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Variations of zero forcing and power domination

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Variations of zero forcing and power domination

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DEDICATION

I would like to dedicate this thesis to my wife Scarlitte, without whose support I would not be in the mathematics program at Iowa State University.

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ABSTRACT

Zero forcing is a propagation process on a graph that turns white vertices into blue vertices. In this process, an initial set of vertices in a graph G are chosen to be blue and all others are colored white, then a color-change rule is iteratively applied until all of G becomes blue. A large amount of research has gone into finding the size of a minimum zero forcing set in graphs as the process has connections in linear algebra, computer science, and even physics. In power networks, the zero forcing process is used to measure the phase throughout the system. This process is known as the power domination process, and it is a well researched area of graph theory. Power domination has since been generalized using different variations of zero forcing. The generalization that will be discussed in this dissertation is known as k-power domination which uses the generalization of zero forcing known as k-forcing. Recently, a new variation of zero forcing known as leaky forcing was introduced to research concerns involving faulty vertices in a system. An ℓ-leaky forcing set is a zero forcing set that is resistant to any ℓ "broken" vertices (leaks) in a graph. In this dissertation, results proving ℓ-leaky forcing sets and ℓ-edge leaky forcing sets are equivalent are provided. Furthermore, bounds for minimum sized k-power dominating sets in hypergraphs are proven.

Keywords zero forcing, leaky forcing, color-change rule, power domination

CHAPTER 1. INTRODUCTION

Be it a social network modeling different friendships between users, or a power network, various real world problems can be modeled by a graph. Moreover, many of the properties discovered in graph theory can be directly applied to solve problems that affect all of us. Two different graph parameters will be discussed in this dissertation, both of which have an underlying connection with zero forcing. Furthermore, these parameters, in some sense, both model control in a dynamical system.

Zero forcing is a propagation process on a graph that increases the number of blue vertices given an initial set of blue vertices, with all other vertices white, and a color-change rule. The *zero forcing color-change rule* states that a blue vertex adjacent to a single white neighbor can force its neighbor blue. Formally, if u is a blue vertex and w is the only white vertex in $N(u)$, then $u \to w$ will be used to denote that u forces w blue. Given a graph G, a *zero forcing set* of G is a subset of vertices from $V(G)$ such that if B is initially colored blue, and the remaining vertices in G are white, then iteratively applying the color-change rule given B results in every vertex in G becoming blue.

The zero forcing parameter was first introduced in [2] as a way to find upper bounds for the maximum nullity for the family of real symmetric matrices whose nonzero off-diagonal entries are described by a graph. Although this is one of the main reasons to research this process, the applications of zero forcing extend further than linear algebra alone. Zero forcing measures control in a system, and by doing so has many real world connections.

With these connections, there are concerns about how a faulty vertex in a network could disrupt the flow of information. This concern was explored by Dillman and Kenter in [10] where they introduced the variation of zero forcing known as *leaky forcing*. Leaky forcing uses the same color-change rule as zero forcing but restricts certain vertices from performing forces. Vertices that are not allowed to perform forces are called *leaks* and an *ℓ-leaky forcing set* for a graph G is a subset of initial blue vertices B such that if any ℓ vertices are chosen to be leaks (after B has been specified), then iteratively applying the color-change rule will force every vertex in G. The *ℓ-leaky forcing number* of G is the size of a minimum ℓ-leaky forcing set and is denoted $\mathrm{Z}_{(\ell)}(G)$. Notice that a 0-leaky forcing set for a graph G is also a zero forcing set, and the 0-leaky forcing number is also the zero forcing number. This notation differs from the notation found in [10]. The term ℓ-leaky forcing will be used instead of the abbreviated ℓ-forcing found in their paper. This change is made in order to prevent confusion between leaky forcing and another generalization of zero forcing called k-forcing.

What if instead of a broken vertex, the connection between adjacent vertices is broken? A natural way to generalize ℓ-leaky forcing is to consider what happens when forces are prohibited from passing over particular edges. An edge xy is an *edge leak* if neither $x \to y$ nor $y \to x$ are allowed. A set of blue vertices B is an *ℓ-edge-leaky forcing set* if B can turn the whole graph G blue given any set of ℓ edge leaks. Denote the ℓ-edge-leaky forcing number of a graph G by $\mathrm{Z}'_{(\ell)}(G)$. A force $v \to u$ is a *specified leak* if v is prohibited from forcing u. A set B is a *specified ℓ-leaky forcing set of* G if B can color G blue when any set of ℓ forces are prohibited. Let $\mathrm{Z}^s_{(\ell)}(G)$ be the minimum size of a specified ℓ-leaky forcing set of G.

Haynes, Hedetniemi, Hedetniemi, and Henning, introduced *power domination* in [12] to study the monitoring process of electrical power networks using graph models. The devices used to measure phase in a power network are known as phase measurement units (PMUs). If a graph is used to model this power network, PMUs will be placed on vertices. In the real world, the least amount of PMUs needed to measure phase in a system is a major problem. In graph theory, this is known as the *power domination problem* and it was shown that the zero forcing process is one of the steps in the power domination process which will be discussed later.

In this dissertation, the generalization of power domination known as *k-power domination* will be explored in both graphs and hypergraphs. Furthermore, domination and k-power domination bounds will be compared and a connection between the two bounds will be discussed.

A *hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (V, E)$ where V is a set of elements called vertices, and E is a set of non-empty subsets of V called edges. An *r-uniform hypergraph* is a hypergraph where each edge contains exactly r vertices. A *dominating set* for a hypergraph $\mathcal{H}$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The *domination number* $\gamma(\mathcal{H})$ is the number of vertices in a smallest dominating set for $\mathcal{H}$. The *white degree* of a vertex v with respect to a set S in $\mathcal{H}$, denoted $deg_w(v, S)$, is the minimum number of edges that contain v and cover the white neighbors of v when each vertex in S is blue and each vertex in $V(\mathcal{H}) \setminus S$ is white. There are two steps in the k-power domination process. The first step is widely known as the domination step. In this step, a set D colors its neighborhood blue. The second step is known as the k-forcing step. Like zero forcing, *k-forcing* is a propagation process that increases the number of blue vertices in the vertex set. The *k-forcing color-change rule for hypergraphs* states that if v is in a blue set of vertices $S \subseteq V(\mathcal{H})$ and $deg_w(v, S) \leq k$, then v colors its neighbors blue. A set of blue vertices B in a hypergraph $\mathcal{H}$ is a *k-forcing set* if iteratively applying the k-forcing color-change rule results in $V(\mathcal{H})$ becoming blue. Finally, given a hypergraph $\mathcal{H}$, a set $D \subseteq V(\mathcal{H})$ is a *k-power dominating set* if $N[D]$ is a k-forcing set. The *k-power domination number* $\gamma_p^k(\mathcal{H})$ is the number of vertices in a smallest k-power dominating set for $\mathcal{H}$.

This variation of power domination was introduced by Chang and Roussel in [9], and although power domination is well researched for simple graphs, few upper bounds have been found for the k-power domination number in hypergraphs. In this dissertation, new bounds will be introduced that resulted from a conjecture from Chang and Roussel. Moreover, using new techniques, a conjecture given by Bjorkman in [4] will be proven.

1.1 Organization

This dissertation is organized in journal paper format and is comprised of three papers currently under review and one paper that is being prepared for submission. Basic notation in graph theory is outlined in Section 1.2. A literature review on zero forcing, leaky forcing, and k-power domination is given in Section 1.3.

Chapter 2 contains the paper "On leaky forcing and resilience" which is currently under review in *Discrete Applied Mathematics*. In Section 2.2 a characterization of ℓ-leaky forcing sets is given. In particular, a necessary and sufficient condition for an $(\ell - 1)$-leaky forcing set to be an ℓ-leaky forcing set is proven. Section 2.3 discusses which structural properties a graph must possess for it to be ℓ-resilient ($\mathrm{Z}_{(0)}(G) = \mathrm{Z}_{(\ell)}(G)$). The ℓ-leaky forcing number for certain graph families is given in Section 2.4.

Chapter 3 contains the paper "Generalizations of Leaky Forcing" which is currently under review in *The Journal of Combinatorial Optimization*. Section 3.2 introduces edge-leaky forcing and uses results from Chapter 2 to prove ℓ-leaky forcing is equivalent to ℓ-edge-leaky forcing. Similarly, Section 3.3 introduces specified leaky forcing and uses results from Chapter 2 to prove ℓ-leaky forcing is equivalent to specified ℓ-leaky forcing. Finally, Section 3.4 introduces mixed leaky forcing and uses results from Chapter 2 to prove ℓ-leaky forcing is equivalent to mixed ℓ-leaky forcing.

Chapter 4 contains the paper "An upper bound for the k-power domination number in r-uniform hypergraphs" which is currently under review in *Discrete Mathematics*. In Section 4.3, an upper bound for the k-power domination number in r-uniform hypergraphs (hypergraphs where each edge contains r vertices) is given. Section 4.4 provides counterexamples to a conjecture given in [9].

Chapter 5 contains the manuscript "A method for finding upper bounds for the k-power domination number using bounds on the domination number" which is being prepared for submission. Section 5.2 introduces general definitions and notation used through the paper.

Section 5.3 introduces and proved various upper bounds for the k-power domination number using results and techniques from Chapter 4.

General conclusions and future work are discussed in Chapter 6.

1.2 Basic graph theory

In this section, the basic tools and definitions used throughout this dissertation are outlined. Note that many of the definitions for graphs also apply to hypergraphs which were defined earlier. A *graph* is a pair $G = (V(G), E(G))$ where $V(G)$ is a set of elements called vertices and $E(G)$ is a set of unordered pairs of elements from $V(G)$. Often, when the graph being discussed is obvious, V and E will be written. In this dissertation, *simple graphs* (graphs with no multiedges or loops) will be used. The *order* of a graph G is the number of vertices in $V(G)$. Two vertices $u, v \in V(G)$ are said to be *adjacent* (or *neighbors*) if $\{u, v\} \in E(G)$ and is often written uv. A vertex $v \in V(G)$ is *incident to* an edge $e \in E(G)$ if e contains v. The *closed neighborhood* of a vertex $v \in V$, $N[v]$, is the set of vertices adjacent to v and v itself. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N[v] \setminus \{v\}$. *The closed neighborhood of a set* S is the set $\bigcup_{v \in S} N[v]$ and is denoted $N[S]$. *The open neighborhood of a set* S is the set $\bigcup_{v \in S} N(v)$ and is denoted $N(S)$. The *degree* of a vertex $v \in V(G)$ is the size of $N(v)$ and is denoted $d(v)$. The *maximum degree* over all vertices in $V(G)$ is denoted $\Delta(G)$ and the *minimum degree* over all vertices in $V(G)$ is denoted $\delta(G)$. A graph H is a *subgraph* of a graph G if H is obtained from G by either deleting edges or vertices. If $H \subseteq V(G)$, the subgraph induced by H is given by $G[H] = G - (V(G) \setminus H)$.

In Chapter 2, Chapter 4, and Chapter 5 (hyper)graph operations will be discussed such as edge and vertex deletion. Given a (hyper)graph G and an edge $e \in E(G)$, the (hyper)graph obtained by removing e is given $G - e = (V(G), E(G) \setminus e)$. Like edge deletion, vertex deletion is also an operation that is used frequently. Let G be a (hyper)graph and let $S \subseteq E(G)$ be a set of edges incident to a vertex $v \in V(G)$. The (hyper)graph obtain by deleting v is given by $G - v = (V(G) \setminus \{v\}, E(G) \setminus S)$.

Another graph operation discussed in Chapter 2 is the *Cartesian product* of two graphs. If G and H are two graphs, their Cartesian product, given by $G\Box H$, is the graph with vertex set $\{(u,v)|u \in V(G) \text{ and } v \in V(H)\}$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.

1.3 Literature review

This section will review the literature on zero forcing, leaky forcing and power domination. Furthermore, this section sets up important questions that are answered later in this dissertation.

1.3.1 Zero forcing and k-forcing

Zero forcing was introduced in [2] to find upper bounds for the maximum nullity for the family of real symmetric matrices whose nonzero off-diagonal entries are described by a graph. In the original paper, the colors used in the zero forcing process were black and white, but in recent literature these colors have changed. The main colors that are used now are blue and white; however, filled and unfilled vertices have also gained popularity where filled has replaced blue and unfilled has replaced white. As stated before, in this dissertation blue and white will be used as it is the convention in most of the recent literature. See Figure 1.1 for an example of a graph with a minimum zero forcing set.

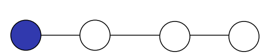

Figure 1.1 The path graph P_4 with a minimum zero forcing set

For an $n \times n$ real symmetric matrix $A = [a_{i,j}]$, $\mathcal{G}(A)$ is the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{ij | i \neq j \text{ and } a_{i,j} \neq 0\}$. Given a graph G of order n, $\mathcal{S}(G)$ is the family of $n \times n$ real symmetric matrices A such that $\mathcal{G}(A)$ is G. The *maximum nullity* of a graph G is the maximum nullity of the matrices in $\mathcal{S}(G)$ and is denoted $\mathrm{M}(G)$. The following proposition fuels much of the research that is done on the zero forcing process.

Proposition 1.3.1. *[2] If G is a graph, then* $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

Because of this fascinating connection, finding minimum zero forcing sets for various graph families has been a major point of research. Furthermore, bounding the zero forcing number has been a concern. Since many graphs can be constructed using graph operations like the Cartesian product, finding general bounds for the zero forcing number for these operations is a motivation for research. The following proposition gives a bound for the Cartesian product of two graphs.

Proposition 1.3.2. *[2] If G and H are graphs, then* $\mathrm{Z}(G \square H) \leq min\{\mathrm{Z}(G)|H|, \mathrm{Z}(H)|G|\}$.

Although zero forcing has a deep connection to linear algebra and matrix theory, connections have been found in other fields such as physics and computer science as well. Connections between fast-mixed search algorithms and zero forcing can be found in [11] and connections in physics can be found in [7]. Since the zero forcing parameter extends to other fields, researching the parameter does not only revolve around finding the maximum nullity of a graph. In fact, researching the structure of a zero forcing set in certain graphs and how they propagate through a system is a major research concern.

Different variations of zero forcing have since been introduced and researched. One generalization of the zero forcing process introduced by Amos, Caro, Davila, and Pepper in [1], is the *k-forcing process.* In this generalization, the color-change rule for the zero forcing process is slightly modified allowing a blue vertex to perform a force if it has at most k white neighbors.

1.3.2 Leaky forcing

A set of vertices having a zero forcing set is enough for the associated dynamical system to be controllable [6]. Leaky forcing was introduced by Dillman and Kenter in [10] to address the concern of broken connections in a system that would stop the zero forcing process from finishing. Simply put, an ℓ-leaky forcing set is a zero forcing set where any ℓ vertices or less can be "broken" and the associated system will still be controllable. Due to leaks making it more difficult for a set of vertices to be a zero forcing set, it is clear that the size of a minimum ℓ-leaky forcing set for a graph G is at least the size of a minimum zero forcing set. The following proposition reflects this fact.

Proposition 1.3.3. *[10] For any graph G of order n,*

$$\mathrm{Z}_{(0)}(G) \leq \mathrm{Z}_{(1)}(G) \leq \cdots \leq \mathrm{Z}_{(n)}(G).$$

Like the zero forcing parameter, finding upper bounds for the ℓ-leaky forcing number is one of the main research problems. Also like the zero forcing parameter, looking at graph operations like the Cartesian product have been considered.

Proposition 1.3.4. *[10] For graphs G and H and a number of leaks $\ell \geq 0$*

$$\mathrm{Z}_{(\ell)}(G \square H) \leq min\{|G| \, \mathrm{Z}_{(\ell)}(H), |H| \, \mathrm{Z}_{(\ell)}(G)\}.$$

An interesting aspect of leaky forcing is that the larger ℓ is, the easier to see what a minimum ℓ-leaky forcing set "looks" like. This is largely due to the fact that the more leaks there are in a graph the more vertices must be in the initial set of blue vertices. This is also reflected in the following lemma.

Lemma 1.3.5. *[10] For a graph G, any ℓ-leaky forcing set will contain at least those vertices in G with degree ℓ or less.*

So depending on the graph and ℓ, there may be a natural starting set of blue vertices. That being said, if a graph G has $\delta(G) \geq j$, and $j > \ell$, then there may not be any intuition on which vertices to initially choose. This can make the $\ell = 1$ case extremely difficult for certain graph families.

One difficult graph of interest is the *grid graph*, $P_n \square P_m$, which is the Cartesian product of the path graph P_n on n vertices and the path graph P_m on m vertices. The following propositions are two of the main results in [10].

Proposition 1.3.6. *[10] If $1 \leq n \leq m$, $\mathrm{Z}_{(1)}(P_n \Box P_m) \leq 2n$.*

Moreover, for certain bounds the following was proven.

Proposition 1.3.7. *[10] If $1 \leq n \leq m$ and $n \leq \lfloor \frac{m}{2} \rfloor + 2$, $\mathrm{Z}_{(1)}(P_n \Box P_m) \leq m$.*

Further analysis prompted the following question from [10] that is answered in Chapter 2.

Question 1.3.8. *[10] If $1 \leq n \leq m$, is it true that $\mathrm{Z}_{(1)}(P_n \Box P_m) \leq min\{m, 2n\}$?*

1.3.3 Power domination and k-power domination

One dynamical system of research interest is the power system. A power system can be represented by a graph where a vertex represents an electrical node and an edge represents a transmission line between two vertices. Haynes et. al researched the problem of locating a smallest set of PMUs to monitor a power system [12]. In this monitoring process a PMU measures the phase for the vertex it is placed on, the edges incident to the vertex it is placed on, and the vertices adjacent to the vertex it is placed on. Once the vertices with PMUs finish this initial monitoring process, the set of monitored vertices follows the zero forcing process. If after the monitoring process finishes and the graph is entirely blue, then the entire system was successfully monitored. The monitoring process itself was introduced in other papers; however, power domination was introduced by [12].

As with the zero forcing process, power domination has been generalized using the k-forcing forcing process. This generalization is known as k-power domination and was introduced by Chang, Dorbec, Montassier, and Raspaud in [8]. In this process, an initial set of blue vertices D colors its neighborhood blue and then $N[D]$ performs the k-forcing process for graphs.

Notice that defining k-forcing, or even zero forcing, for hypergraphs is not entirely clear. This is largely due to there being multiple forcing processes on hypergraphs that coincide with zero forcing when applied to simple graphs. One generalization of zero forcing to hypergraphs introduced by Bergen et. al in [3] is the *infection rule*.

Infection Rule: [3] A non-empty set of blue vertices B can force vertices in an edge e blue if:

1. $B \subset e$, and
2. there are no white vertices v, not contained in e, such that $B \cup \{v\}$ is a subset of an edge.

For the purposes of this dissertation, the color-change rule for hypergraphs previously defined and introduced in [9] for k-power domination will be used.

k-Forcing Rule: [9] A vertex v in a blue set B can force the white vertices in its neighborhood blue if $deg(v, B) \leq k$.

Now k-power domination in hypergraphs can be defined to be an initial blue set of vertices B such that $N[B]$ is a k-forcing set with respect to the previous color-change rule. Notice that for simple graphs, (2-uniform hypergraphs) the white degree of a vertex with respect to a blue set is just the number of white vertices adjacent to it. Therefore, the previous color-change rule coincides with k-forcing in simple graphs and therefore, when $k = 1$, zero forcing in simple graphs.

Like most graph parameters, bounding the k-power domination number is the main problem of interest. For simple graphs there are various bounds for different types of graphs. The following was proven in [8].

Theorem 1.3.9. *[8] If G is a connected graph of order $n \geq k+2$, then*

$$\gamma_p^k(G) \leq \frac{n}{k+2}$$

and this bound is best possible.

Observe that this is closely related to the well-known upper bound for the domination number of $\frac{n}{2}$. For hypergraphs, the following was conjectured in [9] and counterexamples are given in Chapter 4.

Conjecture 1.3.10. *[9] If $\mathcal{H}$ is a connected r-uniform hypergraph and $k + r \leq n$, then*

$$\gamma_p^k(\mathcal{H}) \leq \frac{n}{r+k}$$

with equality if and only if $\mathcal{H}$ is a squid hypergraph of a connected r-uniform hypergraph or $r = 2$ with $\mathcal{H} = K_{(k+2,k+2)}$.

Although the conjecture is false, it is close to being correct. In fact, it is correct for $r \leq 4$ and it closely resembles the domination number upper bound of $\frac{n}{r}$ for similar hypergraphs which was proven by Bujtás, Henning, and Tuza in [5].

Theorem 1.3.11. *[5] If $\mathcal{H}$ is an r-uniform hypergraph of order n with m edges and no isolated vertices, and $r \geq 3$, then*

$$\gamma(\mathcal{H}) \leq \frac{n + \lfloor \frac{r-3}{2} \rfloor m}{\lfloor \frac{3(r-1)}{2} \rfloor}.$$

Setting $r = 3$, or 4, results in the following corollary.

Corollary 1.3.12. *[5] For any r-uniform hypergraph $\mathcal{H}$ on n vertices with no isolated vertices, with $r = 3$ or 4,*

$$\gamma(\mathcal{H}) \leq \frac{n}{r}.$$

It should be of no surprise that known domination bounds for both graphs and hypergraphs are closely related to known k-power domination bounds as 0-power domination is defined to be domination in both graphs and hypergraphs. Depending on how it is viewed, a dominating set can either be a bad, or good, k-power dominating set. The bulk of the results proven in Chapter 5 will show that knowing upper bounds on the domination number can provide a useful starting point for upper bounds on the k-power domination number.

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CHAPTER 2. ON LEAKY FORCING AND RESILIENCE

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Modified from a manuscript submitted to *Discrete Applied Mathematics*

Abstract

A leak is a vertex that is not allowed to perform a force during the zero forcing process. Leaky forcing was recently introduced as a new variation of zero forcing in order to analyze how leaks in a network disrupt the zero forcing process. The ℓ-leaky forcing number of a graph is the size of the smallest zero forcing set that can force a graph despite ℓ leaks. A graph G is ℓ-resilient if its zero forcing number is the same as its ℓ-leaky forcing number. In this paper, we analyze ℓ-leaky forcing and show that if an $(\ell - 1)$-leaky forcing set B is robust enough, then B is an ℓ-leaky forcing set. This provides the framework for characterizing ℓ-leaky forcing sets. Furthermore, we consider structural implications of ℓ-resilient graphs. We apply these results to bound the ℓ-leaky forcing number of several graph families including trees, supertriangles, and grid graphs. In particular, we resolve a question posed by Dillman and Kenter concerning the upper bound on the 1-leaky forcing number of grid graphs.

