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Anti-Ramsey problems on groups and graphs

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by

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DEDICATION

I dedicate my thesis to my family.

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ABSTRACT

Anti-Ramsey theory concerns itself with finding the fewest number of classes into which a large set must be partitioned, so that no small subset is completely partitioned. Typically, this is studied by considering the rainbow number of $\mathcal{F}$ with respect to H, denoted $\mathrm{rb}(H, \mathcal{F})$, which is the smallest integer r such that any r-coloring of the host object H admits a rainbow sub-object in the family $\mathcal{F}$. This thesis studies $\mathrm{rb}(H, \mathcal{F})$ in three settings: where the host object is a simple graph, a tournament, or a cyclic group. In particular, Chapter 2 focuses on the conditions on colorings of different graph families that force rainbow matchings of size $2m$ for some parameter m. Chapters 3 and 4 consider the rainbow number for solution sets to equations in cyclic groups. Chapter 5 takes the host object to be tournaments on n vertices, and determines the rainbow number for some families of directed graphs.

CHAPTER 1. GENERAL INTRODUCTION

1.1 Background and Definitions

This thesis will explore anti-Ramsey theory in graphs and in cyclic groups. Accordingly, there are two sets of basic definitions that will be discussed before motivating anti-Ramsey theory more generally.

1.1.1 Graphs

A *graph* G is an ordered pair (V, E) where V denotes the vertex set of G and E denotes the edge set of G. The vertex set is not subject to any structural requirements, but is generally taken to be a finite set. The edge set is a collection of unordered pairs of vertices. In some contexts it is easier to refer to $V(G)$ and $E(G)$ as the vertex and edge set of G, or G might be entirely suppressed. The size of the vertex and edge sets is denoted $v(G)$ and $e(G)$, respectively.

Two vertices $u, v \in V$ are *adjacent* if $\{u, v\} \in E$. An edge $\{u, v\}$ can also be denoted by uv or vu. It may also be convenient to let a single letter stand for an edge. For example, an edge might be denoted as $e \in E$. The *degree of v in G*, denoted $d(v)$ is the number of edges of G that contain v.

Some common graph families that will be explored throughout this thesis are given here.

- The *complete graph on n vertices*, denoted K_n, has vertex set $[n]$ and edge set $\binom{[n]}{2}$.
- The *path on n vertices*, denoted P_n, has vertex set $[n]$ and edge set $\{\{i, i+1\} : 1 \leq i < n\}$.
- The *cycle on n vertices*, denoted C_n, has vertex set $V = \{0, \ldots, n-1\}$ and edge set $\{\{i, i+1\} : i \in V\}$ where the vertices are taken modulo n.
- The *star on $n+1$ vertices*, denoted S_n, has vertex set $[n+1]$ and edge set $\{1i : 2 \leq i \leq n+1\}$.

- The *matching of size* m, denoted mK_2, has vertex set $[2m]$ and edge set $\{\{2i-1, 2i\} : 1 \leq i \leq m\}$.

A graph G is *isomorphic* to a graph H if there exists a bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. Notice that the isomorphism relationship is transitive, symmetric, and reflexive giving rise to equivalence classes of graphs. In this sense, the notation above for particular graphs (ex: K_n) can be used to represent any graph in an isomorphism class. A graph G is a subgraph of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. A graph G is F*-free* if G does not contain F as a subgraph.

A *k-edge-coloring* of a graph G is a function $c : E(G) \to [k]$. A k-edge-coloring c is *exact* if c is surjective. Suppose that G is a graph with an edge coloring c. A subgraph $F \subseteq G$ is *monochromatic* if there exists an i in the image of c such that $c(e) = i$ for all $e \in E(F)$. A subgraph $F \subseteq G$ is *rainbow* if c is injective when restricted to $E(F)$.

1.1.2 Groups

A group is a set G equipped with a binary operation $+ : G \times G \to G$ such that $+$ is associative ($a + (b + c) = (a + b) + c$ for all $a, b, c \in G$), there exists an identity ($0 + a = a + 0 = a$), and there exist inverses (for all $a \in G$ there exists $b \in G$ such that $a + b = 0$). This thesis only concerns itself with the integers, denoted $\mathbb{Z}$, and *cyclic groups of order* n, denoted $\mathbb{Z}_n$. Notice that these groups have a naturally defined multiplication and can also be thought of as rings. This distinction between $\mathbb{Z}$ as a group and $\mathbb{Z}$ as a ring will not be important, and well defined multiplication will be assumed when needed.

A *k-term arithmetic progression* with common distance $d \geq 1$ starting at a in a group G is a set

$$\{a, a + d, \ldots, a + (k-1)d\} \subseteq G.$$

Given an equation $f(x_1, \ldots, x_k) = 0$, a set $\{a_1, \ldots, a_k\} \subseteq G$ is a *solution set* of f if $f(a_1, \ldots, a_k) = 0$. There are a few equations that are studied in this thesis.

- The Schur equation is given by $x_1 + x_2 - x_3 = 0$ or more commonly $x_1 + x_2 = x_3$.

- The generalized Schur equation for $k \in \mathbb{Z}$ is given by $x_1 + x_2 - kx_3 = 0$ or more commonly $x_1 + x_2 = kx_3$.
- The Sidon equation is given by $x_1 + x_2 - x_3 - x_4 = 0$ or more commonly $x_1 + x_2 = x_3 + x_4$.

Notice that any 3-term arithmetic progression is a solution set for the generalized Schur equation with $k = 2$. In particular,

$$a + (a + 2d) = 2(a + d).$$

Furthermore, any 4-term arithmetic progression is a solution set for the Sidon equation. In particular,

$$a + (a + 3d) = (a + d) + (a + 2d).$$

Arithmetic progressions, the Schur equation, and the Sidon equation are all related and notable because they are used to measure the additive of an arbitrary subset of a group.

For example, a set $A \subset \mathbb{Z}$ is not necessarily a subgroup of $\mathbb{Z}$. However, if A still has some additive structure, then A should have many arithmetic progressions or solution sets to the Schur or Sidon equation. This idea is formally studied in additive combinatorics [10], and will not be addressed in this thesis. For this thesis it suffices to view all of these equations as measures of additivity and, therefore, interest in these equations has a common motivation.

Just as in the case of graphs, we can color groups. A *k-coloring of a group G* is a function $c : G \to [k]$. A k-coloring c is *exact* if c surjective. Similarly, for $A \subseteq G$ and a coloring c, we say that A is *monochromatic* if c is constant on A and that A is *rainbow* if c is injective on A.

1.1.3 Anti-Ramsey Theory

A useful starting point for the discussion of anti-Ramsey theory is the statement of Ramsey's Theorem.

Theorem 1.1.1 (Ramsey's Theorem [7]). *For any integers $k_1, \ldots, k_\ell \geq 2$, there exists $R(k_1, \ldots, k_\ell) \in \mathbb{N}$ such that if the edges of a K_n graph with $n \geq R(k_1, \ldots, k_\ell)$ are colored with ℓ colors, then there exists an i colored K_{k_i} for some $i \in [\ell]$.*

Intuitively, Ramsey's Theorem states that no matter how a large structure is partitioned into ℓ classes, some class must have a nice substructure. This theorem has been generalized to a number of settings. One relevant generalization is Van der Waerden's Theorem.

Theorem 1.1.2 (van der Waerden's Theorem [11])**.** *For any $\ell, k \geq 2$, there exists an $W(\ell, k) \in \mathbb{N}$ such that any ℓ coloring of $[n]$ with $n \geq W(\ell, k)$ contains a monochromatic k-term arithmetic progression.*

While Ramsey theory is primarily concerned with proving the existence of monochromatic substructures, anti-Ramsey theory (and anti-Van der Waerden theory) seeks the existence of rainbow substructures. The corresponding anti-Ramsey number of a graph is defined as follows:

Definition 1.1.3. *The anti-Ramsey number of a graph G with respect to H is the largest integer $k = ar(H, G)$ such that there exists an exact k-coloring of the edges of H with no rainbow copy of G.*

A related notion is the rainbow number of a graph.

Definition 1.1.4. *The rainbow number of a graph G with respect to H is the smallest integer $k = rb(H, G)$ such that any k-coloring of the edges of H admits a rainbow copy of G.*

Notice that $\mathrm{ar}(H, G) + 1 = \mathrm{rb}(H, G)$. This motivates the convention that $\mathrm{rb}(H, G) = e(H)$ when H does not contain G as a subgraph. To accommodate the variety of settings that are explored in this thesis, it may be useful to consider the following vague definition of a rainbow number.

Definition 1.1.5. *The rainbow number of a family of sub-objects $\mathcal{F}$ with respect to a host object H is the smallest integer $k = rb(H, \mathcal{F})$ such that any k-coloring of H (whatever that means) admits a rainbow copy of a sub-object in $\mathcal{F}$.*

Here we can imagine all the possible combinations of host objects and sub-object families: graphs with respect to graphs, solutions sets to an equation with respect to a group or an interval of the integers, and oriented graphs with respect to tournaments. In fact, each of these combinations will be addressed in the chapters ahead.

Unlike Ramsey numbers, there is nothing remarkable about the existence proof of the rainbow number. Any sub-object of a rainbow host will also be rainbow, and since the hosts have a finite size, the rainbow number must also be finite. However, the rainbow number does have a nice dual interpretation to the Ramsey number. The Ramsey number is the minimum size of a host object H so that any attempt to partition H into ℓ classes will fail to destroy every member of a family of sub-objects. On the other hand, the rainbow number is the smallest number k such that if a host object is partitioned into k classes, then some member of a family of sub-objects is completely destroyed. If Ramsey theory is about unavoidable structure, then anti-Ramsey theory is about guaranteed chaos.

1.2 Rainbow Matchings

For a comprehensive survey of anti-Ramsey results on graphs, see [4]. This section will give an account of anti-Ramsey results concerning matchings. The *extremal number of* H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges on n vertices without an H subgraph. In other words, any graph on n vertices with more than $\mathrm{ex}(n, H)$ edges has H as a subgraph. A simple idea relates the anti-Ramsey number of H to the extremal number of H. If a graph G is colored with more than $\mathrm{ex}(n, H)$ colors, then a subgraph of G that contains one edge of each color must contain a rainbow copy of H. This idea is put to work in the following proposition.

Proposition 1.2.1 (Proposition 6 in [8])**.** *For* $n, k \geq 1$*,*

$$ex(n, (k-1)K_2) + 1 \leq ar(K_n, kK_2) \leq ex(n, kK_2).$$

For convenience, see the extremal number for a k-matching next.

Theorem 1.2.2 (Theorem 4.1 in [3])**.** *For* $n, k \geq 1$*,*

$$ex(n, kK_2) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

After a series of improvements, Chen, Li, and Tu proved that the lower bound of Proposition 1.2.1 is essentially correct [2]. Their proof relied on the Gallai-Edmonds Structure Theorem. A simpler proof for the case when $n = 2k$ and $k \geq 3$ was given by Haas and Young in [5].

Theorem 1.2.3 (Theorem 1.3 in [2])**.**

$$ar(K_n, kK_2) = \begin{cases} 4 & n = 4, k = 2 \\ ex(n, (k-1)K_2) + 2 & n = 2k, k \geq 7 \\ ex(n, (k-1)K_2) + 1 & \textit{otherwise} \end{cases}$$

There are two notable features of the optimal anti-Ramsey construction for matchings. First, there is one color that is ambient in the sense that it can be found almost anywhere in the graph. This is a result of taking K_n as the host graph. A similar phenomenon will come up in Chapter 5, which considers rainbow numbers of directed graphs in tournaments.

The second feature, which is more pressing at the moment, is that almost all colors appear on a unique edge, and these uniquely colored edges are packed together so that they do not admit a rainbow matching. This second feature motivates imposing restrictions on colorings that spread out the colors. Chapter 2 explores different restrictions on colorings that spread out the colors in the host graph, making rainbow matchings easier to find.

1.3 Rainbow Solution Sets

A shortfall of Ramsey's and Van der Waerden's Theorems is that they do not specify which color class will contain a monochromatic clique or arithmetic progression. Szemerédi's Theorem addresses this shortfall for Van der Waerden's Theorem. In particular, Szemerédi's Theorem says that any sufficiently dense set should contain arbitrarily long arithmetic progressions.

Theorem 1.3.1 (Szemerédi's Theorem [9])**.** *If $A \subseteq \mathbb{Z}^+$ and*

$$\limsup_{n\to\infty} \frac{|A \cap [n]|}{n} > 0,$$

then A contains a k-term arithmetic progression for all $k \in \mathbb{N}$.

The rainbow analog of Szeméredi's Theorem was proven by Jungić, Licht, Maholian, Nešetřil, and Radoičić.

Theorem 1.3.2 (Theorem 2.1 in [6]). *If c is a 3-coloring of $\mathbb{Z}^+$ with color classes R, B, G such that*

$$\limsup_{n\to\infty} |X \cap [n]| - \frac{n}{6} = \infty$$

for each $X \in \{R, B, G\}$, then c admits a rainbow 3-term arithmetic progression.

The analogous theorem also holds for finite intervals.

Theorem 1.3.3 (Theorem 3 in [1]). *If $n \geq 3$ and c is a 3-coloring of $[n]$ with color classes R, B, G such that*

$$|R|, |B|, |G| > \begin{cases} \lfloor (n+2)/6 \rfloor & n \not\equiv 2 \mod 6 \\ (n+4)/6 & n \equiv 2 \mod 6 \end{cases},$$

then c admits a rainbow 3-term arithmetic progression.

Interestingly, Axenovich and Fon-Der-Flaass showed that for any $k \geq 4$, there exists an equinumerous k-coloring of $[n]$ with no rainbow k-term arithmetic progression [1]. This begs the question of determining how many colors guarantee a k-term arithmetic progression. This question is addressed for various solution sets in Chapters 3 and 4.

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CHAPTER 2. RAINBOW MATCHINGS OF SIZE m IN GRAPHS WITH TOTAL COLOR DEGREE AT LEAST $2mn$

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Abstract

The existence of a rainbow matching given a minimum color degree, proper coloring, or triangle-free host graph has been studied extensively. This paper generalizes these problems to edge colored graphs with given total color degree. In particular, we find that if a graph G has total color degree $2mn$ and satisfies some other properties, then G contains a matching of size m. These other properties include G being triangle-free, C_4-free, properly colored, or large enough.

2.1 Introduction

Given a graph G, let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. If $S \subseteq V$, then $G[S]$ denotes the subgraph induced by the vertices in S. A graph G is an m-matching if G contains exactly m edges, $2m$ vertices, and $e \cap e' = \{\}$ for all edges $e \neq e'$ in $E(G)$. An edge coloring $c : E(G) \to [r] = \{1, \ldots, r\}$ is an assignment of colors to edges. A proper edge coloring of a graph is an edge coloring such that $c(e) \neq c(e')$ whenever $e \cap e' \neq \varnothing$ and $e \neq e'$. The colors used on a graph will be denoted $c(G)$, and R will denote a generic color class. If $X, Y \subseteq V(G)$, then $c(X, Y)$ will denote the set of colors used on edges of the form xy, where $x \in X$, $y \in Y$. A graph G is rainbow under c if c is injective on $E(G)$. In particular, a rainbow matching is a matching where each edge receives a unique color within the matching. The color degree of a vertex v is denoted $\hat{d}_G(v)$, which is the number of colors c assigns to edges incident upon v in G; when it is clear from the context what G is, we will drop the subscript. Let $\hat{d}^R(v)$

denote the number of R colored edges incident upon v. The total color degree of G with respect to c is the sum of all the color degrees in the graph and denoted

$$\hat{d}(G) = \sum_{v \in V(G)} \hat{d}(v).$$

The average color degree of a graph G is obtained by dividing the total color degree by $|V(G)|$, and is an equivalent notion. The minimum color degree of G is denoted $\hat{\delta}(G)$. Finally, let $G - v$ denote the graph G with the vertex v deleted, and $G - R$ denote the graph G with the edges in color class R deleted. When convenient, we will let $c(e)$ denote a color class so that $G - c(e)$ denotes the graph G without the edges in color class containing the edge e.

Rainbow matchings in graphs were originally studied in connection to transversals of Latin squares [9, 10]. However, the existence of rainbow matchings has also been studied in its own right. In [6], Li and Wang conjectured that any graph with $\hat{\delta}(G) \geq m \geq 4$ contains a rainbow matching of size $\lceil \frac{m}{2} \rceil$. This conjecture was partially confirmed in [5], and fully confirmed in [4].

Wang asked for a function f such that any properly edge colored graph G with $|V(G)| \geq f(\hat{\delta}(G))$ contains a rainbow matching of size $\hat{\delta}(G)$ [11]. Diemunsch et al. determined that $|V(G)| \geq \frac{98}{23}\hat{\delta}(G)$ is sufficient [2]. This problem was generalized to find a function f such that any edge colored graph G with $|V(G)| \geq f(\hat{\delta}(G))$ contains a rainbow matching of size $\hat{\delta}(G)$. The authors of [1] found that $|V(G)| \geq \frac{17}{4}\hat{\delta}(G)^2$ sufficed. This was improved to $4\hat{\delta}(G) - 4$ for $\hat{\delta}(G) \geq 4$ in [3] and [8] independently.

Local Anti-Ramsey theory asks Anti-Ramsey type questions with assumptions about the local structure of the host graph. In particular, Local Anti-Ramsey theory is about the minimum k such that any coloring of K_n with $\hat{\delta}(G) \geq k$ contains a rainbow copy of H. In this vein, Wang's question can be posed as follows: given k, what is the smallest N such that any properly edge colored graph G with $|V(G)| \geq N$ and $\hat{\delta}(G) \geq k$ contains a rainbow matching of size k? Furthermore, proper edge-coloring and triangle-free properties play similar roles in restricting the structure of a host graph.

The local assumptions in Anti-Ramsey theory are interesting in so far as they highlight the relationship between a local parameter and the target graph. In much of the rainbow matching literature, there are confounding local assumptions. For example, [2], [7], and [11] all consider host graphs that have a prescribed minimum color degree and are properly edge colored. In this case, an intuitive interpretation is that the minimum color degree and proper edge-coloring properties spread the colors apart in the host graph. As one would expect, this makes it easier to find a large rainbow matching. However, it is unclear whether both the minimum color degree and proper edge coloring property are necessary to find a large matching.

The goal of this paper is to shed light on the relationship between local assumptions and rainbow matchings. Rather than considering host graphs with a prescribed minimum color degree, we will consider host graphs with a prescribed average color degree. This is motivated in part by a question posed during the Rocky Mountain and Great Plains Graduate Research Workshop in Combinatorics in 2017.

Question 2.1.1. *If G is an edge colored graph on n vertices with $\hat{d}(G) \geq 2mn$, does G contain a rainbow matching of size m?*

Section 2.2 considers this question for triangle-free and C_4-free host graphs. In the case of triangle-free graphs, we will prove the slightly stronger statement that if G is a graph with $\hat{d}(G) > 2mn$, then there exists a rainbow matching of size $m+1$. Section 2.3 pertains to properly edge colored host graphs. Finally, Section 2.4 considers edge colored graphs with total color degree $2mn$, but with no further assumptions.

2.2 Triangle-free and C_4-free Graphs

In this section, we consider triangle-free and C_4-free graphs.

Theorem 2.2.1. *Let G be a triangle-free graph on n vertices. Let c be an edge coloring of G with $\hat{d}(G) > 2mn$. Then c admits a rainbow matching of size $m+1$.*

Proof. For the sake of contradiction, let M be a maximum rainbow matching of size $k \leq m$ with edges u_iv_i for $1 \leq i \leq k$, such that the number of colors appearing on $G[V(G) \setminus V(M)] = H$ is maximized. Without loss of generality, suppose that $c(u_iv_i) = i$. Since G is triangle-free,

$\hat{d}(u_i) + \hat{d}(v_i) \leq n$ for all $u_iv_i \in E(M)$. If H has an edge e, then $c(e) \in [k]$. Without loss of generality, suppose that $c(H) = [j]$ for some $0 \leq j \leq k$. Then for all $v \in V(H)$, we have $\hat{d}(v) \leq k + j$. Notice that if there exists an edge $e \in H$ with $c(e) = i$, then we can swap e and u_iv_i to conclude that $\hat{d}(u_i) + \hat{d}(v_i) \leq 2(j + k)$.

Now consider

$$\begin{aligned}
2mn &< \sum_{i=1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \hat{d}_G(v) \\
&\leq \sum_{i=1}^{j} \hat{d}(u_i) + d(v_i) + \sum_{i=j+1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \Big(\hat{d}_H(v) + k \Big) \\
&\leq 2j(k + j) + (k - j)n + (n - 2k)(j + k) \\
&= 2jk + 2j^2 + 2nk - 2jk - 2k^2 \\
&\leq 2j^2 - 2k^2 + 2nk \\
&\leq 2nm.
\end{aligned}$$

This is a contradiction; therefore, $k \geq m + 1$. □

A key element to the proof of Theorem 2.2.1 is the bound $\hat{d}(v) + \hat{d}(u) \leq n$ where uv is an edge in a maximal matching. We can obtain a similar bound in C_4-free graphs in order to prove the next theorem.

Theorem 2.2.2. *Let G be a C_4-free graph on n vertices. Let c be an edge coloring of G with $\hat{d}(G) \geq 2mn$. Then c admits a rainbow matching of size m.*

Proof. For the sake of contradiction, let M be a maximum rainbow matching of size $k < m$ with edges u_iv_i for $1 \leq i \leq k$, such that the number of colors appearing on $G[V(G) \setminus V(M)] = H$ is maximized. Without loss of generality, suppose that $c(u_iv_i) = i$. Since G is C_4-free, $\hat{d}(u_i) + \hat{d}(v_i) \leq n + 1$ for all $u_iv_i \in E(M)$. If H has an edge e, then $c(e) \in [k]$. Without loss of generality, suppose that $c(H) = [j]$ for $0 \leq j \leq k$.

Claim 2.2.3. *If $xy \in E(H)$ with $c(xy) = i \leq j$, then $\hat{d}(u_i) + \hat{d}(v_i) \leq 2j + 2k$.*

Notice that x, y each see at most j colors in H. Since xy can share at most two edges with any edge in M without creating a C_4 subgraph, we have $|c(\{u_i, v_i\}, xy)| \leq 2$ for every $1 \leq i \leq k$. Thus, $\hat{d}(x) + \hat{d}(y) \leq 2j + 2k$. By swapping $u_i v_i$ and xy, we obtain the desired bound on $\hat{d}(u_i) + \hat{d}(v_i)$.

