[ ||Q_2^{-1}M_2Q_2||_1 \leq 0.999977 \text{ for } h_n/h_{n-1} = j/32 \]
\[(j = 1, 2, \ldots, 121).\]

(3) For the 2D-VSF[3],
\[ ||Q_2^{-1}M_3Q_2||_1 \leq 0.999976 \text{ for } h_i/h_{i-1} = j/32 \]
\[(i = n-1, n \quad j = 1, 2, \ldots, 63).\]

With these results, it would seem that letting \( \delta = 1/32 \) and \( \Delta = 47/32 \) would ensure the stability and convergence of the BD-VSVFM with \( K = 3 \). Likewise, letting \( \delta = 1/32 \) and \( \Delta = 63/32 \) would seem to imply that the 2D-VSVFM would also be stable and convergent. Therefore, since these bounds allow for a substantial change in stepsizes, the BD-VSVFM and the 2D-VSVFM with \( K = 3 \) should be quite useful in solve an IVP of the form (1.1).

Remark Since the same \( Q_2 \) works for both the BD-VSVFM and the 2D-VSVFM with \( K = 3 \), combining the formulas from these two methods form another convergent VSVFM.
IV. CONCLUSIONS AND FUTURE WORK

A. Conclusions

A general definition of a VSVFM was given which allows for the use of higher derivative multistep formulas. This definition, unlike the work of Gear and Tu (1974) and Gear and Watanbe (1974), places no restriction on the changes in the stepsize that can occur when a VSVFM is used. Instead, it places restrictions on the formulas which make up the VSVFM. Also, there was no assumption placed on the VSVFM about the boundedness of the coefficients of the formulas in the VSVFM, as was done in the work of Crouziex and Lisbona (1984).

Gear and Tu (1974) gives a definition of stability which requires stability against all perturbations to the stored values which are used to find $y_n$. Here, a slightly more stringent definition was given than that of Gear and Tu in that it also protects against perturbations due to calculating the derivatives of $y$ at $x_n$.

The theorems of Chapter II provide necessary and sufficient conditions for stability and convergence of VSVFMs. These conditions allow for changes in both the stepsize and formula being used. The change in formula can include a change in the number of backpoints needed, a change in the number of derivatives used, and a change from an explicit formula to an implicit formula or vice versa.
In Chapter III, an extension to a theorem by Crouziex and Lisbona (1984) was proved. It gives sufficient conditions for the stability of a VSVFM. In particular, if the stepsize has to be drastically reduced in order to meet the user-supplied error tolerance, these conditions imply that the stability of the VSVFM is ensured if there is a switch to a 1-step formula. VSVFMs based on the Adams-Bashforth formulas, the Adams-Moulton formulas, the backwards differentiation formulas, and one class of second derivative formulas were shown to be convergent. Empirical results were given for the last two of these methods to estimate how much change can be allowed in the stepsize when using a k-step formula with k > 1.

B. Future Work

Some areas deserving further consideration follow:

One question that needs to be answered is whether or not condition (4) of Definition 2.3 is a restriction that needs to be imposed on a general VSVFM to ensure the results obtained in Chapter II. The author believes that condition (4) is not needed.

More work with higher derivative formulas, especially second derivative formulas, should be considered. In a FSFFM, implicit higher derivative formulas have been shown to have better A-stability properties than the implicit first derivative formulas and therefore tend to be better for
solving Stiff ODEs. The major disadvantage of using the higher derivative formulas is the cost of evaluating the derivatives. In a VSVFM, one might be able to take advantage of a implicit higher derivative formula at one step and then use the calculated higher derivatives just obtained as backpoint information in an explicit formula for the next few steps. Thereby, eliminating the need of always having to evaluate the higher derivatives at each step. What this is really leading to is the question of what is meant by a stiffly stable VSVFM and what is necessary to have one.

When using a VSVFM, at almost every step, the coefficients of the formula that is to be used must be calculated. In these calculations, there are bound to be some errors due to round-off. In a practical algorithm, it may be advantageous to only allow the stepsize to be changed by a factor that is exactly machine representable, say for example, by multiples of 1/32. First, this might reduce some of the errors in the coefficients, and second, one would be able to use empirical results, similar to those in Chapter III, to guarantee that a particular method remains stable. In fact, it may even be advantageous to have the coefficients correctly rounded and stored for the algorithm to use.
V. REFERENCES


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