Keywords zero forcing, leaky forcing, color change rule

AMS subject classification 05C57, 05C15, 05C50

2.1 Introduction

Zero forcing is a process by which blue vertices propagate through a simple graph.[1] More formally, we start with an initial set of blue vertices in a graph that color (or force) other vertices blue. A blue vertex v can color (or force) a white vertex w blue if w is the only white neighbor of v. This is called *the zero forcing color change rule.* A set of vertices B is a *zero forcing set of G*, if B can force every vertex in G by iteratively applying the zero forcing color change rule.

[1] We use standard graph theoretic notation as introduced in [14].

The fewest blue vertices needed to turn the entire graph blue is called the *zero forcing number* of the graph. Zero forcing was introduced in [1] to find upper bounds for the maximum nullity for the family of real symmetric matrices whose nonzero off-diagonal entries are described by a graph. However, the connections go beyond linear algebra and graph theory because, more generally, zero forcing is a process that models knowledge or control of systems. In physics, the zero forcing process was used to optimally control quantum systems [4, 13]. In [3], it was shown that if a set of vertices is a zero forcing set, then the associated dynamical system is controllable. In computer science, fast search and mixed search methods are combined to create a fast-mixed search algorithm that is directly related to the zero forcing number [10]. Being able to turn every vertex blue in a graph also models controlling the phase of electricity through an electrical network. In [11], Haynes et al. researched the problem of monitoring an electric power system by placing as few measurement devices as possible and the explicit connection with zero forcing was later made in [5]. For a more robust literature review see [9] and [8].

In light of these applications, there is concern about how a faulty vertex in a network disrupts the flow of information or the ability to control a network. A more recent variation on zero forcing was introduced to address these concerns, in [6] by Dillman and Kenter, called leaky forcing. In this new variation, the following question was explored. What if there is a leak in a system which prevents the zero forcing process from finishing? This paper will further explores this question, and also explore what properties make a network resistant to leaks.

The *leaky forcing color change rule* for a blue set $B \subseteq V(G)$ and a set of $L \subseteq V(G)$ states that a blue vertex u with exactly one white neighbor w can force w to blue if u is not in L. A *leak* in a graph G, a vertex in L from the previous sentence, is a vertex that is not allowed to perform a force. Note that if L is empty the leaky forcing color change rule is just the zero forcing color change rule. An *ℓ-leaky forcing set* for a graph G is a subset of initial blue vertices B such that if any ℓ vertices are chosen to be leaks (after B has been specified), then iteratively applying the color change rule will force every vertex in G. The *ℓ-leaky forcing number* of G is the size of a minimum ℓ-leaky forcing set and is denoted $\mathrm{Z}_{(\ell)}(G)$. Notice that a 0-leaky forcing set for a graph G is also a zero forcing set, and the 0-leaky forcing number is also the zero forcing number. This notation differs from the notation introduced by Dillman and Kenter in [6]. We use the term ℓ-leaky forcing instead of the abbreviated ℓ-forcing. This change is made in order to prevent confusion between leaky forcing and another generalization of zero forcing called k-forcing.

Intuitively, the more leaks there are in a graph, the larger the initial blue set must be. This is captured in the following proposition.

Proposition 2.1.1. *[6] For any graph G,*

$$\mathrm{Z}_{(0)}(G) \leq \mathrm{Z}_{(1)}(G) \leq \cdots \leq \mathrm{Z}_{(n)}(G).$$

A fundamental problem concerning ℓ-leaky forcing is determining when the inequalities in Proposition 2.1.1 are actually equalities (or strict inequalities). To this end, a graph G is said to be *ℓ-resilient* if $\mathrm{Z}_{(0)}(G) = \mathrm{Z}_{(\ell)}(G)$.

Section 2.2 presents results relating $(\ell-1)$-leaky forcing sets to ℓ-leaky forcing sets for general graphs. Theory is then developed for some general bounds on ℓ-leaky forcing numbers. Section 2.3 explores which structural properties a graph G must have for $\mathrm{Z}_{(0)}(G)$ to equal $\mathrm{Z}_{(\ell)}(G)$, and studies how vertex or edge deletion affect the ℓ-leaky forcing number. The ℓ-leaky forcing number for trees, supertriangles, and new bounds for the ℓ-leaky forcing number for grid graphs are given in Section 2.4.

2.2 Characterization of ℓ-leaky forcing sets

In general, the intuition behind ℓ-leaky forcing sets is that the set of initial blue vertices is robust. If ℓ vertices in the graph cannot perform a force, then the rest of the blue vertices can take over those forcing responsibilities. To help formalize this intuition, we will define a few concepts that let us analyze the zero forcing process. In general, let $B \subseteq V(G)$ be an initial set of blue vertices in G. If vertex u colors v blue, then we say that u forces v and denote it by $u \to v$.

The symbol $u \to v$ is called a force. A *chronological ordering of a set of forces F given a blue set B* is an ordering of the elements of F given by $\{u_i \to v_i\}_{i=1}^{|F|}$ such that $u_i \to v_i$ is allowed by the zero forcing color change rule when $B \cup \{v_1, \ldots, v_{i-1}\}$ is blue. A *forcing process F of B in G* is a set of forces such that there exists a chronological ordering of the forces in F that turns the whole graph turns blue. Intuitively, F represents the instructions for how B can force G blue, or provides a proof that B is a zero forcing set. Implicitly, F gives rise to discrete time steps in which sets of white vertices turn blue. Given an initial blue set B and forcing process F, we say that $B \subseteq B' \subseteq V(G)$ is *obtained from B using F* if B can color B' blue using only a subset of F.

The set $B^{[\infty]}$ is the set of blue vertices after the zero forcing rule has been exhaustively applied with B as an initial blue set. Furthermore, $B_L^{[\infty]}$ will be determined after a set of leaks L has been chosen. A sequence of forces $x_1 \to x_2, x_2 \to x_3, \ldots, x_{k-1} \to x_k$ is a *forcing chain* and will be abbreviated by $x_1 \to x_2 \to x_3 \to \cdots \to x_k$. A forcing chain is *maximal* in F (and implicitly given B) if x_1 is in B and x_k does not perform a force in F. Note that $x_1 = x_k$ is possible if $x_1 \in B$ and x_1 does not perform a force in F. Let $\mathcal{F}(B)$ denote the set of all possible forces given a vertex set B. That is $u \to v \in \mathcal{F}(B)$ if there exists a forcing process F of B in G that contains $u \to v$. Given this notation, B is an ℓ-leaky forcing set if for every $L \subseteq V(G)$ with $|L| = \ell$ there exists a forcing process F such that if $u \to v$, then $u \notin L$. This idea can be used to relate an ℓ-leaky forcing set to its set of possible forces $\mathcal{F}(B)$.

Lemma 2.2.1. *Let G be a graph. If B is an ℓ-leaky forcing set, then for all $v \in V(G) \setminus B$, there exists $x_1 \to v, x_2 \to v, \ldots, x_{\ell+1} \to v \in \mathcal{F}(B)$ with $x_i \neq x_j$, $i \neq j$.*

Proof. Suppose that there exists $v \in V(G) \setminus B$ such that v can be forced by at most $x_1, \ldots, x_k$ distinct vertices where $k \leq \ell$. Let $L = \{x_1, \ldots, x_k\}$. By construction, there does not exist a forcing process F of B that avoids L. In particular, if F is a forcing process of B with $x \to v \in F$, then $x \in L$. Therefore, B is not an ℓ-leaky forcing set. $\square$

Lemma 2.2.1 formalizes the intuition that an ℓ-leaky forcing set is robust enough to force every vertex despite ℓ vertices that cannot perform forces. In fact, the set of possible forces can be used to characterize 1-leaky forcing sets. Theorem 2.2.3 shows that the converse of Lemma 2.2.1 is true for 1-leaky forcing. The strategy for proving Theorem 2.2.3 is to construct a forcing process that avoids using particular vertices. This is the subject of Lemma 2.2.2.

Let B be a fixed set of blue vertices with forcing processes F and F'. The idea is to use forcing process F to obtain B' from B, and then continue forcing with process F'. To formalize this idea, suppose $S \subseteq V(G)$ and let

$$F(S) = \{x \to y \in F : y \notin S\}.$$

By extension,

$$F \setminus F(S) = \{x \to y \in F : y \in S\}.$$

The set $F(S)$ is to be understood as a forcing process that follows F after S is blue. Similarly, $F \setminus F(S)$ is the set of forces in F that force into S. Intuitively, if F is some forcing process for a set B, then $F(S)$ is the subset of F that is a forcing process for a blue set $S \supseteq B$. The following lemma proves that abandoning process F to follow process F' creates a new forcing process. The proof of Lemma 2.2.2 is summed up in the following diagram:

$$B \xrightarrow{F} B' \xrightarrow{F'} V(G).$$

The diagram is intended to be read as: obtain B' from B using forcing process F, then obtain $V(G)$ from B' using F'. The fact that going from B' to $V(G)$ actually only uses forces in $F'(B')$ is a technical detail and will generally be suppressed. Instead, we will think of F' as being a process of B' in G.

Lemma 2.2.2. *Let B be a zero forcing set in G with zero forcing processes F and F'. Then $(F \setminus F(B')) \cup F'(B')$ is a forcing process of B for any B' obtained from B using F.*

Proof. Let B' be some set of blue vertices obtained from B using F. Let $v_1 < v_2 < \cdots < v_n$ be an ordering of the vertices of G so that whenever v_i is blue for all $i < k$, then there exists

$j < k$ such that $v_j \to v_k \in F'$ which is allowed by the zero forcing color change rule. Since F' is a forcing process, such an order can be found by simply keeping track of the order in which F' turns vertices in G blue.

For the sake of contradiction, suppose that $F'(B')$ cannot force G blue given initial set B'. In particular, exhaustively apply forces in $F'(B')$ iteratively in order to obtain $B^* \neq V(G)$. This gives us the following set relations: $B \subseteq B' \subseteq B^* \subset V(G)$. Let k be the smallest index such that v_k is not in B^*. This implies that $v_i \in B'$ for all $i < k$. By construction of the ordering of the vertices of G, there exist a $j < k$ such that $v_j \to v_k \in F'$ that can be performed if B^* is blue. Since $v_k \notin B^*$, we know that $v_k \notin B'$. Therefore, $v_j \to v_k \in F'(B')$; which contradicts the fact that all the forces in $F'(B')$ have been exhaustively applied. □

In essence, Lemma 2.2.2 guarantees that two forcing processes for a set B can be combined. However, another interpretation is that we can switch to a new process at any point of an old process. In this way, sets of forces function as instructions that can be exchanged at any time. In the proof of Theorem 2.2.3, we use Lemma 2.2.2 to construct a forcing process that is not obstructed by an arbitrarily fixed leak.

Theorem 2.2.3. *A set B is a 1-leaky forcing set if and only if for all $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}(B)$ with $y \neq x$.*

Proof. If B is a 1-leaky forcing set, then Lemma 2.2.1 gives $x \to v, y \to v \in \mathcal{F}(B)$ with $y \neq x$ for all $v \in V(G) \setminus B$.

Now assume for all $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}(B)$ with $y \neq x$. Let z be a leak. It suffices to show that there exists a forcing process of B that does not contain a force originating at z. Let F be some forcing process of B in G. If z is the end of a forcing chain of F, then B can force G using F. Therefore, assume that z is not the end of a forcing chain in F. This implies that there exists v such that $z \to v \in F$. Let B' be a set of blue vertices obtained from B using F such that $z \to v$ is valid given B', but $v \notin B'$. By assumption, there exists y such that $y \to v \in \mathcal{F}(B)$ and $y \neq z$. Therefore, there exists a forcing process F' of B in G with

$y \to v$. Notice that $N[z] \setminus \{v\} \subseteq B'$. Thus, $z \to u \notin F'(B')$ for any $u \in V(G)$. Therefore, $(F \setminus F(B')) \cup F'(B')$ is a forcing process that does not use a force originating from z. □

A straightforward generalization of Theorem 2.2.3 is false. In particular, the converse of Lemma 2.2.1 does not hold by the following counterexample. Consider $G = K_{\ell+1} \square K_2$ where $\ell \geq 2$ with $V(G) = \{u_i : 1 \leq i \leq \ell+1\} \cup \{v_i : 1 \leq i \leq \ell+1\}$ with u_i's and v_i's both inducing cliques (see Figure 2.1). Let the blue set $B = \{u_i : 1 \leq i \leq \ell+1\}$ and notice that every vertex in $\{v_i : 1 \leq i \leq \ell+1\}$ can be forced in $\ell+1$ ways. However, setting $\{u_2, \ldots, u_{\ell+1}\}$ as leaks will prevent B from coloring the rest of the graph blue. Therefore, B is not an ℓ-leaky forcing set.

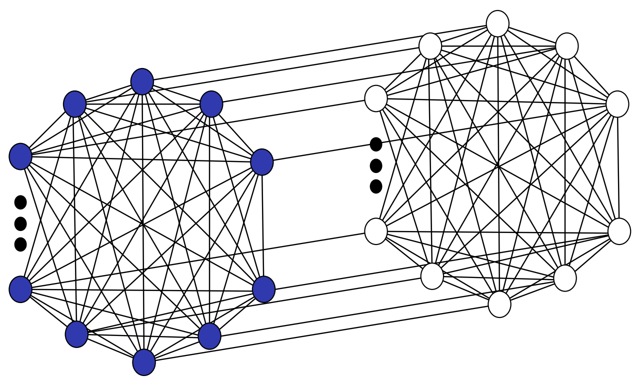

Figure 2.1 Counterexample to an attempted generalization of Theorem 2.2.3.

Theorem 2.2.5 requires new definitions. For a set of leaks L, let $\mathcal{F}_L(B)$ be the set of forces that are possible with initial blue set B and leaks L. In other words, $u \to v \in \mathcal{F}_L(B)$ implies $u \notin L$ and there exists a forcing process F such that $u \to v \in F$. Before we proceed with the proof of Theorem 2.2.5, we need to understand how a zero forcing set is disrupted by leaks.

Lemma 2.2.4. *If B is an $(\ell-1)$-leaky forcing set and L is a set of $k \geq \ell$ leaks, then $|L \setminus B_L^{[\infty]}| \leq k - \ell$.*

Proof. Assume that L is a set of $k \geq \ell$ leaks, and let $|L \setminus B_L^{[\infty]}| \geq k - \ell + 1$. Every vertex in $B_L^{[\infty]}$ has either 0, 1, or at least 2 white neighbors. If $v \in B_L^{[\infty]}$ such that v has one white neighbor, then $v \in L$. Since $|L| = |L \setminus B_L^{[\infty]}| + |L \cap B_L^{[\infty]}|$, it follows that $|L \cap B_L^{[\infty]}| \leq \ell - 1$. Notice that leaks in $L \setminus B_L^{[\infty]}$ did not change the zero forcing behavior of B. Therefore, $L' = L \cap B_L^{[\infty]}$ is a set of at most $\ell - 1$ leaks that prevents B from forcing G. Thus, B is not an $(\ell - 1)$-leaky forcing set. □

Notice that if $k = \ell$, then Lemma 2.2.4 says that an $(\ell - 1)$-leaky forcing set B will always be able to turn any set of ℓ leaks blue. Given an $(\ell - 1)$-leaky forcing set B and a set of ℓ leaks L, we can find a time at which every leak is blue. The proof of Theorem 2.2.3 singles out this time step, and argues that there is enough freedom in our choice of forces to proceed despite the leaks. The proof of Theorem 2.2.5 singles out a time when at least $\ell - 1$ leaks are blue, and recognizes that the constellation of blue and white vertices at this time is very similar to the situation covered in Theorem 2.2.3.

The following diagram roughly depicts how the first part of the proof of Theorem 2.2.5 goes:

$$B \xrightarrow{F} B' \supseteq B^* \xrightarrow{F'} V(G). \tag{2.1}$$

We never explicitly name F' in the proof. Instead, we use Lemma 2.2.4 to find B' and then whittle down G into a subgraph G^* such that $B^* \subseteq B'$ is a 1-leaky forcing set of G'. Using this fact, we implicitly find a forcing process F' of G^* by invoking Theorem 2.2.3. By construction, F' circumvents the only leak that is not blue in G^* and can be appended to F once B' is colored blue using Lemma 2.2.2.

Theorem 2.2.5. *A set B is an ℓ-leaky forcing set if and only if B is an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ leaks L and $v \in V(G) \setminus B$ there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Proof. Let B be an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ leaks L and $v \in V(G) \setminus B$ there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$. Let L' be a set of ℓ leaks.

Since B is an $(\ell - 1)$-leaky forcing set, by Lemma 2.2.4 we can apply forces until all leaks in L' are blue. Let the resulting set of blue vertices be B', let $L'' \subseteq L' \cap B'$ be a set of $\ell - 1$ leaks, and $B^* = B' \setminus L''$. Notice that if a blue vertex is a leak or has performed a force, then it can be safely deleted without altering the zero forcing behavior of the remaining blue vertices.

Consider $G^* = G - L''$. Since B is an $(\ell - 1)$-leaky forcing set, B^* is a zero forcing set of G^*. We know that for all $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_{L''}(B)$ in G. However, since vertices in L'' are not allowed to perform forces, it follows that x, y are not in L''. Thus, $x \to v, y \to v \in \mathcal{F}(B^*)$ in G^*, and B^* is a 1-leaky forcing set of G^* by Theorem 2.2.3. Therefore, B could force G despite L', and B is an ℓ-leaky forcing set of G. For a summary of this part of the proof, see diagram (2.1) above.

To prove the contrapositive of the forward direction, suppose that B is an $(\ell - 1)$-leaky forcing set with a set of $\ell - 1$ leaks L and a vertex $v \in V(G) \setminus B$ such that there does not exist $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$. Since B is an $(\ell - 1)$-leaky forcing set, there must exist exactly one force $x \to v \in \mathcal{F}_L(B)$. However, $L \cup \{x\}$ is a set of ℓ leaks that will prevent B from coloring v blue. Therefore, B is not an ℓ-leaky forcing set. □

Theorem 2.2.5 provides a way of analyzing when G is ℓ-resilient and more generally when $\mathrm{Z}_{(k)}(G) = \mathrm{Z}_{(\ell)}(G)$. In particular, we can determine if a k-leaky forcing set B is in fact an ℓ-leaky forcing set by repeatedly applying Theorem 2.2.5. In this way, Theorem 2.2.5 is a way to build up a characterization of ℓ-leaky forcing sets and which graphs are ℓ-resilient. To illustrate this point, notice that a zero forcing set B is a 2-leaky forcing set if and only if B satisfies conditions in Theorem 2.2.5 for $\ell = 1$ and $\ell = 2$.

Theorem 2.2.3 results in Corollary 2.2.7, but first we introduce some definitions. Let B be a zero forcing set of a graph G. A *reversal* of B given a forcing process F is the set of vertices of G that appear at the ends of maximal forcing chains given F. Equivalently, let the reversal of B given F be denoted by

$$R_F(B) = \{v \in V(G) : v \to y \notin F, \forall y \in V(G)\}.$$

Often, we do not specify a particular forcing process F and would be happy with any forcing process. In this case, we let $R(B)$ denote a reversal of B with the understanding that there is some underlying forcing process F which is unnamed and unspecified.

In [2], Barioli et al. proved the following theorem:

Theorem 2.2.6. *[2] If B is a zero forcing set of G, then so is any reversal of B.*

The proof of Theorem 2.2.6 shows that a reversal $R_F(B)$ is a zero forcing set by reversing the forces in F. That is, if F is a forcing process of B, then $F' = \{x \to y : y \to x \in F\}$ is a forcing process for $R_F(B)$.

Corollary 2.2.7. *For all graphs G, $\mathrm{Z}_{(1)}(G) \leq 2\,\mathrm{Z}_{(0)}(G)$. In particular, if B is a zero forcing set of G and $R(B)$ is a reversal, then $B \cup R(B)$ is a 1-leaky forcing set.*

Proof. Let B be a minimum zero forcing set of G, and let $R(B)$ denote a reversal of B. For all $v \in V(G) \setminus (B \cup R(B))$, there exists $x \neq y$ such that $x \to v, y \to v \in \mathcal{F}(B \cup R(B))$ by proof of Theorem 2.2.6. Therefore, by Theorem 2.2.3, $B \cup R(B)$ is a 1-leaky forcing set of G. Finally, notice that $|R(B)| = |B|$ to conclude that $\mathrm{Z}_{(1)}(G) \leq 2\,\mathrm{Z}_{(0)}(G)$. □

If equality in Corollary 2.2.7 holds for G, then for every minimum zero forcing set B, $v \in B$, and forcing process F, there exists $u \in V(G) \setminus B$ such that $v \to u \in F$. That is, every vertex in a minimum zero forcing set actually performs a force. This follows from the fact that if $v \in B \cap R(B)$, then v is the start and end of a chain in a zero forcing process.

One of the interpretations of a leak is that v is a leak if v must appear at the end of its zero forcing chain. Recall that a reversal of a zero forcing set B consists of the ends of the forcing chains of B given a process F. Therefore, if we have an ℓ-leaky forcing set B and L is a set of ℓ leaks, then $L \subseteq R(B)$. Furthermore, we can use an ℓ-leaky forcing set to build a zero forcing set around an arbitrary set of ℓ vertices by appealing to the reversal.

