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Forcing in set theory and its applications to topology

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FORCING IN SET THEORY AND ITS APPLICATIONS TO TOPOLOGY

Iowa State University

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Forcing in set theory and its applications to topology

by

Kyriakos Keremedis

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

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1. INTRODUCTION

This dissertation consists of two parts. The first part is concerned with the Theory of Sets and the second with Topology. These two parts are linked by the recent developments in Axiomatic Set Theory. In particular, Cohen's forcing method allows the construction of models for various set-theoretically or topologically formulated statements, establishing their relative consistency with ZF [10].

The first part of our dissertation is motivated by the works of Fraenkel, Mostowski (see Abian [4]) and Cohen [10]. We develop a forcing language and use the notion of the hereditarily symmetric sets, to prove the relative consistency of $ZF^- + \neg AC$. The novelty of our method consists in the reduction of a model of $ZF^-$ to that of $ZF^- + \neg AC$. Another novelty of our work is the development of a second forcing language which also preserves the law of the double negation.

The second part of our work is motivated by Martin's and generalized Martin's axioms. We use these axioms to obtain new results in topological spaces, especially in connection with normality and compactness.

In particular in the introductory Section 2, using
Mostowski's collapsing theorem, we consider the sets of ordinals which exist in a standard model of ZF and show that these characterize the model. We also introduce the notion of the minimal standard transitive model of ZF which is indispensable in the sequel.

In Section 3, we develop the concept of the Bijectional models, which we use to prove the consistency of ZF with the existence of an infinite set of atoms. These models are used in Section 4.

In Section 4, we develop a new forcing language which (in contradistinction to the classical forcing language) is used to systematically eliminate certain sets from permutational models to yield models of ZF+$\neg$AC.

In Section 5, we introduce a variant of the unramified forcing language based on the Topology induced by a non-molecular partial order in a suitable countable standard transitive model of ZF.

In Section 6, we develop a new variant of the unramified forcing language where the law of the double negation is preserved. We prove all the theorems required to show that this forcing language can be used to yield various models of ZF such as ZF+$\neg$CH and ZF+MA.

In Section 7, the forcing introduced in Section 5 together with the concept of the hereditarily symmetric
sets in generic extensions are used to construct a model of \( ZF + \neg AC \).

Sections 8, 9, 10 and 11 constitute the Topology part of the dissertation.

In Section 8 based on Martin's axiom, we prove a few combinatorial theorems concerning the existence of a subset of \( \omega \), having finite and infinite intersections with two given elements of \( P(\omega) \). These theorems are used to obtain results related to the sequential compactness and meager sets.

In Section 9, we introduce \( E \)-spaces which satisfy separation conditions weaker than normality. Then using Martin's and generalized Martin's axioms, we obtain some normality results for \( E \)-spaces.

In Section 10, based on cardinality considerations, weaker versions of compactness are developed. Then, various implications among them are obtained using set-theoretical methods.

Finally in Section 11, we consider topologies on infinite products of sets. Bases of these Topologies are defined in terms of cardinality of factor spaces from which proper open sets are chosen. We show that some of the notions of compactness of Section 10 are preserved under products of certain cardinality.
2. MODELS AND STANDARD TRANSITIVE MODELS OF ZF

The present work is in the setting of the Zermelo-Fraenkel Set Theory which is developed in the framework of the First Order Predicate Calculus with equality and which is denoted by ZF. Thus, as expected, the language of ZF is the first order language with equality and with one and only one binary predicate \(\varepsilon(x,y)\) - the "elementhood relation". We denote this language by \(\mathcal{L}\) and we write \(x \varepsilon y\) instead of \(\varepsilon(x,y)\).

The axioms of ZF are: Extensionality, Sumset, Power-set, Infinity Regularity and the axiom scheme of Replacement [13, p. 284]. It is known that ZF is not finitely axiomatizable [9, p. 83].

ZF refers to one and only one type of objects, namely sets. Nevertheless, we also introduce classes in order to avoid the difficulties involved in handling formulas [15, p. 3].

We assume once and for all that ZF is consistent. Hence, by Goedel's completeness theorem there exists a set \(M\) and a binary relation \(R\) on \(M\) such that \((M,R)\) is a model of ZF, i.e., all the axioms of ZF are satisfied in \((M,R)\) when \(\varepsilon\) is interpreted as \(R\). However, by Goedel's Second Incompleteness theorem the existence of \((M,R)\) cannot be proved in ZF (although it can be proved in
a stronger system). On the other hand, based on the consistency of ZF we can prove (in ZF) the existence of a class $V$ such that

$$V = \bigcup_{u \in \text{Ord}^*} V_u$$

where $V_0 = \{\emptyset\}$ and $V_{u+1} = P(V_u)$ and $V_u = \bigcup_{i \in u} V_i$ if $u$ is a limit ordinal.

It can be readily verified that in $(V, \in)$ all the axioms of ZF are satisfied.

Extending the notion of a model to include classes as domains of models, $(V, \in)$ is referred to as von Neumann's model of all well-founded sets. It is believed that all the reasonable mathematics takes place in $(V, \in)$, see, e.g., [21, 41] and [12, pp. 98-101].

In what follows all our constructions are carried out in the model $(V, \in)$ of ZF.

Let $R$ be a binary relation over a subclass or a set $Q$ of $V$. Then, as usual, $(Q, R)$ is called a model of a collection $S$ of statements of $V$ if every statement of $S$ is satisfied in $(Q, R)$ when $\in$ is interpreted by $R$.

As expected, $(Q, R)$ is called a transitive model if $Q$ is a transitive class or a transitive set, i.e., $x \in Q$ and $y \in x$ imply $y \in Q$. Moreover, $(Q, R)$ is called a standard model if $R$ is the restriction of $\in$ to $Q$, in which case, by abuse of notation, we denote $(Q, R)$ by $(Q, \in)$. 
According to the above, \((V, \varepsilon)\) is a standard transitive model of \(ZF\).

In what follows, we consider mainly standard transitive models of \(ZF\). These models have the following important property. Let us call \(\exists u \in x\) and \(\forall u \in x\) restricted quantifiers, and say that a formula \(F\) (naturally of \(L\)) is a restricted formula if it has no other quantifiers than restricted ones. Let \((S, \varepsilon)\) be a standard transitive model of \(ZF\) and \(F\) be a restricted formula. Then for every \(a_1, \ldots, a_n \in S\) it is the case that:

\[
S \models F(a_1, \ldots, a_n) \iff F(a_1, \ldots, a_n)
\]

which implies that in standard transitive models the axiom of Extensionality is satisfied (which is the most desirable property for a model to possess).

In connection with the above, let us observe that not every standard model of \(ZF\) is transitive, however, by Mostowski's collapsing theorem [23, p. 147], every standard model of \(ZF\) is isomorphic to a standard transitive model of \(ZF\).

Besides \((V, \varepsilon)\) there are two other prominent examples of standard transitive models of \(ZF\): Goedel's model \((L, \varepsilon)\) of constructible sets [14] and Cohen's minimal model \((M, \varepsilon)\) of strongly constructible sets [9].
(L, ε) is the least standard transitive model containing all the ordinals (of course of V). In (L, ε) the axiom of Constructibility is satisfied and therefore also the axiom of Choice AC, and, the generalized continuum hypothesis GCH. Thus, (L, ε) provides a proof [15, pp. 109—110] of the consistency of V=L, AC and GCH with ZF.

(M, ε) is the minimal standard transitive model which is a submodel of any standard transitive model of ZF. In (M, ε) also, V=L and therefore AC and GCH are satisfied.

If the domain of a model is a set and not a proper class, we indicate that domain by a small letter. In this connection, we mention the following consequences of Goedel's Completeness theorem [13], where ZFC stands for ZF+AC.

(1) In every standard transitive model of ZFC there exists a set (in fact, even countable) which is a model of ZFC

(2) There is a model of ZFC in which there exists no set which is a model of ZF

We prove below a weaker version of (2):

LEMMA 1. There is a standard transitive model of ZFC
in which there exists no set which is a standard transitive ZF.

**PROOF.** Assume on the contrary. Thus, in model \((L, \varepsilon)\) there exists a set which is a standard model of ZF. But then, based on Mostowski's collapsing theorem, there exists a set \(m_0 \in L\) such that \((m_0, \varepsilon)\) is a standard transitive model of ZFC. Now, starting with \((m_0, \varepsilon)\) instead of \((L, \varepsilon)\) we derive the existence of a set \(m_1 \in m_0\) such that \((m_1, \varepsilon)\) is a standard transitive model of ZFC. Because of the validity of the axiom of Regularity in \((L, \varepsilon)\), the iteration of the above process must stop yielding \((m_n, \varepsilon)\) in which there is no set which is a standard model of ZF.

Let \(M\) and \(SM\) stand for the ZF statements given by:

(3) \(M \equiv \text{there exists a set which is a model of ZF}\) and

(4) \(SM \equiv \text{there exists a set which is a standard model of ZF}\)

Clearly, by Goedel's second Incompleteness theorem, \(M\) and therefore also \(SM\) are unprovable in ZF. On the other hand, it is believed that \(SM\) and also \(M\) are consistent with ZF. In fact, by Goedel's Completeness theorem \(M\) is
is equivalent to Con(ZF).

Obviously, SM implies M, however, the converse does not hold as shown below:

**LEMMA 2.** \( \vdash M \rightarrow SM \)

**PROOF.** Assume to the contrary. Let \((M, \varepsilon)\) be Cohen's minimal model. By (1) we have:

(5) \[ (M, \varepsilon) \models M \]

But then, by our assumption, from (5) it follows that \((M, \varepsilon) \models SM\) which is a contradiction since by minimality of \((M, \varepsilon)\) there is no set in \((M, \varepsilon)\) which is a standard model of ZF. Thus, our assumption is false and the Lemma is proved.

Motivated by a result of [42] we prove below the fact that standard transitive models of ZF are uniquely characterized by the subsets of ordinals that they possess.

First however, we prove:

**LEMMA 3.** Let \(A\) and \(B\) be transitive \(\varepsilon\)-isomorphic classes. Then \(A = B\).

**PROOF.** Let \(f\) be an isomorphism from \(A\) onto \(B\). We prove by transfinite induction on the rank of the elements of \(A\) that \(f(x) = x\) for every \(x \in A\).
Let \( f(y) = y \) for every element \( y \) of \( A \) of rank \( < r \) and let \( x \) be an element of \( A \) of rank \( r \). But then

\[
(y \in x) \iff f(y) \in f(x) \iff y \in f(x)
\]

Thus, \( f(x) = x \), as required.

THEOREM 1. Let \( M \) and \( N \) be standard transitive models of \( ZF \). Let the axiom of Choice be valid in \( M \). If \( M \) and \( N \) have the same sets of ordinals then \( M = N \).

PROOF. Let \( F \) be the canonical map \([2]\) from \( \text{On} \times \text{On} \) onto \( \text{On} \). Let \( P \) be a set of ordered pairs of ordinals. Since \( P = F^{-1}(F[P]) \) we see that if \( P \) is a set of \( M \) then \( P \) is also a set of \( N \) and vice versa. Thus, \( M \) and \( N \) have the same sets of pairs of ordinals.

First we show that \( M \subseteq N \). Since \( M \) and \( N \) are well-founded

\[
M = \bigcup_{r \in \text{On}} M_r \quad \text{and} \quad N = \bigcup_{r \in \text{On}} N_r
\]

where \( M_r \) and \( N_r \) are the von Neumann's \( r \)-th stages of \( M \) and \( N \) respectively. Since \( AC \) is valid in \( M \), there exists an ordinal \( m \) and a 1:1 map \( f \) from \( m \) onto \( M_r \). Let \( E \) be the relation

\[
p E q \iff f(p) \in f(q) \quad \text{for every} \quad p, q \in m
\]

Clearly \( E \) is well-founded and extensional in both
M and N. Thus, by Mostowski's collapsing theorem \((m, E)\) yields a transitive set \(T\) in both \(M\) and \(N\). By Lemma 3, we see that \(T = M_r\). Hence, \(M_r \in N\) and thus \(M \subseteq N\).

Next, we show that \(N \subseteq M\) by induction on the rank of the elements of \(N\). Assume that \(N_r \subseteq M\) and let \(x \in N_{r+1}\). By the induction hypothesis \(x \subseteq M\). Let

\[
w = \sup\{k : k = \text{rank}(y) \text{ for some } y \in x\}
\]

Clearly \(x \subseteq M_w\). Since AC is valid in \(M\) there exists a map \(h\) from \(M_w\) onto an ordinal \(u\). Since \(M \subseteq N\) we see that \(h \in N\). However, \(h[x] \in N\) and because \(M\) and \(N\) have the same sets of ordinals \(h[x] \in M\). But since \(M\) is a model, \(h^{-1}(h[x]) \in M\). But \(h^{-1}(h[x]) = x\) and therefore \(x \in M\) implying that \(N \subseteq M\), as desired.

We prove below another theorem concerning standard models.

**THEOREM 2.** In \(ZF + SM\) it cannot be proved that there exists an uncountable set \(S\) which is a standard model of \(ZF\) in which AC is true and CH is false.

**Proof.** Since \(V=L\) is consistent with \(ZF\) it suffices to show that \(ZF + (V=L) + SM\) implies that there is no such a set \(S\).

Let \(S\) be any uncountable model of \(ZF\) which is a
set in \((L, \varepsilon)\), where \((L, \varepsilon)\) is the Gödel's model of constructible sets satisfying \(\text{SM}\). We show that if \(\text{AC}\) is valid in \(S\) then so is \(\text{OH}\).

Let \(a_0 = \{x : x \in S \text{ and } x \text{ is an ordinal}\}\). We claim that \(a_0\) is uncountable. Assume on the contrary that \(a_0\) is countable and let

\[
S = \bigcup_{r \in a_0} V_r
\]

where \(V_r\) is the von Neumann's \(r\)-th stage relative to \(S\). Since \(\text{AC}\) is valid in \(S\), for every ordinal \(r \in a_0\) there is a 1:1 map \(f_r\) from an ordinal \(w_r \in a_0\) onto \(V_r\). Since \(S \in L\) by the transitivity of \(L\) we have \(f_r \in L\). Hence \(V_r\) is countable which is a contradiction.

Since \(a_0\) is uncountable, \(S\) contains all the countable ordinals of \((L, \varepsilon)\). Let

\[
H = \bigcup_{r \in a_0} L_r
\]

where \(L_r\) is Gödel's \(r\)-th stage. Clearly \(H \subseteq S \subseteq L\).

Since every subset of \(\omega\) in \((L, \varepsilon)\) can be found in some stage of \(H\) with a countable index, we conclude that every subset of \(\omega\) which is in \((L, \varepsilon)\) is also in \((H, \varepsilon)\) as well as in \((S, \varepsilon)\). Hence \(P_H(\omega) = P_S(\omega) = P_L(\omega)\). Clearly, \(\omega_L\) is the same set in \((L, \varepsilon)\), \((S, \varepsilon)\) and \((H, \varepsilon)\)
From (7) it follows that $|\mathcal{P}_H(\omega)| = \omega_1$ and since $H \subseteq S$ we see that $|\mathcal{P}_S(\omega)| = \omega_1$, i.e., CH is valid in $(S, \varepsilon)$, as desired.

It is well-known that the unrestricted scheme 
\{x : P(x)\} of Comprehension is contradictory [44]. In contrast to this the set of all \{x : P(x)\}, where $P(x)$ is a set-theoretic formula (i.e., a formula of \$\$) with one free variable $x$, exists provided $x$ and $P(x)$ are restricted to a given set $A$. This is shown in the following where all the constructions are carried out in the Goedel's model $(L, \varepsilon)$ of constructible sets.

**LEMMA 4.** Given a set $A$ there exists (in ZF) a set $B$ such that $B = \{\ldots, \{x : P(x)\}, \ldots\}_A$, i.e., $B$ is the set of all definable subsets of $A$ with the defining formulas all relativized to $A$.

**PROOF.** Let 
\[ B_0 = \{A\} \cup A \]
\[ B_{n+1} = B_n \cup \{\{a, b\}, \ldots, a - b, \ldots, a \cdot b, \ldots, \text{dom}(a), \ldots, \{(x, y) : (x, y) \varepsilon a \land x \varepsilon y\}, \ldots \]
\[ \ldots, \{(y, z, x) : (x, y, z) \varepsilon a\}, \ldots \]
\[ \ldots, \{(y, x, z) : (x, y, z) \varepsilon a\}, \ldots \]
\[ \ldots, \{(x, z, y) : (x, y, z) \varepsilon a\}, \ldots \]

where $B_n = \{a, b, \ldots\}$. It can be readily verified that
the set
\[(8) \quad L(A) = \left( \bigcup_{n \in \omega^n} R_n \right) \cap P(A)\]
is the desired set \(B\).

**Lemma 5.** Let \(A\) be a set. Then there exists a set \(R(A)\) given by
\[(9) \quad R(A) = \{\ldots, \{y : (\exists x)(x \in a \land f(x, y))\}, \ldots\}\]
where \(f(x, y)\) is a binary predicate functional in \(x\) on a set \(a \in A\), and \(f\) is restricted to \(A\).

**Proof.** By Lemma 4, the set \(B = \{\ldots, \{x : P(x)\}, \ldots\}_A\) exists. By theorem scheme of Separation the set
\[\{x : x \in B \land (\exists f)(\exists a)(f \text{ is a function from } a \text{ into } A \text{ and } x = \text{range}(f))\}\]
is the desired set \(R(A)\).

**Theorem 3.** Let \(M_0\) be a transitive set. Then there is a standard transitive model \(M\) of ZF such that \(M_0 \subseteq M\).

**Proof.** For every ordinal \(\nu \geq 1\) we define the set \(M_\nu\) as follows. Set
\[(10) \quad C_\nu = \bigcup_{\mu \in \nu} M_\mu\]
\[(11) \quad T_\nu = L(C_\nu)\]
(12) \[ M_v = \{ \cup x : x \in C_v \} \cup \{ P(x) : x \in C_v \} \cup R(T_v) \cup \{ \omega \} \cup \omega \]

where \( L(C_v) \) and \( R(T_v) \) are given as in (8) and (9) and \( Ux \) and \( P(x) \) stand as usual for the Sumset and Powerset of \( x \). Let

(13) \[ M = \bigcup_{v \in \text{On}} M_v \]

Clearly \( (M, \varepsilon) \) is a model for the axioms of Powerset and Extensionality. Next we prove that \( (M, \varepsilon) \) is a model for the axiom scheme of Replacement. Let \( P(x, y) \) be a binary predicate functional in \( x \) on a set \( s \in M \). Using the fact that the axiom of Replacement is valid in \( (L, \varepsilon) \), we can find an ordinal \( u \) such that if \( P(x, y) \) is valid in \( M \) for some \( x \in s \) then \( y \in M_u \). An application of the Lowenheim-Skolem theorem [6, p. 82] at this point will yield an ordinal \( w \) such that \( F|M \iff F|M_w \). But then in \( R(M_w) \) there is a set \( m \) consisting precisely of the mates of every \( x \in s \). Thus, \( m \in M_{w+1} \) and the axiom scheme of Replacement is valid in \( (M, \varepsilon) \) as required.

**Lemma 6.** Let \( M_v \) be given as in (12) then \( M_v \) is transitive.

**Proof.** By assumption \( M_0 \) is transitive, it suffices to show that if \( M_v \) is transitive then \( M_{v+1} \) is transitive.
also. This will follow if we show \( M_v \subseteq M_{v+1} \). But this is indeed the case since by (9) the range of the identity function on every element of \( M_v \) is an element of \( M_{v+1} \).

**THEOREM 4.** Let \( M' \) be as given in (13). Let \( M' = \bigcup_{v \in \text{On}(L)} M_v \), where \( M_v \) is given as in (12) and where all the constructions are carried out in \( M \), then \( M' = M \)
i.e.,

\[
M = \bigcup_{v \in \text{On}(L)} M_v = \bigcup_{v \in \text{On}(M)} M_v = M'
\]

**PROOF.** From (13) it follows that all the ordinals of \( M \) are also ordinals of \( L \). If \( L \) and \( M \) have the same ordinals then (14) follows trivially.

It remains to show (14) when \( \text{On}(M) \subseteq \text{On}(L) \). Let \( a_0 \) be the first ordinal of \( L \) not in \( M \) then

\[
U_{v \in \text{On}(M)} M_v = \bigcup_{v \in \text{On}(M)} M_v = M'
\]

Since \( M' \) is a model, from (10), (11) and (12), it follows that

\[
M_{a_0} = U_{v \in a_0} M_v
\]

Similarly from (10), (11) and (14), it follows that

\[
M_{a_0} = U_{v \in \text{On}(L)} M_v
\]
The desired result now follows from (15), (16) and (17).

**COROLLARY 1.** Let \((K, \varepsilon)\) be any standard transitive model of ZF such that \(M_0 \subseteq K\). Then \(M \subseteq K\). Furthermore, if \(M_0\) is countable and \(\text{On}(M) \subseteq \text{On}(K)\) then \(M \subseteq K\), and \(M\) is countable.

**PROOF.** By Theorem 4, we have

\[
\bigcup_{\nu \in \text{On}(K)} M_\nu = \bigcup_{\nu \in \text{On}(M)} M_\nu = M
\]

The fact that \(M \subseteq K\) follows at once from (18). If \(\text{On}(M) \subseteq \text{On}(K)\), let \(a_0\) be the first ordinal of \(K\) not in \(M\). But then from (14) and (15) we have \(M \subseteq K\). Since \(M_0\) is countable, by the Lowenheim-Skolem theorem [11, p. 192] there exists a countable transitive submodel \((M', \varepsilon)\) of \((M, \varepsilon)\), which by (18) implies that \((M, \varepsilon)\) is countable.

**REMARK 1.** The unique model \(M\) described in Theorem 3 and Corollary 1, is called the minimal model. In order to assure the countability of \(M\) we had to assume that \(\text{On}(M) \subseteq \text{On}(L)\). It turns out that this assumption is equivalent to the axiom SM which assures the existence of a set \(S\) which is a standard model of ZF.
3. BIJECTIONAL MODELS OF ZF

In view of Lemma 3 on page 9, no standard transitive model of ZF admits nontrivial automorphisms. However, if \((K, \varepsilon)\) is a standard transitive model of ZF then as shown below any nontrivial bijection can be used to define a nonstandard model.

Let \((K, \varepsilon)\) be a standard transitive model of ZF and let \(f\) be a nontrivial bijection from \(K\) onto \(K\) definable in \((K, \varepsilon)\). We consider the relational system \((K, \varepsilon')\), where \(\varepsilon'\) is the elementhood relation defined by

\[
x \varepsilon' y \leftrightarrow x \varepsilon f^{-1}(y)
\]

We show that \((K, \varepsilon')\) is a model of ZF\(^{-}\), i.e., ZF minus the axiom of Regularity. We assume that in \((K, \varepsilon)\) distinct constants stand for distinct sets. Thus the axiom of extensionality is valid in \((K, \varepsilon')\). In what follows all the primed symbols refer to \((K, \varepsilon')\) system.

THEOREM 5. The axiom of Sumset is valid in \((K, \varepsilon')\).

PROOF. Let \(s \in K\), consider the set \(f^{-1}(s)\) and the binary predicate \(F(x, y) \equiv (y = f^{-1}(x))\). Clearly \(F\) is functional in \(x\) on \(f^{-1}(s)\). Therefore the set

\[
A = \{ y : (x \varepsilon f^{-1}(s)) \land y = f^{-1}(x) \}
\]
exists in \((K,\varepsilon)\) and consequently \(UA\) exists in \((K,\varepsilon)\).

Furthermore,

\[ t \in f(UA) \iff t \in UA \]

\[ \iff (\exists x)(x \in A) \land (t \in x) \]

\[ \iff (\exists y)(y \in f^{-1}(s)) \land (t \in f^{-1}(y)) \]

\[ \iff (\exists y)(y \in s') \land (t \in y) \]

\[ \iff t \in U's \]

Showing that \(f(UA)\) is the Sumset \(U's\) of \(s\) in \((K,\varepsilon')\).

