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A Lax-Wendroff discontinuous Galerkin method for linear hyperbolic systems

by

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this thesis. The Graduate College will ensure this thesis is globally accessible and will not permit alterations after a degree is conferred.

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In this work we develop a Lax-Wendroff discontinuous Galerkin (LxW-DG) method for solving linear systems of hyperbolic partial differential equations (PDEs). The proposed method is a variant of the standard LxW-DG method from the literature. The process by which the standard LxW-DG is obtained can be summarized as follows: (1) compute a truncated Taylor series in time that relates the solution that is being sought to the known solution at the previous time-step; (2) replace all the temporal derivatives in this Taylor expansion by spatial derivatives by repeatedly invoking the underlying PDE; (3) multiply this expansion by appropriate test functions, integrate over a finite element, and perform a single integration-by-parts that places a derivative on the test functions as well as introducing boundary terms; and finally, (4) replace the boundary terms by appropriate numerical fluxes. The key innovation in the newly proposed method is that we replace the single integration-by-parts step by an approach that moves all spatial derivatives onto the test functions; this process introduces many new terms that are not present in the standard LxW-DG approach. We develop this newly proposed approach in both one and two spatial dimensions and compare the regions of stability to the standard LxW-DG method. We show that compared to the standard LxW-DG method, the modified method has a larger region of stability and has improved accuracy. We demonstrate the properties of this new method by applying it to several numerical test cases.
CHAPTER 1. INTRODUCTION

1.1 Overview

In this work we develop a Lax-Wendroff discontinuous Galerkin (LxW-DG) method specifically for linear systems of hyperbolic partial differential equations (PDEs). We consider one dimensional linear hyperbolic PDEs of the form:

\[ q_t + Aq_x = 0, \quad (1.1) \]

where \( q(t, x) : \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}^M \), \( M \) is the number of equations, and the matrix \( A \in \mathbb{R}^{M \times M} \) must be diagonalizable and have only real eigenvalues. We also consider two dimensional linear hyperbolic PDEs of the form:

\[ q_t + Aq_x + Bq_y = 0, \quad (1.2) \]

where \( q(t, x, y) : \mathbb{R}_{\geq 0} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^M \), \( M \) is the number of equations, and the matrices \( A, B \in \mathbb{R}^{M \times M} \) must have the property that the matrix

\[ \cos(\theta)A + \sin(\theta)B \in \mathbb{R}^{M \times M} \quad (1.3) \]

is be diagonalizable with only real eigenvalues for all \( \theta \in \mathbb{R} \).

The Lax-Wendroff discontinuous Galerkin scheme, which was originally developed by Qiu, Dumbser, and Shu [9], can be used to solve (1.1) and (1.2), but can also be used when the system is non-linear. In this work we focus exclusively on the case of linear hyperbolic PDEs, and we show that through appropriate modifications of LxW-DG we are able to improve both the stability and accuracy.
1.2 A simple example

Consider the one dimensional advection equation

\[ q_t + u q_x = 0, \quad q(t = 0, x) = q_0(x), \quad u > 0 \quad x \in (a, b), \quad t \in (0, T) \]  

with periodic boundary conditions:

\[ q(t, x = a) = q(t, x = b). \]  

This equation is a model for the advection (i.e., propagation) of a passive tracer at the constant speed \( u \) in a periodic medium. The advection equation represents the simplest hyperbolic partial differential equation. Indeed, one can obtain an exact solution via the method of characteristics:

\[ q(t, x) = q_0^*(x - ut), \]  

where \( q_0^*(x) \) is the periodic extension of \( q_0(x) \) from \( x \in (a, b) \) to \( x \in \mathbb{R} \).

This problem can also be solved numerically using a variety of numerical methods, the simplest of which is the classical upwind method (e.g., see LeVeque [8]). The upwind method is an explicit one step method in time. The interval \((a, b)\) is divided into \( N \) sub-intervals or elements with length

\[ \Delta x := \frac{b - a}{N}. \]  

The cell centers are defined as

\[ x_i = a + \frac{\Delta x}{2} i \quad \text{for} \quad i = 1, ..., N. \]  

We can define the solution at the \( n \)th time step at the \( i \)th cell center as

\[ Q_i^n \approx q(t^n, x_i) \quad \text{for} \quad i = 1, ..., N. \]  

The upwind method can then be written as follows:

\[ Q_i^{n+1} = Q_i^n - u \frac{\Delta t}{\Delta x} (Q_i^n - Q_{i-1}^n) \quad \text{for} \quad i = 1, ..., N, \]
where the periodic boundary conditions are enforce by setting $Q_{n-1} = Q_N^n$.

In the upwind update formula (1.10) it is evident that the new solution in element $i$ depends only on the old solution in the $i$ and $i-1$ elements. According to the celebrated theory developed by Courant, Friedrichs, and Lewy [7], a necessary condition for linear stability of a numerical update of the form (1.10) is that the non-dimensional speed $u \Delta t / \Delta x$ (i.e., the CFL number) must be less than or equal to unity:

$$\frac{u \Delta t}{\Delta x} \leq 1.$$  \hspace{1cm} (1.11)

Indeed, it can easily be shown using techniques such as von Neumann stability analysis that condition (1.11) is indeed both necessary and sufficient for the linear stability of the upwind method (e.g., see LeVeque [8]).

The upwind method is a first order accurate method. To achieve a higher order one step method one can use a variety of techniques. One option is to make use of the Lax-Wendroff discontinuous Galerkin (LxW-DG) method developed by Qiu, Dumbser, and Shu [9]. Unfortunately, one drawback of these methods is that the $m$th order method has the stability condition of the form

$$\frac{u \Delta t}{\Delta x} \leq \frac{1}{2m - 1}.$$  \hspace{1cm} (1.12)

Thus as the order increases, the maximum allowable stable time-step steadily decreases. The net effect of this is that high-order methods become rapidly more and more expensive.

1.3 Scope of This Work

In the more general one dimensional case of (1.1) the condition that ensures stability of the $m$th order LxW-DG method is

$$CFL := \lambda \frac{\Delta t}{\Delta x} \leq CFL_{max},$$  \hspace{1cm} (1.13)

where $\lambda$ is the spectral radius of $A$. As with the upwind method described above, $m$th order LxW-DG methods only use nearest neighbors in their update, and therefore have a
necessary condition for linear stability given by:

\[ CFL_{\text{max}} = 1. \]  

(1.14)

However, just as described above, it turns out that \( m \)th order LxW-DG methods are only linearly stable under the much more restrictive condition [9]:

\[ CFL_{\text{max}} \approx \frac{1}{2m - 1}. \]  

(1.15)

The goal of the current work is to develop a modified Lax-Wendroff DG method that actually achieves the optimal condition:

\[ CFL_{\text{max}} = 1. \]  

(1.16)

For the advection equation (1.4) such a method already exists: the so-called semi-Lagrangian discontinuous Galerkin method [10]. The semi-Lagrangian method, however, does not generalize directly to linear systems of hyperbolic equations. In this work, we attempt to develop a modified LxW-DG scheme that reduces to the semi-Lagrangian scheme of [10] in the case of the advection equation, but that can be applied to more general linear systems in both one and two spatial dimensions.

The main idea behind the proposed scheme is briefly summarized here. The process by which the standard LxW-DG is obtained can be summarized as follows: (1) compute a truncated Taylor series in time that relates the solution that is being sought to the known solution at the previous time-step; (2) replace all the temporal derivatives in this Taylor expansion by spatial derivatives by repeatedly invoking the underlying PDE; (3) multiply this expansion by appropriate test functions, integrate over a finite element, and perform a single integration-by-parts that places a derivative on the test functions as well as introducing boundary terms; and finally, (4) replace the boundary terms by appropriate numerical fluxes. The key innovation in the newly proposed method is that we replace the single integration-by-parts step by an approach that moves all spatial derivatives onto the test functions; this process introduces many new terms that are not present in the standard
LxW-DG approach. We develop this newly proposed approach in both one and two spatial dimensions and compare the regions of stability to the standard LxW-DG method. We show that compared to the standard LxW-DG method, the modified method has a larger region of stability and has improved accuracy. We demonstrate the properties of this new method by applying it to several numerical test cases.

The outline of this thesis is as follows: in Chapter 2 we recall the classical Lax-Wendroff discontinuous Galerkin scheme developed by Qiu, Dumbser, and Shu [9], although we restrict ourselves to the case of linear hyperbolic systems. In Chapter 3 we develop our newly proposed scheme. In Chapter 4 we compare the two sets of schemes on both one and two dimensional examples. We demonstrate that the newly proposed scheme has both a larger $CFL_{\text{max}}$ and a smaller error when compared to the standard LxW-DG scheme. Finally, in Chapter 5, we draw some conclusions of the current study.
CHAPTER 2. THE STANDARD LAX WENDROFF GALERKIN METHOD

The modern form of discontinuous Galerkin methods were developed in the series of papers by Cockburn, Shu, and their collaborators [1, 2, 3, 4, 5, 6]. In this chapter, we will first review the Legendre polynomial basis and establish the test function space and approximate solution space. We will review the derivation of the standard LxW-DG in one space dimension. Then we will repeat the process for the two dimensional problem.