Furthermore, $\sum_{v \in H} \hat{d}_G(v) \leq (n-2k)(j+k) + k$. The $(n-2k)j$ term comes from the fact that H has $n - 2k$ vertices, each of which can see every color in $[j]$. We will show that there are at most $(n-2k)k + k$ color degrees in H that do not come from a color in $[j]$ by contradiction. Suppose that there are $(n-2k)k + k + 1$ edges from H to M. By the pigeon hole principle, there exists an edge $u_i v_i \in M$ that receives at least $n - 2k + 2$ edges from H. Notice that each vertex in H can send at most two edges to $u_i v_i$. Therefore, there must exist two vertices in H that each send two edges to $u_i v_i$, witnessing a C_4 subgraph; this is a contradiction.

Now consider

$$\begin{aligned}
2mn &\leq \sum_{i=1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \hat{d}_G(v) \\
&\leq \sum_{i=1}^{j} \hat{d}(u_i) + d(v_i) + \sum_{i=j+1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \left(\hat{d}_H(v) + k\right) \\
&\leq j(2k+2j) + (k-j)(n+1) + (n-2k)(j+k) + k \\
&= 2kj + 2j^2 + nk + k - nj - j + nj + nk - 2kj - 2k^2 + k \\
&\leq 2j^2 + 2nk - j + 2k - 2k^2 \\
&\leq 2j^2 - 2k^2 + 2k - j - 2n + 2mn \\
&< 2mn.
\end{aligned}$$

This is a contradiction; therefore, $k \geq m$. □

2.3 Properly Edge Colored Graphs

In this section, we consider properly edge colored graphs. The idea to analyze a greedy algorithm that constructs a matching appears in [2] and [1]. The algorithm employed in this section is similar, with some adjustments to take into account the weaker degree assumption.

Theorem 2.3.1. *Let c be a proper edge coloring of G with $n \geq 8m$ and $\hat{d}(G) \geq 2mn$. Then c admits a rainbow matching of size m.*

Proof. Assume that G is an edge minimal counter example to Theorem 2.3.1. Consider the following algorithm:

1. set $G_0 := G$
2. if there exists $v \in V(G_{i-1})$ with $\hat{d}(v) \geq 3(m-i)+1$, then $G_i = G_{i-1} - v$ and return to 2
3. else, if there exists color class R with $|R| \geq 2(m-i)+1$ in G_{i-1}, then $G_i = G_{i-1} - R$ and return to 2
4. else, if there exists $uv \in E(G_{i-1})$, then $G_i = G_{i-1} - u - v - c(uv)$ and return to 2
5. return $i-1$

Claim 2.3.2. *Suppose the algorithm returns $k \leq m$. Then G_i contains a matching of size $k-i$ for $0 \leq i \leq k$*

We will prove the claim by reverse induction on i. If $i = k$, then G_i is empty, and the claim is true. Assume that the claim is true for i. We will prove the claim for $i-1$. By the induction hypothesis, there exists a matching $M \subseteq G_i$ of size $k-i$. There are three cases:

Case 1: Assume $G_i = G_{i-1} - v$ where $\hat{d}(v) \geq 3(m-i)+1$. By construction, $v \notin V(M)$. Since $\hat{d}(v) \geq 3(m-i)+1$, there exists $u \in N(v)$, such that $u \notin V(M)$ and $c(uv) \notin c(M)$. Then $M' = M \cup \{uv\}$ is a rainbow matching of size $k-i+1$.

Case 2: Assume $G_i = G_{i-1} - R$ for some color R with $|R| \geq 2(m-i)+1$. This implies that $c(e) \neq R$ for all $e \in E(M)$. Since c is a proper coloring and $|R| \geq 2(m-i)+1$, there exist $e \in G_{i-1}$ such that $c(e) = R$ and $M' = M \cup \{e\}$ is a rainbow matching.

Case 3: Assume that $G_i = G_{i-1} - v - u - c(uv)$ for some $uv \in E(G_{i-1})$. By construction $N[u] \cup N[v]$ is disjoint from $V(M)$ and $c(e) \neq c(uv)$ for all $e \in M$. Therefore, $M' = M \cup \{uv\}$ is a rainbow matching.

This concludes the proof of the claim. Since G is an edge minimal counter example, the algorithm applied to G will return $k < m$. We will now derive a contradiction.

Let $W(G_i)$ denote the difference of total color degree between G_i and G_{i-1} under c.

Claim 2.3.3. *For all $1 \leq i \leq k$, we have $W(G_i) \leq 2n$.*

Case 1: Assume $G_i = G_{i-1} - v$ where $\hat{d}(v) \geq 3(m-i)+1$. Notice that v is incident to at most $n-1$ edges. Therefore, deleting v will remove at most $2(n-1)$ color degrees.

Case 2: Assume $G_i = G_{i-1} - R$ for some color R with $|R| \geq 2(m-i)+1$. Because c is proper, $|R| \leq \lfloor n/2 \rfloor$. Deleting all edges of color R reduces the total color degree by at most n.

Case 3: Assume that $G_i = G_{i-1} - v - u - c(uv)$ for some $uv \in E(G_{i-1})$. Since G_i is not constructed by step 2, we know that $\hat{d}(u), \hat{d}(v) \leq 3(m-i)$. Furthermore, since G_i is not constructed by step 3, we know that $|c(uv)| \leq 2(m-i)$. This implies that

$$\begin{aligned} W(G_i) &= 2(\hat{d}(v) + \hat{d}(u)) + 2|c(uv)| \\ &\leq 16(m-i) \\ &\leq 2n. \end{aligned}$$

This concludes the proof of the claim. Now we have

$$2nm \leq \hat{d}(G) = \sum_{i=1}^{k} W(G_i) \leq 2nk,$$

which is a contradiction since $k < m$. Therefore, the theorem is proven. □

2.4 General Edge Colored Graphs

Theorem 2.4.1 provides contrast for Theorems 2.2.1, 2.2.2, and 2.3.1. The proof of Theorem 2.4.1 is similar to the proof of Theorem 2.3.1. However, the greedy algorithm has been modified to accommodate graphs that are not properly colored.

Theorem 2.4.1. *Let c be an edge coloring of G be a graph with $\hat{d}(G) \geq 2mn$ and $n \geq 12m^2 + 4m$. Then c admits a rainbow matching of size m.*

Proof. Assume that G is an edge minimal counter example to Theorem 2.4.1. Since G is edge minimal, no color class can induce a P_4 (path on 4 vertices) or a triangle. This follows from the fact that if a color class R induces a P_4 or triangle, then an edge can be deleted without reducing the total color degree of the graph. Therefore, each color class in G induces a forest of stars. Let $s(R)$ denote the number of components induced by the color class R. Consider the following algorithm:

1. set $G_0 := G$
2. if there exists $v \in V(G_{i-1})$ with $\hat{d}(v) \geq 3(m-i)+1$, then $G_i = G_{i-1} - v$ and return to 2
3. else, if there exists color R with $s(R) \geq 2(m-i)+1$ in G_{i-1}, then $G_i = G_{i-1} - R$ and return to 2
4. else, if there exists a vertex v and a color R such that $\hat{d}^R(v) \geq 3(m-i)+1$ in G_{i-1}, then $G_i = G_{i-1} - v - R$ and return to 2
5. else, if there exists $uv \in E(G_{i-1})$, then $G_i = G_{i-1} - u - v - c(uv)$ and return to 2
6. return $i-1$

Since this algorithm is so similar to the algorithm featured in the proof of Theorem 2.3.1, the only things that remain to be checked are that step 4 lets us extend a matching, and that the bounds on steps 4 and 5 are still good.

Assume that $G_i = G_{i-1} - v - R$ where $\hat{d}^R(v) \geq 3(m-i)+1$. Let M be a rainbow matching of size $k-i$ contained in G_i. Since $v \notin V(G_i)$, $v \notin V(M)$. Furthermore, M does not contain an edge with color R. Since $\hat{d}^R(v) \geq 2(m-i)+1$, there exists an edge uv with $c(uv) = R$ and $u \notin M$. Then $M \cup \{uv\}$ is a rainbow matching of size $k-i+1$ contained in G_{i-1}.

If $G_i = G_{i-1} - v - R$ where $\hat{d}^R(v) \geq 3(m-i)+1$, then 2 and 3 must have been rejected. The color R contributes at most $n - 3(m-i)$ color using edges that are not incident upon v. Since $\hat{d}(v) \leq 3(m-i)$ and $d(v) \leq n$, it follows that
$W(G_i) \leq n - 3(m-i) + \hat{d}(v) + d(v) \leq n - 3(m-i) + 3(m-i) + n = 2n.$

Suppose $G_i = G_{i-1} - v - u - c(uv)$. Then steps 2, 3, and 4 must have been rejected. This implies that $\hat{d}(v), \hat{d}(u) \leq 3(m-i)$. Furthermore, each color at v, u can be represented at most $3(m-i)$ times. Finally, the edges of color $c(uv)$ can induce at most $2(m-i)$ stars with $3(m-i)$ edges each. Therefore, deleting all $c(uv)$ colored edges reduces the color degree by at most $6m^2 + 2m$. Thus, $W(G_i) \leq 24m^2 + 8m \leq 2n$.

Suppose that the algorithm terminates in $k < m$ steps. Now we have

$$2nm \leq \hat{d}(G) = \sum_{i=1}^{k} W(G_i) \leq 2nk,$$

which is a contradiction since $k < m$. Therefore, the theorem is proven. □

2.5 Future Work

Though we were not able to resolve Question 2.1.1 for all graphs, we believe the answer is affirmative:

Conjecture 2.5.1. *All edge colored graphs G with $\hat{d}(G) \geq 2mn$ contain a rainbow matching of size m.*

It would also be interesting to know under which conditions there exists a matching of size $m+1$. It seems that a small improvement in the estimates in the proofs of Theorems 2.2.1 and 2.3.1 could yield this result for edge colored graphs G with $\hat{d}(G) \geq 2mn$. In fact, it may be that the proper question to ask is whether any graph G with $\hat{d}(G) \geq 2mn$ contains a rainbow matching of size $m+1$.

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CHAPTER 3. RAINBOW NUMBERS FOR $x_1 + x_2 = kx_3$ IN $\mathbb{Z}_n$

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Abstract

In this work, we investigate the fewest number of colors needed to guarantee a rainbow solution to the equation $x_1 + x_2 = kx_3$ in $\mathbb{Z}_n$. This value is called the rainbow number and is denoted by $rb(\mathbb{Z}_n, k)$ for positive integer values of n and k. We find that $rb(\mathbb{Z}_p, 1) = 4$ for all primes greater than 3 and that $rb(\mathbb{Z}_n, 1)$ can be determined from the prime factorization of n. Furthermore, when k is prime, $rb(\mathbb{Z}_n, k)$ can be determined from the prime factorization of n for some n.

3.1 Introduction

Let $\mathbb{Z}_n$ be the cyclic group of order n, and let an *r-coloring* of $\mathbb{Z}_n$ be a function $c : \mathbb{Z}_n \to [r]$ where $[r] := \{1, ..., r\}$. In this paper, we assume that each r-coloring is *exact* (surjective). Given an exact r-coloring, we define r *color classes* $C_i = \{x \in \mathbb{Z}_n \mid c(x) = i\}$ for $1 \leq i \leq r$. Occasionally, when convenient, we will use R, G, B, and Y to denote the colors or the color classes red, green, blue, and yellow, respectively. Furthermore, we will use $\text{im}(c)$ to denote the set of colors used by c.

Fix an integer k. Let a *triple* (x_1, x_2, x_3) be any three elements in $\mathbb{Z}_n$ which are a solution to $x_1 + x_2 \equiv kx_3 \mod n$. When $k = 1$, we will call these triples *Schur triples*. Such a triple is called a *rainbow triple* under a coloring c when $c(x_1) \neq c(x_2)$, $c(x_1) \neq c(x_3)$, and $c(x_2) \neq c(x_3)$. Consequently, a coloring will be called *rainbow-free* when there does not exist a rainbow triple in $\mathbb{Z}_n$ under c.

The *rainbow number* of $\mathbb{Z}_n$ given $x_1 + x_2 = kx_3$, denoted $rb(\mathbb{Z}_n, k)$, is the smallest positive integer r such that any r-coloring of $\mathbb{Z}_n$ admits a rainbow triple. By convention, if such an integer does not exist, we set $rb(\mathbb{Z}_n, k) = n + 1$. A *maximum* coloring is a rainbow-free r-coloring of $\mathbb{Z}_n$ where $r = rb(\mathbb{Z}_n, k) - 1$.

For a coloring c of $\mathbb{Z}_{st}$, the i^{th} *residue class* modulo t is the set of all the elements in $\mathbb{Z}_{st}$ which are congruent to $i \mod t$. Denote each residue class as $R_i = \{j \in \mathbb{Z}_{st} | j \equiv i \mod t\}$. We say the i^{th} *residue palette* modulo t is the set of colors which appear in the i^{th} *residue class*, and we will denote each palette as $P_i = \{c(j) | j \equiv i \mod t\}$.

Rainbow numbers for the equation $x_1 + x_2 = 2x_3$, for which the solutions are 3-term arithmetic progressions, have been studied in [1], [2], [3], and [5]. These problems are historically rooted in Roth's Theorem, Szemerédi's Theorem, and van der Waerden's Theorem. The first half of our paper explores the rainbow numbers of $\mathbb{Z}_n$ given the Schur equation, $x_1 + x_2 = x_3$. We rely on the work of Llano and Montenjano in [4], Jungić et al. in [3], and Butler et al. in [2] to prove exact values for $rb(\mathbb{Z}_n, 1)$ in terms of the prime factorization of n. Our results are an extension to the results in [1], [3], and [5].

The motivation for our results is captured in the idea that the rainbow number of $\mathbb{Z}_n$ given $x_1 + x_2 = kx_3$ to the prime factors of n. Theorems 3.1.1 and 3.1.3 confirm that rainbow numbers of depend on the prime factorization.

Theorem 3.1.1. *For a prime $p \geq 5$, $rb(\mathbb{Z}_p, 1) = 4$.*

Remark 3.1.2. *It can be deduced through inspection that $rb(\mathbb{Z}_2, 1) = rb(\mathbb{Z}_3, 1) = 3$.*

Theorem 3.1.1 gives exact values for $rb(\mathbb{Z}_p, 1)$ where p is prime. Therefore, Theorems 3.1.1 and 3.1.3 give exact values for $rb(\mathbb{Z}_n, 1)$. The proof for Theorem 3.1.3 is at the end of Section 3.2.3.

Theorem 3.1.3. *For a positive integer n with prime factorization $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$,*

$$rb(\mathbb{Z}_n, 1) = 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

We continue by considering the equation $x_1 + x_2 = px_3$ for any prime p. Many of the techniques for the $k = 1$ case generalize. However, there are complications. If we let the prime factorization of n be $n = p^{\alpha} \cdot q_1^{\alpha_1} \cdots q_m^{\alpha_m}$, then we can produce a recursive formula for $rb(\mathbb{Z}_n, p)$ detailed in Theorem 3.1.6. To obtain exact results from the recursive formula, we need to know the value of $rb(Z_q, p)$ for prime p and q. These values are determined in Theorems 3.1.4 and 3.1.5.

We would like to note that Theorem 3.1.4 resembles Theorem 3.5 in [3]. In essence, we use the ideas from Theorem 3.5 in [3] to construct a lower bound. The upper bound is obtained by using structural information from Theorem 2 in [4], which we restate as Theorem 3.3.1.

Theorem 3.1.4. *Let p, q be distinct and prime. Then $rb(\mathbb{Z}_q, p) = 4$ if and only if p, q do not satisfy either of the following conditions:*

1. *p generates $\mathbb{Z}_q^*$,*
2. *$|p| = (q-1)/2$ in $\mathbb{Z}_q^*$ and $(q-1)/2$ is odd.*

Otherwise, $rb(\mathbb{Z}_q, p) = 3$.

Theorem 3.1.5. *For $p \geq 3$ prime and $\alpha \geq 1$,*

$$rb(\mathbb{Z}_{p^\alpha}, p) = \begin{cases} 3 & p = 3, \alpha = 1 \\ 4 & p = 3, \alpha \geq 2 \\ \frac{p+1}{2} + 1 & p \geq 5 \end{cases} .$$

The values for $rb(\mathbb{Z}_{2^\alpha}, 2)$ are resolved in [1]. In conjunction with Theorems 3.1.4 and 3.1.5, Theorem 3.1.6 determines exact values for $rb(\mathbb{Z}_n, p)$ when n is a prime power. The proof for Theorem 3.1.6 is at the end of Section 3.3.4.

Theorem 3.1.6. *Let p and q be distinct primes. Then*

$$rb(\mathbb{Z}_{q^\alpha}, p) = 2 + \alpha(rb(\mathbb{Z}_q, p) - 2).$$

3.2 Schur Triples

Section 1 is dedicated to proving Theorem 3.1.3. In Section 3.2.1 we introduce the idea of a dominant color to describe the structural properties of colorings of $\mathbb{Z}_p$. Additionally, we prove Proposition 3.2.4, the Schur triple counterpart of Theorem 3.2 in [3]. We use Proposition 3.2.4 to prove Theorem 3.1.1, concluding Section 1.1. In Section 3.2.2 we show that the lower bound of $rb(\mathbb{Z}_n, 1)$ can be determined by the prime factorization of n. The equivalent upper bound is proved in 3.2.3. Combining Sections 3.2.2 and 3.2.3 proves Theorem 3.1.3.

3.2.1 Schur Triples in $\mathbb{Z}_p$, p prime

Let c be a coloring of $\mathbb{Z}_n$. We say a sequence $S_1, S_2, \dots, S_k$ of colors *appears at position* i if $c(i) = S_1, c(i+1) = S_2, \dots, c(i+k-1) = S_k$. A sequence is *bichromatic* if it contains exactly two colors. A color R is *dominant* if for $S = \{c(x) : i \le x \le j, i < j\}$, $|S| = 2$ implies $R \in S$. That is, R appears in every bichromatic string. Using dominant colors to derive a contradiction is used in [3]. We also use this idea to describe the structure of rainbow-free colorings of $\mathbb{Z}_p$. However, we must show that a dominant color exists.

Lemma 3.2.1. *There exists a dominant color in every rainbow-free coloring of $\mathbb{Z}_n$. Furthermore, $c(1)$ is dominant.*

Proof. Let c be a rainbow-free coloring of $\mathbb{Z}_n$. Note that $(1, i, i+1)$ is a Schur triple for all $i \notin \{0, 1\}$. Since c is rainbow-free, either $c(i) = c(i+1)$, $c(1) = c(i)$, or $c(1) = c(i+1)$. Thus, if $c(i) \ne c(i+1)$, then $c(1)$ must appear on either i or $i+1$. This implies that $c(1)$ is dominant. □

An immediate result from this lemma is that any color which doesn't appear on 1 must be adjacent to itself or the dominant color. Now we can relate the structure of our coloring to the presence of a rainbow triple. Without loss of generality, let $c(1) = R$ be dominant.

Lemma 3.2.2. *Let c be an r-coloring of $\mathbb{Z}_n$ with $r \ge 3$. If BB and GG appears in c, then there exists a rainbow Schur triple in c.*

Proof. Let c be an r-coloring of $\mathbb{Z}_n$ with $r \geq 3$ such that BB and GG appears in c. Without loss of generality, assume R is dominant, and c contains BB and GG. Then, the sequence BBR must appear at some position i and the sequence GGR must appear at some position j.

Consider the Schur triple $(i, j+2, i+j+2)$. Since $c(i) = B$, and $c(j+2) = R$, then either c contains a rainbow Schur triple, or $c(i+j+2)$ is R or B. Assume the second case, and consider the Schur triple $(i+2, j, i+j+2)$. Since $c(i+2) = R$, and $c(j) = G$ then either c contains a rainbow Schur triple or $c(i+j+2)$ is R. Again, assume the second case, and finally consider the triple $(i+1, j+1, i+j+2)$. Since $c(i+1) = B$, $c(j+1) = G$, and $c(i+j+2) = R$, this triple is rainbow. Therefore, c contains a rainbow Schur triple. □

Therefore, if c is a rainbow-free coloring of $\mathbb{Z}_n$ with R dominant, either GG or BB can appear in c, but not both. Next we show that there are ways to re-order colorings while maintaining whether or not Schur triples are rainbow.

Lemma 3.2.3. *Let c be an r-coloring of $\mathbb{Z}_n$. If m is relatively prime to n, then c has a rainbow Schur triple if and only if $\hat{c}(x) := c(mx)$ contains a rainbow Schur triple. Additionally, the cardinality of each color class will be maintained.*

Proof. Let (x_1, x_2, x_3) be a triple in c. By definition, $x_1 + x_2 = x_3$ in $\mathbb{Z}_n$ is equivalent to

$$x_1 + x_2 = sn + r$$
$$x_3 = tn + r,$$

as equations in the integers for some $s, t \in \mathbb{Z}$. Multiply both equations by m to get

$$mx_1 + mx_2 = msn + mr$$
$$mx_3 = mtn + mr$$

Therefore, $mx_1 + mx_2 \equiv mr \mod n$, and $mx_3 \equiv mr \mod n$, so $mx_1 + mx_2 \equiv mx_3 \mod n$. Thus, (mx_1, mx_2, mx_3) is rainbow in $\hat{c}$ if and only if (x_1, x_2, x_3) is rainbow in c.

Finally, the last statement of Lemma 3.2.3 follows from the fact that if m is relatively prime to n, then the map $F : x \mapsto mx$ is a bijection. □

Our next result is the Schur equation counterpart to Theorem 3.2 in [3].