**THEOREM 6.** The axiom of Powerset is valid in \((K,\varepsilon')\).

**PROOF.** Let \(s \in K\). Consider the binary predicate \(P(x,y) = (y = f(x))\). Clearly \(P\) is functional in \(x\) on \(P(f^{-1}(s))\). Hence the set

\[ B = \{ y : (x \in P(f^{-1}(s)) \land y = f(x) \} \]

exists in \((K,\varepsilon)\) and since \(X \subseteq Y \iff f^{-1}(X) \subseteq f^{-1}(Y)\) we have

\[ t \in f(B) \iff t \in B \]

\[ \iff f^{-1}(t) \subseteq f^{-1}(s) \]

\[ \iff t \subseteq s \]

\[ \iff t \in P'(s) \]

Hence \(f(B)\) is the powerset \(P'(s)\) of \(s\) in \((K,\varepsilon')\).
To show that the axiom scheme of Replacement is valid in \((K, \varepsilon')\), we need the following lemma.

**Lemma 7.** For every formula \(F'\), there exists a formula \(F\) such that

\[ F'(a_1, \ldots, a_n) \iff F(a_1, \ldots, a_n). \]

Furthermore, if \(F'\) is a binary predicate functional in \(x\) on a set \(s\) then \(F\) is a binary predicate functional in \(x\) on \(f^{-1}(s)\).

**Proof.** We prove the lemma by induction on the degree of complexity of the formulas. Let \(F'\) be atomic, i.e., \(a \varepsilon' b\) or \(x \varepsilon' a\) or \(a \varepsilon' x\) or \(x \varepsilon' y\). Then clearly the corresponding \(F\) is given by \(a \varepsilon f^{-1}(b)\) or \(x \varepsilon f^{-1}(a)\) or \(a \varepsilon f^{-1}(x)\) or \(x \varepsilon f^{-1}(y)\).

Next let \(F' \equiv \forall G'\) then by the induction hypothesis there exists a formula \(G\) such that

\[ G'(a_1, \ldots, a_n) \iff G(a_1, \ldots, a_n) \]

and therefore

\[ \forall G'(a_1, \ldots, a_n) \iff \forall G(a_1, \ldots, a_n) \]

Hence \(F \equiv \forall G\) is the desired formula.

A similar argument applies to \(F' \equiv (G' \rightarrow H')\).

Now, let
\[ F'(x_1, \ldots, x_n) \equiv (\forall p)G'(p, x_1, \ldots, x_n) \]

and assume that
\[ (K, \varepsilon') \models F'(a_1, \ldots, a_n) \]

i.e.,
\[ (K, \varepsilon') \models G'(p, a_1, \ldots, a_n) \quad \text{for every } p \in K. \]

By the induction hypothesis, there exists a formula \( G \) such that
\[ G'(p, a_1, \ldots, a_n) \iff G(p, a_1, \ldots, a_n) \]

for every \( p \in K \). Thus
\[ (\forall p)G'(p, a_1, \ldots, a_n) \iff (\forall p)G(p, a_1, \ldots, a_n) \]

Let \( F(x_1, \ldots, x_n) \equiv (\forall p)G(p, x_1, \ldots, x_n) \). Clearly \( F \) satisfies the conclusion of the Lemma.

The above proof also ensures the functionality of \( F \), as stated in the conclusion of the lemma.

**THEOREM 7.** The axiom scheme of Replacement is valid in \( (K, \varepsilon') \).

**PROOF.** Let \( F'(x, y) \) be a binary predicate functional in \( x \) on a set \( s \in K \). Consider the binary predicate \( F(x, y) \) functional in \( x \) on \( f^{-1}(s) \) in \( (K, \varepsilon) \) mentioned in Lemma 7. Since the axiom scheme of Replacement is valid in \( (K, \varepsilon) \) the set
exists in \((K, \varepsilon)\). Furthermore,

\[ y \in f(M) \iff y \in M \]

\[ \iff (\exists x)(x \in f^{-1}(s) \land F(x, y)) \]

\[ \iff (\exists x)(x \in s) \land F(x, y) \]

showing that the set \(f(M)\) is the set of mates of \(x \in s\). Consequently the axiom scheme of Replacement is valid in \((K, \varepsilon')\).

Thus \((K, \varepsilon')\) is a model of \(ZF^-\) as promised.

REMARK 2. The assumption made at the beginning of this section, namely that distinct constants stand for distinct sets, is crucial for the validity of the axiom of Extensionality in \((K, \varepsilon')\). The following counterexample shows that bijections need not preserve Extentionality. This, incidentally also shows that the axiom of Extensionality is independent of the rest of the axioms of \(ZF^-\).

Let \((M, \varepsilon)\) be a model of \(ZF\). Let \((M', \varepsilon)\) be the model resulting from \((M, \varepsilon)\) by adjoining a new symbol \(p\) such that

\[ (20) \quad p = \emptyset \]

\[ (21) \quad p \in t \iff \emptyset \in t \]

Clearly \((M', \varepsilon)\) is a model of \(ZF\). We consider the bijec-
tion $f$ from $M'$ onto $M'$ given by

(22) $f(p) = 1$

(23) $f(1) = p$

(24) $f(t) = t$ for $t \neq p$ and $t \neq 1$

Obviously $1 =^* 0$ but $0 \in^* p$ whereas $1 \notin^* p$. Hence the axiom of Extensionality is not valid in $(M', \in^*)$.

Recalling the definition [1] of an atom $a$, i.e., $a = \{a\}$ we prove

THEOREM 8. There exists a model of $ZF^-$ in which there is an infinite set $A$ of atoms.

PROOF. Let $f : K \rightarrow K$ be defined as follows

(25) $f(i) = i$, $i \in \omega$, $i \geq 2$

(26) $f(1) = \{i\}$, $i \in \omega$, $i \geq 2$

(27) $f(x) = x$ otherwise.

Clearly $f$ is a bijection from $K$ onto $K$.

Let us consider the model $(K, \in^*)$ with $\in^*$ defined as in (19). From (19), (25), (26) and (27) it follows that

$2 \in^* 2 \leftrightarrow 2 \in f^{-1}(2) = \{2\}$

$3 \in^* 3 \leftrightarrow 3 \in f^{-1}(3) = \{3\}$

and in general
Thus every \( n \in (\omega - \{0,1\}) \) is an atom of \((K, \varepsilon')\).
Since \( f((\omega - \{0,1\})) = (\omega - \{0,1\}) \) we see that
\( A = (\omega - \{0,1\}) \) is a set of \((K, \varepsilon')\) with infinitely many
atoms, as required.
4. FORCING IN PERMUTATIONAL MODELS

Having established the consistency of \( ZF^- \) with the existence of an infinite set \( A \) of atoms, we construct below a model \((V,\varepsilon)\) of \( ZF^- + A \) in which we consider the class of all symmetric sets (as defined below). Corresponding to that class we introduce a forcing language by means of which we prove the independence of the AC from \( ZF^- + A \). Some motivation for our ideas can be found in \([4, 20, 22, 43]\).

The model \((V,\varepsilon)\) is constructed inductively as follows.

(28) \[ V_0 = A = \{\emptyset, a_0, a_1, \ldots\} \]

(29) \[ V_u = \bigcup_{w \in u} V_w \quad \text{if } u \text{ is a limit ordinal} \]

(30) \[ V_{u+1} = P(V_u) \]

and we let

(31) \[ V = \bigcup_{u \in \text{On}} V_u \]

Let \( p \) be a permutation of the set \( A \) of all atoms.

We define \([p]\) as follows

(32) \[ [p] = \text{The set of all permutations of } A \text{ which keep the same set of atoms elementwise fixed as does } p \]

Let \( P \) be given by
(33) \[ P = \{ [p] : p \text{ is a permutation of A keeping finitely many atoms fixed} \} \]

We consider the partially ordered set \( (P, \leq) \) where \( \leq \) stands for inverse inclusion.

For any element \( c \) of \((V,\varepsilon)\) and \([p] \in P\), we let
\[
[p](c) = \bigcup \{ h(c) : h \in [p] \}
\]

Let us call a set \( b \) of \((V,\varepsilon)\) symmetric iff

\[ \forall p \in [p] \forall A \in [p] ( [p](A) = A) \]

Let \( G \) be a maximal simply ordered subset of \( P \).

Based on the set \( G \) we define in \((V,\varepsilon)\) the relation \( S \) given by

\[ a S b \iff (a \in b) \land (\forall p \in [p] \forall A \in [p] ( [p](A) = A)) \]

For \( b \in V \) we define

\[ \bar{b} = \{ \bar{a} : a S b \} \quad \text{and} \quad \bar{a} = b \text{ if } b \text{ is an atom} \]

We call \( \bar{b} \) the last survivor of \( b \), and \( b \) a progenitor of \( \bar{b} \). Next let

\[ H_u = \{ x : x S V_u \} \]

and

\[ H = \bigcup_{u \in 0} H_u \]

We prove below that \((H,\varepsilon)\) where \( H \) is given as in (38) is a model of \( \text{ZF}^- \). In order to do this we introduce
a suitable forcing language which is an extension of ZFC + A with the set of all symmetric sets of V as new constants.

We define our forcing language as follows

\[ [p] \models a \epsilon b \iff (\exists c)(\exists q)(c \epsilon b \land [q] \geq [p] \land [q](c) = c \land [p] \models a = b) \]

\[ [p] \models a = b \iff (\forall c)[((\exists q)([q] \geq [p] \land [q](c) = c) \to ((c \epsilon a \to (\exists d)(\exists h)([h] \geq [p] \land [h](d) = d \land d \epsilon b \land [p] \models c = d)) \land (c \epsilon b \to (\exists d)(\exists h)([h] \geq [p] \land [h](d) = d \land d \epsilon a \land [p] \models c = d))] \]

and for any formulae E and F we define

\[ [p] \models \neg E \iff \neg ([p] \models E) \]

\[ [p] \models E \lor F \iff ([p] \models E) \lor ([p] \models F) \]

\[ [p] \models (\exists x)E(x) \iff (\exists b)([p] \models E(b)) \]

It must be understood that in the above definitions max(rank(a), rank(b)) \geq 1. For the case where max(rank(a), rank(b)) = 0 we define

\[ [p] \models a \epsilon b \iff a = b, \text{ and } [p] \models a = b \iff a = b \]

**Lemma 8.** If \( \{ (x_1, \ldots, x_n) : F(x_1, \ldots, x_n) \} \), is a class of \( (V, \epsilon) \) then so is \( \{ (x_1, \ldots, x_n) : (\exists p)([p] \models F(x_1, \ldots, x_n)) \} \).
PROOF. We observe that in the definition of forcing $[p] \vdash a = b$ is defined in terms of $[p] \vdash c = d$ where $\max(\text{rank}(c), \text{rank}(d)) < \max(\text{rank}(a), \text{rank}(b))$. Therefore, it follows that if $F$ is a formula of $ZFC + A$ whose constants are symmetric sets of $V$ then so is the formula $(\exists p)([p] \vdash F(x_1, \ldots, x_n))$, from this we see that $\{(x_1, \ldots, x_n) : (\exists p)([p] \vdash F(x_1, \ldots, x_n))\}$ is a class of $(V, \in)$ as required.

LEMMA 9. If $[p] \vdash F$ and $[q] \geq [p]$ then $[q] \vdash F$. Moreover there exists $[g] \in G$ such that $[g] \vdash F$.

PROOF. The first assertion is a straightforward consequence of the above definition, and the second of the maximality of $G$.

LEMMA 10. Let $P \equiv (a = b)$ then
\[(39) \quad H \vdash P \iff (\exists g)([g] \in G \land [g] \vdash F)\]

PROOF. We prove Lemma 10 by induction on $\nu = \max(\text{rank}(a), \text{rank}(b))$. The conclusion of the lemma follows trivially in case $\nu = 0$. We assume that
\[(\forall u)(u < \nu \rightarrow (H \vdash F \iff (\exists g)([g] \in G \land [g] \vdash F)))\]
and show that if $\text{rank}(a)$ or $\text{rank}(b)$ is equal to $\nu$ then the conclusion of the lemma is true. We have
\[(40) \quad H \vdash F \iff (\forall c)((cS a \rightarrow (\exists d)(dS b \land \overline{d} = \overline{c})))\]
\[ (\text{cSb} \rightarrow (\exists d)(\text{dSb} \land d = c)) \]

By (35), Lemma 9 and by the induction hypothesis, since \( \max(\text{rank(c)}, \text{rank(d)}) < v \) we have

\[(41) \quad H \models F \iff (\forall c)([g] \epsilon G \land ([g^1][g_1] \epsilon G \land [g_1](c) = c \land c \epsilon a) \rightarrow ((\exists d)([g_2]([g_2] \epsilon G \land [g_2](d) = d \land d \epsilon b \land [g] \models d = c)) \lor
((\exists g_3)([g_3] \epsilon G \land [g_3](c) = c \land c \epsilon b) \rightarrow ((\exists d)([g_4]([g_4] \epsilon G \land [g_4](d) = d \land d \epsilon a \land [g] \models d = c)))
\]

iff \( (\exists g)([g] \epsilon G \land (\forall c)(([q] \epsilon G \land [q](c) = c) \rightarrow ((c \epsilon a \rightarrow (\exists d)([g_2]([g_2] \epsilon G \land [g_2](d) = d \land d \epsilon b \land [g] \models d = c)) \land (c \epsilon b \rightarrow (\exists d)([g_4]([g_4] \epsilon G \land [g_4](d) = d \land d \epsilon a \land [g] \models d = c)))) \)

iff \( (\exists g)([g] \epsilon G \land [g] \models F) \)

as required.

**Lemma 11.** Let \( F \) be any \( \text{ZFC} + A \) formula then

\[(42) \quad H \models F \iff (\exists g)([g] \epsilon G \land [g] \models F) \]

**Proof.** We prove (43) by induction on the degree of complexity of formulas.
(i) \[ F = (a \in b) \]

Based on the definition of our forcing we have.

(43) \[ H \models F \iff (\exists c)(\exists g)((g \in G \land [g](c) = c \land c \in b \land c \in a) \]

\[ \iff (\exists c)(\exists g)(([g] \in G \land [g](c) = c \land c \in b) \land (\exists r)([r] \in G \land [r] \models c = a)) \]

By Lemma 9, (43) becomes

\[ H \models F \iff (\exists g)([g] \in G \land [g] \models F) \]
as required.

(ii) \[ F \equiv \top_E \]

Similarly, we have

(44) \[ H \models F \iff \top(H \models E) \]

\[ \iff \top(\exists g)([g] \in G \land [g] \models E) \]

\[ \iff (\forall g)([g] \in G \rightarrow \top([g] \models E)) \]

\[ \iff (\exists g)([g] \in G \land [g] \models F) \]
as required.

(iii) \[ F = (\exists x)E \]

Again, clearly

(45) \[ H \models F \iff (\exists b)(H \models E(b)) \]

\[ \iff (\exists b)(\exists g)([g] \in G \land [g] \models E(b)) \]
as required.
(iv) \[ P \equiv (E \lor Q) \]

Again, we have
\[
(46) \quad H \models P \iff (H \models E) \lor (H \models Q) \\
\iff (\exists g)([g] \in G \land [g] \models P)
\]
as required.

The proof of Lemma 11 now follows from (43), (44), (45) and (46).

**Lemma 12.** For every symmetric sets \( a \) and \( b \) and every permutation \( f \), if \( [g] \in P \) then

\[ [g] \models a = b \iff [g] \models f(a) = f(b) \]

**Proof.** We show this by induction on \( r = \max(\text{rank}(a), \text{rank}(b)) \). The conclusion is obvious for \( r = 0 \). Next, we assume

\[
(47) \quad (\forall r)(r < u \rightarrow ([g] \models a = b \leftrightarrow [g] \models f(a) = f(b)))
\]
and we show that if \( r = u \) then the conclusion of the lemma holds. We have

\[
(48) \quad [g] \models a = b \iff (\forall c)(([q] \models [q] \geq [g] \land [q] \models c) \rightarrow \big( (c \in a \rightarrow (\exists d)([h] \geq [g] \land [h] \models d = d) \land (c \in b \rightarrow (\exists h)([h] \geq [g] \land [h] \models c = d)) \big) \land (c \in b \rightarrow (\exists h)([h] \geq [g] \land [h] \models c = d)))
\]
using the induction hypothesis, and the fact that \( x \vDash y \) iff \( f(x) \vDash f(y) \), we see that (48) reduces to

\[
\begin{align*}
(49) \quad [g] \vDash a = b & \iff (\forall c)[((\exists q)([q] \geq [g] \land [q](c) = c \rightarrow (f(c) \vDash f(a) \rightarrow (\exists d)(\exists h)([h] \geq [g] \land [h](d) = d \land \exists \theta)(f(d) \vDash f(b) \land [g] \vDash f(c) = f(d)) \land (f(c) \vDash f(b) \\
& \rightarrow (\exists d)(\exists h)([h] \geq [g] \land [h](d) = d \land \exists \theta)(f(d) \vDash f(a) \land [g] \vDash f(c) = f(d))))] \\
& \iff [g] \vDash f(a) = f(b)
\end{align*}
\]

as required.

**THEOREM 9.** Let \( p S V_\infty \), then the set

\[ p' = \{ x : x \vDash V_\infty \land (\exists q)([q] \vDash P \land [q] \vDash x \vDash p) \} \]

is symmetric.

**PROOF.** Let us assume that \( p \) is symmetric under \([q] \), i.e., \([q](p) = p \). Let \( h \in [q] \), it suffices to show that if \( h(a) = b \) and \( [g] \vDash t = a \) then \( [g] \vDash h(t) = b \). But this follows directly from Lemma 12.

**LEMMA 13.** Let \( M = \bigcup M_i \) be a model of \( ZF \) such that \( u \leq v \) implies \( M_u \leq M_v \). Let \( F(x) = (\exists y)G(x,y) \) then there is an ordinal \( r \) such that
\[(x \in A) \land (\exists y) G(x, y) \iff (x \in A) \land (\exists y)((y \in V^x) \land G(x, y))\]

**PROOF.** Let \(w_x\) be the first ordinal such that there is a \(y\) with \(G(x, y)\) valid in \((V, \varepsilon)\). Let us define
\[H(x, y) \equiv (y = V_{w_x})\]
Clearly, \(H(x, y)\) is a binary predicate functional in \(x\) on any set. Thus the set
\[D = \{y : x \in A \land H(x, y)\}\]
exists in \((V, \varepsilon)\). It is easy to see now that the set \(UD\), is the desired ordinal \(r\).

**THEOREM 10.** Let \(F\) be any formula all of whose constants are symmetric sets, then for every symmetric set \(B\)
\[T = \{x : x \in B \land (\exists g)([g] \in P \land [g] \models F(x))\}\]
is a symmetric set.

**PROOF.** We prove this Lemma by induction on the degree of complexity of the formulas. Assume first that \(F\) is an atomic formula, i.e., \(F \equiv a \in x\) or \(F \equiv x \in a\) where \(a\) is symmetric. Furthermore, let us assume that \(F \equiv x \in a\) and let \(r = \max\{\text{rank}(a), \text{rank}(B)\}\) then in view of Lemma 12, the set
\[a' = \{x : x \in V^r \land [q] \models F(x)\}\]
is a symmetric set. Since \(B \cap a'\) is symmetric and \(T = B \cap a'\) it follows that the conclusion of the theorem
is true in case $F \equiv (x \in a)$. Next we assume that $F \equiv (a \in x)$. Let $[\varphi]$ be such that $[\varphi](B) = B$ and $[\varphi](a) = a$. We prove that $[\varphi](T) = T$. It suffices to show that if $h \in [\varphi]$ and $x \in T$ then $[\varphi] \vdash a \in x$ iff $[\varphi] \vdash a \in h(x)$

We have

$$[\varphi] \vdash a \in x \quad \text{iff} \quad (\exists c)(\exists f)([f] \geq [\varphi] \land [f](c) = c$$

$$\land c \in x \land [\varphi] \vdash a = c)$$

$$\text{iff} \quad (\exists c)(\exists f)([f] \geq [\varphi] \land [f](c) = c$$

$$\land [\varphi] \vdash h(a) = h(c))$$

$$\text{iff} \quad [\varphi] \vdash a \in h(x)$$

as required.

Next we assume that $F \equiv \neg E$. By our induction hypothesis, the set

$$T' = \{ x : x \in B \land [\varphi] \vdash E(x) \}$$

is symmetric. Since the difference of two symmetric sets is symmetric and $T = B - T'$, $T$ is symmetric as required.

Now we assume that $F \equiv (E \lor Q)$. By our induction hypothesis, the sets

$$T_1 = \{ x : x \in B \land [\varphi] \vdash E(x) \}$$

$$T_2 = \{ x : x \in B \land [\varphi] \vdash Q(x) \}$$
are symmetric. Since the union of two symmetric sets is symmetric and $T = T_1 \cup T_2$, $T$ is symmetric as required.

Finally we assume that $F \equiv (\exists y)G(x,y)$. Let $r$ be the ordinal whose existence is assured by Lemma 13. Since $V_r$ is a symmetric set and products of symmetric sets are symmetric, we see that $B \times V_r$ is a symmetric set. By our induction hypothesis, the set

$$E = \{(x,y) : (x,y)\in B \times V_r \wedge [g] \models G(x,y)\}$$

is symmetric. Since $\text{Dom}(E)$ is a symmetric set and $T = \text{Dom}(E)$, $T$ is symmetric and the desired result follows.

It is well-known [40] that a transitive almost universal class in which the axiom scheme of Separation is valid is a model of $ZF$. Thus we will prove that $H$ as given in (38) is a transitive almost universal class in which axiom scheme of Separation is valid. The fact that $H$ is transitive follows from the observation that if $\bar{x} \in \bar{y} \in H$ then $\bar{x} \in H$ because $x$ is clearly a progenitor of $\bar{x}$. In order to prove that $H$ is almost universal we need the following result.

**Lemma 14.** Let $B \in V$ be such that every element $b$ of $B$ is an element of $H$ i.e., $B = \{\bar{a}, \bar{b}, \ldots\}$. Then there exists an element $B'$ of $H$ such that $B \subseteq B'$.

**Proof.** Let
\[ B_1 = \{ y : x \in B \land y = \{ h(x) : h \text{ is a permutation of } A \text{ keeping finitely many atoms fixed} \} \] 

Clearly since axiom scheme of Replacement is valid in \( V \), the set \( B_1 \) exists. It then follows that \( B' = \overline{\bigcup B_1} \) exists and is the desired set.

In view of Theorem 10, it follows that \( H \) is an almost universal class, closed under axiom scheme of Separation. Hence \( (H, \varepsilon) \) is a model of \( ZF^- \) as promised.

Next we show that \( AC \) fails in \( (H, \varepsilon) \) by showing that there is no well ordering of the set of all atoms. Assume on the contrary, and let \([g] \in P\) be such that \([g] \models (\exists f)(\exists r)(f \text{ is 1:1 function from an ordinal } r \text{ onto the set } A \text{ of all atoms})\).

but then in view of our definition of forcing it follows that there exists a \( q \in [g] \) such that \([q](f) = f\). This is impossible, for if we let \( q \in [g] \) be such that \( q(a_i) = a_i \) with \( i = 1, 2, \ldots, n \) and \( g \) leaves \( a_i \)'s fixed and for some \( b, q(b) \neq f(b) \) then the set \( q(f) = \{(q(x), q(y)) : x \in \text{Dom}(f) \land y \in \text{Ran}(f)\} \) is not equal to \( f \). Hence \( AC \) is not valid in \( (H, \varepsilon) \), as promised.
5. A VARIANT OF UNRAMIFIED FORCING

In this section, we introduce a variant of unramified forcing based on which in Section 7 we show the independence of the AC from ZF. It is known [12] that in ZFC it cannot be proved that there is a class C which is a (inner model) model of ZF + ¬AC.

One way to sidestep this difficulty, is to make, say, the further assumption SM, i.e., the existence of a set M which is a standard transitive model of ZF. Thus in view of Corollary 1, page 15, we assume that there is a countable set M in (L, ∈) (the Goedel's model of constructible sets) such that (M, ∈) is a model of ZFC.

Our aim is to adjoin to M a set G ∈ L, to obtain a model M[G] of ZFC such that On(M) = On(M[G]).

Let (P, ≤) be a partial order, for p ∈ P we define

\[ \{p\} = \{q : q \in P \land q \leq p\} \]

It can be easily seen that the set \( T = \{[p] : p \in P\} \) satisfies all the requirements of a base for a topology on P.

A subset G of P is called P-generic iff

i. \((\forall p)(\forall q)((p \in G \land q \in G) \rightarrow (\exists r)(r \in O \land r \in G \land r \leq p \land r \leq q))\)
As expected elements \( p \) and \( q \) of the partial order \((P, \leq)\) are called compatible iff they have a nonzero lower bound, i.e., \( c \neq 0 \) and \( c \leq p \) and \( c \leq q \). We express the fact that \( p \) and \( q \) are compatible by writing \( p \parallel q \). If \( p \) and \( q \) are not compatible, we say that they are incompatible and write \( p \perp q \).

