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# Lateral stability of the driver/vehicle system: analytical results

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LATERAL STABILITY OF THE DRIVER/VEHICLE SYSTEM - ANALYTICAL RESULTS

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# Lateral stability of the driver/vehicle system — analytical results

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## LIST OF SYMBOLS

| | |
|---|---|
| $a_i$ | Coefficients of the quartic |
| $B_{ij}$ | Coefficients of the equations of motion |
| $C_1$, $C_2$ | Cornering stiffness of front and rear tires, respectively |
| $D_Y$, $D_\psi$ | Constant gains of driver associated with lateral displacement and yaw angle, respectively |
| D, E | Expressions in quartic solution |
| $F_{1x}$, $F_{1y}$ | Forces on front tire in the longitudinal and normal directions, respectively |
| $F_{2x}$, $F_{2y}$ | Forces on rear tire in the longitudinal and normal directions, respectively |
| g | Acceleration of gravity |
| $I_z$ | Moment of inertia of the automobile about a vertical axis through the mass center |
| $K_{us}$ | Understeer coefficient |
| $K_1$, $K_2$, $K_3$ | Expressions in the equation for the critical velocity of the driver/vehicle system |
| $L_1$, $L_2$ | Lengths from the mass center to the front and rear axles, respectively |
| m | Mass of the vehicle |
| P, Q | Matrices in the Lyapunov matrix equation |
| R | Expression in quartic solution |
| t | Time |
| $\underline{u}_x$, $\underline{u}_y$ | Unit vectors in the x and y directions, respectively |

| | |
|---|---|
| $V_{CRIT}$ | Critical speed from classical result |
| $V_{CRIT_{D/V}}$ | Critical speed of the driver/vehicle system |
| $\underline{V}_{t1}$, $\underline{V}_{t2}$ | Velocity of the front and rear tires, respectively |
| $V_x$, $V_y$ | Speeds in the x and y directions, respectively |
| $V$ | Trial Lyapunov function |
| $x$, $y$ | Coordinate directions fixed in the vehicle |
| $X$, $Y$ | Absolute coordinate directions |
| $\alpha_1$, $\alpha_2$ | Slip angles of front and rear tires, respectively |
| $\delta$ | Steer angle of front wheels |
| $\Delta$ | Denominator in expressions for the elements of P |
| $\lambda$ | Roots of characteristic equation |
| $\psi$ | Yaw angle |
| $\Omega_z$ | Angular speed |

# I. INTRODUCTION

The subjective evaluation, made by the driver, of the response of the driver/vehicle system is called handling. Handling is extremely difficult to describe quantitatively. When the driver describes handling, he speaks in qualitative terms, not in quantitative terms. Another difficulty is that the driver's evaluation is based on his preferences. In fact, the handling of one automobile may be rated good by one driver and poor by another. However, there is one handling quality that is demanded by all drivers; that is the quality of lateral stability.

The classical analytical vehicle handling result is for the lateral stability of a car with fixed control of steering (the direction of the front wheels is fixed with respect to the rest of the vehicle) and no roll freedom traveling in a straight line at constant forward speed. The classical result is used extensively in the automobile industry and by government, industrial and university researchers. However, the classical model is too simple. It does not provide a completely adequate description of vehicle dynamic behavior. The classical model does not include important features such as roll freedom, driver control, nonlinearities and time-delays.

More complete vehicle models have been developed and are being used for computer simulation. But it is very difficult to establish the validity of such simulations and it is practically impossible to reach general conclusions from simulation results. As Bidwell [1] has said, "Although the development of elaborate computer simulation has permitted computation of vehicle directional behavior over a wide range of circumstances, most of the physical understanding of the relation of design parameters to performance is obtained by studies of much simpler systems."

Design improvements have, historically, been evolutionary in nature. Design changes were small and developed over long periods of time. This is no longer true. Radical design changes must now be made rapidly if the American automobile industry is to survive. Vehicles must become smaller, lighter and more efficient. Design procedures must be improved. Currently they are costly and time consuming. One way to improve vehicle design procedures is to extend the analytical foundation for vehicle design. The objective of this dissertation is to present analytical results for the stability of the driver/vehicle system. These new results will help to fill the gap between the classical analytical result and complex computer simulation results.

In this dissertation a system will be called stable if all the roots of the characteristic equation have negative real parts, with the exception of the statement of the Lyapunov theorem where the words asymptotically stable are used.

# II. LITERATURE REVIEW

Literature concerning the handling of the automobile is volumous. Therefore, the following review is not exhaustive, but rather is intended to be illustrative, dealing with those papers considered to be representative and significant in the field of automobile handling.

The following review is divided into two sections. The first section is devoted to the vehicle. The second section is devoted to the driver.

## A. Vehicle Literature

The initial portion of this section dealing with the vehicle will be concerned with tires. In order to describe the motion of a vehicle, the forces acting on the vehicle must be known. The most significant forces acting on the automobile arise from the tire/road interaction.

The lateral force acting on a rolling tire is a function of the slip angle. The slip angle is the angle between the velocity vector of the wheel center and the vertical plane of the tire. The slip angle is illustrated in Figure 1.

The force-slip angle concept has been attributed to Brouhleit [2]. Among the first to experimentally measure the force-slip angle relationship were Bradley and Allen [3]. They also measured the braking force-slip angle relationship. The effect of rim width, section width and number of plies on tire properties was investigated by Joy and Hartley [4]. Nordeen and Cortese [5] have described both force and moment characteristics of rolling tires. The interrelationship of longitudinal and lateral tire forces has been discussed by

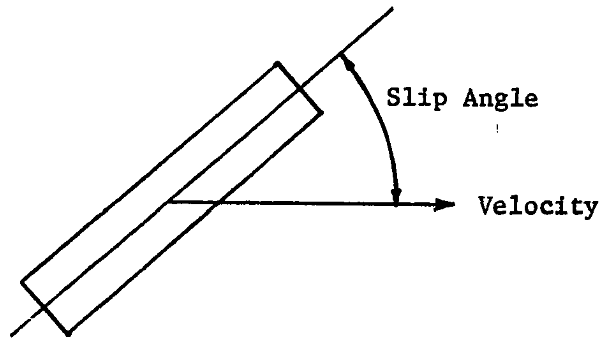


Figure 1. Top view of tire, illustrating the slip angle

Dugoff et al. [6].

The early work with tires concerned itself primarily with steady state properties. More recent work concerns itself with transient behavior. Bergman and Beauregard [7] and Weber and Persch [8] have presented data on the transient behavior of the force-slip angle relationship. An empirical model has been developed by Lippmann and Oblizajek [9] from transient tire test data. Lippmann and Oblizajek concluded from their model that transient effects are small and therefore have insignificant effect on vehicle behavior. But even though small, transient effects may contain important sensory information for the driver.

Calspan Corporation has been involved in tire testing for many years [10]. They have compiled a large amount of tire performance data. In particular, Roland et al. [11] and Schuring et al. [12] have reported work concerning vehicle handling as influenced by tire properties.

Having reviewed briefly some important tire work, now consider the vehicle as a whole. Most steady state behavior of the automobile was understood by 1950. Lind Walker [13] discussed the understanding of vehicle behavior at that time.

Differential equations of motion for the automobile were not developed until 1953. Schilling [14] is credited with being among the first to develop and solve dynamic equations of motion describing the lateral motion of the automobile. Schilling solved his equations for "typical" vehicle parameter values. This work was followed by research at Cornell Aeronautical Laboratory (CAL). An introduction to research at CAL and an excellent historical overview of early automotive development are given by Milliken and Whitcomb [15]. One of the contributions of the research at CAL was the development and experimental substantiation of the linear equations of motion associated with a three degree of freedom model [16].

The classical analytical result for the lateral stability of an automobile for the case of constant speed straight line motion is based on the linear three degree of freedom model. The classical result gives the speed at which an oversteer vehicle will become unstable and shows that an understeer vehicle is stable for all forward speeds [17].

During the early 1960s, several advances were made. One was the incorporation of tire nonlinearities into vehicle models. Analog computer simulations were developed to make use of these new models. Beauvais et al. [18] and Nordeen [19] have reported results of their analog simulations. Another advancement came in the consideration of the effect of aerodynamic forces on handling response.

Bundorf et al. [20] reported experimental results involving aerodynamic forces and compared their results with analytical results. Segel [21] showed that the dynamics of the steering system can influence the lateral stability of the automobile.

In the late 1960s and early 1970s, digital and hybrid simulation were developed [22-29]. These simulations range from the nonlinear seven degree of freedom simulation of Chiesa and Rinonapoli [22] to the thirty-two degree of freedom simulation of Bohn and Keenan [28]. These simulations can include nonlinear effects of tires, suspension components and trailers. They have been used to examine the effect of braking and cornering, spinout, wheel lift and roll over behavior, as well as steady turning, sloams and J-turns.

These complex models of the vehicle do provide a more complete model of the vehicle than did the models used in earlier studies. But it is very difficult to establish the validity of such simulations and it is practically impossible to reach general conclusions from the simulation results.

For example, a hybrid simulation program called the Hybrid Computer Vehicle Handling Program (HVHP) [28] has been developed at the Johns Hopkins University Applied Physics Laboratory. The HVHP requires the use of both analog and digital computers and has a nonlinear seventeen degree of freedom model of a four wheeled vehicle. The HVHP was validated by comparing simulation results with full-scale test data for passenger cars performing specific test procedures, such as straight line braking and sinusoidal steer. To use the HVHP, it is necessary to provide a very large number of vehicle parameters and to carry out the simulation at the Applied Physics Laboratory.