2.1 One dimensional mesh and Legendre polynomial basis

To understand the two discontinuous Galerkin methods we must construct the function space to which the approximate solution belongs. We build the space using the following Legendre polynomial basis:

\[
\Phi = \left\{ 1, \sqrt{3}\xi, \frac{\sqrt{7}}{2}(3\xi^2 - 1), \frac{\sqrt{7}}{2}(5\xi^3 - 3\xi), \frac{3}{8}(35\xi^4 - 30\xi^2 + 3), \ldots \right\}. \tag{2.1}
\]

This basis satisfies the following three-term recurrence relationship for \( k \geq 3 \):

\[
\phi_k = \left( \frac{\sqrt{(2k-3)(2k-1)}}{(k-1)} \right) \xi \phi_{k-1}(\xi) - \left( \frac{(k-2)\sqrt{2k-1}}{(k-1)\sqrt{2k-5}} \right) \phi_{k-2}(\xi).
\]

These polynomials have the following property:

\[
\frac{1}{2} \int_{-1}^{1} \phi_j(\xi) \phi_k(\xi) \, d\xi = \delta_{kj}, \tag{2.2}
\]

where \( \delta_{kj} = 1 \) if \( k = j \) and \( \delta_{kj} = 0 \) if \( k \neq j \).
The space domain is divided into \( N \) elements. Consider \( x \in (a,b) \) and let \( \Delta x = \frac{b-a}{N} \), then the element interfaces (see Figure 2.1) are defined as

\[
x_{i-\frac{1}{2}} = a + (i-1)\Delta x \quad \text{and} \quad x_{i+\frac{1}{2}} = a + i\Delta x \quad \text{for} \quad i = 1, \ldots, N
\]

and the cell centers are defined as

\[
x_i = a + i \left( \frac{\Delta x}{2} \right) \quad \text{for} \quad i = 1, \ldots, N.
\]

We define the canonical variable

\[
x = x_i + \frac{\Delta x}{2} \xi,
\]

with \( \xi \in [-1, 1] \) to conveniently discuss this scheme on each element. The number of basis functions in \( \Phi \) is also the order of accuracy for the scheme. We denote this number by \( m \).

On each element the approximate solution is a linear combination of the elements of \( \Phi \). Thus at the \( n \)th time step and the \( i \)th element, we have

\[
q^n_i(\xi) = q \left( t^n, x_i + \frac{\Delta x}{2} \xi \right) = \sum_{p=1}^{m} Q^n_{i,p} \phi_p(\xi) \quad \text{for} \quad \xi \in (-1,1).
\]

For a discontinuous Galerkin schemes the approximate solutions need not be continuous on the cell interfaces. We now define the space that yields the approximate solution as

\[
Q = \{ q \in L^\infty(a,b) : q \mid_{T_i} \in \mathbb{P}(m-1) \forall T_i \}.
\]

Where \( T_i \) is the \( i \)th element from figure 2.1 and \( \mathbb{P}(m-1) \) is the space of polynomials of degree \( m-1 \) or less.

The initial condition \( q_0 \) is projected onto this space using \( L^2 \) projection

\[
Q^n_{i,k} = \langle q_0, \phi_k \rangle = \frac{1}{2} \int_{-1}^{1} q_0 \left( x_i + \frac{\Delta x}{2} \xi \right) \phi_k(\xi) \, d\xi.
\]
2.2 Standard Lax-Wendroff discontinuous Galerkin method derivation in one dimension

This section outlines the standard Lax-Wendroff discontinuous Galerkin method given in [9] that solves (1.1). Consider the following hyperbolic linear constant coefficient system of PDEs

\[ q_t + A q_x = 0, \tag{2.7} \]
\[ q(0, x) = q_0(x). \tag{2.8} \]

We introduce another canonical variable, \( \tau \in [-1, 1] \), such that \( t = t^n + \Delta t^2 (1 + \tau) \). Using the chain rule we can rearrange (2.7)

\[ q_\tau + \Lambda q_\xi = 0, \tag{2.9} \]

where \( \Lambda = \frac{\Delta t}{\Delta x} A \). Rearranging (2.9) and then integrating from \(-1\) to \(1\) with respect to \( \tau \) we get

\[ q(\tau = 1, \xi) = q(\tau = -1, \xi) - 2 \left( \frac{1}{2} \int_{-1}^{1} \Lambda q(\tau, \xi) d\tau \right)_\xi. \tag{2.10} \]

The integral in the above equation is unknown since the anti-derivative of \( q \) is unknown; however, it can be approximated via a Taylor series expansion. We start with the Taylor expansion of \( q \):

\[ q(\tau, \xi) = q(-1) + (\tau + 1) q_x(-1, \xi) + \frac{(\tau + 1)^2}{2!} q_{xx}(-1, \xi) + \frac{(\tau + 1)^3}{3!} q_{xxx}(-1, \xi) + \ldots \tag{2.11} \]

Using equation (2.9) we write all of the time derivatives of \( q \) in terms of \( \Lambda \) and the space derivatives of \( q \):

\[ q_\tau(\tau, \xi) = -\Lambda q_\xi(\tau, \xi), \]
\[ q_{\tau\tau}(\tau, \xi) = (-\Lambda q_\xi(\tau, \xi))_\tau = -\Lambda q_x(\tau, \xi) = \Lambda^2 q_{xx}(\tau, \xi), \]
\[ q_{\tau\tau\tau}(\tau, \xi) = (\Lambda^2 q_{xx}(\tau, \xi))_\tau = \Lambda^2 q_{xxx}(\tau, \xi) = -\Lambda^3 q_{xxxx}(\tau, \xi), \]
\[ \vdots \]
\[ \partial_\tau^m q(\tau, \xi) = (-1)^m \Lambda^m \partial_\xi^m q(\tau, \xi). \]
Substituting the space derivatives in for the time derivatives (2.10) becomes

\[ q(\tau = 1, \xi) = q(\tau = -1, \xi) - 2 \left( \frac{1}{2} \int_{-1}^{1} \sum_{j=0}^{\infty} (-1)^j A^j \frac{(\tau + 1)^j}{j!} \partial_\xi^j q(-1, \xi) \, d\tau \right) \xi. \]  

(2.12)

Noting that when \( \tau = 1, \ t = t^{n+1} \), and when \( \tau = -1, \ t = t^n \), so it is natural to write \( q(\tau = 1) \) as \( q^{n+1} \) and \( q(\tau = -1) \) as \( q^n \).

The standard Lax-Wendroff discontinuous Galerkin method only requires the first \( m \) terms of (2.12) which we now simplify further as

\[ q^{n+1}(\xi) = q^n(\xi) - 2 \left( \frac{1}{2} \sum_{j=0}^{m} (-1)^j A^j \frac{1}{j!} \partial_\xi^j q^n(\xi) \int_{-1}^{1} (\tau + 1)^j \, d\tau \right) \xi \]

\[ = q^n(\xi) - 2 \left( \frac{1}{2} \sum_{j=0}^{m} (-1)^j A^j \frac{2^{j+1}}{j!(j+1)} \partial_\xi^j q^n(\xi) \right) \xi. \]

We now multiply both sides by a test function \( \phi_k \), the \( k \)th element of \( \Phi \) (2.1)

\[ q^{n+1}(\xi) \phi_k(\xi) = q^n(\xi) \phi_k(\xi) - 2 \left( \frac{1}{2} \sum_{j=0}^{m} (-1)^j A^j \frac{2^{j+1}}{j!(j+1)} \partial_\xi^j q^n(\xi) \right) \phi_k(\xi). \]  

(2.13)

Integrating both sides with respect to \( \xi \) from \(-1\) to \(1\) we get

\[ \int_{-1}^{1} q^{n+1}(\xi) \phi_k(\xi) \, d\xi = \int_{-1}^{1} q^n(\xi) \phi_k(\xi) \, d\xi - 2 \int_{-1}^{1} (F^n(\xi)) \phi_k(\xi) \, d\xi \]  

(2.14)

where

\[ F^n(\xi) = \sum_{j=0}^{m} (-1)^j A^j \frac{2^j}{j!(j+1)} \partial_\xi^j q^n(\xi). \]  

(2.15)

We wish to integrate by parts and pass the \( \xi \)-derivative onto the test function. However the approximate solution is not required to be continuous at the cell interfaces (i.e. when \( \xi = \pm 1 \)) and instead must be replaced with a numerical flux. We use the the Lax-Friedrichs flux

\[ F^n_{i+\frac{1}{2}} = \frac{1}{2} \left( F^-_{i+\frac{1}{2}} + F^+_{i+\frac{1}{2}} - \lambda \left( q^+_{i+\frac{1}{2}} - q^-_{i+\frac{1}{2}} \right) \right) \]  

(2.16)
where

\[ F_{i+\frac{1}{2}}^+ = \lim_{\xi \to -1} F_{i+1}^n(\xi), \]  
\[ F_{i+\frac{1}{2}}^- = \lim_{\xi \to 1} F_i^n(\xi), \]  
\[ q_{i+\frac{1}{2}}^+ = \lim_{\xi \to -1} \hat{q}_{i+1}^n(\xi), \]  
\[ q_{i+\frac{1}{2}}^- = \lim_{\xi \to 1} \hat{q}_i^n(\xi), \]  

and \( \lambda \) is the spectral radius of matrix \( A \) (i.e., largest modulus of all eigenvalues of \( A \)). Here

\[ \hat{q}_i^n(\xi) = \frac{1}{\Delta t} \int_{-1}^{1} q(\tau, \xi) \, d\tau \]
\[ \hat{q}_i^n(\xi) = \frac{1}{2} \int_{-1}^{1} \sum_{j=0}^{m} (-1)^j A^j (\tau + 1)^j j! \partial_\xi^j q_i^n(\xi) \, d\tau \]
\[ \hat{q}_i^n(\xi) = \frac{1}{2} \sum_{j=0}^{m} (-1)^j A^j \partial_\xi^j q_i^n(\xi) \int_{-1}^{1} (\tau + 1)^j \, d\tau \]
\[ \hat{q}_i^n(\xi) = \sum_{j=0}^{m} (-1)^j A^j \frac{2^j}{j!(j+1)} \partial_\xi^j q_i^n(\xi) \]
\[ F_i^n(\xi) = \sum_{j=0}^{m} (-1)^j A^j + \frac{2^j}{j!(j+1)} \partial_\xi^j q_i^n(\xi) \]