A nonzero element \( m \) of the partial order \((P, \leq)\) is called a molecule \([A]\) of \( P \) iff \( x \leq m \) and \( y \leq m \) imply \( x \) and \( y \) are compatible, for every nonzero elements \( x \) and \( y \) of \( P \).

Let \((P, \leq, 1)\) be a partial order with the maximum element \( 1 \), such that \( P \) has no molecule. Let \( G \) be \( P \)-generic over \( M \). We define for every ordinal \( u \) of \( M \)

\[
M_{u+1} = \{ x : x \text{ is a } P \text{ valued relation whose domain is a subset of } M_u \text{ that can be found in } M \}
\]

and

\[
M_u = \bigcup_{v \in u} M_v \quad \text{for limit ordinal } u
\]

Finally let

\[
N = \bigcup_{v \in \text{On}} M_v
\]
REMARK 2. Clearly $M_u \subseteq M_v$ for every ordinal $u \leq v$. Furthermore, if $x \in \text{Dom}(f)$ and $f \in M_u$ then $x \in M_u$.

Based on the notion of the $P$-generic set $G$ we define in $(L, \varepsilon)$ the relation $R$ given by

$$a R b \iff (\exists g)(g \in G \land (a, g) \in b)$$

For $b \in N$ we define

$$\bar{b} = \{\bar{a} : a R b\}$$

We call $\bar{b}$ the value of $b$ and $b$ a name of $\bar{b}$.

Next let

$$M_u[G] = \{\bar{x} : x \in M_u\}$$

and

$$M[G] = \bigcup_{u \in \text{On}} M_u[G]$$

We prove below that $(M[G], \varepsilon)$, where $M[G]$ is given as in (55) is a standard transitive model of ZFC which includes $M$ and contains $G$ and is the smallest such model. In order to do this, we need a new language suitable for discussing $M[G]$ in $M$. This new language customarily is called a forcing language, and is an extension of ZFC with all the sets of $N$ as its constants.

Let $F(x_1, \ldots, x_n)$ be any formula of ZFC with all free variables shown. Let $M, P, M[G]$ be given as above and let $a_1, \ldots, a_n \in N$, then we say that $[p]$ forces $F$. 

and write $[\mathbf{p}] \models F(a_1, \ldots, a_n)$ iff for every set $G$
which is $P$-generic over $M$ it is the case that
$M[G] \models F(\bar{a}_1, \ldots, \bar{a}_n)$.

From the above we see that the notion '[$\mathbf{p}] \models F$' is
defined in $(L, \in)$ and not yet in $(M, \in)$. However, as
shown below, it can be decided in $(M, \in)$ whether or not
$[\mathbf{p}] \models F$.

Let $\mathbf{p} \in P$ and let $F$ be any ZFC formula. The
following five clauses define the notion of $[\mathbf{p}] \models F$.

i. $[\mathbf{p}] \models \mathbf{a} = \mathbf{b}$ iff $\mathcal{cl}_{[\mathbf{p}]}([m : m \epsilon [\mathbf{p}] \wedge (\exists c)(\exists q)
((c, q) \epsilon \mathbf{a} \wedge q \geq m \wedge [m] \models \mathbf{a} = c)) = [\mathbf{p}]$

ii. $[\mathbf{p}] \models \mathbf{a} = \mathbf{b}$ iff $(\forall r)(\forall c)(r \parallel \mathbf{p} \rightarrow \(((c, r) \epsilon \mathbf{a}
\rightarrow (\mathcal{cl}_r([t : t \epsilon [r] \wedge (\exists q)(\exists d)((d, q)
\epsilon \mathbf{b} \wedge q \geq t \wedge [t] \models c = d))) = [r])))
\wedge (((c, r) \epsilon \mathbf{a} \rightarrow (\mathcal{cl}_r([t : t \epsilon [r] \wedge
(\exists q)(\exists d)((d, q) \epsilon \mathbf{a} \wedge q \geq t \wedge [t] \models c
= d))) = [r])))

iii. $[\mathbf{p}] \models \neg F$ iff $(\forall h)(h \epsilon [\mathbf{p}] \rightarrow \neg h \models F)$

iv. $[\mathbf{p}] \models (\exists x)F(x)$ iff $\mathcal{cl}_{[\mathbf{p}]}([m : m \epsilon \mathbf{p} \wedge (\exists b)(m \models
F(b))) = [\mathbf{p}]$
v. \[ [p] \vdash (F \lor Q) \text{ iff } ([p] \vdash F) \lor ([p] \vdash Q) \]

**Lemma 15.** If \( \{(x_1, \ldots, x_n) : F(x_1, \ldots, x_n)\} \), is a class of \((M, \varepsilon)\) then so is \( \{(x_1, \ldots, x_n) : (\exists p)([p] \vdash F(x_1, \ldots, x_n))\} \).

**Proof.** We observe that in the definition of forcing \([p] \vdash a = b\) is defined in terms of \([p] \vdash c = d\) where \(\max(\text{rank}(c), \text{rank}(d)) < \max(\text{rank}(a), \text{rank}(b))\). Therefore, it follows that if \( F \) is a formula of ZFC whose constants are names, then so is the formula \( (\exists p)([p] \vdash f(x_1, \ldots, x_n))\) From this, we see that \( \{(x_1, \ldots, x_n) : (\exists p)([p] \vdash F(x_1, \ldots, x_n))\} \) is a class of \((M, \varepsilon)\) as required.

**Lemma 16.** If \([p] \vdash F\) and \(q \leq p\) then \([q] \vdash F\).

**Proof.** We prove this lemma by induction on the degree of complexity of the formulas.

i. \( F \equiv (a \in b) \)

Let

\[ A = \{ m : m \varepsilon [q] \land (\exists c)(\exists r)((c, r) \varepsilon b \land r \geq m \land [m] \vdash c = a) \} \]

It suffices to show \([q] \subseteq cl_{[q]}(A)\). Assume to the contrary and let \( r \varepsilon [q] \) be such that \([r] \cap A = \emptyset\). By hypothesis \([r] \cap \{ m : m \varepsilon [p] \land (\exists c)(\exists h)((c, h) \varepsilon b \land h \geq m \land [m] \vdash a = c)\} = B \neq \emptyset\). Let \( t \varepsilon B\), then it is easily seen that \( t \varepsilon A\).
Thus \([r] \cap A \neq \emptyset\), which contradicts our assumption. Hence \([q] \models F\) as required.

ii. \(F \equiv (a = b)\)

The result in this case follows from the observation that if \(r \parallel q\) then \(r \parallel p\) also.

iii. \(F \equiv \neg Q\)

The result in this case is a straightforward consequence of the definition of the negation.

iv. \(F \equiv (\exists x)Q(x)\)

The result in this case follows from the induction hypothesis.

v. \(F \equiv (E \lor Q)\)

Again the desired result follows from the induction hypothesis.

**Lemma 17.** Let \(F \equiv (a = b)\) then

\[(56) \quad M[G] \models F \rightarrow (\exists g)(g \in G \land [g] \models F)\]

**Proof.** We prove Lemma 17 by induction on \(v = \max (\text{rank}(a), \text{rank}(b))\). The conclusion of the lemma follows trivially in case \(v = 0\). We assume that

\[(\forall u)(U < v \rightarrow (M[G] \models F \rightarrow (\exists g)(g \in G \land [g] \models F))\]

and show that if \(\text{rank}(a)\) or \(\text{rank}(b)\) is equal to \(v\) then the conclusion of the lemma is true. We have
By the induction hypothesis (57) becomes

\[(58) \quad M[G] \models F \iff (\forall c)(((\exists g_1)(g_1 \in G \land (c, g_1) \in \alpha) \rightarrow (\exists d)(((\exists g_2)(g_2 \in G \land (d, g_2) \in \beta) \land d = c)) \land ((\exists g_3)(g_3 \in G \land (c, g_3) \in \beta) \rightarrow (\exists d)(((\exists g_4)(g_4 \in G \land (d, g_4) \in \alpha) \land d = c))))\]

Let \( g \) be a common extension of \( g_1, g_2, g_3, g_4, g_5 \) and \( g_6 \). Then (58) reduces to

\[(59) \quad M[G] \models F \iff (\exists g)(g \in G \land (\forall c)(((\exists q)(q \geq g \land (c, q) \in \alpha) \rightarrow (\exists d)(((\exists q)(q \geq g \land (d, q) \in \beta) \land [g] \models c = d))) \land ((\exists q)(q \geq g \land (c, q) \in \alpha) \rightarrow (\exists d)(((\exists q)(q \geq g \land (d, q) \in \beta) \land [g] \models c = d))))\]
Next for \( r \leq g \) let
\[
A = \{ t : (c, r) \in A \land t \in [r] \land (\exists d)(\exists q)((d, q) \in A \land q \geq t \land [t] \models c = d) \}
\]
and
\[
B = \{ t : (c, r) \in B \land t \in [r] \land (\exists d)(\exists q)((d, q) \in A \land q \geq t \land [t] \models c = d) \}
\]
We show that \( \text{cl}_{[r]}(A) = \text{cl}_{[r]}(B) = [r] \). Clearly \( \text{cl}_{[r]}(A) \subseteq [r] \), next let \( h \in [r] \), in view of (59) we see that [h] \( \cap A \neq \emptyset \), verifying that \( \text{cl}_{[r]}(A) = [r] \). Similarly we can show that \( \text{cl}_{[r]}(B) = [r] \). In view of the above we see that \( M[G] \models F \rightarrow (\exists g)(g \in G \land [g] \models F) \) as required.

**Lemma 18.** Let \( G \) be a \( P \)-generic set over \( M \). let \( r \in G \) and let \( A \subseteq [r] \) be such that \( \text{cl}_{[r]}(A) = [r] \). Then \( G \cap A \neq \emptyset \).

**Proof.** Let
\[
B = \{ p : p \in P \land (\forall t)(t \in A \rightarrow p \models t) \}
\]
First we show that the set \( D = A \cup B \) is dense in \( P \) i.e., \( \text{cl}(D) = P \). Let \( q \notin D \), clearly there is an \( h \) such that \( h \in A \) and \( h \models q \). Let \( k \) be a lower bound of both \( h \) and \( q \) but then since \( \text{cl}_{[r]}(A) = [r] \), it follows that \( [k] \cap A \neq \emptyset \). Thus \( [q] \cap A \neq \emptyset \), it then follows that every basic open set around every element of \( D^c \) intersects \( D \). Thus \( D \)
is dense as required.

Next we assume that \( G \cap A \neq \emptyset \). Since \( \text{cl}(D) = F \) it follows that \( G \cap D \neq \emptyset \). Since \( r \in G \) we see that \( G \cap B \neq \emptyset \). Hence \( G \cap A \neq \emptyset \) as required.

**LEMMA 19.** Let \( F \) be as in Lemma 17, then

\[
(60) \quad (\exists g)(g \in G \land [g] \vdash F) \implies M[G] \models F
\]

**PROOF.** Again we prove this lemma by induction on \( v = \max(\text{rank}(a), \text{rank}(b)) \). The conclusion of the lemma follows trivially for \( v = 0 \). Next we assume that

\[
(\forall u)(u < v \implies ((\exists g)(g \in G \land [g] \vdash F) \implies M[G] \models F)
\]

and show that if \( \text{rank}(a) \) or \( \text{rank}(b) \) is equal to \( v \) then the conclusion of the lemma is true. Let us assume that \( (c, r) \in a \) with \( r \in G \). Let

\[
A = \{ t : t \in [r] \land (\exists q)((d, q) \in b \land q > t \land [t] \vdash c = d) \}
\]

By Lemma 18 and ii, of the definition of forcing we see that \( G \cap A \neq \emptyset \). Let \( h \in G \cap A \) then since every \( q \) which is greater than \( h \) is an element of \( G \), by our induction hypothesis it now follows that

\[
(c, r) \in a \implies (\exists d)(q \in G \land (d, q) \in b \land \bar{a} = \bar{c})
\]

Similarly one can show that

\[
(c, r) \in b \implies (\exists q)(q \in G \land (d, q) \in a \land \bar{a} = \bar{c})
\]
Thus

\((\forall c)((cRa \rightarrow (\exists d)(dRb \land d = 0)) \land (cRb \rightarrow (\exists d)(dRa
\land d = 0)))\)

and the desired result follows.

**REMARK 3.** The following are equivalent

(i) \((\forall g)(g \in G \rightarrow \neg([g] \models F))\)

(61)

(ii) \((\exists g)(g \in G \land [g] \models \neg F)\)

To see that (ii) \(\rightarrow\) (i), let (ii) be true and let (i) be false, i.e., let \((\exists q)(q \in G \land [q] \models F)\). Let \(r\) be a lower bound of \(g\) and \(q\), then in view of Lemma 16 this is a contradiction.

To see that (i) \(\rightarrow\) (ii), let \(A = \{h : h \in P \land [h] \models F\}\) and \(B = \{t : t \in P \land [t] \models \neg F\}\).

Clearly \(A \cup B\) is a dense subset of \(P\). Indeed let \(q \notin (A \cup B)\) then \(\neg([q] \models \neg F)\) or in view of the definition of the negation of forcing \(\iff (\exists h)(h \in [q] \land [h] \models F)\).

Thus \(h \in A\) and \([q] \cap (A \cup B) \neq \emptyset\). Hence \(A \cup B\) is dense.

**LEMMA 20.** Let \(F\) be any formula of the forcing language, then

(62) \(M[G] \models F \iff (\exists g)(g \in G \land [g] \models F)\)
PROOF. We prove (62) by induction on the degree of complexity of the formulas.

(1) \[ F \equiv (a \& b) \]

Based on (55) and Lemmas 18 and 19 we have:

\[ (63) \quad M[G] \models F \iff (\exists o)(\exists g)((g \in G \land (o, g) \in b \land \overline{c} = \overline{a}) \iff (\exists c)(\exists e)((g \in G \land (c, g) \in b) \land (\exists r) \quad (r \in G \land [r] \vdash c = a)) \]

By Lemma 16, we see that (63) becomes

\[ M[G] \models F \iff (\exists g)(g \in G \land [g] \vdash F) \]

and (1) for (62) has been established.

(11) \[ F \equiv \neg E \]

Again, by the definition of \( M[G] \), and the induction on the degree of complexity of formulas, and Remark 3, we have

\[ (64) \quad M[G] \models F \iff \neg (M[G] \models E) \]

\[ \iff \neg (\exists g)(g \in G \land [g] \vdash E) \]

\[ \iff (\forall g)(g \in G \implies \neg ([g] \vdash E)) \]

\[ \iff (\exists g)(g \in G \land [g] \vdash \neg \neg E) \]

\[ \iff (\exists g)(g \in G \land [g] \vdash F) \]

Thus (62) is established for (11).

(iii) \[ F \equiv (E \lor Q) \]
Again by the definition of forcing we have.

(65) \( M[G] \models F \iff (M[G] \models E) \lor (M[G] \models Q) \)

iff \( (\exists q)(q \in G \land [q] \models E) \lor (\exists g)(g \in G \land [g] \models Q) \)

iff \( (\exists g)(g \in G \land [g] \models F) \)

as required.

(iv) \( F \equiv (\exists x)E(x) \)

Again we have

(66) \( M[G] \models F \iff (\exists b)(M[G] \models E(b)) \)

iff \( (\exists b)(\exists g)(g \in G \land [g] \models E(b)) \)

which in view of (61) is equivalent to \( (\exists g)(g \in G \land [g] \models F) \)

as required. The proof of Lemma 20 now follows from (63), (64), (65) and (66).

**Lemma 21.** \( M[G] \) is a countable set of \( (L, \varepsilon) \)

**Proof.** Since \( M \) is countable in \( (L, \varepsilon) \), it follows that \( N \) as given in (52) is a countable set of \( (L, \varepsilon) \).

Next let \( f : N \rightarrow M[G] \) be defined as follows. For every \( a \in N \) we let \( f(a) = \bar{a} \). Clearly \( f \) is an onto function, showing that \( M[G] \) is a countable set of \( (L, \varepsilon) \) as required.
LEMMA 22. For every set $G$ which is $P$-generic over $M$ it is the case that $M \subseteq M[G]$.

PROOF. Let $a \in M$ and let us define by recursion $\bar{a}$, as follows. $\bar{a} = \{(x, 1) : x \in a\}$. Clearly $a \in N$ and $\bar{a} = a$, showing that $a \in M[G]$ as required.

LEMMA 23. Let $M[G]$ be as given in (55), then $M[G]$ is a transitive set of $(L, \varepsilon)$.

PROOF. Let $\bar{a} \in M[G]$ and $x \in \bar{a}$. Since $\bar{a} = \{y : yRa\}$ it follows that $x = y$ for some $y \in a$, showing that $y$ is a name for the value $x$. Thus $x \in M[G]$ as required.

LEMMA 24. For every set $G$ which is $P$-generic over $M$ it is the case that $G \in M[G]$.

PROOF. Let $Q = \{(p, p) : p \in P\}$, where $P$ is defined as in Lemma 22. It follows easily from (53) that $Q = G$. Thus $G \in M[G]$ as required.

LEMMA 25. Let $M[G]$ be given as in (55). Let $(K, \varepsilon)$ be any standard transitive model with $G \in K$ and $M \subseteq K$. Then $M[G] \subseteq K$.

PROOF. Since $G \in K$, it follows that (53) can be defined in $(K, \varepsilon)$. From this it follows that if $a \in N$, 

then $a \in K$ and $\bar{a} \in K$, showing that $M[G] \subset K$ as required.

Next we show that $M[G]$ is a model of $ZF$. The fact that the axiom of Extensionality is valid in $M[G]$, follows from Lemma 23 (every transitive class is a model of Extensionality). The axiom of Infinity is valid in $M[G]$, because it is valid in $(M, \in)$. The axiom of Regularity is valid in $M[G]$, because it is valid in $(L, \in)$. Next we proceed to show that the remaining axioms of $ZF$ are valid in $M[G]$.

**THEOREM 11.** Let $M[G]$ be as given in (55), then the axiom of Sumset is valid in $M[G]$.

**PROOF.** Let $\bar{A} \in M[G]$. By Lemma 15, the set

$$B = \{(x, p) : (x, p) \in \text{Dom}(\text{UDom}(A)) \land p \land [p] \vdash (\exists y)(y \in A \land x \in y)\}$$

exists. Since

$$\bar{t} \in B \iff (\exists g)(g \in G \land (t, g) \in B)$$

$$\iff (\exists g)(g \in G \land \lnot (\exists y)(y \in A \land t \in y))$$

$$\iff M[G] \models (\exists y)(\exists \bar{a} \land \bar{t} \in \bar{y})$$

$$\iff \bar{t} \in U \bar{A}$$

showing that $B$ is the Sumset of $\bar{A}$ in $M[G]$.
THEOREM 12. Let $M[G]$ be as given in (55), then the axiom of Powerset is valid in $M[G]$.

PROOF. Let $\bar{A} \in M[G]$. By (52) $A \in M_u$ for some ordinal $u$. By Lemma 15 again, the set

$$B = \{(x, p) : (x, p) \in (M_{u+1} \times P) \land [p] \models x \leq A\}$$

exists. We show that if $\bar{C} \subseteq \bar{A}$ then there is a $\bar{D} \in \bar{B}$ such that $\bar{C} = \bar{D}$. Let

$$D = \{(x, p) : (x, p) \in (M_u \times P) \land [p] \models x \in C\}$$

It can be easily seen that $D \in M_{u+1}$. Furthermore, $\bar{D} \subseteq \bar{A}$. Next we show that $\bar{D} = \bar{C}$. Clearly $\bar{D} \subseteq \bar{C}$, on the other hand if $\bar{t} \in \bar{C}$ then $\bar{t} \in \bar{A}$. Hence there is a $y \in M_u$ with $\bar{y} = \bar{t}$, but then $\bar{y} \in \bar{D}$ thus $\bar{C} \subseteq \bar{D}$. Hence $\bar{D} = \bar{C}$ and $\bar{B}$ is the powerset of $\bar{A}$ in $M[G]$, showing that the Powerset axiom is valid in $M[G]$.

THEOREM 13. Let $M[G]$ be as given in (55), then the axiom scheme of Replacement is valid in $M[G]$.

PROOF. Let $F(x, y)$ be a binary predicate functional in $x$ on a set $\bar{A} \in M[G]$. Let

$$H(x, y) \equiv (x \in A \land (\exists y)(y \text{ is of minimum rank } \land (\exists p)([p] \models F(x, y))))$$

Clearly $H$ is a binary predicate functional in $x$ on the set $A$ in $(M, \varepsilon)$. Since the axiom scheme of Replace-
ment is valid in \((M, \varepsilon)\), the set

\[ D = \{y : x \varepsilon A \land H(x, y)\} \]

exists. Let \(u = \sup\{r : y \varepsilon D \land r = \text{rank}(y)\} \). Let

\[ B = \{(y, p) : (y, p) \varepsilon (M_u \times P) \land [p] \models \chi \varepsilon A \land F(x, y)\} \]

Clearly the set \(B\) exists. Furthermore,

\[ t \in \bar{B} \iff (\exists g)(g \in G \land (t, g) \in B) \]

\[ \iff (\exists g)(g \in G \land [g] \models (x \varepsilon A \land F(x, t))) \]

\[ \iff M[G] \models (\bar{x} \in A \land F(\bar{x}, t)) \]

showing that \(\bar{B}\) is the set of the mates of elements of \(A\) under \(F(x, y)\). Hence the axiom scheme of Replacement is valid in \(M[G]\).

In view of the Theorems 11, 12, and 13, it follows that \(M[G]\) is a model of \(ZF\) as promised. Next we show that AC is valid in \(M[G]\).

THEOREM 14. Let \(M[G]\) be as given in (55), then AC is valid in \(M[G]\).

PROOF. Let \(\bar{A} \in M[G]\) and let

\[ A = \{(a, p_1), (a, g_2), (b, g_1), (c, p_2), \ldots,\} \]

Let

\[ \text{Dom}(A) = \{a, b, c, d, \ldots,\} \]
Since AC is valid in \((M, \in)\), there is an ordinal \(v \in M\) and a \(1:1\) function \(f \in M\) from \(v\) onto \(\text{Dom}(A)\).

Let

\[ f = \{(0, a), (b, 1), (c, 2), (d, 3), \ldots\} \]

Next for every \(r \in \text{Dom}(A)\) let

\[ S_r = \{p : p \in P \wedge (r, p) \in A\} \]

Based on \(f\) and \(S_r\) let

\[ g = \{\{(0, 1), ((a, 1), (0, 1)), (0, 1)) \times S_a, \ldots\} \]

Let \(f' = Ug\). Clearly \(h = f'\) is a function in \(M[G]\) from the subset \(\text{Dom}(f')\) of \(v\) onto \(A\). Since \(\text{Dom}(f')\) is isomorphic to an ordinal \(w\), under, say, the isomorphism \(k\), it follows that the function \(f \circ k\) induces a well ordering on \(A\). Thus the axiom of Choice is valid in \(M[G]\).

Finally we show that both models \((M, \in)\) and \((M[G], \in)\) have the same ordinals.

**THEOREM 15.** Let \(M[G]\) be as given in (55), then 
\((M[G], \in)\) has the same ordinals as \((M, \in)\).

**PROOF.** It suffices to show that if \(v\) is an ordinal of \(M[G]\), then \(v\) is an ordinal of \(M\). Assume that \(v \in M[G]\) and let \(n\) be a name for \(v\). Clearly \(\text{rank}(n) \geq \text{rank}(v)\), and since \(n \in M\) it follows that \(v \in M\), as required.
REMARK 4. From Lemmas 21, 22, 23, 24, 25 and Theorems 11, 12, 13, 14, 15 it follows that $M[G]$ is the smallest transitive extension of $M$, such that $M[G]$ is a model of ZFC having $G$ as a set, and having the same ordinals as $M$ does.
6. FORCING VIA COMPATIBILITY

We refer to the notions and notations established in Section 5. For \(a, b \in \mathbb{N}\) it seems natural to define

\[
p \models a \preceq b \iff (\forall h)(h < p \rightarrow (\exists c)(\exists q)(q \models h \land (c, q) \in b \\
\land p \models a = c))
\]

where \(p \models a = c\) is defined appropriately.

However, the following indicates that, it may be the case that \(p \models \top(a \preceq b)\) but \(\top(p \models a \preceq b)\).