Another example is a digital computer program for simulating the directional response of commercial trucks and tractor-trailers developed at the University of Michigan's Highway Safety Research Institute [29]. The simulated vehicle can have up to thirty-two degrees of freedom. Nonlinear effects such as the relation between cornering force, side slip angle and longitudinal slip are included in the simulation. Full-scale tests were conducted as part of the Highway Safety Research Institute study and the results compared with simulation results. In general, the simulation results and full-scale test results compared favorably. However, in some cases, there were considerable unexplained differences. For instance, the lateral acceleration and yaw rate values from the simulation of a loaded tractor-trailer in a steady turn were from 28% to 62% higher than the associated full-scale test values.

The compliance concept is based on the observation that the forces acting on many parts of a vehicle during typical maneuvers do not deviate significantly from static-equilibrium values. Thus, approximate analyses of the deflections of such components can be made on a quasi-static basis. Nedley and Wilson [30] have provided useful definitions for lateral-force compliance steer, lateral-force compliance camber, aligning-torque compliance steer due to slip, aligning-torque compliance steer due to camber, body aligning-torque steer, and brake steer. The cornering compliance concept has been developed by Bundorf and Leffert [31] as an extension of the compliance concept. Cornering compliance for rolling vehicles has been considered by Winsor [32].

## B. Driver Literature

Without question the most difficult part of the driver/vehicle system to model is the driver. Humans have tremendous ability to gather information, analyze situations and generally respond in an effective way. These qualities make the human a good driver, but a complex one. The human driver operates both as an open loop controller and as a closed loop controller. Examples of open loop control are lane changes, avoidance maneuvers, and turning from one street onto another. Closed loop control is used to maintain position within a lane. Most driver models deal with either open loop control or closed loop control, but not both. Several driver models have been developed and are briefly discussed below.

Based on man-machine theory, McRuer and Krendel [33] developed the crossover model. The crossover model views the driver as a feedback controller with many possible loop closures. The underlying idea of the crossover model is that the driver desires a certain performance from the driver/vehicle system and he will adapt his performance to achieve the overall desired performance. Weir and McRuer [34] have presented results of simulator studies to validate the crossover model. They concluded that an outer loop dealing with lateral position was necessary for lateral stability. Also necessary was a loop closure of at least one of the following: path angle, path rate, heading angle or heading rate, with two or more giving more realistic results.

Another type of driver model is the predictive model. Kondo and Ajimine [35] developed the sight point model. In the sight point model, the driver looks down the road some prescribed distance and steers as a function of the difference between the point he is headed toward and the desired path. A similar model was proposed by Yoshimoto [36] in which the driver predicts the future position of the vehicle system using the systems present position, yaw angle and yaw velocity. Again, the driver steers as a function of the difference between the predicted position and the desired position.

Some work has been done in trying to fit driver models to actual driver responses. Weir and Wojcik [37] and McRuer and Klein [38] have reported the results of simulator studies involving the crossover model. In these studies, the crossover model was fitted to actual driver responses. Carson and Weirwille [39] developed a driver model that considers the driver to be a proportional controller with thresholds below which he does not respond. The proportional control operates on the yaw angle and lateral displacement. This model also incorporates a time delay to simulate a real driver's reaction time. This model was fitted to responses of people driving a simulator.

The literature reviewed above indicated that there exists a classical analytical result, a few analog simulations concerned primarily with nonlinear tire properties, a large number of complex computer simulations and several driver models. What is needed is further analytical work to help bridge the gap between the classical analytical result and the complex simulation results. New analytical handling results would provide the much needed foundation for improved automotive design procedures. This dissertation will present new analytical results. These results will deal specifically with the lateral stability of the driver/vehicle system.

## III. CLASSICAL RESULT

The classical result arises from the problem of determining the lateral stability of an automobile in straight line, constant speed motion. The model used for the automobile is shown in Figure 2. It is called the bicycle model. In the bicycle model, the automobile is modeled as a zero-width vehicle, with two wheels per axle assumed. The wheels are located on the center line of the vehicle with the front axle located a distance $L_1$ in front of the center of mass and the rear axle located a distance $L_2$ behind the center of mass. The front axle is steered at an angle $\delta$. The resultant lateral force exerted on the tire by the road is assumed to act perpendicularly to the plane of the wheel, directly below the wheel center. The coordinate system is fixed in the vehicle at its mass center. The equations of motion are written by summing forces in the x and y directions and by summing moments about a vertical axis through the mass center. These equations can be written as:

$$m(\dot{V}_x - V_y\Omega_z) = F_{1x} \cos \delta - F_{1y} \sin \delta + F_{2x} \quad (1)$$

$$m(\dot{V}_y + V_x\Omega_z) = F_{1x} \sin \delta + F_{1y} \cos \delta + F_{2y} \quad (2)$$

$$I_z\dot{\Omega}_z = L_1F_{1x} \sin \delta + L_1F_{1y} \cos \delta - L_2F_{2y} \quad (3)$$

where $F_{1x}$, $F_{2x}$ = forces resulting from braking or tractive efforts and/or rolling resistance, acting parallel to the front and rear wheels, respectively;

$F_{1y}$, $F_{2y}$ = lateral forces acting perpendicularly to the plane of the wheel and directly below the wheel

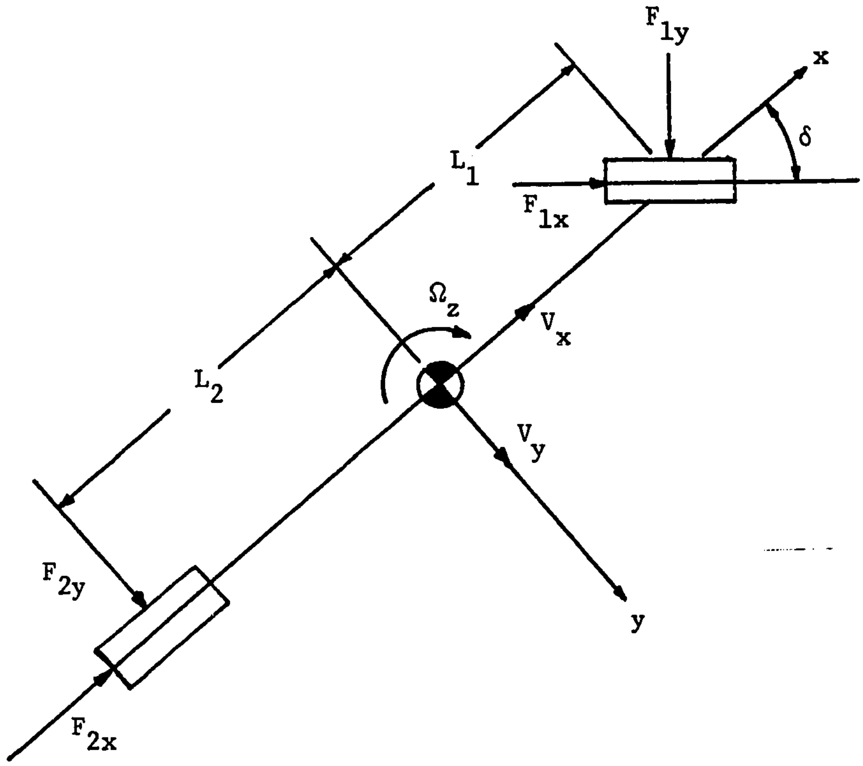


Figure 2. The bicycle model

center, front and rear forces, respectively;

$I_z$ = moment of inertia of the vehicle about the vertical axis;

$L_1$, $L_2$ = lengths from the mass center to the front and rear axles, respectively;

m = mass of the vehicle;

$V_x$ = speed of the vehicle in the x direction;

$V_y$ = speed of the vehicle in the y direction;

$\delta$ = steer angle of the front axle; and

$\Omega_z$ = angular speed of the vehicle about the vertical axis.

The lateral force on a tire can be expressed as the negative of the product of the tire's cornering stiffness and its slip angle. The total lateral force acting on an axle is the sum of the lateral forces of the tires on the axle. Therefore, for two wheels per axle, the lateral force on the front and rear axles, respectively, can be written

$$F_{1y} = - 2 \; C_1 \alpha_1 \tag{4}$$

and

$$F_{2y} = - 2 \; C_2 \alpha_2 \tag{5}$$

where $\alpha_1$, $\alpha_2$ = front and rear slip angles, respectively; and

$C_1$, $C_2$ = front and rear cornering stiffnesses, respectively.