Lemma 2.2.8. *If B is an ℓ-leaky forcing set of G, then any ℓ vertices of G are in a zero forcing set of size $|B|$.*

Proof. Let L be an arbitrary set of ℓ leaks in G. Since B is an ℓ-leaky forcing set, there exists a forcing process F that colors G blue. Let $R_F(B)$ be the reversal of B given F. Notice that $L \subseteq R_F(B)$ since vertices in L do not perform forces but eventually become blue. Furthermore, $R_F(B)$ is a zero forcing set by Theorem 2.2.6. □

The idea in the proof of Lemma 2.2.8 is that given an ℓ-leaky forcing set B, we can cleverly pick leaks to guarantee certain graph properties. For Lemma 2.2.8, clever leak choice lets us find zero forcing sets that contain the leaks. The next theorem shows that clever leak choice leads to a general lower bound for ℓ-leaky forcing sets.

Theorem 2.2.9. *Given a graph G on n vertices with $2 \leq \ell \leq n-3$, we have $\ell + 2 \leq \mathrm{Z}_{(\ell)}(G)$.*

Proof. For the sake of contradiction, let $B = \{x_1, \ldots, x_k\}$ be an ℓ-leaky forcing set with $k \leq \ell+1$. If $k \leq \ell$, then B is a set of leaks which stops all forcing; therefore, it follows that $k = \ell + 1$. Furthermore, we can conclude that each $x_i \in B$ has exactly one neighbor in $V(G) \setminus B$ by setting $B \setminus \{x_i\}$ as leaks.

Set $\{x_2, \ldots, x_{\ell+1}\}$ as leaks to obtain a forcing chain $x_1 \to v_1 \to v_2 \to \ldots \to v_{n-\ell-1}$. This implies that $v_i v_j \in E(G)$ if and only if $j = i+1$.

Let v_i be the neighbor of x_2 in $V(G) \setminus B$. If $1 \leq i \leq n-\ell-2$, then setting $\{v_i, x_3, \ldots, x_{\ell+1}\}$ as the leaks does not allow v_{i+1} to be forced which contradicts the fact that B is an ℓ-leaky forcing set. Therefore, $i = n - \ell - 1$.

Let v_j be the neighbor of x_3 in $V(G) \setminus B$. Without loss of generality, suppose that $|j-1| \leq |n-\ell-1-j|$. Now set $L = \{v_j, x_2, x_4, \ldots x_{\ell+1}\}$ as leaks. Notice that v_{j+1} exists since $n - \ell - 1 \geq 2$. Furthermore, v_{j+1} will not get forced given L and B is not an ℓ-leaky forcing set. This is a contradiction. □

2.3 Resilience and structural results

In this section, structural properties of ℓ-resilient graphs are explored, and various differences between zero forcing sets and ℓ-leaky forcing sets are shown. There is a natural relationship between the structure of a graph, and the behavior of ℓ-leaky forcing sets of the graph. On one hand, the graph structure places restrictions on ℓ-leaky forcing sets. On the other hand, certain forcing behaviors set constraints on the underlying graph.

The following lemma is an example of how graph structures place restrictions on ℓ-leaky forcing sets.

Lemma 2.3.1. *[6] For any graph G, any ℓ-leaky forcing set will contain at least those vertices in G of degree ℓ or less.*

Proposition 2.3.2 shows that if B contains every vertex in G, then $\ell \geq \Delta(G)$.

Proposition 2.3.2. *For any graph G on n vertices, $\mathrm{Z}_{(\ell)}(G) = n$ if and only if $\Delta(G) \leq \ell$.*

Proof. If $\Delta(G) \leq \ell$, then $\mathrm{Z}_{(\ell)}(G) = n$ by Lemma 2.3.1. Therefore, we will focus on the other direction of the statement.

Let v be a vertex with $d(v) \geq \ell + 1$. Let $B = V(G) \setminus \{v\}$ be the blue set, and let L be a set of ℓ leaks. By the pigeonhole principle, there is a vertex $u \in N(v) \setminus L$. Therefore, u can force v. Thus, B is an ℓ-leaky forcing set and $\mathrm{Z}_{(\ell)}(G) < n$. □

A bound for the minimum degree of an ℓ-resilient graph can be given using the fact that a vertex with ℓ neighbors must be blue, and placing ℓ leaks does not prevent the graph from forcing.

Proposition 2.3.3. *If G is ℓ-resilient with no isolated vertices, then $\delta(G) \geq \ell + 1$.*

Proof. Let B be a minimum ℓ-leaky forcing set of G and assume that there exists a vertex x_0 with ℓ or fewer neighbors. By Lemma 2.3.1, $x_0 \in B$. There are two cases: x_0 has no white neighbors given B, or x_0 has at least one white neighbor given B.

Case 1: Assume that x_0 has no white neighbors given B. Let F be a forcing process for B given $N(x_0)$ as a set of leaks. Since no vertex in $N[x_0]$ performs a force in F, $F \cup \{x \to x_0\}$ for $x \in N(v)$ is a forcing process for $B \setminus \{x_0\}$. This contradicts the minimality of B.

Case 2: Assume that x_0 has k white neighbors with $1 \leq k \leq \ell$. Let the white neighbors of v be $\{x_1, \ldots, x_k\}$. Let F be a forcing process of B given leak set $\{x_0, \ldots, x_{k-1}\}$. By definition, no vertex in $\{x_0, \ldots, x_{k-1}\}$ performs a force in F. Let B' be the smallest blue set obtained from B using F such that $\{x_1, \ldots, x_k\}$ is blue. Let $F_1 = \{y_i \to z_i : 1 \leq i \leq r_1\} \subseteq F$ be the forces of F used to obtain B'. Notice that B' is an ℓ-leaky forcing set since it contains B which is an ℓ-leaky forcing set. Therefore, there exists a forcing process $F_2 = \{a_i \to b_i : 1 \leq i \leq r_2\}$ of B' such that no vertex in $\{x_1, \ldots, x_k\}$ perform a force (by setting $\{x_1, \ldots, x_k\}$ as the set of leaks). Furthermore, vertices in $u \in \{v, y_1, \ldots, y_{r_1}\}$ do not preform a force in F_2 since $N[u] \subseteq B'$. Therefore, $F_1 \cup F_2$ is a forcing process of B such that no vertex in $\{x_0, x_1, \ldots, x_k\}$ performs a force. Thus, $F_1 \cup F_2 \cup \{x_1 \to v\}$ is a forcing process of $B \setminus \{v\}$; which is a contradiction since B is a minimal zero forcing set. □

One way of reading Proposition 2.3.3 is that if G is ℓ-resilient, then G cannot contain a small edge-cut surrounding a single vertex. Rather than considering leaks that isolate just a single vertex, we can consider leaks that separate whole portions of a graph. In this sense, Theorem 2.3.4 follows the spirit of Proposition 2.3.3, and shows that if G is ℓ-resilient, then G does not contain a small or dense edge-cut.

Theorem 2.3.4. *If G is ℓ-resilient, then G does not have an edge-cut C such that the edges of C induce a subgraph of $K_{a,b}$ with $a + b \leq \ell$.*

Proof. Let B be a minimum ℓ-leaky forcing set, and let C be a minimum edge-cut such that the edges of C induce a subgraph of $K_{a,b}$. This implies that $|V(C)| \leq \ell$. Since C is a minimum edge-cut, there exists exactly two components G_1, G_2 in $G - C$. Let $L = V(C)$ be a set of leaks. Since B is an ℓ-leaky forcing set, there exists forcing process F of B such that B forces $V(G)$ despite L. Furthermore, $B' = B \cap V(G_1)$ is a zero forcing set of G_1 with forcing process $F' = F(V(G_2))$. Therefore, $R_{F'}(B')$ is a zero forcing set of G_1 that contains $V(C) \cap V(G_1)$ by Lemma 2.2.8. Let $x \in V(C) \cap V(G_1)$. Now $(B \cap V(G_2)) \cup (R_{F'}(B') \setminus \{x\})$ is a zero forcing set of

G. Since $|R_{F'}(B')| = |B'| = |B \cap V(G_1)|$,

$$|(B \cap V(G_2)) \cup (R_{F'}(B') \setminus \{x\})| = |(B \cap V(G_2))| + |B \cap V(G_1)| - 1 = |B| - 1.$$

Thus, G is not ℓ-resilient. □

Notice that a $K_{\ell+2}$ is an ℓ-resilient graph with an edge-cut that induces a subgraph of $K_{1,\ell+1}$. Therefore, Theorem 2.3.4 can potentially be improved.

In [2], the authors considered whether there is a graph G with a vertex v such that v is in every zero forcing set of G. It was shown that no such graph with non-trivial components exists.

Theorem 2.3.5. *[2] If G is a connected graph of order greater than one, then no vertex is in every minimum zero forcing set of G.*

Notice that by changing the question to reflect ℓ-leaky forcing, the answer is affirmative by either Lemma 2.3.1 or Proposition 2.3.2.

Furthermore, [2] considered whether there is a graph with a unique minimum zero forcing set. Since the reversal of a minimum zero forcing set is also a minimum zero forcing set by Theorem 2.2.6, no such non-empty graph exists.

Corollary 2.3.6. *[2] If G is a connected graph of order greater than one, then G does not have a unique minimum zero forcing set.*

However, there do exist graphs with unique minimum ℓ-leaky forcing sets with $\ell \geq 1$. To see this take the complete graph $K_{\ell+2}$ with ℓ leaves at every vertex. Let B be the set of leaves. By Lemma 2.3.1, B has to be in every minimum ℓ-leaky forcing set, and B is an ℓ-leaky forcing set. See Figure 2.2 for an example when $\ell = 1$.

The following theorem from [7] shows how removing an edge from a graph impacts the zero forcing number.

Theorem 2.3.7. *[7] For every graph G and every edge e in G,*

$$-1 \leq \mathrm{Z}(G) - \mathrm{Z}(G - e) \leq 1.$$

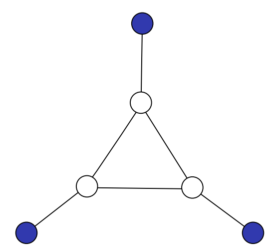

Figure 2.2 An example of a graph with a unique 1-leaky forcing set.

Proposition 2.3.8 investigates how edge-deletion affects the 1-leaky forcing number.

Theorem 2.3.8. *For every graph G and every edge e in G,*

$$-2 \leq \mathrm{Z}_{(1)}(G) - \mathrm{Z}_{(1)}(G-e).$$

Proof. Let B be a minimum 1-leaky forcing set and $e = vw$.

Case 1: Assume both $v, w \in B$. Notice that $\mathcal{F}(B)$ is the same given G and $G - e$, since $v \to w$ and $w \to v$ are not in $\mathcal{F}(B)$ given G. Therefore, B is a 1-leaky forcing set for $G - e$ by Theorem 2.2.3. Thus, $\mathrm{Z}_{(1)}(G-e) \leq \mathrm{Z}_{(1)}(G)$.

Case 2: Without loss of generality, assume $v \in B$ and $w \notin B$. Therefore, there exists $x_1, y_1 \in V(G)$ such that $x_1 \to w, y_1 \to w \in \mathcal{F}(B)$ with $x_1 \neq y_1$ by Theorem 2.2.3. There are two subcases to consider.

Subcase 1: Assume $v \notin \{x_1, y_1\}$. Then B is a 1-leaky forcing set for $G - e$ since every vertex in $V(G-e) \setminus B$ can still be forced in two ways. Thus, $\mathrm{Z}_{(1)}(G-e) \leq \mathrm{Z}_{(1)}(G)$.

Subcase 2: Without loss of generality, assume $v = x_1$. If F is a forcing process of B in G, then $F' = F \setminus \{v \to w\}$ is a forcing process of $B \cup \{w\}$ in $G - e$. Then $B' = B \cup \{w\}$ is a 1-leaky forcing set for $G - e$. Thus, $\mathrm{Z}_{(1)}(G-e) \leq \mathrm{Z}_{(1)}(G) + 1$.

Case 3: Assume $v, w \notin B$. Therefore, there exists $x_1, y_1, x_2, y_2 \in V(G)$ such that $x_1 \to w, y_1 \to w, x_2 \to v, y_2 \to v \in \mathcal{F}(B)$ with $x_1 \neq y_1$ and $x_2 \neq y_2$ by Theorem 2.2.3. There are three subcases to consider.

Subcase 1: Assume $v \notin \{x_1, y_1\}$ and $w \notin \{x_2, y_2\}$. Then B is a 1-leaky forcing set for $G - e$ since v and w (or any other white vertex) can be forced in two ways without e. Thus, $\mathrm{Z}_{(1)}(G - e) \leq \mathrm{Z}_{(1)}(G)$.

Subcase 2: Without loss of generality, assume $v = x_1$ and $w \notin \{x_2, y_2\}$. By deleting e, w can only be forced in one way using B. Therefore, $B' = B \cup \{w\}$ is a 1-leaky forcing set for $G - e$ by a similar argument to subcase 2 of case 2. Thus, $\mathrm{Z}_{(1)}(G - e) \leq \mathrm{Z}_{(1)}(G) + 1$.

Subcase 3: Without loss of generality, assume $v = x_1$ and $w = x_2$. By deleting e, both w and v might only be forced in one way. If F is a forcing process of B in G, then $F' = F \setminus \{v \to w, w \to v\}$ is a forcing process of $B' = B \cup \{v, w\}$ in $G - e$. Therefore, B' is a 1-leaky forcing set for $G - e$. Thus, $\mathrm{Z}_{(1)}(G - e) \leq \mathrm{Z}_{(1)}(G) + 2$. □

Figure 2.3 shows that the inequalities in the proof of Theorem 2.3.8 can be tight, and the examples in Figure 2.4 give some indication that $\mathrm{Z}_{(1)}(G) - \mathrm{Z}_{(1)}(G - e)$ is bounded above by 2.

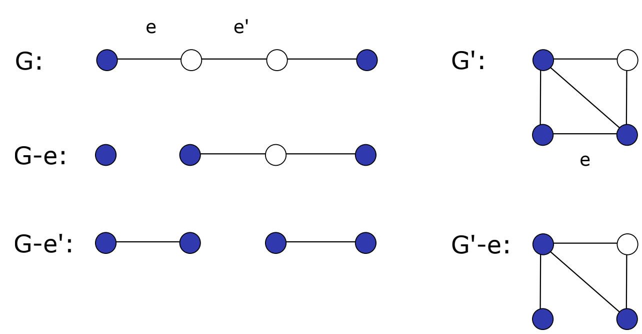

Figure 2.3 Examples where deleting an edge increases the 1-leaky number by zero (G' to $G' - e$), one (G to $G - e$), and two (G to $G - e'$).

Furthermore, it has been shown the behavior of vertex deletion can be used to extract some information about zero forcing sets.

Theorem 2.3.9. *[7] Let G be a graph and $v \in V(G)$. If $\mathrm{Z}(G) - \mathrm{Z}(G - v) = -1$, then $v \notin B$ for all minimum zero forcing sets B of G.*

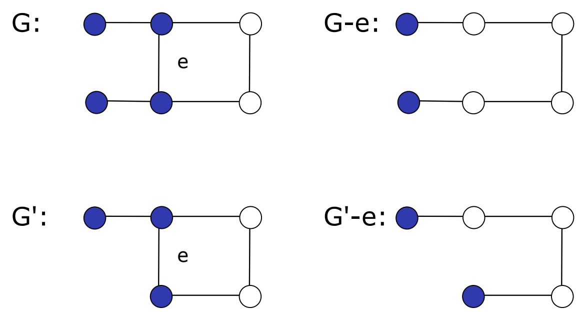

Figure 2.4 Examples where deleting an edge decreases the 1-leaky forcing number by two (G to $G-e$), and one (G' to $G'-e$).

Recall that by Lemma 2.2.8, if G is an ℓ-resilient graph, any ℓ vertices are in a minimum zero forcing set. The following corollary, relates the behavior of vertex deletion with respect to the zero forcing number and ℓ-resilience. Furthermore, Corollary 2.3.10 gives a simple way of determining if a graph is not ℓ-resilient.

Corollary 2.3.10. *If G is ℓ-resilient, then $\mathrm{Z}(G) - \mathrm{Z}(G-v) \neq -1$ for all $v \in V(G)$. Equivalently, if $\mathrm{Z}(G) - \mathrm{Z}(G-v) = -1$ for some $v \in V(G)$, then G is not ℓ-resilient.*

Proof. Let G be an ℓ-resilient graph. By Proposition 2.1.1, G is a 1-resilient graph and by Lemma 2.2.8, every vertex is in some minimum zero forcing set. Hence, by the contrapositive of Theorem 2.3.9, $\mathrm{Z}(G) - \mathrm{Z}(G-v) \neq -1$ for all $v \in V(G)$. □

The following example shows that unlike edge deletion, given a vertex v in a graph G, $\mathrm{Z}_{(1)}(G) - \mathrm{Z}_{(1)}(G-v)$ cannot be bounded below. Consider the graph G constructed by identifying an endpoint of k paths of order $n \geq 3$. Call this identified vertex v. Since G is a tree with k leaves, $\mathrm{Z}_{(1)}(G) = k$. Notice that $G-v$ consists of k disconnected paths of length $n-1$. Therefore, $\mathrm{Z}_{(1)}(G-v) = 2k$ and $\mathrm{Z}_{(1)}(G) - \mathrm{Z}_{(1)}(G-v) = -k$. Recall that k is arbitrary, showing that $\mathrm{Z}_{(1)}(G) - \mathrm{Z}_{(1)}(G-v)$ is unbounded.

2.4 The ℓ-leaky forcing number for certain graph families

Section 2.4 applies theory developed in Sections 2.2 and 2.3 to resolve the ℓ-leaky forcing numbers for various graph families. Recall that Section 2.2 ended with Theorem 2.2.9 which states that $\ell + 2 \leq Z_{(\ell)}(G)$ as long as $2 \leq \ell \leq |V(G)| - 3$. Proposition 2.4.1 applies this bound to restrict the order of a $(k-1)$-resilient graph with zero forcing number k.

Proposition 2.4.1. *If $Z_{(0)}(G) = Z_{(k-1)}(G) = k$ and $|V(G)| \geq 2$, then G is the complete graph K_{k+1}, $K_a \cup \overline{K}_{k+1-a}$ with $a \geq 2$, or $\overline{K}_k$.*

Proof. If $G = \overline{K}_k$, then we are done. Therefore, let G be a graph on n vertices with at least one edge. By Theorem 2.2.9, $n - 2 \leq k - 1$, implying that $n \leq k + 1$. Furthermore, if $n \leq k$, then $Z_{(0)}(G) \leq k - 1$. Thus, $n = k + 1$. By Proposition 2.2 in [12], K_{k+1} is the only connected graph on $k + 1$ vertices with zero forcing number equal to k. Therefore, we will assume that G is disconnected. Notice that G cannot have more than one nontrivial component without contradicting the zero forcing number. Let C be the non-trivial component of G. If $Z_{(0)}(C) < |V(C)| - 1$, then we have a contradiction. Therefore, C is a clique by Proposition 2.2 in [12] and the proof is complete. □

The proof idea from Theorem 2.2.9 can also be applied for the case where $\ell = 1$. However, here we find that it is possible for $Z_{(1)}(G) \leq 2$. In particular, we can combine the idea of clever leak placement with Proposition 2.3.3 to determine which graphs have 1-leaky forcing number equal to 2. Note that Dillman and Kenter found that paths and cycles have 1-leaky forcing number equal to 2 [6].

Theorem 2.4.2. *Let G be a graph on $n \geq 2$ vertices with no isolated vertices. Then, $Z_{(1)}(G) = 2$ if and only if $G = C_n$ with $Z_{(0)}(G) = 2$ or $G = P_n$ with $Z_{(0)}(G) = 1$.*

Proof. The backward direction of the statement is proven in [6], so assume that $Z_{(0)}(G) = Z_{(1)}(G) = 2$. Let $\{x, y\}$ be a 1-leaky forcing set of G. Since $Z_{(0)}(G) = Z_{(1)}(G)$, we can conclude that $\delta(G) \geq 2$ by Proposition 2.3.3. Therefore, $xy \in E(G)$ and $d(x), d(y) = 2$. Set $L = \{y\}$ as a leak to obtain a forcing chain $x \to v_1 \to v_2 \to \ldots \to v_{n-2}$. This implies that $v_iv_j \in E(G)$ if and only if $j = i+1$. Note that y is adjacent to exactly one vertex v_i with $1 \le i \le n-2$. If $i < n-2$, then set $L = \{v_i\}$ as a leak. In this case, $\{x, y\}$ cannot force v_{n-2}, which is a contradiction. Therefore, $i = n-2$ and G is a cycle.

If $\mathrm{Z}_{(0)}(G) = 1$, then G is a path by a result in [12]. □

By considering all the cases where $\mathrm{Z}_{(1)}(G) = 2$ and noting that $\mathrm{Z}_{(1)}(G) > 1$, we can conclude that the general lower bound in Theorem 2.2.9 extends to $\ell = 0, 1$ provided that we rule out paths and cycles.

Corollary 2.4.3. *If G is a graph of order n with no isolated vertices, G is not P_n or C_n, and $0 \le \ell \le n-3$, then $\ell + 2 \le \mathrm{Z}_{(\ell)}(G)$.*

Lemma 2.3.1 states that any ℓ-leaky forcing set must contain all vertices of degree ℓ or less. It is interesting to consider under which conditions

$$B = \{v \in V(G) : d(v) \le \ell\}$$

is an ℓ-leaky forcing set of G. Though we do not have a complete answer to this question, Theorem 2.4.4 confirms that trees belong to this class of graphs. Note that the case where $\ell = 1$ of Theorem 2.4.4 was proven in [6].

Theorem 2.4.4. *If T is a tree and B is the set of vertices in T with degree at most ℓ where $\ell \ge 1$, then $\mathrm{Z}_{(\ell)}(T) = |B|$.*

Proof. Let L be a set of ℓ leaks and note that Lemma 2.3.1 implies $\mathrm{Z}_{(\ell)}(T) \ge |B|$. For the sake of contradiction, suppose that $B_L^{[\infty]} \ne V(T)$. Obtain the graph H by deleting all blue vertices in $B_L^{[\infty]}$ with at most one white neighbor from T. Notice that H is a forest. Let T' be a component of H.

Suppose that T' is an isolated vertex, v. Then v has at least $\ell + 1$ blue neighbors in T, each of which has v as its only white neighbor. Therefore, $N_T(v) \subseteq L$ which is a contradiction. Thus, we may assume that T' is a tree on at least two vertices.