It is important to note the difference between \( \hat{q}_i^n(\xi) \) and \( q_i^n(\xi) \). Using this flux and integrating by parts in (2.14) we get

\[ \int_{-1}^{1} q_{i+1}^{n+1}(\xi) \phi_k(\xi) \, d\xi = \int_{-1}^{1} q_i^n(\xi) \phi_k(\xi) \, d\xi 
- \hat{F}_{i+\frac{1}{2}}^n \phi_k(1) + \hat{F}_{i-\frac{1}{2}}^n \phi_k(-1) + \int_{-1}^{1} F_i^n(\xi) \phi_k(\xi) \, d\xi. \]  

(2.22)

Substituting the approximate solution (2.4) into (2.22) and making use of (2.2), we have

\[ Q_{i,k}^{n+1} = Q_{i,k}^n + \frac{1}{2} \left( \hat{F}_{i-\frac{1}{2}}^n \phi_k(-1) - \hat{F}_{i+\frac{1}{2}}^n \phi_k(1) + \int_{-1}^{1} F_i^n(\xi) \phi_k(\xi) \, d\xi \right). \]  

(2.23)

Note that this calculation includes substituting the approximate solution (2.4) in the defi-
initions of $F^n_i$ (2.16) and $\hat{q}^n_i$ (2.21):

$$\hat{q}^n_i = \sum_{j=0}^{m} \sum_{p=1}^{m} (-1)^j A^j \frac{2^j}{(j+1)!} Q^n_{i,p} \partial_x^j \phi_p(\xi), \quad (2.24)$$

$$F^n_i = \sum_{j=0}^{m} \sum_{p=1}^{m} (-1)^j A^j+1 \frac{2^j}{(j+1)!} Q^n_{i,p} \partial_x^j \phi_p(\xi). \quad (2.25)$$

### 2.3 Two dimensional mesh and Legendre polynomial basis

Similar to the one dimensional case the approximate solution belongs to a basis of Legendre polynomials. In two dimensions we consider two basis and their tensor product:

$$\Phi_1 = \left\{ 1, \sqrt{3} \xi, \frac{\sqrt{5}}{2} (3\xi^2 - 1), \frac{\sqrt{7}}{2} (5\xi^3 - 3\xi), \frac{3}{8} (35\xi^4 - 30\xi^2 + 3), \ldots \right\}, \quad (2.26)$$

$$\Phi_2 = \left\{ 1, \sqrt{3} \eta, \frac{\sqrt{5}}{2} (3\eta^2 - 1), \frac{\sqrt{7}}{2} (5\eta^3 - 3\eta), \frac{3}{8} (35\eta^4 - 30\eta^2 + 3), \ldots \right\}, \quad (2.27)$$

$$\Phi_1 \otimes \Phi_2 = \begin{bmatrix}
1 & \sqrt{3} \eta & \cdots & \phi_{2,m}(\eta) & \cdots \\
\sqrt{3} \xi & 3\xi \eta & \cdots & \sqrt{3} \xi \phi_{2,m}(\eta) & \cdots \\
\vdots & \vdots & \ddots & \vdots & \cdots \\
\phi_{1,n}(\xi) & \phi_{1,n}(\xi) \sqrt{3} \eta & \cdots & \phi_{1,n}(\xi) \phi_{2,m}(\eta) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \quad (2.28)$$

All of the elements in the above matrix make up the Q-basis. The approximate solution space uses the P-basis. The P-basis contains all the elements in the upper left triangle of the above matrix. For example the second order P-basis is $\{ 1, \sqrt{3} \xi, \sqrt{3} \eta \}$. This paper deals with the first and second order Lax-Wendroff DG method and our improved method and the two bases we will use are

$$\Phi_{m=1} = \{ 1 \},$$

$$\Phi_{m=2} = \left\{ 1, \sqrt{3} \xi, \sqrt{3} \eta \right\}. \quad (2.29)$$

The space domain is divided into $N_x$ times $N_y$ rectangular elements. Analogous to the definitions in section 2.1 if $y \in (c, d)$, then $\Delta y = \frac{d - c}{N_y}$ and the centers of the element in
the $y$ direction are $y_j = c + \frac{\Delta y}{2} j$ for $j = 1, \ldots, N_y$. The interfaces in the $y$ direction are

$$y_{j-\frac{1}{2}} = c + (j - 1)\Delta y \quad \text{and} \quad y_{j+\frac{1}{2}} = c + j\Delta x$$

for $j = 1, \ldots, N_y$.

Thus the $ij$ element is

$$\left( x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right) \times \left( y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}} \right).$$

We continue to use the canonical variables $\tau$ where $t = t^n + \frac{\Delta t}{2}(1 + \tau)$ and $\xi$ where $x = x_i + \frac{\Delta x}{2}\xi$ and introduce a third canonical variable $\eta$ where $y = y_j + \frac{\Delta y}{2}\eta$, $\tau, \xi, \eta \in [-1, 1]$.

Consider the two dimensional system of

$$q_t + Aq_x + Bq_y = 0 \quad (2.30)$$

$$q(0, x, y) = q_0(x, y). \quad (2.31)$$

The canonical form of (2.30) is

$$q_\tau + Aq_\xi + Bq_\eta = 0, \quad A = \frac{\Delta t}{\Delta x} A, \quad B = \frac{\Delta t}{\Delta x} B. \quad (2.32)$$

### 2.4 Standard Lax-Wendroff discontinuous Galerkin method derivation in two dimensions

We now derive the two dimensional version of the Lax Wendroff discontinuous Galerkin method that solves (1.2). We use a slightly different approach compared to the derivation in section 2.2. In the case of advection they are equivalent, however in the case of a system they result in slightly different methods.

This method can be derived for arbitrary order but we will focus on the second order derivation. We start with a Taylor series expansion in time and keep the terms with a second derivative or less

$$q^{n+1} = q^n + 2q^n_\tau + 2q^n_{\tau\tau}. \quad (2.33)$$

Using (2.32) we can derive the following relationships

$$q_\tau = -Aq_\xi - Bq_\eta,$$

$$q_{\tau\tau} = A^2 q_{\xi\xi} + A B q_{\eta\xi} + B A q_{\xi\eta} + B^2 q_{\eta\eta}.$$
Equation (2.33) can now be expressed in terms of the variables in space $\xi$ and $\eta$:

$$q^{n+1} = q^n - 2A q_\xi - 2B q_\eta + 2A^2 q_{\xi\xi} + 2A B q_{\eta\xi} + 2B A q_{\xi\eta} + 2B^2 q_{\eta\eta}. \quad (2.34)$$

Next we use (2.29) as a test function space and multiply every term in (2.34) by a test function:

$$q^{n+1} \phi_k = q^n \phi_k - 2A q_\xi \phi_k - 2B q_\eta \phi_k + 2A^2 q_{\xi\xi} \phi_k + 2A B q_{\eta\xi} \phi_k + 2B A q_{\xi\eta} \phi_k + 2B^2 q_{\eta\eta} \phi_k. \quad (2.35)$$

We integrate (2.35) with respect to both variables in space, $\xi$ and $\eta$. For convenience we use Euler notation for derivatives:

$$\int_{-1}^{1} \int_{-1}^{1} q^{n+1} \phi_k \, d\xi \, d\eta = \int_{-1}^{1} \int_{-1}^{1} q^n \phi_k \, d\xi \, d\eta - \int_{-1}^{1} \int_{-1}^{1} 2A \partial_\xi q^n \phi_k \, d\xi \, d\eta$$

$$- \int_{-1}^{1} \int_{-1}^{1} 2B \partial_\eta q^n \phi_k \, d\xi \, d\eta + \int_{-1}^{1} \int_{-1}^{1} 2A^2 \partial_\xi^2 q^n \phi_k \, d\xi \, d\eta + \int_{-1}^{1} \int_{-1}^{1} 2A B \partial_{\xi\eta} q^n \phi_k \, d\xi \, d\eta$$

$$+ \int_{-1}^{1} \int_{-1}^{1} 2B A \partial_{\xi\eta} q^n \phi_k \, d\xi \, d\eta + \int_{-1}^{1} \int_{-1}^{1} 2B^2 \partial_\eta^2 q^n \phi_k \, d\xi \, d\eta. \quad (2.36)$$