The slip angle is the angle between the velocity vector of the wheel center and the vertical plane of the tire. The velocities of the wheel centers at the front and rear are, respectively,

$$\underline{V}_{t1} = V_x \underline{u}_x + (V_y + L_1\Omega_z)\, \underline{u}_y \tag{6}$$

and

$$\underline{V}_{t2} = V_x \underline{u}_x + (V_y - L_2\Omega_z)\, \underline{u}_y \quad . \tag{7}$$

Relationships between the slip angles $\alpha_1$ and $\alpha_2$, the steer angle $\delta$, and the quantities $V_x$, $V_y$ and $\Omega_z$ can be found by taking the ratio of the speed of the wheel center in the y direction to the speed of the wheel center in the x direction. For small angles

$$\alpha_1 + \delta \approx \tan\,(\alpha_1 + \delta) = \frac{V_y + L_1\Omega_z}{V_x} \tag{8}$$

$$\alpha_2 \approx \tan\,\alpha_2 = \frac{V_y - L_2\Omega_z}{V_x} \quad . \tag{9}$$

For the classical problem of constant speed, straight line motion, four assumptions are made: 1) the steer angle $\delta$ is zero; 2) there are no braking, tractive or rolling resistance forces ($F_x$'s = 0); 3) the speed in the x direction is constant; and 4) all second or higher order terms of the variables are small compared to first order terms and can be neglected ($\cos\theta = 1 \quad \sin\theta = \theta$). If Equations (4), (5), (8) and (9) are combined with Equations (1), (2) and (3), Equation (1) vanishes and Equations (2) and (3) can be written as

$$m\dot{V}_y + (mV_x + \frac{2L_1C_1 - 2L_2C_2}{V_x})\Omega_z + (\frac{2C_1 + 2C_2}{V_x})V_y = 2C_1\delta \tag{10}$$

$$I_z\dot{\Omega}_z + \left(\frac{2L_1^2C_1 + 2L_2^2C_2}{V_x}\right)\Omega_z + \left(\frac{2L_1C_1 - 2L_2C_2}{V_x}\right)V_y = 2L_1C_1\delta \tag{11}$$

Writing Equations (10) and (11), with $\delta = 0$, in matrix form gives

$$\begin{bmatrix} m & 0 \\ 0 & I_z \end{bmatrix} \begin{Bmatrix} \dot{V}_y \\ \dot{\Omega}_z \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} V_y \\ \Omega_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{12}$$

where

$$B_{11} = \frac{2C_1 + 2C_2}{V_x} \quad ; \tag{13}$$

$$B_{12} = mV_x + \frac{2L_1C_1 - 2L_2C_2}{V_x} \quad ; \tag{14}$$

$$B_{21} = \frac{2L_1C_1 - 2L_2C_2}{V_x} \quad ; \tag{15}$$

and

$$B_{22} = \frac{2L_1^2C_1 + 2L_2^2C_2}{V_x} \quad . \tag{16}$$

Premultiplying Equation (12) by the inverse of the mass-inertia matrix of Equation (12) and substituting the following assumed solution

$$\begin{Bmatrix} V_y \\ \Omega_z \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} e^{\lambda t} \quad , \tag{17}$$

results in the following equation:

$$\begin{bmatrix} \lambda + \frac{B_{11}}{m} & \frac{B_{12}}{m} \\ \frac{B_{21}}{I_z} & \lambda + \frac{B_{22}}{I_z} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} e^{\lambda t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (18)$$

Equation (18) yields a nontrival solution only if the determinant of the matrix is zero. Setting the determinant equal to zero gives

$$\lambda^2 + (\frac{B_{11}}{m} + \frac{B_{22}}{I_z})\ \lambda + (\frac{B_{11}B_{22} - B_{12}B_{21}}{m\ I_z}) = 0\ . \qquad (19)$$

This is the characteristic equation of the system and its roots are the eigenvalues. They are

$$\lambda_1 = \frac{-(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}) + \sqrt{\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)^2 - 4(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\ I_z})}}{2} \qquad (20)$$

and

$$\lambda_2 = \frac{-(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}) - \sqrt{\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)^2 - 4(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\ I_z})}}{2}\ . \qquad (21)$$

The eigenvectors associated with these eigenvalues are, respectively,

$$Q_1 = \begin{Bmatrix} \frac{-B_{12}}{m} \\ \lambda_1 + \frac{B_{11}}{m} \end{Bmatrix} \qquad (22)$$

and

$$Q_2 = \begin{Bmatrix} \dfrac{-B_{12}}{m} \\ \lambda_2 + \dfrac{B_{11}}{m} \end{Bmatrix} . \tag{23}$$

The transient solution of the equation of motion can be written in terms of the eigenvalues and eigenvectors. The transient solution is

$$\begin{Bmatrix} V_y \\ \Omega_z \end{Bmatrix} = A_1 Q_1 e^{\lambda_1 t} + A_2 Q_2 e^{\lambda_2 t} \tag{24}$$

where $A_1$ and $A_2$ are constants which depend on the initial conditions.

For the system to be stable, the roots, $\lambda_1$ and $\lambda_2$, of the characteristic equation (Equation 19) must have negative real parts. The roots of a quadratic equation will have negative real parts if all the coefficients are positive. The coefficient of the second order term is one, so it is clearly positive. The coefficient of the first order term is also positive as can be shown by recalling Equations (13) and (16) and noting that all the parameters describing the physical system are positive. Thus, if the zeroth order term is positive, the system is stable.

An expression that the zeroth order term be positive can be written in terms of the vehicle parameters as

$$(2C_1 + 2C_2)(2L_1^2 C_1 + 2L_2^2 C_2) - (mV_x^2 + 2L_1C_1 - 2L_2C_2)(2L_1C_1 - 2L_2C_2) > 0 . \tag{25}$$

Performing the indicated products yields:

$$4L_1^2C_1^2 + 4L_2^2C_1C_2 + 4L_1^2C_1C_2 + 4L_2^2C_2^2 - mV_x^2(2L_1C_1 - 2L_2C_2)$$
$$- (4L_1^2C_1^2 - 4L_1L_2C_1C_2 + 4L_2^2C_2^2) > 0 \quad . \tag{26}$$

Simplifying gives

$$4C_1C_2(L_1 + L_2)^2 + mV_x^2(2L_2C_2 - 2L_1C_1) > 0 \quad . \tag{27}$$

If the difference, $2L_2C_2 - 2L_1C_1$, is positive, the condition is satisfied and the system is stable. However, if the difference, $2L_2C_2 - 2L_1C_1$, is negative, the condition that the zeroth order term be positive will be satisfied only if

$$V_x < \sqrt{\frac{-4C_1C_2(L_1 + L_2)^2}{m(2L_2C_2 - 2L_1C_1)}} \quad . \tag{28}$$

Dividing both numerator and denominator by $4C_1C_2(L_1 + L_2)g$ gives

$$V_x < \sqrt{\frac{-(L_1 + L_2)g}{\frac{mg}{L_1 + L_2}\left(\frac{L_2}{2C_1} - \frac{L_1}{2C_2}\right)}} \quad . \tag{29}$$

Now $\frac{mgL_2}{L_1 + L_2}$ and $\frac{mgL_1}{L_1 + L_2}$ are the normal loads, $W_1$ and $W_2$, on the front and rear axles, respectively. The condition may then be written as

$$V_x < \sqrt{\frac{-(L_1 + L_2)g}{\left(\frac{W_1}{2C_1} - \frac{W_2}{2C_2}\right)}} \tag{30}$$

or

$$V_x < \sqrt{\frac{-(L_1 + L_2)g}{K_{us}}} \tag{31}$$

where $K_{us} = (\frac{W_1}{2C_1} - \frac{W_2}{2C_2})$ and is called the understeer coefficient.

If the understeer coefficient, $K_{us}$, is positive, Equation (27) is satisfied and the vehicle will be stable for all speeds, $V_x$. But if the understeer coefficient is negative, the vehicle will be stable only at speeds less than a critical speed. The critical speed is

$$V_{CRIT} = \sqrt{\frac{-(L_1 + L_2)g}{K_{us}}} \quad . \tag{32}$$

This is the classical result for lateral stability of an automobile in constant speed, straight line motion.

The classical results give conditions under which the vehicle will be stable. But what does stability mean? For the classical result, stability means that if the system is disturbed the lateral velocity and yaw velocity will return to zero. Physically this means the vehicle will return to straight line motion. But this resulting straight line motion is not the original straight line motion. The classical approach to the problem uses coordinates fixed in the vehicle. The equations of motion resulting from this approach are in terms of velocities. No displacement terms appear in the equations of motion. Thus, the classical approach yields only a subset of the complete set of equations of motion and stability of this subset does not imply stability of the complete set of equations of motion. Displacements are very important because the physical system, the automobile, travels on a road of limited width. If the position of the vehicle exceeds the boundaries of the road, the vehicle is off the road. Thus, it is necessary to use an absolute coordinate system (i.e. a system fixed in the road) so that displacement and orientation will be included in the analysis.

A coordinate system fixed in the road is illustrated in Figure 3. The equations of motion for the vehicle in the absolute reference frame can be obtained from the previous equations of motions (those in the moving reference frame) by adding two additional equations. These two equations relate the velocities of the vehicle in the moving reference frame to the absolute coordinates or their time derivatives. The two additional equations are

$$\dot{\psi} = \Omega_z \tag{33}$$

and

$$\dot{Y} = V_y + V_x\psi \tag{34}$$

where $\psi$ = the yaw angle with respect to the X axis of the absolute reference frame, and

$Y$ = the lateral displacement of the vehicle's mass center in the absolute reference frame.

The equations of motion in the absolute reference frame may be written as

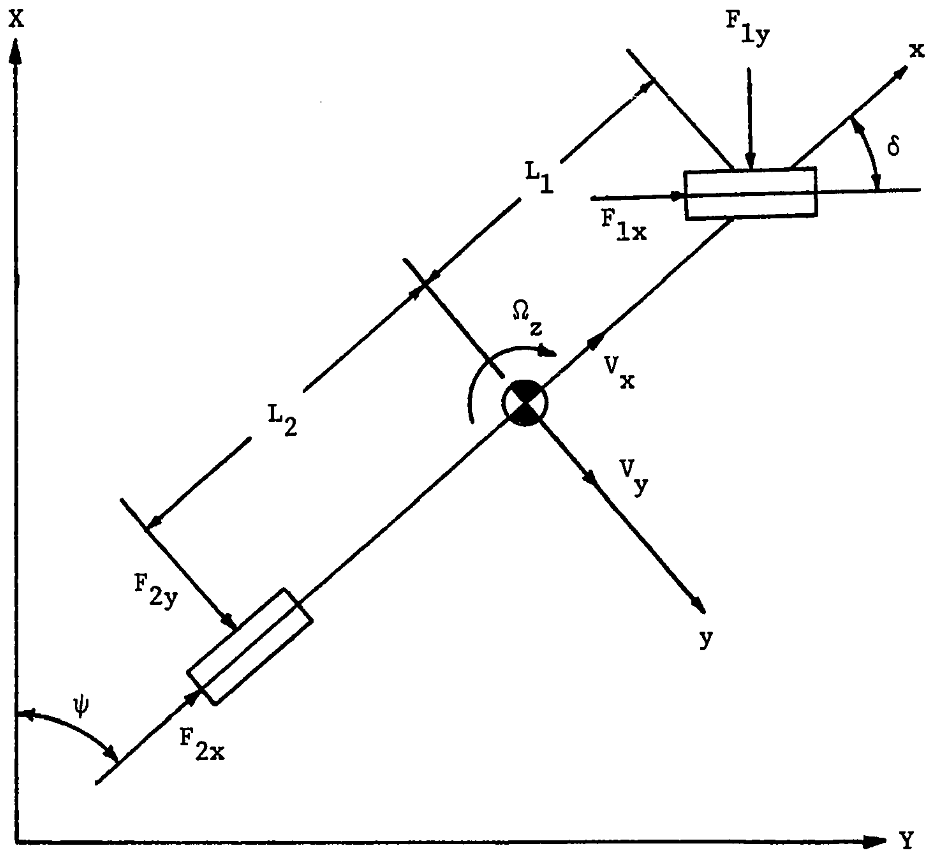


Figure 3. Bicycle model and absolute coordinates

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{V}_y \\ \dot{\Omega}_z \\ \dot{\psi} \\ \dot{Y} \end{Bmatrix}$$