Notice that all of the leaves of T' are white. Furthermore, T' has at least two leaves u and v. Therefore, u has at least ℓ blue neighbors in T each of which has u as its only white neighbor. The same holds for v. It follows that $(N_T(u) \cup N_T(v)) \cap B_L^{[\infty]} \subseteq L$, which is a contradiction. Thus, $B_L^{[\infty]} = V(T)$ and B is an ℓ-leaky forcing set. □

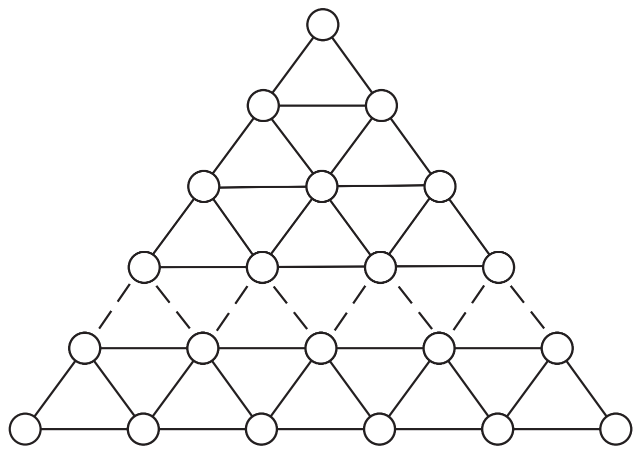

Figure 2.5 The supertriangle, T_n.

The n^{th} *supertriangle*, T_n, is an equilateral triangular grid with n vertices on each side. Supertriangles are another example that the implied lower bound in Lemma 2.3.1 can be tight. In particular, Theorem 2.4.5 shows that if $\ell = 4, 5$, then the vertices of a T_n with degree less than or equal to ℓ do form an ℓ-leaky forcing set. Notice that the supertriangle has three sides and a natural orientation. Let the sides be given by S_0, R_0, M_0 where S_0 is the left side, R_0 is the right side, and M_0 is the bottom side in Figure 2.5. Furthermore, let S_i be the left side of $T_n - \bigcup_{j<i} S_j$. In this sense, we can consider the left diagonal "rows" of T_n indexed from the left. Define R_i and M_i similarly. Furthermore, use the depiction of the supertriangle in Figure 2.5 to partition the edges of T_n into three sets: the set of horizontal edges H, the set of edges pointing from the upper left corner of the page to the lower right corner of the page SE, and the edges pointing from the lower left corner of the page to the upper right corner of the page NE.

Theorem 2.4.5. *For T_n,*

1. $\mathrm{Z}_{(0)}(T_n) = \mathrm{Z}_{(1)}(T_n) = n$,
2. $\mathrm{Z}_{(2)}(T_n) \leq \mathrm{Z}_{(3)}(T_n) \leq 2n - 1$, *and*
3. $\mathrm{Z}_{(4)}(T_n) = \mathrm{Z}_{(5)}(T_n) = 3n - 3$.

We will refer to the parts of Theorem 2.4.5 as claims.

Proof. Claim 1. In [1], it was shown that $\mathrm{Z}_{(0)}(T_n) = n$ by taking S_0, M_0, or R_0 as a blue set of T_n. We will show that $\mathrm{Z}_{(1)}(T_n) = n$ by finding a zero forcing set of size n that forces every vertex in two ways. Let $B = M_0$. Notice that vertices in M_1 can be forced by two distinct vertices in M_0 along NE edges or along SE edges. That is, every vertex in M_1 can be forced in two ways. By way of induction, vertices in M_{i+1} can be forced by two distinct vertices in M_i along NE edges or along SE edges. Therefore, B is a 1-leaky forcing set by Theorem 2.2.3. Recall that $\mathrm{Z}_{(0)}(T_n) \leq \mathrm{Z}_{(1)}(T_n)$ by Proposition 2.1.1. Thus, $\mathrm{Z}_{(1)}(T_n) = n$.

Claim 2. Let $B = S_0 \cup R_0$. Clearly, B is a 1-leaky forcing set. Let $x \in V(T_n)$ be a leak. Notice that $x \in S_i \cap R_j$ for some i and j. The initial blue set B can color vertices $B' = \bigcup_{\substack{y \leq i \\ z \leq j}} S_y \cup R_z$ blue in two ways without being inhibited by the leak x. Notice that B' is a zero forcing set of T_n such that x does not have any white neighbors and every remaining white vertex can be forced in two ways. Therefore, by Theorem 2.2.5, B is a 2-leaky forcing set and $\mathrm{Z}_{(2)}(T_n) \leq 2n - 1$.

Now let $x_1, x_2 \in V(T_n)$ be a pair of distinct leaks. Let i be the smallest index such that S_i contains a leak. Similarly, let j be the smallest index such that R_j contains a leak. The initial blue set B can color vertices $B' = \bigcup_{\substack{y \leq i \\ z \leq j}} S_y \cup R_z$ blue in two ways without being inhibited by x_1, x_2. There are two cases: either $S_i \cup R_j$ contains both x_1 and x_2, or $S_i \cup R_j$ contains only x_1 (without loss of generality).

Case 1: Assume that $S_i \cup R_j$ contains both x_1 and x_2. Let $S_i \cap R_j = \{x\}$. Notice that $S_i \setminus \{x\}$ and $R_j \setminus \{x\}$ each contain at most one leak. Therefore, S_i can force the remaining white vertices using only edges in SE and H (where forces using edges in H go from left to right). Similarly, R_j can force the remaining white vertices using only edges in NE and H (where forces using edges in H go from right to left). Thus, B can force every vertex in $V(T_n) \setminus B$ in two ways despite leaks x_1 and x_2.

Case 2: Assume that $S_i \cup R_j$ contains only x_1 (without loss of generality). Therefore, $S_i \cap R_j = \{x_1\}$ and $N(x_1)$ is blue. Since B' is a 2-leaky forcing set of T_n, we know that every vertex $v \in V(T_n) \setminus B'$ can be forced in two ways by Theorem 2.2.5 despite the leak x_2.

In either case, B can force all white vertices in two ways despite leaks x_1 and x_2. Therefore, B is a 3-leaky forcing set by Theorem 2.2.5. Thus, $\mathrm{Z}_{(2)}(T_n) \leq \mathrm{Z}_{(3)}(T_n) \leq 2n-1$.

Claim 3. Since S_0, R_0, and M_0 contain all the vertices of T_n with degree 4 or less, we know that $3n-3 \leq \mathrm{Z}_{(4)}(T_n)$ by Lemma 2.3.1. We will proceed by showing that $B = S_0 \cup R_0 \cup M_0$ is a 4-leaky forcing set and then show B is a 5-leaky forcing set.

Clearly, B is a 3-leaky forcing set. Let L be a set of three leaks. Let i, j, k be the smallest indices such that each of S_i, R_j, M_k contain leaks. Notice that B can force the vertices in

$$B' = \bigcup_{\substack{x \leq i \\ y \leq j \\ z \leq k}} S_x \cup R_y \cup M_z$$

in two ways. There are two cases: $S_i \cup R_j \cup M_k$ contains exactly three leaks, or $S_i \cup R_j \cup M_k$ contains exactly two leaks. In any case, $S_i \setminus (R_j \cup M_k)$ contains at most one leak and $R_j \setminus (S_i \cup M_k)$ contains at most one leak (without loss of generality).

Case 1.1: If $S_i \cup R_j \cup M_k$ contains three leaks, then the remaining vertices can be forced in two ways despite the leaks in L as discussed in Case 1 in the argument for claim 2.

Case 2.1: If $S_i \cup R_j \cup M_k$ contains two leaks, then let $i' > i$ and $j' > j$ be the smallest indices such that $S_{i'}$ and $R_{j'}$ contain a leak. As we have seen in Case 2 of the argument for claim 2, S_i and R_j can force the vertices in $B'' = \bigcup_{\substack{i < y \leq i' \\ j < z \leq j'}} S_y \cup R_z$ in two ways despite the leaks in L. Furthermore, $B' \cup B''$ will force the remaining white vertices in two ways despite leaks in L. In either case, B is a 4-leaky forcing set by Theorem 2.2.5.

Let L be a set of four leaks. Let i, j, k be the smallest indices such that S_i, R_j, M_k contain leaks. Notice that B can force the vertices

$$B' = \bigcup_{\substack{x \leq i \\ y \leq j \\ z \leq k}} S_x \cup R_y \cup M_z$$

in two ways. There are three cases: $S_i \cup R_j \cup M_k$ contains either two, three, or four leaks. In any case, without loss of generality, $S_i \setminus (R_j \cup M_k)$ and $R_j \setminus (S_i \cup M_k)$ contain at most one leak each.

Case 1.2: If $S_i \cup R_j \cup M_k$ contains four leaks, then S_i and R_j can force every remaining white vertex despite the leaks in L in two ways as discussed in Case 1 in the argument of claim 2.

Case 2.2: Assume that $S_i \cup R_j \cup M_k$ contains three leaks. Following the argument in Case 2.1 of claim 3, we see that B' can force the remaining white vertices in two ways despite the leaks in L.

Case 3.2: Assume that $S_i \cup R_j \cup M_k$ contains two leaks. Let $i' > i$, $j' > j$, and $k' > k$ be the smallest indices such that $S_{i'}$, $R_{j'}$, and $M_{k'}$ contain a leak. In this is situation, $S_{i'} \cup R_{j'} \cup M_{k'}$ necessarily contains two leaks. Let

$$B'' = \bigcup_{\substack{i < x \leq i' \\ j < y \leq j' \\ k < z \leq k'}} S_x \cup R_y \cup M_z$$

and notice that B' can force each vertex in B'' in two ways despite the leaks in L. Finally, $B' \cup B''$ can force the remaining white vertices as discussed in Case 1 in the argument for claim 2.

In any case, B can force each white vertex in two ways despite the leaks in L. Therefore, B is a 5-leaky forcing set by Theorem 2.2.5. Thus, $\mathrm{Z}_{(4)}(T_n) = \mathrm{Z}_{(5)}(T_n) = 3n - 3$. □

A *rectangular grid graph* is the Cartesian product $P_n \square P_m$. Let $V(P_n \square P_m) = \{(x, y) : 1 \leq x \leq n, 1 \leq y \leq m\}$ and $(x, y)(a, b) \in E(G)$ if and only if $x = a$ and $y \pm 1 = b$ or $y = b$ and $x \pm 1 = a$. We say that (x, y) is the vertex in row x and column y. The motivation for studying these graphs is due to a question posed by Dillman and Kenter [6]. However, the grid graph has also been studied extensively in more general zero forcing contexts [1].

Question 2.4.6. *[6] If $1 \leq n \leq m$, is there an $n \geq 10$, such that $\mathrm{Z}_{(1)}(P_n \square P_m) > \min\{2n, m\}$?*

In pursuit of answering Question 2.4.6, Dillman and Kenter established several upper bounds on $\mathrm{Z}_{(1)}(P_n \square P_m)$. The upper bound arguments in [6] are based on describing a set of initially colored vertices and doing case analysis to show the set is a 1-leaky forcing set, e.g. the proof of Theorem 2.4.7 uses this technique. A generalization of the 1-leaky forcing set used in Theorem 2.4.7 is used to establish Theorem 2.4.10 by applying Theorem 2.2.3. This shows that the answer Question 2.4.6 is “No.”

Theorem 2.4.7. *[6] If $1 \leq n \leq m$ and $n \leq \left\lfloor \frac{m}{2} \right\rfloor + 2$, then $\mathrm{Z}_{(1)}(P_n \square P_m) \leq m$.*

Theorem 2.4.7, Propositions 2.4.8, and 2.4.9 allow the focus of the proof of Theorem 2.4.10 to be when $6 \leq n < m$. Note that the bound in Proposition 2.4.8 can also be achieved using Corollary 2.2.7.

Proposition 2.4.8. *[6] If $1 \leq n \leq m$, then $\mathrm{Z}_{(1)}(P_n \square P_m) \leq 2n$.*

Proposition 2.4.9. *[6] If $1 \leq n$, then $\mathrm{Z}_{(1)}(P_n \square P_n) = n$.*

The argument for Theorem 2.4.10 describes a zero forcing set B with m vertices such that each white vertex can be forced in two different ways. The next two definitions help describe a set of forcing chains of B. Define the set of *boundary* vertices of $P_n \square P_m$, denoted $\mathcal{B}_0$, as the set of vertices of degree three or less. Define the set of y-*interior* vertices of $P_n \square P_m$, denoted $\mathcal{B}_y$, as the set of vertices whose distance to a nearest boundary vertex is y for $1 \leq y \leq \lceil n/2 \rceil - 1$. In addition, define $\mathcal{C}_y = (P_n \square P_m)[\mathcal{B}_y]$. Note that $\mathcal{C}_y$ is an induced cycle except for the largest value of y. See Figure 2.6 for an example of a decomposition of $P_n \square P_m$ into $\mathcal{C}_y$. We say that a graph H contains a forcing chain C if $V(C) \subseteq V(H)$ and $x \to y \in C$ implies that $xy \in E(H)$. Similarly, we say a forcing chain C contains a graph H if $V(H) \subseteq V(C)$ and $xy \in E(H)$ implies $x \to y \in C$. The purpose of this language is to describe the relative layout of forcing chains in a graph.

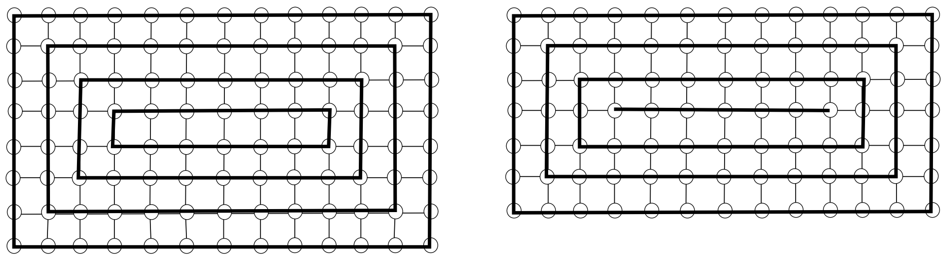

Figure 2.6 Decompositions of grid graphs into $\mathcal{C}_y$ when n is even (left image) and when n is odd (right image).

Theorem 2.4.10. *If* $1 \leq n \leq m$, *then* $\mathrm{Z}_{(1)}(P_n \Box P_m) \leq \min\{2n, m\}$.

Proof. If $n = m$, then $\mathrm{Z}_{(1)}(P_n \Box P_m) = n$, by Proposition 2.4.9, so it will be assumed that $n < m$. If $n \in \{1, 2, 3, 4, 5\}$, Theorem 2.4.7 and Proposition 2.4.8 give the desired result. This allows the focus of the remainder of the proof to be the cases when $6 \leq n < m$.

Let x be the smallest integer such that $n \leq 2x + 3 \leq m$. Since $n + 1 \leq m$, such an integer will always exist. Define $k = \left\lfloor \frac{m - 2x}{2} \right\rfloor$, and $j = m - 2x - k$. Note that $j \geq 1$ and $k \geq 1$ by construction. Let

$$B_1 = \{(i, x + k), (x + 1, x + p) \,:\, 1 \leq i \leq x,\, 1 \leq p \leq k\},$$

$$B_2 = \{(i, x + k + 1), (x + 1, x + k + p) \,:\, 1 \leq i \leq x, 1 \leq p \leq j\},$$

and $B = B_1 \cup B_2$. Note that $|B| = m$.

The argument will show that B_1 or B_2 can each force the entire graph without vertices in B_2 or B_1, respectively, performing a force. More importantly, the description of the forcing chains shows that every vertex in $V(P_n \Box P_m) \setminus B$ can be forced in two different ways. In particular, we will find two disjoint forcing processes given in Figure 2.7.

Case 1: Assume that n is even. If vertices in B_1 do not perform forces, then there is a forcing process that creates forcing chains originating with vertices in B_2. These forcing chains can be found in $\mathcal{C}_y$. More particularly, the forcing chain in $\mathcal{C}_{i-1}$ includes $(i, x+k+1), (i, x+k+2)$ and $(i, x + k - 1)$ and is oriented in a clockwise direction (see Figure ??) for $1 \leq i \leq x$, and the final forcing chain in $\mathcal{C}_{n/2-1}$ includes $(x + 1, x + k + j), (x + 2, x + k + j)$, and $(x + 2, x + 1)$.

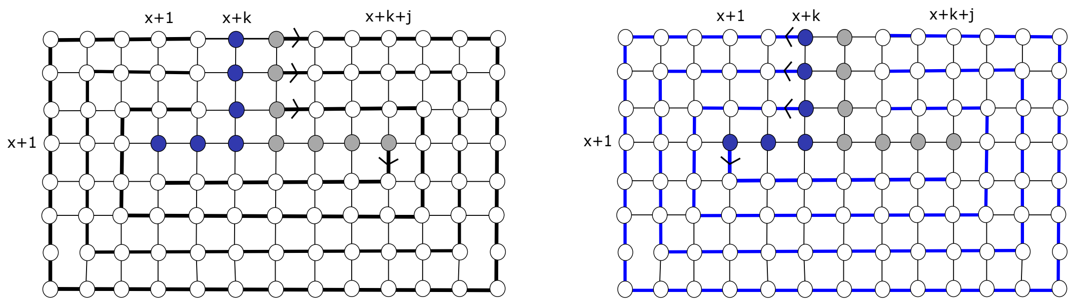

Figure 2.7 Forcing chains originating from B_2 (grey vertices, left image) and from B_1 (dark vertices, right image) in $P_8\Box P_{13}$.

If vertices in B_2 do not perform forces, then the forcing chains originating with vertices in B_1 include $(i, x+k)$, $(i, x+k-1)$, and $(i, x+k+2)$ and are oriented in a counter-clockwise direction for $1 \le i \le x$, and the final forcing chain includes $(x+1, x+1), (x+2, x+1)$ and $(x+2, x+j+k)$.

Case 2: Assume that n is odd. Define $p = \lceil n/2 \rceil$. If vertices in B_1 or B_2 do not perform forces, then the description of the forcing chains that reside in $\mathcal{C}_y$ for $0 \le y \le p-3$ are the same as the first case. If vertices in B_1 do not perform forces, then the forcing chain originating from vertices in B_2 that is in $\mathcal{C}_{p-2}$ includes $(x+1, x+k+j), (x+3, x+k+j)$, $(x+3, x+1)$, and $(x+2, x+1)$ and is oriented in a clockwise direction. The final forcing chain is not in $\mathcal{C}_{p-1}$, but contains $\mathcal{C}_{p-1}$. In particular, the final forcing chain starts at $(x+1, x+k+j-1)$, which is in B_1 since $j \ge 2$ when n is odd. The forcing chain resumes so that it includes $(x+2, x+k+j-1)$ and continues to $(x+2, x+2)$. If vertices in B_2 do not perform forces, then the forcing chains we just described are augmented and reversed as in case 1.

The two cases establish that for each $v \in V(P_n \Box P_m) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}(B)$ with $y \ne x$. Thus, Theorem 2.2.3 gives that B is a 1-leaky forcing set and $\mathrm{Z}_{(1)}(P_n \Box P_m) \le m$. Combining this with Proposition 2.4.8 cited earlier from [6] establishes that $\mathrm{Z}_{(1)}(P_n \Box P_m) \le \min\{2n, m\}$. □

As remarked in [6], it is believed that $\mathrm{Z}_{(1)}(P_n \Box P_m) = \min\{2n, m\}$. Theorem 2.4.11 gives some evidence that this is true.

Theorem 2.4.11. *If $G = P_n \Box P_m$ where $m \geq 2n^2$, then $\mathrm{Z}_{(1)}(G) = 2n$.*

Proof. Recall that the columns of G are indexed by $[m]$ and the rows of G are indexed by $[n]$. Let $G[i, j]$ be the subgraph of G induced by the vertices in columns i through j.

Assume that $B \subseteq V(G)$ is a set of blue vertices such that $|B| \leq 2n - 1$. By the pigeon hole principle, there exists k such that $M = G[(k-1)n + 1, kn]$ does not contain a blue vertex. Notice that if $k = 1$ or $kn = m$, then setting $(1, n+1)$ or $(1, m-n)$ as a leak, respectively, shows that B is not a 1-leaky forcing set. Therefore, we will assume that $k \neq 1$ and $kn \neq m$.

Let $S = G[1, (k-1)n]$ and $R = G[kn + 1, m]$. Without loss of generality, assume that $|V(S) \cap B| \leq n - 1$. Consequently, $|V(R) \cap B| \geq n$. Notice that $V(S) \cap B$ is not a zero forcing set of S, since $\mathrm{Z}_{(0)}(S) = n$. Exhaustively apply the zero forcing rule until the only remaining valid forces are $x \to y$ with $y \in M$. Notice that at this point in the process $G[(k-1)n - 1, (k-1)n]$ must contain a white vertex, otherwise, $V(S) \cap B$ could force all of S. This implies that there exists a vertex $(i, (k-1)n)$ that cannot perform a force either because it is white or because it has a white neighbor in $G[(k-1)n - 1, (k-1)n]$. Since $(i, (k-1)n)$ has a white neighbor that $V(S) \cap B$ cannot force, $(i, (k-1)n)$ essentially acts like a leak (it will never perform a force without help from a forcing chain originating in $V(R) \cap B$). Without loss of generality, assume that $i \leq n/2$. Therefore, $V(S) \cap B$ cannot color all of M blue. Furthermore, notice that any leak in column $kn + 1$ shows that $V(R) \cap B$ cannot color M blue. In particular, assume that $(n + 1 - i, kn + 1)$ is a leak.

Let B'_S be the set of blue vertices in M obtained by allowing $V(S) \cap B$ to force until failure. Similarly, let B'_R be the set of blue vertices in M obtained by allowing $V(R) \cap B$ to force until failure given a leak at $(n + 1 - i, kn + 1)$. Recall that neither $V(S) \cap B$ nor $V(R) \cap B$ can color all of the vertices in M blue. Therefore, B is a 1-leaky forcing set only if there exists $u \in B'_S$ and $v \in B'_R$ such that u is adjacent to v in G.

Suppose that $u \in B'_S$ and $v \in B'_R$. There are three cases: (1) u and v are in the same row, (2) u and v are in the same column, or (3) u and v are not adjacent. We will show that (1) or (2) implies (3). For some intuition, consult Figure 2.8.

