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Cospectral constructions and spectral properties of variations of the distance matrix

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Cospectral constructions and spectral properties of variations of the distance matrix

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DEDICATION

I would like to dedicate this work to my incredible parents, Janice and Chris, whose love, support and laughter reaches beyond Pluto.

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ACKNOWLEDGMENTS

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ABSTRACT

A graph is a collection of objects (vertices) and connections between the objects (edges). Graphs can be associated with matrices by assigning matrix entries corresponding to the graph structure. As the graph grows large so does the matrix making it difficult to understand the graph's properties. The spectrum (multi-set of eigenvalues) of a matrix for a graph gives a snapshot of the graph structure independent of labeling. We know not all structural properties are captured by the spectrum by the existence of pairs of graphs that share a spectrum (cospectral graphs). In this dissertation, we investigate cospectrality for several graph matrices as well as discuss spectral properties of two recent matrix variants.

CHAPTER 1. GENERAL INTRODUCTION

In the most general terms, graph theory is the study of objects (vertices) and their relations (edges). Models of network systems use graphs, such as communication networks, where two communication centers are connected if they can send signals to each other. They also model networks like the internet, where webpages are connected if there is a clickable link from one to another. As these networks grow large, it becomes more difficult and more cumbersome to understand structural information. One way to gain an understanding of large graphs is by linear algebra tools; this requires a way to describe graphs with matrices.

Graphs can be associated with matrices by assigning matrix entries corresponding to the graph structure. For example, assign 1 in the i, j entry if there is an edge between vertex i and vertex j and 0 otherwise (this gives the adjacency matrix). *Spectral graph theory* is the study of relating the *spectrum* (multi-set of eigenvalues) of a matrix to structural properties of a graph.

1.1 Notation and Definitions

1.1.1 Graph Theory

Let $G = (V, E)$ be graph with a vertex set V and edge set E. We define an edge to be a two element subset of the vertices: $e = \{u, v\} = uv$. Two vertices are *adjacent*, denoted $u \sim v$, if $uv \in E$. A vertex and an edge are *incident* if v is in the edge e. The *neighborhood* of v, denoted $N(v)$, is the set of vertices adjacent to v. The *degree* of a vertex, denoted $\deg(v)$, is the number of edges incident to v. Note $|N(v)| = \deg(v)$. A graph is called $k-$*regular* if for all vertices $v \in V$, $\deg(v) = k$ for some constant k.

A *path* of length k is a sequence of distinct vertices $(v_1, \ldots, v_{k+1})$ such that $v_i \sim v_{i+1}$ for $1 \leq i \leq k$. If vertices are allowed to repeat, then the sequence is called a *walk*. A *cycle* of length k

is a sequence of vertices $(v_1, \ldots, v_{k+1})$ such that $v_i \sim v_{i+1}$ where $v_1 = v_{k+1}$ and all other vertices are unique. Note that the length of a path, walk, and cycle is counted by the number of edges.

The *distance*, denoted $\text{dist}_G(u, v)$, between two vertices in a graph G is the length of the shortest path. If no such path exists then we say $\text{dist}_G(u, v) = \infty$. If context is clear, we will write $\text{dist}(u, v)$. The *transmission* of a vertex is defined as $\text{tr}(v) = \sum_{u \in V} \text{dist}(u, v)$. A graph is *transmission regular* if for all vertices $\text{tr}(v) = c$ for some constant c.

A graph is *connected* if for every two vertices there exists a path connecting them. If a graph is not connected, then we can consider the connected components of a graph.

There are times when we want to consider a subgraph of a graph G. A *subgraph* G' of G is a graph which has vertex set $V' \subseteq V$ and edge set $E' \subseteq E \cap \binom{V'}{2}$ (a subset of edges that only contain vertices in V'). When $E' = E \cap \binom{V'}{2}$, we say that the subgraph, denoted $G[V']$, is *induced*.

We say that a graph is bipartite if $V = A \cup B$ and $E \subseteq \{(a, b) | a \in A, b \in B\}$. In other words, all the edges go between the parts A and B. An equivalent definition is that there are no odd cycles.

A graph has an *equitable partition* if the vertices can be partitioned into sets $C_1, \ldots, C_k$ such that the induced graph on C_i is a regular graph and for $v \in C_i$, $|N(v_i) \cap C_j| = n_{ij}$ for all i, j. In other words, each vertex of C_i has the same number of adjacent vertices in each C_j. For example, consider the Petersen Graph given in Figure 1.1 which can be partitioned where $C_1 = \{v_1\}$, $C_2 = \{v_2, v_5, v_6\}$, and $C_3 = \{v_3, v_4, v_7, v_8, v_9, v_{10}\}$. This forms an equitable partition with $n_{11} = 0$, $n_{12} = 1$, $n_{13} = 0$, $n_{21} = 1$, $n_{22} = 0$, $n_{23} = 2$, $n_{31} = 0$, $n_{32} = 1$, and $n_{33} = 2$.

The graph on n vertices with all possible edges is called the complete graph, denoted K_n. The bipartite graph with all possible edges between the parts A, B is called the complete bipartite graph, denoted $K_{|A|,|B|}$. A connected graph without a cycle is called a *tree*. A graph where each component is a tree is called a *forest*. For other terms and results about graphs undefined here, see [25].

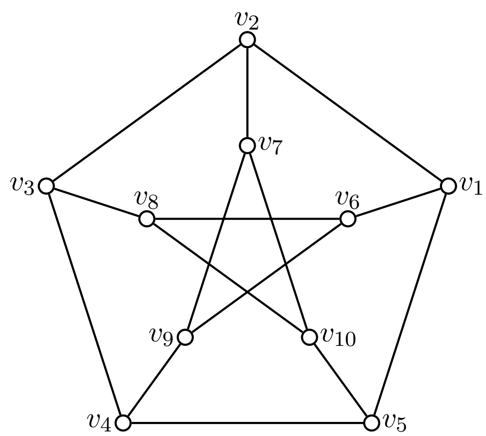


Figure 1.1 The Petersen Graph is a 3−regular graph on 10 vertices.

1.1.2 Linear Algebra

A $n \times m$ *matrix* M is an arrangement of numbers into n rows and m columns. We denote the entry in the ith row and jth column as $M_{i,j}$. A matrix is a square matrix if $n = m$.

A *submatrix* of M is a matrix formed by taking a subset of the rows and columns of M. We write $M[\alpha|\beta]$ to indicate the submatrix of M using rows in the set α and columns in the set β. We write $M(\alpha|\beta)$ to indicate the submatrix of M not using rows in the set α and not using columns in the set β. A *principal submatrix* is a square submatrix in which the subset of rows and columns are the same.

For a $n \times n$ matrix M, the *determinant* is a scalar value that can be computed as

$$\det M = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}.$$

The determinant can also be defined recursively using co-factor expansion.

$$\det M = \sum_{i=1}^{n} M_{i,j}(-1)^{i+j} \det(M(i|j))$$

for some arbitrary j in the index.

For a square matrix M, the *characteristic polynomial* is $p_M(x) = \det(xI - M)$. The roots of the characteristic polynomial form a multiset called the *spectrum*, denoted $\operatorname{spec}(M) = \{\lambda_1^{(m_1)}, \ldots, \lambda_k^{(m_k)}\}$ where m_i is the multiplicity of λ_i. The spectrum is also known as the *eigenvalues* of M. If for some vector $\vec{y}$ such that $M\vec{y} = \lambda\vec{y}$, then we say that $(\lambda, \vec{y})$ are an eigenpair where λ is an eigenvalue and $\vec{y}$ is a eigenvector.

Example 1.1.1. *Let* $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. *Consider the vector* $\vec{v} = [1, 1]^T$. *By computation* $M\vec{v} = \vec{v}$. *Therefore,* $\vec{v}$ *is an eigenvector of* M *for the eigenvalue* 1.

The *trace* of a matrix M denoted trace(M), is the sum of the diagonal entries. In addition, the trace is the sum of the eigenvalues.

$$\begin{aligned}\text{trace}(M) &= \sum_{i=1}^{n} M_{i,i} \\ &= \sum_{\lambda_i \in \text{spec}(M)} \lambda_i\end{aligned}$$

A square matrix N is called *diagonal* is all the nonzero entries occur on the diagonal $N = \text{diag}(N_{1,1}, N_{2,2}, \cdots, N_{m,m})$. A matrix is symmetric if $M_{i,j} = M_{j,i}$. A real symmetric matrix M is called *positive semi-definite* if for all $\vec{y}$, $\vec{y}^T M\vec{y} \geq 0$. The eigenvalues of a positive semi-definite matrix are non-negative real numbers. A real symmetric M is called *positive definite* if for all $\vec{y} \neq \vec{0}$, $\vec{y}^T M\vec{y} > 0$. The eigenvalues of a positive semi-definite matrix are positive real numbers.

The *inertia* of a matrix M with real eigenvalues is the tuple inertia$(M) = (n_-, n_0, n_+)$, where n_- is the number of negative eigenvalues, n_0 is the number of eigenvalues which are zero, and n_+ is the number of positive eigenvalues. Note that for any real symmetric matrix M of order m, $n_- + n_0 + n_+ = m$

For other terms and results about linear algebra undefined here, see [21].

1.2 Common Matrices in Spectral Graph Theory

There are many ways to associate a graph with a matrix in spectral graph theory. We will discuss the following matrices in this work, all of which have the rows and columns indexed by vertices.

- **Adjacency matrix:** A is the 0-1 matrix with a 1 in the i, j entry if $i \sim j$ and 0 otherwise.
- **Laplacian matrix:** L is the matrix with the degree of vertex i on diagonal entry i, i and a -1 in the i, j entry if $i \sim j$ and 0 otherwise. This can be written as $L = D - A$ where D is the degree diagonal matrix.
- **Signless Laplacian matrix:** $|L| = D + A$.
- **Normalized Laplacian matrix:** $\mathcal{L} = D^{-1/2}(D - A)D^{-1/2}$ (assuming no isolated vertices).
- **Distance matrix:** $\mathcal{D}$ is the matrix with $\text{dist}(i, j)$ in the i, j entry.
- **Distance Laplacian matrix:** $\mathcal{D}^L = \mathcal{T} - \mathcal{D}$ where $\mathcal{T}$ is the diagonal matrix with the transmissions along the diagonal.
- **Signless Distance Laplacian matrix:** $|\mathcal{D}^L| = \mathcal{T} + \mathcal{D}$ where $\mathcal{T}$ is the diagonal matrix with the transmissions along the diagonal.
- **Exponential distance matrix:** $\mathcal{D}^q$ is the matrix with $q^{\text{dist}(i,j)}$ in the i, j entry for some variable q.