Whenever possible in (2.36), we integrate by parts once

$$\int_{-1}^{1} \int_{-1}^{1} q^{n+1} \phi_k \, d\xi \, d\eta = \int_{-1}^{1} \int_{-1}^{1} q^n \phi_k \, d\xi \, d\eta - 2A \int_{-1}^{1} q^n \phi_k (\xi = 1) - q^n \phi_k (\xi = -1) \, d\eta$$

$$+ 2A \int_{-1}^{1} \int_{-1}^{1} q^n \partial_\xi \phi_k \, d\xi \, d\eta - 2B \int_{-1}^{1} q^n \phi_k (\eta = 1) - q^n \phi_k (\eta = -1) \, d\xi$$

$$+ 2B \int_{-1}^{1} \int_{-1}^{1} q^n \partial_\eta \phi_k \, d\xi \, d\eta + 2A^2 \int_{-1}^{1} \partial_\xi q^n \phi_k (\xi = 1) - \partial_\xi q^n \phi_k (\xi = -1) \, d\eta$$

$$- 2A^2 \int_{-1}^{1} \int_{-1}^{1} \partial_\xi q^n \partial_\xi \phi_k \, d\xi \, d\eta + 2A B \int_{-1}^{1} \partial_\eta q^n \phi_k (\xi = 1) - \partial_\eta q^n \phi_k (\xi = -1) \, d\xi$$

$$- 2A B \int_{-1}^{1} \int_{-1}^{1} \partial_\eta q^n \partial_\eta \phi_k \, d\xi \, d\eta + 2B^2 \int_{-1}^{1} \partial_\eta q^n \phi_k (\eta = 1) - \partial_\eta q^n \phi_k (\eta = -1) \, d\xi$$

$$- 2B^2 \int_{-1}^{1} \int_{-1}^{1} \partial_\eta q^n \partial_\eta \phi_k \, d\xi \, d\eta. \quad (2.37)$$

Since our approximate solution is not defined at the interfaces (when $\xi = \pm 1$ or when $\eta = \pm 1$) we must define them.
The following factorization of $A$ is given in [8]. Recall that $A$ must be diagonalizable so we consider the eigenvalue decomposition $A = RAR^{-1}$. If the eigenvalues of $A$ are $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_{Meqn}$ then we define $\Lambda^+$ and $\Lambda^-$ as

$$
\Lambda^+ = \begin{bmatrix}
\lambda_1^+ & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_{Meqn}^+
\end{bmatrix}
$$

and

$$
\Lambda^- = \begin{bmatrix}
\lambda_1^- & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_{Meqn}^-
\end{bmatrix},
$$

(2.38)

where $\lambda_m^+ = \max(0, \lambda_m)$ and $\lambda_m^- = \min(0, \lambda_m)$. $\Lambda^+$ retains all positive eigenvalues and replaces the negative eigenvalues with zeros, and $\Lambda^-$ retains all negative eigenvalues and replaces the positive eigenvalues with zeros. Using these two matrices we define $A^+$ and $A^-$ as

$$
A^+ = RA^+R^{-1} \quad \text{and} \quad A^- = RA^-R^{-1}.
$$

Using these definitions we can define $q^n$ at the interfaces. The definition of $q^n$ at the right $(i + \frac{1}{2})$ interface is

$$
2A \int_{-1}^{1} q^n \phi_k(\xi = 1) \, d\eta = \\
2A^- \int_{-1}^{1} q_{i+1,j}^n(\xi = -1) \phi_k(\xi = 1) \, d\eta + 2A^+ \int_{-1}^{1} q_{i,j}^n(\xi = 1) \phi_k(\xi = 1) \, d\eta.
$$

(2.39)

The definition for the left $(i - \frac{1}{2})$ interface is analogous and the same process can be carried out for the $j \pm \frac{1}{2}$ interfaces.
We rewrite (2.37) using the above definition for $q^n$ at the interfaces

$$
\int_{-1}^{1} \int_{-1}^{1} q_{ij}^{n+1} \phi_k \, d\xi \, d\eta = \int_{-1}^{1} \int_{-1}^{1} q_{ij}^{n} \phi_k \, d\xi \, d\eta \\
- 2A^- \int_{-1}^{1} q_{i+1j}^{n} \phi_k(\xi = -1) \, d\eta - 2A^+ \int_{-1}^{1} q_{ij}^{n} \phi_k(\xi = 1) \, d\eta \\
+ 2A^- \int_{-1}^{1} q_{ij}^{n} \phi_k(\xi = -1) \, d\eta + 2A^+ \int_{-1}^{1} q_{i-1j}^{n} \phi_k(\xi = -1) \, d\eta \\
+ 2A^- \int_{-1}^{1} q_{ij}^{n} \partial \xi \phi_k \, d\xi \, d\eta - 2B^- \int_{-1}^{1} q_{ij+1}^{n} \phi_k(\xi = -1) \, d\xi \\
- 2B^+ \int_{-1}^{1} q_{ij}^{n} \phi_k(\eta = 1) \, d\xi + 2B^- \int_{-1}^{1} q_{ij}^{n} \phi_k(\eta = -1) \, d\xi \\
+ 2B^+ \int_{-1}^{1} q_{ij-1}^{n} \phi_k(\eta = 1) \, d\xi + 2B^- \int_{-1}^{1} q_{ij}^{n} \partial \eta \phi_k \, d\xi \, d\eta \\
+ 2A^- B^- \int_{-1}^{1} q_{i+1j}^{n} \phi_k(\xi = -1) \, d\eta + 2B^- \int_{-1}^{1} q_{ij}^{n} \phi_k(\xi = 1) \, d\eta \\
- 2A^+ B^- \int_{-1}^{1} q_{i-1j}^{n} \phi_k(\xi = -1) \, d\eta + 2A^- B^+ \int_{-1}^{1} q_{ij}^{n} \phi_k(\xi = 1) \, d\eta \\
- 2B^- A \int_{-1}^{1} q_{ij+1}^{n} \phi_k(\eta = -1) \, d\xi + 2B^+ A \int_{-1}^{1} q_{ij}^{n} \phi_k(\eta = 1) \, d\xi \\
- 2B^- A \int_{-1}^{1} q_{ij-1}^{n} \phi_k(\eta = -1) \, d\xi + 2B^+ A \int_{-1}^{1} q_{ij}^{n} \phi_k(\xi = -1) \, d\xi \\
- 2B A \int_{-1}^{1} q_{ij}^{n} \partial \xi \phi_k \, d\xi \, d\eta + 2B^- B^+ \int_{-1}^{1} q_{ij+1}^{n} \phi_k(\eta = -1) \, d\xi \\
+ 2B^+ B^- \int_{-1}^{1} q_{ij}^{n} \phi_k(\eta = 1) \, d\xi - 2B^+ B^- \int_{-1}^{1} q_{ij}^{n} \phi_k(\eta = -1) \, d\xi \\
- 2B^+ B^- \int_{-1}^{1} q_{ij}^{n} \partial \eta \phi_k \, d\xi \, d\eta. \quad (2.40)
$$

Next we substitute $q_{ij}^{n} = Q_{ij}^{n,1} + \sqrt{3} \xi Q_{ij}^{n,2} + \sqrt{3} \eta Q_{ij}^{n,3}$ and integrate to arrive at the discretized...
second order Lax-Wendroff method:

\[ Q_{ij}^{n+1,1} = -A^- Q_{i+1,j}^n + \left( \sqrt{3} A^- + \sqrt{3} A^2^- \right) Q_{i+1,j}^n + \sqrt{3} A^- B Q_{i+1,j}^n - B^- Q_{ij}^{n+1} \]

\[ + \sqrt{3} B^- A Q_{ij}^n + \left( \sqrt{3} B^- + \sqrt{3} B^2^- \right) Q_{ij}^n + (I - A^+ + A^- - B^+ + B^-) Q_{ij}^n \]

\[ + \left( -\sqrt{3} A^+ - \sqrt{3} A^- + \sqrt{3} A^2^+ - \sqrt{3} B^2^- + \sqrt{3} A^+ B - \sqrt{3} A^- B \right) Q_{ij}^n \]

\[ + A^+ Q_{i-1,j}^n + \left( \sqrt{3} A^+ - \sqrt{3} A^2^+ \right) Q_{i-1,j}^n - \sqrt{3} A^+ B Q_{i-1,j}^n \]

\[ + B^+ Q_{ij-1}^n - \sqrt{3} B^+ A Q_{ij-1}^n + \left( \sqrt{3} B^+ - \sqrt{3} B^2^+ \right) Q_{ij-1}^n, \quad (2.41) \]

\[ Q_{ij}^{n+1,2} = -\sqrt{3} A^- Q_{i+1,j}^n + (3 A^- + 3 A^2^-) Q_{i+1,j}^n + 3 \sqrt{3} A^- B Q_{i+1,j}^n - \sqrt{3} A^- B Q_{ij}^{n+1} \]

\[ + \left( -\sqrt{3} A^+ - \sqrt{3} A^- + 2 \sqrt{3} A \right) Q_{ij}^n \]

\[ + (I - 3 A^+ + 3 A^- - B^+ + B^- + 3 A^2^- + 3 A^2^- - 6 A^2) Q_{ij}^n \]

\[ + (3 A^+ B + 3 A^- B - 6 A B) Q_{ij}^n - \sqrt{3} A^+ Q_{i-1,j}^n + (-3 A^+ + 3 A^2^+) Q_{i-1,j}^n \]

\[ + 3 A^+ B Q_{i-1,j}^n + B^+ Q_{ij-1}^n, \quad (2.42) \]

\[ Q_{ij}^{n+1,3} = -A^- Q_{i+1,j}^n - \sqrt{3} B^- Q_{ij+1}^n + 3 B^- A Q_{ij+1}^n + (3 B^- + 3 B^2^-) Q_{ij+1}^n \]

\[ + \left( -\sqrt{3} B^+ - \sqrt{3} B^- + 2 \sqrt{3} B \right) Q_{ij}^n + (3 B^+ A + 3 B^- A - 6 B A) Q_{ij}^n \]

\[ + (I - A^+ + A^- - 3 B^+ + 3 B^- + 3 B^2^+ + 3 B^2^- - 6 B^2) Q_{ij}^n + A^+ Q_{i-1,j}^n \]

\[ - \sqrt{3} B^+ Q_{ij-1}^n + A^+ B Q_{ij-1}^n + (-3 B^+ + 3 B^2^+) Q_{ij-1}^n. \quad (2.43) \]
CHAPTER 3. THE MODIFIED LAX-WENDROFF DISCONTINUOUS GALERKIN METHOD

In this chapter we derive the modified Lax-Wendroff discontinuous Galerkin method. The difference between the two methods is that the newly proposed approach uses not just a single integration-by-parts, but instead we use maximal integration-by-parts.