Let \(P = \{(p, p), (t, p), (t, t), (h, p), (h, h)\}\) and

\(b = \{(c, t), (a, h)\}\)

\(a = \{(2, p), (2, h)\}\)

\(c = \{(2, t)\}\)

Clearly, \(p \models \top(a \preceq b)\) but \(\top(p \models a \preceq b)\).

On the other hand, if we define the notion of forcing by the following clauses, it will turn out that \(p \models a \preceq b\)

\[
\text{iff } p \models \top(a \preceq b) \text{ and the Truth Lemma 20, on page 45 remains valid.}
\]

(67) \(p \models a \preceq b\) iff \((\forall h)(h < p \rightarrow (\exists c)(\exists m < h)(\exists q \geq m)\\
[((c, q) \in b \land m \models a = c))
\)
(68) \( p \models a = b \iff (\forall h)(h < p \rightarrow (\exists t)(t < h \land (\forall m)(m < t \rightarrow \exists \alpha)(m \models (c\alpha A b) \lor (c\beta A a)))) \)

(69) \( p \models \neg F \iff (\forall h)(h < p \rightarrow \neg (h \models F)) \)

(70) \( p \models (\exists x)F(x) \iff (\forall h)(h < p \rightarrow (\exists m)(m < h \land (\exists b)(m \models F(b)))) \)

and finally for any formulas \( E \) and \( Q \) we define

(71) \( p \models (E \lor Q) \iff (p \models E) \lor (p \models Q) \)

As in Section 5, the seeming circularities in (67) and (68) are removed via (69) by the reduction of the ranks of \( a \) and \( b \) in the definition of \( p \models a = b \).

The motivation in (68), as expected, is the fact that in \( ZF \) we have

\[ a = b \iff (\forall x)(x \models a \iff x \models b) \]

\[ \iff \neg (\exists x)((x \models a \land x \not\models b) \lor (x \models b \land x \not\models a)) \]

which yields (68) in view of (69) and (70).

As shown below (68) can be stated equivalently in a simpler form. However, first we prove

**Lemma 26. (Definability).** If \( ((x_1, \ldots, x_n) : F(x_1, \ldots, x_n)) \) is a class then so is \( ((x_1, \ldots, x_n) : (\exists p)(p \models \ldots) \).
\[ F(x_1, \ldots, x_n) \].

PROOF. The above clearly follows from the fact that in view of (67) to (71), forcing is defined for every formula of our forcing language.

LEMMA 27. (Extension)

(72) \(^{(p \models F \land q \leq p) \rightarrow q \models F}\)

PROOF. Since in (68), (69) and (70) all the definitions start with \((\forall h)(h \leq p \ldots )\). The conclusion of the Lemma follows trivially.

LEMMA 28. (68) is equivalent to

(73) \( p \models a = b \iff (\forall h)(h \leq p \rightarrow \forall (\exists c)(h \models ((c \epsilon a \land c \epsilon b) \lor (c \epsilon a \land c \epsilon b)))) \)

PROOF. First we prove that (68) \(\rightarrow\) (73). Assume on the contrary and let (68) be true and (73) be false. Then there exists \(h \leq p\) such that \((\exists c)(h \models P(c))\) where \(P(c)\) stands for the scope of \((\exists c)(h \models \ldots )\). But then by Lemma 27 this contradicts (68).

Next we show that (73) \(\rightarrow\) (68). Assume on the contrary and let (73) be true and (68) be false. Then

\((\exists h)(h \leq p \land (\forall t)(t < h \rightarrow \exists m)(m \leq t \land (\exists c)(m \models P(c))))\)
which contradicts (73). Thus Lemma 26 is proved.

**THEOREM 16.** With forcing defined by (67) to (71), for every formula \( F \) of the forcing language we have:

\[(74) \quad p \forces F \iff p \forces \top \top F\]

**PROOF.** We consider several cases

(i) \( F \equiv (a = b) \)

From (69) we have

\[p \forces \top \top (a = b) \iff (\forall h) (h \leq p \rightarrow (\exists m) (m \leq h \land m \forces (a = b)))\]

We first show that \( p \forces \top \top (a = b) \rightarrow p \forces a = b \).

Let us assume on the contrary that there exists \( h \leq p \) and there exists \( c \) such that \( h \forces ((ca \land cb) \lor (ca \land ca)) \). However, by the hypothesis and Lemma (27) no such \( h \) exists. Hence \( p \forces a = c \).

Next we show that \( p \forces a = b \rightarrow p \forces \top \top (a = b) \). Let us assume on the contrary that there exists \( h \leq p \) such that for all \( m \leq h \) it is the case that \( \top (m \forces a = b) \). But this in view of the hypothesis and Lemma 27 is a contradiction. Thus (74) is established for (i).

(ii) \( F \equiv a \varepsilon b \)

Again, from (69) we have
We first show that $p \models \forall (a \in b) \iff (\forall h)(h < p \rightarrow (\exists m)(m < h \land m \models a \in b))$.

Next we show that $p \models a \in b \rightarrow p \models \forall (a \in b)$. Let $h < p$ then by hypothesis $(\exists m)(m < h \land (\forall t)(t < m \rightarrow (\exists c)(\exists q)(c, q) \in b \land n \models a = c))$. From this by (67) it follows that $p \models a \in b$.

We first show that $p \models \forall (a \in b) \rightarrow p \models a \in b$. Let $h < p$ then by hypothesis $(\exists m)(m < h \land (\forall t)(t < m \rightarrow (\exists c)(\exists q)(c, q) \in b \land n \models a = c))$. From this by (67) it follows that $p \models a \in b$.

Next we show that $p \models a \in b \rightarrow p \models \forall (a \in b)$. Let us assume on the contrary that there exists $h < p$ such that for all $m < h$ it is the case that $\forall (m \models a \in b)$. But this in view of the hypothesis and Lemma 27 is a contradiction. Thus (74) is established for (ii).

(iii) $F \equiv \forall Q$

Again, from (69) we have

$p \models \forall F \iff (\forall h)(h < p \rightarrow \forall (h \models \forall Q))$.

Now invoking proof by the induction on the degree of complexity of formulas, the above reduces to

$p \models \forall F \iff (\forall h)(h < p \rightarrow \forall (h \models Q))$.

Thus (74) is established for (iii).

(iv) $F \equiv (\exists x)Q(x)$

Again, from (69) we have.

$p \models \forall F \iff (\forall h)(h < p \rightarrow (\exists m)(m < h \land m \models F))$.
We first show that \( p \vdash \forall F \rightarrow p \vdash F \). Let us assume on the contrary that there exists \( h \leq p \) such that for all \( m \leq h \), it is the case that \( \exists (\exists b) (m \vdash F(b)) \). However, by hypothesis, we have \( (\exists m) (m < h \land (\exists t)(t < m \rightarrow (\exists n)(n < t \land (\exists b)(n \vdash F(b)))) \). Thus \( n < h \) and \( n \vdash F(b) \) contradicting our assumption.

Hence \( p \vdash \forall F \rightarrow p \vdash F \).

Next we show that \( p \vdash F \rightarrow p \vdash \forall F \). Let us assume on the contrary that there exists \( h \leq p \) such that for all \( m \leq h \) it is the case that \( \forall (m \vdash F) \). But this in view of the hypothesis and Lemma 27, is a contradiction.

Thus (74) is established for (iv).

Finally for any formulas \( E \) and \( Q \) let

\[(v) \quad F \equiv (E \lor Q)\]

From (71), we have,

\[p \vdash \forall F \text{ iff } p \vdash (\forall E \lor \forall Q)\]
\[\text{iff } p \vdash \forall E \lor p \vdash \forall Q\]
\[\text{iff } p \vdash E \lor p \vdash Q\]
\[\text{iff } p \vdash (E \lor Q)\]
\[\text{iff } p \vdash F\]

From the above we see that for any formula \( F \), it is the case that \( p \vdash F \) iff \( p \vdash \forall F \) and Theorem 16 is proved.
Let $F(x_1, \ldots, x_n)$ be a formula of the set-theoretical language $L$, with all its free variables shown. Let $a_1, \ldots, a_n \in \mathbb{N}$. It is expected that $M[G] \models F(a_1, \ldots, a_n)$ if and only if there exists $g \in G$ such that $g \vdash F(a_1, \ldots, a_n)$.

The above is indeed the case as shown by Theorem 17 below (usually referred to as the 'Truth Lemma').

First however, we make some preliminary remarks.

REMARK 5. Let $F$ be any formula of $L$. Then by (67) to (71) it follows that no $p$ forces $F$ and $\neg F$. On the other hand, there exists $q \leq p$ such that $q \vdash F$ or $q \vdash \neg F$. Moreover, given a formula $F$ there exists $g \in G$ such that $g \vdash F$ or $g \vdash \neg F$.

THEOREM 17. Let $F$ be a formula of $L$. Then

$$M[G] \models F \iff (\exists g)(g \in G \land g \vdash F)$$

PROOF. We prove this theorem by induction on the degree of complexity of the formulas:

(i) $F \equiv (a = b)$

We first show $M[G] \models F \iff (\exists g)(g \in G \land g \vdash a = b)$. We prove this result by induction on $\nu = \max(\text{rank}(a), \text{rank}(b))$. The conclusion follows trivially for $\nu = 0$. We assume that

$$(\forall u)(u < \nu \rightarrow (M[G] \models F \iff (\exists g)(g \in G \land g \vdash F))$$
and show that if rank(a) or rank(b) is equal to v then the conclusion is true. We have by (55)

(76) \[ M[G] \models P \iff \neg (\exists c)(\neg (\exists d) A \neg \exists e A \neg \exists f) \]

Clearly

(77) \[ M[G] \models \exists c \exists c \exists c \iff (\exists d)(\exists e)(\exists f)(\neg (\exists c) A (d, e, f) \models a \land \overline{c} = \overline{c}) \]

By the induction hypothesis (77) becomes

(78) \[ M[G] \models \exists c \exists c \exists c \iff (\exists d)(\exists e)(\exists f)(\neg (\exists c) A (d, e, f) \models a \land (\exists h)(\neg (\exists h) A \models a \iff \overline{c} = \overline{c}) \]

Let \( r \) be a common extension of both \( g \) and \( h \). Then (78) reduces to

(79) \[ M[G] \models \exists c \exists c \exists c \iff (\exists r)(\exists c)(\exists q)(\exists d, q) (\models a \land \overline{r} \iff \overline{c} = \overline{c}) \]

In view of (67), we see that (79) becomes

(80) \[ M[G] \models \exists c \exists c \exists c \iff (\exists g)(\neg (\exists c) A \models a \iff \overline{c} = \overline{c}) \]

Clearly in view of (80) we see that (76) becomes

(81) \[ M[G] \models P \iff \neg (\exists c)(\neg (\exists d) A \neg \exists e A \neg \exists f) \land \neg (\exists c)(\neg (\exists d) A \neg \exists e A \neg \exists f) \]

\[ \land \neg (\exists c)(\neg (\exists d) A \neg \exists e A \neg \exists f) \land \neg (\exists c)(\neg (\exists d) A \neg \exists e A \neg \exists f) \]

Let \( r \in G \) be a common extension of \( g_1, g_2, g_3, g_4 \). One can readily verify that (81) yields
\[ M[G] \models F \iff \neg (\exists \alpha)(\forall x)(xG \land x \models F(c)) \]

where \( F(c) = ((\alpha \land \alpha \land \beta) \lor (\alpha \land \delta \land \beta)) \).

One can check that the above is equivalent to

\[ M[G] \models F \iff (\forall x)(xG \rightarrow \neg (\exists \alpha)(x \models F(c))) \]

Let

\[ (82) \quad (\forall x)(xG \rightarrow \neg (\exists \alpha)(x \models F(c))) \]

and

\[ (83) \quad (\exists x)(xG \land (\exists h)(h \prec x \rightarrow \neg (\exists \alpha)(h \models F(c)))) \]

To complete the proof it suffices to show that \((82) \iff (83)\). First we show that \((83) \rightarrow (82)\)
Assume on the contrary, and let \((83)\) be true but \((82)\) be false, i.e., \((\exists x)(xG \land (\exists \alpha)(x \models F(c)))\) which, letting \(h = x\) in \((83)\) yields a contradiction.

Next we show that \((82) \rightarrow (83)\). Assume on the contrary, and let \((82)\) be true but \((83)\) be false, i.e.,

\[ (84) \quad (\forall x)(xG) \rightarrow (\exists h)(h \prec x \land (\exists \alpha)(h \models F(c))) \]

Let

\[ A = \{ h : h \in P \land (\exists \alpha)(h \models F(c)) \} \]

and

\[ B = \{ t : (\forall h)(h \in A \rightarrow h \models t) \} \]

\(A \cup B\) is a dense subset of \(P\), because if \(p \in P\) and \(p \notin (A \cup B)\) then for some \(h \in A\) it is the case that \(h \models p\).
Let \( m \) be a lower bound of both \( h \) and \( p \). Clearly in view of Lemma 27, it follows that \( m \in A \), and since \( m \leq p \) we see that \( A \cup B \) is dense in \( F \).

However, from the above it also follows that \( G \cap A = \emptyset \) because of (82). Moreover, \( G \cap B = \emptyset \) because if not, for some \( t \in G \cap B \) it would follow from (84) that for some \( h \leq t \) it is the case that \((\exists c)(h \vdash F(c))\). Thus \( h \in A \) and \( t \vdash h \) and \( h \leq t \), which is a contradiction.

But since \( A \cup B \) is dense in \( F \) we see that \( G \cap A = \emptyset \) and \( G \cap B = \emptyset \), i.e., a contradiction, implying that our assumption is false and therefore \((82) \implies (83)\). Thus (75) is established for (i).

(ii) \( F \equiv (a \in b) \)

Based on (55) and (i) we have.

\[(85) \quad M[G] \models F \iff (\exists c)(\exists g)((g \in G \land (c, g) \in b) \land \overline{c} = \overline{a}) \]

\[\iff (\exists c)(\exists g)((g \in G \land (c, g) \in b) \land (\exists r)(r \in G \land r \vdash c = a)) \]

Let \( q \) be a common extension of \( g \) and \( r \), then by Lemma 27 and (67) we see that (85) becomes

\[M[G] \models F \iff (\exists q)(q \in G \land q \vdash F) \]

Thus, (75) is established for (ii).

(iii) \( F \equiv (\exists E) \)
Again, by (55) and by the induction on the degree of complexity of the formulas, we have

(86) \( M[G] \not\models F \iff \neg(M[G] \models E) \)

\[ \iff \neg(\exists g \in G \land \forall a \in A, g(a) \models \varphi) \]

\[ \iff (\forall g \in G \to \neg(g \models E)) \]

and by Remark 5, we see that (86) becomes

\( M[G] \models F \iff (\exists g \in G \land g \models F) \)

Thus, (75) is established for (iii).

(iv) \( F \equiv (E \lor Q) \)

Again, by (55) and by the induction on the degree of complexity of the formulas, we have

(87) \( M[G] \models F \iff (M[G] \models E) \lor (M[G] \models Q) \)

\[ \iff (\exists h \in G \land h \models E) \lor (\exists t \in G \land t \not\models Q) \]

and letting \( g \) be a common extension of \( h \) and \( t \), (87) becomes

\( M[G] \models F \iff (\exists g \in G \land g \models F) \)

Thus, (75) is established for (iv).

Finally we consider

(v) \( F \equiv (\exists x)E(x) \)
By (55) and by the induction on the degree of complexity of the formulas, we have

\[(88) \quad M[G] \vDash F \iff (\exists b)(M[G] \vDash E(b)) \]
\[\iff (\exists b)(\exists g)(g \in G \land g \vDash E(b)) \]
\[\iff (\exists g)(g \in G \land (\exists b)(g \vDash E(b)))\]

We show that (70) is equivalent to the above, i.e., we show

\[(89) \quad (\exists g)(g \in G \land (\exists b)(g \vDash E(b))) \iff \]
\[(90) \quad (\forall h)(h \in G \rightarrow (\exists m)((m \leq h) \land (\exists b)(m \vDash E(b))))\]

Clearly by Lemma 27 we have (89) \(\rightarrow\) (90). Let

\[A = \{h : h \in P \land (\exists b)(h \vDash E(b))\}\]

and

\[B = \{t : (\forall h)(h \in A \rightarrow h \vDash t)\}\]

Clearly \(A \cup B\) is dense in \(P\). Indeed, let \(q \notin (A \cup B)\) then \(q \vDash h\) for some \(h \in A\). Let \(r\) be a lower bound of both \(q\) and \(h\), then clearly \(r \in A\) implying that \((A \cup B)\) is dense in \(P\).

We show that (90) \(\rightarrow\) (89). From the above it follows that \(G \cap (A \cup B) \neq \emptyset\). If \(G \cap A \neq \emptyset\) then trivially (89) is true. On the other hand we observe that \(G \cap A = \emptyset\) yields a contradiction. Indeed, if \(G \cap A = \emptyset\) then \(G \cap B \neq \emptyset\). Let \(t \in (G \cap B)\) and let \(r\) be a lower bound of \(t\) and \(g\), then by (90) we see that \(\exists m \leq r\) and \((\exists b)(m \vDash E(b)))\).
(75) is established for (v).

The proof of Theorem 17 now follows from the proofs of (i) to (v).

Since (53) and (55), have the same meaning as in Section 5, it follows that Lemmas 21, 22, 23, 24, 25 remain valid, i.e., \( M[G] \) is the smallest transitive extension of \( M \).

Again, the axiom of Extensionality is valid in \( M[G] \) because \( M[G] \) is a transitive set. The axiom of Infinity is valid in \( M[G] \) because \( M \subseteq M[G] \) and \( M \) is a model of Infinity. The axiom of Regularity is valid in \( M[G] \) because it is valid in \( (L, \in) \). Next we proceed to show that the remaining axioms of ZF are valid in \( M[G] \).

**THEOREM 18.** Let \( M[G] \) be as given in (55), then the axiom of Sumset is valid in \( M[G] \).

**PROOF.** Let \( A \in M[G] \). By Lemma 26, the set

\[ B = \{ (x, p) : (x, p) \in \text{Dom}(\text{UDom}(A)) \land p \models (\exists y)(y \in A \land x \in y) \} \]

exists. The rest of the proof is similar to that of Theorem 11 on page 49.

**THEOREM 19.** Let \( M[G] \) be as given in (55), then the axiom of Powerset is valid in \( M[G] \).
PROOF. Let $A \in M[G]$. By (52), we see that $A \in M_u$ for some ordinal $u$. By Lemma 26 again, the set

$$B = \{(x, p) : (x, p) \in (M_{u+1} \times P) \land p \models x \subseteq A\}$$

exists. We show that if $\mathcal{D} \subseteq A$, then there is a $\mathcal{D} \in B$ such that $\mathcal{D} = \mathcal{D}$. Let

$$D = \{(x, p) : (x, p) \in (M_u \times P) \land p \models x \in \mathcal{D}\}$$

It can be easily seen that $D \in M_{u+1}$. The rest of the proof is similar to that of Theorem 12 on page 50.

THEOREM 20. Let $M[G]$ be as given in (55), then the axiom scheme of Replacement is valid in $M[G]$.

PROOF. Let $F(x, y)$ be a binary predicate functional in $x$ on a set $A \in M[G]$. Let

$$H(x, y) \equiv (x \in A \land (\exists y)(y \text{ is of minimum rank } \land (\exists p)(p \models F(x, y)))$$

Clearly $H$ is a binary predicate functional in $x$ on the set $A$ in $(M, \varepsilon)$. Since the axiom scheme of Replacement is valid in $(M, \varepsilon)$, the set

$$D = \{y : x \in A \land H(x, y)\}$$

exists. Let

$$u = \sup\{r : y \in D \land r = \text{rank}(y)\}$$

and let
\[ B = \{ (y, p) : (y, p) \in (M \times P) \land p \models x \in A \land F(x, y) \} \]

Clearly the set \( B \) exists. The rest of the proof is similar to that of Theorem 13 on page 50.

**THEOREM 21.** Let \( M[G] \) be as given in (55), then AC is valid in \( M[G] \).

**PROOF.** Same as that of Theorem 14 on page 51.

**THEOREM 22.** Let \( M[G] \) be as given in (55), then \( (M[G], \varepsilon) \) has the same ordinals as \( (M, \varepsilon) \).

**PROOF.** Same as that of Theorem 15 on page 52.

**REMARK 5.** From Theorems 18, 19, 20, 21, 22, it follows that \( M[G] \) is the smallest transitive extension of \( M \), such that \( M[G] \) is a model of ZFC having \( G \) as a set and having the same ordinals as \( M \) does.
In this section, we construct a submodel \( (H, \varepsilon) \) of a generic extension \( (M[G], \varepsilon) \) of Cohen's minimal model \( (M, \varepsilon) \), using hereditarily symmetric names of \( (M, \varepsilon) \). We show that \( (H, \varepsilon) \) is a model of \( ZF + \neg AC \).

In \( M \), we consider the partially ordered set \( (P, \leq) \) where \( P \) is the set of all finite functions from \( \omega \times \omega \) into \( \{0, 1\} \), and \( \leq \) is taken as usual to be functional extension.

Clearly, \( (P, \leq) \) is nonmolecular. Let \( G \) be a \( P \)-generic set over \( M \). As is well known \( UG \) is a function from \( \omega \times \omega \) onto \( \{0, 1\} \), as shown below:

\[
\begin{array}{ccccccccc}
5 & \rightarrow & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
4 & \rightarrow & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
3 & \rightarrow & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
2 & \rightarrow & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & \rightarrow & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & \rightarrow & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\]

\( \bar{x}_0 \bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \bar{x}_5 \bar{x}_6 \)
LEMMA 29. For every \( i \in \omega \) let \((UG_i^1)(n) = (UG)(i,n)\) and let \((UG_i^2)(n) = (UG)(n,i)\). Then

\[
\text{(91)} \quad (UG_i^1) \neq (UG_j^1) \text{ for } i \neq j, \ i, j \in \omega
\]

\[
\text{(92)} \quad (UG_i^2) \neq (UG_j^2) \text{ for } i \neq j, \ i, j \in \omega
\]

Moreover, \((UG_i^1) \in M\) and \((UG_i^2) \in M\) for every \( i \in \omega \).

PROOF. First we prove (91). Let

\[
D_{i,j} = \{ p : p \in P \land (\exists n)(n \in \omega \land (i, n) \in \text{Dom}(p) \land (j, n) \in \text{Dom}(p) \land p(i, n) \neq p(j, n)) \}
\]

Clearly, \( D_{i,j} \) is a dense subset of \( P \). Thus \( G \cap D_{i,j} \neq \emptyset \).

Let \( q \in G \cap D_{i,j} \), since \( UG \) is an extension of \( q \) it follows that \( (UG)(i,n) \neq (UG)(j,n) \) and consequently \( (UG_i^1) \neq (UG_j^1) \).

Similarly, we can prove (92). Next we show that \((UG_i^1) \in M\). For any function \( h \in M \) from \( \omega \) onto \( \{0, 1\} \) let

\[
D_{h,i} = \{ p : p \in P \land (\exists n)(n \in \omega \land h(n) \neq p(i, n)) \}
\]

Clearly, \( D_{h,i} \) is a dense subset of \( P \). Thus \( G \cap D_{h,i} \neq \emptyset \).

Let \( q \in G \cap D_{h,i} \), since \( UG \) is an extension of \( q \), it follows that \( (UG)(i,n) \neq h(n) \) and consequently \( UG_i \neq h \).

Thus, \( UG_i \in M \).

Similarly we can show that \( UG_i^2 \in M \).
Next, let us consider a permutation \( z \) of \( \omega \) onto \( \omega \), say given by
\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
3 & 6 & 0 & 2 & 7 & 8 & 1 & \ldots 
\end{pmatrix}
\]

We define the effect of \( z \) on an element \( p \) of \( P \) as follows. If \( p = ((0,3,1),(5,9,1),(2,2,1)) \), then \( z(p) = ((3,3,1),(8,9,1),(0,2,1)) \), i.e., \( z(p) \) is obtained from \( p \) by changing only the first coordinates of the elements of \( p \) according to the dictates of \( z \).