Proposition 3.2.4. *Let p be prime. Then every 3-coloring c of $\mathbb{Z}_p$ with $\min(|R|, |G|, |B|) > 1$ contains a rainbow Schur triple.*

Proof. For the sake of contradiction, assume that c is a rainbow-free 3-coloring of $\mathbb{Z}_p$ and $\min(|R|, |G|, |B|) > 1$. Without loss of generality, assume that $|R| = \min(|R|, |G|, |B|)$. Since there are at least two elements of $\mathbb{Z}_p$ colored R, there exists a minimal element $1 \leq i \leq p-1$ such that $c(i) = R$. Because p is prime, i is relatively prime to p and i has a multiplicative inverse. Let $\hat{c}(x) := c(ix)$ so that $\hat{c}(1) = R$. Therefore, by Lemma 3.2.1, R is dominant in $\hat{c}$. By Lemma 3.2.2, BB and GG cannot both appear in $\hat{c}$. Without loss of generality, assume that GG does not appear in $\hat{c}$. Because R is dominant, R must follow each G, so $|R| \geq |G|$. Furthermore, BR must appear in $\hat{c}$. This implies that $|R| \geq |G| + 1$ in $\hat{c}$ which implies $|R| \geq |G| + 1$ in c by Lemma 3.2.3. This contradicts our assumption that $|R| = \min(|R|, |G|, |B|)$. □

Lemma 3.2.5. *If c is a rainbow-free r-coloring of $\mathbb{Z}_p$ for a prime p with $r > 2$, then $c(x) = c(-x)$.*

Proof. Let c be a rainbow-free r-coloring of $\mathbb{Z}_p$. For the sake of contradiction, assume that there exists $i, -i$ with $c(i) \neq c(-i)$. Without loss of generality, let $c(i) = R$ and $c(-i) = G$. Now, let $\hat{c}(x) := c(ix)$ and let $\bar{c}(x) := c(-ix)$. By Lemma 3.2.3, $\hat{c}$ and $\bar{c}$ are both rainbow-free. Since $\hat{c}(1) = c(i) = R$ and $\bar{c}(1) = c(-i) = G$, R is dominant in $\hat{c}$, and G is dominant in $\bar{c}$. Notice that $\hat{c}(x) = \bar{c}(-x)$, so if two colors are adjacent at some position in $\hat{c}$, then they are also adjacent at some position in $\bar{c}$. Thus, since G is dominant in $\bar{c}$, G must also appear in every bichromatic sequence in $\hat{c}$, and, consequently, G is also dominant in $\hat{c}$. If both R and G are dominant in $\hat{c}$, then $\hat{c}$ must only contain R and G, and $r = 2$; this is a contradiction. □

Note that this lemma shows that the coloring from 1 to $p-1$ must be symmetric in a rainbow-free coloring of $\mathbb{Z}_p$.

Remark 3.2.6. *For any prime $p \geq 5$, $\mathbb{Z}_p$ can be colored with three colors by coloring zero uniquely and coloring 1 to $p-1$ with two colors in any way such that $c(x) = c(-x)$ for all x. This* *coloring is rainbow-free since any three group elements which witness three colors must contain* 0*, and in order to make a Schur triple of three distinct elements where one of the elements is* 0 *the other two elements must be* x *and* $-x$ *for some* x *(see also Corollary 2 in [4]).*

Now we have enough information about the structure of rainbow-free colorings to prove Theorem 3.1.1. A color class C is *singleton* if $|C| = 1$.

Proof of Theorem 3.1.1. For the sake of contradiction, suppose that $r + 1 = rb(\mathbb{Z}_p, 1) > 4$ for a prime $p \geq 5$, and let c be a rainbow-free r-coloring of $\mathbb{Z}_p$ with $r > 3$. Note that since c is rainbow-free, at least one of the color classes in c must contain more than one element. Partition the color classes of c into three sets to define $\hat{c}$, an exact 3-coloring of $\mathbb{Z}_p$. We use the union of the color classes within each part of the partition as the color classes for $\hat{c}$. Since we are concatenating colors, $\hat{c}$ is also rainbow-free. By Proposition 3.2.4, regardless of how the color classes of c are partitioned, there exists some color class in $\hat{c}$ with exactly one element. If $r \geq 5$, then there exists a partition of the five or more color classes such that each color class has more than one element. Therefore, $r = 4$.

Furthermore, if two or more color classes are not singleton, then there would exist a partition of the color classes that yields no singleton color classes in $\hat{c}$. Therefore, all but one of the four color classes in c must be singleton.

If there are three singleton color classes in c, then there exists an $x \neq 0$ such that $c(x) \neq c(-x)$. This contradicts Lemma 3.2.5, and c cannot be rainbow-free.

Thus, there does not exist an exact rainbow-free r-coloring of $\mathbb{Z}_p$ for $r > 3$ and $p \geq 5$. □

3.2.2 Lower bound

In order to prove the lower bound for $rb(\mathbb{Z}_n, 1)$, we examine the relationship between Schur triples in $\mathbb{Z}_n$ and $\mathbb{Z}_{\frac{n}{m}}$ where m divides n.

Lemma 3.2.7. *If there exists a Schur triple of form* (x_1, x_2, x_3) *in* $\mathbb{Z}_n$ *where* $m|x_1, x_2, x_3$ *for some* $m|n$*,* $m, n \in \mathbb{Z}$*, then there exists a Schur triple of the form* $(x_1/m, x_2/m, x_3/m)$ *in* $\mathbb{Z}_{\frac{n}{m}}$*.*

Proof. By definition, $x_1 + x_2 = x_3$ in $\mathbb{Z}_n$ implies that in the integers

$$x_1 + x_2 = qn + r$$
$$x_3 = tn + r,$$

for some $q, t \in \mathbb{Z}$. Divide both equations by m to get

$$\frac{x_1}{m} + \frac{x_2}{m} = q\frac{n}{m} + \frac{r}{m}$$
$$\frac{x_3}{m} = t\frac{n}{m} + \frac{r}{m}.$$

Now we must check that $\frac{r}{m}$ is an integer. Since $m|(x_1 + x_2 - qn)$, we know $m|r$.

By definition, this means that there exists a Schur triple of the form $(x_1/m, x_2/m, x_3/m)$ in $\mathbb{Z}_{\frac{n}{m}}$. □

This shows that Schur triples can be "projected" from the cyclic group $\mathbb{Z}_n$ to a subgroup $\mathbb{Z}_{\frac{n}{m}}$. Next, we will show another property of Schur triples related to the divisibility of a triple's elements by a prime.

Lemma 3.2.8. *For a positive integer n and a prime p, if $x_1 + x_2 \equiv x_3 \mod np$, then p cannot divide exactly two of (x_1, x_2, x_3).*

Proof. If $x_1 + x_2 \equiv x_3 \mod np$, then there exist integers c_1, c_2, and r_0 such that $x_1 + x_2 = c_1np + r_0$ and $x_3 = c_2np + r_0$.

Assume that p divides x_1 and x_2. Then there exist integers c_3 and c_4 such that $x_1 = c_3p$ and $x_2 = c_4p$. We know there exist integers c_5 and r_1 with $0 \leq r_1 < p$ such that $x_3 = c_5p + r_1$, so we want to show $r_1 = 0$. Immediately, we see that $c_3p + c_4p = c_1np + r_0$ and $c_5p + r_1 = c_2np + r_0$, which, after substituting for r_0, shows us $c_3p + c_4p = c_1np + c_5p + r_1 - c_2np$. Solving for r_1 gives us

$$r_1 = c_3p + c_4p - c_1np - c_5p + c_2np$$
$$= p(c_3 + c_4 - c_1n - c_5 + c_2n)$$

This means that p divides r_1, forcing $r_1 = 0$. Thus, p divides x_3.

Now assume p divides x_1 and x_3, i.e. there exist integers c_6 and c_7 such that $x_1 = c_6p$ and $x_3 = c_7p$. We know there exist integers c_8 and r_2 with $0 \le r_2 < p$ such that $x_2 = c_8p + r_2$, so we want to show $r_2 = 0$. Immediately, we see that $c_6p + c_8p + r_2 = c_1np + r_0$ and $c_7p = c_2np + r_0$, which, after substituting for r_0, shows us $c_6p+c_8p+r_2 = c_1np+c_7p-c_2np$. Solving for r_2 gives us

$$\begin{aligned} r_2 &= c_1np + c_7p - c_2np - c_6p - c_8p \\ &= p(c_1n + c_7 - c_2n - c_6 - c_8). \end{aligned}$$

This means that p divides r_2, forcing $r_2 = 0$. Thus, p divides x_2. By symmetry, this case is identical to the case where p divides x_2 and x_3.

Therefore, we can see that if p divides two elements in (x_1, x_2, x_3), then p must also divide the third. □

Lemma 3.2.9. *Let p, t be positive integers with p prime. If there exists a rainbow-free r-coloring of $\mathbb{Z}_t$, then there exists a rainbow-free $r + rb(\mathbb{Z}_p, 1) - 2$-coloring of $\mathbb{Z}_{pt}$.*

Proof. Let t, p be positive integers such that p is a prime. Assume $\hat{c}$ is a rainbow-free r-coloring of $\mathbb{Z}_t$. Then let c be an exact $(r + rb(\mathbb{Z}_p, 1) - 2)$-coloring (if $p = 2$ or $p = 3$, then c is an exact $(r + 1)$-coloring. Otherwise, c is an exact $r + 2$ coloring) of $\mathbb{Z}_{pt}$ as follows:

$$c(x) := \begin{cases} \hat{c}(x/p) & x \equiv 0 \mod p \\ r+1 & x \equiv 1 \text{ or } p-1 \mod p \cdot \\ r+2 & \text{otherwise} \end{cases}$$

Notice that if (x_1, x_2, x_3) is a Schur triple in $\mathbb{Z}_{pt}$, then there are three cases by Lemma 3.2.8: p divides exactly one of (x_1, x_2, x_3), p divides each of (x_1, x_2, x_3), or p divides none of (x_1, x_2, x_3).

Case 1: The two terms x_i, x_j where $i, j \in \{1, 2, 3\}$ that are not divisible by p are either additive inverses modulo p or are equal modulo p. Thus, $c(x_i) = c(x_j)$ and (x_1, x_2, x_3) does not form a triple.

Case 2: The coloring of each x_i is inherited from $\hat{c}$. Since $\hat{c}$ does not admit rainbow triples, we know that this triple will not be rainbow by Lemma 3.2.7.

Case 3: The three integers in the triple will be colored from $\{r+1, r+2\}$, so the triple will not be rainbow. In each case, c is a rainbow-free $r + rb(\mathbb{Z}_p, 1) - 2$-coloring of $\mathbb{Z}_{pt}$. □

Proposition 3.2.10. *For any positive integer $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$,*

$$rb(\mathbb{Z}_n, 1) \geq 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

Proof. If n is prime, there is nothing to show. Suppose that the claim holds true for n where n has N prime factors.

Assume that $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ where $\alpha_1 + \cdots + \alpha_m = N + 1$. By the induction hypothesis, there exists a rainbow-free r-coloring of $\mathbb{Z}_{n/p_1}$ where

$$r = 1 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big) - rb(\mathbb{Z}_{p_1}, 1) + 2.$$

Therefore, by Lemma 3.2.9, there exists a rainbow-free $r + rb(\mathbb{Z}_{p_1}, 1) - 2$ coloring of $\mathbb{Z}_n$. Thus, by induction

$$rb(\mathbb{Z}_n, 1) \geq 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

□

3.2.3 Upper bound

To establish the upper bound for $rb(\mathbb{Z}_n, 1)$, we consider residue classes and their corresponding residue palettes under c. Lemma 3.2.11 lets us create a well-defined reduction of a coloring of $\mathbb{Z}_{st}$ to a coloring of $\mathbb{Z}_t$. We use the coloring described in Lemma 3.2.12 to prove an upper bound for $rb(\mathbb{Z}_{st}, 1)$.

Lemma 3.2.11. *Let $R_0, R_1, \ldots, R_{t-1}$ be the residue classes modulo t for $\mathbb{Z}_{st}$, and let $P_0, P_1, \cdots, P_{t-1}$ be the corresponding residue palettes under rainbow-free c. Then $|P_i \setminus P_0| \leq 1$ for $1 \leq i \leq t-1$.*

Proof. Assume that $|P_i \setminus P_0| \geq 2$. Then R_i must contain at least two elements which receive colors that do not appear in P_0. Without loss of generality, let G and B denote two colors in

$P_i \setminus P_0$. Then there exists two integers m and n such that $c(mt+i) = G$ and $c(nt+i) = B$. Consider the Schur triple $(mt-nt, nt+i, mt+i)$. Notice that $mt - nt \equiv 0 \mod t$, $c(mt-nt) \neq G, B$. Thus, we have a rainbow triple under c in $\mathbb{Z}_{st}$, which is a contradiction. Therefore, $|P_i \setminus P_0| \leq 1$ for $1 \leq i \leq t-1$. □

Lemma 3.2.12. *Let s and t be positive integers. Let $R_0, R_1, \ldots, R_{t-1}$ be the residue classes modulo t for $\mathbb{Z}_{st}$ with corresponding residue palettes P_i. Suppose c is a coloring of $\mathbb{Z}_{st}$ where $|P_i \setminus P_0| \leq 1$. Let $\hat{c}$ be a coloring of $\mathbb{Z}_t$ given by*

$$\hat{c}(i) := \begin{cases} P_i \setminus P_0 & \text{if } |P_i \setminus P_0| = 1 \\ \alpha & \text{otherwise} \end{cases}$$

where $\alpha \notin P_i$ for $0 \leq i \leq t$. If $\hat{c}$ contains a rainbow Schur triple, then c contains a rainbow Schur triple.

Proof. Suppose (x_1, x_2, x_3) is a rainbow Schur triple in $\hat{c}$. Then, at least two of x_1, x_2, x_3 must receive a color other than α. We consider the following two cases.

Case 1: Neither x_1 nor x_2 receive color α.

Without loss of generality, assume that $c(x_1) = G$ and $C(x_2) = B$. This implies that there exist n, m such that $c(nt + x_1) = G$ and $c(mt + x_2) = B$. There is a Schur triple of the form $(nt + x_1, mt + x_2, (n+m)t + (x_1 + x_2))$ in $\mathbb{Z}_{st}$. Since $x_1 + x_2 \equiv x_3 \mod t$, $(n+m)t + (x_1 + x_2)$ is in the residue class R_{x_3}. As $\hat{c}(x_3) \neq G, B$, we have $G, B \notin P_{x_3}$. Therefore, the triple $(nt + x_1, mt + x_2, (n+m)t + (x_1 + x_2))$ is rainbow.

Case 2: One of x_1 or x_2 is colored α.

Without loss of generality, assume that $c(x_1) = \alpha$, $c(x_2) = B$, and $c(x_3) = G$. Then $c(nt + x_2) = B$ for some n, and $c(mt + x_3) = G$ for some m. There is a Schur triple of the form $((m-n)t + (x_3 - x_2), nt + x_2, mt + x_3)$ in $\mathbb{Z}_{st}$. Since $x_1 + x_2 \equiv x_3 \mod t$, $(m-n)t + (x_3 - x_2)$ is in the residue class R_{x_1}. As $\hat{c}(x_1) = \alpha$, we have $G, B \notin P_{x_1}$. Therefore, the triple $((m-n)t + (x_3 - x_2), nt + x_2, mt + x_3)$ is rainbow.

Hence, if $\hat{c}$ has a rainbow Schur triple, then c has a rainbow Schur triple. □

Proposition 3.2.13. *Let s and t be positive integers. Then $rb(\mathbb{Z}_{st}, 1) \leq rb(\mathbb{Z}_s, 1) + rb(\mathbb{Z}_t, 1) - 2$.*

Proof. Let c be an exact r-coloring of $\mathbb{Z}_{st}$, and let $\hat{c}$ be a coloring constructed from c as in Lemma 3.2.12. Notice that the set of colors used in c is comprised of the colors in R_0 and each color used in $\hat{c}$ other than α. Thus, $r = |P_0| + |\hat{c}| - 1$, where $|\hat{c}|$ is the number of colors appearing in $\hat{c}$. If c is a rainbow-free coloring of $\mathbb{Z}_{st}$, then R_0 is a rainbow-free coloring of $\mathbb{Z}_s$. Thus, $|P_0| \leq rb(\mathbb{Z}_s, 1) - 1$. Also, $\hat{c}$ is a rainbow-free coloring of $\mathbb{Z}_t$, so $|\hat{c}| \leq rb(\mathbb{Z}_t, 1) - 1$. Thus, $r \leq rb(\mathbb{Z}_s, 1) + rb(\mathbb{Z}_t, 1) - 3$. If we let c be the maximum rainbow-free coloring of $\mathbb{Z}_{st}$, then $r = rb(\mathbb{Z}_{st}, 1) - 1$. This shows that $rb(\mathbb{Z}_{st}, 1) \leq rb(\mathbb{Z}_s, 1) + rb(\mathbb{Z}_t, 1) - 2$. □

Using both the upper bound we just established and the lower bound established in Proposition 3.2.10 of Section 3.2.2, we prove Theorem 3.1.3.

Proof of Theorem 3.1.3. Recursively applying Proposition 3.2.13 to prime factors of n yields

$$rb(\mathbb{Z}_n, 1) \leq 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

Since this is identical to the lower bound from Proposition 3.2.10 in Section 3.2.2, we can conclude

$$rb(\mathbb{Z}_n, 1) = 2 + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{p_i}, 1) - 2) \Big).$$

□

3.3 Triples for $x_1 + x_2 = px_3$, p prime

Section 3.3 is dedicated to proving Theorem 3.1.6. In Section 3.3.1, we establish exact values for $rb(\mathbb{Z}_q, p)$ where $p \neq q$ are prime. Finding an exact value for $rb(\mathbb{Z}_p, p)$ is more difficult, and is the subject of Section 3.3.2. Some properties of rainbow-free colorings of $\mathbb{Z}_q$ are used in the construction of the general lower bound in Section 3.3.3. The equivalent upper bound is proved in Section 3.3.4. Combining Sections 3.3.3 and 3.3.4 proves Theorem 3.1.6.

3.3.1 Exact values for $rb(\mathbb{Z}_q, p)$, $p \neq q$ prime

Lemmas 3.3.3, 3.3.4, 3.3.5, 3.3.6 establish the upper bound $rb(\mathbb{Z}_q, p) \leq 4$. These lemmas are proven by assuming that there exists a rainbow-free r-coloring c with $r \geq 4$, and reducing c to a 3-coloring $\hat{c}$. In each case, we find that $\hat{c}$ does not conform to the structure of a rainbow-free 3-coloring outlined in Theorem 3.3.1 proven in [4]. For convenience, we include Theorem 3.3.1 and the necessary definitions from [4].

For a subset $X \subseteq \mathbb{Z}_q^*$ and $a \in \mathbb{Z}_q^*$ define $aX := \{ax \mid x \in X\}$, $X + a := \{x + a \mid x \in X\}$, and $X - a := X + (-a)$. We say the set aX is the *dilation* of X by a. Let $\langle x \rangle \leq \mathbb{Z}_q^*$ denote the subgroup multiplicatively generated by x. A subset $X \in \mathbb{Z}_q^*$ is H*-periodic* if X is the union of cosets of H, where $H \leq \mathbb{Z}_p^*$. In the case that X is $\langle -1 \rangle$-periodic, we say that X is *symmetric*. This coincides with the notion that X is symmetric if and only if $X = -X$.

Theorem 3.3.1. *[[4], Theorem 2] A 3-coloring $\mathbb{Z}_q = A \cup B \cup C$ with $1 \leq |A| \leq |B| \leq |C|$ is rainbow-free for $x_1 + x_2 = kx_3$ if and only if, up to dilation, one of the following holds.*

1. *$A = \{0\}$ and both B and C are symmetric and $\langle k \rangle$-periodic subsets.*

2. *$A = \{1\}$ for*

   (i) $k = 2 \mod q$, $(B - 1)$ and $(C - 1)$ are symmetric and $\langle 2 \rangle$-periodic subsets.

   (ii) $k = -1 \mod q$, $(B \setminus \{2\}) + 2^{-1}$, $(C \setminus \{2\}) + 2^{-1}$ are symmetric subsets.

3. *$|A| \geq 2$, for $k = -1 \mod q$ and $A, B,$ and C are arithmetic progressions with difference 1 such that $A = [a_1, a_2 - 1]$, $B = [a_2, a_3 - 1]$, and $C = [a_3, a_1 - 1]$, with $(a_1 + a_2 + a_3) = 1$ or 2.*

Suppose that $q \geq 5$ is prime. Let c be a coloring of $\mathbb{Z}_q$ with color classes $C_1, \ldots, C_r$ with $1 \leq |C_1| \leq |C_2| \leq \cdots \leq |C_r|$ and $r \geq 4$. Theorem 3.3.1 tells us that rainbow-free colorings have very particular structure, up to the rearrangement of the elements of $\mathbb{Z}_q$. The overarching goal of Lemmas 3.3.3, 3.3.4, 3.3.5, and 3.3.6, is to show that if a coloring has too many colors, then some color classes can be combined to violate the structure of a rainbow-free coloring.

Observation 3.3.2. *If $C_1 = \{0\}$ and $C_2 = \{x\}$, then $(x, -x, 0)$ is a rainbow triple for $x \neq 0$.*

Therefore, if c has two or more singleton color classes, we can assume that $\{0\}$ is not a color class. Furthermore, since dilation preserves the rainbow-free property, we can assume that if $|C_2| = 1$, then $C_1 = \{1\}$.

Lemma 3.3.3. *If $p \not\equiv -1 \mod q$ and $|C_2| = 1$, then c admits a rainbow triple.*

Proof. Consider the coloring $\hat{c}$ given by the color classes $C_1, C_2, \bigcup_{i=3}^{r} C_i$. If $\hat{c}$ admits a rainbow triple, then c also admits a rainbow triple and we are done. If $\hat{c}$ does not admit a rainbow triple, then $\hat{c}$ must conform to case 2.(i) in Theorem 3.3.1. Therefore, $p \equiv 2 \mod q$. In this case, triples satisfying $x_1 + x_2 = kx_3$ in $\mathbb{Z}_q$ are 3-term arithmetic progressions. In [2], Proposition 3.5 establishes that $rb(\mathbb{Z}_q, 2) \leq 4$. Therefore, there exists a rainbow triple under c. □

Lemma 3.3.4. *If $p \equiv -1 \mod q$ and $|C_3| = 1$, then c admits a rainbow triple.*

Proof. Let $C_2 = \{x\}, C_3 = \{y\}$. For the sake of contradiction, assume that c is rainbow free.

If $x = 2$, then $(x, -3, 1)$ is a rainbow triple. The same argument for y shows that $x, y \neq 2$.

Consider the coloring $\hat{c}$ given by the color classes $C_1, C_2, \bigcup_{i=3}^{r} C_i$. Then by Theorem 3.3.1 we must have $C_2 \setminus \{2\} + 2^{-1}$ is symmetric and so $x + 2^{-1} = -2^{-1} - x$. Solving for x gives that $x = -2^{-1}$. Considering the coloring given by $C_1, C_3, C_2 \cup \bigcup_{i=4}^{r} C_i$ gives that $y = -2^{-1}$, which is a contradiction. □

Lemma 3.3.5. *If $p \not\equiv -1 \mod q$, and $|C_2| \geq 2$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. Consider the coloring $\hat{c}$ given by $C_1 \cup C_2, C_3, \bigcup_{i=4}^{r} C_i$. Since $|C_3| \geq |C_2| \geq 2$, notice that $\hat{c}$ does not have a singleton color class and is rainbow-free. This contradicts Theorem 3.3.1. □

Lemma 3.3.6. *If $p \equiv -1 \mod q$ and $|C_3| \geq 2$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. There are two cases: $|C_2| \geq 2$, or $|C_2| = 1$.