For the distance, distance Laplacian, and signless distance Laplacian matrix, we assume that the graphs are connected. The spectrum of each of these matrices gives different structural information about the graph.

For example, the spectrum of the adjacency matrix does not preserve the number of connected components as the Saltire Pair ($K_{1,4}$ and $K_1 \cup C_4$), shown in Figure 1.2(a), are two cospectral graphs. However, the spectrum of the adjacency matrix does determine the number of edges.

Theorem 1.2.1. *[8] Let λ_i be the eigenvalues of the adjacency matrix of graph G. Then $\sum \lambda_i^2 = 2|E|$.*

On the other hand, the spectrum of the combinatorial Laplacian matrix does preserve the number of connected components of a graph.

Theorem 1.2.2. *[6] Let ζ be the multiplicity of eigenvalue 0 for the Laplacian matrix of graph G. Then G has ζ connected components.*

However, the spectrum of the combinatorial Laplacian does not determine whether a graph is bipartite.

The spectrum of the normalized Laplacian does determine bipartite-ness.

Theorem 1.2.3. *[6] Let G be a graph without isolated vertices and with $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ the eigenvalues of the normalized Laplacian. Then the following are equivalent:*

1. *G is bipartite*

2. *G has i components and $\lambda_j = 2$ for $n - i \leq j \leq n - 1$*

3. *For each i, $2 - \lambda_i = \lambda_{n-1-i}$*

These examples show how structural properties determined by the spectrum differ between matrices. A large area of spectral graph theory is connecting structural properties to spectrum of matrices to exhibit their strengths.

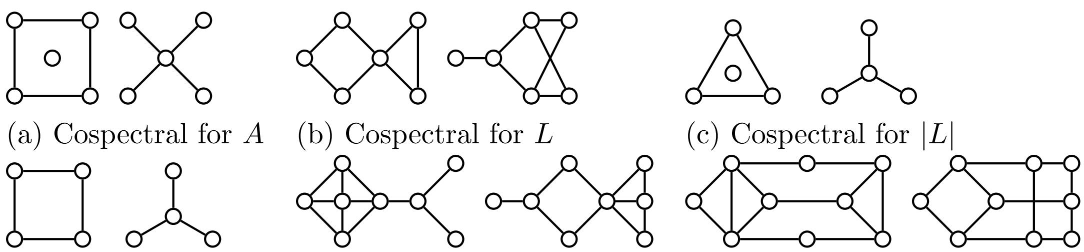
Comprehending these strengths is necessary when using graph matrices in application settings. For example, the distance matrix was originally studied to solve the addressing problem of networks. In an efficient manner, scientists were trying to develop a binary labeling system so that computers could send messages to each other in a network. Using tools of spectral graph theory and graph decompositions, length of this address had to be at least $\max(n_-, n_+)$ (the maximum of number of negative or positive eigenvalues of the distance matrix) bits long.

Understanding the relationship between the structure of a graph and the spectrum of a matrix is a main goal of spectral graph theory. To find connections or strengths of a matrix, we use linear algebra and graph theory. To find weakness of a matrix, we often turn to examples of pairs of graphs with the same spectrum.

1.3 Constructing Cospectral Graphs

Graphs G_1 and G_2 that share the same spectrum with respect to a matrix M, i.e. $\text{spec}_M(G_1) = \text{spec}_M(G_2)$ are called *cospectral*. When G_1 and G_2 are not isomorphic, this is an interesting relationship as it shows some structure not being captured by the spectrum.

Given a matrix M, examples of cospectral graphs are easily found on a small number of vertices by exhaustive search. For example, Figure 1.2 gives pairs of graphs that are cospectral for the adjacency matrix, combinatorial Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix, distance matrix, and distance Laplacian matrix. Additionally, Table 1.1 gives the number of graphs with a cospectral mate for different matrices.



(a) Cospectral for A (b) Cospectral for L (c) Cospectral for $|L|$

(d) Cospectral for $\mathcal{L}$ (e) Cospectral for $\mathcal{D}$ (f) Cospectral for $\mathcal{D}^L$

Figure 1.2 Examples of cospectral graphs for A, L, $|L|$, $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{D}^L$

Pairs of cospectral graphs are viewed two ways. The first is that cospectrality is an algebraic accident. This view is the spectral graph theory equivalent of being struck by lightening. However, as we can observe in Table 1.1, we seem to be struck by lightening quite often. This observation leads us to the second view of pairs of cospectral graphs as patterns of weakness of the matrix.

With this mindset, we can generalize these patterns of cospectrality into cospectral constructions. Cospectral constructions give in-depth reasons why structural information is lost by the spectrum.

For example, Godsil and McKay in [13] proved a cospectral construction that involved a local edge switching operation.

Theorem 1.3.1. *[13] Let G be a graph with a partition $C_1, \ldots, C_m, S$ such that the set of C_i forms an equitable partition and each vertex of S is connected to all, half, or none of the vertices*

Table 1.1 Number of graphs with a cospectral mate for various matrices

#vertices	#graphs	A	$L = D - A$	$\|L\| = D + A$	$\mathcal{L}$	$\mathcal{D}$	$\mathcal{D}^L$
1	1	0	0	0	0	0	0
2	2	0	0	0	0	0	0
3	4	0	0	0	0	0	0
4	11	0	0	2	2	0	0
5	34	2	0	4	4	0	0
6	156	10	4	16	14	0	0
7	1044	110	130	102	52	22	43
8	12346	1722	1767	1201	201	658	745
9	274668	51039	42595	19001	1092	25058	19778

in each C_i. Let H be the graph resulting from each vertex in S that is connected to half of the vertices in C_i switch adjacencies to the other half of C_i. Then G and H are cospectral for the adjacency matrix.

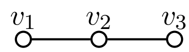
Modifications of this construction method have generated versions for the combinatorial and signless Laplacian matrices. Additionally, this method explains a majority of cospectral graphs on these matrices [19] and has been used to generate many examples of cospectral mates [3, 7, 9, 10, 11, 12, 17, 18, 23, 24].

For almost all well-known graph matrices, there exists cospectral construction methods. More details of construction methods can be found in Chapter 3 which is a survey of cospectral constructions. Developing cospectral construction methods gives us insight into the limitations of graph matrices.

1.4 Spectral Proprieties of the Distance Matrix

Recall that the distance matrix of a graph G, $\mathcal{D}$, has entries $\mathcal{D}_{i,j} = \text{dist}(v_i, v_j)$. For an example of the distance matrix, see Figure 1.3.

The spectral properties of the distance matrix were originally studied for the addressing problem by Graham and Pollak [15]. When setting up the internet and networks of computers sending messages to one another, scientists understood that not every computer would be linked



$$\mathcal{D} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

(a) (b)

Figure 1.3 (a) The path graph on three vertices (b) The distance matrix of the path graph on three vertices.

to every other computer. Instead, information would travel from one computer to the next until the intended recipient received the information. To achieve this, there needed to be a way to efficiently address each computer so that (1) each binary address was unique and (2) messages were efficiently transmitted along the network. This goal was inspired by Hamming codes on the hypercube graphs on 2^n vertices (an example is shown in Figure 1.4) where the vertices are labeled with binary strings of length n such that the labels of two adjacent vertices differ by one bit. Therefore, the distance (and shortest path) between any two vertices is calculated by finding the number of bits different in their labels.

Graham and Pollak [15] used the distance matrix to solve this problem for any graph using graph decompositions. They found that the length of the address needs to be at least $\max(n_-, n_+)$ where n_- (n_+) are the number of negative (positive) eigenvalues. Winkler [26] later showed that the address is at most $n-1$ in length.

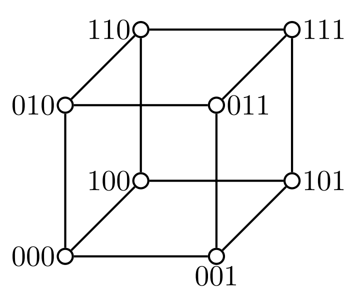


Figure 1.4 The hypercube in three dimensions with Hamming code labeling. Each adjacent vertices have exactly one different bit in their labels.

Since then the spectrum of the distance matrix has been studied. One of the first results about the distance matrix is that for any tree T on n vertices $\det(\mathcal{D}) = (-1)^{n-1}(n-1)2^{n-2}$ [15]. In other words, all trees on n vertices have the same determinant. In this same work, Graham and Pollak found the inertia of trees.

Graham and Lovász [16] found the inverse of the distance matrix of a tree, giving the coefficients of the characteristic polynomial classified in terms of subgraphs. Graham, Hoffman, and Hosoya [14] generalized the determinant result to non-trees by showing that $\det(\mathcal{D}(G))$ is determined by $\det(\mathcal{D}(G_i))$ where G_i's are the blocks of G.

Bapat et al. [2] extended these results to weighted trees and also found the inertia of unicyclic graphs.

Theorem 1.4.1. *[2] Let G be a unicyclic graph with cycle length $2k+1$ and $2k+1+m$ vertices. Then inertia$(\mathcal{D}(G)) = (2k+m, 0, 1)$.*

Theorem 1.4.2. *[2] Let G be a unicyclic graph with cycle length $2k$ and $2k+m$ vertices. Then inertia$(\mathcal{D}(G)) = (k+m, k-1, 1)$.*

The determinant of the distance matrix for other families of graphs shows that the determinant does not preserve many structural properties. Cheng and Lin [5] show that the determinant and inertia is independent of structure for clique-path graphs.

In addition to spectral properties of the distance matrix, the distance matrix has also been studied in terms of cospectrality. The first cospectral construction proved by McKay [22] uses matrix properties to switch out a special subgraph without changing the spectrum of a tree. This results leads to the conclusion that almost all trees have a cospectral mate for the distance matrix.

Heysse [20], provides other more recent cospectral constructions for the distance matrix by use of eigenvector perturbation to construct cospectral graphs with a different number of edges and a local switching method that classifies all distance cospectral graphs on seven vertices.

The distance matrix provides interesting connections between the spectrum and the structure of the graph. Even as a dense matrix (minimal entries are zero), as previous results have shown, it does not detect many structural differences between graphs.