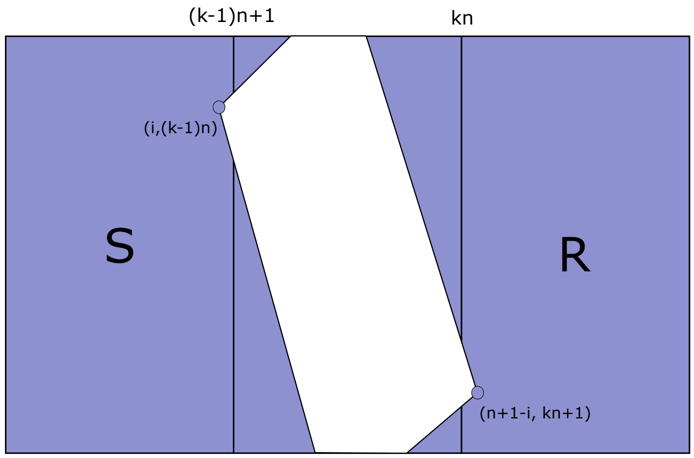

Figure 2.8 Relative layout of blue vertices and white vertices (not to scale) after forces have been exhaustively applied in the proof of Theorem 2.4.11.

Case 1: Suppose that u and v are in row j. In particular, let $u = (j, u')$ and $v = (j, v')$. Since $(i, (k-1)n)$ cannot perform any forces given $V(S) \cap B$, we know that $u' < (k-1)n + |i-j|$. Similarly, the leak at $(n+1-i, kn+1)$ implies that $kn - |n+1-i-j| < v'$.

However,

$$|n+1-i-j| + |i-j| \leq n$$

implies that

$$(k-1)n + |i-j| \leq kn - |n+1-i-j|.$$

By applying the bounds on u' and v', we get

$$u' < (k-1)n + |i-j| \leq kn - |n+1-i-j| < v'.$$

Therefore, u and v are not adjacent.

Case 2: Suppose that u and v are in column j. In particular, let $u = (u', j)$ and $v = (v', j)$. Since $(i, (k-1)n)$ cannot perform any forces given $V(S) \cap B$, we know that

$$u' \in [n] \setminus [i-j+1, i+j-1].$$

Since u' exists, we know that $j \leq n-i$. Similarly, the leak at $(n+1-i, kn+1)$ implies that

$$v' \in [n] \setminus [n+1-i-(n-j), n+1-i+n-j].$$

Since v' exists, we know that $j \geq i + 1$. Therefore, $u' \in [n] \setminus [1, i + j - 1] = [i + j, n]$ and $v' \in [n] \setminus [1 - i + j, n] = [1, j - i]$. Notice that $j - i$ and $j + i$ differ by at least 2. Therefore, u and v are not adjacent.

Since there does not exist $u \in B'_S$ that is adjacent to $v \in B'_R$, B is not a 1-leaky forcing set of G. Thus, $2n \leq \mathrm{Z}_{(1)}(G)$. The corresponding upper bound is obtained by Theorem 2.4.10. □

2.5 Closing remarks

One proof technique we tried to develop is extending minimum zero forcing sets to minimum 1-leaky forcing sets. However, this is not always possible. Consider $P_4 \square P_5$ and let B be a side of length 4. This set is a minimum zero forcing set for this graph. The 1-leaky forcing number for this graph is 5, but there is no way to add one blue vertex to B to get a minimum 1-leaky forcing set. Therefore, there is a graph with a minimum zero forcing set that cannot be extended to a minimum 1-leaky forcing set.

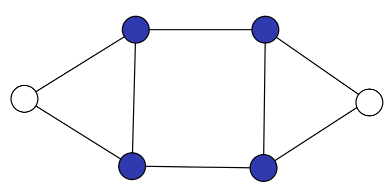

Figure 2.9 Graph G with a minimum 1-leaky forcing set that does not contain a minimum zero forcing set.

Similarly, minimum 1-leaky forcing sets cannot always be reduced to a minimum zero forcing set. Consider the graph G depicted in Figure 2.9. The zero forcing number for G is two, and the blue set shown above is a minimum 1-leaky forcing set. One can check that removing any two vertices from this blue set will not result in a minimum zero forcing set. Therefore, there exists a graph with a minimum 1-leaky forcing set that does not contain a minimum zero forcing set.

Notice that if the top row of vertices are colored blue in G in Figure 2.9, there is a way to remove two blue vertices to yield a minimum zero forcing set. With these examples in mind, the following conjecture is proposed.

Problem 2.5.1. *For every graph G, does there exists a minimum ℓ-leaky forcing set B such that B contains a minimum zero forcing set?*

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CHAPTER 3. GENERALIZATIONS OF LEAKY FORCING

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Modified from a manuscript submitted to *Graphs and Combinatorics*

Abstract

Vertex leaky forcing was recently introduced as a new variation of zero forcing in order to show how vertex leaks can disrupt the zero forcing process in a graph. An edge leak is an edge that is not allowed to be forced across during the zero forcing process. The ℓ-edge-leaky forcing number of a graph is the size of a smallest zero forcing set that can force the graph blue despite ℓ edge leaks. This paper contains an analysis of the effect of edge leaks on the zero forcing process instead of vertex leaks. Furthermore, specified ℓ-leaky forcing is introduced. The main result is that ℓ-leaky forcing, ℓ-edge-leaky forcing, and specified ℓ-leaky forcing are equivalent. Furthermore, all of these different kinds of leaks can be mixed so that vertex leaks, edge leaks, and specified leaks are used. This mixed ℓ-leaky forcing number is also the same as the (vertex) ℓ-leaky forcing number.

Keywords zero forcing, leaky forcing, color change rule

AMS subject classification 05C57, 05C15, 05C50

3.1 Introduction

Zero forcing was introduced by the AIM Minimum Rank and Special Graphs Work Group in [2], in order to find upper bounds for the maximum nullity for the family of real symmetric matrices whose off-diagonal entries are described by a graph. The zero forcing process uses a set of blue vertices in a graph that color other vertices blue given a color change rule. Given a graph G and a blue vertex $v \in V(G)$, if v has one white neighbor w, then v forces w (v colors w blue). Formally, this process is known as the *zero forcing color-change rule.* A *zero forcing set* for G is an initial set of blue vertices B such that after iteratively and exhaustively applying the zero forcing color-change rule, every vertex in G is blue. The *zero forcing number* of a graph is the size of a minimum zero forcing set, and is denoted $\mathrm{Z}(G)$.

Zero forcing has shown up as a way to control quantum systems [4, 7]. In fact, it was shown that if a set of vertices is a zero forcing set, then the associated quantum system is controllable [3]. Another system that utilizes the zero forcing process is the electric power system. In [6], Haynes et al. looked into the problem of monitoring an electric power system by placing as few measurement devices as possible. These applications of zero forcing lead to a natural questions: What if something breaks in the system? Is there a way to keep control? These questions were the main focus in [1] and [5]. In [5], Dillman and Kenter introduced *leaky forcing*, which is a variation on zero forcing that focuses on when vertices in a graph are not able to force. Leaky forcing uses the same color-change rule as zero forcing, but certain vertices are not allowed to perform forces.

Given a graph G, a *vertex leak* (also referred to as a leak) is a vertex in G that is not able to perform a force. An *ℓ-leaky forcing set*, is a zero forcing set such that for any set of ℓ vertex leaks in G, exhaustively applying the color-change rule results in every vertex in G becoming blue. The *ℓ-leaky forcing number* for a graph G is the size of a minimum ℓ-leaky forcing set, and is denoted by $\mathrm{Z}_{(\ell)}(G)$. Notice that $\mathrm{Z}(G) = \mathrm{Z}_{(0)}(G)$. Furthermore, the notion of how resilient a graph is to leaks, and which structures need to be circumvented in a graph for a zero forcing set to be an ℓ-leaky forcing set were explored in [1]. The notation used in this paper will follow the notation introduced in [1]. The rest of this section contains results from [1] which are useful for exploring variations of leaky forcing.

In general, let $B \subseteq V(G)$ be an initial set of blue vertices in G. If vertex u colors v blue, then u forces v and denote it by $u \to v$. The symbol $u \to v$ is called a force. A *set of forces F of B in G* is a set of forces such that there is a chronological ordering of the forces in F where each force is valid and the whole graph turns blue. When the set B is clear form context, F may be referred to as a forcing process of B or a forcing process F (suppressing the reference to B). Intuitively, F represents the instructions for how B can force G blue, or provides a proof that B is a zero forcing set. Implicitly, F gives rise to discrete time steps in which sets of white vertices turn blue. A set B' such that $B \subseteq B' \subseteq V(G)$ is *obtained from B using F* if B can color B' blue using only a subset of forces in a forcing process F. More generally, B' is obtained from B if there is some forcing process F by which B can color B' blue.

The set $B^{[\infty]}$ is the set of blue vertices after the zero forcing rule has been exhaustively applied with B as an initial blue set. Furthermore, $B_L^{[\infty]}$ will be determined after a set of leaks L has been chosen. In particular, $B_L^{[\infty]}$ is the set of blue vertices obtained from B with leaks L after the zero forcing rule has been exhaustively applied. Let $\mathcal{F}(B)$ denote the set of all possible forces given a vertex set B. That is, $u \to v \in \mathcal{F}(B)$ if there exists a set of forces F of B in G that contains $u \to v$. Given this notation, B is an ℓ-leaky forcing set if for every $L \subseteq V(G)$ with $|L| = \ell$ there exists a forcing process F such that if $u \to v \in F$, then $u \notin L$.

Suppose $S \subseteq V(G)$ and F is a forcing process. Let

$$F(S) = \{x \to y \in F : y \notin S\}.$$

By extension,

$$F \setminus F(S) = \{x \to y \in F : y \in S\}.$$

The following lemma proves that abandoning process F to follow process F' creates a new forcing process.

Lemma 3.1.1. *[1] Let B be a blue set in G with forcing processes F and F'. Then $(F \setminus F(B')) \cup F'(B')$ is a forcing process of B for any B' obtained from B using F.*

The next lemma shows that for any $(\ell - 1)$-leaky forcing set B and set of ℓ vertex leaks L, there exists a time when all ℓ leaks in L are blue. Furthermore, there is also a time when all but one of the ℓ leaks in L are blue.

Lemma 3.1.2. *[1] If B is an $(\ell - 1)$-leaky forcing set and L is a set of $k \geq \ell$ vertex leaks, then $|L \setminus B_L^{[\infty]}| \leq k - \ell$.*

The previous two lemmas are used to prove Theorem 3.1.3. The gist of the proof is to use a forcing process that turns all but one of the leaks blue. This is possible by Lemma 3.1.2. At this point, the forcing process is abandoned for a process that will completely force the graph despite the remaining leak. Switching forcing processes is justified by Lemma 3.1.1.

Theorem 3.1.3. *[1] A set B is an ℓ-leaky forcing set if and only if B is an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ vertex leaks L and $v \in V(G) \setminus B$ there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Edge-leaky forcing is introduced in Section 3.2. The main result of this section is that the ℓ-edge-leaky forcing number is the same as the ℓ-leaky forcing. Section 3.3 introduces specified leaks, and shows that preventing a directional force is also equivalent to vertex leaky forcing. In Section 3.4, vertex leaks, edge leaks, and specified leaks are mixed in the leak set without changing the underlying behavior of leaky forcing. Furthermore, in Section 3.5, sets of leaks with a particular underlying structure are explored. In general, analogs of Lemma 3.1.2 will be used to conclude that the condition in Theorem 3.1.3 applies for ℓ-edge-leaky forcing and specified ℓ-leaky forcing.

3.2 On edge-leaky forcing

A natural generalization of ℓ-leaky forcing is to consider what happens when forces are prohibited from passing over particular edges. An edge xy is an *edge leak* if neither $x \to y$ nor $y \to x$ are allowed. A set of blue vertices B is an ℓ-edge-leaky forcing set if B can turn the whole graph G blue given any set of ℓ edge leaks. Denote the ℓ-edge-leaky forcing number of a graph G by $\operatorname{Z}'_{(\ell)}(G)$. Setting both x and y as vertex leaks is a strictly stronger constraint on the zero forcing process than setting xy as an edge leak. However, setting x as a vertex leak is not obviously as strong as setting xy as an edge leak, since setting x as a vertex leak still allows $y \to x$. This makes the following result somewhat surprising.

Theorem 3.2.1. *A set B is a* 1*-edge-leaky forcing set if and only if for all $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}(B)$ with $y \neq x$.*

Proof. Assume that B is a 1-edge-leaky forcing set. This implies that B is a zero forcing set with forcing process F. Let $v \in V(G) \setminus B$ and $x \to v \in F$. Since B is a 1-edge-leaky forcing set, there exists a forcing process F' by which B turns G blue despite setting xv as an edge leak. Therefore, F' must contain a force $y \to v$ where $y \neq x$. Thus $x \to v, y \to v \in \mathcal{F}(B)$, proving the forward direction.

Assume that for all $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}(B)$ with $y \neq x$. Clearly, this implies that B is a zero forcing set of G with forcing process F. Let xv be an arbitrary edge leak. If neither $x \to v$ nor $v \to x$ are in F, then there is nothing to show. Therefore, without loss of generality, assume that $x \to v \in F$. Let B' be a set of blue vertices obtained from B using F such that $x \to v$ is valid given F (were it not for xv being an edge leak), and $v \notin B'$. By assumption, there exists $y \to v \in \mathcal{F}(B)$ where $y \neq x$. This implies that there exists a set of forces F' of B in G with $y \to v$. Since $x \in B'$, it follows that $v \to x \notin F'(B')$. Therefore, $(F \setminus F(B')) \cup F'(B')$ is a forcing process of B that does not use xv. □

With Theorem 3.2.1, its not as surprising that the ℓ-edge-leaky forcing number is equivalent to the ℓ-leaky forcing number for all $\ell \geq 0$. The next lemma finds an appropriate time to switch forcing processes and controls how edge leaks and forcing sets interact. Let $L - S$ where $S \subseteq V(G)$ denote the edges in L that do not have vertices in S. Explicitly,

$$L - S = \{xy \in L : x, y \notin S\}.$$

By extension,

$$L \setminus (L - S) = \{xy \in L : x \in S \text{ or } y \in S\}.$$

Lemma 3.2.2. *If B is an $(\ell - 1)$-edge-leaky forcing set and L is a set of $k \geq \ell$ edge leaks, then $|L - B_L^{[\infty]}| \leq k - \ell$.*

Proof. Assume that L is a set of $k \geq \ell$ edge leaks, and let $|L - B_L^{[\infty]}| \geq k - \ell + 1$. Furthermore, let $L' = L \setminus (L - B_L^{[\infty]})$. Since $|L'| = |L| - |L - B_L^{[\infty]}|$, it follows that $|L'| \leq k - k + \ell - 1 = \ell - 1$. Notice that edge leaks $uv \in L - B_L^{[\infty]}$ did not change the zero forcing behavior of B. In particular, these edge leaks never played a role in stopping B from propagating because their endpoints never were forced. Therefore, L' is a set of at most $\ell - 1$ edge leaks which shows that B is not an $(\ell - 1)$-edge-leaky forcing set. □

As in the vertex leaky setting, Lemma 3.2.2 says that if B is an $(\ell - 1)$-edge-leaky forcing set and L is a set of ℓ edge leaks, then B forces at least one vertex in every edge leak.

Theorem 3.2.3. *A set B is an ℓ-edge-leaky forcing set if and only if B is an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ vertex leaks L and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Proof. Proceed by induction on ℓ. Notice Theorem 3.2.1 is the base case when $\ell = 1$. Assume that the claim holds for all $r < \ell$.

Let B be an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ vertex leaks L and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$. Clearly, B is an $(\ell - 2)$-leaky forcing set such that for every set of $\ell - 2$ vertex leaks L and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$. Therefore, by the induction hypothesis, B is an $(\ell - 1)$-edge-leaky forcing set.

Let L be a set of ℓ edge leaks. By Lemma 3.2.2, it is possible to apply forces one by one until every edge in L contains a blue vertex. Let B' be the resulting set of blue vertices. Notice that B' is an $(\ell - 1)$-edge-leaky set since $B \subseteq B'$. Therefore, if B' contains an edge in L, then there is nothing left to show. Thus, assume that every edge in L contains at exactly one blue vertex in B'.

Let $A \subseteq \{x \in B' : xy \in L\}$ such that $|L - A| \leq 1$ and $|A| \leq \ell - 1$. Notice that vertices in A can only perform a force if an edge in L has two blue end points. Therefore, assume that vertices in A never perform a force; otherwise, there is nothing left to show.

Let $G^* = G - A$. Since B is an $(\ell - 1)$-leaky forcing set, it follows that $B^* = B' \setminus A$ is a zero forcing set of G^*. At this point there is at most one edge leak from L in G^*. Let $v \in V(G^*) \setminus B^*$. By assumption, there exists $x \to v, y \to v \in \mathcal{F}_A(B)$ in G with $y \neq x$.

Notice that $x, y \notin A$ since v is a white neighbor of both x and y. Therefore, $x, y \in V(G^*)$ and $x \to v, y \to v \in \mathcal{F}_A(B^*)$. By Theorem 3.2.1, B^* is a 1-edge-leaky forcing set of G^*. Thus, B^* can color G^* blue, demonstrating that B is an ℓ-edge-leaky forcing set.

To prove the contrapositive of the forward direction, assume that B is an $(\ell-1)$-leaky forcing set, $L = \{x_1, \ldots, x_{\ell-1}\}$ is a set of leaks, and $v \in V(G) \setminus B$ such that if $x \to v, y \to v \in \mathcal{F}_L(B)$, then $x = y$. Since B is an $(\ell-1)$-leaky forcing set, there exist $x_0 \to v \in \mathcal{F}_L(B)$. Let $L' = L \cup \{x_0\}$ be a set of ℓ vertex leaks and exhaustively apply the zero forcing rule so that $B_{L'}^{[\infty]}$ is blue. Notice that B is not an ℓ-leaky forcing set by Theorem 3.1.3; so $B_{L'}^{[\infty]} \subset V(G)$ (strictly contained). However, by Lemma 3.1.2, $L' \subseteq B_{L'}^{[\infty]}$.

To complete the proof, the set of vertex leaks L' will be converted into a set of edge leaks. Notice that every vertex $x_i \in L'$ has exactly one white neighbor y_i; otherwise, it is possible to remove a vertex from L' to conclude that B is not an $(\ell-1)$-leaky forcing set. Let $L^* = \{x_i y_i : 0 \leq i \leq \ell - 1\}$. Now L^* demonstrates that B is not an ℓ-edge-leaky forcing set. □

Corollary 3.2.4. *For any graph G and $\ell \geq 0$,*

$$\mathrm{Z}_{(\ell)}(G) = \mathrm{Z}'_{(\ell)}(G).$$

In particular, B is an ℓ-leaky forcing set if and only if B is an ℓ-edge-leaky forcing set.

The combination of Theorems 3.1.3 and 3.2.3 provide insight into how leaks interact with the zero forcing rule. In particular, vertex leaks are nicer than edge leaks. Once a vertex leak turns blue, it can safely be deleted from the graph and disregarded for the rest of the process. Edge leaks do not afford us the same luxury. Even if an endpoint of an edge leak turns blue, the vertex cannot be deleted without further care, since it might perform a force later.

3.3 On specified-leaky forcing

Throughout this section, $v \to u$ is a *specified leak* if v is prohibited from forcing u. In this sense, setting a vertex v as a leak represents the set of specified leaks $\{v \to u : u \in N(v)\}$, and setting an edge uv as a leak represents the set of specified leaks $\{v \to u, u \to v\}$. It seems as though prohibiting $v \to x$ and $v \to y$ is not more restrictive than prohibiting just $v \to x$ or $v \to y$, but not both. This is more intuitive after considering the fact that in any particular forcing process F, only setting $v \to x$ or $v \to y$ as a specified leak poses a problem since $v \to x$ and $v \to y$ are not both in F. Furthermore, the strength of leaks being picked after the initial blue sets makes a single leak $v \to x$ as devastating as two leaks $v \to x, v \to y$.

To formalize this intuition a little more, consider the following definitions. A set B is a *specified ℓ-leaky forcing set of G* if B can color G blue when any set of ℓ forces are prohibited. Let $\mathrm{Z}^s_{(\ell)}(G)$ be the minimum size of a specified ℓ-leaky forcing set of G.

In Section 3.2, Lemma 3.2.2 is used to control the interaction between an initial blue set and a set of edge leaks. However, the proof of Theorem 3.3.1 does not require a lemma analogous to Lemma 3.2.2 even thought the statements of the two theorems are similar. Proposition 3.3.3 is analogous to Lemma 3.2.2, and will be proven at the end of this section.

Consider the following definitions before proceeding with the proof of Theorem 3.3.1: If $v \to u$ is a specified leak, the v is called *the tail of the leak* $v \to u$ and u is the *head of the leak* $v \to u$. Let $T(L) = \{x : x \to y \in L\}$ be the set of tails of L and $H(L) = \{y : x \to y \in L\}$ be the set of heads of L.

Theorem 3.3.1. *A set B is a specified ℓ-leaky forcing set if and only if B is an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ vertex leaks L and $v \in V(G) \setminus B$, there exist $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Proof. Let L be a set of ℓ specified leaks that shows that B is not a specified ℓ-leaky forcing set. Notice that $|T(L)| \leq \ell$. Therefore, $T(L)$ demonstrates that B is not an ℓ-leaky forcing set. Thus, by Theorem 3.1.3, there exist $v \in V(G) \setminus B$ such that if $x \to v, y \to v \in \mathcal{F}_{T(L)}(B)$ then $y = x$.