Case 1: Assume that $|C_2| \geq 2$ and $C_1 = \{x\}$. By Theorem 3.3.1, the coloring $C_1 \cup C_2, C_3, \bigcup_{i=4}^{r} C_i$ is of the form

$$C_1 \cup C_2 = [a_1, a_2 - 1],$$

$$C_3 = [a_2, a_3 - 1],$$
$$\bigcup_{i=4}^{r} C_i = [a_3, a_1 - 1].$$

Notice that x is not adjacent to at least one of C_3 or $\bigcup_{i=4}^{r} C_i$. Without loss of generality, assume x is not adjacent to C_3 (the other case follows the same argument). Consider the coloring $\hat{c}$ given by $C_2, C_1 \cup C_3, \bigcup_{i=4}^{r} C_i$. Notice that $\hat{c}$ can only be dilated by 1 or -1 to preserve the interval structure of $\bigcup_{i=4}^{r} C_i$. However, dilating by 1 or -1 will not make $C_1 \cup C_3$ an arithmetic progression with difference 1. This is a contradiction.

Case 2: Assume that $|C_2| = 1$. Consider the coloring $\hat{c}$ given by

$$C_1 \cup C_2, C_3, \bigcup_{i=4}^{r} C_i.$$

By Theorem 3.3.1, $\hat{c}$ is of the form

$$C_1 \cup C_2 = [a_1, a_2 - 1],$$

$$C_3 = [a_2, a_3 - 1],$$
$$\bigcup_{i=4}^{r} C_i = [a_3, a_1 - 1]$$

with $a_1 + a_2 + a_3 \in \{1, 2\}$. Since every set is an arithmetic progression with difference 1, we have $a_2 - 1 = a_1 + 1$. This implies that $a_3 \in \{-2a_1 - 1, -2a_1\}$. This implies that $c(-2a_1 - 1) \neq c(a_1), c(a_1 + 1)$. Therefore, triple $(-2a_1 - 1, a_1, a_1 + 1)$ is rainbow, which is a contradiction. □

Lemmas 3.3.3, 3.3.4, 3.3.5, and 3.3.6 form a case analysis and road map for finding rainbow solutions. In particular, these lemmas translate the structural properties of rainbow-free 3-colorings, to an upper bound on the rainbow number. The rest of the work in the proof of

Theorem 3.1.4 determines the relationship between p and q that makes rainbow-free 3-colorings possible.

Proof of Theorem 3.1.4. By Lemmas 3.3.3, 3.3.4, 3.3.5, and 3.3.6, we know that $rb(\mathbb{Z}_q, p) \leq 4$. Therefore, it suffices to show that there exists a rainbow-free 3-coloring of $\mathbb{Z}_q$ if and only if p, q do not satisfy either condition 1 or 2. First we will prove that if there exists a rainbow-free 3-coloring, then p, q do not satisfy conditions 1 and 2.

Let c be a rainbow-free 3-coloring. There are two cases, $p \not\equiv -1 \mod q$ or $p \equiv -1 \mod q$.

Case 1: By Theorem 3.3.1, either 0 is uniquely colored, or $p \equiv 2 \mod q$.

Suppose 0 is uniquely colored and $c(1) = R$. Notice that if $c(x) = R$, then $c(px) = R$ and $c(-x) = R$. If p, q satisfy either 1 or 2, then $\{p^i, -p^i \mid i \in \mathbb{Z}\} = \mathbb{Z}_q^*$, which contradicts the fact that c is a 3-coloring.

Suppose $p \equiv 2 \mod q$. Then neither 1 nor 2 are satisfied by Theorem 3.5 in [3].

Case 2: Suppose $p \equiv -1 \mod q$. Then $|p| = 2$. If $(q-1)/2$ is odd, then $(q-1)/2 \neq 2$. Therefore, neither 1 nor 2 are satisfied.

To prove the reverse direction, suppose that p, q do not satisfy either 1 or 2. Let c be given by

$$C_1 = \{0\}, C_2 = \{p^i, -p^i \mid i \in \mathbb{Z}\}, C_3 = \mathbb{Z}_q^* \setminus C_2.$$

Since p, q do not satisfy either 1 or 2, C_3 is non-empty. Notice that any rainbow triple must contain 0 and some element $y \in C_2$. However, if $0, y, z$ is a triple, then $z \in C_2$. Therefore, c is rainbow-free. □

The following corollary is used in Section 3.3.3 to prove a general lower bound for $rb(\mathbb{Z}_n, p)$.

Corollary 3.3.7. *There exists a maximum rainbow-free coloring of $\mathbb{Z}_q$ where 0 is uniquely colored and the color classes are symmetric.*

3.3.2 Exact values for $rb(\mathbb{Z}_{p^\alpha}, p)$, p prime

In order to determine the rainbow numbers for equations of the form $x_1 + x_2 = px_3$ for prime $p \geq 3$, we still need to determine $rb(\mathbb{Z}_{p^\alpha}, p)$ for $\alpha \geq 1$. We will prove Theorem 3.1.5 using induction. Observation 3.3.8 and Propositions 3.3.9, 3.3.10, and 3.3.11 provide the lower bound and base case for our induction argument. Lemmas 3.3.12 and 3.3.13 provide the basic structure of a rainbow-free coloring of $\mathbb{Z}_{p^\alpha}$. Lastly, Lemmas 3.3.14, and 3.3.15 exploit the structure to derive a contradiction by forcing a rainbow triple. Throughout this section, for $0 \leq k \leq p-1$, recall that the k^{th} residue class mod p is the set $R_k = \{j \in \mathbb{Z}_{p^\alpha} : j \equiv k \mod p\}$ and that the k^{th} residue palette P_k is the set of colors which appear on R_k.

Observation 3.3.8. *Notice* $rb(\mathbb{Z}_3, 3) = 3$ *and* $rb(\mathbb{Z}_9, 3) = 4$.

Proposition 3.3.9. *Let* $p \geq 3$ *be prime. Then* $rb(\mathbb{Z}_p, p) = \frac{p+1}{2} + 1$.

Proof. To prove the lower bound, consider the following coloring:

$$c(x) = \begin{cases} x & 0 \leq x \leq \frac{p+1}{2} \\ -x & \text{otherwise} \end{cases}.$$

Notice that $c(x) = c(-x)$ for all $x \in \mathbb{Z}_p$. Furthermore, if (x_1, x_2, x_3) is a triple, then $x_1 = -x_2$. Thus, c is a rainbow-free $\frac{p+1}{2}$ coloring, and $rb(\mathbb{Z}_p, p) > \frac{p+1}{2}$.

To prove the upper bound, assume that c is an $\frac{p+1}{2} + 1$ coloring of $\mathbb{Z}_p$. By the pigeonhole principle, there exists $x \in \mathbb{Z}_p$ such that $x \neq 0$ and $c(x) \neq c(-x)$. Since $p \geq 3$, $x \neq -x$, and there exist $y \neq x, -x$ such that $c(y) \neq c(x), c(-x)$. Therefore, $(x, -x, y)$ is a rainbow-triple, and $rb(\mathbb{Z}_p, p) \leq \frac{p+1}{2} + 1$. □

For the rest of the section, we will assume that $\alpha \geq 2$.

Proposition 3.3.10. *For* $\alpha \geq 2$,

$$rb(\mathbb{Z}_{3^\alpha}, 3) > 3.$$

Proof. Suppose that $\alpha \geq 3$ and $\bar{c}$ is a rainbow-free 3-coloring of $\mathbb{Z}_9$. Let c be a 3-coloring of $\mathbb{Z}_{p^\alpha}$ given by $c(i) := \bar{c}(i \mod 9)$. Assume that x_1, x_2, x_3 is a triple in $\mathbb{Z}_{3^\alpha}$. Then x_1, x_2, x_3 is a triple in $\mathbb{Z}_9$ and cannot be rainbow. □

Proposition 3.3.11. *For prime $p \geq 5$ and $\alpha \geq 1$,*

$$rb(\mathbb{Z}_{p^\alpha}, p) \geq \frac{p+1}{2} + 1.$$

Proof. Color all of R_i, R_{p-i} color i for $0 \leq i \leq \frac{p+1}{2}$. Suppose $x_1 + x_2 = px_3$ and $x_1 \equiv j \mod p$ for $0 \leq j \leq p-1$. Then $x_2 \equiv p-j \mod p$, and x_1, x_2, x_2 is not rainbow. ☐

Lemma 3.3.12. *If c does not admit a rainbow triple, then*

$$P_i = P_{p-i}$$

when $0 < i < p$.

Proof. For the sake of contradiction, suppose that there exists $0 < i < p$ with $G \in P_i \setminus P_{p-i}$. Then there exists an element $px + i$ with color G in R_i. Let $py + p - i$ be an element in R_{p-i}. Notice that

$$x_1 = p(py - x + p - 1 - i) + p - i$$

$$x_2 = px + i$$

$$x_3 = py + p - i$$

is a triple. Since $G \notin P_{p-i}$, we have $c(x_3) = c(x_1)$. Furthermore, $x_1 - x_3 = p(py - x + p - 1 - i) + p - i - py - p + i = p(y(p-1) - x + p - 1)$. Since $py + p - i$ was arbitrary, we can choose y so that $y(p-1) - x + p - 1 \not\equiv 0 \mod p$. Since $y(p-1) - x + p - 1 \not\equiv 0 \mod p$, we know that $y(p-1) - x + p - 1$ is an additive generator of $\mathbb{Z}_{p^{\alpha-1}}$. This implies that $P_{p-i} = \{B\}$.

Let $pz + j$ be an element with $c(pz + j) \notin \{G, B\}$. Then

$$x_1 = p(pz - x + j - 1) + p - i$$

$$x_2 = px + i$$

$$x_3 = pz + j$$

is a rainbow triple, which is a contradiction. ☐

Notice that by Lemma 3.3.12, it is sufficient to only consider the structure of R_i for $0 < i < \frac{p+1}{2}$.

Lemma 3.3.13. *Suppose c does not admit a rainbow triple. If there exists $0 < i < p$ such that $|P_i \setminus P_0| \geq 1$, then $|P_0| = 1$.*

Proof. Since c does not admit a rainbow triple, $P_i = P_{p-i}$. Without loss of generality, suppose that $G \in P_i \setminus P_0$ and let $c(pa_1 + i) = c(pa_2 + p - i) = G$. Let $pb \in R_0$ be arbitrary. Consider the following triple:

$$x_1 = pb$$
$$x_2 = p(pa_1 + i - b)$$
$$x_3 = pa_1 + i.$$

Since c is rainbow-free, $c(x_1) = c(x_2)$. Next, consider the following triple:

$$x_1' = p(pa_1 + i - b)$$
$$x_2' = p(pa_2 + p - i - pa_1 - i + b)$$
$$x_3' = pa_2 + p - i.$$

Since c is rainbow-free, $c(x_1') = c(x_2')$. This implies that

$$c(pb) = c(p(pa_2 + p - i - pa_1 - i + b)).$$

Notice that difference in position between x_2' and pb, given by $pa_2 + p - i - pa_1 - i + b - b$, does not depend on b. Furthermore, $pa_2 + p - i - pa_1 - i + b - b$ is relatively prime to $p^{\alpha-1}$. Therefore, all elements in R_0 receive the same color. □

Lemma 3.3.14. *Let p be prime with $p \geq 5$. If there exists $0 < i < \frac{p+1}{2}$ such that $|P_i \setminus P_0| \geq 2$ and $G \notin P_i \cup P_0$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. Since $p \geq 5$ and $|P_0| = 1$, there exists $j \neq i$ such that $0 < j < p$ and $G \in P_j \setminus (P_i \cup P_0)$. By Lemma 3.3.12,

$P_j = P_{p-j}$ and $P_i = P_{p-i}$. Let $c(pa_1 + j) = c(pa_2 + p - j) = G$. Let $pb + i \in R_i$ be arbitrary. Consider the following triple:

$$x_1 = pb + i$$
$$x_2 = p(pa_1 + j - b - 1) + p - i$$
$$x_3 = pa_1 + j.$$

Then $c(x_1) = c(x_2)$. Next consider the following triple:

$$x_1' = p(pa_1 + j - b - 1) + p - i$$
$$x_2' = p(pa_2 + p - j - pa_1 - j + b) + i$$
$$x_3' = pa_2 + p - j$$

Then $c(x_1') = c(x_2')$. This implies that

$$c(pb + i) = c(p(pa_2 + p - j - pa_1 - j + b) + i).$$

Notice that the difference in position between x_2' and $pb+i$, given by $pa_1 + p - j - pa_1 - j + b - b$, does not depend on b. Furthermore, $pa_2 + p - j - pa_1 - j + b - b$ is relatively prime to $p^{\alpha-1}$. Therefore, all elements in R_i receive the same color. This is a contradiction, since $|P_i| \geq 2$. □

Lemma 3.3.15. *If $p \geq 5$, $\mathbb{Z}_{p^\alpha}$ is colored with at least 4 colors, and there exists $0 < i < \frac{p+1}{2}$ with $im(c) = P_i \cup P_0$ and $|P_i \setminus P_0| \geq 2$, then c admits a rainbow triple.*

Proof. For the sake of contradiction, suppose that c does not admit a rainbow triple. By Lemma 3.3.13, let $P_0 = \{R\}$. By Lemma 3.3.12, $P_i = P_{p-i}$. Since P_i contains all colors except possibly R, there exists a, b, d such that $c(pa + i) = G$, $c(pb + p - i) = B$ and $c(pd + i) = B$. Consider the following triple:

$$x_1 = pa + i$$
$$x_2 = p(pb + p - i - a - 1) + p - i$$
$$x_3 = pb + p - i.$$

Then $c(x_2) \in \{B, G\}$. Let $x \in \{a, d\}$ such that $c(px+i) \neq c(x_2)$ and consider the following triple:

$$x_1' = p(pb - p - i - a - 1) + p - i$$

$$x_2' = p(px - pb + p + 2i + a) + i$$

$$x_3' = px + i.$$

Notice that $c(x_2') \in \{B, G\}$. Furthermore, the difference in position between x_2' and $pa + i$, given by $px - pb + p + 2i \equiv 2i \mod p$, does not depend on a, b, d modulo p. Therefore, for any $x \in \mathbb{Z}_p$ there exists $a \equiv x$ such that $c(pa + i) \in \{B, G\}$.

Since P_{p-i} contains all colors of c except for possibly R, there exists y such that $c(py + p - i) = Y$. Select $a \equiv -1 - y \mod p$ such that $c(pa + i) \in \{B, G\}$. Then the triple $(py + p - i, pa + i, a + y + 1)$ is rainbow since $a + y + 1 \in R_0$. □

Proof of Theorem 3.1.5. Proposition 3.3.10 provides the lower bound for $p = 3$, $\alpha \geq 2$. Observation 3.3.8 covers the case when $p = 3, \alpha = 1, 2$.

We will proceed by induction on α. Suppose that $rb(\mathbb{Z}_{p^{\alpha-1}}, 3) = 4$ for some $\alpha \geq 3$. Let c be a 4 coloring of $\mathbb{Z}_{3^\alpha}$. For the sake of contradiction, suppose that c does not admit a rainbow triple. If $|P_0| = 4$, then c admits a rainbow triple by the induction hypothesis. Therefore, $|P_0| \leq 3$ and there exits $0 < i < p$ such that $|P_i \setminus P_0| \geq 1$. By Lemma 3.3.13, $|P_0| = 1$. This implies that $\text{im}(c) = |P_i \setminus P_0|$. By Lemma 3.3.15, c admits a rainbow triple. This completes the case when $p = 3$.

Let $p \geq 5$. With Proposition 3.3.9 as the base case, we will proceed by induction on α. Suppose that $rb(\mathbb{Z}_{p^{\alpha-1}}, p) = \frac{p+1}{2} + 1$ for some $\alpha \geq 2$. For the sake of contradiction, suppose that c does not admit a rainbow triple. If $|P_0| = \frac{p+1}{2} + 1$, then c admits a rainbow triple by the induction hypothesis. Therefore, $|P_0| \leq \frac{p+1}{2}$ and there exists $0 < j < p$ such that $|P_j \setminus P_0| \geq 1$. By Lemma 3.3.13, $P_0 = \{R\}$. By the pigeon hole principle, there exists $0 < i < \frac{p+1}{2}$ such that $|P_i \setminus P_0| \geq 2$. Notice that one of the following must hold:

1. $G \notin P_i \cup P_0$ for some color $G \neq R$,

2. $\text{im}(c) = P_i \cup P_0$.

Therefore, by Lemmas 3.3.14 and 3.3.15, c must admit a rainbow triple. This completes the case when $p \geq 5$. □

3.3.3 Lower bound for $rb(\mathbb{Z}_n, p)$, p prime

Since p is the coefficient of the equation that we are considering, we will use q to denote a prime other than p. Using values for $rb(\mathbb{Z}_q, k)$, we establish a lower bound for $rb(\mathbb{Z}_n, p)$. In order to proceed in a similar manner as with the Schur equation, Lemmas 3.3.16 and 3.3.17 are about the structure of triples. Lemma 3.3.18 exploits this structural information to construct a coloring that witnesses the lower bound for Proposition 3.3.19.

Lemma 3.3.16. *If $x_1 + x_2 = kx_3$ is a triple in $\mathbb{Z}_n$ where $m|x_1, x_2, x_3$ for some $m|n$, $m, n \in \mathbb{Z}$, then there exists a triple of the form $x_1/m + x_2/m = kx_3/m$ in $\mathbb{Z}_{\frac{n}{m}}$.*

Proof. By definition $x_1 + x_2 = kx_3$ in $\mathbb{Z}_n$ implies:

$$x_1 + x_2 = qn + r$$

$$kx_3 = tn + r$$

Divide both equations by m to get:

$$\frac{x_1}{m} + \frac{x_2}{m} = q\frac{n}{m} + \frac{r}{m}$$
$$k\frac{x_3}{m} = t\frac{n}{m} + \frac{r}{m}$$

Now we must check that $\frac{r}{m}$ is an integer. Since $m|(x_1 + x_2 - qn)$, we know $m|r$. By definition, this means there exists a triple of the form $x_1/m + x_2/m = kx_3/m$ in $\mathbb{Z}_{\frac{n}{m}}$. □

Next, we show that q cannot divide exactly two terms of a triple.

Lemma 3.3.17. *Let (x_1, x_2, x_3) be a triple of the form $x_1 + x_2 = kx_3$ in $\mathbb{Z}_{qn}$. If q is relatively prime to k and q divides two of the terms in (x_1, x_2, x_3) then q must divide the third term in (x_1, x_2, x_3).*

Proof. We consider the case where q divides x_1, x_2 and the case where q divides x_1, x_3.

Case 1: Assume q divides x_1, x_2. By definition the equation $x_1 + x_2 = kx_3$ in $\mathbb{Z}_{qn}$ means:

$$x_1 + x_2 = c_1 qn + r$$
$$k \cdot x_3 = c_2 qn + r$$

We rearrange the first equation to get q divides $x_1 + x_2 - c_1 qn$, which implies that q divides r. Thus, q divides $c_2 qn + r$, which implies q divides kx_3. We know q and k are relativity prime, therefore, q must divide x_3.

Case 2: Similarly, assume q divides x_1, x_3. By definition the equation $x_1 + x_2 = kx_3$ in $\mathbb{Z}_{qn}$ means:

$$x_1 + x_2 = c_1 qn + r$$
$$k \cdot x_3 = c_2 qn + r$$

From the second equation we get q divides $kx_3 - c_2 qn$, which implies that q divides r. Thus, q divides $x_1 - c_1 \cdot qn - r$, which implies q divides x_2. □

Notice that Lemmas 3.3.16 and 3.3.17 are stated for the equation $x_1 + x_2 = kx_3$ without the stipulation that k is prime. We can use the above lemmas to find our lower bound.

Lemma 3.3.18. *Let q, t be positive integers with q prime, and $q \neq p$. If there exists a rainbow-free r-coloring of $\mathbb{Z}_t$, then there exists a rainbow-free $(r + rb(\mathbb{Z}_q, p) - 2)$-coloring of $\mathbb{Z}_{qt}$.*

Proof. Let $q, t \in \mathbb{Z}$ such that q is prime, and $q \neq p$. Let $\hat{c}$ be a rainbow-free r-coloring for $\mathbb{Z}_t$ and let $\bar{c}$ be a maximum coloring of $\mathbb{Z}_q$ such that 0 is uniquely colored and the other color classes are symmetric subsets, as described in Corollary 3.3.7. Let c be an exact $(r+1)$-coloring of $\mathbb{Z}_{qt}$ if $rb(\mathbb{Z}_q, p) = 3$ or an exact $(r+2)$-coloring of $\mathbb{Z}_{qt}$ if $rb(\mathbb{Z}_q, p) = 4$ as follows:

$$c(x) = \begin{cases} \hat{c}(\frac{x}{q}) & x \equiv 0 \mod q \\ r + \bar{c}(x \mod q) & \text{otherwise.} \end{cases}$$

Since q and p are distinct primes, q and p are relatively prime. By Lemma 3.3.17, since q is relatively prime to p, q cannot divide exactly two of the terms in (x_1, x_2, x_3) for the equation $x_1 + x_2 = px_3$. Therefore, for all triples in $\mathbb{Z}_{qt}$, q can divide all three elements, no elements, or exactly one element of the triple.

Case 1: If q divides all three terms in (x_1, x_2, x_3), then by the constructions of c, the triple has the same colors as the triple $(\frac{x_1}{q}, \frac{x_2}{q}, \frac{x_3}{q})$ in $\hat{c}$. By Lemma 3.3.16, if (x_1, x_2, x_3) is a triple in $\mathbb{Z}_{qt}$ and $q|x_1, x_2, x_3$, then $(\frac{x_1}{q}, \frac{x_2}{q}, \frac{x_3}{q})$ is a triple in $\mathbb{Z}_t$. Thus, since $\hat{c}$ is a rainbow-free coloring, triples where all three elements are divisible by q cannot be rainbow in c.

Case 2: Suppose q divides none of the terms in (x_1, x_2, x_3). There is a maximum of two colors added on terms not divisible by q. Thus, there are at most two colors coloring the elements in any such triple, and triples of the form (x_1, x_2, x_3) with each x_i not divisible by q are not rainbow.