We define the effect of \( z \) on an element \( s \) of \( M \) in an obvious way and as follows. Let
\[
s = (5,(6,3),((2,3,1),2),((3,2,9),(5,6)))
\]
then
\[
z(a) = (5,(6,3),((0,3,1),2),((2,2,9),(5,6)))
\]

A set \( s \) of \( M \) is called symmetric iff there exists a natural number \( n \), say, 4 such that for every permutation \( z' \) which leaves the first \( n \) natural numbers, say, 0, 1, 2, 3 fixed, i.e.,
\[
z' = \begin{pmatrix}
0 & 1 & 2 & 3 & \ldots
0 & 1 & 2 & 3 & \ldots
\end{pmatrix}
\]
we have \( z'(s) = s \).

A set \( s \) of \( M \) is called hereditarily symmetric iff
every element of every element of every element ... of s is symmetric.

For every ordinal \( u \in M \) let \( H_u \) be the set of all the hereditarily symmetric sets of \( M_u \) defined by (50) and (51), i.e.,

\[
H_{u \uparrow} = \{ x : x \in M_{u+1} \text{ and } x \text{ is hereditarily symmetric} \}
\]

and

\[
H_u = \{ x : x \in M_u \text{ and } x \text{ is hereditarily symmetric} \}
\]

for \( u \) a limit ordinal of \( M \).

**Theorem 23.** Let \( H_v \) be given as in (93) or (94) then

\[
H = \bigcup_{u \in On} H_u
\]

is a model of \( ZF \).

**Proof.** In view of [J] it suffices to show that \( H \) is closed under the Eight Goedel's operations and that \((H, \in)\) is an almost universal, transitive class of \( M[G] \).

First we show that if \( A \) and \( B \) are elements of \( H \) then so is their unordered pair \( (A, B) \). Since \( A, B \) are hereditarily symmetric names, then so is the set \( \{(A, 1), (B, 1)\} \) but then \( (A, B) = ((A, 1), (B, 1)) \).
Hence $(\bar{A}, \bar{B}) \in H$ as required.

Next we show that if $\bar{A}$ and $\bar{B}$ are elements of $H$ then so is $\bar{A} \cdot \bar{B}$. First however, we show that if $A$ is hereditarily symmetric set, then so is the set

\[(96) \ A' = \{(x, h) : (x, h) \in \mathbb{N}_u \times \mathbb{P} \land \neg x \in A\}\]

where $A \in \mathbb{M}_u$ and $\mathbb{N}_u$ is the set of all hereditarily symmetric elements of $\mathbb{M}_u$. Since $A$ is symmetric, there is a natural number $n$ such that for every permutation $z$ which leaves the first $n$ natural numbers fixed we have, $z(A) = A$. It is easy to see that $h \models x \in A$ iff $z(h) \models z(x) \in z(A)$ and if $x \in \mathbb{N}_u$ then $z(x) \in \mathbb{N}_u$. From the above observation it follows that $z(A') = A'$, showing that $A'$ is a hereditarily symmetric name. Similarly, we can show that if $B \in \mathbb{N}_w$ then

\[(97) \ B' = \{(x, h) : (x, h) \in \mathbb{N}_w \times \mathbb{P} \land \neg x \in B\}\]

where $r = \max\{u, w\}$, is a hereditarily symmetric name of $M$.

Since $A'$ and $B'$ are hereditarily symmetric names, it can be easily shown that $A' - B'$ is a hereditarily symmetric name too. Clearly $\bar{A}' = \bar{A}$ and $\bar{B}' = \bar{B}$. Since $\xi \in \bar{A}$ iff $(\exists x)(x \in \mathbb{N}_r \land \xi \models x \in A)$ for some $g \in G$, it follows from (96) and (97) that $(\bar{A}' - \bar{B}') = \bar{A}' - \bar{B}' = \bar{A} - \bar{B}$ Hence $\bar{A} - \bar{B} \in H$ as required.

Next we show that if $\bar{A}$ and $\bar{B}$ are elements of $H$, then so is $\bar{A} \times \bar{B}$.
Since $A$ and $B$ are hereditarily symmetric names, the set

$$T = \{ (((a, b), (a, c)), (b, c)) : (a, b) \in A \times B \}$$

is hereditarily symmetric. Clearly, $T = A \times B$. Thus, $A \times B \in H$ as required.

Next, we show that if $\bar{A}$ and $\bar{B}$ are elements of $H$, then so is the set

$$E = \{ (x, y) : (x, y) \in (\bar{A} \times \bar{B}) \text{ and } x \in y \}$$

Let

$$C = \{ ((x, y), h) : (x, y) \in \text{Dom}(A) \times \text{Dom}(B) \}$$

Clearly $C$ is a hereditarily symmetric name with $\bar{C} = E$. Thus $E \in H$ as required.

Next, we show that if $\bar{A} \in H$, then $\text{Dom}(\bar{A}) \in H$. Let

$$A' = \{ ((x, y), h) : (x, y) \in A \times P \}$$

Clearly $A'$ is a hereditarily symmetric name with $\text{Dom}(A') = \bar{A}'$. Thus, $\text{Dom}(\bar{A}) \in H$ as required.

Finally, we show that if $\bar{A} \in H$, then so are the sets

$$A_1 = \{ (q, t, p) : (p, q, t) \in \bar{A} \}$$

$$A_2 = \{ (t, q, p) : (p, q, t) \in \bar{A} \}$$

$$A_3 = \{ (p, t, q) : (p, q, t) \in \bar{A} \}$$

Let $\bar{A} \in N_u$, we show that $A_1 \in H$. Let

$$A_1' = \{ ((q, t, p), h) : (q, t, p) \in N_u \times P \}$$

Let $\bar{A} \in N_u$, we show that $A_1 \in H$. Let

$$A_1' = \{ ((q, t, p), h) : (q, t, p) \in N_u \times P \}$$
Since $A$ is a hereditarily symmetric name, it follows that $A_1$ is a hereditarily symmetric name as well, and by the Truth Lemma it follows that $\bar{A}_1 = A_1$ showing that $A_1 \in H$ as required.

Next we show that $H$ is a transitive class of $M[G]$. Let $\bar{x} \in \bar{y} \in H$. Since $\bar{x} \in \bar{y}$ it follows that $(\exists t)(t \in y \wedge \bar{x} = t)$ showing that $\bar{x} \in H$ as required.

Finally, we show that $(H, \varepsilon)$ is almost universal. Let $\bar{B} \in M[G]$ be such that every element $b \in \bar{B}$ is an element of $H$ i.e., $\bar{B} = \{\bar{a}, \bar{b}, \bar{c}, \ldots\}$ where $a$, $b$, $c$, ... are hereditarily symmetric names. We show that there exists an element $\bar{B}'$ of $H$ such that $\bar{B} \subseteq \bar{B}'$. Let $B \in \mathcal{N}_u$ where $\mathcal{N}_u$ is the set of all the hereditarily symmetric names of $M$ of rank $u$. Clearly the set $\mathcal{N}_u' = \{(x, 1) : x \in \mathcal{N}_u\}$ is a hereditarily symmetric name of $M$ and it follows that $\bar{B} \subseteq \mathcal{N}_u'$ showing that $(H, \varepsilon)$ is almost universal as required. Hence $(H, \varepsilon)$ is a model of $ZF$, as promised.

Let

(98) \[ X_1^i = \{(i, p) : i \in \omega \wedge p \in P \wedge p(i, j) = 1\} \]

(99) \[ X_1 = \bigcup\{X_1^i : i \in \omega\} \]

(100) \[ X = \{(X_1^i, 1) : i \in \omega\} \]

Clearly $X$ is a hereditarily symmetric name of $M$.

We show that in $(H, \varepsilon)$ there is no one-to-one mapping $\bar{F}$ from $\omega$ into $\bar{x}$.
Assume on the contrary and let \( F \) be such a mapping. Since \( F \) is hereditarily symmetric, it follows that there exists a natural number \( n \) such that for every permutation \( z \) which leaves the first \( n \) natural numbers fixed, we have \( z(F) = F \). Let \( k \) be a natural number strictly greater than \( n \). By our assumption there exists \( p \in F \) such that

\[ p \upharpoonright F(i) = X_k \quad \text{for some } i \in \omega \]

Let \( m \) be a natural number strictly greater than \( n \) and different from both \( k \) and any of the first coordinates of the elements of \( p \).

Finally let us consider the permutation

\[ \hat{w} = \begin{pmatrix} \cdots & k & \cdots & m & \cdots \\ \cdots & m & \cdots & k & \cdots \end{pmatrix} \]

It can be readily shown that

\[ w(p) \upharpoonright w(F(i)) = w(X_k) \]

It can be easily shown that \( w(X_k) = X_w(k) \). Thus, in view of the above it follows that

\[ w(p) \upharpoonright F(i) = X_m \]

It is also easy to verify that the elements \( p \) and \( w(p) \) have a common extension \( q \). Thus \( q \upharpoonright F(i) = X_m \) and \( q \upharpoonright F(i) = X_k \), which is a contradiction. Thus, our assumption is false and AC fails in \((H, \varepsilon)\), as required.
8. MARTIN'S AXIOM

Martin's Axiom (MA) asserts the existence of certain filters $F$ of a poset $(P, \leq)$ which intersect a given list $\{D_i : i \in k\}$ of dense subsets $D_i$ of $(P, \leq)$.

Clearly if $\{D_i : i \in \omega\}$ is a given list of dense subsets of a poset $(P, \leq)$ then the filter generated by $\{d_0, d_1, \ldots\}$ where $d_{i+1} \leq d_i \in D_i$ for every $i \in \omega$ is the desired set $F$. However, if $\{D_i : i \in k\}$ is such that the cardinal $k > \omega$ then the existence of $F$ cannot be proved in general within ZF. In this case an additional axiom (such as Martin's Axiom) is needed which will ensure the existence of $F$ for certain posets and certain lists of their dense subsets.

A poset $(P, \leq)$ is said to satisfy c.i.c. (in the literature is known as c.c.c.) iff any set of pairwise incompatible elements of $(P, \leq)$ is countable.

MARTIN'S AXIOM: Let $(P, \leq)$ be a c.i.c. poset and $\{D_i : i \in k\}, k < 2^\omega$ be a list of dense subsets of $(P, \leq)$. Then there exists a filter $F$ such that $F$ intersects every $D_i$.

D. A. Martin and R. M. Solovay proved in [19] that the axiom system $\text{ZFC} + \text{MA} + \text{CH}$ is relatively consistent, i.e., if $\text{ZFC}$ is consistent, then it remains consistent.
if we add \( MA + \neg CH \). Their proof required a considerable extension of the technique of forcing developed in Section 5.

We show that the two conditions mentioned in the above axiom are needed for the conclusion of the axiom as the following two examples indicate.

**EXAMPLE 1.** Let \((M, \varepsilon)\) be a model for ZF such that \(2^\omega > \omega_1\). Let \(P\) be the set of all functions from finite subsets of \(\omega\) into \(\omega_1\). We make \(P\) into a partial order \((P, \preceq)\) by requiring \(p \preceq q\) if \(p\) is an extension of \(q\). Clearly the subset \(A = \{(0, 1), (0, 2), \ldots, (0, \omega^2)\}\) of \(P\) consists of pairwise incompatible elements. Thus, \((P, \preceq)\) does not satisfy the c.i.c. condition. Let \(D_i\) be the set of all elements of \(P\) whose range contains \(i, i \in \omega_1\). Clearly \(D_i\) is a dense subset of \(P\) for every \(i \in \omega_1\). Consider the list \(\{D_i : i \in \omega_1\}\). It is easily seen that no filter \(F\) of \((P, \preceq)\) can intersect every \(D_i\) for otherwise \(UF\) would be a function from a subset of \(\omega\) onto \(\omega_1\) which is a contradiction.

**EXAMPLE 2.** Let \((P, \preceq)\) be the poset of all functions from finite subsets of an infinite set \(A\) into 2 ordered by extension. \((P, \preceq)\) satisfies the c.i.c. condition. To see this, let \(C\) be an antichain of \(P\). Let us partition \(C\) into equivalence classes \(E_i, i \in \omega\) such that every two
elements of \( E_i \) have the same number of elements. Since the elements of each class are pairwise incompatible, each class has finitely many elements and consequently \( C \) is countable.

Let \( H \) be the set of all functions from \( A \) into \( 2 \). For every \( h \in H \) let

\[
D_h = \{ f : f \in P \text{ and } f \text{ differs from } h \text{ on some } a \in A \}
\]

Clearly \( D_h \) is dense in \((P, \leq)\). If there existed a filter \( F \) intersecting every member of the above mentioned list, then \( UF \) would be a function from \( A \) to \( 2 \) which is different from each \( h \in H \), which is a contradiction.

A Topological space \( X \) has the c.i.c. property iff there is no uncountable family of pairwise disjoint non-empty open sets.

Martin's Axiom can be defined topologically as the assertion that no compact Hausdorff space with the c.i.c. property is the union of \( < 2^\omega \) closed nowhere dense sets.

Let \( F, I \) be collections of subsets of \( \omega \). An \((F, I)\) - set is a subset \( C \) of \( \omega \) such that \( C \cap A \) is finite for every \( A \in F \) and \( C \cap B \) and \( C^c \cap B \) are infinite for every \( B \in I \).
A set $D \subseteq \omega$ is \textbf{F-small} if there are sets $A_1, \ldots, A_n$ in $F$ such that $D \cap (A_1 \cup \ldots \cup A_n)^c$ is finite. If we assume that a $(F, I)$ - set exists, then no set in $I$ is $F$ - small, for if $C$ is an $(F, I)$ - set, then for every $B \in I$ and every $A_1, \ldots, A_n \in F$, $C \cap B \cap (A_1 \cup \ldots \cup A_n)^c$ is finite. Since $B \cap (A_1 \cup \ldots \cup A_n)^c$ is a superset of $C \cap B \cap (A_1 \cup \ldots \cup A_n)^c$ and $C \cap B \cap (A_1 \cup \ldots \cup A_n)^c$ is infinite, it follows that no $B \in I$ is $F$ - small.

\textbf{THEOREM 24.} \cite{19}. Assume that MA is valid. Let $F, I$ be collections of subsets of $\omega$ such that $|F| < 2^\omega$ and $|I| < 2^\omega$. If no set in $I$ is $F$ - small then there exists a $(F, I)$ - set $C$.

\textbf{PROOF.} Let $(P, \leq)$ be defined as follows.

$$P = \{ f : f : S \rightarrow \{0, 1\} \text{ and } S \text{ is } F \text{- small} \text{ and } |f^{-1}(1)| < \omega \}$$

We let $f \leq g$ hold, if $f$ is an extension of $g$.

First we show that $P$ satisfies the c.i.c. condition. Clearly if $f$ and $g$ are incompatible, then $f^{-1}(1) \subseteq g^{-1}(1)$ and since there are countably many finite subsets of $\omega$, a pairwise incompatible subset of $(P, \leq)$ must be countable. Next for every $A \in F$ let

$$D_A = \{ f : f \in P \text{ and } \text{Dom}(f) \text{ contains } A \}$$
and for every \( B \in I, n \in \omega \) let

\[
D_{B, n} = \{ f : f \in P \text{ and } |B \cap f^{-1}(1)| \geq n \text{ and } |B \cap f^{-1}(0)| \geq n \}
\]

\( D_A \) is a dense subset of \((P, \leq)\). To see this let \( f \in P \). The function \( f' \) which extends \( f \) to \( \text{Dom}(f) \cup (A - \text{Dom}(f)) \) by \( f'[A - \text{Dom}(f)] = 0 \), is clearly an element of \( P \) and consequently an element of \( D_A \), showing that \( D_A \) is a dense subset of \((P, \leq)\). Next we show that \( D_{B, n} \) is a dense subset of \((P, \leq)\). Let \( f \in P \), since \( \text{Dom}(f) \) is \( F \)-small and \( B \) is not \( F \)-small, it follows that we can find \( 2n \) numbers \( i_1, \ldots, i_n, j_1, \ldots, j_n \) in \( B - \text{Dom}(f) \). We extend \( f \) by \( f' \) such that \( f'(i_k) = 0 \) \( k = 0, \ldots, n \), \( f'(j_k) = 1 \) \( k = 0, \ldots, n \). Clearly \( f' \in D_{B, n} \) showing that \( D_{B, n} \) is a dense subset of \((P, \leq)\). Since

\[
|\{D_A : A \in F\} \cup \{D_{B, n} : B \in I \text{ and } n \in \omega\}| < 2^\omega
\]

it follows by MA that there is a filter \( G \) of \((P, \leq)\) which intersects every \( D_A \) and every \( D_{B, n} \). Clearly \( g = \cup G \) is a function from \( \omega \) to \( \{0, 1\} \) which extends every member of \( F \). Let \( C = g^{-1}(1) \). If \( A \in G \), \( g \) extends some \( f \in D_A \) and consequently \( |C \cap A| < \omega \). If \( B \in I \) and \( n \in \omega \), \( g \) extends some member of \( D_{B, n} \). Hence \( C \cap B \) and \( C^c \cap B \) have at least \( n \) members. Since this holds for all \( n \in \omega \), it follows that \( C \cap B \) and \( C^c \cap B \) are
infinite subsets of \( \omega \). Thus, \( C \) is a \((F, I)\)-set as required.

**THEOREM 25.** [33]. Assume that \( MA \) is valid. Let \( X \) be a Topological space, let \( x \) be a limit point of the sequence \( \{x_n : n \in \omega\} \). Suppose that there is a nhhood base at \( x \) having cardinality \( < 2^\omega \). Then there is a subsequence of \( \{x_n : n \in \omega\} \) converging to \( x \).

**PROOF.** Let \( \{V_i : i \in J\} \) be a nhhood base at \( x \) with \( |J| < 2^\omega \). Let \( A_i = \{n : x_n \in V_i\} \) and \( F = \{A_i : i \in J\} \) and \( I = \{\omega\} \). We show that \( \omega \) is not \( F \)-small. Assume on the contrary and let \( \omega \cap (A_1 \cup \ldots \cup A_m)^C \) be a finite set for some \( A_1 \ldots A_m \in F \). Since \( x \) is a limit point of \( \{x_n : n \in \omega\} \) and \( \cap \{V_i : i \in m\} \) is a nhhood of \( x \), it follows that infinitely many \( x_n \)'s are in \( \cap \{V_i : i \in m\} \). Thus, infinitely many \( n \)'s are in \( (A_1 \cup \ldots \cup A_m)^C \) and consequently \( \omega \cap (A_1 \cup \ldots \cup A_m)^C \) is infinite, which is a contradiction. Thus, \( \omega \) is not \( F \)-small. By Theorem 24 it follows that there is an infinite set \( C \) such that \( C \cap A_i \) is finite for all \( i \in J \). Hence, the subsequence \( \{x_i : i \in C\} \) converges to \( x \) as required.

**COROLLARY 3.** [33]. Assume \( MA \) is valid. The product of \( < 2^\omega \) compact, first countable spaces is sequentially
compact.

PROOF. Let \( \{X_i : i \in I\} \) be the collection of spaces, \( X \) their product. Since \( X \) is compact, every sequence in \( X \) has a limit point. In view of Theorem 25 it is enough to show that \( X \) has a nhood base at each of its points of cardinality \( < 2^\omega \). Let \( x = \{x_i : i \in I\} \) be a point of \( X \). Since \( X_i \) is first countable, \( x_i \) has a countable nhood base for every \( i \in I \). Since a basic nhood \( V \) of \( x \) is of the form \( V = p_{a_1}^{-1}(V_{a_1}) \cap \cdots \cap p_{a_n}^{-1}(V_{a_n}) \), where \( V_{a_1}, \ldots, V_{a_n} \) are basic nhoods of \( x_{a_1}, \ldots, x_{a_n} \), it follows that there is a nhood base at \( x \in X \) of cardinality \( < 2^\omega \), as required.

COROLLARY 4. Assume \( MA + \neg CH \). The product of \( \omega_1 \) sequentially compact, first countable spaces is sequentially compact.

PROOF. Let \( \{X_i : i \in \omega_1\} \) be the collection of spaces and let \( X \) be their product. By Theorem 5.5 in [29] it follows that \( X \) is countably compact and consequently every sequence \( \{x_n : n \in \omega\} \) has a limit point. In view of Theorem 25 it is enough to show that \( X \) has a nhood base of cardinality \( < 2^\omega \) at each of its points. But this follows at once from the proof of Corollary 3. Thus, \( X \) is sequentially compact as required.
A family $U = \{0_i : i \in I\}$ of open sets of a space $X$ is a $\sigma$-base, if for every open set $0 \subseteq X$ there is an $i \in I$ such that $0_i \subseteq 0$.

**THEOREM 26.** [33]. Assume that MA is valid. Let $X$ be a space with a countable $\sigma$-base. Let $\{M_a : a \in n\}$ $n < 2^\omega$ be a family of nowhere dense sets. Then there exists a family $\{H_i : i \in \omega\}$ of nowhere dense sets with $U\{M_a : a \in n\} \subseteq U\{H_i : i \in \omega\}$.

**PROOF.** Fix a countable $\sigma$-base, $U$, for $X$ and let $G_1, G_2 \ldots$ be a sequence of elements of $U$, such that every element of $U$ appears infinitely many times. Let

$A_a = \{j : G_j \cap M_a \neq \emptyset\}$

and

$B_i = \{j : G_j \subseteq G_i\}$

Finally let $F = \{A_a : a \in n\}$ and $I = \{B_i : i \in \omega\}$. We show that no $B_i$ is $F$-small. Assume on the contrary and let $L = B_i \cap (A_1 \cup \ldots \cup A_m)^C$ be a finite set for some $A_1 \ldots A_m \in F$. Let $M' = M_1 \cup \ldots \cup M_m$. Since unions of finitely many nowhere dense sets are nowhere dense, it follows that $M'$ is a nowhere dense set of $X$. Clearly $A_1 \cup \ldots \cup A_m = \{j : G_j \cap M' \neq \emptyset\}$. If $G_i \cap M' = \emptyset$, then
i \not \in A_1 \cup \ldots \cup A_m$. Since $G_i$ appears with infinitely indices, it follows that $L$ is infinite, which is a contradiction. Next we assume that $G_i \cap M' \neq \emptyset$. Since $M'$ is nowhere dense, it follows that there is a $v \in \omega$, such that $G_v \subseteq G_i$ and $G_v \cap M' = \emptyset$. Again, since $G_v$ appears with infinitely many indices we conclude that $L$ must be infinite. Thus, no element of $I$ is $F$-small and consequently in view of Theorem 24 there is an $(F, I)$ set $C$. Based on the set $C$ we define.

$$N_i = \bigcup \{G_m : m \geq i \text{ and } m \in \omega\}$$

$$H_i = N_i^C$$

We show that $H_i$ is a (closed) nowhere dense set. Let $G_v \cap H_i \neq \emptyset$. Since $C \cap B_v$ is infinite, it follows that there is a $q \geq i$ such that $G_q \subseteq G_v$ but then clearly $G_q \subseteq N_i$. Hence $G_q \cap H_i = \emptyset$, showing that $H_i$ is nowhere dense.

Next we show that $U\{M_a : a \in \omega\} \subseteq U\{H_i : i \in \omega\}$. Let $t \in M_a$. Then since $M_a$ is a nowhere dense set, it can be easily seen that there is a $q \in \omega$ such that $G_q \cap M_a = \emptyset$. Since $C \cap A_a$ is finite and $C \cap B_q$ is infinite we can find an $h \in C$ such that $h$ is greater than any member of $C \cap A_a$ and $G_h = G_q$. It follows that $t \in G_h^C$ from which it follows at once that $t \in H_q$, in fact more
is true, namely $M_a \subseteq H_\omega^a$. Hence

$$\bigcup \{M_a : a \in \omega\} \subseteq \bigcup \{H_i : i \in \omega\}$$

and the desired result follows.

COROLLARY 4. [33]. Assume that MA is valid. Let

$$\{M_a : a \in \omega\} \subseteq 2^\omega$$

be a family of nowhere dense sets of a second countable space $X$. Then there exists a family $\{H_i : i \in \omega\}$ of nowhere dense sets of $X$ with

$$\bigcup \{M_a : a \in \omega\} \subseteq \bigcup \{H_i : i \in \omega\}.$$ 

Let $X$ be a Topological space. A set $A \subseteq X$ is

**Meager**, provided $A$ is the union of countably many nowhere dense sets.

A set $A \subseteq X$ has the property of **Baire** if there is an open set $B$ such that $A - B$ and $B - A$ are meager.

COROLLARY 5. [33]. Assume MA is valid. If $X$ is a Topological space having a countable base, then the union of $< 2^\omega$ meager sets is meager.

COROLLARY 6. Assume MA is valid. If $X$ is a Topological space having a countable $c$-base, then the union of $< 2^\omega$ meager sets is a meager set.