Suppose that B is an $(\ell - 1)$-leaky forcing set, $L = \{x_1, \ldots, x_{\ell-1}\}$ is a set of $\ell - 1$ leaks, and $v_0 \in V(G) \setminus B$ such $x \to v_0, y \to v_0 \in \mathcal{F}(L)$ implies $y = x$. Since B is an $(\ell - 1)$-leaky forcing set, there exists $x_0 \to v_0 \in \mathcal{F}_L(B)$. Let $L' = L \cup \{x_0\}$. By Lemma 3.1.2,

$$|L' \setminus B^{[\infty]}_{L'}| = 0.$$

Notice that if there exists $y \in L'$ with at least two white neighbors, then $L' \setminus \{y\}$ would show that B is not an $(\ell-1)$-leaky forcing set. Therefore, each vertex $x_i \in L'$ has exactly one white neighbor $v_i \in V(G) \setminus B_{L'}^{[\infty]}$. The set of specified leaks $\{x_i \to v_i : 0 \le i \le \ell - 1\}$ shows that B is not a specified ℓ-leaky forcing set. □

Corollary 3.3.2. *For any graph G and $\ell \geq 0$,*

$$\mathrm{Z}^s_{(\ell)}(G) = \mathrm{Z}_{(\ell)}(G).$$

In particular, B is an ℓ-leaky forcing set if and only if B is a specified ℓ-leaky forcing set.

As previously noted, Proposition 3.3.3 controls the interaction between a specified $(\ell-1)$-leaky forcing set and a set of specified leaks L. Suppose that L is a set of specified leaks and let $S \subseteq V(G)$. Let

$$L - S = \{x \to y \in L : x \notin S\}$$

and

$$L \setminus (L - S) = \{x \to y \in L : x \in S\}.$$

The next Proposition controls how many leaks are required to halt a specified $(\ell-1)$-leaky forcing set. This is an analogous result to Lemma 3.2.2.

Proposition 3.3.3. *If B is an $(\ell-1)$-leaky forcing set and L is a set of $k \geq \ell$ specified leaks, then $|L - B_L^{[\infty]}| \leq k - \ell$.*

Proof. Assume that L is a set of $k \geq \ell$ specified leaks, and let $|L - B_L^{[\infty]}| \geq k - \ell + 1$. If $v \in B_L^{[\infty]}$ and v has exactly one white neighbor u, then $v \to u \in L$ by the leaky forcing rule. Furthermore, let $L' = L \setminus (L - B_L^{[\infty]})$ and notice that $|L'| \leq \ell - 1$. Specified leaks $x \to y \in L - B_L^{[\infty]}$ did not change the zero forcing behavior of B. In particular, these specified leaks never played a role in stopping B from propagating because the tail never turned blue. Therefore, $T(L')$ is a set of at most $\ell - 1$ vertex leaks which show that B is not an $(\ell-1)$-leaky forcing set. □

3.4 On mixed-leaky forcing

This section investigates what happens when a system has various types of leaks preventing the zero forcing process from finishing. A set $B \subseteq V(G)$ is a *mixed ℓ-leaky forcing set* of a graph G if B can color G blue despite any set of ℓ vertex leaks, edge leaks, or specified leaks (refer to these collectively as *leaks*). Let $\operatorname{Z}_{(\ell)}^{m}(G)$ be the minimum size of a mixed ℓ-leaky forcing set.

Lemma 3.4.1. *Let $L = L_1 \cup L_2 \cup L_3$ be a set of $k \geq \ell$ leaks where L_1 is the set of vertex leaks, L_2 is the set of edge leaks, and L_3 is the set of specified leaks. If B is a mixed $(\ell-1)$-leaky forcing set, then $|L_1 \setminus B_L^{[\infty]}| + |L_2 - B_L^{[\infty]}| + |L_3 - B_L^{[\infty]}| \leq k - \ell$.*

Proof. To prove the contrapositive, assume that $|L_1 \setminus B_L^{[\infty]}| + |L_2 - B_L^{[\infty]}| + |L_3 - B_L^{[\infty]}| \geq k - \ell + 1$. Every vertex in $B_L^{[\infty]}$ has either 0, 1, or at least 2 white neighbors. If $v \in B_L^{[\infty]}$ such that v has exactly one white neighbor u, then either $v \in L_1$, $vu \in L_2$, or $v \to u \in L_3$. Let $L' = [L_1 \setminus (L_1 \setminus B_L^{[\infty]})] \cup [L_2 \setminus (L_2 - B_L^{[\infty]})] \cup [L_3 \setminus (L_3 - B_L^{[\infty]})]$. Since this is a disjoint union,

$$
\begin{aligned}
|L'| &= |L_1 \setminus (L_1 \setminus B_L^{[\infty]})| + |L_2 \setminus (L_2 - B_L^{[\infty]})| + |L_3 \setminus (L_3 - B_L^{[\infty]})| \\
&\leq \ell - 1.
\end{aligned}
$$

Notice that any leak in either $L_1 \setminus B_L^{[\infty]}$, $L_2 - B_L^{[\infty]}$, or $L_3 - B_L^{[\infty]}$ did not change the zero forcing behavior of B. In particular, these leaks never played a role in stopping B from propagating because the vertex leaks were never forced blue, the tails of the specified leaks were never forced blue, and the endpoints of the edge leaks were never forced blue. Therefore, L' is a set of at most $\ell - 1$ leaks which shows B is not a mixed $(\ell-1)$-leaky forcing set. □

Notice that if L_2 and L_3 are empty, then Lemma 3.1.2 is recovered. With this more general formulation of leaky forcing, the next theorem can be proven.

Theorem 3.4.2. *A set B is a mixed ℓ-leaky forcing set if and only if B is an $(\ell-1)$-leaky forcing set such that for every $\ell - 1$ vertex leaks L and $v \in V(G) \setminus B$, there exist $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$.*

Proof. Proceed by induction on ℓ. Notice either Theorem 3.1.3, Theorem 3.2.1, or Theorem 3.3.1 handles the base case when $\ell = 1$. Assume the claim holds for all $r < \ell$.

Let B be an $(\ell - 1)$-leaky forcing set such that for every set of $\ell - 1$ vertex leaks L and $v \in V(G)$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$. Clearly, B is an $(\ell - 2)$-leaky forcing set such that for every set of $\ell - 2$ vertex leaks L and $v \in V(G) \setminus B$, there exists $x \to v, y \to v \in \mathcal{F}_L(B)$ with $y \neq x$. Therefore by the induction hypothesis, B is a mixed $(\ell - 1)$-leaky forcing set.

Let $L = L_1 \cup L_2 \cup L_3$ be a set of ℓ leaks where L_1 is the set of vertex leaks, L_2 is the set of edge leaks, and L_3 is the set of specified leaks. By Lemma 3.4.1, it is possible to apply forces one by one until every vertex leak in L_1 is blue, every edge leak in L_2 contains a blue vertex, and the tails of specified leaks in L_3 are blue. Let B' be the resulting set of blue vertices. Notice that B' is a mixed $(\ell - 1)$-leaky forcing set since $B \subseteq B'$. If B' contains an edge from either L_2 or L_3, then there is nothing left to show. Therefore, assume that the edges in L_2 are incident to one blue vertex, and only the tails of forces in L_3 are blue.

Let $L' \subset L$ be a set of $\ell - 1$ leaks. Notice that blue vertices in $L_1 \cap L'$, blue vertices incident to edges in $L_2 \cap L'$, and the tails of specified leaks in $L_3 \cap L'$ can be deleted. Let A be the set of these vertices.

Consider $G^* = G - A$. Since B is an $(\ell - 1)$-leaky forcing set, it follows that $B^* = B' \setminus A$ is a zero forcing set for G^*. At this point there is at most one leak from L in G^*. Let $v \in V(G^*) \setminus B^*$. By assumption there exists $x \to v, y \to v \in \mathcal{F}_A(B)$ in G with $y \neq x$. Since $x, y \notin A$, it follows that $x, y \in V(G^*)$ and $x \to v, y \to v \in \mathcal{F}(B^*)$. By Theorems 3.1.3, 3.2.3, and 3.3.1, B^* is a 1-edge-leaky forcing set, a 1-leaky forcing set, and a specified 1-leaky forcing set. Thus, B^* can color G^* blue, demonstrating that B is a mixed ℓ-leaky forcing set.

To prove the contrapositive of the forward direction, assume B is an $(\ell - 1)$-leaky forcing set, $L = \{x_1, \ldots, x_{\ell-1}\}$ is a set of vertex leaks and $v \in V(G) \setminus B$ such that if $x \to v, y \to v \in \mathcal{F}_L(B)$, then $x = y$. Since B is an $(\ell-1)$-leaky forcing set, there exists $x_0 \to v \in \mathcal{F}_L(B)$. Let $L' = L \cup \{x_0\}$

be a set of ℓ vertex leaks. Notice that L' demonstrates that B is not an ℓ-leaky forcing set. Thus, B is also not a mixed ℓ-leaky forcing set. □

Corollary 3.4.3. *For any graph G and $\ell \geq 0$,*

$$\mathrm{Z}_{(\ell)}(G) = \mathrm{Z}^m_{(\ell)}(G).$$

In particular, B is an ℓ-leaky forcing set if and only if B is a mixed ℓ-leaky forcing set.

3.5 Independent sets of specified leaks

Corollaries 3.2.4 and 3.3.2 suggest that the strength of a set of specified leaks is somewhat independent of the number of leaks or their structure as a subgraph. In particular, a set of ℓ vertex leaks or ℓ edge leaks is at most as strong as a set of ℓ specified leaks even though ℓ vertex or edge leaks corresponds to more than ℓ specified leaks. Thus, arranging specified leaks into sets of out-stars or 2-cycles is in some sense inefficient. The goal of this section is to formally develop what it means for out-stars and 2-cycles in a set of specified leaks to be inefficient.

Let L be a set of specified leaks on $V(G)$. Notice that a specified leak $v \to u$ can be thought of as a directed edge from v to u. Therefore, L naturally corresponds to the edge set of a directed graph on the vertex set $V(G)$. This gives rise to a notion of isomorphic sets of specified leaks. Let L_1 and L_2 be sets of specified leaks on $V(G)$. A set of specified leaks L_1 is *isomorphic* to L_2 if there exists a bijection $\phi : V(G) \to V(G)$ such that $x \to y \in L_1$ if and only if $\phi(x) \to \phi(y) \in L_2$.

A set B is an L-leaky forcing set if B can turn G blue despite any set of L_1 leaks that is isomorphic to L_2 where $L_2 \subseteq L$. Correspondingly, the L-leaky forcing number of G, denoted $\mathrm{Z}_{(L)}(G)$, is the size of the smallest L-leaky forcing set. It is implicitly assumed that L is a set of specified leaks on $V(G)$.

A set of specified leaks L is a *set of independent leaks* if for all $x \to y, v \to u \in L$, it follows that $x \neq v$ and $y \neq v$. Equivalently, L is independent if $|T(L)| = |L|$ and $T(L) \cap H(L) = \varnothing$. Let $I(L)$ denote the size of the largest set of independent leaks contained by L.

These definitions let us abstract away general structure of a set of specified leaks L and focus on the parameter of L that seems to matter. In particular, a set of specified leaks L is no stronger than a maximum set of independent leaks contained in L.

Theorem 3.5.1. *Let L be a set of specified leaks on $V(G)$ and let $\ell = I(L)$. If B is a specified ℓ-leaky forcing set then B is an L-leaky forcing set. That is,*

$$\mathrm{Z}_{(L)}(G) \leq \mathrm{Z}^{s}_{(\ell)}(G).$$

To prove Theorem 3.5.1, consider *active leaks*. The set of active leaks given a blue set B and a set of specified leaks L is the set of leaks in L that actively prevents B from performing a force. Formally the set of active leaks is given by

$$A(B, L) = \{x \to y \in L : x \in B, \{y\} = N(x) \setminus B\}.$$

Proof. First, consider the contrapositive of the desired result. Suppose that B is not an L-leaky forcing set. Therefore, there exists a set of specified leaks L' which is isomorphic to a subset of L that prevents B from coloring all of G blue. Let $B^{[\infty]}_{L'}$ be the set of blue vertices obtained from B given L' by exhaustively applying forces. Notice that $B^{[\infty]}_{L'} \neq V(G)$, and let $A = A(B^{[\infty]}_{L'}, L')$. Since A is a set of independent leaks, it follows that

$$|A| \leq I(L') \leq I(L).$$

Furthermore, A demonstrates that B is not a specified ℓ-leaky forcing set. □

The converse of Theorem 3.5.1 holds when $I(L) = 1$.

Proposition 3.5.2. *Let L be a set of specified leaks on $V(G)$ such that $1 = I(L)$. A set B is an L-leaky forcing set if and only if B is a specified 1-leaky forcing set.*

Proof. The backward direction Proposition 3.5.2 is covered by Theorem 3.5.1. Therefore, assume that B is not a specified 1-leaky forcing set. This implies that there exists $L' = \{x \to y\}$ that stops B from turning G blue. By assumption, L has a set of independent leaks of size 1. Therefore, L' is isomorphic to a subset of L. Therefore, B is not an L-leaky forcing set. □

The proof of Proposition 3.5.2 relies on the fact that, up to isomorphism, there is only one set of independent leaks. Proving the converse of Theorem 3.5.1 fails since an arbitrary set of ℓ independent leaks cannot always be injected into L when $I(L) = \ell$. To illustrate this point, consider the following example. Let $G = K_{\ell+1} \square K_2$, $\ell \geq 2$ with vertex set $V(G) = \{x_1, \ldots, x_{\ell+1}, y_1, \ldots, y_{\ell+1}\}$ where the sets $\{x_i : 1 \leq i \leq \ell+1\}, \{y_i : 1 \leq i \leq \ell+1\}$ induce cliques, and $\{x_i y_i : 1 \leq i \leq \ell+1\}$ induces a matching. Let $L_1 = \{x_i \to y_i : 1 \leq i \leq \ell+1\}$, and $L_2 = \{x_i \to x_{\ell+1} : 1 \leq i \leq \ell\}$. Suppose that $B = \{x_i : 1 \leq i \leq \ell+1\}$. First, notice that B is not a specified 2-leaky forcing set, since $L = \{x_1 \to y_1, x_2 \to y_2\}$ prevents B from turning y_1, y_2 blue. This also shows that B is not an L_2-leaky forcing set. However, B is an L_1 leaky forcing set. Since $I(L_1) = I(L_2) = \ell$, this example shows that the converse of Theorem 3.5.1 is false for $\ell \geq 2$.

3.6 Closing remarks

Though vertex leaky forcing and edge leaky forcing are natural generalizations of the zero forcing process, its relationship to the linear algebra roots of zero forcing is less clear. Consider the following system of equations below:

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + a_{1,4}x_4 = 0$$
$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + 0x_4 = 0$$
$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + 0x_4 = 0$$
$$a_{4,1}x_1 + 0x_2 + 0x_3 + a_{4,4}x_4 = 0$$

or equivalently as the matrix representation,

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & 0 & 0 & a_{4,4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here it is required that $a_{i,j} = a_{j,i} \neq 0$ for $i \neq j$. Zero forcing studies the minimum number of entries in the x vector that need to be set to 0 before one can conclude that the whole x vector is identically 0. In particular, determining that $x_2 = x_4 = 0$ implies $x_1 = x_3 = 0$ is equivalent to seeing that $\{x_2, x_4\}$ is zero forcing set in Figure 3.1. In this setting x_4x_1 as an edge leak corresponds to assuming that $a_{1,4}$ and $a_{4,1}$ are zero divisors. That is, if $a_{4,1}$ is a zero divisor and $x_4 = 0$, then $a_{4,1}x_1 + a_{4,4}x_4 = 0$ does not imply that $x_1 = 0$. Equivalently, if x_1x_4 is an edge leak, then x_1 cannot be used to turn x_4 blue.

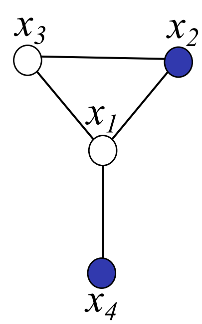

Figure 3.1 The paw graph with a corresponding zero forcing set.

Let G be an arbitrary graph and A a symmetric matrix with nonzero pattern corresponding to G with 2ℓ zero divisors in the off-diagonal entries (ℓ zero divisors in the upper off-diagonal entries). Under the interpretation in the previous paragraph, $\mathrm{Z}'_{(\ell)}(G)$ corresponds to the minimum number of 0 entries in x that force $x = 0$ vector under the condition that $Ax = 0$.

Up to this point, the authors are unaware of generalizations of the notions dimension, rank, and spectrum for modules over rings with zero divisors and linear transformations thereof. As a general problem, and a curiosity well beyond the scope of this paper, the authors would be very interested to see if the edge-leaky forcing number can be used as a tool to analyze some appropriate notion of minimum rank or maximum nullity for R-linear transformations (homomorphisms) of modules M over ring R with zero divisors.

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CHAPTER 4. AN UPPER BOUND FOR THE K-POWER DOMINATION NUMBER IN R-UNIFORM HYPERGRAPHS

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Abstract

The generalization of power domination, k-power domination, is a graph parameter denoted γ_p^k. The study of power domination was introduced to study electric power grids. Chang and Roussel introduced k-power domination in hypergraphs and conjectured the upper bound for the k-power domination number for r-uniform hypergraphs on n vertices was $\frac{n}{r+k}$. This upper bound was shown to be true for simple graphs ($r = 2$) and it was further conjectured that only a family of hypergraphs, known as the squid hypergraphs attained this upper bound. In this paper, the conjecture is proven to hold for hypergraphs with $r = 3$ or 4; but is shown to be false, by a counterexample, for $r \geq 7$. A new upper bound is also proven for $r \geq 3$.

Keywords zero forcing, power domination, hypergraphs

AMS subject classification 05C57, 05C15, 05C50, 05C65

4.1 Introduction

Power domination was first introduced by Haynes, Hedetniemi, Hedetniemi and Henning in [5] to study the monitoring process of electrical power networks by placing as few measurement devices as possible. This was done by defining the power domination problem in graph theoretic terms. In particular, Phase Measurement Units (PMUs) were placed at a set of initial vertices and then certain rules were applied. These rules consisted of a domination step followed by the zero forcing process.

Zero forcing was first introduced in [1] as a way to find upper bounds for the maximum nullity of real symmetric matrices whose nonzero off-diagonal entries are described by a graph. It was later discovered to be one of the processes in power domination. Zero forcing has since been generalized, and specifically in hypergraphs, there have been multiple generalizations. In this paper, the generalization introduced in [4] as a process in k-power domination in hypergraphs will be used. Moreover, this generalization also is consistent with the definition of k-power domination in simple graphs which was introduced by Chang, Dorbec, Montassier, and Raspaud in [3].

4.2 Preliminaries

A *hypergraph*, $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, is a set of vertices $V(\mathcal{H})$ combined with a set of edges $E(\mathcal{H})$ such that $E(\mathcal{H})$ is a subset of the power set of $V(\mathcal{H})$. When it is obvious which hypergraph is being used, V and E will be written. A hypergraph $\mathcal{H}$ is said to be *r-uniform* when each edge in E has order k.

The *closed neighborhood* of a vertex $v \in V$, $N[v]$, is the set of vertices adjacent to v and v itself. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N[v] \setminus \{v\}$. The closed (or open) neighborhood of a set S is the set $\bigcup_{v \in S} N[v]$ (or $\bigcup_{v \in S} N(v)$) and is denoted $N[S]$ (or $N(S)$).

Define the *white degree* of a vertex v with respect to a set S in $\mathcal{H}$, denoted $deg_w(v, S)$, to be the minimum number of edges that cover v and the white neighbors of v when each vertex in S is blue and each vertex in $V(\mathcal{H}) - S$ is white. Denote the *maximum white degree* by $\Delta_w(S)$. The set of *external private neighbors* of a vertex v with respect to a set of vertices S in a hypergraph $\mathcal{H}$, denoted $epn(v, S)$, is the set of vertices not in S that are adjacent to v, but not to any other vertex is S.

A set of vertices D is known as a *dominating set* of a hypergraph $\mathcal{H}$ if $\bigcup_{v \in D} N[v] = V$. The size of a minimum dominating set in a hypergraph $\mathcal{H}$ is denoted $\gamma(\mathcal{H})$ and is called the *domination number*. In [4], Chang and Roussel defined a *k-power dominating set* on a hypergraph $\mathcal{H}$, as a set $D \subseteq V$ which colors vertices in V blue with respect to the following rules: the vertices in $N[D]$ are colored blue; and if a blue vertex v is incident to a set of at most k unobserved edges that contain all of the white neighbors of v, all vertices in the unobserved edge are colored blue. If iteratively applying these rules results in all vertices in $\mathcal{H}$ becoming blue, then D is a *k-power dominating set* of $\mathcal{H}$. The *k-power dominating number* of $\mathcal{H}$, denoted $\gamma_p^k(\mathcal{H})$, is the minimum cardinality of a k-power dominating set of $\mathcal{H}$. In this definition, an edge is *unobserved* if the edge contains at least one white vertex. Notice the following inequality observed by Chang et al. in [3] still applies to k-power domination for hypergraphs. Given a hypergraph $\mathcal{H}$,

$$\gamma(\mathcal{H}) = \gamma_p^0(\mathcal{H}) \geq \gamma_p^1(\mathcal{H}) \geq \cdots \geq \gamma_p^k(\mathcal{H}) \geq \gamma_p^{k+1}(\mathcal{H}) \geq \ldots.$$

With the definition of white degree, the color change rule for k-forcing in hypergraphs can be redefined: if v is a vertex in a set of blue vertices S of a hypergraph $\mathcal{H}$ with $deg_w(v, S) \leq k$, then change the color of the neighbors of v to blue. If applying this rule results in all vertices in $\mathcal{H}$ being colored blue, then S is a k-forcing set of $\mathcal{H}$. Therefore the definition of a k-power dominating set in [4], is equivalent to a set D such that once every vertex in D and its neighborhood are colored blue, $N[D]$ is a k-forcing set.