Case 3: Suppose q divides exactly one of (x_1, x_2, x_3). First assume q divides x_1. Notice that if $x_1 + x_2 \equiv px_3 \mod qt$ then $x_1 + x_2 \equiv px_3 \mod q$. Since 0 is uniquely colored in $\bar{c}$, the rainbow-free coloring of $\mathbb{Z}_q$, any triple in $\mathbb{Z}_q$ of the form $0 + x_2 \equiv px_3 \mod q$ is colored so that x_2 and x_3 receive the same color. In this case, $c(x_2) = r + \bar{c}(x_2 \mod q)$ and $c(x_3) = r + \bar{c}(x_3 \mod q)$, so (x_1, x_2, x_3) is not rainbow under c. If q divides either x_2 or x_3 the argument proceeds the same way. □

Proposition 3.3.19. *Let p be prime and let n be an integer with prime factorization $n = p^\alpha \cdot q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdots q_m^{\alpha_m}$ where q_i is prime, $q_i \neq q_j$ for $i \neq j$ and $\alpha_i \geq 0$. Then,*

$$rb(\mathbb{Z}_n, p) \geq rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i(rb(\mathbb{Z}_{q_i}, p) - 2)\Big)$$

Proof. If n is a power of p, then there is nothing to show. Suppose that the claim holds true for n where n has N prime factors that are not p.

Assume that $n = p^\alpha \cdot q_1^{\alpha_1} \cdot q_2^{\alpha_2} \cdots q_m^{\alpha_m}$ where $\alpha_1 + \cdots + \alpha_m = N + 1$. By the induction hypothesis, there exists a rainbow-free r-coloring of $\mathbb{Z}_{n/q_1}$ where

$$r = rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i(rb(\mathbb{Z}_{q_i}, p) - 2)\Big) - rb(\mathbb{Z}_{q_1}, p) + 2.$$

Therefore, by Lemma 3.3.18 there exists a rainbow-free $r + rb(\mathbb{Z}_{q_1}, p) - 2$ coloring of Z_n. Thus, by induction

$$rb(\mathbb{Z}_{p^\alpha}, p) + \sum_{i=1}^{m} \Big(\alpha_i (rb(\mathbb{Z}_{q_i}, p) - 2)\Big).$$

□

3.3.4 Upper bound for $rb(\mathbb{Z}_{q^\alpha}, p)$, $p \neq q$ prime

In this section we prove an upper bound matching Proposition 3.3.19 when n is a power of a prime. The main idea of Theorem 3.1.6 is to decompose $rb(\mathbb{Z}_{q^\alpha}, p)$ where p, q are prime into $2 + \alpha \cdot (rb(\mathbb{Z}_q, p) - 2)$. There is a nice case that follows the same ideas as for the upper bound of Theorem 3.1.3. This case is when the residue class R_0 receives enough colors to guarantee that $|P_i \setminus P_0| \leq 1$. However, this case is not forced. It is possible that a residue class other than R_0 receives the most colors. In this case, we create auxiliary colorings of residue classes that violate Theorem 3.3.1. Lemmas 3.3.20 and 3.3.21 show that the auxiliary colorings we use preserve rainbow solutions. Lemma 3.3.23 handles the cases when $p \not\equiv -1, 2 \mod q$, while Lemma 3.3.24 handels the case whne $p \equiv 1, 2 \mod q$. The proof of Theorem 3.1.6 comes together by combining Lemmas 3.3.23, and 3.3.24.

Suppose the following for the rest of this section: Let $q \neq p$ be prime. Let c be a coloring of $\mathbb{Z}_{q^\alpha}$ where $\alpha \geq 2$. Let $R_0, R_1, \cdots, R_{q-1}$ be the residue classes modulo q for $\mathbb{Z}_{q^\alpha}$ with corresponding residue palettes $\{P_i\}$. We will let G, B denote two colors that are not in P_0.

The next two lemmas have the exact same proof.

Lemma 3.3.20. *Suppose $\hat{c}$ is a 3-coloring of $\mathbb{Z}_q$ such that $X \subseteq P_i$ implies $c(i) \in X$ where X is a nonempty subset of $\{G, B\}$, and $c(i) \in \{G, B\}$ implies $c(i) \in P_i$, and $c(i) = \beta$ otherwise. If $\hat{c}$ contains a rainbow triple, then c contains a rainbow triple.*

Lemma 3.3.21. *Suppose $|P_i \setminus P_j| \leq 1$. Let $\hat{c}$ be a coloring of $\mathbb{Z}_q$ such that:*

$$\hat{c}(i) := \begin{cases} P_i \setminus P_j & \text{if } |P_i \setminus P_j| = 1 \\ \beta & \text{otherwise} \end{cases}$$

If $\hat{c}$ contains a rainbow triple then c contains a rainbow triple.

Proof of Lemmas 3.3.20 and 3.3.21. Suppose that (x_1, x_2, x_3) is a rainbow triple in $\mathbb{Z}_q$ under $\hat{c}$. There are two cases: $\hat{c}(x_3) = \beta$, or $\hat{c}(x_3) \neq \beta$.

Case 1: If $\hat{c}(x_3) = \beta$, then $\beta \neq \hat{c}(x_1), \hat{c}(x_2)$. Without loss of generality, suppose that x_1 and x_2 are colored G and B, respectively. This implies that there exists u, v such that $c(qu + x_1) = G$ and $c(qv + x_2) = B$. We must find an integer s such that

$$u + v - ps \equiv \begin{cases} 1 \mod q^{\alpha-1} & x_1 + x_2 \geq q \\ 0 \mod q^{\alpha-1} & x_1 + x_2 < q \end{cases}.$$

Since p and q are relatively prime, we can always solve for s. Therefore, there exists a rainbow triple in $\mathbb{Z}_{q^\alpha}$ under c.

Case 2: Assume $\hat{c}(x_3) \neq \beta$. Without loss of generality, $\hat{c}(x_1) \neq \beta$, and there exist u, v such that $c(qu + x_1) = G$ and $c(qv + x_3) = B$ where $G, B \notin P_{x_2}$. Notice that $pqv - qu + px_3 - x_1 \in R_{x_2}$. Therefore, there exists a rainbow triple in $\mathbb{Z}_{q^\alpha}$ under c. □

Lemma 3.3.22. *Let c be rainbow-free with $|P_i \setminus P_0| \leq 1$. Then*

$$|c| \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 3.$$

Proof. Suppose c is an r-coloring. Let $\hat{c}$ be a coloring constructed from c as described in Lemma 3.3.21. Notice that the set of colors used in c is comprised of the colors in R_j and each color used in $\hat{c}$ other than α. Thus, we know that $r = |P_j| + |\hat{c}| - 1$, where $|\hat{c}|$ is the number of colors appearing in $\hat{c}$.

Since c is a rainbow-free coloring of $\mathbb{Z}_{q^\alpha}$, we know $c|_{R_0}$ must be a rainbow-free coloring of $\mathbb{Z}_q$, so $|P_0| \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) - 1$. Furthermore, $\hat{c}$ is a rainbow-free coloring of $\mathbb{Z}_q$, implying that $|\hat{c}| \leq rb(\mathbb{Z}_q, p) - 1$. Therefore, $r \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 3$. □

From Lemma 3.3.22, we can conclude that if c has too many colors, then either c has a rainbow triple, or $|P_i \setminus P_0| \geq 2$ for some i. In the next lemma, we show that if c has too many colors and $|P_j \setminus P_0| \geq 2$ for some j, then we can still conclude that c has a rainbow triple.

Lemma 3.3.23. *Suppose c has $r = rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 2$ colors and $p \not\equiv 2, -1 \mod q$. Then c is not rainbow-free.*

Proof. By Lemma 3.3.22, there exists P_j such that $|P_j \setminus P_0| \geq 2$, or we are done. For the sake of contradiction, suppose c is rainbow-free. Without loss of generality, $G, B \in P_j \setminus P_0$. Since $p \not\equiv 2 \mod q$, we have a x such that $2j \equiv px \mod q$ and $x \not\equiv j \mod q$. Without loss of generality, this implies that $G \in P_x$. Furthermore, P_{-j} cannot contain either B or G.

Define a coloring $\hat{c}$ of $\mathbb{Z}_q$ by

$$\hat{c}(i) = \begin{cases} B & i = j \\ G & i = x \\ B & B \in P_i, i \neq j, x \\ G & G \in P_i, i \neq j, x \\ \beta & \text{otherwise} \end{cases}$$

where the ambiguity if $G, B \in P_i$ and $i \neq j, x$ is resolved arbitrarily. Notice that $\hat{c}(0) = \beta, \hat{c}(j) = B$, and $\hat{c}(x) = G$. Therefore, $\hat{c}$ is always a 3-coloring. Now, if we find that $\hat{c}$ has a rainbow triple, then we have reached a contradiction via Lemma 3.3.20. By Theorem 3.3.1, there are three ways $\hat{c}$ can be rainbow-free: 0 is uniquely colored, $p \equiv -1 \mod q$, or $p \equiv 2 \mod q$. However, none of these three situations hold. The element 0 is not uniquely colored since $\hat{c}(-j) = \beta$. Furthermore, $p \not\equiv 2 \mod q$ and $p \not\equiv -1 \mod q$ by assumption. Thus, we can find a rainbow triple. $\square$

Lemma 3.3.24. *Suppose c has $r = rb(\mathbb{Z}_{q^{\alpha-1}}, p) + rb(\mathbb{Z}_q, p) - 2$ and $p \equiv -1, 2 \mod q$. Then c is not rainbow-free.*

Proof. For the sake of contradiction, suppose that c is rainbow-free. By Theorem 3.1.4, $rb(\mathbb{Z}_q, p) = 4$. Notice that $|P_0| \leq rb(\mathbb{Z}_{q^{\alpha-1}}, p) - 1$, which implies that there are at least 3 colors not represented in P_0. Without loss of generality, we will let these colors be denoted by G, B, and Y. Given an ordering of colors $X_1 < X_2 < X_3$, consider the coloring $\hat{c} : \mathbb{Z}_q \to \{X_1, X_2, X_3, \beta\}$

given by

$$\hat{c}(x) = \begin{cases} \min\{X_i | X_i \in P_x\} & \{X_1, X_2, X_3\} \cap P_x \neq \{\} \\ \beta & \text{otherwise} \end{cases}.$$

Suppose $\hat{c}$ contains a rainbow triple x, y, z with colors $\hat{c}(x) < \hat{c}(y) < \hat{c}(z)$ (where $\beta > X_i$ for all i). Then there exists $sq + x \in R_x$, $tq + y \in R_y$ with $c(sq + x) = \hat{c}(x)$ and $c(tq + y) = \hat{c}(y)$. By construction of $\hat{c}$, we have $\hat{c}(x), \hat{c}(y) \notin P_z$. This implies completing $sq + x$ and $tq + y$ with an element in R_z will provide a rainbow triple. Therefore, if there exists an ordering of G, B, Y such that $\hat{c}$ is a 4-coloring of $\mathbb{Z}_q$, then we are done.

Claim: If $|P_x \setminus P_0| \geq 1$, then $|P_x \setminus P_0| \geq 2$.

Without loss of generality, let $G < B < Y$ be an ordering that maximizes the number of colors used by $\hat{c}$. Since $G \notin P_0$, $\hat{c}$ is at least a 2-coloring. Furthermore, $\hat{c}$ always uses β on 0. We may also assume that $\hat{c}$ is not a 4-coloring. This implies that $\hat{c}$ does not use Y. Now if $Y \in P_x$, then either $G \in P_x$ or $B \in P_x$. By reordering the colors, we can conclude that if $B \in P_x$, then either $G \in P_x$ or $Y \in P_x$, and if $G \in P_x$, then either $B \in P_x$ or $Y \in P_x$. In particular, if $|P_x \setminus P_0| \geq 1$, then $|P_x \setminus P_0| \geq 2$. This concludes the proof of the claim.

Let P_j be such that $|P_j \setminus P_0| \geq 2$. By applying the $\phi(x) = j^{-1}x$ to $\mathbb{Z}_q$, we can assume that $j = 1$. Furthermore, we have that $P_{-1} \subseteq P_0$. By selecting two elements in R_1 with different colors not in P_0, we can conclude that $|P_{-2} \setminus P_0| \geq 2$. In particular, the set $I = \{i : |P_i \setminus P_0| \geq 2\}$ contains $(-2)^k$ for any non-negative integer k. Notice that if $q = 5$, then I contains 4 which gives us a rainbow triple since $P_{-1} \subseteq P_0$. Furthermore, if $q \geq 17$, then $1, -2 = q - 2, 4, -8 = q - 8$ are distinct modulo q. The fact that $1, -2, 4, -8$ are distinct for primes $7, 11, 13$ is true by inspection. Therefore, $|I| \geq 4$ and $q \geq 7$.

For any four distinct elements in I there exists indices x_1, x_2, x_3, x_4 such that (without loss of generality for the colors) $G \in P_{x_1} \cap P_{x_2}$ and $B \in P_{x_3} \cap P_{x_4}$. Since $|I| \geq 4$ we can define a coloring

$\hat{c}$ of $\mathbb{Z}_q$ by

$$\hat{c}(i) = \begin{cases} G & i = x_1, x_2 \\ B & i = x_3, x_4 \\ B & B \in P_i, i \neq x_1, x_2, x_3, x_4 \\ G & G \in P_i, i \neq x_1, x_2, x_3, x_4 \\ \beta & \text{otherwise} \end{cases}$$

where the ambiguity if $G, B \in P_i$ and $i \neq x_1, x_2, x_3, x_4$ is resolved arbitrarily. If $\hat{c}$ has a rainbow solution, then c has a rainbow solution by Lemma 3.3.20.

Notice that every color class of $\hat{c}$ has size at least 2. Therefore, if $p \equiv 2 \mod q$, then there exists a rainbow solution in $\mathbb{Z}_q$ given $\hat{c}$. Thus, by Theorem 3.3.1, up to dilation, $\beta = [a_3, a_1 - 1]$, $G = [a_1, a_2 - 1]$, and $B = [a_2, a_3 - 1]$ with $a_1 + a_2 + a_3 \equiv 0, 1 \mod q$ and $0 < a_1 < a_2 < a_3 \leq q$. Furthermore, $[a_1, a_3 - 1]$ must be closed under multiplication by -2, since we can find a rainbow triple otherwise. We will conclude that this structure is impossible. Let $\succ$ be an ordering on $\mathbb{Z}_q$, where $a > b$ if $a' > b'$ where a' (resp. b') is a representative of the equivalence class a (resp. b) such that $0 \leq a' \leq q$.

First, assume that $a_1 = 1$. This implies that $a_3 - 1 \succeq -2$ and $a_3 = -1$. However, this is a contradiction since

$$a_1 + a_2 + a_3 \equiv 1 + a_2 - 1 \equiv 0, 1 \mod q$$

and $q > a_2 > 1$. Therefore, we can conclude that $a_1 > 1$ (and $a_1 \succ 1$). Notice that if $q - \frac{q+1}{2} \in [a_1, a_3 - 1]$, then we have reached a contradiction, since $-2(q - \frac{q+1}{2}) \equiv 1 \mod q$. We will complete the proof by assuming that $a_1 \succ q - \frac{q+1}{2}$ or $q - \frac{q+1}{2} \succ a_3 - 1$, and deriving a contradiction in each case.

Case 1: Assume that $q - \frac{q+1}{2} \succ a_3 - 1$. If $a_1 \preceq \lfloor q/4 \rfloor$, then $q - 2a_1 \succeq q - \frac{q+1}{2}$, which is a contradiction. Therefore, $a_3 - 1 \succ \lfloor q/4 \rfloor$. Now if $a_3 - 1 \succeq \lfloor q/3 \rfloor$, then $4(a_3 - 1) \succ a_3 - 1$. Furthermore, if $\lfloor q/3 \rfloor \succ a_3 - 1$, then $-2(a_3 - 1) \succ a_3 - 1$. Either one of the previous two possibilities results in a contradiction since $[a_1, a_3 - 1]$ is closed under multiplication by -2.

Case 2: Assume that $q - \frac{q+1}{2} \prec a_1$. If $a_3 - 1 \succeq 3q/4$, then $q - 2(a_3 - 1) \preceq q - \frac{q+1}{2}$, which is a contradiction. Therefore, $a_1 \prec \lfloor 3q/4 \rfloor$. Now if $a_1 \succeq \lceil 2q/3 \rceil$, then $-2a_1 \prec a_1$. Furthermore, if $\lceil 2q/3 \rceil \succ a_1$, then $4a_1 \prec a_1$. Either one of the previous two possibilities results in a contradiction since $[a_1, a_3 - 1]$ is closed under multiplication by -2.

Since both cases result in contradictions, we conclude that c is not rainbow-free. □

Lemmas 3.3.23 and 3.3.24 cover all possible cases for the value of $p \mod q$.

Proof of Theorem 3.1.6. Iteratively applying Lemmas 3.3.23 and 3.3.24 gives the desired result by matching the lower bound in Proposition 3.3.19. □

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CHAPTER 4. RAINBOW SOLUTIONS TO THE SIDON EQUATION IN CYCLIC GROUPS AND INTERVALS

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Abstract

Given a coloring of group elements, a rainbow solution to an equation is a solution where every element is assigned a different color. The rainbow number of $X \in \{\mathbb{Z}_n, [n]\}$ for an equation eq, denoted $rb(X, eq)$, is the smallest number of colors r such that every exact r-coloring of X admits a rainbow solution to the equation eq. We prove that for every exact 4-coloring of $\mathbb{Z}_p$, where $p \geq 3$ is prime, there exists a rainbow solution to the Sidon equation $x_1 + x_2 = x_3 + x_4$. Furthermore, we determine the rainbow number of $\mathbb{Z}_n$ and $[n]$ for the Sidon equation.

4.1 Introduction

An *r-coloring* of a set X is a function $c : X \to [r]$, where $[r] = \{1, 2, \ldots, r\}$. An r-coloring is *exact* if the function c is surjective. In this paper we focus on r-colorings of the cyclic group $\mathbb{Z}_n$ and the interval $[n]$, and all r-colorings are assumed to be exact. A subset $A \subseteq \mathbb{Z}_n$ or $A \subseteq [n]$ is called *rainbow* (under the coloring c) if each element of A is colored distinctly. Given an equation eq, the *rainbow number* of $X \in \{\mathbb{Z}_n, [n]\}$ for an equation eq, denoted $rb(X, eq)$, is the smallest number of colors r such that every exact r-coloring of X admits a rainbow solution to the equation eq. By convention, if no such integer r exists we set $\mathrm{rb}(X, eq) = |X| + 1$. We say that a coloring c is *rainbow eq-free* if there is no rainbow solution to eq under c.

Rainbow numbers of $\mathbb{Z}_n$ and $[n]$ for the equation $x_1 + x_2 = 2x_3$, for which the solutions are 3-term arithmetic progressions, are known as anti-van der Waerden numbers. Rainbow arithmetic progressions have been studied extensively in [2, 3, 5, 8, 14]. Generalizing the equation $x_1 + x_2 = 2x_3$, Bevilacqua et al. in [4] determined the rainbow number of $\mathbb{Z}_n$ for the equation $x_1 + x_2 = kx_3$ for some n when $k = 1$ (which is known as the Schur equation) or when k is prime. Ansaldi et.al. determined the rainbow number of $\mathbb{Z}_p$ for the equation $a_1x_1 + a_2x_2 + a_3x_3 = b$, and established the rainbow number of $\mathbb{Z}_n$ for this equation under certain conditions on the coefficients in [1]. Structures of rainbow-free colorings for various equations have been studied in [7, 9, 10].

In this paper, we establish the rainbow number of $\mathbb{Z}_n$ and $[n]$ for the Sidon equation $x_1 + x_2 = x_3 + x_4$. The Sidon equation is a classical object in additive number theory and is used to measure the additive energy of a set (see [11, 12]). Fox, Mahdian, and Radoičić showed in [6] that for every 4-coloring of $[n]$, where each color class has cardinality more than $\frac{n+1}{6}$, there exists a rainbow solution to the Sidon equation. The lower bound on a color class cardinality is tight. Taranchuk and Timmons in [13] studied the maximum number of rainbow solutions to the Sidon equation for a fixed number of colors.

In this paper we denote the Sidon equation by S. We determine $\mathrm{rb}(\mathbb{Z}_p, S)$, where p is prime in Section 4.2. Furthermore, we determine $\mathrm{rb}(\mathbb{Z}_n, S)$ in Section 4.3. Finally, we determine $\mathrm{rb}([n], S)$ in Section 4.4. Notice that $\mathrm{rb}(\mathbb{Z}_2, S) = 3$ and $\mathrm{rb}(\mathbb{Z}_3, S) = 4$ by convention. Our main results are as follows:

Theorem 4.1.1. *Let $p \geq 3$ be a prime. Then $rb(\mathbb{Z}_p, S) = 4$.*

Theorem 4.1.2. *Let $n = p_1 \cdots p_k$ be a prime factorization such that $p_i \leq p_j$ whenever $i < j$. Let m be the smallest index such that $p_m \geq 3$ (or $m = k$ if this index does not exist), $f_1 = |\{p_i : p_i \leq 3, i \neq m\}|$, and $f_2 = |\{p_i : p_i \geq 5, i \neq m\}|$. Then*

$$rb(\mathbb{Z}_{p_m}, S) + f_1 + 2f_2 = rb(\mathbb{Z}_n, S).$$

Theorem 4.1.3. *For any integer $n \geq 2$,*

$$rb([n], S) = \lfloor \log_2(n-1) \rfloor + 3.$$

4.2 Rainbow numbers for $\mathbb{Z}_p$

The section is structured as follows: First, we prove lemmas that let us make some assumptions about the structure of a rainbow Sidon-free coloring of $\mathbb{Z}_p$. In particular, Lemma 4.2.2 shows that i-dominant colors must exist for all $i \in \mathbb{Z}_p^*$ and Lemma 4.2.3 lets us translate and scale colorings, where $\mathbb{Z}_p^*$ is the cyclic multiplicative group. The remainder of the lemmas prove various structural results about rainbow Sidon-free 4-colorings on $\mathbb{Z}_p$. In fact, not all of the structural requirements of a rainbow Sidon-free 4-coloring on $\mathbb{Z}_p$ can be satisfied. In this sense, the entire section is to be read as a proof by contradiction, in which the contradiction is found in the proof of Theorem 4.1.1.

As an interesting note, we would like to point out that Lemmas 4.2.5 and 4.2.8 have analogous results in the context of the interval $[n]$ as shown in [6] by Fox et al. In these cases, our proofs are similar, but expedited by the assumptions Lemmas 4.2.2 and 4.2.3 afford us. Curiously, Lemma 4.2.6 contrasts with Lemma 3 in [6], where it is shown that a YY-string must exist.