3.1 Modified Method in one dimension

We start the derivation of the modified method with an equation closely resembling (2.13). However the summation requires the first $2m - 2$ terms:

$$
q^{n+1}(\xi)\phi_k(\xi) = q^n(\xi)\phi_k(\xi) - 2 \left( \frac{1}{2} \sum_{j=0}^{2m-2} (-1)^j A_j^{j+1} \frac{2^{j+1}}{(j+1)!} \partial_{\xi}^j q^n(\xi) \right) \phi_k(\xi)
$$

$$
= q^n(\xi)\phi_k(\xi) + \sum_{j=0}^{2m-2} (-1)^j A_j^{j+1} \frac{2^{j+1}}{(j+1)!} \partial_{\xi}^j q^n(\xi)\phi_k(\xi)
$$

$$
= q^n(\xi)\phi_k(\xi) + \sum_{s=1}^{2m-1} (-1)^s A_s^s \frac{2^s}{s!} \partial_{\xi}^s q^n(\xi)\phi_k(\xi). \quad (3.1)
$$
We integrate both sides of the equation above with respect to $\xi$ and integrate by parts the maximum amount of times:

$$
\int_{-1}^{1} q^{n+1}(\xi) \phi_k(\xi) \, d\xi = \int_{-1}^{1} q^n(\xi) \phi_k(\xi) \, d\xi + \int_{-1}^{1} \sum_{s=1}^{2m-1} (-1)^s A^s \frac{2^s}{s!} \partial_{\xi}^s q^n(\xi) \phi_k(\xi) \, d\xi,
$$

$$
\int_{-1}^{1} q^{n+1}(\xi) \phi_k(\xi) \, d\xi = \int_{-1}^{1} q^n(\xi) \phi_k(\xi) \, d\xi + \sum_{s=1}^{2m-1} (-1)^s A^s \frac{2^s}{s!} \int_{-1}^{1} \partial_{\xi}^s q^n(\xi) \phi_k(\xi) \, d\xi,
$$

$$
\int_{-1}^{1} q^{n+1}(\xi) \phi_k(\xi) \, d\xi = \int_{-1}^{1} q^n(\xi) \phi_k(\xi) \, d\xi
$$

$$
+ \sum_{s=1}^{2m-1} (-1)^s A^s \frac{2^s}{s!} \sum_{l=1}^{s} (-1)^l \partial_{\xi}^{s-l} q^n \partial_{\xi}^{l-1} \phi_k(1) - (-1)^l \partial_{\xi}^{s-l} q^n \partial_{\xi}^{l-1} \phi_k(1)
$$

$$
+ \sum_{s=1}^{2m-1} A^s \frac{2^s}{s!} \int_{-1}^{1} q^n(\xi) \partial_{\xi}^s \phi_k(\xi) \, d\xi,
$$

(3.2)

The approximate solution $q^n$ is not defined when $\xi = \pm 1$. We define $q^n$ at the right $(i + \frac{1}{2})$ interface as

$$
A^s \frac{2^s}{s!} \sum_{l=1}^{s} (-1)^l \partial_{\xi}^{s-l} q^n \partial_{\xi}^{l-1} \phi_k(1)
$$

$$
A^s \frac{2^s}{s!} \sum_{l=1}^{s} (-1)^l \partial_{\xi}^{s-l} q^n \partial_{\xi}^{l-1} \phi_k(1) + A^s \frac{2^s}{s!} \sum_{l=1}^{s} (-1)^l \partial_{\xi}^{s-l} q^n \partial_{\xi}^{l-1} \phi_k(1),
$$

(3.3)

where $A^{s\pm} = (A^{\pm})^s$. The definition for the left $(i - \frac{1}{2})$ interface is analogous. We rewrite (3.2) using the above definition for $q^n$, noting that $\partial_{\xi}^s \phi_k(\xi) = 0$ for $s > m$ thus changing the indexing on the last sum in (3.2)

$$
\int_{-1}^{1} q^{n+1}_i(\xi) \phi_k(\xi) \, d\xi = \int_{-1}^{1} q^n_i(\xi) \phi_k(\xi) \, d\xi
$$

$$
+ \sum_{s=1}^{2m-1} (-1)^s A^s \frac{2^s}{s!} \sum_{l=1}^{s} (-1)^l \left( A^{s+} \partial_{\xi}^{s-l} q^n_{i-1}(1) + A^{s-} \partial_{\xi}^{s-l} q^n_i(-1) \right) \partial_{\xi}^{l-1} \phi_k(1)
$$

$$
- \sum_{s=1}^{2m-1} (-1)^s A^s \frac{2^s}{s!} \sum_{l=1}^{s} (-1)^l \left( A^{s+} \partial_{\xi}^{s-l} q^n_i(1) + A^{s-} \partial_{\xi}^{s-l} q^n_{i+1}(-1) \right) \partial_{\xi}^{l-1} \phi_k(1)
$$

$$
+ \sum_{s=1}^{m} A^s \frac{2^s}{s!} \int_{-1}^{1} q^n_i(\xi) \partial_{\xi}^s \phi_k(\xi) \, d\xi.
$$
Substituting the approximate solution (2.4) into the above equation. We have

\[
\int_{-1}^{1} \sum_{p=1}^{m} Q_{i,p}^{n+1} \phi_p(\xi) \phi_k(\xi) \, d\xi = \int_{-1}^{1} \sum_{p=1}^{m} Q_{i,p}^{n} \phi_p(\xi) \phi_k(\xi) \, d\xi \\
+ \sum_{s=1}^{2m-1} \frac{2s}{s!} \sum_{l=1}^{s} (-1)^{s+l} \left( A^{s+} \sum_{p=1}^{m} Q_{i-1,p}^{n} \phi_p(1) + A^{s-} \sum_{p=1}^{m} Q_{i,p}^{n} \phi_p(-1) \right) \partial_{\xi}^{l-1} \phi_k(-1) \\
- \sum_{s=1}^{2m-1} \frac{2s}{s!} \sum_{l=1}^{s} (-1)^{s+l} \left( A^{s+} \sum_{p=1}^{m} Q_{i+1,p}^{n} \phi_p(1) + A^{s-} \sum_{p=1}^{m} Q_{i,p}^{n} \phi_p(-1) \right) \partial_{\xi}^{l-1} \phi_k(1) \\
+ \sum_{s=1}^{m} A^{s} \frac{2s}{s!} \int_{-1}^{1} \sum_{p=1}^{m} Q_{i,p}^{n} \phi_p(\xi) \partial_{\xi}^{s} \phi_k(\xi) \, d\xi.
\] (3.4)

Making use of (2.2) along with rearrangements, the above equation can be simplified as

\[
Q_{i,k}^{n+1} = Q_{i,k}^{n} \\
+ \sum_{s=1}^{2m-1} \sum_{l=1}^{s} \frac{12s}{2s!} (-1)^{s+l} \partial_{\xi}^{l-1} \phi_k(-1) \left( A^{s+} \sum_{p=1}^{m} Q_{i-1,p}^{n} \partial_{\xi}^{s-} \phi_p(1) + A^{s-} \sum_{p=1}^{m} Q_{i,p}^{n} \partial_{\xi}^{s-} \phi_p(-1) \right) \\
- \sum_{s=1}^{2m-1} \sum_{l=1}^{s} \frac{12s}{2s!} (-1)^{s+l} \partial_{\xi}^{l-1} \phi_k(1) \left( A^{s+} \sum_{p=1}^{m} Q_{i+1,p}^{n} \partial_{\xi}^{s-} \phi_p(1) + A^{s-} \sum_{p=1}^{m} Q_{i,p}^{n} \partial_{\xi}^{s-} \phi_p(-1) \right) \\
+ \sum_{s=1}^{m} A^{s} \sum_{p=1}^{m} Q_{i,p}^{n} \frac{12s}{2s!} \int_{-1}^{1} \phi_p(\xi) \partial_{\xi}^{s} \phi_k(\xi) \, d\xi.
\] (3.5)
For convenience we define the following:

\[
\alpha_{l,p}^1 = \partial_{\xi}^{l-1} \phi_p(1), \quad \beta_{l,p}^1 = \partial_{\xi}^{l-1} \phi_p(-1), \quad \begin{cases} 
    l \in [1,2m-1] \\
    p \in [1,m]
\end{cases}, 
\]