1.5 Thesis Organization

The next three chapters of this thesis are papers that are either submitted or being prepared to submit to journals. Chapter 2 contains the paper "Cospectral constructions for several graph matrices using cousin vertices" submitted for publication in *Special Matrices*. In this paper, I present a new method of constructing cospectral graphs using an expanded definition found in [4] (a paper I was a co-author on and worked on the cospectral construction). Not only does this generalize the cospectral construction found in [4], but it also shows necessary graph conditions so that it creates cospectral graphs for almost all well-studied matrices. This construction is the only one for both the adjacency and distance matrices when $\mathrm{diam}(G) > 2$ and for non-trees. This chapter also contains an unpublished section that gives the algorithms used for the computations presented in the paper.

Chapter 3 contains the paper "Survey of cospectral constructions for several graph matrices" that is being prepared to be submitted for publication. This survey examines standard tools to construct cospectral graphs.

Chapter 4 contains the paper "Spectral properties of variations of the distance matrix" that is being prepared to be submitted for publication. This paper presents results on the inertia of trees, cycles, unicylic, and multipartite graphs for the exponential distance matrix. I also present a cospectral construction where the graphs are cospectral for exactly one value of q. Additionally, I present evidence to the conjecture that trees are spectrally determined for the distance Laplacian matrix. This result extends work presented previously [1] where they made such a conjecture.

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CHAPTER 2. COSPECTRAL CONSTRUCTIONS FOR SEVERAL GRAPH MATRICES

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Modified from a manuscript under review in Special Matrices

2.1 Abstract

Graphs can be associated with a matrix according to some rule and we can find the spectrum of a graph with respect to that matrix. Two graphs are cospectral if they have the same spectrum. Constructions of cospectral graphs help us establish patterns about structural information not preserved by the spectrum. We generalize a construction for cospectral graphs previously given for the distance Laplacian matrix to a larger family of graphs. In addition, we show that with appropriate assumptions this generalized construction extends to the adjacency matrix, combinatorial Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix, and distance matrix. We conclude by enumerating the prevelance of this construction in small graphs for the adjacency matrix, combinatorial Laplacian matrix, and distance Laplacian matrix.

2.2 General Introductions

Let $G = (V, E)$ be a graph. Two vertices $u, v \in V$ are adjacent, denoted $u \sim v$, if $uv \in E$. The neighborhood of v, denoted $N(v)$, is the set of vertices adjacent to v. Let M a matrix associated with the graph. Then the spectrum (multiset of eigenvalues) of M, $\text{spec}_M(G)$, is referred to as the spectrum of G with respect to M. If M is clear, we will say the spectrum of G, denoted $\text{spec}(G)$. If two graphs G_1, G_2 share the same spectrum with respect to M, i.e. $\text{spec}_M(G_1) = \text{spec}_M(G_2)$, then we say that G_1, G_2 are cospectral graphs. When G_1 is not isomorphic to G_2, this is an interesting relationship between the two graphs because it gives us information into what structural properties of a graph are not preserved by M. There are many possible choices for M, and each gives us a different insight into the graph.

Given M, examples of cospectral graphs are (usually) easy to find for a small number of vertices by exhaustive search. To find families of cospectral graphs on a large number of vertices, we need constructions. These constructions help us understand how information about the structural proprieties of a graph are not determined by the spectrum and demonstrate the weaknesses of a matrix.

Cospectral constructions have been studied for the adjacency matrix [6], combinatorial Laplacian [8, 12], signless Laplacian [8], normalized Laplacian [4], distance matrix [9], and to a lesser extent distance Laplacian matrix [2, 3]. The adjacency matrix A has a 1 in the i, j entry if there is an edge between vertices i, j, and 0 otherwise. The combinatorial Laplacian is defined as $L = D - A$, where D is the diagonal matrix with the degrees of the vertices down the diagonal. The signless Laplacian is defined as $|L| = D + A$. The normalized Laplacian is defined as $\mathcal{L} = D^{-1/2} L D^{-1/2}$. These matrices off-diagonal zero-nonzero pattern is that of the adjacency matrix; therefore, we will refer to these matrices as adjacency matrices.

The distance matrix $\mathcal{D}$ has entries $\mathcal{D}_{i,j} = \text{dist}(i, j)$ where the distance is the length of the shortest path between vertex i and vertex j. The distance Laplacian $\mathcal{D}^L = T - \mathcal{D}$ where T is the diagonal matrix with the transmission of the vertices (sum of the distances to a particular vertex) down the diagonal. When dealing with distance matrices, the graph is assumed to be connected.

The most well known cospectral construction for graphs is Godsil-McKay switching for the adjacency matrix [6]. The construction is a special case of Seidel switching. Seidel switching on a graph G with switching set S produces a new graph G' on the vertices of G by keeping edges with both of the endpoints in either S or $G \backslash S$, and adds an edge between vertices in S and $G \backslash S$ if and only if that edge was not in G. An example of this switching is shown in Figure 2.1.

Godsil-McKay switching puts conditions on the graph G and the switching set to create a pair of cospectral graphs. Their proof consists of showing that a matrix $\mathcal{C} = \text{diag}(\frac{1}{k} J - I, I)$ is a similarity matrix for the adjacency matrix of the two cospectral graphs. Haemers and Spence

Figure 2.1 Two graphs that are cospectral for the adjacency matrix by Godsil-McKay switching about $S = \{v\}$.

extended the construction to L and $|L|$ by introducing the GM*-property and its relaxation for graphs [8]. The GM*-property is a set of sufficient graph conditions such that $\mathcal{C}$ is a similarity matrix for matrices of the form $M = \alpha A + \beta I + \gamma D$. The relaxation of the GM*-property relaxes the graph conditions if the matrix has constant row sums. This extension is one of the most well known cospectral construction for the Laplacian and signless Laplacian matrices.

This construction and others for well-studied matrices can be thought of as a *switch* of a set of edges [1, 6, 8, 9, 13]. In this paper, we present a construction that *swaps* a set of edges. The difference between these two concepts is in a switch exactly one endpoint of an edge change while in a swap, both endpoints of an edge change.

A well known construction of cospectral pairs of graphs with no cycles, called trees, uses swapping [10, 11]. The construction identifies special rooted trees T_1, T_2 with root vertices u_1, u_2 that create a pair of cospectral trees when alternately merging u_1, u_2 with a vertex v of another arbitrary graph. In other words, the subtree T_1 is swapped for subtree T_2 to create a cospectral pair of graphs. The construction presented in this paper differs by almost always introducing a cycle to the graph (hence cospectral pair are not trees), and, for several matrices, there are numerous options for subgraphs to be swapped.

Our result broadens a construction for the distance Laplacian matrix and extends it to the distance matrix, matrices of the form $M = \alpha A + \beta I + \gamma D$, and the normalized Laplacian. We do this by demonstrating graph conditions such that $\text{diag}(\frac{1}{k}J - \hat{I}, I)$ is a similarity matrix where $\hat{I}$ is the matrix with ones along the anti-diagonal and zeros elsewhere. Our result is the only known cospectral construction for both distance and adjacency matrices (when the diameter of the graph is greater than two and not a tree).

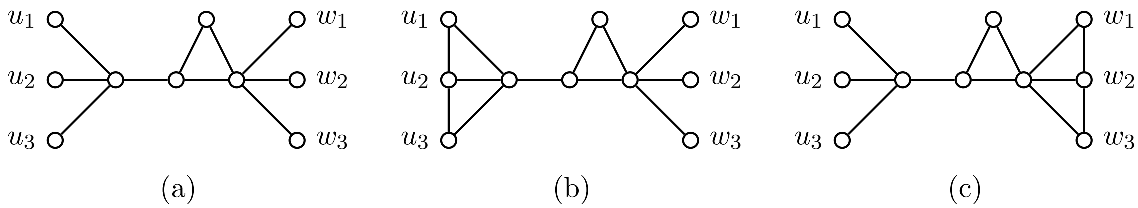

Figure 2.2 (a) A graph with isolated twin set $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$ that are the same size and with degree 1. By adding 2 edges into each set, we produced cospectral graphs ((b) and (c)) for the Laplacian matrix.

This construction exploits well known spectral properties of twin vertices in both distance and adjacency matrices. *(Isolated) Twin vertices* are two vertices in a graph that have the same neighborhood (and thus are not connected). In a distance or adjacency matrix representation of a graph, the columns (and rows) corresponding to twin vertices, v_1, v_2, have the same entries in each row (and column) corresponding to the other vertices in the graph. In other words, $M_{v_1,u} = M_{v_2,u}$ for all $u \in V \backslash \{v_1, v_2\}$. This allows $[1, -1, 0, \cdots, 0]^T$ to be an eigenvector of our matrix. Since these matrices are real symmetric matrices, our remaining eigenvectors can be chosen such that the first two entries are the same.

This leads to a very natural cospectral construction, which is formally given for the combinatorial Laplacian matrix in [5] and the distance Laplacian matrix in [3]. For the combinatorial Laplacian matrix, the construction says that if we have two sets of isolated twin vertices of the same size and have the same degree, then adding k edges into one set is cospectral to adding k edges into the other set. In other words, these k edges can be *swapped* from one twin set to another. An example of this construction is shown in Figure 2.2.

We will show that the twin condition can be relaxed for the combinatorial Laplacian matrix when constructing cospectral graphs and can be broadened to include all adjacency and distance matrices. This relaxation is an extension of a cospectral construction for the distance Laplacian given in [3] and an example is shown in Figure 2.3.

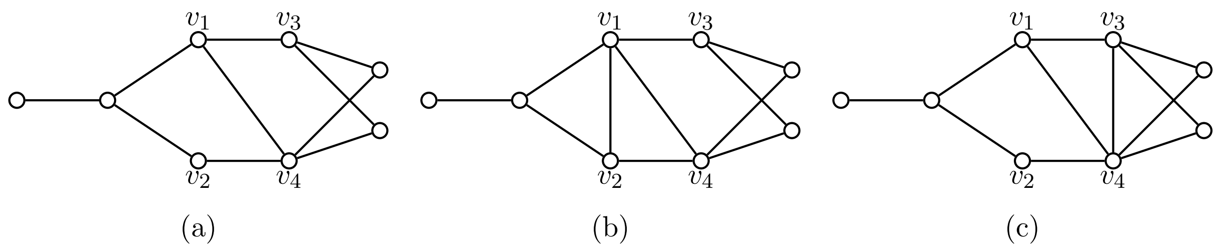

Figure 2.3 (a) A graph G with co-transmission cousin pair $\{v_1, v_2\}, \{v_3, v_4\}$. The graphs shown in (b) and (c) are the two graphs formed by *swapping* edges to create a pair of cospectral graphs using the construction given in [3].