As noted in the introduction, $\mathrm{rb}(\mathbb{Z}_2, S) = 3$ and $\mathrm{rb}(\mathbb{Z}_3, S) = 4$ by convention. The following observation is a trivial lower bound for $\mathrm{rb}(\mathbb{Z}_p, S)$ when $p \geq 5$.

Observation 4.2.1. *Let $p \geq 3$ be a prime. Then $rb(\mathbb{Z}_p, S) > 3$.*

To prove the corresponding upper bound, we show that any 4-coloring of $\mathbb{Z}_p$ with $p \geq 5$ prime admits a rainbow solution to the Sidon equation. Suppose $c : \mathbb{Z}_p \to \{R, Y, G, B\}$ is a rainbow Sidon-free 4-coloring. We say that a color X is *dominant* if for any pair of elements $x, x+1 \in \mathbb{Z}_p$, either $c(x) = c(x+1)$ or $X \in \{c(x), c(x+1)\}$. More generally, we say a color X is *i-dominant* if for any pair of elements $x, x+i \in \mathbb{Z}_p$, either $c(x) = c(x+i)$ or $X \in \{c(x), c(x+i)\}$. We should note that the idea of dominant colors has appeared in [6].

For a color X, an interval of integers $[i, i+j]$ is an *X-string* if $c([i, i+j]) = \{X\}$. Similarly, given two colors X_1, X_2, an interval $[i, i+j]$ is an *X_1X_2-string* if $c([i, i+j]) = \{X_1, X_2\}$ (these strings are also called bichromatic). An X or X_1X_2-string $[i, i+j]$ is *maximal* if $c(i-1), c(i+j+1) \notin \{X\}$ or $c(i-1), c(i+j+1) \notin \{X_1, X_2\}$, respectively. A pattern (sequence) of colors $X_0X_1X_2\cdots X_k$ *appears at position* j if $c(j+i)=X_i$ for $0\le i\le k$; if such a j does not exist, then $X_0X_1X_2\cdots X_k$ *does not appear.* A string A is i-*periodic* if for all $x, x+i\in A$ we have $c(x)=c(x+i)$. Often we will abuse notation and identify a string A with its induced pattern of colors.

Lemma 4.2.2. *Every rainbow Sidon-free* 4*-coloring* c *of* $\mathbb{Z}_p$ *has an* i*-dominant color for any* $i\in\mathbb{Z}_p\setminus\{0\}$.

Proof. Let $i\in\mathbb{Z}_p\setminus\{0\}$. Form a graph H on $V(H)=\{R,Y,G,B\}$ where $X_1X_2\in E(H)$ if and only if there exists $x\in\mathbb{Z}_p$ such that $\{c(x),c(x+i)\}=\{X_1,X_2\}$. Notice that $\delta(H)\ge 1$, since $\mathbb{Z}_p$ does not have any proper subgroups generated by i. By construction, if H contains a $2K_2$ subgraph, then c admits a rainbow solution to the Sidon equation. Therefore, H is $2K_2$-free. Since $\delta(H)\ge 1$ and H is $2K_2$-free, H must be isomorphic to $K_{1,3}$. Let $X\in V(H)$ such that $d(X)=3$. Notice that X is an i-dominant color. □

The next lemma shows that the rainbow Sidon-free property of colorings is preserved by translating and scaling colorings.

Lemma 4.2.3. *Let* c *be a coloring of* $\mathbb{Z}_p$*,* $i\in\mathbb{Z}_p^*$*, and* $j\in\mathbb{Z}_p$*. Let* $c_{i,j}$ *be given by* $c_{i,j}(x)=c(ix+j)$*. The coloring* c *is rainbow Sidon-free if and only if* $c_{i,j}$ *is rainbow Sidon-free.*

Proof. Let $A=\{x_1,x_2,x_3,x_4\}\subset\mathbb{Z}_p$. Notice that A is rainbow under c if and only if $A_{i,j}=\{ix_1+j,ix_2+j,ix_3+j,ix_4+j\}$ is rainbow under $c_{i,j}$. Furthermore, $x_1+x_2=x_3+x_4$ if and only if $ix_1+j+ix_2+j=ix_3+j+ix_4+j$. □

It should be noted that Proposition 3.5 in [5] and Theorem 3.5 in [8] together determine when $\mathrm{rb}(\mathbb{Z}_p,eq)=3$ and when $\mathrm{rb}(\mathbb{Z}_p,eq)=4$, where eq is the equation $x_1+x_2=2x_3$. Since we use Proposition 3.5 from [5], we have stated it below.

Proposition 4.2.4 (Proposition 3.5 in [5])**.** *Let* eq *be the equation* $x_1+x_2=2x_3$*. For every prime number* p*,* $3\le rb(\mathbb{Z}_p,eq)\le 4$*.*

Let $x, x+i, x+2i$ be a rainbow 3-term arithmetic progression in $\mathbb{Z}_p$ under the coloring c. Without loss of generality, let $c(x+i) = R$. Notice that R is dominant under the coloring $c_{i^{-1},0}$. Furthermore, R is not 2-dominant given $c_{i^{-1},0}$. Since c and $c_{i^{-1},0}$ have the same behavior with respect to rainbow Sidon solutions, we will always assume that R is dominant, and R is not 2-dominant. In particular, we will assume that the pattern YRB appears in c (otherwise, we can find a rainbow 3-term arithmetic progression, and scale/translate the coloring to put ourselves in this position). Furthermore, since a 2-dominant color must exist (and it is either Y or B), we can assume that Y is 2-dominant. From this point forward in this section, we will use these assumptions to prove structural results about c. Ultimately, these structures will lead to a contradiction, in the sense that we will find a rainbow solution to the Sidon equation in the proof of Theorem 4.1.1.

Lemma 4.2.5. *Let $X \in \{B, G\}$. Every maximal RX-string is 2-periodic with exactly one element colored by X within every substring of length 2. In particular, BB and GG do not appear.*

Proof. Let A be a maximal RG-string. Recall that we assume the pattern YRB appears under the coloring c. Therefore, if A contains patterns RRG, GRR, RGG, or GGR, then c admits a rainbow solution to the Sidon equation. Thus, A must be of the form $RGR \cdots RGR$.

Now let A be a maximal RB-string. Since Y is 2-dominant, we know that patter GRY must appear under c. This implies that A cannot contain RRB, BRR, BBR, RBB. Thus, A must be of the form $RBR \cdots RBR$. □

The next lemma extends Lemma 4.2.5 to shows that the pattern YY cannot appear under the coloring c. It will be very useful since it implies that if $c(x) = Y$, then $c(x-1) = c(x+1) = R$.

Lemma 4.2.6. *The pattern YY does not appear under the coloring c.*

Proof. Let i and $i+m$ be colored G and B by c. Suppose the pattern RYY exists at position j, and consider colors of $j+m, j+m+1, j+m+2$. There are three cases: $c(j+m)$ in $\{Y\}$, $\{B, G\}$, or $\{R\}$. If $c(j+m) = Y$, then $\{i, j+m, i+m, j\}$ is a rainbow Sidon solution. If

$c(j+m) \in \{B, G\}$, then $c(j+m+1) = R$ by Lemma 4.2.5 and $\{i, j+m+1, i+m, j+1\}$ is a rainbow Sidon solution. Therefore, $c(j+m) = R$.

Next consider the color of $j+m+2$. There are three case: $c(j+m+2)$ is in $\{R\}$, $\{B, G\}$, or $\{Y\}$. If $c(j+m+2) = R$, then $\{i, j+m+2, i+m, j+2\}$ is a rainbow Sidon solution. If $c(j+m+2) \in \{B, G\}$, then $c(j+m+1) = R$ by Lemma 4.2.5 and $\{i, j+m+1, i+m, j+1\}$ is a rainbow Sidon solution. Therefore, $c(j+m+2) = Y$.

It follows that $c(j+m+1) \notin \{B, G\}$ since $c(j+m+2) = Y$ and R is dominant. Furthermore, $c(j+m+1) \neq R$, else $\{i, j+m+1, i+m, j+1\}$ is a rainbow Sidon solution. Therefore, $c(j+m+1) = Y$.

Thus if there is an RYY-string starting at j, then there is an RYY at $j+m$. Since m is a generator of $\mathbb{Z}_p$, Case 3.3 implies that $c(x) = R$ for all $x \in \mathbb{Z}_p$, which is a contradiction. □

The following lemma proves that any pair of elements with colors B and G must have an R-string in the middle between them.

Lemma 4.2.7. *Suppose that X_1RY appears at i and YRX_2 appears at $i+j-2$ where j is odd and $\{X_1, X_2\} = \{B, G\}$. Then $c(i + \frac{j-1}{2}) = c(i + \frac{j+1}{2}) = R$.*

Proof. Let $y = i + \frac{j-1}{2}$. For the sake of contradiction, suppose that $\{c(y), c(y+1)\} \neq \{R\}$. By the symmetry of the problem, there are three cases: $Y \in \{c(y), c(y+1)\}$, $c(y) = X_1$, and $c(y) = X_2$.

If $c(y) = Y$, then by the dominance of R and Lemma 4.2.6 $c(y+1) = R$, hence $\{i, i+j, y, y+1\}$ is a rainbow Sidon solution. If $c(y) = X_1$, then $c(y+1) = R$ and $c(y+2) \in \{X_1, Y\}$. Depending on the value of $c(y+2)$, either $\{i+1, i+j, y, y+2\}$ or $\{i+2, i+j, y+1, y+2\}$ is a rainbow Sidon solution. If $c(y) = X_2$, then $c(y-1) = R$ and $\{i, i+j-2, y, y-1\}$ is a rainbow Sidon solution.

In each case, we get a contradiction. □

Lemma 4.2.8. *The patterns BRB and GRG do not appear under the coloring c.*

Proof. Without loss of generality, suppose that BRB exists. Let $i_b, \ldots, i_b + j_b$ be a maximal BR-string that starts and ends with B (we are truncating the R colored element at the start and end of a maximal BR-string). Note that this string alternates colors between B and R. This implies that $j_b \geq 2$ and j_b is even. Let $i_g, \ldots, i_g + j_g$ be a maximal GR-string that starts and ends with G. This implies that j_g is even.

Since p is odd, either $i_g - i_b - j_b$ or $i_b - i_g - j_g$ is odd. Without loss of generality, suppose that $j = i_g - i_b - j_b$ is odd. Let $i = i_b + j_b$ so that $i + j = i_g$. In particular, we have chosen i and j such that $BRBRY$ appears at $i - 2$ and YRG appears at $i + j - 2$, where j is odd.

Let $y = i + \frac{j-1}{2}$. Notice that $c(y) = c(y+1) = R$ by Lemma 4.2.7. Let k be the smallest integer such that either $c(y-k) = Y$ or $c(y+1+k) = Y$. Notice that if $\{c(y-k), c(y+1+k)\} = \{R, Y\}$, then $\{i, i+j, y-k, y+1+k\}$ is a rainbow Sidon solution. Therefore, $c(y-k) = c(y+1+k) = Y$. Notice that $c(i-2) = B$ and $c(y+1+k-2) = R$. However,

$$(y-k) + (y+1+k-2) = 2i + j - 2 = (i-2) + (i+j).$$

Therefore, $\{i-2, i+j, y-k, y+1+k-2\}$ is a rainbow Sidon solution, a contradiction. Therefore, neither BRB nor GRG can appear under the coloring c. □

Lemma 4.2.9. *All R-strings have length* 1 *or* 3.

Proof. Let elements i and $i + m$ be colored with B and G, respectively. Since $m \geq 1$ is a generator of $\mathbb{Z}_p$, either (1) there exists YRR at some position j such that YRR does not appear at $j + m$, or (2) every R-string has length 1 and the lemma is proven. We now assume (1) holds, and will show that $YRRRY$ appears at position j. We will proceed by considering the colors of $j + m, j + m + 1, j + m + 2$.

First we will conclude that $c(j + m) = Y$ by ruling out other three options. If $c(j + m) = R$, then $\{i, j + m, i + m, j\}$ is a rainbow Sidon solution. If $c(j + m) = B$, then $\{i + 2, j + m, i + m, j + 2\}$ is a rainbow Sidon solution because $c(i + 2) = Y$ by Lemma 4.2.8. If $c(j + m) = G$, then $\{i, j + m, i + m - 2, j + 2\}$ is a rainbow Sidon solution because $c(i + m - 2) = Y$ by Lemma 4.2.8. Therefore, $c(j + m) = Y$. Furthermore, by Lemma 4.2.6, $c(j + m + 1) = R$ since R is dominant.

Next consider the color of $j+m+2$. Notice that if $c(j+m+2)=Y$, then $\{i, j+m+2, i+m, j+2\}$ is a rainbow Sidon solution. Furthermore, recall that j was selected so that YRR does not appear at $j+m$. Therefore, $c(j+m+2) \neq R$. In particular, $c(j+m+2) \in \{B, G\}$.

Now $c(j+4) \neq R$, else either $\{i, j+m+2, i+m-2, j+4\}$ or $\{i+2, j+m+2, i+m, j+4\}$ is a rainbow Sidon solution depending on the color of $j+m+2$. Therefore, $c(j+4)=Y$ and $c(j+3)=R$.

Thus, we have shown that if there exists a YRR at position x, then there exists a $k \geq 1$ such that YRX with $X \in \{B, G\}$ appears at position $x+km$, $YRRRY$ appears at position $x+(k-1)m$ and YRR appears at $x+\ell m$ for $0 \leq \ell < k$. We will show that this behavior propagates backwards from j. In particular, suppose that $YRRRY$ appears at position j, and that YRR appears at $j-m$. Notice that $c(j-m+3), c(j-m+4) \in \{R, Y\}$ since Y is 2-dominant. If $c(j-m+4)=R$, then $\{i+m, j-m+4, i, j+4\}$ is a rainbow Sidon solution. Therefore, $c(j-m+4)=Y$, and so by Lemma 4.2.6 we have $c(j-m+3)=R$. Hence $YRRRY$ appears at position $j-m$.

In summary, if there exists a YRR at position x, then there exists a $k \geq 1$ such that YRX with $X \in \{B, G\}$ appears at position $x+km$, $YRRRY$ appears at position $x+(k-1)m$ and YRR appears at $x+\ell m$ for $0 \leq \ell < k$. By the argument in the previous paragraph, $YRRRY$ appears at $x+\ell m$ for every $0 \leq \ell < k$. Thus, every R-string has length 1 or 3. □

With no further ado:

Theorem 4.1.1. *The coloring c does not exist. In particular, $rb(\mathbb{Z}_p, S)=4$ for all prime $p \geq 3$.*

Proof. Since c is a 4-coloring, there must exist i and j such that $c(i)=B$ and $c(j)=G$. Furthermore, since p is odd, there are distinct $m_1, m_2 \in \mathbb{Z}_p$ such that $i+m_1=j$ and $i-m_2=j$ where either m_1 is odd or m_2 is odd. Without loss of generality, suppose that m_1 is odd. By Lemma 4.2.7, let $y=i+\frac{j-1}{2}$ and $c(y)=c(y+1)=R$. Notice that Lemma 4.2.9 guarantees that the R-string containing y has length 3. Therefore, either $c(y-1)=Y$ and $c(y+2)=R$ or

$c(y-1) = R$ and $c(y+2) = Y$. In either case, $\{i, y-1, y+2, j\}$ is a rainbow Sidon solution. This contradicts the assumption of the section, that c is a rainbow Sidon-free 4-coloring of $\mathbb{Z}_p$. □

4.3 Rainbow numbers for $\mathbb{Z}_n$

This section is broken into two subsections. The lower bound for $\mathrm{rb}(\mathbb{Z}_n, S)$ is shown in the first subsection, which will provide insight into where (and how) to look for rainbow solutions to the Sidon equation in the upper bound argument. The upper bound is shown in the second subsection.

4.3.1 Lower Bound

We construct a lower bound coloring for $\mathbb{Z}_n$ by expanding a coloring for $\mathbb{Z}_{n/p}$. Essentially, we insert $p-1$ elements between each pair of elements in $\mathbb{Z}_{n/p}$ (taken in their natural cyclic ordering), and color the elements appropriately. The method for coloring the new inserted elements is specifically chosen to maintain i-dominant colors.

Lemma 4.3.1. *Let c be a rainbow Sidon-free r-coloring of $\mathbb{Z}_n$ and p be prime. If $p \leq 3$, then there exists a rainbow Sidon-free $(r+1)$-coloring of Z_{pn}. If $p \geq 5$, then there exists a rainbow Sidon-free $(r+2)$-coloring of $\mathbb{Z}_{pn}$.*

Proof. Assume that $p \leq 3$. Let

$$\hat{c}(x) = \begin{cases} c(x/p) & x \equiv 0 \mod p \\ r+1 & \text{otherwise} \end{cases}$$

be an $(r+1)$-coloring of $\mathbb{Z}_{pn}$. Let $X = \{x_1, x_2, x_3, x_4\} \subseteq \mathbb{Z}_{pn}$ such that $x_1 + x_2 = x_3 + x_4$.

Without loss of generality, if $x_1, x_2, x_3 \in \langle p \rangle$, then $x_4 \in \langle p \rangle$ where $\langle p \rangle$ is the subgroup of $\mathbb{Z}_{pn}$ generated by p. In this case, X is not rainbow, since c is a rainbow free coloring of $\langle p \rangle \cong \mathbb{Z}_n$. Furthermore, if $x_i, x_j \notin \langle p \rangle$ for $1 \leq i < j \leq 4$, then $\hat{c}(x_i) = \hat{c}(x_j)$ and X is not rainbow. Thus, $\hat{c}$ is a rainbow Sidon-free $(r+1)$-coloring of $\mathbb{Z}_{pn}$.

Assume that $p \geq 5$. Let

$$\hat{c}(x) = \begin{cases} c(x/p) & x \equiv 0 \mod p \\ r+1 & x \equiv 1, p-1 \mod p \\ r+2 & \text{otherwise} \end{cases}$$

be an $(r+2)$-coloring of $\mathbb{Z}_{pn}$. Let $X = \{x_1, x_2, x_3, x_4\} \subseteq \mathbb{Z}_{pn}$ such that $x_1 + x_2 = x_3 + x_4$.

As in the previous case, c is a rainbow free coloring of $\langle p \rangle \cong \mathbb{Z}_n$. Therefore, if any three elements in X are in $\langle p \rangle$, then X is not rainbow. Thus, assume that exactly two elements in X are in $\langle p \rangle$. In particular, assume that $x_i, x_j \in \langle p \rangle$.

Without loss of generality, if $i = 1$ and $j = 2$, then $x_3 + x_4 = s_1 p$ for some integer s_1. Therefore, $x_3 \equiv -x_4 \mod p$ and $\hat{c}(x_3) = \hat{c}(x_4)$.

Without loss of generality, if $i = 1$ and $j = 3$, then $x_2 \equiv x_4 \mod p$. Therefore, $\hat{c}(x_2) = \hat{c}(x_4)$.

In either case, X is not rainbow. Thus, $\hat{c}$ is a rainbow Sidon-free $(r+2)$-coloring of $\mathbb{Z}_{pn}$. □

Repeatedly applying Lemma 4.3.1 gives a lower bound for $\text{rb}(\mathbb{Z}_n, S)$. Notice that the statement of Proposition 4.3.2 selects p_m to maximize the number of colors used in the construction.

Proposition 4.3.2. *Let $n = p_1 \cdots p_k$ be a prime factorization such that $p_i \leq p_j$ whenever $i < j$. Let m be the smallest index such that $p_m \geq 3$ (or $m = k$ if this index does not exist), $f_1 = |\{p_i : p_i \leq 3, i \neq m\}|$, and $f_2 = |\{p_i : p_i \geq 5, i \neq m\}|$. Then*

$$rb(\mathbb{Z}_{p_m}, S) + f_1 + 2f_2 \leq rb(\mathbb{Z}_n, S).$$

Proof. By starting with a rainbow free $(\text{rb}(\mathbb{Z}_{p_m}, S) - 1)$-coloring of $\mathbb{Z}_{p_m}$, we can construct a rainbow free r-coloring with

$$r = \text{rb}(\mathbb{Z}_{p_m}, S) - 1 + f_1 + 2f_2$$

by repeatedly applying Lemma 4.3.1 for primes p_i, $i \neq m$. □

4.3.2 Upper bound

Since the lower bound construction expands colorings depending on primes p, it is intuitive that an upper bound argument would reduce colorings modulo p until a rainbow Sidon solution can be found. Let t be a be a positive integer that divides n. Let $R_i = i + \langle t \rangle$ be the i^{th} coset of the subgroup generated by t in $\mathbb{Z}_n$. This is notation that is consistent with work in [4]. Lemma 4.3.3 identifies which coset of t we want to narrow in on.

Lemma 4.3.3. *Let $t = n/p$ (p prime) divide n and c be a rainbow Sidon-free coloring of $\mathbb{Z}_n$. Consider the cosets of $\langle t \rangle$, R_i with $0 \leq i < t$. There exists an index j such that $|c(R_i) \setminus c(R_j)| \leq 1$ for all $0 \leq i < t$.*

Proof. Let j be the index that maximizes $|c(R_j)|$. For the sake of contradiction, assume that $|c(R_i) \setminus c(R_j)| \geq 2$ for index i. This implies that there exists $x_1, x_2 \in R_i$ such that $c(x_1) \neq c(x_2)$ and $c(x_1), c(x_2) \notin c(R_j)$. Let $x_3 \in R_j$. Notice that

$$x_1 = s_1 t + i$$

$$x_2 = s_2 t + i$$

$$x_3 = s_3 t + j.$$

Therefore,

$$x_4 = x_1 + x_3 - x_2 = (s_1 - s_2 + s_3)t + j.$$

Since c is rainbow Sidon-free, $c(x_4) = c(x_3)$. Notice that $x_4 - x_3 = t(s_1 - s_2)$ and that $|R_j| = p$. Since $s_1 - s_2 \neq 0$, and $s_1 - s_2$ is relatively prime to p, we can conclude that $|c(R_j)| = 1$. This contradicts our choice of index j. □

The next lemma focuses our attention on the coset with the most number of colors under a coloring c. Furthermore, it controls the number of colors lost in the process.