(3.6)

\[
\alpha_{s,l,k}^2 = \frac{1}{2} \frac{2^s}{s!} (-1)^{s+l} \alpha_{l,k}^1, \quad \beta_{s,l,k}^2 = \frac{1}{2} \frac{2^s}{s!} (-1)^{s+l} \beta_{l,k}^1, \quad \begin{cases} 
    s \in [1,2m-1] \\
    l \in [1,2m-1] \\
    k \in [1,m]
\end{cases}, 
\]

(3.7)

\[
\mu_{s,k,p} = \frac{1}{2} \frac{2^s}{s!} \int_{-1}^{1} \phi_p(\xi) \partial_{\xi}^s \phi_k(\xi) \, d\xi, \quad \begin{cases} 
    s \in [1,m] \\
    k \in [1,m] \\
    p \in [1,m]
\end{cases}, 
\]

(3.8)

\[
F_{s,l,n}^{i-\frac{1}{2}} = \mathcal{A}_s^{s+} \sum_{p=1}^{m} \alpha_{s-l+1,p}^1 Q_{i-1,p}^n + \mathcal{A}_s^{s-} \sum_{p=1}^{m} \beta_{s-l+1,p}^1 Q_{i,p}^n, \quad \begin{cases} 
    l \in [1,s] \\
    s \in [1,2m-1]
\end{cases}, 
\]

(3.9)

\[
F_{i}^{s,k,n} = \sum_{l=1}^{s} \beta_{s,l,k}^2 F_{s,l,n}^{i-\frac{1}{2}} - \alpha_{s,l,k}^1 F_{s,l,n}^{i+\frac{1}{2}}, \quad \begin{cases} 
    s \in [1,2m-1] \\
    k \in [1,m] \\
    l \in [1,s]
\end{cases}, 
\]

(3.10)

\[
I_{i}^{s,k,n} = \mathcal{A}_s^{s} \sum_{p=1}^{m} \mu_{s,k,p} Q_{i,p}^n, \quad \begin{cases} 
    s \in [1,m] \\
    k \in [1,m]
\end{cases}. 
\]

(3.11)

Using these definitions we can rewrite (3.5) as follows:

\[
Q_{i,k}^{n+1} = Q_{i,k}^n + \sum_{s=1}^{m} I_{i}^{s,k,n} + \sum_{s=1}^{2m-1} F_{i}. 
\]

(3.12)

### 3.2 Two Dimensional Modified Lax Wendroff discontinuous Galerkin Method

The first order modified method in two dimensions is given in [8] as the Corner-Transport Upwind method. The second order modified method we derive uses dimensional splitting
discussed in [11]. Let \( N_A \) be the one-step method that solves the problem \( q_t + Aq_x = 0 \) and \( N_B \) be the one-step method that solves the problem \( q_t + Bq_y = 0 \). The second order dimensional splitting is

\[
Q^* = N_A \left( Q^n, \frac{\Delta t}{2} \right),
\]

\[
Q^{**} = N_B (Q^*, \Delta t),
\]

\[
Q^{n+1} = N_A \left( Q^{**}, \frac{\Delta t}{2} \right).
\]

The one dimensional methods \( N_A \) and \( N_B \) are derived similarly to the methods in section 2.3. The difference being that (2.29) is used as the basis for the approximate solution space and the test function space. To derive \( N_A \) we start with (3.1) where \( m = 2 \)

\[
q^{n+1} \phi_k = q^n \phi_k - 2A \partial_\xi q^n \phi_k + 2A^2 \partial^2_\xi q^n \phi_k - \frac{4}{3} A^3 \partial^3_\xi \phi_k.
\]

Integrating the above equation with respect to \( \xi \) and \( \eta \) yields:

\[
\int_{-1}^{1} \int_{-1}^{1} q^{n+1} \phi_k \, d\xi d\eta = \int_{-1}^{1} \int_{-1}^{1} q^n \phi_k \, d\xi d\eta
\]

\[
-2A \int_{-1}^{1} \int_{-1}^{1} \partial_\xi q^n \phi_k \, d\xi d\eta + 2A^2 \int_{-1}^{1} \int_{-1}^{1} \partial^2_\xi q^n \phi_k \, d\xi d\eta - \frac{4}{3} A^3 \int_{-1}^{1} \int_{-1}^{1} \partial^3_\xi \phi_k \, d\xi d\eta.
\]

Now integrating each term by parts the maximal amount of times and use (3.3) to write
the above equation as:

\[
\int_{-1}^{1} \int_{-1}^{1} q_{ij}^{n+1} \phi_k \, d\xi \, d\eta = \int_{-1}^{1} \int_{-1}^{1} q_{ij}^{n} \phi_k \, d\xi \, d\eta \\
- 2A^{-} \int_{-1}^{1} q_{i+1j}^{n}(\xi = -1) \phi_k(\xi = 1) \, d\eta - 2A^{+} \int_{-1}^{1} q_{ij}^{n}(\xi = 1) \phi_k(\xi = 1) \, d\eta \\
+ 2A^{-} \int_{-1}^{1} q_{ij}^{n}(\xi = -1) \phi_k(\xi = -1) \, d\eta + 2A^{+} \int_{-1}^{1} q_{i-1j}^{n}(\xi = 1) \phi_k(\xi = -1) \, d\eta \\
+ 2A \int_{-1}^{1} \int_{-1}^{1} q_{ij}^{n} \partial_\xi \phi_k \, d\xi \, d\eta \\
+ 2A^{-} \int_{-1}^{1} \partial_\xi q_{i+1j}^{n}(\xi = -1) \phi_k(\xi = 1) \, d\eta + 2A^{+} \int_{-1}^{1} \partial_\xi q_{ij}^{n}(\xi = 1) \phi_k(\xi = 1) \, d\eta \\
- 2A^{-} \int_{-1}^{1} \partial_\xi q_{ij}^{n}(\xi = -1) \phi_k(\xi = -1) \, d\eta - 2A^{+} \int_{-1}^{1} \partial_\xi q_{i-1j}^{n}(\xi = 1) \phi_k(\xi = -1) \, d\eta \\
- 2A^{-} \int_{-1}^{1} \partial_\xi q_{i+1j}^{n}(\xi = -1) \partial_\xi \phi_k(\xi = 1) \, d\eta - 2A^{+} \int_{-1}^{1} \partial_\xi q_{ij}^{n}(\xi = 1) \partial_\xi \phi_k(\xi = 1) \, d\eta \\
+ 2A^{-} \int_{-1}^{1} \partial_\xi q_{ij}^{n}(\xi = -1) \partial_\xi \phi_k(\xi = -1) \, d\eta + 2A^{+} \int_{-1}^{1} \partial_\xi q_{i-1j}^{n}(\xi = 1) \partial_\xi \phi_k(\xi = -1) \, d\eta \\
+ 4A^{-} \int_{-1}^{1} \partial_\xi q_{i+1j}^{n}(\xi = -1) \partial_\xi \phi_k(\xi = 1) \, d\eta + 4A^{+} \int_{-1}^{1} \partial_\xi q_{ij}^{n}(\xi = 1) \partial_\xi \phi_k(\xi = 1) \, d\eta \\
- 4A^{-} \int_{-1}^{1} \partial_\xi q_{ij}^{n}(\xi = -1) \partial_\xi \phi_k(\xi = -1) \, d\eta \\
- 4A^{+} \int_{-1}^{1} \partial_\xi q_{i-1j}^{n}(\xi = 1) \partial_\xi \phi_k(\xi = -1) \, d\eta.
\]

We substitute the approximate solution \( q_{ij}^{n} = Q_{ij}^{n,1} + \sqrt{3} \xi Q_{ij}^{n,2} + \sqrt{3} \eta Q_{ij}^{n,3} \) into the above
equation and $\mathcal{N}_A$ becomes

$$Q_{ij}^{n+1,1} = Q_{ij}^{n,1} - A^- Q_{i+1,j}^{n,1} + \sqrt{3} A^- Q_{i+1,j}^{n,2} - A^+ Q_{ij}^{n,1} - \sqrt{3} A^+ Q_{ij}^{n,2} + A^- Q_{ij}^{n,1} - \sqrt{3} A^- Q_{ij}^{n,2}$$

$$+ A^+ Q_{i-1,j}^{n,1} + \sqrt{3} A^+ Q_{i-1,j}^{n,2} + \sqrt{3} A^2 Q_{i+1,j}^{n,2} + \sqrt{3} A^2 Q_{ij}^{n,2} - \sqrt{3} A^2 Q_{ij}^{n,2} - \sqrt{3} A^2 Q_{i-1,j}^{n,2},$$

$$Q_{ij}^{n+1,2} = Q_{ij}^{n,2} - \sqrt{3} A^- Q_{i+1,j}^{n,1} + 3 A^- Q_{i+1,j}^{n,2} - \sqrt{3} A^+ Q_{ij}^{n,1} - \sqrt{3} A^+ Q_{ij}^{n,2} - \sqrt{3} A^- Q_{ij}^{n,1}$$

$$+ 3 A^- Q_{ij}^{n,2} - \sqrt{3} A^+ Q_{i-1,j}^{n,1} - 3 A^+ Q_{i-1,j}^{n,2} + 2 \sqrt{3} A Q_{ij}^{n,1} + 3 A^2 Q_{i+1,j}^{n,2} + 3 A^2 Q_{ij}^{n,2}$$

$$+ 3 A^2 Q_{ij}^{n,2} + 3 A^2 Q_{i-1,j}^{n,2} - \sqrt{3} A^2 Q_{i+1,j}^{n,1} + 3 A^2 Q_{i-1,j}^{n,2} - \sqrt{3} A^2 Q_{ij}^{n,1} - 3 A^2 Q_{ij}^{n,2}$$

$$+ \sqrt{3} A^2 Q_{ij}^{n,1} - 3 A^2 Q_{ij}^{n,2} + \sqrt{3} A^2 Q_{i-1,j}^{n,1} + 3 A^2 Q_{i-1,j}^{n,2} + 2 A^3 Q_{i+1,j}^{n,2} + 2 A^3 Q_{ij}^{n,2}$$

$$- 2 A^3 Q_{ij}^{n,2} - 2 A^3 Q_{i-1,j}^{n,2},$$

$$Q_{ij}^{n+1,3} = Q_{ij}^{n,3} - A^- Q_{i+1,j}^{n,3} - A^+ Q_{ij}^{n,3} + A^- Q_{ij}^{n,3} + A^+ Q_{i-1,j}^{n,3}.$$
CHAPTER 4. NUMERICAL RESULTS

In this chapter we demonstrate both the order of accuracy and the stability of both the standard Lax-Wendroff discontinuous Galerkin scheme and the newly proposed method.