COROLLARY 7. [33]. Assume MA is valid. If $X$ is a Topological space having a countable base, then the union of $< 2^\omega$ sets in $X$ having the property of Baire, has the
property of Baire.

COROLLARY 7. Assume MA is valid. If $X$ is a Topological space having a countable c - base, then the union of $< 2^\omega$ sets in $X$ having the property of Baire has the property of Baire.

THEOREM 27. [26] Assume MA. If $X$ is a c.i.c. compact Hausdorff space, then $X$ cannot be written as a union of $< 2^\omega$ many nowhere dense sets.

PROOF. Assume on the contrary and let

$X = \bigcup\{N_i : i \in k, k < 2^\omega \text{ and } N_i \text{ is nowhere dense}\}$

Let

$P = \{O : O \text{ is a nonempty open subset of } X\}$

We make $P$ into a partial order by letting $0 \leq F$ iff $0 \subseteq P$. Since any two elements of $P$ are incompatible iff they are disjoint and since $X$ is c.i.c., it follows that $(P, \leq)$ satisfies the c.i.c. condition. For every $i \in k$ let

$D_i = \{F : F \in P \text{ and } F \cap N_i = \emptyset\}$

We show that $D_i$ is dense in $(P, \leq)$ for every $i \in k$. Let $0 \in P$. Since $N_i$ is nowhere dense, it follows that there exists a nonempty open set $U$ of $X$, such that $\overline{F} \subseteq U$, verifying that $D_i$ is dense in $(P, \leq)$. By MA,
there exists a filter \( G \) of \((P, \leq)\) intersecting every dense set \( D_i, i \in k \). Let \( \{F_i : i \in k\} \) be a family of open sets such that \( F_i \in G \cap D_i \). Since \( G \) is a filter and \( X \) is a compact space, it follows that \( \cap\{F_i : i \in k\} \) is non-empty. We have,

\[
\cap\{F_i : i \in k\} = \cap\{F_i : i \in k\} \cap X \\
= \cup\{\cap\{F_i : i \in k\} \cap N_j : j \in k\} \\
= \emptyset
\]

Which is a contradiction. Thus, \( X \) cannot be written as a union of \(< 2^\omega\) nowhere dense sets, as required.

**COROLLARY 8.** Assume MA is valid. If \( X \) is a c.i.c compact Hausdorff space such that \(|X| > \omega\), then \(|X| \geq 2^\omega\).

Let \( X \) be a Topological space, the character of a point \( p \in X \), is the smallest cardinal of a family of sets which is a neighborhood base at \( p \).

A regular space \( X \) is called \( c \)-complete, provided it has a family \( \{U_i : i \in k\}, k < 2^\omega \) of \( c \)-bases for the topology on \( X \), such that whenever \( G \subseteq U\{U_i : i \in k\} \) is a regular filter (i.e., if \( F_1, F_2 \in G \), then there exists \( F \in G \) such that \( F \subseteq \text{int}(F_1 \cap F_2) \)) with \(|G| < 2^\omega\) and \( G \cap U_i \neq \emptyset \) for every \( i \in k \), then \( \cap G \neq \emptyset \).

**THEOREM 28.** [16]. Assume MA is valid. No \( c \)-complete, c.i.c. Topological space \( X \) can be
written as a union of $< 2^\omega$ nowhere dense sets.

**Proof.** Assume on the contrary and let

$$X = \bigcup\{N_i : i \in k, k < 2^\omega \text{ and } N_i \text{ is nowhere dense}\}$$

Let \(\{U_j : j \in J\}\) be the family required for \(c\)-completeness and let \(U = \bigcup\{U_j : j \in J\}\). We make \(U\) into a partial order by letting \(p \leq q\) iff \(\overline{p} \subseteq q\). Clearly \((U, \leq)\) satisfies the c.i.c. condition. Let

$$D_{i, j} = \{0 : 0 \in U_j \text{ and } 0 \cap N_i = \emptyset\}$$

Clearly \(D_{i, j}\) is dense in \((U, \leq)\) for every \(i \in k\) and \(j \in J\). By MA there is a filter \(G\) in \((U, \leq)\) such that \(G \cap D_{i, j} \neq \emptyset\) for every \(i \in k\) and \(j \in J\). Clearly \(G\) is a regular filter. Next let \(G_i, j = G \cap D_{i, j}\) and let \(E^0 = \{G_i, j : i \in k, j \in J\}\). By the AC let \(f_0\) be a choice function of \(E^0\) and let

$$F_1 = \{f_0(G_i, j) : i \in k, j \in J\}$$

Clearly \(|F_1| < 2^\omega\). Let

$$F_1^1 = \{\text{finite intersections of members of } F_1\}$$

For every \(z \in F_1^1\) let

$$E_z^1 = \{0 : 0 \in G \text{ and } \overline{0} \subseteq z\}$$

and

$$E^1 = \{E_z^1 : z \in F_1^1\}$$
Let $f_1$ be a choice function of $E^1$ and let

$$F_2 = F_1 \cup F_1 \cup \{f_1(E_2^1) : z \in F_1^1\}$$

Clearly $|F_2| < 2^\omega$. We continue this process countably many times. Finally let

$$F = \cup\{F_i : i \in \omega\}$$

Clearly $F$ is a regular filter with $|F| < 2^\omega$. By c-completeness we have $\cap F \neq \emptyset$. But

$$\cap F = \cap F \cap X$$
$$= \cup (\cap F \cap N_i)$$
$$= \emptyset$$

Which is a contradiction. Thus, $X$ cannot be written as a union of $< 2^\omega$ nowhere dense sets.

COROLLARY 9. Assume MA is valid. The cardinality of a c-complete, c.i.c. Topological space $X$ with $|X| > \omega$, is $\geq 2^\omega$. 
9. MARTIN'S AND GENERALIZED MARTIN'S AXIOMS
AND E-SPACES

A $T_1$ space $X$ is called $E_1$-space iff for any two disjoint closed sets $A$, $B$ there are regularly closed disjoint sets $U$, $V$ such that $A \subseteq U$ and $B \subseteq V$.

A $T_1$ space $X$ is called $E_2$-space iff for any two disjoint closed sets $A$, $B$ with nonempty interior, there are disjoint open sets $U$, $V$ such that $A \subseteq U$ and $B \subseteq V$.

A $T_1$ space $X$ is called $E_3$-space iff for any two disjoint regularly closed sets $A$, $B$ there are disjoint open sets $U$, $V$ such that $A \subseteq U$ and $B \subseteq V$.

Let $(P, \leq)$ be a poset. A subset $Q \subseteq P$ is centered provided each finite subset of $Q$ has a common lower bound.

Let $k$ be an infinite cardinal, then a poset $(P, \leq)$ is $k$-centered, provided $P$ is the union of $k$ centered subcollections.

The Generalized Martin's Axiom $GMA$ is the statement:
Let $(P, \leq)$ be an $\omega_1$-centered poset such that every countable centered subset of $P$ has a lower bound. Then if $\{D_i : i \in k\}$, $k < 2^{\omega_1}$ is a list of dense subsets of $(P, \leq)$, then there exists a filter $F$ of $(P, \leq)$ such that $F$ intersects every $D_i$. 
THEOREM 29. A regular $E_2$ space is $T_4$.

PROOF. Let $A, B$ be any two closed disjoint subsets of $X$ and $t \in B$. Since $X$ is regular, there are disjoint open sets $O_1, O_2$ with $A \subseteq O_1$ and $t \in O_2$. Let $C = B \cup O_2$. If $x \in A$, by regularity there is an open set $O_3$, such that $x \in O_3 \subseteq O_3 \subseteq O_1$. Let $D = A \cup O_3$. Then $C, D$ are closed disjoint sets with nonempty interior and consequently there are disjoint open sets $U, V$ with $C \subseteq U$ and $D \subseteq V$. Thus, $X$ is $T_4$ as required.

THEOREM 30. [26]. Assume MA is valid. Let $F$ be a collection of subsets of $\omega$ of cardinality $< 2^\omega$. Assume that whenever, $B \subseteq F$ is finite, $|\cap B| = \omega$. Then there is an infinite set $S \subseteq \omega$ such that $S - A$ is finite for all $A \in F$.

PROOF. Let $F' = \{ \omega - A : A \in F \}$ and $I = \{ \omega \}$. Clearly $\omega$ is not $F'$-small. Thus, in view of Theorem 24 it follows that there is a $(F', I)$-set $S$. We have $S - A = (\omega - A) \cap S$, which is finite for every $A \in F$ and the desired result follows.

COROLLARY 10. [45]. Assume MA is valid. Let $X$ be a separable, countably compact, $T_1$ space. Let $U = \{ U_i : i \in I \}$, $|I| < 2^\omega$ be an open cover of $X$. Then
there is a finite subfamily \( V \subset U \) such that \( UV \) is a dense subset of \( X \).

**Proof.** Assume that for no finite subfamily \( V \) of \( U \) is it the case that \( \overline{UV} = X \). Let \( S \) be a countable dense subset of \( X \) and let

\[
F = \{ S - U_1, S - U_2, \ldots \}
\]

Clearly \( F \) satisfies the hypotheses of Theorem 30. Thus, there is an infinite set \( D \subset S \) such that \( D - (S - U_i) \) is finite for every \( i \in I \). Since \( U \) is a cover it follows that \( D \) cannot have a limit point in \( X \), which contradicts the countable compactness of \( X \). Hence, there is a finite subfamily \( V \) of \( U \) such that \( \overline{UV} = X \).

For every ordinal \( \nu \), let \( P_\nu \) be the set-theoretic statement: Given a collection of \( < 2^{\omega_\nu} \) subsets of \( \omega_\nu \) such that the intersection of any subcollection of cardinality \( \omega_\nu \) has cardinality \( \omega_\nu \), then there is a subset \( S \subset \omega_\nu \) of cardinality \( \omega_\nu \) such that for each element \( A \) of the collection it is the case that \( |S - A| < \omega_\nu \).

William Weiss has shown in [45] that the \( \text{GUA} + \text{CH} \) is equivalent to the set-theoretic statement \( P_1 \).

Let \( X \) be a topological space, \( p \in X \) is an \( A \)-point of \( S \subset X \), provided for every nhhood \( V \) of \( p \), \( |V \cap S| > |V \cap S| \).
THEOREM 31. Assume GMA + CH. Let $X$ be an $\omega_1$-cap compact space (see page 113), let $S$ be a subspace, with $|S| = \omega_1$ and $\overline{S} = X$. If $U = \{U_i : i \in I\}$, $|I| < 2^{\omega_1}$ is an open cover of $X$. Then there is a countable subfamily $V$ of $U$ with $\overline{\cup V} = X$.

PROOF. Let $S$ be a dense subset of $X$ of cardinality $\omega_1$. We assume that no countable subfamily $V$ of $U$ is dense in $X$ and arrive at a contradiction. Let

$$F = \{S - U_1, S - U_2, \ldots\}$$

Clearly the intersection of any countable subfamily of $F$ is of cardinality $\omega_1$. Thus, by $P_1$ there is an uncountable set $B \subseteq S$, such that $B \cap U_i$ is countable for every $i \in I$. Since $U$ is a cover, $B$ cannot have a cap in $X$, which is a contradiction. Thus, there is a countable subfamily $V$ of $U$, such that $\cup V$ is dense in $X$.

THEOREM 32. Assume MA + $\neg$CH. Let $X$ be a regular, $E_3$, hereditarily separable, countably compact and finally $2^\omega$-compact (see page 112). Then $X$ is a $T_4$ space.

PROOF. It suffices to show that $X$ is an $E_1$ space. Let $A, B$ be any two closed disjoint subsets of $X$. 


Clearly A, B with the relative Topology are separable, countably compact and finally $2^\omega$-compact subspaces of X.

For every point $a \in A$, let $U_a$ be an open set in X, such that $a \in U_a$ and $\overline{U_a} \cap B = \emptyset$. Similarly for every point $b \in B$ let $V_b$ be an open set in X containing $b$ and such that $\overline{V_b} \cap A = \emptyset$. Let

$$U' = \{U_a \cap A : a \in A\} \text{ and } V' = \{V_b \cap B : b \in B\}$$

Since X is finally $2^\omega$-compact space, it follows that there is a subcover $U''$ of $U'$ of cardinality $< 2^\omega$.

By Corollary 10, there is a finite subfamily $U$ of $U''$, say, $U_{a_1} \cap A, \ldots, U_{a_n} \cap A$ whose union is dense in A, i.e.,

$$\text{cl}_A(U_{a_1} \cup \ldots \cup U_{a_n}) = A.$$ From this it follows that $A \subset (\bigcup \{U_{a_i} : i \in n\})$. Similarly we can find $V_{b_1}, i \in m$ such that $B \subset (\bigcup \{V_{b_i} : i \in m\})$. Let

$$0_1 = U(V_{b_i} : i \in m) - U(U_{a_i} : i \in n)$$

$$0_2 = U(U_{a_i} : i \in n) - U(V_{b_i} : i \in m)$$

Clearly $0_1, 0_2$ are open sets in X such that $B \subset \overline{0_1}$ and $A \subset \overline{0_2}$. Thus, X is $E_1$ and consequently $T_4$.

COROLLARY 11. Assume MA+CH. Let X be a $T_3$ hereditarily separable, countably compact, finally $2^\omega$-compact space. Then X is a $E_1$-space.

A space X is called k-sporadic iff every closed
subset of $X$ has a dense subset of cardinality $< k$.

**THEOREM 33.** Assume GMA + CH. Let $X$ be a regular, $E_2, \omega_1$-cap compact, finally $2^{\omega_1}$-compact space. If every uncountable subspace of $X$ has a dense subset of cardinality $\omega_1$, then $X$ is $T_4$.

**PROOF.** It suffices to show that $X$ is a $E_1$ space. Let $A, B$ be any two disjoint closed subsets of $X$. For every $a \in A$, by regularity, let $U_a$ be an open set of $X$ such that $a \in U_a$ and $U_a \cap B = \emptyset$. Similarly for every $b \in B$ let $V_b$ be an open set of $X$ such that $b \in V_b$ and $V_b \cap A = \emptyset$. Let

$$U' = \{U_a \cap A : a \in A\} \quad \text{and} \quad V' = \{V_b \cap B : b \in B\}$$

Since $X$ is finally $2^{\omega_1}$-compact, we see that there is a subcover $U''$ of $U'$, of cardinality $< 2^{\omega_1}$. Thus, by Theorem 31 it follows that there is a countable subfamily $\{U_{a_i} : i \in \omega\}$ of $U''$ such that $\bigcup \{U_{a_i} : i \in \omega\}$ is a dense subset of $A$. Similarly, there is a countable subfamily $\{V_{b_i} : i \in \omega\}$ of $V'$ such that $\bigcup \{V_{b_i} : i \in \omega\}$ is a dense subset of $B$. Now we construct open sets $Q_n$ and $P_n$ inductively as follows.

$$Q_1 = U_{a_1} \quad \quad \quad \quad P_1 = V_{b_1} - \overline{Q_1}$$

$$Q_2 = U_{a_2} - \overline{P_1} \quad \quad \quad \quad P_2 = V_{b_2} - (Q_1 \cup Q_2)$$

........................

........................
It is easily seen that \( Q = \overline{U^n_0} \) and \( P = \overline{U^n_1} \) are disjoint regularly closed sets containing \( A \) and \( B \) respectively. Hence \( X \) is \( T_4 \) as required.

A space \( X \) has \textit{depth} \( k \) at \( x \in X \) iff \( x \) has a nhood base with the property the intersection of \(< k \) many members of the nhood base contains a member of the nhood base.

A space \( X \) has \textit{depth} \( k \) iff \( X \) has depth \( k \) at each of its points.

Let \( X \) be a topological space, let \( C(X) \) denote the collection of all open sets of \( X \). A subset \( A \subseteq X \) is called \( G^k_{d,k} \) iff there is a subfamily \( U \) of \( C(X) \) such that \(|U| \leq k \) and \( A = \bigcap U \). The complement of a \( G^k_{d,k} \) set is called \( F^k_{d,k} \). We put

\[
C^k(X) = \{A : A \subseteq X \text{ and } A \text{ is } G^k_{d,k}\}
\]

If \( X \) is a \( T_1 \) space and \( t \in X \) then we define

\[
ps(t, X) = \min\{k : \{t\} \in C^k(X)\}
\]

Let \( k \) be any infinite cardinal. A sequence of type \( k \) or a \( k \)-sequence is a function on the set \( k \). The value of a sequence \( s \) at \( v \in k \) is denoted by \( s_v \). Let \( A \) be a set. Then a \( k \)-sequence \( s \) is in \( A \) iff \( s_v \in A \) for every ordinal \( v \in k \). We say that the \( k \)-sequence \( s \) is \underline{eventually} in \( A \) iff there is an ordinal \( v \in k \) such that \( s_u \in A \) whenever \( v \in u \). We say that the \( k \)-sequence
s converges to \( x \) iff \( s \) is eventually in every nhhood of \( x \).

Let \( s \) be a k-sequence. We say that \( t \) is a k-subsequence of \( s \) iff there is a k-sequence \( n \) such that \( t = s \circ n \) and for each ordinal \( w \in k \) there is an ordinal \( u \) such that \( n^u \geq w \) whenever \( u \in v \).

We say that a k-sequence \( s \) is frequently in \( A \) iff for every \( v \in k \) there is a \( w \in k \) such that \( v \in w \) and \( s_w \in A \).

A point \( x \) is a cluster point of the k-sequence \( s \) iff \( s \) is frequently in every nhhood of \( x \).

**Theorem 34.** Assume \( \text{GMA} + \text{CH} \). Let \( X \) be a space of character \( < 2^{\omega_1} \) and of depth \( \omega_1 \) at \( x \in X \). If \( x \) is a cluster point of the \( \omega_1 \)-sequence \( s = \{ s_i : i \in \omega_1 \} \). Then there exists a \( \omega_1 \)-subsequence \( t \) of \( s \) converging to \( x \).

**Proof.** Let \( \{ U_i : i \in I \} \), \( |I| < 2^{\omega_1} \) be the required nhhood base at \( x \). Let \( A_i = s \cap U_i \), \( i \in I \) and \( F = \{ A_i : i \in I \} \). Clearly any countable intersection of members of \( F \) is uncountable. By \( P_1 \) it follows that there is an uncountable subset \( t \subset s \), \( t = \{ t_v : v \in \omega_1 \} \) such that for all \( i \in I \) \( t - A_i \) is at most countable. Thus, \( t \) converges to \( x \) as required.
THEOREM 35. Assume GMA + CH. Let $X$ be any space of character $< 2^\omega_1$ and of depth $\omega_1$ at $t$. Let $S \subseteq X$, $|S| = \omega_1$. If $t$ is a cap of $S$ not in $S^*$, then there is an $\omega_1$-sequence of points of $S$ converging to $t$.

PROOF. Let $\{U_i : i \in I\}, |I| < 2^\omega_1$ be a neighborhood base at $t$. Let $A_i = S \cap U_i, i \in I$ and $F = \{A_i : i \in I\}$. Clearly any countable intersection of members of $F$ is uncountable. Thus, by there is a set $B \subseteq S, |B| = \omega_1$, such that $B - U_i$ is countable for every $i \in I$. Let $b = \{b_i : i \in \omega_1\}$ be an enumeration of $B$. Clearly $b$ is an $\omega_1$-sequence converging to $t$, as required.

THEOREM 36. Assume GMA + CH. Let $X$ be a $T_3$ of depth $\omega_1, \omega_1$-cap compact space. Let $F, |B| = \omega_1$ be a subspace of $X$. Let $t$ be a cap of $B$ not in $B$ with $ps(t, X) < 2^\omega_1$. Then there is an $\omega_1$-sequence $s$ of points of $B$ converging to $t$.

PROOF. Let $\{U_i : i \in I\}, |I| < 2^\omega_1$ be the family of neighborhoods of $t$ such that $\cap \{U_i : i \in I\} = \{t\}$. Since $X$ is a regular, $T_3$ space, it follows that $\cap \{\overline{U}_i : i \in I\}$ is equal to $\{t\}$. Let $F = \{B \cap \overline{U}_i : i \in I\}$. Clearly any countable intersection of members of $F$ is uncountable.
Thus, by $P_1$ there is a set $S \subseteq B$, $|S| = \omega_1$ such that $S - \overline{U_i}, i \in I$ is at most countable. Let $s = \{s_i : i \in \omega_1\}$ be an enumeration of $S$. Since $t$ is the unique cap of $S$, (every element $x$ of $X - t$ has a neighborhood $V$ with $|V \cap S|$ equal $\omega$) it follows easily that $t$ is an $A$-point of $S$ and consequently, $s$ converges to $t$, as required.

THEOREMS 35 and 36 are generalizations of the following two theorems which are due to Malyhin and Sapirovski [16, p. 509].

THEOREM 37. Assume MA. Let $X$ be any topological space, $A \subseteq X$, $|A| = \omega$. Let $s$ be a limit point of $A$ not in $A$. If the character of $X$ at $s$ is $< 2^\omega$, then there exists a sequence $t$ of points of $A$ converging to $s$.

PROOF. Similar to the proof of Theorem 35.

THEOREM 38. Assume MA. Let $X$ be a regular, $T_1$, countably compact topological space. Let $A \subseteq X$, $|A| = \omega$ and let $t$ be a limit of $A$ not in $A$. Let $ps(t, X) < 2^\omega$. Then there is a sequence $s$ of points of $A$ converging to $t$.

PROOF. Similar to the proof of Theorem 37.
X is called $\mathbb{F}_k$-space iff the union of any family of cardinality $k$ of closed sets is a closed set.

THEOREM 39. Assume GMA. Let $X$ be a space and $H, K$ any two separated subsets of $X$, such that $H \cup K$ can be written as a union of $< 2^{\aleph_1}$ Lindelöf sets in $X$. Let $F_1, F_2$ and $F_3$ be families of subsets of $X$ with the following properties:

(i) Both $F_1$ and $F_2$ are families of closed subsets of $X$, closed under countable unions.

(ii) Every point $h \in H$ ($k \in K$) has a neighborhood base consisting of members of $F_1$ ($F_2$ respectively)

(iii) $F_3$ is of cardinality $\omega_1$ and separates disjoint members of $F_1$ and $F_2$.

Then $H$ and $K$ have disjoint neighborhoods in $X$.

PROOF. Let

$P = \{ (A, B) : A \in F_1, B \in F_2, A \cap B = A \cap \overline{K} = B \cap \overline{H} = \emptyset \}$

We make $P$ into a partial order by letting $(C, D) \leq (A, B)$ iff $A \subseteq C$ and $B \subseteq D$. Since $(A, B), (C, D) \in P$ are compatible iff there exists $(N, M) \in P$ such that $A \subseteq N$, $C \subseteq N$ and $D \subseteq M$, $B \subseteq M$, it follows that if $(A, B)$ and $(C, D)$ are compatible, then $(A \cup C) \cap (B \cup D) = \emptyset$. 
Next we show that \((P, \leq)\) is an \(\omega_1\)-centered poset, such that every countable subset of it which is centered, has a lower bound. Let \(\{C^i : i \in \omega_1\}\) be an enumeration of \(F_3\) and let

\[ C^i = \{(A, B) : (A, B) \in P, A \subseteq C^i \subseteq X - B\} \]

Clearly \(C^i\) is a centered subset of \(P\), in fact more is true, namely every countable subset of \(C^i\), has a common lower bound. Since \(P = \bigcup\{C^i : i \in \omega_1\}\) and every centered subset of \(P\) is contained in some \(C^i\), we see that \(P\) is an \(\omega_1\)-centered poset, such that every countable, centered subset of it has a common lower bound.

For every Lindelöf subset \(k \subseteq X (h \subseteq H)\) let

\[ D_k = \{(A, B) \in P : k \subseteq B\} \]
\[ D_h = \{(A, B) \in P : h \subseteq A\} \]

We show that \(D_k (D_h)\) is a dense subset of \((P, \leq)\). Let \((A, B) \in P\) be such that \(k \neq B\). For every \(x \in k\), let \(F_x \in F_2\) be a basic neighborhood of \(x\) such that \(F_x \cap (A \cup \overline{B}) = \emptyset\). Since \(k\) is Lindelöf and \(k \cup (F_x : x \in k)\), it follows that \(k \subseteq \bigcup (F_{x_i} : i \in \omega)\). Let \(B' = \bigcup (F_{x_i} : i \in \omega) \cup B\). Clearly \((A, B') \in D_k\) and \((A, B') \leq (A, B)\) showing that \(D_k\) is dense in \((P, \leq)\). Similarly, we can show that \(D_h\) is a dense subset of \((P, \leq)\). Since there are only \(< 2^{\omega_1}\) dense subsets, it follows by GMA that there is a
filter $G$ intersecting every $D_k$, $k \in K$ and every $D_h$, $h \in H$. Let

$$U = \text{int}(\cup \{A : (A, B) \in G \text{ for some } B\})$$
$$V = \text{int}(\cup \{B : (A, B) \in G \text{ for some } A\})$$

Clearly $U$ and $V$ are disjoint open sets containing $K$ and $H$ respectively.