Although k-power domination on simple graphs has been, and continues to be, a well studied area in graph theory, little is known about upper bounds for the k-power domination number of hypergraphs. In [4], an upper bound for r-uniform hypergraphs was conjectured and it was believed the squid hypergraphs were the only hypergraphs attaining this bound. A *squid hypergraph* can be defined from any r-uniform hypergraph as follows. For every vertex v, add $(k+1)+(r-2)$ vertices $v_1, v_2, \ldots, v_{k+1}, v_1', v_2', \ldots, v_{r-2}'$ and $k+1$ edges $e_{v,i} = \{v, v_1', v_2', \ldots, v_{r-2}', v_i\}$ for $1 \leq i \leq k+1$. See

Lemma 4.2.1. *Let $\mathcal{H}$ be a r-uniform hypergraph and $\mathcal{H}'$ be the squid hypergraph of $\mathcal{H}$. Then $\gamma_p^k(\mathcal{H}') = |\mathcal{H}|$.*

Conjecture 4.2.2. *[4] If $\mathcal{H}$ is a connected r-uniform hypergraph and $k + r \leq n$, then*

$$\gamma_p^k(\mathcal{H}) \leq \frac{n}{r+k}$$

with equality if and only if $\mathcal{H}$ is a squid hypergraph of a connected r-uniform hypergraph or $r = 2$ with $\mathcal{H} = K_{(k+2,k+2)}$.

In Section 4.3, it is shown that the first part of the Conjecture 4.2.2 holds for hypergraphs with $k + r \leq n$ vertices and $r \leq 4$. It is also shown that for $r \geq 3$, $\gamma_p^k(\mathcal{H}) \leq \frac{n+\lfloor\frac{r-3}{2}\rfloor m}{\lfloor\frac{3(r-1)}{2}\rfloor+k}$. In Section 4.4 a counterexample to Conjecture 4.2.2 is given for $k < r$ and $r \geq 7$.

4.3 An upper bound for r-uniform hypergraphs

In [3], Chang and Roussel first proved that if G is a connected graph of order $n \geq 2$, then $\gamma_p^k(G) \leq \frac{n}{2+k}$. Since simple graphs are 2-uniform hypergraphs, Conjecture 4.2.2 holds for $r = 2$. In this section, the upper bound in the conjecture is proven for $r \leq 4$, and a new upper bound is proven for $r \geq 3$.

Theorem 4.3.1. *[3] If $\mathcal{H}$ is a connected 2-uniform hypergraph (simple graph) with $k + 2 \leq n$ vertices, then*

$$\gamma_p^k(\mathcal{H}) \leq \frac{n}{2+k}.$$

The above theorem was proven using the following lemma from [3].

Lemma 4.3.2. *[3] If G is a connected graph with maximum degree at least $k + 2$, then there exists a k-power dominating set containing only vertices of degree at least $k + 2$.*

In order to prove the conjecture for $r \leq 4$, the following lemma is needed.

Lemma 4.3.3. *Let $\mathcal{H}$ be a connected hypergraph with $k+3 \leq n$ vertices. There exists a minimum k-power dominating set D such that $k + 1 \leq deg_w(v, N[D\backslash\{v\}])$ for all $v \in D$.*

Proof. Let $\mathcal{H}$ be a connected hypergraph with $n \geq k + 3$ vertices. Given a set of vertices S, a vertex $v \in S$ is *attached in S* if $|N(v) \cap S| \geq 1$. Let D be a k-power dominating set such that the number of attached vertices is maximized. Moreover, choose D such that out of all k-power dominating sets with maximum number of attached vertices, it has the minimum number of vertices v with property $deg_w(v, N[D \setminus \{v\}]) \leq k$.

First note, if $v \in D$ and $deg_w(v, N[D \backslash \{v\}]) \leq k$, then for any vertex $u \in N(v)$ the set $(D \backslash \{v\}) \cup \{u\}$ is also a minimum k-power dominating set. To see this let $u \in N(v)$. In the domination step of $(D \backslash \{v\}) \cup \{u\}$, $N[D \backslash \{v\}]$ becomes blue and u colors v blue. Furthermore, v can color its neighbors blue since $deg_w(v, N[D \backslash \{v\}]) \leq k$. Thus $N[D]$ becomes blue and $(D \backslash \{v\}) \cup \{u\}$ is a minimum k-power dominating set.

Now assume that $v \in D$ is a vertex with $deg_w(v, N[D \backslash \{v\}]) \leq k$. If there exists a $u \in N(v) \cap D$, then $D \backslash \{v\}$, by the previous note, is a k-power dominating set. But this is a contradiction since D is a minimum set. So assume v has no neighbors in D, and note this means v is not attached.

Since $\mathcal{H}$ is connected, there exists a path $v = u_0, u_1, \ldots, u_\ell = v'$ to any vertex v' in D. Pick v' so that v' is a vertex in D closest to v; this implies that u_i is not in D for all $i \in \{1, \ldots, \ell - 1\}$, and that no u_i is adjacent to a vertex in $D \backslash \{v\}$, except for $u_{\ell-1}$.

Let $D_i = D \setminus \{v\} \cup \{u_i\}$ for $1 \leq i \leq \ell$, and note that since $u_\ell = v' \in D$,

$$|D_\ell| = |D \setminus \{v\} \cup \{u_\ell\}| = |D \setminus \{v\} \cup \{v'\}| = |D| - 1.$$

Furthermore, notice that if D_i is a minimum k-power dominating set and $deg_w(u_i, N[D_i \backslash \{u_i\}]) \leq k$ for each i, $0 \leq i \leq \ell - 1$, then D_{i+1} is a minimum k-power dominating set. However, D_ℓ is not a k-power dominating set because $|D_\ell| < |D|$ by definition. Therefore, there has to exist a first u_j on the path (starting from v) with $deg_w(u_j, N[D_j \backslash \{u_j\}]) \geq k + 1$, and $deg_w(u_i, N[D_i \backslash \{u_i\}]) \leq k$ for all $i < j$. Observe that D_j is a k-power dominating set.

If $j = \ell - 1$, then D_j is a minimum k-power dominating set that contradicts the maximality condition of D, since u_j is adjacent to $v' \in D_j$.

If $j = \ell - 2$ and $d \in D_j$ has the property that $deg_w(d, N[D_j \backslash \{d\}]) \leq k < deg_w(d, N[D \backslash \{d\}])$, then there exists a vertex x adjacent to u_j and d (if not, the path we chose was not minimum). Let $D_x = (D_j \setminus \{d\}) \cup x$. Then, D_x is a k-power dominating set that contradicts the maximality property of D since x is adjacent to u_j. If no vertex d has the property that $deg_w(d, N[D_j \backslash \{d\}]) \leq k < deg_w(d, N[D \backslash \{d\}])$, then D_j contradicts the minimality property of D.

If $j \leq \ell - 3$, then u_j is not adjacent to any neighbor of a vertex in D, or any vertex in D. Now, D_j has one less vertex v with property $deg_w(v, N[D_j \setminus \{v\}]) \leq k$ compared to D and both have the same number of attached vertices, a contradiction.

Thus, no vertex $v \in D$ has the property $deg_w(v, N[D \backslash \{v\}]) \leq k$ as every case reaches a contradiction.

□

Lemma 4.3.4. *Let $\mathcal{H}$ be a hypergraph and let D be a minimum k-power dominating set of $\mathcal{H}$ such that $k + 1 \leq deg_w(v, N[D \setminus \{v\}])$. Then $k + 1 \leq deg_w(v, N[D \setminus \{v\}]) \leq |epn(v, D)|$, for all $v \in D$.*

Proof. Using Lemma 4.3.3, let D be a minimum k-power dominating set of $\mathcal{H}$ such that $k + 1 \leq deg_w(v, N[D \backslash \{v\}])$ for all $v \in D$. Color v and $N[D \backslash \{v\}]$ blue, for some $v \in D$. Then the external private neighborhood of v is left white, which implies $deg_w(v, N[D \setminus \{v\}]) \leq epn(v, D)$. □

In [2], Bujtás, Henning, and Tuza proved the following upper bound for dominating sets in hypergraphs.

Theorem 4.3.5. *[2] If $\mathcal{H}$ is an r-uniform hypergraph of order n with m edges and no isolated vertices, and $r \geq 3$, then*

$$\gamma(\mathcal{H}) \leq \frac{n + \lfloor \frac{r-3}{2} \rfloor m}{\lfloor \frac{3(r-1)}{2} \rfloor}.$$

Setting $r = 3$, or 4, results in the following corollary.

Corollary 4.3.6. *[2] For any r-uniform hypergraph $\mathcal{H}$ on n vertices with no isolated vertices, with $r = 3$ or 4,*

$$\gamma(\mathcal{H}) \leq \frac{n}{r}.$$

With the Lemma 4.3.3, Lemma 4.3.4 and Theorem 4.3.5, the following main result is proven.

Theorem 4.3.7. *If $\mathcal{H}$ is a connected, r-uniform hypergraph of order n, $k + r \leq n$, with m edges, and $r \geq 3$, then*

$$\gamma_p^k(\mathcal{H}) \leq \frac{n + \lfloor \frac{r-3}{2} \rfloor m}{\lfloor \frac{3(r-1)}{2} \rfloor + k}.$$

Proof. Let $\mathcal{H}$ be a connected, r-uniform hypergraph. By Lemma 4.3.4, let D be a minimum k-power dominating set such that for all $v \in D$, $k + 1 \leq deg_w(v, N[D \setminus \{v\}]) \leq |epn(v, D)|$.

For sake of contradiction, assume that $|V(\mathcal{H})| < \lfloor \frac{3(r-1)}{2} \rfloor |D| - \lfloor \frac{r-3}{2} \rfloor m + k|D|$. From the vertex set and edge set of $\mathcal{H}$, an auxiliary graph $\mathcal{H}'$ will be constructed by removing at least $k|D|$ vertices. This hypergraph will be constructed such that it will have a dominating set that dominates a k-forcing set of $\mathcal{H}$, which leads to a contradiction since this set will be a k-power dominating set in $\mathcal{H}$ that is smaller than D.

Let $\{v_1, v_2, \ldots, v_{|D|}\}$ be an ordering of the vertices in D. For each v_i, let P_{v_i} be a set of k vertices from the external private neighborhood of v_i. Let P be the union of each P_{v_i} and let $B = N[D] \setminus P$. Observe the B is a k-forcing set for $\mathcal{H}$.

To constuct $\mathcal{H}'$, each P_{v_i} will be iteratively removed. First consider $\mathcal{H} \setminus P_{v_1}$. This hypergraph may have isolated vertices since removing vertices from $\mathcal{H}$ also removes edges with $r - 1$ or fewer vertices. Let I_1 be the set of vertices that become isolated after removing P_{v_1}. For every vertex $x \in I_1$, pick exactly one vertex from P_{v_1} that is in an edge with x and v_1 in $\mathcal{H}$. Call this set of chosen vertices T_1. Let $|T_1| = t$.

In $\mathcal{H}$, if $deg_w(v_1, B \setminus I_1) \leq k$, let $I_1 = I_{v_1}$ and let $H_1 = \mathcal{H} \setminus P_{v_1}$. If $deg_w(v_1, B \setminus I_1) \geq k + 1$, the following process is done on $\mathcal{H} \setminus P_{v_1}$ to construct H_1. Since $deg_w(v_1, B \setminus I_1) \geq k + 1$, then $deg_w(v_1, N[D] \setminus (I_1 \cup T_1)) \geq t + 1$. Since $deg_w(v_1, N[D] \setminus (I_1 \cup T_1)) \geq t + 1$, there exists at least $t + 1$ vertices in I_1, each in a distinct edge containing v_1 and some vertex from T_1 in $\mathcal{H}$. Let A_1 be the set of these vertices in I_1 and let A_2 be the set of distinct edges containing these vertices.

Let $x \in A_1$, $e' \in A_2$ with $x \in e'$, and $C \subseteq A_1 \setminus \{x\}$ with $|C| = |e' \cap T_1|$, then construct the new edge $e = \{v_1, e' \setminus (e' \cap T_1), C\}$ in $\mathcal{H} \setminus P_{v_1}$ and note that this edge contains r vertices. Iteratively do this for every vertex in A_1. Once this process finishes, every vertex in I_1 is contained in one of the newly added edges. In this case, let $I_{v_1} = \{\emptyset\}$. Hence, in $\mathcal{H}$, $deg_w(v_1, B \setminus I_{v_1}) \leq k$. Let H_1 be the hypergraph constructed after this process is done on $\mathcal{H} \setminus P_{v_1}$.

The same process is then done on $H_1 \setminus P_{v_2}$ to get H_2 and continued until $H_{|D|}$ is constructed. Let $I = \bigcup I_{v_i}$ and $\mathcal{H}' = H_{|D|} \setminus I$. Let m' be the number of edges in $\mathcal{H}'$ and note that $m' \leq m$. Observe that by construction $B \setminus I$ is a k-forcing set for $\mathcal{H}$.

Through this construction $|V(\mathcal{H}')| \leq |V(\mathcal{H})| - k|D| < \lfloor\frac{3(r-1)}{2}\rfloor|D| - \lfloor\frac{r-3}{2}\rfloor m$ therefore by Theorem 4.3.5,

$$\begin{aligned}\gamma(\mathcal{H}') &< \frac{\lfloor\frac{3(r-1)}{2}\rfloor|D| - \lfloor\frac{r-3}{2}\rfloor m + \lfloor\frac{r-3}{2}\rfloor m'}{\lfloor\frac{3(r-1)}{2}\rfloor} \\ &\leq \frac{\lfloor\frac{3(r-1)}{2}\rfloor|D| - \lfloor\frac{r-3}{2}\rfloor m + \lfloor\frac{r-3}{2}\rfloor m}{\lfloor\frac{3(r-1)}{2}\rfloor} \\ &= |D|.\end{aligned}$$

This is a contradiction since a minimum dominating set of $\mathcal{H}'$ dominates $B \setminus I$ in $\mathcal{H}$. Therefore,

$$\gamma_p^k(\mathcal{H}) \leq \frac{n + \lfloor\frac{r-3}{2}\rfloor m}{\lfloor\frac{3(r-1)}{2}\rfloor + k}.$$

□

Setting $r = 3$, or 4, results in the following corollary.

Corollary 4.3.8. *For any connected r-uniform hypergraph $\mathcal{H}$ of order n, $k + r \leq n$, with no isolated vertices, with $r = 3$ or 4,*

$$\gamma_p^k(\mathcal{H}) \leq \frac{n}{r+k}.$$

The squid hypergraphs show that the bound in Theorem 4.3.7 is tight for $r = 3$ and $r = 4$ by Lemma 4.2.1.

The following construction from [2] is used to show for any $r \geq 5$ there exists a k such that $\frac{n+\lfloor\frac{r-3}{2}\rfloor m}{\lfloor\frac{3(r-1)}{2}\rfloor+k} = \gamma_p^k(\mathcal{H})$. Let the hypergraph $\mathcal{H}$ be defined as follows. The vertex set of $\mathcal{H}$ is partitioned as $V = V_1 \cup V_2 \cup V_3 \cup \{v_{12}, v_{13}, v_{23}\}$ where $|V_1| = \lfloor\frac{r-1}{2}\rfloor$ and $|V_2| = |V_3| = \lceil\frac{r-1}{2}\rceil$. Let

V_2' be a subset of V_2 of size $\lfloor \frac{r-1}{2} \rfloor$. Let the edge set of $\mathcal{H}$ consist of the edges $e_1 = V_1 \cup V_2 \cup \{v_{12}\}$, $e_2 = V_1 \cup V_3 \cup \{v_{13}\}$, and $e_3 = V_2' \cup V_3 \cup \{v_{23}\}$.

Notice that with this construction, $\gamma_p^k(\mathcal{H}) = 1$ for $k \geq 1$. Setting $k = \lfloor \frac{3(r-1)}{2} \rfloor$,

$$\frac{n + \lfloor \frac{r-3}{2} \rfloor m}{\lfloor \frac{3(r-1)}{2} \rfloor + k} = 1 = \gamma_p^k(\mathcal{H}).$$

4.4 Counterexamples

Although Conjecture 4.2.2 holds for $r \leq 4$, it is not true for $r \geq 7$. A counterexample is now given for $r \geq 7$. These hypergraphs are related to the hypergraphs constructed in [2] that show the domination number upper bound is tight. Let $X = \{x_1, x_2, \ldots, x_{k+1}\}$, $Y = \{y_1, y_2, \ldots, y_{k+1}\}$, and $Z = \{z_1, z_2, \ldots, z_{k+1}\}$, let A_1, A_2, and A_3 be sets of $k+2$ vertices, and let B_1 and B_2 be sets of ℓ vertices, $\ell \geq 0$. Let $\mathcal{H}$ be the hypergraph constructed with the following edge sets:

$$A_1 \cup B_1 \cup \{x_i\} \cup A_2,\ A_2 \cup B_1 \cup \{z_i\} \cup A_3 \text{ and } A_1 \cup B_2 \cup \{y_i\} \cup A_3,\ 1 \leq i \leq k+1.$$

$\mathcal{H}$ is a $(5 + 2k + \ell)$-uniform hypergraph with $9 + 6k + 2\ell$ vertices. If the conjecture is true, then

$$\gamma_p^k(\mathcal{H}) \leq \frac{9 + 6k + 2\ell}{5 + 3k + \ell} < 2.$$

Therefore $\gamma_p^k(\mathcal{H}) = 1$. If any vertex v in A_1, A_2, A_3, B_1 or B_2 is chosen as the k-power dominating set, or if $\{x_i\}, \{y_i\}$ or $\{z_i\}$ is chosen as the k-power dominating set, for some $i \in \{1, 2, \ldots, k+1\}$, then everything in $N[v]$ (these are the vertices that are forced blue in the first step of the process) will be incident to $k+1$ or more edges containing white vertices. Therefore, $\gamma_p^k(\mathcal{H}) \geq 2$ which contradicts the conjecture.

Originally, in [4], Chang and Roussel conjectured that the only connected r-uniform hypergraphs, $\mathcal{H}$, where $\gamma_p^k(\mathcal{H}) = \frac{n}{r+k}$, were the squid hypergraphs. We will show that there are many such hypergraphs by extending the original definition of a squid hypergraph.

Given integers $d \geq 1$, $k \geq 1$ and $r \geq 2$ and a vector of positive integers $x = (x_1, \dots, x_d)$ where $1 \leq x_i \leq r-1$, construct a (d, k, r, x)-squid as follows. Let

$$s_{1,1}, \dots, s_{1,r-x_1}, s_{2,1}, \dots, s_{2,r-x_2}, \dots, s_{d,1}, \dots, s_{d,r-x_d}$$

be called the *strong vertices* and let

$$w_{1,1}, \dots, w_{1,k+x_1}, w_{2,1}, \dots, w_{2,k+x_2}, \dots, w_{d,1}, \dots, w_{d,k+x_d}$$

be called the *weak vertices*. Call the set of all vertices with first index i, the *i-th spine*. Within each spine, add $k+1$ edges containing r vertices between the strong vertices and weak vertices such that no subset of them cover the weak vertices in the spine. These edges will contain every vertex in the strong portion of the spine. Finally, add any edges as desired among the strong vertices (even amongst different spines), so long as the hypergraph is connected and r-uniform.

Proposition 4.4.1. *Let $\mathcal{H}$ be a (d, k, r, x)-squid with $d(r+k) = |V(\mathcal{H})|$, then $\gamma_p^k(\mathcal{H}) = d$.*

Proof. To show that $\gamma_p^k(\mathcal{H}) = d$, observe that any set that intersects the strong portion of each spine is a dominating set and hence a k-power dominating set. It suffices to show that any k-power dominating set must necessarily have one vertex within each spine. Suppose to the contrary that there is a k-power dominating set that does not intersect one of the spines. Since edges containing weak vertices must be contained with that spine, in order to color the weak vertices within that spine, one of the corresponding strong vertices must force. However, observe that since the spine requires $k+1$ edges to cover its weak vertices, no strong vertex can force. This completes the proof. □

4.5 Conclusion

We have shown that Conjecture 4.2.2 does not hold for $r \geq 7$, and have shown a new upper bound for the k-power domination number that is related to the upper bound of the domination number given in [2]. The following questions still remain.

Question 4.5.1. *Does Conjecture 4.2.2 hold for* $r = 5$ *or* 6*?*

It is interesting to note that if $\frac{n}{r+k}$ is replaced with $\left\lceil \frac{n}{r+k} \right\rceil$ in Conjecture 4.2.2, then that equation would hold for the counterexamples given in the previous section.

Question 4.5.2. *Is Theorem 4.3.7 tight?*

As of now, there is no known example of a hypergraph with $\gamma_p^k = \frac{n+\lfloor \frac{r-3}{2} \rfloor m}{\lfloor \frac{3(r-1)}{2} \rfloor + k}$ for all r and k.

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CHAPTER 5. A METHOD FOR FINDING UPPER BOUNDS FOR THE K-POWER DOMINATION NUMBER IN HYPERGRAPHS

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Abstract

In this paper the generalization of power domination, known as k-power domination, is explored in hypergraphs. We give new upper bounds for the k-power domination number, and in doing so, prove a conjecture given by Bjorkman that states the power domination number in a hypergraph $\mathcal{H}$ on at least four vertices with minimum edge size three is at most $\frac{|V(\mathcal{H})|}{4}$. This results further suggests that given a known upper bound for the domination number for certain types of hypergraphs, there exists an analogous upper bound for the k-power domination number.

Keywords zero forcing, domination, power domination, hypergraphs

AMS subject classification 05C57, 05C15, 05C50, 05C65

5.1 Introduction

Power domination was first introduced by Haynes, Hedetniemi, Hedetniemi and Henning in [8] to study the monitoring process of electrical power networks by placing as few measurement devices as possible. In particular, Phase Measurement Units (PMUs) were placed at a set of initial vertices and then certain rules were applied. These rules consisted of a domination step followed by the zero forcing process.

Zero forcing was first introduced in [2] as a way to find upper bounds for the maximum nullity of real symmetric matrices whose nonzero off-diagonal entries are described by a graph. Zero forcing has since been generalized, and specifically in hypergraphs, there have been multiple generalizations. In this paper, the generalization introduced in [6] as a process in k-power domination in hypergraphs will be used. Moreover, this generalization also is consistent with the definition of k-power domination in simple graphs.

Chang and Roussel introduced k-power domination in hypergraphs as a generalization of power domination. In this paper, new upper bounds for the k-power domination number are given and a conjecture from [4] is proven. In [4], Bjorkman conjectured the following.