$$+ \begin{bmatrix} B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -V_x & 0 \end{bmatrix} \begin{Bmatrix} V_y \\ \Omega_z \\ \psi \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{35}$$

where $B_{11}$, $B_{12}$, $B_{21}$ and $B_{22}$ are given by Equations (13) - (16).

The characteristic equation of the equation of motion is

$$\lambda^2 \left[ \lambda^2 + \left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right) \lambda + \left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\, I_z}\right) \right] = 0 \ . \tag{36}$$

The roots of this quartic equation are:

$$\lambda_1 = \frac{-\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right) + \sqrt{\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)^2 - 4\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\, I_z}\right)}}{2} \tag{37}$$

$$\lambda_2 = \frac{-\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right) - \sqrt{\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)^2 - 4\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\, I_z}\right)}}{2} \tag{38}$$

$$\lambda_3 = 0 \tag{39}$$

$$\lambda_4 = 0 \ . \tag{40}$$

The roots $\lambda_1$, $\lambda_2$ and $\lambda_3$ give rise to solutions

$$Q_1 = \begin{Bmatrix} \dfrac{-B_{12}\lambda_1}{B_{11} + m\lambda_1} \\ \lambda_1 \\ 1 \\ \dfrac{V_x}{\lambda_1} - \dfrac{B_{12}}{B_{11} + m\lambda_1} \end{Bmatrix} e^{\lambda_1 t} , \quad Q_2 = \begin{Bmatrix} \dfrac{-B_{12}\lambda_2}{B_{11} + m\lambda_1} \\ \lambda_2 \\ 1 \\ \dfrac{V_x}{\lambda_2} - \dfrac{B_{12}}{B_{11} + m\lambda_2} \end{Bmatrix} e^{\lambda_2 t} ,$$

$$Q_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} . \tag{41}$$

The solution associated with the root $\lambda_4 = 0$ has the form

$$Q_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} te^0 + \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} e^0 . \tag{42}$$

Substituting this assumed solution into Equation (35) yields:

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{V_x} \\ 1 \end{Bmatrix} . \tag{43}$$

The desired solution associated with $\lambda_4 = 0$ is then

$$Q_4 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} t + \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{V_x} \\ 1 \end{Bmatrix} . \tag{44}$$

The general solution of the equations of motion is therefore

$$\begin{Bmatrix} V_y \\ \Omega_z \\ \psi \\ Y \end{Bmatrix} = A_1Q_1 + A_2Q_2 + A_3Q_3 + A_4Q_4 \tag{45}$$

where $A_1$, $A_2$, $A_3$ and $A_4$ are constants dependent on initial conditions.

The characteristic equation has two zero roots. This results in a solution that will grow with time. Thus, the automobile is laterally unstable. That is, it will not remain on a straight road. In order for the vehicle to remain on the road, steering control is needed. This control is generally provided by a driver. The driver/vehicle system will be considered in the next section.

# IV. DRIVER/VEHICLE SYSTEM RESULTS

## A. Driver/Vehicle System

In the classical approach to the problem of the lateral stability of an automobile, the front wheels were fixed at $\delta = 0$. Now a driver will be introduced. Carson and Weirwille [39] have modeled the driver as a proportional controller responding to lateral displacement and yaw angle, where thresholds on the lateral displacement and yaw angle and a time delay must be satisfied before the driver responds. A simpler model will be used here, where the driver will be modeled simply as a proportional controller responding to lateral displacement and yaw angle expressed as

$$\delta = - D_\psi \psi - D_Y Y \tag{46}$$

where $\delta$ = the front wheel steer angle;

$\psi$ = the yaw angle of the vehicle;

$Y$ = the lateral displacement of the vehicle; and

$D_\psi$, $D_Y$ = positive constants.

If this driver model is substituted into the equations of motion, in the absolute reference frame, Equations (10) and (11), the result is

$$m \dot{V}_y + \left(\frac{2C_1 + 2C_2}{V_x}\right) V_y + \left(mV_x + \frac{2L_1C_1 - 2L_2C_2}{V_x}\right) \Omega_z$$

$$= 2C_1(-D_\psi \psi - D_Y Y) \tag{47}$$

$$I_z\dot{\Omega}_z + \left(\frac{2L_1C_1 - 2L_2C_2}{V_x}\right)V_y + \left(\frac{2L_1^2C_1 + 2L_2^2C_2}{V_x}\right)\Omega_z$$

$$= 2L_1C_1(-D_\psi\psi - D_Y Y) \tag{48}$$

$$\dot{\psi} = \Omega_z \tag{49}$$

$$\dot{Y} = V_y + V_x\psi \;. \tag{50}$$

The driver terms can be moved to the left side of the equations since they are functions of $\psi$ and Y. The resulting equations of motion can be written in matrix form as

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{V}_y \\ \dot{\Omega}_z \\ \dot{\psi} \\ \dot{Y} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -V_x & 0 \end{bmatrix} \begin{Bmatrix} V_y \\ \Omega_z \\ \psi \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{51}$$

where

$$B_{13} = 2C_1D_\psi; \tag{52}$$

$$B_{14} = 2C_1D_Y; \tag{53}$$

$$B_{23} = 2L_1C_1D_\psi; \tag{54}$$

$$B_{24} = 2L_1C_1D_Y; \text{ and} \tag{55}$$

$B_{11}$, $B_{12}$, $B_{21}$ and $B_{22}$ are defined by Equations (13)-(16).

Premultiplying Equation (51) by the inverse of the mass-inertia matrix of Equation (51) yields

$$\begin{Bmatrix} \dot{V}_y \\ \dot{\Omega}_z \\ \dot{\psi} \\ \dot{Y} \end{Bmatrix} + \begin{bmatrix} \frac{B_{11}}{m} & \frac{B_{12}}{m} & \frac{B_{13}}{m} & \frac{B_{14}}{m} \\ \frac{B_{21}}{I_z} & \frac{B_{22}}{I_z} & \frac{B_{23}}{I_z} & \frac{B_{24}}{I_z} \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -V_x & 0 \end{bmatrix} \begin{Bmatrix} V_y \\ \Omega_z \\ \psi \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{56}$$

The characteristic equation for this system is

$$\begin{aligned} \lambda^4 &+ \left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)\lambda^3 + \left(\frac{B_{11}B_{22}}{m\,I_z} - \frac{B_{12}B_{21}}{m\,I_z} + \frac{B_{14}}{m} + \frac{B_{23}}{I_z}\right)\lambda^2 \\ &+ \left(\frac{B_{11}B_{23}}{m\,I_z} - \frac{B_{13}B_{21}}{m\,I_z} + \frac{B_{14}B_{22}}{m\,I_z} - \frac{B_{12}B_{24}}{m\,I_z} + \frac{B_{24}}{I_z} V_x\right)\lambda \\ &+ \left(\frac{B_{14}B_{23}}{m\,I_z} - \frac{B_{13}B_{24}}{m\,I_z} + \frac{B_{11}B_{24}}{m\,I_z} V_x - \frac{B_{14}B_{21}}{m\,I_z} V_x\right) = 0\ . \end{aligned} \tag{57}$$

In terms of the vehicle and driver parameters, the characteristic equation simplifies to

$$\lambda^4 + [\frac{2C_1 + 2C_2}{mV_x} + \frac{2L_1^2C_1 + 2L_2^2C_2}{I_zV_x}]\lambda^3$$

$$+ [\frac{4C_1C_2(L_1 + L_2)^2}{mI_zV_x^2} - \frac{2L_1C_1 - 2L_2C_2}{I_z} + \frac{2C_1D_Y}{m} + \frac{2L_1C_1D_\psi}{I_z}]\lambda^2 \tag{58}$$

$$+[\frac{4C_1C_2(L_1 + L_2)}{mI_zV_x}(D_\psi + L_2D_Y)]\lambda + [\frac{4C_1C_2(L_1 + L_2)D_Y}{mI_z}] = 0 \quad .$$

The characteristic equation can be written as

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{59}$$

where

$$a_3 = \frac{2C_1 + 2C_2}{mV_x} + \frac{2L_1^2C_1 + 2L_2^2C_2}{I_zV_x} \; ; \tag{60}$$

$$a_2 = \frac{4C_1C_2(L_1 + L_2)^2}{mI_zV_x^2} - \frac{2L_1C_1 - 2L_2C_2}{I_z}$$

$$+ \frac{2L_1C_1D_\psi}{I_z} + \frac{2C_1D_Y}{m} \; ; \tag{61}$$

$$a_1 = \frac{4C_1C_2(L_1 + L_2)}{mI_zV_x}(D_\psi + L_2D_Y) \; ; \text{ and} \tag{62}$$

$$a_0 = \frac{4C_1C_2(L_1 + L_2)D_Y}{mI_z} \quad . \tag{63}$$

Solutions for the roots of a quartic equation do exist and may be found in theory of equations texts and some mathematical handbooks. The following is one such solution [40].