We will describe this construction in Section 2.3 and give generalized definitions of sets of twin vertices called *cousins*. These definitions allow us to show that a matrix $\mathcal{S} = \text{diag}(\frac{1}{k}J - \hat{I}, I)$ is a similarity matrix between graphs where we perform a *swap* on the edges between the cousin vertices.

This construction method demonstrates a loss of information about the graph structure from many different matrix representations. To show this construction, we will first introduce *cousin* vertices and prove some simple results about *gluing* edges within cousins. Then we will prove some linear algebra results about real symmetric matrices to show that our graph construction preserves the spectrum of adjacency and distance matrices. Then, we will enumerate the prevalence of this construction on a small number of vertices for the adjacency matrix, combinatorial Laplacian matrix, and distance Laplacian matrix. Finally, we will conclude with some remaining questions about the frequency for which this construction occurs and complementary cospectral constructions.

2.3 Extension of Cousin Cospectral Construction

The cospectral construction given in [3] which we wish to extend starts with a set of vertices with special proprieties called *cousins*.

Definition 2.3.1. *[3] In a graph G, the set of vertices $C = \{v_1, v_2\} \cup \{v_3, v_4\}$ are called a set of* co-transmission cousins *if*

1. *For all* $u \in V(G)\backslash C$*,* $dist(v_1,u) = dist(v_2,u)$ *and* $dist(v_3,u) = dist(v_4,u)$*;*

2. $\sum_{u\in V(G)\backslash C} dist(v_1,u) = \sum_{u\in V(G)\backslash C} dist(v_3,u)$.

We can think of them as two sets of twin vertices with the same transmission if we ignore adjacencies between our special four vertices. The construction of cospectral graphs for the distance Laplacian *swaps* K_2 with $\overline{K_2}$ between $\{v_1, v_2\}$ and $\{v_3, v_4\}$ with some conditions to create a pair of cospectral graphs. An example of this construction is shown in Figure 2.3.

Our construction extends *swapping* of edges from a pair of two vertex sets to a pair of m vertex sets. Therefore, we will first extend the definition of cousins.

Definition 2.3.2. *In a graph* G*, the pair of sets of vertices* U *and* W *are called* cousins *if*

1. $|U| = |W| = m$*;*

2. $dist(u_i,v) = dist(u_j,v)$ *and* $dist(w_i,v) = dist(w_j,v)$ *for all* $u_i,u_j \in U$*, all* $w_i,w_j \in W$*, and all* $v \in V(G)\backslash\{U \cup W\}$*.*

   Additionally, we would call a pair co-transmission cousins *if*

3. $$\sum_{v\in V(G)\backslash\{U\cup W\}} dist(u_i,v) = \sum_{v\in V(G)\backslash\{U\cup W\}} dist(w_j,v) \text{ for all } u_i \in U \text{ and all } w_j \in W.$$

These definitions about pairs of sets of vertices will be used for our construction for the distance and distance Laplacian matrices. The next definitions about pairs of sets of vertices will be used for our construction for the adjacency, combinatorial Laplacian, and signless Laplacian matrices.

Definition 2.3.3. *In a graph* G*, the pair of sets of vertices* U *and* W *are called* relaxed cousins *if*

1. $|U| = |W| = m$*;*

2. $u_i \sim v$ *if and only if* $u_j \sim v$ *and* $w_i \sim v$ *if and only if* $w_j \sim v$ *for all* $u_i,u_j \in U$*, all* $w_i,w_j \in W$*, and all* $v \in V(G)\backslash\{U \cup W\}$*.*

   Additionally, we would call a pair co-degree (relaxed) cousins *if*

3. $|N(u_i)\backslash W| = |N(w_j)\backslash U|$ *for all* $u_i \in U$ *and all* $w_j \in W$*.*

We will note that a pair of relaxed cousins encompasses a pair of (co-transmission) cousins. Figure 2.4 gives a Venn diagram of the interactions between the definitions. If there are no edges between sets U, W, then these must be a pair of sets of twin vertices.

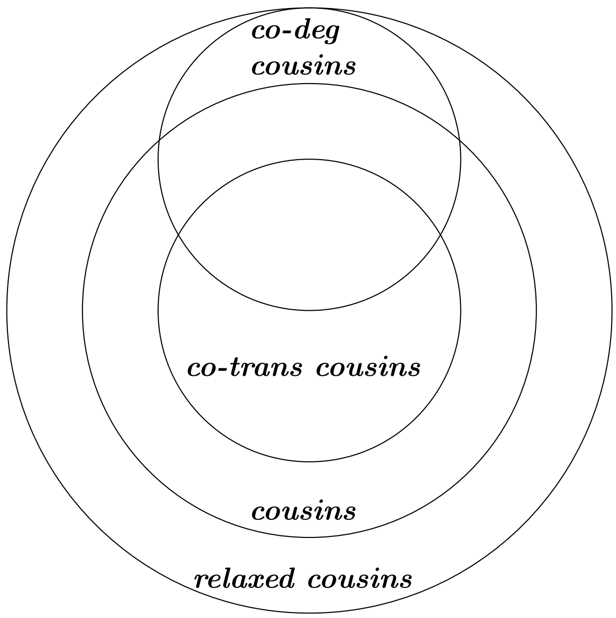

Figure 2.4 A Venn diagram of the definitions for cousin cospectral constructions.

For our cospectral constructions, we will start with a base graph and *glue* in two graphs two different ways creating cospectral graphs. This operation can formally be defined using maps and the following example demonstrates *gluing* by using maps.

Example 2.3.4. *Let* G *be an empty graph on four vertices and let* H *be the complete graph on three vertices. So* $V(G) = \{v_1, v_2, v_3, v_4\}$ *with* $E(G) = \emptyset$ *and* $V(H) = \{u_1, u_2, u_3\}$ *with* $E(H) = \{u_1u_2, u_2u_3, u_3u_1\}$.

Let $\phi : V(H) \to V(G)$ be an injective function such that $\phi(u_i) = v_i$. ϕ will be our gluing *map. A new graph*

$$\begin{aligned} G' &= G + \phi(E(H)) \\ &= G + \phi(\{u_1u_2, u_2u_3, u_3u_1\}) \\ &= G + \{\phi(u_1u_2), \phi(u_2u_3), \phi(u_3u_1)\} \\ &= G + \{v_1v_2, v_2v_3, v_3v_1\} \end{aligned}$$

is the graph K_3 union with an isolated vertex. We would say that G' is G with H glued into G with respect to ϕ.

An interesting property about cousins is that gluing into U or W does not change the length of a shortest path between two vertices u, v where v is not in U or W.

Lemma 2.3.5. *Let G be a graph with cousins V_1, V_2 on m vertices. Let H_1, H_2 be any two graphs on m vertices and $\phi_{i,j}$ be a bijective mapping from H_i to V_j for $i, j \in \{1, 2\}$. Let $G_1 = G + \phi_{1,1}(E(H_1)) + \phi_{2,2}(E(H_2))$ and $G_2 = G + \phi_{2,1}(E(H_2)) + \phi_{1,2}(E(H_1))$.*

Then for all $u \notin V_1 \cup V_2$, $v \in V(G)$, $dist_{G_1}(u, v) = dist_{G_2}(u, v)$.

Proof. Let $u \notin V_1 \cup V_2$ and $v \in V(G)$. If a shortest path in G_1 does not contain an edge with both its endpoints in V_1 nor V_2 and a shortest path in G_2 does not contain an edge with both its endpoints in V_1 nor V_2, then the distances are the same, i.e. $\text{dist}_{G_1}(u, v) = \text{dist}_{G_2}(u, v)$ since G_1 and G_2 only differ by edges that have both of their endpoints in V_1 or V_2.

Let $\mathcal{P}$ be a shortest path in G_1 between u, v that contains, without loss of generality, edge w_1w_2 where $w_1, w_2 \in V_1$. We know for any subpath $\mathcal{P}' \subseteq \mathcal{P}$ that starts at vertex x and ends at vertex y, then $\mathcal{P}'$ is a shortest path between x and y.

This implies that a shortest path between u and w_2 uses edge w_1w_2. Moreover, $\text{dist}(u, w_1) + 1 = \text{dist}(u, w_2)$ but every $u \notin V_1 \cup V_2$ must be equidistant from vertices in V_1 (and V_2) since they are cousins. Therefore, there is no shortest path $\mathcal{P}$ that contains an edge with both of its endpoints in V_1. There is an analogous argument for V_2 and G_2.

Thus the distance between u, v is preserved from G_1 to G_2. □

An additional interesting property of gluing is when we glue into a bipartite graph $G = (A \cup B, E)$ that has an automorphism π where $\pi(A) = B$ and $\pi^2 = id$, then gluing any graph H into A with map ϕ is isomorphic to gluing H into B with map $\pi(\phi)$.

Lemma 2.3.6. *Let $G = (A \cup B, E)$ be a bipartite graph with independent sets A, B on m vertices such that there exists a graph automorphism π where $\pi(A) = B$, $\pi(B) = A$, and $\pi^2 = id$.*

Let H_A and H_B be any two graphs on m vertices, ϕ_A be a bijective mapping from $V(H_A)$ to $V(A)$, and ϕ_B be a bijective mapping from $V(H_B)$ to $V(B)$.

Then $G_1 = G + \phi_A(E(H_A)) + \phi_B(E(H_B))$ is isomorphic to $G_2 = G + \pi(\phi_A(E(H_A))) + \pi(\phi_B(E(H_B)))$.