4.1 One dimensional error analysis

In one dimension the exact solution to (1.1) is known. In order to measure the error between the exact solution and the approximate solution, the exact solution needs to be projected onto (2.5) using the $L^2$ projection similar to (2.6). We denote the exact solution at the final time as $p$ then the projection of $p$ onto (2.5) is

$$p_i(\xi) = p\left(x_i + \frac{\Delta x}{2} \xi \right) = \sum_{\ell=1}^{m+1} P_{i,\ell} \phi_{\ell}(\xi) \text{ for } \xi \in (-1,1). \quad (4.1)$$

Noting the first $m + 1$ terms of the projection are used in order to compare it to the approximate solution at the final time. We take the difference of the projection of the exact solution and the approximate solution at the final time

$$q(\xi) - p(\xi) = \sum_{\ell=1}^{N} \left( Q_{i,\ell} - P_{i,\ell} \right) \phi_{\ell} - P_{i,N+1} \phi_{N+1}. \quad (4.2)$$

Consider the following $L^2$ norm:

$$\|q - p\|_{L^2(a,b)}^2 = \int_{a}^{b} (q(x) - p(x))^2 dx. \quad (4.3)$$

We write the integral in (4.3) as a sum of integrals over the $N$ elements and substitute (4.2) for $p - q$

$$\frac{\Delta x}{2} \sum_{i=1}^{N} \int_{-1}^{1} (q(\xi) - p(\xi))^2 d\xi = \frac{\Delta x}{2} \sum_{i=1}^{N} \int_{-1}^{1} \left( \sum_{\ell=1}^{N} \left( Q_{i,\ell} - P_{i,\ell} \right) \phi_{\ell} - P_{i,N+1} \phi_{N+1} \right)^2 d\xi. \quad (4.4)$$
Using the relationship 2.2 we rewrite the equation above to gain the following relationship

\[ \|q - p\|^2_{L^2(a,b)} = \Delta x \sum_{i=1}^{N} \sum_{\ell=1}^{m} \left( Q_i^{\ell} - P_i^{\ell} \right)^2 + \left( P_i^{N+1} \right)^2. \]  
(4.5)

Similarly we write the \( L^2 \) norm of the exact solution as

\[ \|p\|^2_{L^2(a,b)} = \Delta x \sum_{i=1}^{N} \sum_{\ell=1}^{m+1} \left( P_i^{\ell} \right)^2. \]  
(4.6)

Using (4.6) and (4.5) we write out the formula for the normalized \( L^2 \) error as

\[ \frac{\|q - p\|^2_{L^2(a,b)}}{\|p\|^2_{L^2(a,b)}} = \frac{\sum_{i=1}^{N} \sum_{\ell=1}^{m} \left( Q_i^{\ell} - P_i^{\ell} \right)^2 + \left( P_i^{N+1} \right)^2}{\sum_{i=1}^{N} \sum_{\ell=1}^{m+1} \left( P_i^{\ell} \right)^2}. \]  
(4.7)

### 4.2 One dimensional Example

In this section we present a one dimensional scalar example and a comparison of the standard Lax-Wendroff discontinuous Galerkin method and the modified method. We compare the regions of stability and discuss the accuracy and rate of convergence for the two methods. Consider the following problem,

\[ q_t + q_x = 0, \quad x \in (-5, 5), \quad t \in [0, 2], \]
\[ q(0, x) = e^{-x^2}, \]  
(4.8)

with periodic boundary conditions \( q(t, -5) = q(t, 5) \).

Using (4.7) we construct convergence tables to compare the two schemes at first, second, and third order accuracy and present graphs of the second order solutions at time \( t = 2 \). The methods all use a uniform mesh.

Shown in Figures 4.1 and 4.2 are solutions to the 1D advection equation with the second-order versions of the standard and the modified Lax-Wendroff schemes, respectively. The modified scheme is run with \( CFL = 1 \), while the standard scheme is run at \( CFL = 0.25 \), since it is stable only up to a CFL number of 1/3.

Shown in Tables 4.1, 4.2, 4.3, 4.4, 4.5, and 4.6 are convergence tests for both the standard and modified Lax-Wendroff schemes. These results highlight two important features of our
modified scheme: (1) the modified scheme is always stable up to a CFL number of one, while the standard method has an ever decreasing maximum CFL number as the order increases; and (2) even with a larger time step, the modified scheme produces solutions with better accuracy compared to the standard method.

Figure 4.1 Second order standard Lax-Wendroff method with $CFL = .25$ for example (4.8) with $N = 200$. 
Figure 4.2 Second order modified Lax-Wendroff method with $CFL = 1$ for example (4.8) $N = 200$.

Table 4.1 Convergence study for the first order standard Lax-Wendroff discontinuous Galerkin method with $CFL = 1$ for the one dimensional advection equation example (4.8).

<table>
<thead>
<tr>
<th>number of elements</th>
<th>$L^2$ error</th>
<th>order of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.141762868729989</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>0.071833512495503</td>
<td>0.980750736978811</td>
</tr>
<tr>
<td>80</td>
<td>0.036042237082713</td>
<td>0.994968502102683</td>
</tr>
<tr>
<td>160</td>
<td>0.018036924023108</td>
<td>0.998735234084784</td>
</tr>
<tr>
<td>320</td>
<td>0.009020440782396</td>
<td>0.999683488189371</td>
</tr>
<tr>
<td>640</td>
<td>0.004510467720353</td>
<td>0.99992088397685</td>
</tr>
</tbody>
</table>
Table 4.2  Convergence table for the first order modified Lax-Wendroff discontinuous Galerkin method with $CFL = 1$ for the one dimensional advection equation example (4.8).

<table>
<thead>
<tr>
<th>number of elements</th>
<th>$L^2$ error</th>
<th>order of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.141695499133613</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>0.071832539019644</td>
<td>0.980084517132654</td>
</tr>
<tr>
<td>80</td>
<td>0.036042169691962</td>
<td>0.994951648318818</td>
</tr>
<tr>
<td>160</td>
<td>0.018036912123287</td>
<td>0.998733488387300</td>
</tr>
<tr>
<td>320</td>
<td>0.009020437293993</td>
<td>0.999683094296180</td>
</tr>
<tr>
<td>640</td>
<td>0.004510466389390</td>
<td>0.999920756190738</td>
</tr>
</tbody>
</table>

Table 4.3  Convergence study for the second order standard Lax-Wendroff discontinuous Galerkin method with $CFL = .25$ for the one dimensional advection equation example (4.8).

<table>
<thead>
<tr>
<th>number of elements</th>
<th>$L^2$ error</th>
<th>order of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.036680840533453</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>0.008903672703251</td>
<td>2.042554232690053</td>
</tr>
<tr>
<td>80</td>
<td>0.002192978201732</td>
<td>2.021509088899123</td>
</tr>
<tr>
<td>160</td>
<td>0.000544797913582</td>
<td>2.009098388282101</td>
</tr>
<tr>
<td>320</td>
<td>0.000135824048039</td>
<td>2.003982242743170</td>
</tr>
<tr>
<td>640</td>
<td>0.000033912874312</td>
<td>2.001833964111390</td>
</tr>
</tbody>
</table>
Table 4.4 Convergence study for the second order modified Lax-Wendroff discontinuous Galerkin method with $CFL = 1$ for the one dimensional advection equation example (4.8).

<table>
<thead>
<tr>
<th>number of elements</th>
<th>$L^2$ error</th>
<th>order of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.015783129539519</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>0.004011937783488</td>
<td>1.976012158683084</td>
</tr>
<tr>
<td>80</td>
<td>0.001007183801201</td>
<td>1.993972479151714</td>
</tr>
<tr>
<td>160</td>
<td>0.000252059434081</td>
<td>1.998491127455036</td>
</tr>
<tr>
<td>320</td>
<td>0.000063031342298</td>
<td>1.99962661150520</td>
</tr>
<tr>
<td>640</td>
<td>0.000015758866061</td>
<td>1.999905657756788</td>
</tr>
</tbody>
</table>

Table 4.5 Convergence study for the third order standard Lax-Wendroff discontinuous Galerkin method with $CFL = 0.1665$ for the one dimensional advection equation example (4.8).