Given the quartic

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \text{ ,} \tag{64}$$

the resolvent cubic is

$$z^3 - a_2 z^2 + (a_3 a_1 - 4a_0)\, z - a_3^2\, a_0 + 4a_2 a_0 - a_1^2 = 0 \text{ .} \tag{65}$$

Let

$$R = \sqrt{\frac{a_3^2}{4} - a_2 + z} \tag{66}$$

where z is a root of the resolvent cubic. If $R \neq 0$,

$$D = \sqrt{\frac{3a_3^2}{4} - R^2 - 2a_2 + \frac{4a_3 a_2 - 8a_1 - a_3^3}{4R}} \tag{67}$$

and

$$E = \sqrt{\frac{3a_3^2}{4} - R^2 - 2a_2 - \frac{4a_3 a_2 - 8a_1 - a_3^3}{4R}} \text{ .} \tag{68}$$

If $R = 0$,

$$D = \sqrt{\frac{3a_3^2}{4} - 2a_2 + 2\sqrt{z^2 - 4a_0}} \tag{69}$$

and

$$E = \sqrt{\frac{3a_3^2}{4} - 2a_2 - 2\sqrt{z^2 - 4a_0}} \text{ .} \tag{70}$$

The roots of the quartic are:

$$\lambda_1 = -\frac{a_3}{4} + \frac{R}{2} + \frac{D}{2} ; \tag{71}$$

$$\lambda_2 = -\frac{a_3}{4} + \frac{R}{2} - \frac{D}{2} ; \tag{72}$$

$$\lambda_3 = -\frac{a_3}{4} - \frac{R}{2} + \frac{E}{2} ; \quad \text{and} \tag{73}$$

$$\lambda_4 = -\frac{a_3}{4} - \frac{R}{2} - \frac{E}{2} . \tag{74}$$

For a quartic equation with real coefficients, the roots will be either four real roots or two real roots and a complex conjugate pair of roots or two pairs of complex conjugate roots.

## B. Routh-Hurwitz Result

Stability is the question at hand. The system will be stable if all the roots of the characteristic equation have negative real parts. For a linear system, a necessary condition for stability is that all the coefficients of the characteristic equation be positive. Sufficient conditions for a linear system to be stable are given by the Routh-Hurwitz criterion. If the necessary condition that all the coefficients of the characteristic equation be positive is met, then the Routh-Hurwitz criterion for stability is that all the elements of the first column of the Routhian array must be positive. For the quartic of the form

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 , \tag{75}$$

the first column of the Routhian array is

$$a_3$$

$$a_2a_3 - a_1$$

$$(a_2a_3 - a_1)a_1 - a_0a_3^2$$

$$a_0 \quad .$$

Consider the first term of the array, $a_3$, approaching zero. Note that the second and third terms will become negative before $a_3$ equals zero. Consider the second term, $a_2a_3 - a_1$, approaching zero. Note that the third term will become negative before $a_2a_3 - a_1$ equals zero. Based on these considerations, the third term will become zero before either of the first two. Thus if the system becomes unstable, it will not be because either of the first two terms have become zero.

The fourth term, $a_0$, could become zero or negative. Thus, there are two possibilities for system instability. Either $(a_2a_3 - a_1)a_1 - a_0a_3^2$ has become negative or $a_0$ has become negative.

As noted earlier, a quartic equation with real coefficients will have either four real roots or two real roots and a pair of complex conjugate roots or two pairs of complex conjugate roots. If the system becomes unstable, either a real root has become positive or the real part of a complex conjugate pair has become positive.

It is easy to show that if $a_0$ becomes zero the quartic has a zero root. The quartic would appear as

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda = 0 \tag{76}$$

which has $\lambda$ as a factor. Therefore, zero is a root of the equation. If $a_0$ becomes negative, the root will become positive. This is the case of a real root crossing the imaginary axis of the complex plane.

It is more difficult to show that the crossing of a complex conjugate pair into the right half plane is associated with the other Routh-Hurwitz condition, namely $(a_2a_3 - a_1)a_1 - a_3^2a_0$, becoming zero. Consider the root

$$\lambda = -\frac{a_3}{4} + \frac{R}{2} + \frac{D}{2} \tag{77}$$

of the quartic solution. Suppose D is imaginary. Then for the real part of $\lambda$ to equal zero requires $R = \frac{a_3}{2}$. From the definition of R,

$$R = \sqrt{\frac{a_3^2}{4} - a_2 + z} \quad , \tag{78}$$

$R = \frac{a_3}{2}$ implies that $z = a_2$. If $z = a_2$ is substituted into the resolvent cubic, the result is

$$a_2^3 - a_2(a_2)^2 + (a_3a_1 - 4a_0)a_2 - a_3^2a_0 + 4a_2a_0 - a_1^2 = 0 \tag{79}$$

which reduces to

$$a_3a_2a_1 - a_3^2a_0 - a_1^2 = 0 \tag{80}$$

which is the Routh-Hurwitz condition. One should check to be sure that D is indeed imaginary if $R = \frac{a_3}{2}$. Substituting $R = \frac{a_3}{2}$ into the definition of D yields

$$D = \sqrt{\frac{3a_3^2}{4} - \frac{a_3^2}{4} - 2a_2 + \frac{4a_3a_2 - 8a_1 - a_3^3}{2a_3}} \tag{81}$$

which simplifies to

$$D = \sqrt{-4\,\frac{a_1}{a_3}}\ . \tag{82}$$

In the case of the driver/vehicle system, $a_1$ and $a_3$ are always positive, so D must be imaginary. Thus not only has it been shown that the crossing of the imaginary axis by a complex conjugate pair is associated with the Routh-Hurwitz condition $(a_2a_3 - a_1)a_1 - a_3^2a_0 = 0$, but also that the roots have imaginary parts of magnitude $2\sqrt{\frac{a_1}{a_3}}$ when they cross.

Recall the coefficients of the characteristic equation for the driver/vehicle system

$$a_3 = \frac{2C_1 + 2C_2}{mV_x} + \frac{2L_1^2C_1 + 2L_2^2C_2}{I_zV_x}\ ; \tag{83}$$

$$a_2 = \frac{4C_1C_2(L_1 + L_2)^2}{mI_zV_x^2} - \frac{2L_1C_1 - 2L_2C_2}{I_z} + \frac{2L_1C_1D_\psi}{I_z} + \frac{2C_1D_Y}{m}\ ; \tag{84}$$

$$a_1 = \frac{4C_1C_2(L_1 + L_2)(D_\psi + D_YL_2)}{mI_zV_x}\ ; \text{ and} \tag{85}$$

$$a_0 = \frac{4C_1C_2(L_1 + L_2)D_Y}{mI_z}\ . \tag{86}$$

Consider the conditions for stability. The necessary condition is that all the coefficients be positive. Since all the parameters describing the driver and vehicle are positive, $a_3$, $a_1$ and $a_0$ are positive. Only $a_2$ could be negative or zero. The two sufficient conditions are $a_0 > 0$ and $(a_2a_3 - a_1)a_1 - a_0a_3^2 > 0$. The first of these, $a_0 > 0$, is satisfied.

The last condition can be rearranged to isolate $a_2$ as

$$a_2 > \frac{a_0 a_3}{a_1} + \frac{a_1}{a_3} \quad . \tag{87}$$

Since $a_3$, $a_1$ and $a_0$ are positive, this condition is more restrictive than the necessary condition that $a_2 > 0$. Thus the only condition that must be met to insure the lateral stability of the driver/vehicle system is

$$a_2 > \frac{a_0 a_3}{a_1} + \frac{a_1}{a_3} \quad . \tag{88}$$

Writing $a_2$ in terms of driver and vehicle parameters yields

$$\frac{4C_1 C_2 (L_1 + L_2)^2}{m I_z V_x^2} + \frac{2L_2 C_2 - 2L_1 C_1}{I_z} + \frac{2L_1 C_1 D_\psi}{I_z} + \frac{2C_1 D_Y}{m} > \frac{a_0 a_3}{a_1} + \frac{a_1}{a_3} \; . \tag{89}$$

Subtracting $\frac{a_0 a_3}{a_1}$ and $\frac{a_1}{a_3}$ from both sides and multiplying by $m I_z V_x^2$ gives

$$4C_1 C_2 (L_1 + L_2)^2 + [(\frac{2L_2 C_2 - 2L_1 C_1}{I_z} + \frac{2L_1 C_1 D_\psi}{I_z} + \frac{2C_1 D_Y}{m}$$

$$- \frac{a_0 a_3}{a_1} - \frac{a_1}{a_3}) m I_z] \; V_x^2 > 0 \; . \tag{90}$$

The first term, $4C_1 C_2 (L_1 + L_2)^2$, is positive; so if the bracketed term multiplying $V_x^2$ is positive, the condition is satisfied and the system is stable for all speeds. If, however, the bracketed term is negative, the condition will be satisfied only for speeds less than a critical speed. ˜nat critical speed is given by

$$V_{\substack{CRIT\\D/V}} = \sqrt{\frac{-4C_1C_2(L_1 + L_2)^2}{\left(\frac{2L_2C_2 - 2L_1C_1}{I_z} + \frac{2L_1C_1D_\psi}{I_z} + \frac{2C_1D_Y}{m} - \frac{a_0a_3}{a_1} - \frac{a_1}{a_3}\right)mI_z}} \;. \tag{91}$$