Proof. Let π be a graph automorphism of $G = A \cup B$ such that $\pi(A) = B$, $\pi(B) = A$, and $\pi^2 = id$. Since π is a graph automorphism of G, then $E(G) = \pi(E(G))$. Consider

$$\begin{aligned}
\pi\left[E(G_1)\right] &= \pi\left[E(G) + \phi_A(E(H_A)) + \phi_B(E(H_B))\right] \\
&= \pi\left[E(G)\right] + \pi\left[\phi_A(E(H_A))\right] + \pi\left[\phi_B(E(H_B))\right] \\
&= E(G) + \pi[\phi_A(E(H_A))] + \pi[\phi_B(E(H_B))] \\
&= E(G_2),
\end{aligned}$$

and

$$\begin{aligned}
\pi\left[E(G_2)\right] &= \pi\left[E(G) + \pi(\phi_A(E(H_A))) + \pi(\phi_B(E(H_B)))\right] \\
&= \pi\left[E(G)\right] + \pi\left[\pi(\phi_A(E(H_A)))\right] + \pi\left[\pi(\phi_B(E(H_B)))\right] \\
&= E(G) + \phi_A(E(H_A)) + \phi_B(E(H_B)) \\
&= E(G_1).
\end{aligned}$$

Therefore π is a graph isomorphism of G_1, G_2. □

In our construction method we will be using this idea of gluing graphs H_A, H_B into U, W in a base graph G two different ways to create a pair of cospectral graphs. If there are initially no edges within U or W, then the induced subgraph of G on vertices U, W, denoted $G[U \cup W]$, is a bipartite graph. As demonstrated in Lemma 2.3.6, in a bipartite graph G, gluing graphs into G in two different ways can create a graph isomorphism between the two new graphs. Therefore, when we glue H_A, H_B in two different ways, the $2m \times 2m$ submatrices representing the vertices of $G[U \cup W]$ are permutation similar. We can always choose our labeling of the two new graphs such that the $2m \times 2m$ submatrix representing the vertices of $G[U \cup W]$ in their corresponding matrices are permutation similar by an anti-diagonal reflection.

2.3.1 Linear Algebra Results

We now discuss some linear algebra tools developed to operate with anti-diagonal reflections. When a $n \times n$ matrix M is reflected along its anti-diagonal, we denote this with ${}^{T}M$ and has (i, j)-entry $m_{n-j+1,n-i+1}$. Let $\widehat{I}$ be the $n \times n$ matrix with ones along the anti-diagonal and zeros elsewhere.

Lemma 2.3.7. *Let M be a matrix in $\mathbb{F}_{n\times n}$. Then ${}^{T}M = \widehat{I}M^{T}\widehat{I}$ and ${}^{T}M$ is similar to M.*

Proof. Consider $x = [x_1, \ldots, x_n]$. Therefore

$$\widehat{I}x^{T} = [x_n, \ldots, x_1]^{T}$$
$$x\widehat{I} = [x_n, \ldots, x_1].$$

In other words, $\widehat{I}$ reverses the order of a column or row vector when multiplied from the right or left respectively. Therefore $\widehat{I}M$ is the matrix M but where each column is in reverse order (horizontal reflection). And $M\widehat{I}$ is the matrix M but where each row is in reverse order (vertical reflection).

Therefore, $\widehat{I}M\widehat{I}$ is the matrix M but with all the columns and rows in reverse order.

$$\widehat{I}M\widehat{I} = \begin{bmatrix} m_{n,n} & \cdots & m_{n,1} \\ \vdots & \ddots & \vdots \\ m_{1,n} & \cdots & m_{1,1} \end{bmatrix}$$

To have M be reflected along its anti-diagonal (thus have the anti-diagonal be stationary), we must take the transpose of $\widehat{I}M\widehat{I}$. Therefore ${}^{T}M = (\widehat{I}M\widehat{I})^T = \widehat{I}M^T\widehat{I}$ since $\widehat{I}$ is its own transpose.

Note that $(\widehat{I})^2 = I$ therefore $\widehat{I}$ is its own inverse. So M^T is similar to ${}^{T}M$ and we know that M is similar to M^T. Thus M is similar to ${}^{T}M$. □

The next result gives an explicit matrix that only reflects a submatrix along its anti-diagonal given some conditions about the initial matrix.

Lemma 2.3.8. *Let* $M = \begin{bmatrix} N & Q \\ Q^T & B \end{bmatrix}$ *be a symmetric* $m \times m$ *matrix with* $2k \times 2k$ *submatrix* N *having constant row sums and every column of* Q *is of the form* $[p, \ldots, p, r, \ldots, r]^T$ *where* p, r *occur* k *times each. Then,* M *is similar to* $M' = \begin{bmatrix} {}^{T}N & Q \\ Q^T & B \end{bmatrix}$.

Proof. Let M be a matrix with the proprieties stated in the hypothesis. Consider the matrix

$$\mathcal{S} = \begin{bmatrix} \frac{1}{k}J_{2k} - \widehat{I_{2k}} & 0 \\ 0 & I_{m-2k} \end{bmatrix} \tag{2.1}$$

where J is the all ones matrix. Observe that $\mathcal{S}$ is its own inverse and transpose. We will show that $\mathcal{S}$ is a similarity matrix for M and M'.

Therefore, we will show

$$\begin{bmatrix} \frac{1}{k}J - \widehat{I} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} N & Q \\ Q^T & B \end{bmatrix} \begin{bmatrix} \frac{1}{k}J - \widehat{I} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} {}^{T}N & Q \\ Q^T & B \end{bmatrix}.$$

First, we will show that $\left(\frac{1}{k}J - \widehat{I}\right) N \left(\frac{1}{k}J - \widehat{I}\right) = {}^{T}N$ and then $\left(\frac{1}{k}J - \widehat{I}\right) Q = Q$.

We know that N is a symmetric $2k \times 2k$ matrix with constant row and column sums equal to α. Additionally, since $\widehat{I}$ reverses the columns or rows of a matrix, it follows that $J\widehat{I} = J$ and $\widehat{I}J = J$. Using these facts and Lemma 2.3.7, consider the following.

$$\left(\frac{1}{k}J-\widehat{I}\right)N\left(\frac{1}{k}J-\widehat{I}\right)=\frac{1}{k^2}JNJ-\frac{1}{k}JN\widehat{I}-\frac{1}{k}\widehat{I}NJ+\widehat{I}N\widehat{I}$$
$$=\frac{2k\alpha}{k^2}J-\frac{\alpha}{k}J-\frac{\alpha}{k}J+\widehat{I}N\widehat{I}$$
$$=\widehat{I}N\widehat{I}$$
$$=\widehat{I}N^T\widehat{I}$$
$$= {}^TN.$$

The columns of Q all have the form $[p,\dots,p,r,\dots,r]^T$ where p,r occur k times each. Therefore

$$(\frac{1}{k}J-\widehat{I})[p,\dots,p,r,\dots,r]^T=\frac{1}{k}J[p,\dots,p,r,\dots,r]^T-\widehat{I}[p,\dots,p,r,\dots,r]^T$$
$$=\frac{1}{k}[k(p+r),\dots,k(p+r)]^T-[r,\dots,r,p,\dots,p]^T$$
$$=[p,\dots,p,r,\dots,r]^T.$$

□

We now are ready to provide graph structural conditions that create cospectral graphs that have $\mathcal{S}$ (as defined in (1)) as a similarity matrix.

2.3.2 Construction Method

The method of creating cospectral graphs for the distance Laplacian given in [3] not only generalizes to larger families of graphs, it also extends to other matrices. This is because the proof used the fact that the distance Laplacian has constant row sums equal to zero. Thus it is very natural to extend it to the combinatorial Laplacian.

The constructions outline necessary graph conditions such that Lemma 2.3.8 can be applied to the matrix. Therefore, our argument will be that the columns of Q are of the appropriate form and that gluing our graphs H_1, H_2 into G in two ways results in the anti-diagonal flip of N.

Recall that $G[U]$ is the induced subgraph of G on the vertices of $U \subseteq V(G)$.

Theorem 2.3.9. *Let G be a graph containing two vertex sets V_1, V_2 each on m vertices such that*

1. *$G[V_1], G[V_2]$ are empty subgraphs;*

2. *there exists a graph automorphism π for $G[V_1 \cup V_2]$ such that $\pi(V_1) = V_2$, $\pi(V_2) = V_1$, and $\pi^2 = id$.*

Let H_1 and H_2 be any two graphs on m vertices and ϕ_i be a bijective mapping from H_i to V_i for $i \in \{1, 2\}$. Let $G_1 = G + \phi_1(E(H_1)) + \phi_2(E(H_2))$ and $G_2 = G + \pi(\phi_1(E(H_1))) + \pi(\phi_2(E(H_2)))$.

- *If V_1 and V_2 are co-transmission cousins, then G_1 and G_2 are cospectral for the distance Laplacian matrix.*

- *If V_1 and V_2 are cousins and $G_1[V_1 \cup V_2]$ is a transmission regular graph, then G_1 and G_2 are cospectral for the distance matrix.*

- *If V_1 and V_2 are co-degree cousins, then G_1 and G_2 are cospectral for the combinatorial Laplacian matrix.*

- *If V_1 and V_2 are co-degree cousins and $G_1[V_1 \cup V_2]$ is a regular graph, then G_1 and G_2 are cospectral for the signless Laplacian matrix.*

- *If V_1 and V_2 are relaxed cousins and $G_1[V_1 \cup V_2]$ is a regular graph, then G_1 and G_2 are cospectral for the adjacency matrix.*

Proof. First consider the cases for the distance matrices where V_1, V_2 are cousins (since co-transmission cousins are a special case of cousins). By Lemma 2.3.5, we know that no shortest path uses an edge with both of its endpoints in V_1 or V_2. Thus no distance between pairs of vertices where at least one is not in V_1 nor V_2 changes.

In the case of relaxed or co-degree cousins, we note that no adjacency changes between a pair of vertices where at least one is not in V_1 nor V_2 when we add edges with both of its endpoints in V_1 or V_2.

So we can partition the respective matrix M of the two graphs G_1, G_2 into

$$\begin{bmatrix} M_1 & Q \\ Q^T & B \end{bmatrix} \textit{ and } \begin{bmatrix} M_2 & Q \\ Q^T & B \end{bmatrix}$$

where M_1, M_2 are $2m \times 2m$ submatrices that are indexed by the vertices in V_1 followed by the vertices in V_2.

We claim that with the appropriate labeling M_1 and M_2 are permutation similar submatrices where the permutation is an anti-diagonal reflection. In other words, ${}^{T}M_1 = M_2$.

Let π be a graph automorphism of $G[V_1 \cup V_2]$ such that $\pi(V_1) = V_2$, $\pi(V_2) = V_1$, and $\pi^2 = id$. We can relabel the vertices of V_2 in G such that $\pi(v_{1,i}) = v_{2,m-i+1}$ and $\pi(v_{2,i}) = v_{1,m-i+1}$ for $v_{1,i} \in V_1$ and $v_{2,i} \in V_2$ since π is an involution. By Lemma 2.3.6 π is a graph isomorphism of $G_1[V_1 \cup V_2], G_2[V_1 \cup V_2]$ and M_1, M_2 are permutation similar matrices. Since we chose π such that $\pi(v_{1,i}) = v_{2,m-i+1}$ and $\pi(v_{2,i}) = v_{1,m-i+1}$, it follows that ${}^{T}M_1$ and M_2 are equivalent.