<table>
<thead>
<tr>
<th>number of elements</th>
<th>$L^2$ error</th>
<th>order of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.003885079286536</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>0.000419815385744</td>
<td>3.210117095593912</td>
</tr>
<tr>
<td>80</td>
<td>0.000050873105459</td>
<td>3.044779971916876</td>
</tr>
<tr>
<td>160</td>
<td>0.000006184071420</td>
<td>3.040274277442799</td>
</tr>
<tr>
<td>320</td>
<td>0.000000768003515</td>
<td>3.009372162949333</td>
</tr>
<tr>
<td>640</td>
<td>0.000000095369402</td>
<td>3.009514531729327</td>
</tr>
</tbody>
</table>

4.3 Two dimensional example

In this section we consider a two dimensional system of equations and compare the standard Lax-Wendroff discontinuous Galerkin method to our modified Lax-Wendroff discontinuous Galerkin method. Consider the two dimensional example:

$$q_t + Aq_x + Bq_y = 0, \quad x \in (-5,5), \quad y \in (-5,5), \quad t \in [0,2],$$

$$q_1(0,x,y) = e^{-x^2-y^2}, \quad q_2(0,x,y) = q_3(0,x,y) = 0,$$

(4.9)
Table 4.6  Convergence study for the third order modified Lax Wendroff discontinuous Galerkin method $CFL = 1$ for the one dimensional advection equation example (4.8).

<table>
<thead>
<tr>
<th>number of elements</th>
<th>$L^2$ error</th>
<th>order of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.001488388703174</td>
<td>-</td>
</tr>
<tr>
<td>40</td>
<td>0.000189451982223</td>
<td>2.973847205746399</td>
</tr>
<tr>
<td>80</td>
<td>0.000023789528587</td>
<td>2.993433646027272</td>
</tr>
<tr>
<td>160</td>
<td>0.000002977079911</td>
<td>2.998356831368865</td>
</tr>
<tr>
<td>320</td>
<td>0.000000372240991</td>
<td>2.999589109121571</td>
</tr>
<tr>
<td>640</td>
<td>0.000000046533438</td>
<td>2.999897262156332</td>
</tr>
</tbody>
</table>

along with double periodic boundary conditions, where the constant matrices $A$ and $B$ are given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Shown in Figures 4.3 and 4.4 are surface plots of the numerical solution for the first equation of $q$: $q_1$. Note that they both solve the problem numerically but the modified method can use a much large time step.

Figure 4.3  Second order standard Lax-Wendroff method with $CFL = .25$ for example (4.9) with a total number of elements of $50^2 = 2500$. 
Figure 4.4 Second order modified Lax-Wendroff method with CFL = 1 for example (4.9) with a total number of elements of $50^2 = 2500$.

4.4 Von Neumann analysis

In the previous sections we showed numerical examples of both the standard and the newly proposed Lax-Wendroff discontinuous Galerkin methods. In this section, we aim to give a theoretical justification as to why the newly proposed scheme is stable up to a CFL number of one, while the standard method is not. To this aim, we present the von Neumann analysis (e.g., see LeVeque [8]) for the second order one dimensional methods below.

Consider the advection equation

$$q_t + u q_x = 0, \quad u > 0.$$ 

For this problem the standard Lax-Wendroff discontinuous Galerkin method can be reduced to the following update formula:

$$
\begin{bmatrix}
Q_1^{i} \\
Q_2^{i}
\end{bmatrix}^{n+1} = 
\begin{bmatrix}
U & \sqrt{3}U - \sqrt{3}U^2 \\
-\sqrt{3}U & 3U^2 - 3U
\end{bmatrix}
\begin{bmatrix}
Q_1^{i-1} \\
Q_2^{i-1}
\end{bmatrix}^n +
\begin{bmatrix}
1 - U & \sqrt{3}U^2 - \sqrt{3}U \\
\sqrt{3}U & 1 - 3U - 3U^2
\end{bmatrix}
\begin{bmatrix}
Q_1^{i} \\
Q_2^{i}
\end{bmatrix}^n.
$$
where $U = |u| \frac{4}{\Delta t}$. Applying the von Neumann stability analysis to the method above with $Q^{t,n}_i = Q^{t,n} e^{i \xi \Delta x}$ and $\zeta = e^{-i \xi \Delta x} \ (I = \sqrt{-1})$ yields:

$$
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}
+ 1 + U \begin{bmatrix}
1 \quad \sqrt{3} U (1 - U) (\zeta - 1) - \sqrt{3} U^2 \\
\sqrt{3} U (1 - \zeta) \quad 1 + 3 U \zeta (U - 1) - 3 U (1 + U)
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix} =
\begin{bmatrix}
Q_1^n + U (\zeta - 1) \sqrt{3} U (1 - U) (\zeta - 1) - \sqrt{3} U^2 \\
\sqrt{3} U (1 - \zeta) \quad 1 + 3 U \zeta (U - 1) - 3 U (1 + U)
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}.
$$

It suffices to examine the amplification matrix when $\zeta = 1$ yielding

$$
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}
+ 1 + U \begin{bmatrix}
1 \quad 0 \\
0 \quad 1 - 6 U
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix} =
\begin{bmatrix}
Q_1^n + U (\zeta - 1) \sqrt{3} U (1 - U) (\zeta - 1) - \sqrt{3} U^2 \\
\sqrt{3} U (1 - \zeta) \quad 1 + 3 U \zeta (U - 1) - 3 U (1 + U)
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}.
$$

The eigenvalues are 1 and $1 - 6 U$. The method is stable when the maximum absolute eigenvalue is less than one. Thus if $U \in [0, \frac{1}{3}]$ then the method is stable but if $U \in (\frac{1}{3}, 1]$ the method is unstable.

For the same problem the modified Lax-Wendroff method can be reduced to

$$
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}
+ 1 + U \begin{bmatrix}
1 \quad 0 \\
0 \quad 1 + 6 U^2 - 6 U
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix} =
\begin{bmatrix}
Q_1^n + U (\zeta - 1) \sqrt{3} U (1 - U) (\zeta - 1) - \sqrt{3} U^2 \\
\sqrt{3} U (1 - \zeta) \quad 1 + 3 U \zeta (U - 1) - 3 U (1 + U)
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}.
$$

Applying the von Neumann stability analysis to the modified method yields:

$$
\begin{bmatrix}
Q_1^n + U (\zeta - 1) \sqrt{3} U (1 - U) (\zeta - 1) - \sqrt{3} U^2 \\
\sqrt{3} U (1 - \zeta) \quad 1 + 3 U \zeta (U - 1) - 3 U (1 + U)
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix} =
\begin{bmatrix}
Q_1^n + U (\zeta - 1) \sqrt{3} U (1 - U) (\zeta - 1) - \sqrt{3} U^2 \\
\sqrt{3} U (1 - \zeta) \quad 1 + 3 U \zeta (U - 1) - 3 U (1 + U)
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}.
$$

It suffices to examine the amplification matrix when $\zeta = 1$ yielding

$$
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}
+ 1 + U \begin{bmatrix}
1 \quad 0 \\
0 \quad 1 + 6 U^2 - 6 U
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix} =
\begin{bmatrix}
Q_1^n + U (\zeta - 1) \sqrt{3} U (1 - U) (\zeta - 1) - \sqrt{3} U^2 \\
\sqrt{3} U (1 - \zeta) \quad 1 + 3 U \zeta (U - 1) - 3 U (1 + U)
\end{bmatrix}
\begin{bmatrix}
Q_1^n \\
Q_2^n
\end{bmatrix}.
$$

The eigenvalues for the modified method are 1 and $1 + 6 U^2 - 6 U$. The difference between the two methods is the $6 U^2$ term. This is the term that allows the larger region of stability. The modified method is stable if $U \in [0, 1]$. 
CHAPTER 5. CONCLUSION

5.1 Summary of the current work

In this work we have developed a modified Lax-Wendroff discontinuous Galerkin method for solving linear systems of hyperbolic partial differential equations. This method improves the standard Lax-Wendroff discontinuous Galerkin method from the literature. The key achievement of this newly proposed scheme is that it is stable up to $CFL = 1$, unlike the standard LxW-DG scheme, which is stable only up to $CFL \approx 1/(2m - 1)$, where $m$ is the order of accuracy of the scheme.

In the case of one space dimension, the method was presented for arbitrary order of accuracy. In the two dimensional case the first and second order methods were developed. In the current work, the two dimensional method were implemented using dimensional splitting, which means that the proposed method is not a one step method. However, it still has a larger region of stability compared to the standard Lax-Wendroff discontinuous Galerkin method.

The standard and modified LxW-DG schemes were implemented and compared. It was shown that the modified methods exhibit higher accuracy and a larger region of stability. Von Neumann stability analysis was provided to prove that the proposed scheme has better stability properties.

5.2 Future work

In the one dimensional case we developed an automatic way to generate the scheme for arbitrary orders of accuracy. In the two dimensional case we only have the first and second
order modified schemes and desire a way to automate the generation of the schemes for arbitrary order. In particular, we wish to remove the need to use dimensional splitting the two dimensional case. In future work, these schemes will be extended further to also handle the three dimensional case.

Another point of improvement that we seek is to implement the proposed scheme on parallel computer architectures. The framework of using a fully one-step method with a relatively large CFL number should lend itself well to modern parallel machines. These issues will be explored in future work.
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