This may be rewritten in terms of the understeer coefficient by multiplying numerator and denominator under the radical by

$$\frac{g}{4C_1C_2(L_1 + L_2)}$$

and recognizing

$$\frac{mg\ L_2}{2C_1(L_1 + L_2)} - \frac{mg\ L_1}{2C_2(L_1 + L_2)} = K_{us} \quad . \tag{92}$$

This yields

$$V_{\substack{CRIT\\D/V}} = \sqrt{\frac{-(L_1 + L_2)g}{K_{us} + K_1 - K_2 - K_3}} \tag{93}$$

where

$$K_{us} = \frac{mg\ L_2}{2C_1(L_1 + L_2)} - \frac{mg\ L_1}{2C_2(L_1 + L_2)}; \tag{94}$$

$$K_1 = \frac{mg\ L_1D_\psi}{2C_1(L_1 + L_2)} + \frac{I_zgD_Y}{2C_2(L_1 + L_2)} \;; \tag{95}$$

$$K_2 = \frac{a_0a_3mgI_z}{4C_1C_2(L_1 + L_2)a_1} \;; \quad \text{and} \tag{96}$$

$$K_3 = \frac{a_1 mgI_z}{4C_1C_2(L_1 + L_2)a_3} \quad . \tag{97}$$

$K_2$ and $K_3$ can be written in terms of only driver and vehicle parameters by substituting for $a_3$,$a_1$ and $a_0$ from Equations (83), (85) and (86), to give

$$K_2 = \frac{[(2C_1 + 2C_2)I_z + (2L_1^2C_1 + 2L_2^2C_2)m]g\ D_Y}{4C_1C_2(L_1 + L_2)(D_\psi + D_YL_2)} \tag{98}$$

and

$$K_3 = \frac{mI_zg(D_\psi + D_YL_2)}{(2C_1 + 2C_2)I_z + (2L_1^2C_1 + 2L_2^2C_2)m} \quad . \tag{99}$$

Equation (93) is a new analytical result concerning the lateral stability of the driver/vehicle system. This new result will be compared with the classical result for the lateral stability of an automobile and discussed in Section V.

## C. Lyapunov's Second Method Results

Another method for determining the stability of the driver/vehicle system is Lyapunov's second method [41]. Briefly, for a system of differential equations

$$\{\dot{x}\} = [B]\ \{x\} \tag{100}$$

if a function V(x) can be found such that 1) its value is positive for all $x \neq 0$ and zero for $x = 0$ (V is positive definite) and 2) its total derivative with respect to time, $\dot{V}(x)$, is negative along every trajectory of the system ($\dot{V}$ is negative definite), then the solution x = 0 is asymptotically stable. A function meeting these two conditions is called a Lyapunov function. Hereafter asymptotically stable will be called stable.

Whereas the Routh-Hurwitz criterion is limited to linear problems, Lyapunov's second method is applicable to a very wide range of problems including nonlinear problems. If a Lyapunov function could be found for the linear driver/vehicle system, it might also be a Lyapunov function for a nonlinear driver/vehicle system and even if it was not, it would serve as a good starting place for the search for such a function.

The major difficulty with the use of Lyapunov's second method is the finding of a Lyapunov function. For a conservative, holonomic dynamical system the Hamiltonian is a good candidate for the Lyapunov function [41]. However, because of the nonconservative tire/road interaction forces vehicle dynamics problems are nonconservative.

First, consider the classical lateral stability problem again. Try the kinetic energy associated with the variables $V_y$ and $\Omega_z$ as a possible Lyapunov function. So,

$$V = \frac{1}{2} m V_y^2 + \frac{1}{2} I_z \Omega_z^2 . \qquad (101)$$

The requirement that V be positive definite is clearly satisfied. Differentiating V with respect to time yields

$$\dot{V} = m V_y \dot{V}_y + I \Omega_z \dot{\Omega}_z . \qquad (102)$$

Substituting expressions for $\dot{V}_y$ and $\dot{\Omega}_z$ from Equation (12), $\dot{V}$ may be written as

$$\dot{V} = m\,V_y\left(-\frac{B_{11}}{m}V_y - \frac{B_{12}}{m}\Omega_z\right) + I_z\Omega_z\left(-\frac{B_{21}}{I_z}V_y - \frac{B_{22}}{I_z}\Omega_z\right) \tag{103}$$

where $B_{11}$, $B_{12}$, $B_{21}$ and $B_{22}$ are defined by Equations (13)-(16). For stability, $\dot{V}$ must be negative definite. Equation (103) can be rewritten in matrix form as

$$\dot{V} = -\{x\}^T [B] \{x\} \tag{104}$$

where $\{x\}^T = \{V_y \;\Omega_z\}$ and (105)

$$[B] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} . \tag{106}$$

In order that $\dot{V}$ be negative definite, the quadratic form $\{x\}^T[B]\{x\}$ must be positive definite. The positive definiteness of the quadratic form may be checked by means of Sylvester's theorem [41] provided the matrix [B] is symmetric. However, the matrix is not symmetric. A quadratic form with a symmetric matrix may be obtained, without altering the quadratic form, by replacing the matrix [B] with one-half the sum of the matrix [B] and its transpose. Thus, the equation for $\dot{V}$ can be written as

$$\dot{V} = -\{V_y\Omega_z\} \begin{bmatrix} B_{11} & \dfrac{B_{12}+B_{21}}{2} \\ \dfrac{B_{12}+B_{21}}{2} & B_{22} \end{bmatrix} \begin{Bmatrix} V_y \\ \Omega_z \end{Bmatrix} \tag{107}$$

According to Sylvester's theorem, the necessary and sufficient conditions that a quadratic form be positive definite are that all the principal minor determinants of the symmetric matrix of the quadratic form be greater than or equal to zero. The requirements that the successive principal minor determinants be greater than or equal to zero are

$$B_{11} > 0 \ , \quad \text{and} \tag{108}$$

$$B_{11}B_{22} - \left(\frac{B_{12} + B_{21}}{2}\right)^2 > 0 \ . \tag{109}$$

If these two inequalities are satisfied stability is assured. The first inequality is satisfied for all forward speeds since $B_{11}$ is always positive (see Equation (13)). The second inequality can be rewritten as

$$B_{11}B_{22} - B_{12}B_{21} > \left(\frac{B_{12} - B_{21}}{2}\right)^2 \ . \tag{110}$$

This inequality leads to a conservative result for the critical speed since the quantity $\left(\frac{B_{12} - B_{21}}{2}\right)^2$ is positive and the condition for stability from the Routh-Hurwitz criterion is

$$B_{11}B_{22} - B_{12}B_{21} > 0 \ . \tag{111}$$

It is clearly desirable to find a Lyapunov function that will yield a less conservative result.

For linear systems the standard approach is to seek a function of the form

$$V = \{x\}^T \; [P] \; \{x\} \tag{112}$$

where the matrix [P] is symmetric. Differentiating with respect to time gives

$$\dot{V} = \{\dot{x}\}^T \; [P] \; \{x\} + \{x\}^T \; [P] \; \{\dot{x}\} \; . \tag{113}$$

Substituting for $\{\dot{x}\}$ from the differential equations describing the system

$$\{\dot{x}\} = [B] \; \{x\} \tag{114}$$

gives

$$\dot{V} = \{x\}^T \; [B]^T \; [P] \; \{x\} + \{x\}^T \; [P] \; [B] \; \{x\} \tag{115}$$

or

$$\dot{V} = \{x\}^T([B]^T \; [P] + [P] \; [B]) \; \{x\} \quad . \tag{116}$$

Since $\dot{V}$ is desired to be negative semidefinite, let

$$[B]^T \; [P] + [P] \; [B] = - \; [Q] \tag{117}$$

where Q is positive definite. Equation (117) is called the Lyapunov matrix equation. If the matrices in Equation (117) are n x n, then there are n(n+1)/2 unknown elements of the matrix [P]. If the indicated multiplications and addition are performed on the left side of Equation (117), the elements of the resulting matrix can be equated to the corresponding elements of the matrix -[Q]. This will result in n(n+1)/2 independent equations in the n(n+1)/2 unknown elements of the matrix [P]. MacFarlane [42] has developed a systematic method for writing down the elements of the matrix on the left side of Equation (117) which is useful for higher order problems.