It is obvious that in Theorem 39, the condition that $H \cup K$ be the union of $< 2^\omega$ Lindelof sets in $X$, is satisfied in case $H \cup K$ can be written, as a union of $< 2^\omega$ compact sets, or $|H \cup K| < 2^\omega$.

Theorem 39 is the analogue of the following theorem, which is due to I. Juhasz and W. Weiss.

THEOREM 40. [17]. Assume MA. Let $X$ be a space and $H$, $K$ any two separated subsets of $X$ such that $H \cup K$ can be written as a union of $< 2^\omega$ compact sets in $X$. Let $F_1$, $F_2$ and $F_3$ be families of subsets of $X$ with the following properties:

(i) Both $F_1$ and $F_2$ are families of closed subsets of $X$ closed under finite unions.

(ii) Every point $h \in H$ ($k \in K$) has a nhood base consisting of members of $F_1$ ($F_2$ respectively).

(iii) $F_3$ is countable and separates disjoint members of $F_1$ and $F_2$. 
Then $H$ and $K$ have disjoint neighborhoods in $X$.

**PROOF.** Let

$$P = \{(A, B) : A \in F_1, B \in F_2, A \cap B = A \cap K = B \cap H = \emptyset\}$$

We make $P$ into a partial order by letting $(C, D) \leq (A, B)$ iff $A \subseteq C$ and $B \subseteq D$. We can show, as in Theorem 39 that $X$ is c.i.c., the rest of the proof is similar to that of Theorem 39.

**COROLLARY 12.** Assume MA. Let $X$ be a space and $H, K$ any two separated subsets of $X$, such that $H \cup K$ can be written as a union of $< 2^\omega$ Lindelof subspaces of $X$ with the following properties:

(i) Both $F_1$ and $F_2$ are families of closed subsets of $X$, closed under countable unions.

(ii) Every point $h \in H$ ($k \in K$) has a neighborhood consisting of members of $F_1$ ($F_2$ respectively).

(iii) $F_3$ is countable and separates disjoint members of $F_1$ and $F_2$.

Then $H$ and $K$ have disjoint neighborhoods in $X$.

**PROOF.** Similar to that of Theorem 40.
THEOREM 41. [17]. Assume MA. Let X be a set and let T, F be topologies on X with the following properties:

(i) F is $T_2$
(ii) F $\subseteq$ T
(iii) Every $x \in X$ has a T-nhood base consisting of F-compact sets
(iv) There is a countable family $\mathcal{C}$ of sets which separates disjoint F-compact sets

If H and K are any T-separated sets in X whose union can be written as a union of $< 2^\omega$ F-compact sets, then H and K have disjoint nhoods in $(X, T)$

PROOF. Let H and K be T-separated sets which can be written as a union of $< 2^\omega$ F-compact sets; let $F_1 = F_2 = \{\text{all F-compact sets}\}$ and let $F_3 = \mathcal{C}$. Then clearly $F_1$ and $F_2$ are families of closed sets of $(X, T)$, closed under finite unions, such that every point $h \in H$ ($k \in K$) has a nhood base consisting of members of $F_1$ ($F_2$ respectively) and $F_3$ is countable. Thus, by Theorem 40, H and K have disjoint nhoods in $(X, T)$.

COROLLARY 13. Assume MA. Let T, F be topologies
on a set $X$ with the following properties:

(i) $F$ is $T_2$
(ii) $(X, F)$ is a $P_\omega$-space
(iii) $F \subseteq T$
(iv) Every $x \in X$ has a $T$-neighborhood base, consisting of $F$-closed Lindelof spaces
(v) There is a countable family $C$ which separates disjoint $F$-closed Lindelof spaces

If $H$ and $K$ are $T$-separated sets of $(X, T)$ whose union can be written as a union of $< 2^\omega$ $F$-Lindelof sets, then $H$ and $K$ have disjoint neighborhoods in $(X, T)$ as required.

**PROOF.** Let $H$ and $K$ be $T$-separated sets which can be written as a union of $< 2^\omega$ $F$-Lindelof sets and let $F_1 = F_2 = \{\text{all } F \text{-closed Lindelof sets}\}$, let $F_3 = C$, clearly Corollary 12 can be applied. Hence $H$ and $K$ have disjoint neighborhoods in $(X, T)$.

**COROLLARY 14.** Assume GMA. Let $X$ be a space, let $T, F$ be any two topologies on $X$ with the following properties:

(i) $F$ is $T_2$
(ii) \((X, F)\) is a \(P_\omega\)-space

(iii) \(F \subseteq T\)

(iv) Every \(x \in X\) has a \(T\)-nhood base, consisting of \(F\)-closed Lindelof spaces

(v) There is a family of cardinality \(\omega_1\) of subsets of \(X\), which separates disjoint \(F\)-closed Lindelof sets.

If \(H\) and \(K\) are \(T\)-separated sets of \((X, T)\) whose union can be written as a union of \(< 2^{\omega_1}\) \(F\)-Lindelof sets, then \(H\) and \(K\) have disjoint nhoods in \((X, T)\).

**PROOF.** Let \(H\) and \(K\) be \(T\)-separated sets which can be written as a union of \(< 2^{\omega_1}\) \(F\)-Lindelof sets and let \(F_1 = F_2 = \{\text{all } F\text{-closed Lindelof sets}\}\), let \(F_3\) be the separating family. Clearly Theorem 39 can be applied. Thus, \(H\) and \(K\) have disjoint nhoods in \((X, T)\).

We recall that an **outer base** of the subspace \(Y\) in a space \(X\), is a family of open sets in \(X\) such that, for every \(y \in Y\) and every nhood \(U\) of \(y\) in \(X\) there is a \(V \in B\) with \(y \in V \subset U\). The smallest cardinal of such an outer base, is the **outer weight** of \(Y\) in \(X\), denoted by \(w(Y|X)\).

**THEOREM 42.** Assume MA. (GMA). Let \(X\) be a regular \((P_\omega)\) space, \(H, K\) any two separated subsets of \(X\) with
Then \( H \) and \( K \) have disjoint nhoods in \( X \).

**THEOREM 43.** Assume MA + \( ICH \). Let \( X \) be a \( T_3 \) space of cardinality \( \omega_1 \); let \( a \) be an \( A \)-point of \( X \), such that \( X - a \) is first countable. Then \( X \) is a \( T_4 \) space.

**PROOF.** Let \( H, K \) be any two disjoint closed sets in \( X \). It suffices to show that if \( a \notin K \), then \( H \) and \( K \) have disjoint nhoods in \( X \). Since \( a \notin K \), \( |K| \leq \omega \) and \( w(K|X) \leq \omega \). Thus, Theorem 42 is applicable, yielding two disjoint nhoods of \( H \) and \( K \). Hence, \( X \) is \( T_4 \).

**THEOREM 44.** Let \( m \) be a regular cardinal \( > \omega \); let \( X \) be a space of cardinality \( m \). Then \( ps(a, X) = m \) for every \( A \)-point \( a \) of \( X \).

**PROOF.** Assume on the contrary and let \( V, |V| = k \), \( k < m \), be a family of nhoods at \( a \), with \( \cap V = \{a\} \). From this, it follows that \( |\bigcup \{U^c : U \in V\}| = m \), which is a
contradiction. Thus, ps(a, X) = m, as required.

A family \( B \) of open sets, in a space \( X \) is called a \( p - \) base (strong \( p - \) base) for \( X \), if for any two distinct points \( x, y \in X \) there is a \( V \in B \), such that \( x \in V \) and \( y \notin V \) (\( y \notin \overline{V} \) respectively).

If \( T \) and \( F \) are two topologies on a set \( X \). Then \( T \) is locally \( F - \) compact (locally \( F - \) Lindelof) iff every \( x \in X \) has a \( T - \) nhood base consisting of \( F - \) compact sets (\( F - \) Lindelof sets).

We denote by \( S(k, v) \) the class of all topological spaces \( X \) which satisfy the following property: whenever \( A \) and \( B \) are separated subsets of \( X \) such that \( A \) can be written as the union of \( \leq k \) compact sets and \( B \) can be written as the union of \( \leq v \) compact sets, then \( A \) and \( B \) have disjoint nhoods in \( X \).

We denote by \( L(k, v) \) the analogue of \( S(k, v) \), when compact is replaced by Lindelof.

THEOREM 45. Assume MA. (GMA). Let \( T, F \) be two topologies on a set \( X \) with the following properties:

(i) \( F \subseteq T \)

(ii) \( (X, T) \) is locally \( F - \) compact ((\( X, F \) is a \( P_\omega \) space and \( (X, T) \) is locally \( F - \) closed Lindelof)

(iii) \( (X, F) \) has a countable (of cardinality \( \omega_1 \)) strong
Then \((X, T) \in S(2^\omega, 2^\omega) (\langle X, T \rangle \in L(2^{\omega_1}, 2^{\omega_1}))\)

**PROOF.** Since \((X, F)\) has a strong \(p\) - base, it follows that \((X, F)\) is \(T_2\). Let \(F_1 = F_2 = \{\text{all } F\) - compact sets\} (\{all \(F\) - closed Lindelof sets\}), \(F_3 = \{\text{finite unions of finite intersections of members of the strong } p\) - base\} (\{countable intersections of countable intersections of members of the strong } p\) - base\}). Clearly \(F_1, F_2, F_3\) satisfy the hypotheses of Theorem 41 (Corollary 13) and consequently \((X, T) \in S(2^\omega, 2^\omega) (\langle X, T \rangle \in L(2^{\omega_1}, 2^{\omega_1}))\), as required.

It is known [24, pp. 212-213] that if \(X\) is the countable union of certain of its subspaces \(X_n\) and if the topology of \(X\) is coherent with the spaces \(X_n\), then if each \(X_n\) is normal, so is \(X\).

**THEOREM 46.** Assume MA. (GMA). Let \(X = \bigcup\{X_i : i \in \omega\}\) and let \(T, F\) be two topologies on \(X\) such that

(i) \(T\) is coherent with the spaces \(X_i\) and \((X, T)\) is locally \(F\) - compact ((\(X, F\) is a \(P_\omega\) - space and \((X, T)\) is locally \(F\) - closed Lindelof)

(ii) \(F \subseteq T\)

(iii) \(X_i\) is the union of \(< 2^\omega F\) - compact sets (\(< 2^{\omega_1} F\) - Lindelof sets) and is \(T\) - closed
(iv) $(X, P|X)$ has a countable strong $p$-base (of cardinality $\omega_1$ strong $p$-base).

Then $(X, T)$ is $T_4$.

**PROOF.** It follows from Theorem 45 that each $(X_1, T|X_1)$ is $T_4$ and consequently by the remark before the Theorem, $(X, T)$ is $T_4$, as required.

**COROLLARY 15.** Assume MA. Let $X, P$ and $T$ be as in Theorem 46, satisfying (i), (ii) and (iv). If $(X, F)$ is a $P_\omega$-space and $(X, T)$ is locally $F$-closed Lindelof and $X_1$ is the union of $< 2^\omega$ $F$-Lindelof spaces. Then $(X, T)$ is a $T_4$ space.

**THEOREM 47.** Assume MA. (GMA). Let $X$ be an $E_\omega$, $< 2^\omega (\leq 2^{\omega_1})$ topological space. Let $F_1$ and $F_2$ be families of subsets of $X$ with the following properties:

(i) $F_1$ is a family of closed sets in $X$, closed under finite (countable) unions.

(ii) Every point $x \in X$ has a neighborhood base consisting of members of $F_1$

(iii) $F_2$ is countable (of cardinality $\omega_1$) and separates disjoint members of $F_1$

Then $X$ is $T_4$.
PROOF. It suffices to show that $X$ is an $E_1$-space. Let $H$ and $K$ be any two closed disjoint sets and $(P, \leq)$ be as in Theorem 39. Furthermore, let $D_H$, $D_K$ be subsets of $X$ with $|D_H| < 2^\omega$ ($< 2^{\omega_1}$), $|D_K| < 2^\omega$ ($< 2^{\omega_1}$) and $\bar{D}_H = H$ and $\bar{D}_K = K$. Let
\[ D_H^d = \{(A, B) \in P : d \in D_H \cap A\} \]
\[ D_K^d = \{(A, B) \in P : d \in D_K \cap B\} \]
Again, it follows that $D_H^d$ and $D_K^d$ are dense subsets of $(P, \leq)$. Since there are $< 2^\omega$ ($< 2^{\omega_1}$) dense subsets, it follows that there is a filter $G$ intersecting every $D_H^d$, $d \in D_H$ and $D_K^d$, $d \in D_K$. Let
\[ U = \text{int}(\{A : (A, B) \in G \text{ for some } B\}) \]
\[ V = \text{int}(\{B : (A, B) \in G \text{ for some } A\}) \]
Clearly $U$ and $V$ are open sets, such that $\bar{U} \cap \bar{V} = \emptyset$ and $H \subseteq \bar{U}$ and $K \subseteq \bar{V}$. Thus, $X$ is $E_1$ as required.
10. SOME WEAKER VERSIONS OF COMPACTNESS

Let \( m \) be an infinite cardinal. A space \( X \) is \([m, m]\) - compact, provided every open cover of \( X \) of cardinality \( m \) has a subcover of cardinality \( < m \).

A space \( X \) is initially \( m \) - compact or \([\omega, m]\) - compact, provided every open cover of \( X \) of cardinality \( \leq m \) has a finite subcover.

A space \( X \) is finally \( m \) - compact, provided every open cover has a subcover of cardinality \( < m \).

A space \( X \) is \([m, n]\) - compact, provided every open cover of \( X \) of cardinality \( \leq n \) has a subcover of cardinality \( < m \).

Clearly if a space \( X \) is both, initially \( m \) - compact and finally \( m \) - compact, then \( X \) is compact and conversely if a space \( X \) is compact, then \( X \) is initially \( m \) - compact and finally \( m \) - compact.

A space \( X \) is \( m \) - chain compact, provided every \( m \) - sequence \( s \) has a convergent \( m \) - subsequence (in the sense that there is a subset \( m' \subset m \) with \( \sup(m') = m \) and \( s' = s|_{m'} \) converges).

A space \( X \) is initially \( m \) - chain compact, provided \( X \) is \( k \) - chain compact for every cardinal \( k \leq m \).

A space \( X \) is finally \( m \) - chain compact, provided
X is m - chain compact for every cardinal $k \geq m$.

A space $X$ is weakly m - chain compact, provided every m - sequence $s$ in $X$ has a cluster point.

A space $X$ initially weakly m - chain compact, provided $X$ is weakly $k$ - chain compact for every cardinal $k \leq m$.

A space $X$ is finally weakly m - chain compact, provided $X$ is weakly $k$ - chain compact for every cardinal $k \geq m$.

A space $X$ is m - limit point compact, provided every subset $S$ of $X$ of cardinality $m$ has a limit point.

A space $X$ is initially m - limit point compact, provided $X$ is $k$ - limit point compact for every cardinal $k \leq m$.

A space $X$ is limit point compact, provided every infinite subset $S$ of $X$ has a limit point.

Clearly an $\omega$ - limit point compact space $X$ is limit point compact and conversely.

A point $a \in X$ is a cap of $X$ iff for every nhood $O$ of $a$ it is the case that $|O| = |X|$.

A space $X$ is m - cap compact, provided every subset $S$ of $X$ of cardinality $m$ has a cap $a$.

A space $X$ is initially m - cap compact, provided $X$ is $k$ - cap compact for every ordinal $k \leq m$. 
A space $X$ is **finally $m$-cap compact**, provided $X$ is $k$-cap compact for every ordinal $k \geq m$.

A space $X$ is **$m$-$A$-compact**, provided every subset $S$ of $X$ of cardinality $m$ has an $A$ point $a$.

A space $X$ is **initially $m$-$A$-compact**, provided $X$ is $k$-$A$-compact for every cardinal $k \leq m$.

A space $X$ is **finally $m$-$A$-compact**, provided $X$ is $k$-$A$-compact for every cardinal $k \geq m$.

A space $X$ is **$m$-bounded**, provided every $m$-sequence $a'$ of $X$ is contained in an $m$-compact subspace of $X$.

A space $X$ is **initially $m$-bounded**, provided $X$ is $k$-bounded for every cardinal $k \leq m$.

A space $X$ is **finally $m$-bounded**, provided $X$ is $k$-bounded for every cardinal $k \geq m$.

A space $X$ is **weakly $m$-bounded**, provided every $m$-sequence $s$ of $X$ has an $m$-subsequence $t$ contained in an $m$-compact subspace $A$ of $X$.

A space $X$ is **initially weakly $m$-bounded**, provided $X$ is weakly $k$-bounded for every cardinal $k \leq m$.

A space $X$ is **finally weakly $m$-bounded**, provided $X$ is weakly $k$-bounded for every cardinal $k \geq m$.

An **$m$-tower** of a space $X$ is a pair $(F, f)$ where
F \subseteq P(X) and f : m \rightarrow P is a 1:1 onto function
such that f(i) \subseteq f(j) for i > j and f(i) is a closed
subset of X for every i \in m.

A space X is \( R_m \)-compact, provided that the
intersection of any \( m \)-tower \( B \) of X is nonempty.

A space X is initially \( R_m \)-compact, provided X
is \( B_k \)-compact for every cardinal \( k \leq m \).

A space X is finally \( R_m \)-compact, provided X is
\( B_k \)-compact for every cardinal \( k \geq m \).

It is easy to see that a space X is always \( |X|^{+} \)-compact, \( |X|^{+} \)-limit point compact, \( |X|^{+} \)-cap compact
and \( B|X|^{+} \)-compact. Consequently, eventual compactness
of the above mentioned forms of compactness is
uninteresting for cardinals \( m > |X| \). Hence in what
follows, we will confine ourselves to cardinals \( m \) not
greater than the cardinality of the spaces involved.

It is very interesting to know whether any of the
above mentioned properties are preserved under arbitrary
products, or at least under products of certain
cardinality.

DIAGRAM 1. This diagram indicates the various
implications that hold among the above mentioned notions
of compactness, they are proved in Theorem 51.
Diagram 1. Implications among weaker forms of compactness

A family of sets $U$ satisfies the m.i.p. (m-intersection property), provided the intersection of any subfamily $W \subseteq U$ of cardinality $< m$ is nonempty.
LEMMA 30. Let $W$ be a maximal family of sets satisfying the $k.i.p$. If $U \subseteq W$ and $|U| < k$ then $\cup U \in W$.

PROOF. Assume on the contrary and let $U \subseteq W$ be a subfamily of $W$ such that $|U| < k$ and $\cup U \notin W$. It follows that there exists a subfamily $V$ of $W$ of cardinality $< k$ such that $\cup U \cap \cup V = \emptyset$ or $\cup (U \cup V) = \emptyset$, which is a contradiction, for $|U \cup V| < k$ and by hypothesis $W$ satisfies the $k.i.p$. Thus, $\cup U \in W$ as required.

LEMMA 31. Let $W$ be a maximal family of sets satisfying the $k.i.p$. If $S$ is a set having a non-empty intersection with every member of $W$ then $S \in W$.

PROOF. The conclusion of the Lemma follows immediately from the observation that $W \cup \{S\}$ is again a family of sets satisfying the $k.i.p$ and the fact that $W$ is a maximal such a family.

The next theorem is characterization of final $m$-compactness of a space $X$ in terms of families of sets satisfying the $k.i.p$.

THEOREM 48. A space $X$ is finally $m$-compact iff the intersection of any family of closed subsets of $X$
satisfying the m.i.p. is nonempty.

PROOF. First we show that if $X$ is finally $m$-compact space, then the intersection of any family of closed subsets of $X$ satisfying the m.i.p. is nonempty. Assume on the contrary and let $W$ be a family of subsets of $X$ satisfying the above properties and having empty intersection. It follows that $\bigcup\{F^C : F \in W\} = X$. Since $X$ is finally $m$-compact, there is a subfamily $V \subseteq W$, $|V| < m$ with $\bigcup\{F^C : F \in V\} = X$, from which it follows that $\bigcap\{F : F \in V\} = \emptyset$ which is a contradiction. Hence $W \neq \emptyset$ as required.

Next we show that if the intersection of any family of closed subsets of $X$ satisfying the m.i.p. is nonempty, then $X$ is finally $m$-compact. Assume on the contrary and let $W$ be any cover of $X$ with no subcover of cardinality $< m$. It follows that $\{F^C : F \in W\}$ is a family of closed subsets of $X$ satisfying the m.i.p. and consequently $\bigcap\{F^C : F \in W\} \neq \emptyset$, from which it follows that $\bigcup\{F : F \in W\} \neq X$ contradicting the fact that $W$ is a cover of $X$. Thus, $X$ is finally $m$-compact as required.

THEOREM 49. A space $X$ is $[m, n]$-compact iff the intersection of any family of cardinality $\leq n$ of
closed subsets of $X$ satisfying the m.i.p. is non-empty.

**PROOF.** Similar to that of Theorem 48.

**THEOREM 50.** Closed subspaces of $m$-compact spaces are $m$-compact.

**PROOF.** Let $A$ be a closed subspace of the $m$-compact space $X$ and let $F = \{F_i : i \in m\}$ be any family of closed sets in $A$ satisfying the m.i.p. Clearly $F$ is a family of subsets of $X$ having the above mentioned properties and consequently $\cap F \neq \emptyset$. Thus, in view of Theorem 49 it follows that $A$ is $m$-compact space as required.

In the next theorem we prove all the implications indicated in Diagram 1.

**THEOREM 51.** Let $X$ be a Topological space then the following hold.

(i) Let $m$ be a regular cardinal: if $X$ is $m$-compact then $X$ is $B_m$-compact.

(ii) If $X$ is $B_m$-compact then $X$ is $m$-compact.

(iii) Let $m$ be a regular cardinal: if $X$ is $B_m$-compact then $X$ is weakly $m$-chain compact.
(iv) If $X$ is weakly $m$-chain compact then $X$ is $B_{m}$-compact.

(v) If $X$ is $m$-compact then $X$ is $m$-limit point compact.

(vi) If $X$ is a $T_{1}$, $m$-limit point compact then $X$ is $m$-compact.

(vii) Let $m$ be an infinite cardinal number: if $X$ is $m$-bounded then $X$ is $m$-compact. In particular an $\omega$-bounded space $X$ is countably compact.

(viii) If $X$ is $m$-cap compact then $X$ is $m$-compact.

(ix) Let $m$ be a regular cardinal: if $X$ is $m$-compact then $X$ is $m$-cap compact.

(x) Let $m$ be a regular cardinal: if $X$ is weakly $m$-bounded then $X$ is $m$-compact. In particular if $X$ is weakly $\omega$-bounded then $X$ is countably compact.

(xi) If $X$ is $m$-$\mathcal{A}$-compact then $X$ is $m$-compact.

**Proof.** (1) Assume on the contrary and let $P = \{F_{i} : i \in m\}$ be an $m$-tower of $X$ with $\cap P = \emptyset$. It follows that $\cup(F_{i}^{c} : i \in m) = X$ and consequently there is a cardinal $n < m$ with $\cup(F_{k_{i}} : i \in n) = X$ from which it follows that $\cap(F_{k_{i}} : i \in n) = \emptyset$. Let
\( p = \sup\{k_i : i \in \mathbb{n}\} \). Since \( m \) is a regular cardinal it follows that \( p < m \). Let \( j \in m, j > p \). Clearly \( F_j \subset F_{k_i} \) for every \( i \in \mathbb{n} \) and consequently \( F_j \subset \cap \{F_i : i \in \mathbb{n}\} \) which is a contradiction. Thus, \( X \) is \( \mathbb{R}_m \)-compact as required.