For each case, we claim that M_1 (and M_2) is a symmetric $2m \times 2m$ matrix with constant row and column sums and the columns of Q all have the form $[p, \ldots, p, r, \ldots, r]^T$ where p, r appear m-times.

- For the distance Laplacian matrix, we know V_1, V_2 are co-transmission cousins which means that Q has constant row sums and has columns of the form $[p, \ldots, p, r, \ldots, r]^T$ where p, r appear m-times. And the distance Laplacian has row sums equal to zero, therefore M_1 has constant row sums.

- For the distance matrix, we know V_1, V_2 are cousins which means that Q has columns of the form $[p, \ldots, p, r, \ldots, r]^T$ where p, r appear m-times. Since $G_1[V_1 \cup V_2]$ is transmission regular, it follows that M_1 has constant row sums.

- For the combinatorial Laplacian matrix, we know V_1, V_2 are co-degree cousins which means that Q has constant row sums and has columns of the form $[p, \ldots, p, r, \ldots, r]^T$ where $p, r \in \{0, -1\}$ appear m-times. And the combinatorial Laplacian has row sums equal to zero, therefore M_1 has constant row sums.

- For the signless Laplacian matrix, we know V_1, V_2 are co-degree cousins which means that Q has constant row sums and has columns of the form $[p, \ldots, p, r, \ldots, r]^T$ where $p, r \in \{0, 1\}$ appear m-times. Since $G_1[V_1 \cup V_2]$ is a regular graph, the sum of the rows using only the non-diagonal entries of M_1 is constant. We know that the diagonal entries of M_1 are the sums of the non diagonal entries of $[M_1, Q]$. This is a constant because the sum of the non-diagonal entries of M_1 is constant and Q has constant row sums. Therefore M_1 has constant diagonal entries and moreover constant row sums.

- For the adjacency matrix, we know V_1, V_2 are relaxed cousins which means that Q has columns of the form $[p, \ldots, p, r, \ldots, r]^T$ where $p, r \in \{0, 1\}$ appear m-times. Since $G_1[V_1 \cup V_2]$ is a regular graph, it follows that M_1 has constant row sums.

Thus by Lemma 2.3.8, $\mathcal{S}$ is a similarity matrix for M of G_1, G_2. Therefore G_1, G_2 are cospectral for M. □

This allows us to create many different cospectral graphs on several matrices. Figure 2.6 gives a Venn diagram of the conditions to be cospectral for a matrix. Next we present some examples of this construction, some will be simultaneously for several matrices.

Example 2.3.10. *Consider the graphs in Figure 2.5. The graph in (a) is a graph G with co-transmission cousins $V_1 = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}\}$ and $V_2 = \{v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}\}$. Let π be a map from $V_1 \cup V_2 \to V_1 \cup V_2$ where $\pi(v_{1,j}) = v_{2,5-j}$ and $\pi(v_{2,j}) = v_{1,5-j}$. For our graph G we can see that this is a graph automorphism for $G[V_1 \cup V_2]$.*

Therefore, by Theorem 2.3.9 we can glue any two graphs on four vertices into V_1, V_2 and then into V_2, V_1 with respect to π to create a pair of cospectral graphs for the distance Laplacian.

In (b) and (c) we have glued the paw graph and K_2 into the cousin sets in two different ways to create cospectral graphs.

Example 2.3.11. *Consider the graphs in Figure 2.7. The graph in (a) is a graph G with co-degree cousins $V_1 = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}, v_{1,6}\}$ and $V_2 = \{v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, v_{2,5}, v_{1,6}\}$. Let π*

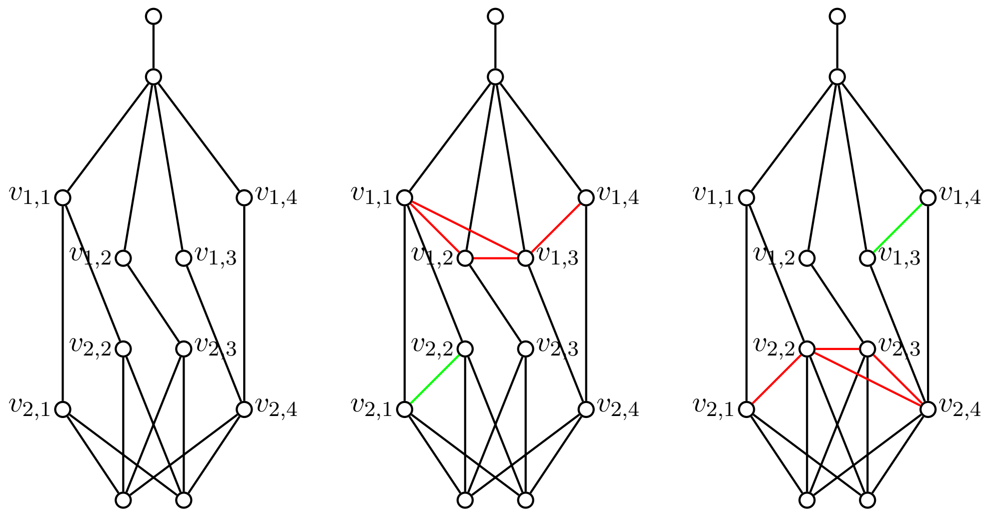

Figure 2.5 (a) A graph G with $V_1 = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}\}$ a set of co-transmission cousins to $V_2 = \{v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}\}$. (b) The graph constructed by G with H_1 (paw graph) glued into V_1 and $H_2 = K_2 + \{v_3, v_4\}$ glued into V_2. (c) The graph constructed by G with H_2 glued into V_1 and H_1 glued into V_2. By Theorem 2.3.9 the graphs show in (b) and (c) are cospectral for the distance Laplacian.

be a map from $V_1 \cup V_2 \to V_1 \cup V_2$ where $\pi(v_{1,j}) = v_{2,7-j}$ and $\pi(v_{2,j}) = v_{1,7-j}$. For our graph G we can see that this a graph automorphism for $G[V_1 \cup V_2]$.

Therefore, by Theorem 2.3.9 we can glue any two graphs on six vertices into V_1, V_2 and then into V_2, V_1 with respect to π to create a pair of cospectral graphs for the combinatorial Laplacian.

Since we glued in two non-isomorphic 3-regular graphs such that $G_1[V_1 \cup V_2]$ in (b) is a regular graph, we also create a pair of cospectral graphs for the signless Laplacian and adjacency matrix. Additionally, since $G_1[V_1 \cup V_2]$ has diameter 2 and is regular, it is also transmission regular. Therefore, this pair of graphs in (b) and (c) are also cospectral for the distance matrix.

We can have pairs of cospectral graphs using Theorem 2.3.9 for the adjacency matrix without being cospectral for the signless Laplacian. Figure 2.8 gives such a pair.

Since the conditions for the signless Laplacian construction are a special case of the conditions for the adjacency matrix and a special case of the combinatorial Laplacian construction (see Figure 2.6), it follows that anytime we have a pair of graphs that are cospectral for the signless

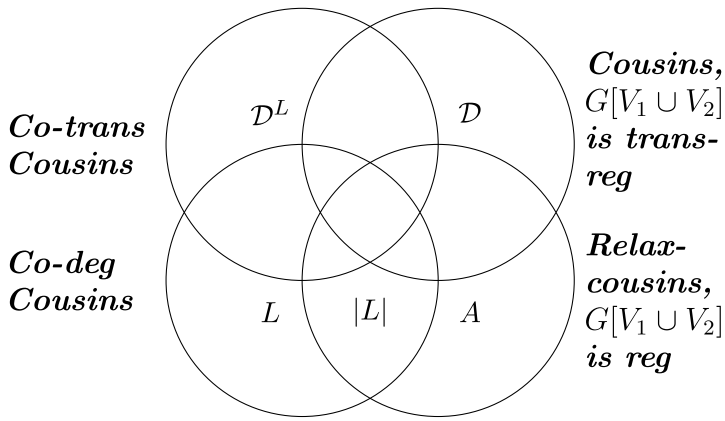

Figure 2.6 A Venn diagram of the conditions on the vertices for Theorem 2.3.9.

Laplacian using Theorem 2.3.9 these graphs are also cospectral for the adjacency matrix and the combinatorial Laplacian.

The adjacency matrix, combinatorial Laplacian, and signless Laplacian can be related using the *generalized characteristic polynomial.* The generalized characteristic polynomial of a graph G, denoted $\phi_G(\lambda, r)$ is the determinant of the matrix

$$N_G(\lambda, r) = \lambda I_n - A_G + rD_G. \tag{2.2}$$

It can be beneficial to use this matrix because we can write our adjacency, combinatorial Laplacian, signless Laplacian, and normalized Laplacian in terms of this matrix. For example if $p_M(\lambda)$ is the characteristic polynomial of M, then

$$\begin{aligned} p_A(\lambda) &= \phi_G(\lambda, 0) \\ p_L(\lambda) &= \phi_G(-\lambda, 1) \\ p_{|L|}(\lambda) &= \phi_G(\lambda, -1) \\ p_{\mathcal{L}}(\lambda) &= \frac{(-1)^{|V|}}{\det(D_G)} \phi_G(0, -\lambda + 1). \end{aligned}$$

Therefore if $\phi_G(\lambda, r) = \phi_H(\lambda, r)$, then graphs G, H are cospectral for the adjacency, Laplacian, signless Laplacian, and normalized Laplacian. We have seen our construction classify pairs of graphs that are cospectral for three of these matrices and now we extend it to the generalized characteristic polynomial.