For the classical problem, the Lyapunov matrix equation, with the matrix [Q] set equal to the identity, is

$$\begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{21}}{I_z} \\ -\frac{B_{12}}{m} & -\frac{B_{22}}{I_z} \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{12}}{m} \\ -\frac{B_{21}}{I_z} & -\frac{B_{22}}{I_z} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{118}$$

Performing the indicated multiplications and addition, and equating the elements yields three independent equations which can be written in matrix form as

$$\begin{bmatrix} -2\frac{B_{11}}{m} & -2\frac{B_{21}}{I_z} & 0 \\ -\frac{B_{12}}{m} & -(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}) & -\frac{B_{21}}{I_z} \\ 0 & -2\frac{B_{12}}{m} & -2\frac{B_{22}}{I_z} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ -1 \end{Bmatrix} \tag{119}$$

This matrix equation can be solved by premultiplying both sides of the equation by the inverse of the 3 x 3 matrix. The result is

$$P_1 = \frac{\left[\frac{2\,B_{22}}{I_z}\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right) - \frac{2B_{12}B_{21}}{m\,I_z}\right] + \frac{2\,B_{21}^{\,2}}{I_z^{\,2}}}{4\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\,I_z}\right)} \;; \tag{120}$$

$$P_2 = \frac{\frac{2\,B_{12}B_{22}}{m\,I_z} + \frac{2\,B_{11}B_{21}}{m\,I_z}}{4\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\,I_z}\right)} \;; \quad \text{and} \tag{121}$$

$$P_3 = \frac{\frac{2\,B_{12}^{\,2}}{m^2} + \left[\frac{2\,B_{11}}{m}\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right) - \frac{2B_{12}B_{21}}{m\,I_z}\right]}{4\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\,I_z}\right)} \tag{122}$$

If the matrix [P] whose elements were just found is positive definite, then

$$V = \{V_y \Omega_z\}\;[P]\begin{Bmatrix} V_y \\ \Omega_z \end{Bmatrix} \tag{123}$$

is a Lyapunov function. The positive definiteness of the matrix [P] can be checked using Sylvester's theorem. The conditions for the matrix [P] to be positive definite are

$$P_1 > 0 \tag{124}$$

and

$$P_1 P_3 - P_2^{\,2} > 0 \;. \tag{125}$$

By rearranging the expression for $P_1$ given as Equation (120), the first condition can be written as

$$\frac{\left[\frac{2\,B_{11}B_{22} - 2\,B_{12}B_{21}}{m\,I_z} + \frac{2\,B_{22}^2}{I_z^2} + \frac{2\,B_{21}^2}{I_z^2}\right]}{4\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\,I_z}\right)} > 0 \quad . \qquad (126)$$

This condition is satisfied if

$$B_{11}B_{22} - B_{12}B_{21} > 0 \quad . \qquad (127)$$

The second condition after some manipulation of the expressions given as Equations (120)-(122) can be written as

$$\frac{4\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\,I_z}\right)\left(\frac{2\,B_{11}B_{22} - 2\,B_{12}B_{21}}{m\,I_z} + \frac{B_{11}^2}{m^2} + \frac{B_{12}^2}{m^2} + \frac{B_{21}^2}{I_z^2} + \frac{B_{22}^2}{I_z^2}\right)}{\left[4\left(\frac{B_{11}}{m} + \frac{B_{22}}{I_z}\right)\left(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\,I_z}\right)\right]^2} > 0. \qquad (128)$$

This condition is also satisfied if the Inequality (127) is satisfied. In fact, the Inequality (127) is the Routh-Hurwitz condition for stability in this problem. Thus, a Lyapunov function has been found for the classical lateral stability problem which yields the same result as the Routh-Hurwitz criterion.

Now suppose a driver who responds proportionally to yaw angle is added to the system. The equations of motion can be expressed as

$$\begin{Bmatrix} \dot{V}_y \\ \dot{\Omega}_z \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{12}}{m} & -\frac{B_{13}}{m} \\ -\frac{B_{21}}{I_z} & -\frac{B_{22}}{I_z} & -\frac{B_{23}}{I_z} \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} V_y \\ \Omega_z \\ \psi \end{Bmatrix} \tag{129}$$

where $B_{13}$ and $B_{23}$ are defined in Equations (52) and (54). If this system is stable, the driver will cause the disturbed system to achieve a path parallel to the original path.

Applying the standard method with [Q] equal to the identity, the resulting Lyapunov matrix equation is

$$\begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{21}}{I_z} & 0 \\ -\frac{B_{12}}{m} & -\frac{B_{22}}{I_z} & 1 \\ -\frac{B_{13}}{m} & -\frac{B_{23}}{I_z} & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 & P_4 \\ P_2 & P_3 & P_5 \\ P_4 & P_5 & P_6 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 & P_4 \\ P_2 & P_3 & P_5 \\ P_4 & P_5 & P_6 \end{bmatrix} \begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{12}}{m} & -\frac{B_{13}}{m} \\ -\frac{B_{21}}{I_z} & -\frac{B_{22}}{I_z} & -\frac{B_{23}}{I_z} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{130}$$

Performing the multiplications and addition on the left side of Equation (130) and equating the elements of the resulting matrix to the elements of the matrix on the right side results in six independent equations which can be written as

$$\begin{bmatrix} -2\frac{B_{11}}{m} & -2\frac{B_{21}}{I_z} & 0 & 0 & 0 & 0 \\ -\frac{B_{12}}{m} & -(\frac{B_{11}}{m}+\frac{B_{22}}{I_z}) & -\frac{B_{21}}{I_z} & 1 & 0 & 0 \\ 0 & -2\frac{B_{12}}{m} & -2\frac{B_{22}}{I_z} & 0 & 2 & 0 \\ -\frac{B_{13}}{m} & -\frac{B_{23}}{I_z} & 0 & -\frac{B_{11}}{m} & -\frac{B_{21}}{I_z} & 0 \\ 0 & -\frac{B_{13}}{m} & -\frac{B_{23}}{I_z} & -\frac{B_{12}}{m} & -\frac{B_{22}}{I_z} & 1 \\ 0 & 0 & 0 & -2\frac{B_{13}}{m} & -2\frac{B_{23}}{I_z} & 0 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \end{Bmatrix} \quad (131)$$

This matrix equation can be solved by premultiplying both sides by the inverse of the 6 x 6 matrix. Barnett and Storey [43] have shown that by introducing a skew-symmetric matrix that the largest matrix to be inverted can be reduced to one of order n(n-1)/2. Finding the inverse of the 6 x 6 is not difficult, but it is somewhat tedious. The resulting elements of the matrix [P] are

$$P_1 = 4\{a_0[a_1 + \frac{B_{21}^2}{I_z^2} + \frac{B_{22}^2}{I_z^2} - \frac{2B_{23}}{I_z}] + a_2[\frac{B_{23}^2}{I_z^2} + \frac{B_{21}^2}{I_z^2}]\}/\Delta \; ; \qquad (132)$$

$$P_2 = 4\{a_0[\frac{B_{13}}{m} - \frac{B_{11}B_{21} + B_{12}B_{22}}{m\ I_z}]$$

$$+ a_2[- \frac{B_{11}B_{21} + B_{13}B_{23}}{m\ I_z}]\}/\Delta \; ; \qquad (133)$$

$$P_3 = 4\{a_0[a_1 + \frac{B_{11}^2}{m^2} + \frac{B_{12}^2}{m^2} + 1] + a_2[\frac{B_{11}^2}{m^2} + \frac{B_{13}^2}{m^2}]\}/\Delta \; ; \qquad (134)$$

$$P_4 = 4\{a_0[\frac{B_{21}B_{23}}{I_z^2} - \frac{B_{12}B_{23}}{m\ I_z} + \frac{B_{21}}{I_z}]$$

$$+ a_2[(\frac{B_{13}B_{22} - B_{23}B_{12}}{m\ I_z})(- \frac{B_{23}}{I_z})$$

$$+ (\frac{B_{22}B_{23}}{I_z^2} - \frac{B_{13}B_{21}}{m\ I_z})(\frac{B_{12}B_{21} - B_{11}B_{22}}{m\ I_z})]\}/\Delta \; ; \qquad (135)$$

$$P_5 = 4\{a_o[\frac{B_{22}}{I_z} - \frac{B_{13}B_{21}}{m\ I_z} + \frac{B_{12}B_{13}}{m^2}] + a_2[(- \frac{B_{11}}{m})(\frac{B_{12}B_{21} - B_{11}B_{22}}{m\ I_z})$$

$$+ (- \frac{B_{12}B_{23}}{m\ I_z})(\frac{B_{13}}{m})]\}/\Delta \; ; \qquad (136)$$

and

$$P_6 = 4\{a_0[a_1 + \frac{B_{11}^2}{m^2} + \frac{B_{13}^2}{m^2} + \frac{B_{22}^2}{I_z^2} + \frac{B_{23}}{I_z^2} + \frac{2B_{12}B_{21}}{m\ I_z}]$$

$$+ a_2[(\frac{B_{11}B_{22} - B_{12}B_{21}}{m\ I_z})^2 + (\frac{B_{12}B_{23} - B_{13}B_{22}}{m\ I_z})^2$$

$$+ (\frac{B_{11}B_{23} - B_{13}B_{21}}{m\ I_z})^2\ ]\}/\Delta \qquad (137)$$

where

$$a_0 = \frac{B_{11}B_{23} - B_{13}B_{21}}{m\ I_z}\ ; \qquad (138)$$

$$a_1 = \frac{B_{11}B_{22} - B_{12}B_{21}}{m\ I_z} + \frac{B_{23}}{I_z}\ ; \qquad (139)$$

$$a_2 = \frac{B_{11}}{m} + \frac{B_{22}}{I_z}\ ; \qquad (140)$$

and

$$\Delta = 8\ a_0(a_1a_2 - a_0)\ . \qquad (141)$$

The elements of the matrix [P] for this case are considerably more complicated than those for the classical problem. But note that $a_0$, $a_1$ and $a_2$ are the coefficients from the characteristic equation. Also note that $\Delta$ is the product of the elements of the first column of the Routhian array. This suggests that there may be a relationship between the coefficients of the characteristic equation and the elements of the matrix [P].

The positive definiteness of the matrix [P] must still be shown. The conditions which will assure positive definiteness of the matrix [P] are

$$P_1 > 0 \; ; \tag{142}$$

$$P_1P_3 - P_2^2 > 0 \; ; \quad \text{and} \tag{143}$$

$$P_1P_3P_6 + 2\,P_2P_4P_5 - P_1P_5^2 - P_2^2P_6 - P_4^2P_3 > 0 \quad . \tag{144}$$

If the expressions for the elements of the matrix [P] from Equations (132)-(137) are substituted into these inequalities, the result is so complicated that there is little hope of obtaining a reasonable expression that will assure the positive definiteness of the matrix [P].