(ii) Assume on the contrary and let \( U = \{0^i : i \in m\} \) be an open cover of \( X \) with no subcover of cardinality \( < m \). We construct by transfinite induction on \( m \) an \( m \)-tower \( B \) as follows. Let

\[
F_0 = 0_g(o) \quad \text{with} \quad g(o) = 0
\]

for \( k = v + 1 \) a nonlimit ordinal of \( m \), let

\[
F_k = F_v \cup O^g(k)
\]

where \( g(k) \) is the first \( i \in \mathbb{n} \) with \( 0^i \notin F_v \). If no such \( i \) existed then \( F_v \) would have been the whole space \( X \) and consequently \( U' = \{0^g(k) : k \in v\} \) would have been a subcover of cardinality \( < m \). For \( k \) a limit ordinal we let

\[
F_k = U\{F_i : i \in k\} \cup O^g(k)
\]

where \( g(k) \) is the first \( i \) with \( 0^i \notin U\{F_v : v \in k\} \). Such an \( i \) always exists, for otherwise \( U\{F_v : v \in k\} \) would have been the whole space \( X \) and consequently
U' = \{g(i) : i \in k\} would have been a subcover of X of cardinality < m.

It can be easily seen that

\[ B = \{F_i^C : i \in m\} \]

is an m-tower of X. Since X is \( B_m \)-compact, it follows that \( \cap\{F_i^C : i \in m\} \neq \emptyset \) or equivalently \( U\{F_i : i \in m\} \neq X \) which is a contradiction. Thus, X is \( m \)-compact as required.

(iii) Assume on the contrary and let s be an m-sequence of X without a cluster point. For every \( v \in m \) let

\[ S_v = cl\{(s_u : u \geq v)\} \]

We construct by transfinite induction on m an m-tower B as follows. Let

\[ B_0 = S_{g(o)} \quad \text{where} \quad g(o) = 0 \]

for \( k = v + 1 \) a nonlimit ordinal in m, let

\[ B_k = S_{g(k)} \]

where \( g(k) \) is the first \( i \in m \) for which \( s_{g(v)} \notin S_i \). Such an \( i \) always exists for \( s_{g(v)} \) is not a cluster point of s. Finally for \( k \) a limit ordinal let

\[ B_k = S_{g(k)} \]
where \( g(k) = \sup\{g(i) : i \in k\} \). Clearly \( g(k) \neq m \) and 
\[
B = \{B_k : k \in m\}
\]
is an \( m \)-tower of \( X \). Since \( X \) is \( B_m \)-compact, it follows that \( \bigcap B \neq \emptyset \), but then every point in \( \bigcap B \) is a cluster point of \( s \), which is a contradiction. Thus, our assumption is false and \( s \) has a cluster point.

(iv) Let \( G = \{G_v : v \in m\} \) be an \( m \)-tower of \( X \). Let \( s_v \in G_v \) for every \( v \in m \). Then \( s = \{s_v : v \in m\} \) is an \( m \)-sequence. Since \( X \) is weakly \( m \)-chain compact, \( s \) has a cluster point \( r \). Clearly \( r \) is a limit point of every \( G_v, v \in m \). Since \( G_v \) is closed, \( r \in G_v \) for every \( v \in m \). Thus \( r \in \bigcap G \) and \( X \) is \( B_m \)-compact as required.

(v) Let \( A \) be a subset of \( X \) of cardinality \( m \). If \( A \) has no limit point, then \( A \) is a closed relatively discrete subspace of \( X \) and consequently in view of Theorem 50, \( A \) is \( m \)-compact which is a contradiction \((U = \{a : a \in A\} \) is an open cover of \( A\)). Thus, \( X \) is \( m \)-limit point compact as required.

(vi) Assume on the contrary and let \( U = \{O_i : i \in m\} \) be an open cover of \( X \) with no subcover of cardinality \(< m \). We construct by transfinite induction on \( m \) a set \((\text{in fact an } m \text{-sequence}) A \) of cardinality \( m \) with
no limit point. Let
\[ x_0 \in O_g(o) \text{ where } g(o) = 0 \]
for every ordinal \( k \in m \) let

\[ x_k \in O_g(k) - U\{O_g(i) : i \in k\} \]

where \( g(k) \) is the first \( i \in m \) such that

\[ O_i - U\{O_g(j) : j \in k\} \neq \emptyset \]

such an \( i \) always exists, for otherwise \( \{O_g(j) : j \in k\} \) would have been a subcover of \( U \) of cardinality \( < m \).

Let \( A = \{x_k : k \in m\} \). Clearly \( |A| = m \). We claim that \( A \) has no limit point. First we show that the family \( \{O_g(i) : i \in m\} \) is a cover of \( X \). To see this let \( x \in X \) be such that \( x \notin O_g(i) \) for all \( i \in m \). Since \( U \) is a cover of \( X \), it follows that \( x \in O_v \) for some ordinal \( v \in m \). Let

\[ A = \{j : U\{O_g(i) : g(i) \leq v\} \subseteq U\{O_g(i) : i \in j\}\} \]

and

\[ z = \cap A \]

We claim that the first ordinal \( w \) with

\[ O_w - U\{O_g(i) : i \in z\} \neq \emptyset \]

is the ordinal \( v \). For if \( t < v \) with \( O_t - U\{O_g(i) : i \in z\} \neq \emptyset \), then \( g(z) = t, g(z) < v \) and consequently
Assume that \( A \) has a limit point \( t \), then \( t \in O_g(v) \) for some ordinal \( v \in m \). Since \( X \) is \( T_1 \) there is a neighborhood \( V_t \) of \( t \) such that \( x_v \notin V_t \) but then clearly

\[ V = O_g(v) \cap V_t \]

is a neighborhood of \( t \) such that \( V \cap A = \emptyset \) contradicting the fact that \( t \) is a limit point of \( A \). Thus, \( A \) has no limit point in \( X \), which in view of the hypothesis of the Theorem is a contradiction. Thus \( X \) is \( m \)-compact as required.

(vii) Assume on the contrary and let \( \mathcal{U} = \{O_i : i \in m\} \) be an open cover of \( X \) with no subcover of cardinality \( < m \). We construct by transfinite induction on \( m \) an \( m \)-sequence \( s \) which is not contained in any \( m \)-compact subspace \( A \) of \( X \). Let

\[ s_0 \in O_{g(0)} \quad \text{where} \quad g(0) = 0 \]

for any ordinal \( k \in m \) let

\[ s_k \in O_{g(k)} = \bigcup \{O_i : i \in k\} \]

where \( g(k) \) is the first \( j \in m \) with

\[ O_j \setminus \bigcup \{O_i : i \in k\} \neq \emptyset \]

such a \( j \) always exists, for otherwise \( \{O_i : i \in k\} \) would have been a subcover of \( X \). Let
Since $X$ is $m$-bounded, it follows that

$$s = \{s_i : i \in m\}$$

where $A$ is an $m$-compact subspace of $X$. Since

$$\{O_g(i) : i \in m\}$$

is an open cover of $X$, it follows that

$$\{O_g(i) \cap A : i \in m\}$$

is a cover of $X$ and consequently, there is a subcover of cardinality $< m$, which is a contradiction.

(viii) Assume on the contrary and let $U = \{O_i : i \in m\}$ be an open cover of $X$ with no subcover of cardinality $< m$. We construct by transfinite induction on $m$ a set $A$ with no cap. Let

$$x_0 \in O_{g(o)} \quad \text{where } g(o) = 0$$

for $k \in m$ any ordinal, let

$$x_k \in O_{g(k)} - U\{O_{g(i)} : i \in k\}$$

where $g(k)$ is the first $i \in m$ with

$$O_i - U\{O_{g(j)} : j \in k\} \neq \emptyset$$

such an $i$ always exists, for otherwise $\{O_{g(i)} : i \in k\}$ would have been a subcover of $U$ of cardinality $< m$. 

Let

\[ A = \{x_k : k \in m\} \]

Clearly \(|A| = m\). It follows exactly as in part (vi) of this Theorem that \(A\) has no cap, which is a contradiction. Thus, \(X\) is \(m\) - compact as required.

(ix) Let \(A = \{a_i : i \in m\}\) be a subset of \(X\) of cardinality \(m\). In view of part (iii) of this Theorem it follows that the sequence \(\{a_i : i \in m\}\) has a cluster point \(t\). Since \(m\) is a regular cardinal and every neighborhood of \(t\) contains a subset of \(A\) cofinal to \(m\), we see that \(t\) is a cap of \(A\). Thus \(X\) is \(m\) - cap compact as required.

(x) Similar to that of part (vii) of this Theorem.

(xi) Assume on the contrary and let \(U = \{U_i : i \in m\}\) be an open cover of \(X\) with no subcover of cardinality < \(m\). We construct exactly as in part (viii) of this Theorem, a set \(A\) with no \(A\) - point, which contradicts the fact that \(X\) is \(m\) - \(A\) - compact. Thus, \(X\) is \(m\) - compact as required.

**THEOREM 53.** The following are equivalent for a topological space \(X\) and infinite cardinal number \(m\):
(i) \( X \) is initially \( m \)-compact

(ii) \( X \) is initially \( B_m \)-compact

(iii) \( X \) is initially weakly \( m \)-chain compact

(iv) \( X \) is \( T_1 \) and initially \( m \)-limit point compact

(v) \( X \) is initially \( m \)-cap compact

(vi) \( X \) is initially weakly \( m \)-bounded

(vii) \( X \) is initially \( m \)-bounded

PROOF. We show that (i) \( \iff \) (ii), (the other implications follow similarly). In view of Theorem 51 part (ii), it follows that (ii) \( \implies \) (i). Next, we show that (i) \( \implies \) (ii). Assume on the contrary and let \( F = \{ F_i : i \in m \} \) be an \( m \)-tower of \( X \) with \( \cap F = \emptyset \).

In view of Theorem 51 part (i), it follows that \( \cap \{ F_i : k \in \text{cf}(m) \} = \emptyset \). Since \( \text{cf}(m) \) is a regular cardinal, it follows by Theorem 51 part (ii) that \( X \) is not \( \text{cf}(m) \)-compact, which is a contradiction. Thus, \( X \) is \( B_m \)-compact and (i) \( \implies \) (ii), as required.

THEOREM 54. A space \( X \) is compact iff every subset \( A \subseteq X \) is contained in a \( |A| \)-compact subspace.

PROOF. In view of Theorem 51 part (vii), it follows that \( X \) is \( m \)-compact for every cardinal \( m \). Thus \( X \) is compact as required.
EXAMPLE 3. Let $X = \omega_0$ with the order Topology. Since $X = \bigcup \{\omega_i : i \in \omega\}$ and each $\omega_i$ is Lindelof (in fact compact) it follows that $X$ is Lindelof, i.e., $\omega_1$-compact and in particular $\omega_0$-compact. $X$ is not $B_{\omega_1}$-compact for $\mathcal{A}(F_i : i \in \omega_0) = \emptyset$, where $F_i = \{v : v \in \omega_0 \text{ and } v \geq i\}$.

Example 3 shows that regularity in Theorem 51 part (i) is needed.

EXAMPLE 4. Let $X = \omega_\omega - \{\omega\}$ with the order Topology. Clearly $X$ is $B_{\omega_\omega}$-compact. For every $v \in \omega_\omega$ let $a_v = u$ where $|v| = \omega_\omega$. Clearly $\{a_v : v \in \omega_\omega\}$ is an $\omega_\omega$-sequence with no cluster point.

Example 4 shows that regularity is needed in Theorem 51 part (iii).

EXAMPLE 5. Let $X = \omega_1$ with the order Topology. Clearly $X$ is countably compact. But $X$ is not $\omega_1$-cap compact.

A space $X$ is $m$-monotone at $x \in X$, provided $m$ is the smallest cardinal for which $x$ has a nhood base $\{V_i^X : i \in m\}$ with $V_i^X \subseteq V_j^X$, $i > j$.

A space $X$ is $m$-monotone, provided $X$ is $m$-monotone at any point $x \in X$. 
LEMMA 33. Let $X$ be an $m$-monotone space at $a$. If $a$ is a cluster point of an $m$-sequence $s$ in $X$, then there is an $m$-subsequence $t$ of $s$ converging to $a$.

PROOF. Let $V = \{V_i : i \in m\}$ be the nested neighborhood base at $a$. We construct an $m$-subsequence $t$ of $s$ by transfinite induction on $m$. Let

$$t_0 = s_{g(o)}$$

where $g(o)$ is the first $i \in m$ with $s_i \in V_0$. In general for $k = v + 1$ a nonlimit ordinal in $m$ we let

$$t_k = s_{g(k)}$$

where $g(k)$ is the first $i \in m$ with $s_i \in V_k$. Such an $i$ always exists, for $|V_k \cap s| = m$. Finally for $k$ a limit ordinal in $m$ we let

$$t_k = s_{g(k)}$$

where $g(k)$ is the first $i \geq \sup\{g(j) : j \in k\}$ with $s_i \in V_k$. Such an $i$ always exists, for $\sup\{g(j) : j \in k\}$ is less than $m$ and $|V_k \cap s| = m$. Let

$$t = \{t_j : j \in m\}$$

Clearly $t$ is an $m$-subsequence of $s$ converging to $t$ as required.
COROLLARY 16. Let $X$ be an $m$-monotone and weakly $m$-chain compact space. Then $X$ is $m$-chain compact.

EXAMPLE 6. Let $m$ be a regular cardinal; let $X = m$ and let $X$ have the Topology in which closed sets are $X$ and subsets of $X$ of cardinality $< m$. Clearly every point of $X$ is an $A$-point. Furthermore, $X$ is of depth $m$ and $m$-monotone. First we show that $X$ is of depth $m$ (in fact $P_m$). Let $\{V_i : i \in n\}$, $n \leq m$ be a family of neighborhoods at $x \in X$. We show that $\cap\{V_i : i \in n\}$ is again, a neighborhood of $x$. Let $k_i = \sup\{j : j \in V_i^o\}$ clearly $k_i \neq m$ for all $i \in n$ and consequently $k = \sup\{k_i : i \in n\}$ is different from $m$. It follows that $V = \{x\} \cup \{j : j \geq k\}$ is a neighborhood of $x$ contained in $\cap\{V_i : i \in n\}$.

Next we show that $X$ is $m$-monotone. Clearly the family $\{(x : x \geq i) \cup \{y : i \in m\}$ is a nested neighborhood base at any point $y \in X$.

Let $s$ be any $m$-sequence in $X$. If $|s| < m$ then some value $a_v$ of $s$ is repeated $m$-times and consequently $a_v$ is a cluster point of $X$. If $|s| = m$ then $s$ converges to every point $x \in X$. It follows from Lemma 33 that $X$ is $m$-chain compact.
Next we give an example of a compact space $X$ which is not $m$-chain compact. This example is a straight generalization of an example [29] due to C. T. Scarborough and A. H. Stone.

**EXAMPLE 7.** Let $m$ be an infinite cardinal and let

$$A = \{a_i : i \in 2^m\}$$

be an enumeration of all subsets of $m$ of cardinality $m$. We show that the space $X = 2^{2^m}$ where each coordinate space is taken with the discrete topology, fails to be $m$-chain compact. For every $i \in m$ let $a^0_i$ and $a^1_i$ be two disjoint subsets of $a_i$ with union $a_i$ and of cardinality $m$. We identify $a^0_i = 0$ and $a^1_i = 1$ and $\{a^0_i, a^1_i\} = 2$. Let

$$s = (s_j : j \in m)$$

be an $m$-sequence of $X$ defined as follows. $p_i(s_j) = 1$ if $j \in a^1_i$ and 0 otherwise. We show that $s$ has no convergent $m$-subsequence. Assume on the contrary and let

$$t = (t_{kj} : j \in m)$$

be a subsequence of $s$ converging to $r \in X$. Let

$$B = \{k_j : j \in m\}$$
Clearly \( B = a_v \) for some \( v \in 2^m \). If \( r_v \equiv 1 \) then \( p_v^{-1}(1) \) is a neighborhood of \( r \) leaving out \( m \) - many terms (for any \( j \in a_v^0, p_v(s_j) = 0 \)) and consequently \( t \) cannot converge to \( r \), which is a contradiction. Thus \( X \) is not \( m \) - chain compact as required.

**COROLLARY 17.** Let \( m \geq \omega \) be a cardinal, let \( \{X_i : i \in 2^m\} \) be a family of \( T_1 \) - spaces, each having more than one point. Then \( X = \prod \{X_i : i \in 2^m\} \) is never \( m \) - chain compact.

**PROOF.** For every \( i \in 2^m \) let \( S_i = \{a_i, b_i\}, a_i \in X_i \) and \( b_i \in X_i - \{a_i\} \). Clearly \( S = \prod \{S_i : i \in 2^m\} \) is a closed subspace of \( X \). By example 7, it follows that \( S \) is not \( m \) - chain compact, which is a contradiction as it can be easily seen that a closed subspace of an \( m \) - chain compact space is \( m \) - chain compact again. Hence \( X \) is not \( m \) - chain compact space as required.

**LEMMA 34.** Let \( X \) be a \( T_1 \) Topological space of depth \( m \); let \( A \subset X, |A| = m \). If \( t \in \bar{A} \) and \( W \) is any neighborhood of \( t \) then \( |W \cap A| = m \).

**PROOF.** Assume on the contrary and let \( W \) be a neighborhood of \( t \) such that \( |W \cap A| < m \). For every \( a \in W \cap A \) let
\( V_a \) be a neighborhood of \( t \) missing \( a \) and contained in \( W \). Clearly \( \bigcap \{ V_a : a \in W \cap A \} \) is a neighborhood of \( t \) missing \( A \) which is a contradiction. Hence \( |W \cap A| = m \) as required.

In view of Example 7, an \( m \)-limit point compact space \( X \) need not be \( m \)-chain compact. However, the following holds.

**Theorem 54.** Let \( m \) be a regular cardinal; let \( X \) be a \( T_1 \), \( m \)-monotone and of depth \( m \), space. Then the following are equivalent.

(i) \( X \) is \( m \)-compact

(ii) \( X \) is \( m \)-limit point compact

(iii) \( X \) is \( m \)-chain compact

**Proof.** In view of Theorem 51 it suffices to show (ii) \( \rightarrow \) (iii). In view of Lemma 33 it suffices to show that \( X \) is weakly \( m \)-chain compact. Let \( s \) be an \( m \)-sequence in \( X \). If \( |s| < m \) then clearly \( s \) has a cluster point. If \( |s| = m \) then since \( X \) is \( m \)-limit point compact, it follows that \( s \) (as a set) has a limit point \( t \). By Lemma 34, it follows that \( t \) is a cluster point of \( s \). Thus, \( X \) is \( m \)-chain compact space as required.
REMARK 6. Clearly a $T_1$, $\omega$-monotone space $X$ is first countable and consequently for $m = \omega$, Theorem 52 reduces to the following well-known equivalence.

(i) $X$ is countably compact
(ii) $X$ is limit point compact
(iii) $X$ is sequentially compact.
11. BETWEEN THE PRODUCT AND THE BOX TOPOLOGY

Let \( v \) be a cardinal, let \( \{(X_k, T_k) : k \in v\} \) be a collection of topological spaces and let \( X = \prod\{X_k : k \in v\} \) be their Cartesian product. The \( u \)-topology for every infinite cardinal \( u < v \) on \( X \) is obtained by taking as a base for the open sets, all the sets of the form \( \prod O_a \), where

(i) \( O_a \) is open in \( X_a \) for each \( a \in v \)
(ii) For all but \( u \)-many coordinates, \( O_a = X_a \).

**THEOREM 55.** Let \( m \) be a regular cardinal and \( X = \prod\{X_i : i \in v\}, v \leq m \) be a product of \( m \)-chain compact spaces. Then \( X \) with any \( u \) (\( u < m \)) topology is \( m \)-chain compact.

**PROOF.** Let \( s = \{s_i : i \in m\} \) be any \( m \)-sequence of \( X \), of cardinality \( m \). We construct by transfinite induction on \( v \) a convergent \( m \)-subsequence \( t \) of \( s \) as follows. Let \( p_i \) denote the \( i \)-th projection of \( X \) and

\[
A_1 = m \quad \text{and} \quad t_1 = s_{g_1(1)}
\]

where \( \{p_i(s_{g_1(i)}) : i \in A_1\} \) is an \( m \)-subsequence of
\[ \{ p_i(s_i) : i \in A_i \}, \text{ converging to } x_i \in X_i. \]

In general for \( w = k + 1 \) a nonlimit ordinal of \( v \) we let

\[ A_w = \{ g_k(i) : i \in A_k \} \]

and

\[ t_w = s g_w(w) \]

where \( \{ p_w(s g_w(i)) : i \in A_w \} \) is an \( m \) - subsequence of \( \{ p_w(s_i) : i \in A_w \} \), converging to \( x_w \in X_w. \)

Finally for \( w \) a limit ordinal of \( v \), we let

\[ A_w = \{ \cup \{ g_k(j) - \text{th element of } A_k \} : k \in w \} \]

and

\[ t_w = s g_w(w) \]

where \( \{ p_w(s g_w(i)) : i \in A_w \} \) is an \( m \) - subsequence of \( \{ p_w(s_i) : i \in A_w \} \), converging to \( x_w \in X_w. \) This is possible because \( |A_w| = m \) (\( m \) is a regular cardinal).

If \( v < m \), then the sequence

\[ t = \{ s g_v(i) : i \in A_v \} \]

is an \( m \) - subsequence of \( s \) converging to \( x = \{ x_i : i \in v \}. \) To see this, let

\[ U = \cap \{ p_k^{-1}(0_{k_j}) : j \in k \} \]

be a nhood of \( x. \) Since the \( m \) - sequence \( \{ p(s g_{k_j}(h)) : h \in A_{k_j} \} \) converges to \( x_{k_j} \) for every \( j \in k \), it follows
that
\[ |(P_{k_j}^{-1}(O_{k_j}))^c \cap \{ s_{g_{k_j}}(h) : h \in A_{k_j} \}| < m \]

Since \( t \) is an \( m \) - subsequence of \( \{ s_{g_{k_j}}(h) : h \in A_{k_j} \} \)
we see that
\[ v_j = |(P_{k_j}^{-1}(O_{k_j}))^c \cap t| < m \]
for every \( j \in k \). Thus since \( m \) is a regular cardinal
it follows that
\[ |U^c \cap t| \leq U(v_j : j \in k) \]
\[ < m \]
showing that \( t \) converges to \( x \), as required.

If \( V = m \), then the \( m \) - sequence
\[ t = \{ t_i : i \in m \} \]
is clearly an \( m \) - subsequence of \( s \) and it can be shown just like before, that \( t \) converges to \( x \). Thus \( X \) is \( m \) - chain compact space, as required.

**Corollary 18.** Any countable product of sequentially compact spaces, is sequentially compact.

**Corollary 19.** An \( \omega_1 \) - product of \( \omega_1 \) - chain compact spaces with the product or the \( \omega \) - topology, is \( \omega_1 \) - chain compact space.

In view of Corollary 17, it follows that any \( 2^{\omega_1} \) -
product of $T_1$, $\omega_1$-chain compact spaces is never $\omega_1$-chain compact. However, in view of Theorem 34, it follows

**THEOREM 56.** Assume GMA + CH. Let $X = \prod \{X_i : i \in k\}, k < 2^{\omega_1}$ be a product of compact spaces. If the character of each $X_i$ is $v < 2^{\omega_1}$, then $X$ with the product or the $\omega$-topology, is $\omega_1$-chain compact.

**PROOF.** Since $X$ is compact, every $\omega_1$-sequence has a cluster point (an $m$-sequence is always a net, and consequently it has a cluster point). In view of Theorem 34 it is enough to show that $X$ has a nhhood base at each of its points of cardinality $< 2^{\omega_1}$. Let $x = \{x_i : i \in k\}$ be a point of $X$. Since $X_i$ is of character $v$, $x_i$ has a nhhood base of cardinality $v$ for every $i \in k$. Since a basic nhhood $V$ of $x$ is determined by a finite, or a countable intersection of open strips and since there are $\max\{v, k\}$ many open strips, it follows that $\text{ch}(x, X) < 2^{\omega_1}$, as required.

**COROLLARY 20.** Let $m$ be a regular cardinal, let $X = \prod \{X_i : i \in v\}, v \leq m$ be a product of $m$-bounded and $m$-monotone spaces. Then $X$ with any $u$ ($u < m$) topology is $m$-chain compact.
COROLLARY 21. Let $m$ be a regular cardinal, let $X = \prod_{i \in \nu} X_i$, $\nu \leq m$ be a product of $m$-compact and $m$-monotone spaces. Then $X$ with any $u$ ($u < m$) topology is $m$-chain compact.
12. BIBLIOGRAPHY


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