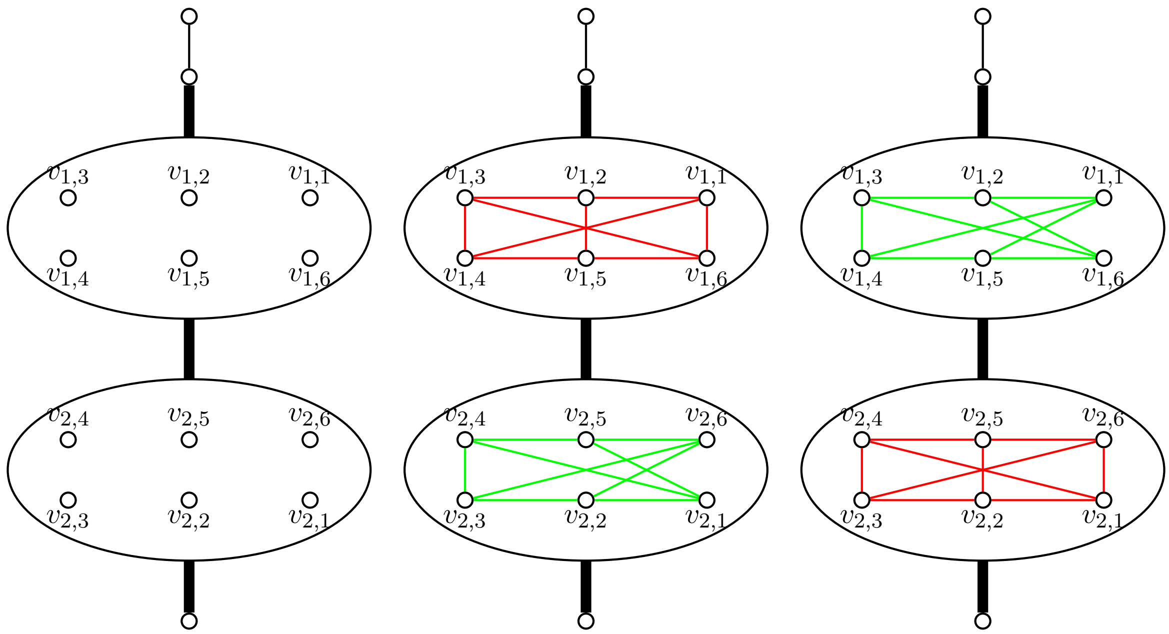

Figure 2.7 Let the thick edges represent all possible edges between a vertex and a cluster of vertices. (a) A graph G with $V_1 = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}, v_{1,6}\}$ and $V_2 = \{v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, v_{2,5}, v_{1,6}\}$ which are (co-degree) cousins. (b) A graph G_1 constructed from G such that $G_1[V_1 \cup V_2]$ is a regular graph. Therefore, it is has the same generalized characteristic polynomial as G_2 shown in (c) by Corollary 2.3.12. Since $G_1[V_1 \cup V_2]$ is a regular graph with $diam = 2$, it is transmission regular. Therefore this pair is cospectral for the distance matrix by Theorem 2.3.9.

Corollary 2.3.12. *Let G be a graph containing two vertex sets V_1, V_2 each on m vertices such that*

1. *$G[V_1], G[V_2]$ are empty subgraphs;*
2. *there exists a graph automorphism π for $G[V_1 \cup V_2]$ such that $\pi(V_1) = V_2$, $\pi(V_2) = V_1$, and $\pi^2 = id$.*

Let H_1 and H_2 be any two graphs on m vertices and ϕ_i be a bijective mapping from H_i to V_i for $i \in \{1, 2\}$. Let $G_1 = G + \phi_1(E(H_1)) + \phi_2(E(H_2))$ and $G_2 = G + \pi(\phi_1(E(H_1))) + \pi(\phi_2(E(H_2)))$.

If V_1 is a set of co-degree cousins to V_2 and $G_1[V_1 \cup V_2]$ is a regular graph, then $\phi_{G_1}(\lambda, r) = \phi_{G_2}(\lambda, r)$ where $\phi(\lambda, r)$ is the generalized characteristic polynomial.

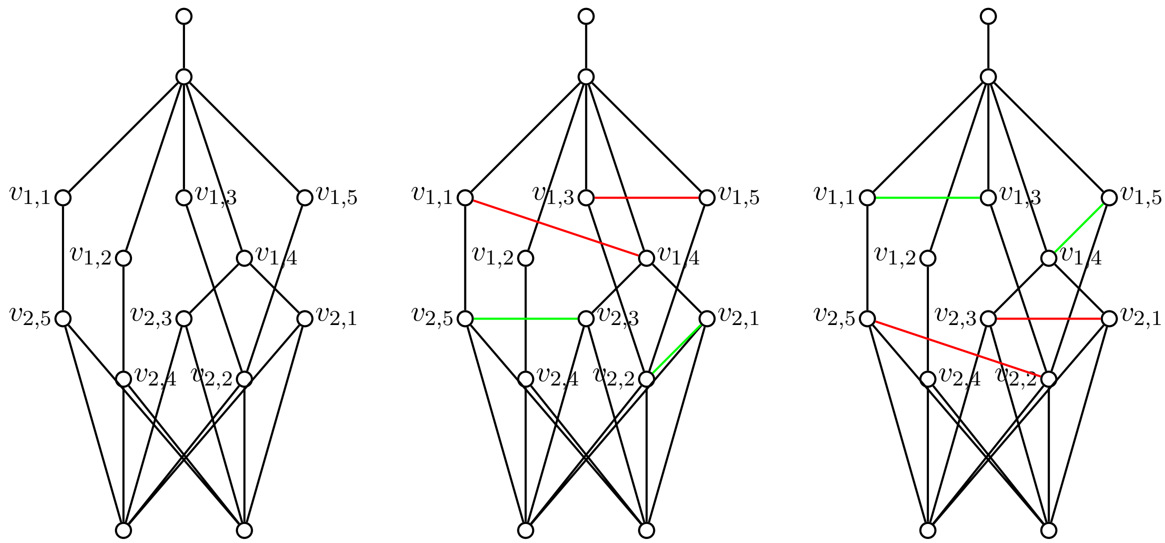

Figure 2.8 (a) A graph G with $V_1 = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{1,5}\}$ a set of relaxed cousins to $V_2 = \{v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, v_{2,5}\}$. In this case, they are also co-transmission cousins. (b) A graph G_1 constructed from G with $H_1 = H_2 = K_2 \cup K_2$ glued into V_1 and V_2. (c) A graph G_2 constructed from G with $H_1 = H_2 = K_2 \cup K_2$ glued into V_1 and V_2. In (b) $G_1[V_1 \cup V_2]$ is a regular graph, therefore by Theorem 2.3.9 (b) and (c) are cospectral for the adjacency matrix and distance Laplacian matrix.

Proof. Since our off-diagonal entries of $N_G(\lambda, r)$ are the off-diagonal entries of A_G, it follows that no adjacency changes between a pair of vertices where at least one is not in V_1 nor V_2 when we add edges with both of its endpoints in V_1 or V_2. Therefore we can partition the respective matrix $N_{G_1}(\lambda, r), N_{G_2}(\lambda, r)$ of the two graphs into

$$\begin{bmatrix} M_1 & Q \\ Q^T & B \end{bmatrix} \textit{and} \begin{bmatrix} M_2 & Q \\ Q^T & B \end{bmatrix}$$

where M_1, M_2 are $2m \times 2m$ submatrices that are indexed by the vertices in V_1 followed by the vertices in V_2.

In an analogous argument in the proof of Theorem 2.3.9, we know that that we can always label the graphs such that ${}^TM_1 = M_2$.

We claim that M_1 (and M_2) is a symmetric $2m \times 2m$ matrix with constant row and column sums and the columns of Q all have the form $[p, \ldots, p, r, \ldots, r]^T$ where p, r appear m-times.

We know that M_1 is $\lambda I_n - A + rD$ restricted to the vertices V_1, V_2 and Q is similarly defined. Since I_n, D are both diagonal matrices, the entries of Q are only from the adjacency matrix of G_1.

We know V_1, V_2 are co-degree cousins which means that Q has constant row sums and has columns of the form $[p, \ldots, p, r, \ldots, r]^T$ where $p, r \in \{0, -1\}$ appear m-times.

Since $G_1[V_1 \cup V_2]$ is a regular graph, the sum of the rows using only the non-diagonal entries of M_1 is constant since these entries are only from the adjacency matrix. We know that the diagonal entries of M_1 are λ plus r times the sums of the non-diagonal entries of $[M_1, Q]$. This is a constant because the sum of the non-diagonal entries of M_1 is constant and Q has constant row sums. Therefore M_1 has constant diagonal entries and moreover constant row sums.

Thus by Lemma 2.3.8, $\mathcal{S}$ is a similarity matrix for $N(\lambda, r)$ of G_1, G_2. Therefore $\phi_{G_1}(\lambda, r) = \phi_{G_2}(\lambda, r)$. □

This allows us to state our construction for the normalized Laplacian matrix since the characteristic polynomial of the normalized Laplacian can be written in terms of the generalized characteristic polynomial.

Corollary 2.3.13. *Let G be a graph containing two vertex sets V_1, V_2 each on m vertices such that*

1. *$G[V_1], G[V_2]$ are empty subgraphs;*
2. *there exists a graph automorphism π for $G[V_1 \cup V_2]$ such that $\pi(V_1) = V_2$, $\pi(V_2) = V_1$, and $\pi^2 = id$.*

Let H_1 and H_2 be any two graphs on m vertices and ϕ_i be a bijective mapping from H_i to V_i for $i \in \{1, 2\}$. Let $G_1 = G + \phi_1(E(H_1)) + \phi_2(E(H_2))$ and $G_2 = G + \pi(\phi_1(E(H_1))) + \pi(\phi_2(E(H_2)))$.

If V_1 is a set of co-degree cousins to V_2 and $G_1[V_1 \cup V_2]$ is a regular graph, then G_1 and G_2 are cospectral for the Normalized Laplacian.

Proof. This follows immediately from Corollary 2.3.12. □

Figure 2.7 gives an example of a graph G and graphs to glue in that meet the hypothesis of Corollary 2.3.12, therefore the graphs given in Figure 2.7 (b) and (c) have the same generalized characteristic polynomial and are cospectral for the normalized Laplacian.

2.4 Enumeration

Examples demonstrate the existence of a construction, while this section helps to show the prevalence of the cospectral construction for several graph matrices. We implement an algorithm for identifying (relaxed) cousin vertices (sets A, B) in a graph. We then find a function from A to B such that the matrix described in (2.1) is a similarity matrix between a graph and its cospectral mate. If no such function is found, then the algorithm returns that the two graphs are not formed using the construction as described in Theorem 2.3.9.

This algorithm was run over cospectral pairs on $7, 8$ vertices for the adjacency matrix, combinatorial Laplacian matrix, and distance Laplacian matrix. The results are tabulated below.