Recall the equations of motion for the driver/vehicle system with the driver acting as a proportional controller responding to both yaw angle and lateral displacement. The equations can be written as

$$\begin{Bmatrix} \dot{V}_y \\ \dot{\Omega}_z \\ \dot{\psi} \\ \dot{Y} \end{Bmatrix} = \begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{12}}{m} & -\frac{B_{13}}{m} & -\frac{B_{14}}{m} \\ -\frac{B_{21}}{I_z} & -\frac{B_{22}}{I_z} & -\frac{B_{23}}{I_z} & -\frac{B_{24}}{I_z} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & V_x & 0 \end{bmatrix} \begin{Bmatrix} V_y \\ \Omega_z \\ \psi \\ Y \end{Bmatrix} \quad . \tag{145}$$

For this set of equations, the Lyapunov matrix equation with [Q] equal to the identity is

$$\begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{21}}{I_z} & 0 & 1 \\ -\frac{B_{12}}{m} & -\frac{B_{22}}{I_z} & 1 & 0 \\ -\frac{B_{13}}{m} & -\frac{B_{23}}{I_z} & 0 & V_x \\ -\frac{B_{14}}{m} & -\frac{B_{24}}{I_z} & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 & P_4 & P_7 \\ P_2 & P_3 & P_5 & P_8 \\ P_4 & P_5 & P_6 & P_9 \\ P_7 & P_8 & P_9 & P_{10} \end{bmatrix}$$

$$+ \begin{bmatrix} P_1 & P_2 & P_4 & P_7 \\ P_2 & P_3 & P_5 & P_8 \\ P_4 & P_5 & P_6 & P_9 \\ P_7 & P_8 & P_9 & P_{10} \end{bmatrix} \begin{bmatrix} -\frac{B_{11}}{m} & -\frac{B_{12}}{m} & -\frac{B_{13}}{m} & -\frac{B_{14}}{m} \\ -\frac{B_{21}}{I_z} & -\frac{B_{22}}{I_z} & -\frac{B_{23}}{I_z} & -\frac{B_{24}}{I_z} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & V_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \qquad (146)$$

This matrix equation leads to ten independent equations for the elements of the matrix [P] which can be written as

$$- 2 \frac{B_{11}}{m} P_1 - 2 \frac{B_{21}}{I_z} P_2 + 2 P_3 = -1 ; \tag{147}$$

$$- \frac{B_{12}}{m} P_1 - (\frac{B_{11}}{m} + \frac{B_{22}}{I_z}) P_2 - \frac{B_{21}}{I_z} P_3 - P_4 + P_8 = 0 ; \tag{148}$$

$$- \frac{B_{13}}{m} P_1 - \frac{B_{23}}{I_z} P_2 - \frac{B_{11}}{m} P_4 - \frac{B_{21}}{I_z} P_5 - V_x P_7 + P_9 = 0 ; \tag{149}$$

$$- \frac{B_{14}}{m} P_1 - \frac{B_{24}}{I_z} P_2 - \frac{B_{11}}{m} P_7 - \frac{B_{21}}{I_z} P_8 = 0 ; \tag{150}$$

$$- 2 \frac{B_{12}}{m} P_2 - 2 \frac{B_{22}}{I_z} P_3 + 2 P_5 = -1 ; \tag{151}$$

$$- \frac{B_{13}}{m} P_2 - \frac{B_{23}}{I_z} P_3 - \frac{B_{12}}{m} P_4 - \frac{B_{22}}{I_z} P_5 + P_6 + V_x P_8 = 0 ; \tag{152}$$

$$- \frac{B_{14}}{m} P_2 - \frac{B_{24}}{I_z} P_3 - \frac{B_{12}}{m} P_7 - \frac{B_{22}}{I_z} P_8 + P_9 = 0 ; \tag{153}$$

$$- 2 \frac{B_{13}}{m} P_4 - 2 \frac{B_{23}}{I_z} P_5 + 2 V_x P_9 = -1 ; \tag{154}$$

$$- \frac{B_{14}}{m} P_4 - \frac{B_{24}}{I_z} P_5 - \frac{B_{13}}{m} P_7 - \frac{B_{23}}{I_z} P_8 + V_x P_{10} = 0 ; \tag{155}$$

and

$$- 2 \frac{B_{14}}{m} P_7 - 2 \frac{B_{24}}{I_z} P_8 = -1 \ . \tag{156}$$

Whether the elements of the matrix [P] are found by inverting the 10 x 10 above or by following the suggestion of Barnett and Storey [43] which introduces a skew-symmetric matrix reducing the problem to inverting a 6 x 6, the resulting elements of the matrix [P] will be extremely complicated expressions. These expressions must then be used to determine under what conditions the matrix [P] is positive definite. These conditions would be difficult to use because of their extreme algebraic complexity.

## V. DISCUSSION

In the Classical Result Section, it was shown that the automobile considered in an absolute reference frame is unstable. The classical result deals with the system's ability to return to straight line motion after a disturbance, but not to the original straight line motion. In order for the automobile to return to the original path, a driver is needed.

In the Driver/Vehicle Results Section, a driver which acts as a proportional controller responding to yaw angle and lateral displacement in the absolute reference frame was introduced. Equations of motion for the driver/vehicle system were written and the Routh-Hurwitz criterion was applied resulting in an expression for the critical speed of the driver/vehicle system.

It will now be shown by example that the classical result may be a nonconservative estimate of the critical speed of the driver/vehicle system. Consider a specific driver/vehicle system with the following parameter values:

$C_1$ = 30,000 N/rad

$C_2$ = 30,000 N/rad

$D_Y$ = 0.0016 rad/m

$D_\psi$ = 0.060 rad/rad

$g$ = 9.81 $m/s^2$

$I_z$ = 2000 $kg\text{-}m^2$

$L_1$ = 1.4 m

$L_2$ = 1.3 m

$$m = 1200 \text{ kg} .$$

From Equation (32) the classical result for the critical speed is 60 m/s. The critical speed of the driver/vehicle system from Equation (93) is 40 m/s. This illustrates that the classical result is nonconservative. In fact, for this oversteer vehicle, the classical result is 50% higher than the driver/vehicle system result.

Consider another specific driver/vehicle system. This time the vehicle is understeer with the following parameter values:

$$C_1 = 30{,}000 \text{ N/rad}$$

$$C_2 = 30{,}000 \text{ N/rad}$$

$$D_Y = 0.0016 \text{ rad/m}$$

$$D_\psi = 0.060 \text{ rad/rad}$$

$$g = 9.81 \text{ m/s}^2$$

$$I_z = 4300 \text{ kg-m}^2$$

$$L_1 = 1.6 \text{ m}$$

$$L_2 = 1.7 \text{ m}$$

$$m = 2100 \text{ kg} .$$

In this case the classical result gives no critical speed; that is the vehicle is stable at all forward speeds. But the critical speed of the driver/vehicle system is 59 m/s. Thus, the classical result implies the vehicle is always stable whereas the driver/vehicle system becomes unstable at a finite speed.

The classical result has proven to be a useful design tool through the years. The new result for the critical speed of the driver/vehicle system should also prove useful. The algebraic expression for the critical speed can be easily evaluated on a hand-held calculator.

Current design practice is to make automobiles understeer vehicles. This is based on experience. The automobile needs to be understeer to perform acceptably. The new result for the driver/vehicle system predicts that the vehicle needs to be understeer to be laterally stable.

Also in the Driver/Vehicle Results Section, a Lyapunov function was found for the classical problem which yields the Routh-Hurwitz condition for stability. In attempting to find a Lyapunov function for the driver/vehicle problem, it was tremendously difficult to perform the algebraic manipulations to carry out the standard method because of the complexity of the expressions.

In the case of only yaw control, the matrix [P] of the Lyapunov matrix equation was found, but it was not shown under what conditions the matrix was positive definite. It is suspected that if the Routh-Hurwitz conditions are satisfied, the matrix will be positive definite. This was the case for the classical problem.

The Lyapunov matrix equation for the driver/vehicle system with the driver responding to both yaw angle and lateral displacement was written. The resulting ten equations to be solved for the ten elements of the matrix [P] were also written, but not solved. They were not solved because based on experience with the third order problem where the elements of the matrix [P] were complicated the expressions for the elements of the matrix [P] for the fourth order problem promised to be overwhelming. Even if they were obtained, it is unlikely that the conditions for the matrix [P] to be positive definite could be found.

And if those conditions were found, the possibility of using the Lyapunov function is very small because of its great complexity.

# VI. CONCLUSIONS

In this dissertation it has been shown that the automobile considered in the absolute reference frame is unstable. If a driver responding proportionally to yaw angle and lateral displacement is added to the system, the system may be made stable. The critical speed of the driver/vehicle system is given by Equation (93). It would be interesting to perform parameter studies using Equation (93). The effect on the critical speed of changes in mass, moment of inertia, mass center location, cornering stiffness and driver gains could be studied quickly and inexpensively.

Lyapunov's second method, though powerful, is difficult to use. Finding a suitable function may be very time consuming. Two Lyapunov functions were found for the second order problem (the classical problem). One is very simple, but results in a very conservative value for the critical speed. The other is more complicated but yields the same result as the Routh-Hurwitz criterion.

A third order problem (yaw control only) was considered. The Lyapunov matrix equation was solved. The resulting function was very complicated and was not shown to be positive definite. However, it was noted that the function could be expressed more simply by looking for terms with factors equal to the coefficients of the characteristic equation and grouping terms containing those coefficients. It was also found that the denominator of the elements of the matrix in the quadratic form was equal to the product of the elements of the first column of the Routhian array. If the results of general 2 x 2, 3 x 3 and 4 x 4 systems could be found, it is suspected that a general method for writing a Lyapunov function in terms of the coefficients of the original system equations could be established.

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