Conjecture 5.1.1. *[4] For any connected hypergraph $\mathcal{H}$ on at least $n \geq 4$ vertices with edge size at least 3 for all $e \in E(\mathcal{H})$, $\gamma_{P_I}(\mathcal{H}) \leq \gamma_p^1(\mathcal{H}) \leq \frac{n}{4}$.*

The lower bound for this conjecture is called the *infectious power domination number* which was introduced by Bjorkman in [4] and uses a different coloring process introced by Bergen et al. in [3]. The first inequality was shown to be true in [4] and the upper bound will be proven in Section 5.3 by proving a more general bound on the k-power domination number. Two main statements will be proven, and in doing so, various upper bounds related to the domination number will be given as corollaries.

5.2 Preliminaries

A *hypergraph*, $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, is a set of vertices $V(\mathcal{H})$ combined with a set of edges $E(\mathcal{H})$ such that $E(\mathcal{H})$ is a subset of the power set of $V(\mathcal{H})$. When it is obvious which hypergraph is being used, V and E will be written. A hypergraph $\mathcal{H}$ is said to be *k-uniform* when each edge in E has order k.

The *closed neighborhood* of a vertex $v \in V$, $N[v]$, is the set of vertices adjacent to v and v itself. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N[v] \setminus \{v\}$. The closed (or open) neighborhood of a set S is the set $\bigcup_{v\in S} N[v]$ (or $\bigcup_{v\in S} N(v)$) and is denoted $N[S]$ (or $N(S)$).

Define the *white degree* of a vertex v with respect to a set S in $\mathcal{H}$, denoted $deg_w(v, S)$, to be the minimum number of edges that cover v and the white neighbors of v when each vertex in S

is blue and each vertex in $V(\mathcal{H}) - S$ is white. The set of *external private neighbors* of a vertex v with respect to a set of vertices S in a hypergraph $\mathcal{H}$, denoted $epn(v, S)$, is the set of vertices not in S that are adjacent to v, but not to any other vertex is S.

A *transversal* in a hypergraph $\mathcal{H}$ is a subset of vertices in $V(\mathcal{H})$ that has a nonempty intersection with every edge in $E(\mathcal{H})$. The *transversal number* of a hypergraph $\mathcal{H}$, denoted $\tau(\mathcal{H})$, is the minimum size of a transversal. A set of vertices D is known as a *dominating set* of a hypergraph $\mathcal{H}$ if $\bigcup_{v \in D} N[v] = V$. The size of a minimum dominating set in a hypergraph $\mathcal{H}$ is denoted $\gamma(\mathcal{H})$ and is called the *domination number*.

The *k-power domination process* in a hypergraph $\mathcal{H}$, introduced by Chang and Roussel in [6], uses the following color-change rule:

k-Forcing Rule: [6] A vertex v in a blue set B can force the white vertices in its neighborhood blue if $deg(v, B) \leq k$.

A *k-forcing set* in a hypergraph $\mathcal{H}$ is a set of blue vertices B such that after iteratively applying the k-forcing color-change rule, all of $\mathcal{H}$ becomes blue. Therefore, a set D is a *k-power dominating set* of a hypergraph $\mathcal{H}$ if $N[D]$ is a k-forcing set. The size of a minimum k-power dominating set in a hypergraph $\mathcal{H}$ is denoted $\gamma_p^k(\mathcal{H})$ and is called the *k-power domination number*.

5.3 Main results

In this section the following statements will be proven:

1. If $\gamma(\mathcal{H}) \leq \frac{c_1 n + c_2 m + j}{\ell}$ for every hypergraph $\mathcal{H}$ on n vertices and m edges, where $\ell \geq 1$, $j \geq 0$, $c_1 \geq 1$ and $c_2 \geq 0$, then $\gamma_p^k(\mathcal{H}') \leq \frac{c_1 n' + c_2 m + j}{\ell + k}$ for every hypergraph $\mathcal{H}'$ on n' vertices.

2. If $\gamma(\mathcal{H}) \leq \frac{c_1 n + c_2 m + j}{\ell}$ for every hypergraph $\mathcal{H}$ of order n with m edges, no isolated vertices and edges of size at least three, where $\ell \geq 1$, $j \geq 0$, $c_1 \geq 1$, and $c_2 \geq 0$, then $\gamma_p^k(\mathcal{H}') \leq \frac{c_1 n' + c_2 m + j}{\ell + k}$ for every hypergraph $\mathcal{H}'$ of order $n' \geq k + 3$ with no isolated vertices and edges of size at least three.

Various upper bounds will then be given as corollaries using known bounds on the domination number.

To prove these statements, a similar method introduced by Alameda et al. in [1] will be used. In this method, an auxiliary hypergraph will be created from the vertex set and edge set of the original hypergraph by removing at least $k\gamma_p^k(\mathcal{H})$ vertices. This hypergraph will be constructed such that it has a dominating set that colors a k-forcing set of $\mathcal{H}$ and by the domination number bound assumption, it will be a smaller k-power dominating set than originally chosen. To be able to do this, there has to be at least $k\gamma_p^k(\mathcal{H})$ vertices that can be deleted. To guarantee this happens, the following lemmas from [1] will be used.

Lemma 5.3.1. *[1] Let $\mathcal{H}$ be a connected hypergraph with $k+3 \leq n$ vertices. There exists a minimum k-power dominating D set such that $k+1 \leq deg_w(v, N[D \backslash \{v\}])$ for all $v \in D$.*

Lemma 5.3.2. *[1] Let $\mathcal{H}$ be a hypergraph and let D be a minimum k-power dominating set of $\mathcal{H}$ such that $k+1 \leq deg_w(v, N[D \setminus \{v\}])$. Then $k+1 \leq deg_w(v, N[D \setminus \{v\}]) \leq |epn(v, D)|$, for all $v \in D$.*

Note that $k+1 \leq |epn(v, D)|$ for some k-power dominating set was first observed by Chang and Roussel in [6].

Theorem 5.3.3. *If $\gamma(\mathcal{H}) \leq \frac{c_1 n + c_2 m + j}{\ell}$ for every hypergraph $\mathcal{H}$ on n vertices and m edges, where $\ell \geq 1$, $j \geq 0$, $c_1 \geq 1$ and $c_2 \geq 0$, then $\gamma_p^k(\mathcal{H}') \leq \frac{c_1 n' + c_2 m + j}{\ell + k}$ for every hypergraph $\mathcal{H}'$ on n' vertices.*

Proof. Assume that $\gamma(\mathcal{H}) \leq \frac{c_1 n + c_2 m + j}{\ell}$ for every hypergraph $\mathcal{H}$ on n vertices, where $\ell \geq 1$, $j \geq 0$, $c_1 \geq 1$ and $c_2 \geq 0$. For the sake of contradiction, assume there exists a hypergraph $\mathcal{H}$ on n' vertices, such that $n' < \frac{\ell|D|}{c_1} + \frac{k|D|}{c_1} - \frac{j}{c_1} - \frac{c_2 m}{c_1}$ where D is a minimmum k-power dominating set. By Lemma 5.3.2, choose D such that for all $v \in D$, $k+1 \leq deg_w(v, N[D \backslash \{v\}]) \leq |epn(v, D)|$.

Let $\{v_1, v_2, \ldots, v_{|D|}\}$ be an ordering of the vertices in D. For each v_i, let P_{v_i} be a set of k vertices from the external private neighborhood of v_i. Let P be the union of each P_{v_i} and let $B = N[D] \setminus P$. Notice that B is a k-forcing set in $\mathcal{H}$.

Iteratively start removing P_{v_i} from $\mathcal{H}$. If there are isolated vertices after removing P_{v_i} from $\mathcal{H}$ that were adjacent to v_i in $\mathcal{H}$, add an edge between these vertices and v_i.

Let H_1 be the hypergraph constructed after removing P_{v_1} from $\mathcal{H}$ and going through the above step, let H_2 be the hypergraph constructed after removing P_{v_2} from H_1 and going through the above step. Continue until $H_{|D|}$ is constructed and let $\mathcal{H}'$ be the hypergraph constructed from $H_{|D|}$ after removing any remaining isolated vertices. Note $|V(\mathcal{H}')| = n' - k|D| < \frac{\ell|D|}{c_1} + \frac{k|D|}{c_1} - \frac{j}{c_1} - \frac{c_2 m}{c_1} - k|D| < \frac{\ell|D|-j-c_2m}{c}$ and $\mathcal{H}'$ has $m' \leq m$ edges. By assumption,

$$\begin{aligned}\gamma(\mathcal{H}') &< \frac{c_1\frac{\ell|D|-j-c_2m}{c_1} + j + c_2m'}{\ell} \\ &\leq \frac{c_1\frac{\ell|D|-j-c_2m}{c_1} + j + c_2m}{\ell} \\ &= |D|.\end{aligned}$$

This is a contradiction since a minimum dominating set of $\mathcal{H}'$ dominates a k-forcing set of $\mathcal{H}$. In particular, this dominating set colors B blue. □

In some sense, Theorem 5.3.3 is the most general form of a statement like this. Notice that if restrictions are placed on the initial hypergraph (uniformity, minimum degree, etc.), then the only part of the proof that needs to change is the construction of the auxiliary hypergraph (or graph).

Theorem 5.3.4. *If $\gamma(\mathcal{H}) \leq \frac{c_1n+c_2m+j}{\ell}$ for every hypergraph $\mathcal{H}$ of order n with m edges, no isolated vertices and edges of size at least three, where $\ell \geq 1$, $j \geq 0$, $c_1 \geq 1$, and $c_2 \geq 0$, then $\gamma_p^k(\mathcal{H}') \leq \frac{c_1n'+c_2m+j}{\ell+k}$ for every hypergraph $\mathcal{H}'$ of order $n' \geq k+3$ with no isolated vertices and edges of size at least three.*

Proof. Assume that $\gamma(\mathcal{H}) \leq \frac{c_1n+c_2m+j}{\ell}$ for every hypergraph $\mathcal{H}$ of order $n \geq q$ with no isolated vertices and edges of size at least three, where $\ell \geq 1$, $j \geq 0$, $c_1 \geq 1$, and $c_2 \geq 0$. For the sake of contradiction, assume there exists a hypergraph $\mathcal{H}$ on $n' \geq k+3$ vertices with no isolated vertices and edges of size at least three, such that $n < \frac{\ell|D|}{c_1} + \frac{k|D|}{c_1} - \frac{j}{c_1} - \frac{c_2m}{c_1}$ where D is a minimmum k-power dominating set. By Lemma 5.3.2, choose D such that for all $v \in D$, $k+1 \leq deg_w(v, N[D \backslash \{v\}]) \leq |epn(v, D)|$.

Let $\{v_1, v_2, \ldots, v_{|D|}\}$ be an ordering of the vertices in D. For each v_i, let P_{v_i} be a set of k vertices from the external private neighborhood of v_i. Let P be the union of each P_{v_i} and let $B = N[D] \setminus P$. Notice that B is a k-forcing set in $\mathcal{H}$ and observe that each P_{v_i} can be chosen such that v_i is in a connected component if P_{v_i} is removed since $k+1 \leq deg_w(v_i, N[D \setminus \{v_i\}])$ for each i.

Iteratively start removing P_{v_i} from $\mathcal{H}$. At each step, there are two cases to consider, either there is one vertex adjacent to v_i in $\mathcal{H}$ that becomes isolated after removing P_{v_i}, or there are two or more vertices that are adjacent to v_i in $\mathcal{H}$ that become isolated after removing P_{v_i}.

Case 1: There is one vertex adjacent to v_i in $\mathcal{H}$ that becomes isolated after removing P_{v_i}.

Now assume there is one vertex u adjacent to v_i in $\mathcal{H}$ that becomes isolated after removing P_{v_i}. In this case, delete u.

Observation 5.3.5. *Observe that in $\mathcal{H}$, $B \setminus \{u\}$ is still a k-forcing set.*

Case 2: There are two or more vertices that are adjacent to v_i in $\mathcal{H}$ that become isolated after removing P_{v_i}.

In this case make an edge between v_i and the remaining isolated vertices.

Let H_1 be the hypergraph constructed after removing P_{v_1} from $\mathcal{H}$ and going through the above cases, let H_2 be the hypergraph constructed after removing P_{v_2} from H_1 and going through the above cases. Continue until $H_{|D|}$ is constructed and let $\mathcal{H}'$ be the hypergraph constructed from $H_{|D|}$ after removing any remaining isolated vertices. Let I be the set of vertices that were removed in Case 1 and note that $B \setminus I$ is a k-forcing set for $\mathcal{H}$. Notice $|V(\mathcal{H}')| = n' - k|D| < \frac{\ell|D|}{c_1} + \frac{k|D|}{c_1} - \frac{j}{c_1} - \frac{c_2 m}{c_1} - k|D| < \frac{\ell|D| - j - c_2 m}{c}$ and that $\mathcal{H}'$ has at most as many edges as $\mathcal{H}$. By assumption,

$$\begin{aligned}\gamma(\mathcal{H}') &< \frac{c_1 \frac{\ell|D|-j-c_2 m}{c_1} + j + c_2 m'}{\ell} \\ &\leq \frac{c_1 \frac{\ell|D|-j-c_2 m}{c_1} + j + c_2 m}{\ell} \\ &= |D|.\end{aligned}$$

This is a contradiction since a minimum dominating set of $\mathcal{H}'$ dominates a k-forcing set of $\mathcal{H}$ even if any of the above cases were applied. In particular, this dominating set colors $B \setminus I$ blue. □

Now a known upper bound from [10] will be used to give an upper bound on the k-power domination number.

Theorem 5.3.6. *[10] If $\mathcal{H}$ is a hypergraph on n vertices and m edges with all edges of size at least three, then $\tau(\mathcal{H}) \leq \frac{n+m}{4}$.*

Corollary 5.3.7. *If $\mathcal{H}$ is a connected hypergraph with no isolated vertices on $n \geq k+3$ vertices and m edges with all edges of size at least three, then $\gamma_k^p(\mathcal{H}) \leq \frac{n+m}{4+k}$.*

Proof. Observe that if $\mathcal{H}$ is a hypergraph with no isolated vertices, then $\gamma(\mathcal{H}) \leq \tau(\mathcal{H})$. Therefore, if $\mathcal{H}$ is a hypergraph with no isolated vertices, then $\gamma(\mathcal{H}) \leq \frac{n+m}{4}$ by Theorem 5.3.6. Set $c_1 = 1$, $c_2 = 1$, $\ell = 4$ and $j = 0$ in Theorem 5.3.4. □

Even more bounds for the k-power domination number can be found using known bounds for the domination number.

Theorem 5.3.8. *[9] If $\mathcal{H}$ is a hypergraph of order n with no isolated vertices and edges of size at least three, then $\gamma(\mathcal{H}) \leq \frac{n}{3}$.*

Corollary 5.3.9. *If $\mathcal{H}$ is a connected hypergraph of order $n \geq k+3$ with no isolated vertices and edges of size at least three, then $\gamma_k^p(H) \leq \frac{n}{3+k}$.*

Proof. Use Theorem 5.3.8 and let $c_1 = 1$, $c_2 = j = 0$, and $\ell = 3$ in Theorem 5.3.4. □

Now letting $k = 1$, Conjecture 5.1.1 is proven as a corollary to Corollary 5.3.9. Furthermore, this bound is tight by various different hypergraphs.

Corollary 5.3.10. *For any connected hypergraph $\mathcal{H}$ on at least $n \geq 4$ vertices with edge size at least 3 for all $e \in E(\mathcal{H})$, $\gamma_{P_I}(\mathcal{H}) \leq \gamma_p^1(\mathcal{H}) \leq \frac{n}{4}$.*

One family of hypergraphs introduced by Chang and Roussel in [6], known as the squid hypergraphs, achieve the bound in Corollary 5.3.9. Figure 5.1 is an example of a 3-uniform squid hypergraph on 12 vertices. The following is the construction of a 3-uniform squid hypergraph.

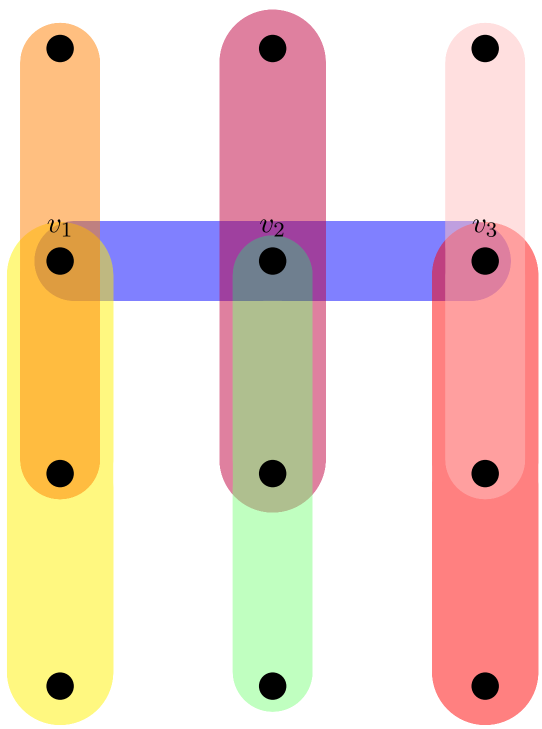

Figure 5.1 A 3-uniform squid hypergraph

Construction: Let $\mathcal{H} = (V, E)$ be any 3-uniform hypergraph. For any vertex $v \in V(\mathcal{H})$, add $k+2$ vertices $v_1, v_2, \ldots, v_{k+1}, v_1'$ and $k+1$ edges $e_{v,i} = \{v, v_1', v_i\}$ for $1 \leq i \leq k+1$. Call this constructed hypergraph $\mathcal{H}' = (V', E')$ the squid hypergraph of $\mathcal{H}$.

First note that $|V(\mathcal{H})| = \frac{|V(\mathcal{H}')|}{k+3}$ To see that this achieves the upper bound, notice that every vertex in $\{v_1, v_2, \ldots, v_{k+1}\}$ has all of its neighbors in $\{v, v_1'\}$ and v and v_1' are adjacent to every vertex in $\{v_1, v_2, \ldots, v_{k+1}\}$ in $k+1$ distinct edges. So a minimum k-power dominating set will contain one vertex from $\{v, v_1', v_1, v_2, \ldots, v_{k+1}\}$ for every $v \in V(\mathcal{H})$. Hence $\gamma_p^k(\mathcal{H}') = \frac{|V(\mathcal{H}')|}{k+3}$.

But what about requiring other restrictions? In [1], the following bound was proven for r-uniform hypergraphs.

Theorem 5.3.11. *[1] If $\mathcal{H}$ is a connected, r-uniform hypergraph of order n, $k+r \leq n$, with m edges, and $r \geq 3$, then*

$$\gamma_p^k(\mathcal{H}) \leq \frac{n + \lfloor \frac{r-3}{2} \rfloor m}{\lfloor \frac{3(r-1)}{2} \rfloor + k}.$$

The above result was proven using the related bound from [5] on the domination number in r-uniform hypergraphs.

Theorem 5.3.12. *[5] If $\mathcal{H}$ is an r-uniform hypergraph of order n with m edges and no isolated vertices, and $r \geq 3$, then*

$$\gamma(\mathcal{H}) \leq \frac{n + \lfloor \frac{r-3}{2} \rfloor m}{\lfloor \frac{3(r-1)}{2} \rfloor}.$$

This suggests there should be a theorem like Theorem 5.3.3 and Theorem 5.3.4 for r-uniform hypergraphs.

In [7], Erdős and Tuza proved the following.

Theorem 5.3.13. *[7] If $\mathcal{H}$ is a connected hypergraph on n edges and m edges with all edges of size at least two, then $\tau(\mathcal{H}) \leq \frac{2(n+m+1)}{7}$.*

Therefore, the following conjecture is given.

Conjecture 5.3.14. *If $\mathcal{H}$ is a connected hypergraph on n edges and m edges with all edges of size at least two, then $\gamma_p^k(\mathcal{H}) \leq \frac{2(n+m+1)}{7+k}$.*

In the proofs of Theorem 5.3.3 and 5.3.4 after removing vertices, it did not matter that the hypergraph became disconnected. The bounds for the domination number still applied. In order to prove the above conjecture, there needs to be a way to reconnect a disconnected hypergraph without affecting the domination in the original hypergraph.

5.4 Concluding remarks

Although it has been suggested that given some bound on the domination number, an analogous bound on the k-power domination number can also be given, there is still much to show. To prove a statement like this, we need to prove the following.

- If $\gamma(\mathcal{H}) \leq \frac{c_1 n + c_2 m + j}{\ell}$ for every hypergraph $\mathcal{H}$ given ANY restriction, where $\ell \geq 1$, $j \geq 0$, $c_1 \geq 1$, and $c_2 \geq 0$, then $\gamma_p^k(\mathcal{H}) \leq \frac{c_1 n + c_2 m + j}{\ell + k}$.

Another interesting question to look at would be to see if giving a tight bound on the domination number results in an analogous tight bound for the k-power domination number. Most of the domination bounds used in this paper are not known to be tight. That being said, the bound given by [9] is tight and the resulting bound in Corollary 5.3.9 is as well.

5.5 References

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CHAPTER 6. GENERAL CONCLUSIONS

Section 1.2 provided basic theory and tools that were used throughout this thesis. Then, in Section 1.3, a review of the literature on the zero forcing process, leaky forcing process, and power domination process was given. Chapter 2 provided a characterization for ℓ-leaky forcing sets; furthermore, ℓ-resilience was introduced and the ℓ-leaky forcing number for certain graph families was explored. Through the characterization of ℓ-leaky forcing sets in Chapter 2, edge variants of leaky forcing were then characterized in Chapter 3. The k-power domination number in r-uniform hypergraphs was then explored in Chapter 4. In this chapter, a new upper bound was given for r-uniform hypergraphs, and counterexamples to a conjecture by Chang and Roussel in [2] were also provided. Finally, in Chapter 5, the domination number and k-power domination were compared and a conjecture by Bjorkman in [1] was proven using techniques from Chapter 4.

For future research, it would be interesting to merge both leaky forcing and power domination together. Since zero forcing is a major step in power domination, it would make sense to explore what happens when leaks are present in a power network. Although k-power domination was mostly explored in hypergraphs, it would also be interesting to research if the techniques used in Chapter 4 and Chapter 5 work well with simple graphs. Moreover, there are many known domination number bounds for simple graphs which could provide a useful starting point for further k-power domination research in the area.

6.1 References

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