Lemma 4.3.4. *Let p be a prime divisor of n and $pt = n$. Then*

$$rb(\mathbb{Z}_n, S) \leq rb(\mathbb{Z}_p, S) + rb(\mathbb{Z}_t, S) - 2.$$

Proof. Let c be a rainbow Sidon-free $(\mathrm{rb}(\mathbb{Z}_n, S) - 1)$-coloring of $\mathbb{Z}_n$. Let R_i be the i^{th} coset of $\langle t \rangle$ in $\mathbb{Z}_n$. By Lemma 4.3.3, there exists index j such that $|c(R_i) \setminus c(R_j)| \leq 1$ for $0 \leq i < t$. Notice that c is rainbow Sidon-free if and only if c' given by $c'(x) = c(x + j)$ is rainbow Sidon-free. Since $R_0 \cong \mathbb{Z}_p$ is a subgroup of $\mathbb{Z}_n$, it follows c' must be a rainbow Sidon-free coloring of $\mathbb{Z}_p$. Furthermore, $|c'(R_i) \setminus c'(R_0)| \leq 1$ for $0 \leq i < t$.

Let

$$\hat{c}(x) = \begin{cases} i & \{i\} = c'(R_x) \setminus c'(R_0) \\ \alpha & c'(R_x) \subseteq c'(R_0) \end{cases}$$

be a coloring of $\mathbb{Z}_t$ (so that α is a color not used by c'). For the sake of contradicion, let $\{x_1, x_2, x_3, x_4\} \subseteq \mathbb{Z}_t$ be rainbow given $\hat{c}$ such that $x_1 + x_2 = x_3 + x_4$. Without loss of generality, assume that $\hat{c}(x_1), \hat{c}(x_2), \hat{c}(x_3) \neq \alpha$. Therefore, there exists $y_i \in R_{x_i}$ such that $c'(y_i) = \hat{c}(x_i)$ for $1 \leq i \leq 3$. Notice that $y_4 = y_1 + y_2 - y_3 \in R_{x_4}$ and that $c'(y_4) \neq c'(y_1), c'(y_2), c'(y_3)$. Therefore, $\{y_1, y_2, y_3, y_4\}$ is a rainbow Sidon solution in $\mathbb{Z}_n$ given c'; this is a contradiction. Thus, $\hat{c}$ is a rainbow Sidon-free coloring of $\mathbb{Z}_t$.

We can combine all this information to bound the number of colors of used by c. In particular,

$$c(\mathbb{Z}_n) = c'(\mathbb{Z}_n) = (c'(R_0) \cup \hat{c}(\mathbb{Z}_t)) \setminus \{\alpha\}.$$

This implies that

$$\mathrm{rb}(\mathbb{Z}_n, S) - 1 = |c(\mathbb{Z}_n)| = |c'(R_0)| + |\hat{c}(\mathbb{Z}_t)| - 1 \leq \mathrm{rb}(\mathbb{Z}_p, S) - 1 + \mathrm{rb}(\mathbb{Z}_t, S) - 2,$$

completing the proof. □

Proposition 4.3.5 is the result of repeatedly applying Lemma 4.3.4. In this sense, it reverses the process used in the construction of a rainbow Sidon-free coloring in the proof of Proposition 4.3.2. It is helpful to recall that $\mathrm{rb}(\mathbb{Z}_p, S) = 4$ for all primes $p \geq 3$ while comparing these two propositions.

Proposition 4.3.5. *Let $n = p_1 \cdots p_k$ be a prime factorization of n. Then*

$$rb(\mathbb{Z}_n, S) \leq 2(1 - k) + \sum_{i=1}^{k} rb(\mathbb{Z}_{p_i}, S).$$

Proof. Recursively apply Lemma 4.3.4. □

Notice that the upper bound given in Proposition 4.3.5 does not meet the lower bound in Proposition 4.3.2. In particular, if $3^k|n$ for $k \geq 2$, then the upper bound exceeds the lower bound by at least $k-1$. This suggests that Lemma 4.3.4 is too generous when $p = 3$.

To improve the upper bound, we want to focus on the case when $n = 3t$. Suppose that c is an r-coloring of $\mathbb{Z}_n$. Since the goal is to show that c admits a rainbow solution to the Sidon equation, we will assume that c is rainbow Sidon-free and pursue a contradiction. Partition $\mathbb{Z}_n$ into cosets of $\langle t \rangle$ denoted R_i, $1 \leq i \leq t$. By Lemma 4.3.3, there exists an index j such that $|c(R_i) \setminus c(R_j)| \leq 1$ for all $0 \leq i < t$. By shifting the coloring, we can assume that $j = 0$.

Lemma 4.3.6. *If* $|c(R_i) \setminus c(R_0)| = 1$ *and* $|c(R_0)| = 3$*, then* $|c(R_i)| = 1$*.*

Proof. Notice that $R_0 = \{0, t, 2t\}$, while $R_i = \{i, t+i, 2t+i\}$. Without loss of generality, suppose that $c(st+i) \notin c(R_0)$ for some $0 \leq s \leq 2$. For the sake of contradiction, suppose that $c((s \pm 1)t + i) \neq c(st+i)$ (where $(s \pm 1)t + i$ is taken modulo n). By assumption, there exists $s't, (s'+1)t \in R_0$ such that $c((s \pm 1)t + i) \notin \{c(s't), c((s' \pm 1)t)\}$. However, $(s' \pm 1)t + st + i = s't + (s \pm 1)t + i$. Since $\{s't, (s \pm 1)t + i, (s' \pm 1)t, st + i\}$ is rainbow under c, a contradiction. □

Rainbow numbers for the Schur equation $x_1 + x_2 = x_3$ will be useful in analyzing the rainbow number for the Sidon equation when 9 divides n. For convenience, we state the relevant results below. Let $\mathrm{rb}(\mathbb{Z}_n, 1)$ denote the fewest number of colors that guarantee a rainbow solution to $x_1 + x_2 = x_3$ in $\mathbb{Z}_n$.

Theorem 4.3.7 (Theorem 1 in [4]). *For a prime* $p \geq 5$, $rb(\mathbb{Z}_p, 1) = 4$.

Remark 4.3.8. *It can be deduced through inspection that* $rb(\mathbb{Z}_2, 1) = rb(\mathbb{Z}_3, 1) = 3$.

This result is important for us because $\mathrm{rb}(\mathbb{Z}_p, S) = \mathrm{rb}(\mathbb{Z}_p, 1)$ except when $p = 3$. In the case that $p = 3$, $\mathrm{rb}(\mathbb{Z}_p, 1) = \mathrm{rb}(\mathbb{Z}_p, S) - 1$.

Theorem 4.3.9 (Theorem 2 in [4]). *For a positive integer n with prime factorization $n = p_1 \cdot p_2 \cdots p_k$,*

$$rb(\mathbb{Z}_n, 1) = 2(1-k) + \sum_{i=1}^{k} rb(\mathbb{Z}_{p_i}, 1).$$

The following fact is immediate from Theorems 4.3.7 and 4.3.9, and Proposition 4.3.2.

Observation 4.3.10. *If 3 divides n, then $rb(\mathbb{Z}_n, S) \geq 1 + rb(\mathbb{Z}_n, 1)$.*

We use this fact to find solutions to $x_1 + x_2 = x_3$ in the proof of Lemma 4.3.11. To our knowledge, this is the first time in the literature on rainbow solutions to equations in $\mathbb{Z}_n$ where previously known rainbow numbers for different equation are employed in the proof. This method may be useful in proving rainbow numbers for more general 4-term equations.

Lemma 4.3.11. *Let 9 be a divisor of n and $3t = n$. Then*

$$rb(\mathbb{Z}_n, S) \leq rb(\mathbb{Z}_3, S) + rb(\mathbb{Z}_t, S) - 3.$$

Proof. Suppose that c is a rainbow Sidon-free r-coloring of $\mathbb{Z}_n$ where $r = \mathrm{rb}(\mathbb{Z}_3, S) + \mathrm{rb}(\mathbb{Z}_t, S) - 3$. We choose not to evaluate $\mathrm{rb}(\mathbb{Z}_3, S)$ to 4 for conceptual clarity. It is easier to keep track of and compare the number of colors used in the un-evaluated form. Without loss of generality, we assume that $|c(R_i) \setminus c(R_0)| \leq 1$ and that $|c(R_0)| \geq |c(R_i)|$ for all i (otherwise, we can shift the coloring to put ourselves in this position). Let

$$\hat{c}(x) = \begin{cases} i & \{i\} = c(R_x) \setminus c(R_0) \\ \alpha & c(R_x) \subseteq c(R_0) \end{cases}$$

be a coloring of $\mathbb{Z}_t$ (so that α is a color not used by c). Notice that $\hat{c}$ is a rainbow Sidon-free $(\mathrm{rb}(\mathbb{Z}_t, S) - 1)$-coloring of $\mathbb{Z}_t$. This implies that $|c(R_0)| = 3$. By Observation 4.3.10, $\hat{c}$ admits a rainbow solution to $x_1 + x_2 = x_3$ in $\mathbb{Z}_t$. In particular, suppose that $i + j = k$ such that $\{i, j, k\}$ is rainbow under $\hat{c}$. Without loss of generality, $\hat{c}(i) \neq \alpha$. By construction, if $\hat{c}(j), \hat{c}(k) \neq \alpha$, then there exists $s_j t + j$ and $s_k t + k$ such that $c(s_j t + j) = \hat{c}(j)$ and $c(s_k t + k) = \hat{c}(k)$. If either $\hat{c}(j) = \alpha$ or $\hat{c}(k) = \alpha$, then let $s_j t + j$ (resp. $s_k t + k$) be some element in R_j (resp. R_k). Furthermore, there exists $s_0t \in R_0$ such that $c(s_0t) \notin \{c(s_jt+j), c(s_kt+k), \hat{c}(i)\}$. By construction, there exists s_i such that

$$s_kt + k + s_0t - s_jt - j = s_it + i \in R_i.$$

Furthermore, $|c(R_i)| = 1$ by Lemma 4.3.6. In particular, $c(s_it+i) = \hat{c}(i)$ and $\{s_it+i, s_jt+j, s_kt+k, s_0t\}$ is a rainbow solution to the Sidon equation under c. This is a contradiction; therefore, any r-coloring of $\mathbb{Z}_n$ is not rainbow Sidon-free. □

Notice that Lemma 4.3.11 is a refinement of Lemma 4.3.4. This refinement lets us improve the upper bound in Proposition 4.3.5 enough to meet the lower bound in Proposition 4.3.2.

Theorem 4.1.2. *Let $n = p_1 \cdots p_k$ be a prime factorization such that $p_i \le p_j$ whenever $i < j$. Let m be the smallest index such that $p_m \ge 3$ (or $m = k$ if this index does not exist), $f_1 = |\{p_i : p_i \le 3, i \ne m\}|$, and $f_2 = |\{p_i : p_i \ge 5, i \ne m\}|$. Then*

$$rb(\mathbb{Z}_{p_m}, S) + f_1 + 2f_2 = rb(\mathbb{Z}_n, S).$$

Proof. The lower bound is provided by Proposition 4.3.2. Notice that if 9 does not divide n, then the upper bound in Proposition 4.3.5 meets the lower bound. Therefore, assume that 9 divides n.

Let α be the largest integer such that 3^α divides n. To prove the upper bound, iterativly apply Lemma 4.3.11 $\alpha - 1$ times to conclude that

$$\mathrm{rb}(\mathbb{Z}_n, S) \le (\alpha-1)(\mathrm{rb}(\mathbb{Z}_3, S) - 3) + \mathrm{rb}(\mathbb{Z}_{n/3^{\alpha-1}}, S).$$

By Proposition 4.3.5,

$$\mathrm{rb}(\mathbb{Z}_n, S) \le (\alpha-1)(\mathrm{rb}(\mathbb{Z}_3, S) - 3) + \mathrm{rb}(\mathbb{Z}_m, S) + 2(-k+\alpha) + \sum_{\substack{p_i \ne 3 \\ i \ne m}} \mathrm{rb}(\mathbb{Z}_{p_i}, S).$$

Let β be the largest integer such that 2^β divides n. Notice that $\alpha + \beta - 1 = f_1$. By regrouping terms and evaluating $\mathrm{rb}(\mathbb{Z}_{p_i}, S)$,

$$\begin{aligned}\mathrm{rb}(\mathbb{Z}_n, S) &\le \mathrm{rb}(\mathbb{Z}_m, S) + (\alpha-1)(\mathrm{rb}(\mathbb{Z}_3, S) - 3) + \beta(\mathrm{rb}(\mathbb{Z}_2, S) - 2) + \sum_{\substack{p_i \ne 2,3 \\ i \ne m}} (\mathrm{rb}(\mathbb{Z}_{p_i}, S) - 2) \\ &= \mathrm{rb}(\mathbb{Z}_m, S) + f_1 + 2f_2.\end{aligned}$$

This concludes the proof. □

4.4 Rainbow numbers for $[n]$

First, we use the the lower bound construction of $\mathrm{rb}(\mathbb{Z}_n, S)$ to prove our lower bound. Then we prove lemmas that give structure of a rainbow Sidon-free coloring of $[n]$, leading to the proof of the upper bound.

Given a rainbow Sidon-free r-coloring $c : [n] \to [r]$, we say that a color X is *dominant* if for any pair of elements $x, x+1 \in [n]$, either $c(x) = c(x+1)$ or $X \in \{c(x), c(x+1)\}$. More generally, we say a color X is *i-dominant* if for any pair of elements $x, x+i \in [n]$, either $c(x) = c(x+i)$ or $X \in \{c(x), c(x+i)\}$. This definition of i-dominance is analogous to the definition previously given for cyclic groups.

Lemma 4.4.1. *There is a rainbow Sidon-free k-coloring of $[2^{k-1}]$ where every color appears on elements in $[2^{k-2}+1]$.*

Proof. The idea is to start coloring $[2]$ with two distinct colors, say R and B, and iteratively at every stage insert a new color between every two previously colored integers.

Let c be a rainbow Sidon-free k-coloring of $[2^{k-1}]$, and define an $(k+1)$-coloring of $[2^k]$ as follows:

$$\hat{c}(x) = \begin{cases} c(\frac{x+1}{2}) & \text{if } x \text{ is odd} \\ k+1 & \text{otherwise.} \end{cases}$$

Notice that $\hat{c}$ is rainbow Sidon-free. Let $\{x_1, x_2, x_3, x_4\}$ be a Sidon solution in $[2^k]$. If all x_i's are even, then this solution is not rainbow since c is rainbow Sidon-free. If some x_i is odd, then there are at least two integers of the solution are odd and are colored $k+1$; hence the solution is not rainbow.

By construction, $c(1) = R$ and $c(2^{k-1}+1) = B$ and these colors appear uniquely on these two integers. Furthermore, every other color must appear between 1 and $2^{k-1}+1$. Therefore, the claim is proven by induction. □

Lemma 4.4.1 is a slight adjustment of Proposition 4.3.2 and gives a rainbow Sidon-free coloring of $\mathbb{Z}_{2^{k-1}}$ with k colors such that every color appears on the elements $2^{k-2}+1$. This gives the following proposition.

Proposition 4.4.2. *For all $n > 0$, $\lfloor \log_2(n-1) \rfloor + 2 < rb([n], S)$.*

Proof. Choose an integer k such that $2^{k-2}+1 \leq n \leq 2^{k-1}$. This k represents the most colors that Lemma 4.4.1 lets us use to color n elements. Solving for k gives the lower bound. □

In order to prove an upper bound on $\mathrm{rb}([n], S)$, suppose that c is a rainbow Sidon-free k-coloring of $[n]$. Notice that we can restrict ourselves to the smallest sub-interval of $[n]$ that contains all k colors, since any rainbow solution to the Sidon equation contained in a sub-interval of $[n]$ will exist in $[n]$. Furthermore, if $[a, b]$ is the smallest sub-interval of $[n]$ that contains all k colors, then $c(a)$ and $c(b)$ are unique within $[a, b]$. Therefore, without loss of generality, we will assume that $c(1)$ and $c(n)$ are uniquely colored (if this is not the case, we can use $[b-a+1]$ colored by $\hat{c} := c(x+a-1)$). This has a very important consequence given by the next proposition.

Proposition 4.4.3. *Suppose that c is a rainbow Sidon-free k-coloring such that $c(1)$ and $c(n)$ are uniquely colored. Then there is a d-dominant color for every $1 \leq d \leq n-1$. Furthermore, a d-dominant color is uniquely determined by d for $1 \leq d \leq n-2$.*

Proof. Let $1 \leq d < n-1$. Notice that $c(1+d) = c(n-d) = R$. For the sake of contradiction, suppose that there exists $x \in [n]$ such that $c(x) \neq c(x+d)$ and $c(x), c(x+d) \neq R$. In this implies that either $\{x, 1+d, 1, x+d\}$ or $\{x, n, x+d, n-d\}$ is a rainbow solution to the Sidon equation. Therefore, R is d-dominant by definition. Notice that R also uniquely d-dominant.

If $d = n-1$, then both $c(1)$ and $c(n)$ are trivially $(n-1)$-dominant. □

We will maintain the assumption that $c(1)$ and $c(n)$ are uniquely colored. In order to proceed, we will construct two sequences: one of distances $\{d_i\}$ and one of colors $\{X_i\}$. The idea is to keep track of the smallest distance such that a "new" color becomes dominant, and to order colors according to the smallest distance at which they are dominant. Then we will analyze the sequence $\{d_i\}$ to get a lower bound on n depending on the number of colors k. Equivalently, this will give an upper bound on k depending only on n.

Let X_0 be a 1-dominant color and $d_0 = 1$. Let

$$d_i = \min\{|x - y| : c(x) \neq c(y) \text{ and } c(x), c(y) \notin \{X_0, \ldots, X_{i-1}\}\}.$$

Let X_i be a d_i-dominant color. Notice that X_i and d_i are defined for $0 \leq i \leq k-2$. Furthermore, X_i is uniquely determined by d_i for $0 \leq i \leq k-3$, and $d_i < d_j$ whenever $i < j$.

Lemma 4.4.4. *Suppose that c is a rainbow Sidon-free k-coloring such that $c(1)$ and $c(n)$ are uniquely colored. For $0 \leq i \leq k-2$, we have $d_i \geq 2^i$.*

Proof. We will proceed by induction on i. The base case is true by inspection since $d_0 = 1 = 2^0$. By the induction hypothesis, suppose that $d_{i-1} \geq 2^{i-1}$. Notice that by definition of d_i, there exists x such that, without loss of generality, $c(x) = X_i$ and $c(x + d_i) = B \notin \{X_0, \ldots, X_i\}$. Furthermore, $c((x, x + d_i)) \subseteq \{X_0, \ldots, X_{i-1}\}$.

Let $y = x + d_{i-1}$. Since X_{i-1} is d_{i-1}-dominant and $d_{i-1} < d_i$, it follows that $c(y) = X_{i-1}$. By definition of d_{i-1}, every element in $(y - d_{i-1}, y + d_{i-1})$ receives a color in $\{X_0, \ldots, X_{i-1}\}$. In particular, $y - d_{i-1} = x$ and $x + d_i \geq y + d_{i-1}$. Consider,

$$\begin{aligned} d_i &= x - x + d_i \\ &= y - x + x + d_i - y \\ &\geq d_{i-1} + d_{i-1} \\ &\geq 2^i. \end{aligned}$$

Therefore, the claim is proven. □

Lemma 4.4.4 suggests that the "first" $k-1$ colors require 2^{k-2} spaces. In particular, $2^{k-2} + 1 \leq n$.

Theorem 4.1.3. *Any rainbow Sidon-free coloring of* $[n]$ *uses at most* $\lfloor \log_2(n-1) \rfloor + 2$ *colors. In particular,*

$$rb([n], S) = \lfloor \log_2(n-1) \rfloor + 3.$$

Proof. Let c be a rainbow Sidon-free coloring of $[n]$ with k colors. Then there exists a minimum interval $[a, b]$ with $b - a + 1 = n' \leq n$ that contains all k colors. By Lemma 4.4.4,

$$2^{k-2} + 1 \leq d_{k-2} + 1 \leq n' \leq n.$$

Therefore,

$$\begin{aligned} 2^{k-2} + 1 &\leq n \\ 2^{k-2} &\leq n - 1 \\ k - 2 &\leq \lfloor \log_2(n-1) \rfloor \\ k &\leq \lfloor \log_2(n-1) \rfloor + 2. \end{aligned}$$

This gives a matching upper bound for Proposition 4.4.2, and completes the proof. □

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CHAPTER 5. RAINBOW NUMBERS IN TOURNAMENTS

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Abstract

The rainbow number of a oriented graph D with respect to a tournament T is the fewest number of colors k such that any k edge coloring of T admits a rainbow copy of D. The main results in this chapter explore the largest rainbow number of D with respect to tournaments on n vertices. The families that are studied are transitive triangles, directed paths, and directed stars.

5.1 Introduction

Let G denote a *simple graph* with vertex set $V(G)$ and edge set $E(G)$. Additionally $v(G)$ is the number of vertices in G and $e(G)$ is the number of edges in G. Similarly, T will denote a *tournament* (oriented complete graph) with the corresponding notions of $V(T), E(T), v(T)$. An *oriented graph* is a simple graph where each edge is an ordered pair. In particular, there are no loops, and each vertex pair can support at most one edge. Let c denote edge colorings of graphs, tournaments, or oriented graphs depending on context. In this way, c can be though of as a function on the edge set of a graph, but $c(G)$ will also be used to denote the number of colors used on G. Often c will be described according to the graph structure and only explicitly used to count colors.

We will use $\mathrm{rb}(T, D)$ to denote the smallest integer k such that any surjective k-coloring of T admits a rainbow copy of D. By extension, $\mathrm{rb}(n, D)$ is the smallest integer k such that any subjectively k-colored tournament on n vertices contains a rainbow copy of D.

In 2014, Li, Ning, Xu, and Zhang proved the following two theorems:

Theorem 5.1.1 (Theorem 1 in [1]). *Let G be an edge colored graph on n vertices. If*

$$e(G) + c(G) \geq \frac{n(n+1)}{2},$$

then G contains a rainbow triangle.

Let $d^c(v)$ be the number of colors used on the edges incident to v. This is known as the color degree of v.

Theorem 5.1.2 (Theorem 2 in [1]). *Let G be an edge-colored graph on n vertices. If*

$$\sum_{v \in V(G)} d^c(v) \geq \frac{n(n+1)}{2},$$

then G contains a rainbow triangle.

As a corollary, Theorem 5.1.2 confirms a conjecture by Li and Wang which states that an edge-colored graph G contains a rainbow triangle if $d^c(v) \geq \frac{n+1}{2}$ for all vertices $v \in V(G)$ [2].

5.2 Transitive Triangles

The intuition for Theorem 5.1.1 is that rainbow triangles can be suppressed in a graph G by either using few colors, or removing edges. Additionally, the number of colors required to find a rainbow triangle is inversely proportional to the number of edges in a graph. The study of rainbow transitive triangles in edge-colored tournaments is motivated by the the fact that rainbow triangles can be suppressed in simple graphs by removing edges. In the sense that removing an edge can destroy a triangle in a simple graph, flipping the orientation of an oriented edge in a tournament destroys transitive triangles. However, flipping an oriented edge is weaker than removing a simple edge since flipping an oriented edge might also create transitive triangles. Nonetheless, it is intuitive the think that a result analogous to Theorem 5.1.1 should hold for tournaments where instead of considering the number of edges in G.

Notice that if T is a transitive tournament (or oriented graph), then every triangle is transitive as we can apply Theorem 5.1.1 to find a rainbow triangle.

Corollary 5.2.1. *If T_n is a transitive tournament on n vertices, then $rb(T_n, T_3) = n$.*

Proof. The upper bound follows from Theorem 5.1.1. For the lower bound, let $c(xy) = x$ for each $xy \in E(T_n)$. Since every T_3 contains two edges with the same source, c is rainbow T_3-free. Furthermore, $n-1$ vertices of T_n have positive out degree, making c an $n-1$ edge-coloring. □

If a tournament is not transitive, then more colors are required to guarantee a rainbow T_3. In particular, we can blow up vertices of a rainbow T_3-free oriented graph with a rainbow T_3 graph T to obtain a larger rainbow T_3-free oriented graph. For the ease of exposition, assume that D and T are oriented edge colored graphs with disjoint color sets. Let $B = B(D, v, T)$ be a oriented graph on $v(D)$ vertices, where v is replaced by a copy of T such that $xy \in E(B)$ for $x \in T \subset D, y \in D$ if and only if $vx \in E(D)$ and $xy \in E(B)$ for $x \in T \subset D, y \in D$ if and only if $vx \in E(D)$ Furthermore, we let these new edges inherit the their colors from D. Consider the following lemma. To remember this notation, $B(D, v, T)$ can be read as the blow-up operation preformed on graph D at vertex v using graph T.

Lemma 5.2.2. *Suppose that D is a rainbow T_3-free oriented graph with $c(D)$ colors. Then $B(D, v, C_3)$ is rainbow T_3-free and $c(B(D, v, C_3)) = c(D) + 3$, where C_3 is rainbow.*

Proof. Clearly, $B(D, v, C_3)$ has three more colors than D since no colors are lost by blowing up v, and the edges of the C_3 replacing v carry 3 new colors.

Let T be a triangle in $B(D, v, C_3)$. If $T \subseteq V(D)$, then T is not a rainbow transitive triangle by assumption. If $T \subseteq V(C_3)$, then T is not transitive. Therefore, we must only concern ourselves with the cases where T contains at least one vertex in C_3 and at least one vertex in D.

Case 1: Suppose T has two vertices $x, y \in V(C_3)$ and one vertex $z \in V(D)$. Notice that the arcs on x, z and y, z have the same color by construction. Therefore, T is not rainbow.

Case 2: Suppose T has two vertices $x, y \in V(D)$ and one vertex $z \in V(C_3)$. Notice that the arcs on x, z and y, z inherit their color and orientation from the arcs on x, v and y, v. Therefore, T is rainbow T_3 if and only if $\{v, x, y\}$ is a rainbow T_3 in D.

In either case, T is not a rainbow T_3. □

Lemma 5.2.2 creates lower bound constructions for $\mathrm{rb}(n, T_3)$, the fewest number of colors need to guarantee a rainbow T_3 on any n-vertex tournament.

Proposition 5.2.3. *For positive integer n,*

$$rb(n, T_3) > \begin{cases} \frac{3n-3}{2} & n \text{ odd} \\ \frac{3n-4}{2} & n \text{ even} \end{cases}.$$

Proof. For the odd case, start with a single vertex and repeatedly apply Lemma 5.2.2. For the even case, start with a single arc and repeatedly apply Lemma 5.2.2. □

Notice that Lemma 5.2.2 holds in generality.

Lemma 5.2.4. *Let D_1, D_2 be a rainbow T_3-free oriented graphs with $c(D_1)$ and $c(D_2)$ colors respectively. Then $B(D_1, v, D_2)$ is rainbow T_3-free and $c(B(D_1, V, D_2)) \leq c(D_1) + c(D_2)$.*

Proof. Clearly, $B(D_1, v, C_2)$ has $c(D_1) + c(D_2)$ since no colors are lost by blowing up v, and the edges of the D_2 replacing v carry at most $c(D_2)$ new colors.

Let T be a triangle in $B(D_1, v, D_2)$. If $T \subseteq V(D_1)$ or if $T \subseteq V(D_2)$, then T is not a rainbow transitive triangle by assumption. Therefore, we must only concern ourselves with the cases where T contains at least one vertex in D_1 and at least one vertex in D_2.

Case 1: Suppose T has two vertices $x, y \in V(D_2)$ and one vertex $z \in V(D_1)$. Notice that the arcs on x, z and y, z have the same color by construction. Therefore, T is not rainbow.

Case 2: Suppose T has two vertices $x, y \in V(D_1)$ and one vertex $z \in V(D_2)$. Notice that the arcs on x, z and y, z inherit their color and orientation from the arcs on x, v and y, v. Therefore, T is rainbow T_3 if and only if $\{v, x, y\}$ is a rainbow T_3 in D_1.

In either case, T is not a rainbow T_3. □

To make use of Lemma 5.2.4, we would like to find an edge-colored oriented graph or tournament D where $\frac{c(D)}{v(D)-1}$ is as large as possible. In the case of a rainbow C_3, this ratio is $\frac{3}{2}$; this fact determines the leading constant in Proposition 5.2.3.

A simple argument provides an upper bound on $\mathrm{rb}(n, T_3)$. Let $d^s(v)$ denote the number of colors X such that every edge with color X is incident upon v; this is known as the saturated color degree of v, or the number of colors saturated by v. More generally, for $S \subseteq V(D)$, let $d^s(S)$ be the number of colors X such that every edge with color X is incident to a vertex in S.

Proposition 5.2.5. *For positive integer n,*

$$rb(n, T_3) \leq 2n.$$

Proof. Notice that a tournament T on 5 vertices contains a T_3 subgraph and has 10 arcs. Therefore, $2n$ colors suffice to find a rainbow T_3 in T. For the sake of contradiction, suppose that T is a tournament on n vertices that is a vertex minimal counterexample to the proposition. Suppose that there exists a vertex v such that $d^s(v) \leq 2$. Then $c(T - v) \geq 2n - 2$ and T is not a vertex minimal counter example. Therefore, $d^s(v) \geq 3$ for all v. Without loss of generality, by the pigeon hole principle there exist two arcs vx, vy such that $c(vx) \neq c(vy)$ are saturated at v. Then v, x, y is a rainbow T_3, which is a contradiction. □

Up until this point, we are unaware of constructions that beat the lower bound in Proposition 5.2.3. Furthermore, using the idea that $d^s(v) = 2$ for all vertices in a tournament and recursively building a rainbow T_3 tournament seems to produce graphs similar to those as constructed in Proposition 5.2.3. Therefore, we have the following conjecture.

Conjecture 5.2.6. *For positive integer n,*

$$rb(n, T_3) \sim \frac{3n}{2}.$$

In particular, $rb(n, T_3) \leq \frac{3n}{2}$.

To disprove this conjecture, it would suffice to find a tournament T such that $\frac{c(T)}{v(T)-1} > \frac{3}{2}$. The following proposition highlights some structural features of a rainbow T_3-free coloring with $3n/2$ colors.

Proposition 5.2.7. *If T is a vertex minimal rainbow T_3-free tournament on n vertices with at least $\frac{3n}{2}$ colors, then*

1. for every vertex $v \in V(T)$, $d^s(v) = 2$,

2. for every distinct pair $u, v \in V(T)$, $d^s(\{u, v\}) \geq 4$, and

3. for every distinct triple $u, v, w \in V(T)$, $d^s(\{u, v, w\}) \geq 6$.

Proof. **Claim 1:** For every vertex $v \in V(T)$, $d^s(v) = 2$.

Suppose that there exists a vertex $v \in V(T)$ with $d^s(v) \leq 1$. Then

$$c(T - v) \geq \frac{3n}{2} - 1 = \frac{3n-2}{2} > \frac{3(n-1)}{2}.$$

This contradicts the minimality of T, proving claim 1.

Claim 2: For every distinct pair $u, v \in V(T)$, $d^s(\{u, v\}) \geq 4$.

Suppose that there exists a pair $u, v \in V(T)$ with $d^s(\{u, v\}) \leq 3$. Then

$$c(T - u - v) \geq \frac{3n}{2} - 3 = \frac{3n-6}{2} = \frac{3(n-2)}{2}.$$

This contradicts the minimality of T, proving claim 2.

Claim 3: For every distinct triple $u, v, w \in V(T)$, $d^s(\{u, v, w\}) \geq 6$.

This claim follows from solving an integer program minimizing the number of colors saturated by $\{u, v, w\}$ and meeting the conditions of claims 1 and 2.

Let x_i be the number of colors saturated by vertex i and no other vertex. Let y_{ij} is 1 if the edge on $\{i, j\}$ is uniquely colored, and therefore, saturated by both i and j. Let z_{ij} denote the number of colors saturated by $\{i, j\}$ that are not saturated by either i or j. Notice that all these variables are non-negative integers. The objective function is given by

$$x_u + x_v + x_w + y_{uv} + y_{vw} + y_{uw} + z_{uv} + z_{vw} + z_{uw}.$$

The constraints of the integer program are

$$x_u + y_{uv} + y_{uw} = 2$$

$$x_v + y_{uv} + y_{vw} = 2$$

$$x_w + y_{vw} + y_{uw} = 2$$

$$x_u + x_v + z_{uv} + y_{uv} + y_{uw} + y_{vw} \geq 4$$

$$x_v + x_w + z_{vw} + y_{uv} + y_{uw} + y_{vw} \geq 4$$

$$x_u + x_w + z_{uw} + y_{uv} + y_{uw} + y_{vw} \geq 4$$

where $x_u + y_{uv} + y_{uw} = 2$ states that u must saturate 2 colors, and $x_u + x_v + y_{uv} + z_{uv} + z_{uw} + z_{vw} \geq 4$ states that $\{u, v\}$ must saturate at least 4 colors. Solving this integer program shows that the minimum value of the objective function is 6. □

A brute force computer search for rainbow T_3-free tournaments with up to 6 vertices and up to 7 colors gives many examples. It ***seems*** that none of these examples satisfy the structure implied by the previous proposition. This presents some computational support for Conjecture 5.2.6, but is not claimed to be a proof.

5.3 Oriented Paths

Let P_n denote the oriented path on n vertices. Oriented paths are easy to find in tournaments. In fact, every tournament on n vertices has a directed path on n vertices. To see this, define a *median order* of a tournament T to be an ordering of the vertex set $v_1 \leq \cdots \leq v_n$ that maximizes the number of edges $xy \in E(T)$ with $x \leq y$. In other words, a median order of a tournament is an ordering of the vertex set that maximizes the number of "forward" edges. Notice that if $v_1 \leq \cdots \leq v_n$ is a median ordering of T, then $v_1 v_2 \cdots v_n$ is a directed path on n vertices. This suggests that the rainbow number for a directed path in a tournament should depend on the rainbow number of a sufficiently large tournament.

Consider the following generalization of Theorem 5.1.1:

Theorem 5.3.1. *(Xu et al. [4]) If G is an edge-colored graph on n vertices and $n \geq k \geq 4$ such that $e(G) + c(G) \geq \binom{n}{2} + ex(K_{k-1}, n) + 2$, then G contains a rainbow K_k.*

Theorem 5.3.1 is stated for simple graphs and cliques of size k. However, if we let G be a tournament, and then Theorem 5.3.1 gives us the rainbow number for a rainbow tournament on k vertices. This idea can be used to find the rainbow number of directed paths in tournaments.

Proposition 5.3.2. *If $n \geq k \geq 4$, then $rb(n, P_k) = ex(K_{k-1}, n) + 2$.*

Proof. To see the lower bound, partition the vertices of the transitive tournament T_n into $k-2$ parts with consecutive vertices as equally as possible. Each edge between two parts receives a unique color, while all edges within a part receive the same color.

Suppose a tournament T is colored with $\mathrm{ex}(K_{k-1}, n) + 2$ colors. By Theorem 5.3.1, there exists a rainbow tournament T' on k vertices. Inspecting a median order of T' will yield a rainbow P_k. □

5.4 Oriented Stars

Let S_k^+ denote the oriented star with k edges oriented away from the center. Similarly, let S_k^- denote the oriented star with k arcs oriented toward the center. Let $S_{a,b}$ denote a star with a out edges, and b in edges. Additionally, let $d^+(v)$ denote the number of edges oriented away from v, and let $d^-(v)$ denote the number of edges oriented toward v. These two parameters of a vertex are called the in-degree and out-degree, respectively.

Proposition 5.4.1. *For $n \geq 1$ and tournament T on n vertices,*

$$rb(T, S_k^+) = 1 + \sum_{v \in V(T)} \min\{d^+(v), k-1\}.$$

Furthermore,

$$rb(T, S_k^-) = 1 + \sum_{v \in V(T)} \min\{d^-(v), k-1\}.$$

Proof. To see the lower bound, color the out edges at v with $\min\{d^+(v), k-1\}$ colors. Since no vertex sees at least k colors on out edges, this coloring does not admit a rainbow S_k^+.

Suppose that T is colored with $1 + \sum_{v \in V(T)} \min\{d^+(v), k-1\}$ colors. By the pigeon hole principle, there exists a vertex v that sees at least $\min\{d^+(v), k-1\} + 1$ colors on its out edges. Notice that v cannot see $d^+(v) + 1$ colors on its out edges, since v only has $d^+(v)$ out edges. Therefore, $d^+(v) \geq k$ and v is the center of a rainbow S_k^+.

The proof for S_k^- is identical. □

Corollary 5.4.2. *For $n \geq 2k - 1$,*

$$rb(n, S_k^+) = rb(n, S_k^-) = n(k-1) + 1.$$

Notice that the extremal construction for a S_k^+ will have a rainbow S_k^- if there exists a vertex with in-degree at least k. In particular, if $n - 1 \geq k$, then an averaging argument shows that a copy of S_k^- is unavoidable. A natural question to ask in this situation is how many colors do we need to guarantee a rainbow copy of either S_k^- or S_k^+.

Proposition 5.4.3. *For $n \geq 2(k-2) + 3 = r + 2$,*

$$n(k-2) + 3 \leq rb(n, \{S_k^+, S_k^-\}).$$

Proof. Notice that we can orient the edges of a K_r such that $d^+(v) = d^-(v) = k - 2$ for all vertices. Arbitrarily blow up the vertices of K_r to create an r-partite oriented graph on n vertices. Give each edge a unique color, and single out two vertices u, v which are not adjacent. Add the edges from u to v with color R. Add all other non-edges with color B, taking care that $d^+(u) = k - 1$ and $d^-(v) = k - 1$ (note that there exists at least one non-edges by our assumption on n). Notice that $d^+(v), d^-(v) \leq k - 1$ for all vertices, and that we have used $n(k-2) + 2$ colors. □

Another interesting question concerning rainbow stars is to determine $\mathrm{rb}(n, S_{a,b})$. To facilitate these the next proof, let $d^{s+}(v)$ and $d^{s-}(v)$ denote the number of colors saturated by v on out edges and in edges, respectively.

Proposition 5.4.4. *For sufficiently large n,*

$$rb(n, S_{a,b}) \geq h(n) = \frac{(n+a-b)(n+b-a)}{4} + \frac{(n+a-b)(a-1)}{2} + \frac{(n+b-a)(b-1)}{2} + 1.$$

Proof. Let G be a tournament on x vertices with $x(a-1)$ colors and no rainbow S_a^+. Similarly, let H be a tournament on $n-x$ vertices with $(n-x)(b-1)$ colors and no rainbow S_b^-. Notice that G and H exists for sufficiently large n and x by Corollary 5.4.2.

Let T be a tournament on $V(G)\dot{\cup}V(H)$ with edges

$$E(G) \cup E(H) \cup \{(x,y) : x \in V(H), y \in V(G)\}.$$

Let the edges in the copies of G and H inherit their colors from G and H respectively. Furthermore, let edges in $\{(x,y) : x \in V(H), y \in V(G)\}$ be uniquely colored.

Notice that T does not have a rainbow $S_{a,b}$ since vertices in $V(G)$ do not support a rainbow S_a^+ and vertices in $V(H)$ do not support a rainbow S_b^-. Furthermore, T has

$$c(T) = f(x) = x(n-x) + x(a-1) + (n-x)(b-1).$$

Maximizing f with respect to x shows that a maximum is achieved at $x = \frac{n+a-b}{2}$. Notice that $f\left(\frac{n+a-b}{2}\right) = h(n) - 1$. □

The corresponding upper bound is proven using progressive induction. Progressive induction is an method similar to induction that avoids explicitly proving a base case. For convenience, a paraphrased version of the progressive induction lemma that was proven in [3] is stated next.

Lemma 5.4.5 (Lemma of The Progressive Induction in [3])**.** *Let $\mathcal{U}$ be a universe of objects that are partitioned into finite sets $\mathcal{U}_n$ for $n \in \mathbb{N}$. Let $f : \mathcal{U} \to \mathbb{N}$ such that there exists $N \in \mathbb{N}$ so that if $g \in \mathcal{U}_n$ with $n > N$ and $f(g) > 0$, then there exists $g' \in \mathcal{U}_{n'}$ where $n/2 < n' < n$ and $f(g') > f(g)$. Then there exists $N' \in \mathbb{N}$ such that $f(g) = 0$ for all $g \in \mathcal{U}_n$ for $n \geq N'$.*

Intuitively, progressive induction states that if the $\max\{f(g) : g \in U_n\}$ eventually decreases fast enough in n, then eventually f is always 0. In graph theory, $\mathcal{U}_n$ is typically the set of graphs on n vertices. The function $f(g)$ is typically defined to be $x(g) - b(g)$ where x is the graph parameter of interest, and b is the desired upper bound. The proof method involves assuming that $f(g) > 0$ for a large enough graph g. If f does not satisfy the hypothesis of progressive induction for g, then some local structure should become apparent since $f(g - v) \leq f(g)$ for all $v \in V(g)$. Ideally, this local structure leads to a contradiction. This proof technique resembles both induction, and proof by minimal counterexample.

Theorem 5.4.6. *For $a, b \geq 2$, there exists an $N \in \mathbb{N}$ such that*

$$h(n) = \frac{(n+a-b)(n+b-a)}{4} + \frac{(n+a-b)(a-1)}{2} + \frac{(n+b-a)(b-1)}{2} + 1 \geq rb(n, S_{a,b})$$

for all $n \geq N$. In particular, $rb(n, S_{a,b}) = h(n)$ for sufficiently large n.

Proof. This proposition will be proven by way of progressive induction. Let $\mathcal{U}_n$ be the set of edge-colored tournaments on n vertices that are rainbow $S_{a,b}$-free. Let

$$f(T) = \max\{c(T) - h(v(T)), 0\}.$$

Suppose that $T \in \mathcal{U}_n$ with $f(T) > 0$. If there exists a vertex $v \in T$ such that v saturates less than $n/2$ colors, then $f(T - v) > c(T) - n/2 - h(n-1) \geq c(T) - h(n)$. Therefore, either T conforms to the hypothesis of progressive induction, or every vertex in T saturates at least $n/2$ colors.

We will now assume that every vertex of T saturates at least $n/2$ colors, and conclude that $f(T) = 0$. Notice that for every $v \in T$, either $d^{s-}(v) \geq n/2 - a + 1$ or $d^{s+}(v) \geq n/2 - b + 1$. Let $A \subseteq V(T)$ be the set of vertices T such that $d^{s-}(v) \geq n/2 - a + 1$. Let $B \subseteq T$ be the set of vertices of T such that $d^{s+}(v) \geq n/2 - b + 1$. Notice that $A \dot{\cup} B = V(T)$. If $T[A]$ contains a rainbow S_a^+ or $T[B]$ contains a rainbow S_b^-, then we can find a rainbow $S_{a,b}$. Therefore, $T[A]$ and $T[B]$ contain at most $|A|(a-1)$ and $|B|(b-1)$ colors respectively by Proposition 5.4.1. In particular,

$$c(T) \leq c(T[A, B]) + c(T[A]) + c(T[B]) \leq |A||B| + |A|(a-1) + |B|(b-1).$$

Letting $|A| = x$, we have

$$c(T) \leq x(n-x) + x(a-1) + (n-x)(b-1) \leq h(n) - 1.$$

Thus, $f(T) = 0$, which is a contradiction. ∎

5.5 References

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CHAPTER 6. GENERAL CONCLUSION

This thesis explored anti-Ramsey theory in graphs and in cyclic groups. Though graphs and cyclic groups are inherently different structures, some overarching ideas stand out: inductively constructed lower abounds, and reductive upper bounds.

Chapter 2 focused on the conditions on colorings of different graph families that force rainbow matchings of size $2m$ for some parameter m. Chapters 3 and 4 considered the rainbow number for solution sets to equations in cyclic groups. Chapter 5 took the host object to be tournaments on n vertices, and determined the rainbow number for some families of directed graphs.

6.1 Future Work

Though I have taken care to mention conjectures and open problems throughout this work, there is one question I want to draw special attention to now. Let f be a k variable linear function. A general problem is to determine the smallest integer r such that any surjective r-coloring of $[n]$ admits a rainbow solution set to $f = 0$, denoted $\text{rb}([n], f = 0)$. This problem has been considered for the Schur equation and the Sidon equation for sufficiently equitable colorings. For example, it has been shown that if c is a 4-coloring of $[n]$ where each color class has cardinality more than $\frac{n+1}{6}$, then c admits a rainbow solution to the Sidon equation [1]. With this in mind I conjecture the following:

Conjecture 6.1.1. *Let f be a k variable linear function such that each coeffient is either 1 or -1, and there is at least one positive and one negative coefficent. If c is a surjective k-coloring of $[n]$ such that each color class has cardinality more than $\frac{n+1}{2(k-1)}$, then c admits a rainbow solution to $f = 0$.*

In support of this conjecture, I offer a lower bound construction. Let c be a coloring of $[n]$ such that all odd integers receive the same color, and the remaining even integers are assigned to the remaining $k-1$ colors equitably. In this case, each color class has at least roughly $\frac{n}{2(k-1)}$ elements. Furthermore, any rainbow solution set S of $f=0$ must use $k-1$ even elements. Since f is linear, this implies that all elements of S are even, which is a contradiction.

6.2 References

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