Table 2.1 A table of all graph with a cospectral mate for their respective matrices and graphs whose cospectral mate is found by a cousin cospectral construction.

n	A	L	$\mathcal{D}^L$	Cousin-A	Cousin-L	Cousin-$\mathcal{D}^L$
7	110	130	43	0	80	26
8	1,722	1,767	745	0	922	460

From Table 2.1, we can see that for small number of vertices this construction is not prevalent in the adjacency matrix. This is reasonable as the construction requirements for the adjacency matrix are very restrictive.

For the combinatorial Laplacian the number of graphs that have a cospectral pair using this construction is quite large. Combined with data about the prevalence of Godsil-McKay switching in [8], these two constructions account for more graphs than there are graphs with a cospectral mate! This would suggest that there are some graphs with a cospectral mate explained by more than one cospectral construction method.

2.5 General Conclusions

We have presented an extension of a construction method and applied it to many matrices. A natural question about this construction method is what fraction of cospectral graphs does this explain for each matrix as the number of vertices increases? In addition, there are smaller examples than those shown in this paper for some matrices, but it is unknown if there is a smaller example for the distance matrix. Is there an example of a pair of graphs that are cospectral using this construction for all six matrices discussed here?

There is also some evidence that the graph automorphism π as described in Theorem 2.3.9 does not need $\pi(V_1) = V_2$ and $\pi(V_2) = V_1$ for a similar construction shown in [3]. Finding other conditions or cases when π is some other graph automorphism where the spirit of Theorem 2.3.9 holds is an interesting open problem.

Cospectral constructions have now been shown for adjacency matrices using $\text{diag}(\frac{1}{k}J - I, I)$ ([7, 8]) and $\text{diag}(\frac{1}{k}J - \hat{I}, I)$ (Theorem 2.3.9) as similarity matrices. Since both I and $\hat{I}$ are symmetric permutation matrices, is there a cospectral construction for every symmetric permutation matrix P where $\text{diag}(\frac{1}{k}J - P, I)$ is the similarity matrix? This problem has been studied for orthogonal matrices in [1, 13] where the construction method is switching.

2.6 References

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2.7 Appendix: Enumeration

To enumerate the number of graphs explained by this new construction, we count the number of graph matrices that are similar using the special matrix.

First, we defined a method where given two similar matrices and a partition of the vertices into cousin sets A, B, and other vertices U, we create all possible special similarity matrices by iterating over all possible ways to map A to B. If one of these matrices is a similarity matrix, then we stop. If none of these matrices is a similarity matrix, then it is concluded that this choice of cousins does not meet the structural requirements of our theorem.

This method is described in Algorithm 1.

What this leaves in our main method is to iterate over all possible cousin sets. Since we need to create a pair of cousin sets, we check all subsets of vertices on an even number of vertices. These vertices are sorted into two groups based on common neighborhood of the non-selected

Algorithm 1: `SimilarityMatrixBuilder`$(M1, M2, A, B, U)$

Input: Two similar matrices $M1, M2$, a partition of $M1$ into cousin (ordered) sets A, B and remaining vertices U.

Output: Boolean whether $M1, M2$ have particular matrix as similarity matrix

```
m = |A|
result=0
foreach p ∈ S_m do
    foreach s ∈ S_m do
        C = matrix(QQ, n)
        foreach v ∈ U do
            C[v,v]=1
        foreach (i, j) ∈ ([m], [m]) do
            C[A[p[i]],A[p[j]]]= C[A[p[j]],A[p[i]]]= C[B[s[i]],B[s[j]]]= C[B[s[j]],B[s[i]]]=1/m
            if i=j then
                C[A[p[i]],B[s[i]]]= C[B[s[i]],A[p[i]]]=-1/m
            else
                C[A[p[i]],B[s[j]]]= C[B[s[j]],A[p[i]]]=1/m
        if C * M1 * C = M2 then
            result=1
            return result
return result
```

vertices in the graph. If the two groups are the same size, then the graph with the partition is passed through the similarity matrix check. If the matrix check comes back true, we look to the next pair of graphs. If the matrix check is false, then we continue to find cousin pairs in our current graph. If we have tried all possible cousins then we conclude that this pair of graphs is not explained by our construction and we look to the next pair of graphs.

This method is displayed in Algorithm 2.

Algorithm 2: `isCousinConstruct`$(G1, G2)$

```
Input: A pair of cospectral graphs G1, G2.
Output: Boolean whether G1, G2 can be formed by the cousin construction.
M1 = G1.matrixFunction()
M2 = G2.matrixFunction()
result=0
foreach m ∈ [2, ..., ⌊n/2⌋ + 1] do
    foreach S ⊆ [n], |S| = 2m do
        A = ∅, B = ∅, U = V(G1)\S
        foreach v ∈ S do
            if A is empty then
                add v to A
            else
                a ∈ A
                if M1[v,U]=M1[a,U] then
                    add v to A
                    else if B is empty then
                        add v to B
                    else
                        b ∈ B
                        if M1[v,U]=M1[b,U] then
                            add v to B
        if |A| = |B| = m then
            result=SimilarityMatrixBuilder(M1,M2,A,B,U)
        if result then
            break all
return result
```

Having a common neighborhood of non-selected vertices is a condition of all of the cospectral conditions. For the distance matrix, this condition was extended to the same distance of non-selected vertices but we note that this gives an upper bound of cospectral graphs with this method.

CHAPTER 3. SURVEY OF COSPECTRAL CONSTRUCTIONS

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3.1 Abstract

Graphs can be associated with a matrix according to a specific rule. The spectrum of a graph with respect to a matrix is the set of eigenvalues of the associated matrix. Many structural graph properties are directly related to the spectrum. Two graphs are cospectral if they have the same spectrum for a matrix. Although this phenomenon might initially seem like a coincidence, constructions of cospectral graphs provide extensive insight into the particular matrix's weaknesses. Here we survey constructions of cospectral graph for several graph matrices and highlight standard techniques.

3.2 General Introductions

A graph G is a collection of objects called the vertices, $V(G)$, and the connections between them called the edges, $E(G)$. If two vertices have an edge between them, we call them *adjacent*.

Let G be a graph with an associated matrix representation M according to some rule. The spectrum (multiset of eigenvalues) of M is referred to as the spectrum of G with repsect to M. If two (non-isomorphic) graphs, G_1, G_2, share a spectrum with respect to M then G_1, G_2 are called *cospectral* with respect to M.

There are many ways to represent a graph with a matrix. For example, the *adjacency matrix*, A is the matrix with rows and columns index by the vertices with $A_{i,j} = 1$ if i is adjacent to j and $A_{i,j} = 0$ otherwise. Cospectral graphs helps us understand the weaknesses of a matrix.

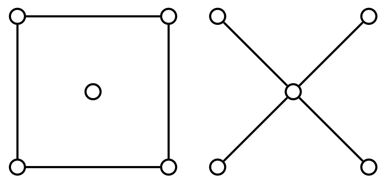

Figure 3.1 Two graphs with spectrum $\{-2, 0^{(3)}, 2\}$ for the adjacency matrix.

For example, consider the graphs pictured in Figure 3.1. These two are called the Saltire pair and are one of the first examples of cospectral graphs for the adjacency matrix found by Cvetković [14]. The Saltire pair have different structural properties. For example, G_1 is a disconnected graph while G_2 is connected; this allows us to conclude that the adjacency matrix spectrum does not always determine the number of connected components.

The existence of a pair of cospectral graphs can be understood as an algebraic accident, the spectral graph theory equivalent of being struck by lightning. This line of thinking is desirable and there are many results finding when a graph is determined by its spectrum (for a survey of results, see [52]). Haemers and Spence [30] conjectured that the spectrum determines most graphs. Additionally, we know that the spectrum stores many structural graph features (for example, the spectrum of the adjacency matrix calculates the number of edges), which furthers the idea that cospectrality is an easily dismissed phenomenon.

Interestingly, we seem to be struck by lightning quite often as there is a nontrivial number of graphs with a cospectral mate (see [7, 29, 30] for an exhaustive search of graphs on a small number of vertices). For connected graphs with no cycles (trees), we know that almost all graphs have a cospectral mate for most matrices [47]. These results lead us to believe that cospectral graphs indicates the weakness of matrices and information lost in the spectrum instead of accidents. Finding constructions bolsters this idea because they patterns of cospectrality and can give insight into the weakness of matrices.

When building constructions, there are two general approaches to take. The first is to investigate how the eigenvalues or characteristic polynomial is related to a graph's features like having a cut vertex or the number of matchings. This approach allows us to determine how the spectrum relates to the graph and some of its blind spots. Once we have the spectrum in terms of these smaller structures of the graph, we can change the small structure in a way that the spectrum does not detect. For example, with graph operations like Cartesian product, we know that the spectrum of the product for some matrices is in terms of the two original base graphs' spectrum. Therefore, we can in some cases swap a cospectral mate of one of the base graphs to create a larger cospectral mate. This method is a linear algebra centered approach to building cospectral graphs and was typical before the widespread use of computers.

The second approach is a graph centered. With computers' aid, we easily generate examples of cospectral graphs on a small number of vertices. In addition, using algorithms, it is easier to visualize patterns in structural changes from one graph to the next in a pair. Examples of this technique include identifying the largest common subgraph on small examples. In general, these observable changes tend to be local edge switching or subgraph swapping. Once we find patterns between graph structural change and cospectrality, the problem becomes finding a linear algebra technique to prove the result. This approach leads to many advances in specialized linear algebra tools and more computational proofs. In general, this approach finds and explains less obvious cospectral constructions.

When constructing cospectral graphs, we are trying to show two matrices with a prescribed structure have the same set of eigenvalues. In general, there are three techniques for showing matrices A, B have the same eigenvalues. (Other general methods for showing two matrices are cospectral can be found in a survey of results in [34].)

1. Find a similarity matrix for A, B
2. Show the characteristic polynomials of A, B are the same
3. Find eigenvectors of B in terms of eigenvectors of A and use them to directly compute eigenvalues.

These techniques all show that two matrices have the same spectrum, so some construction methods are structurally very similar but proof technique-wise different. In general, constructions deal with modifying a small part of the graph or modifying the graph in a more global sense. Many of these construction methods are more suited to specific linear algebra tools.

For example, small changes such as a local edge switching are typically proven with a similarity matrix or eigenvector perturbation. Additionally, more global changes like swapping out a subgraph are typically proven with computing coefficients of the characteristic polynomial.

Cospectral constructions help us understand the limitations of the matrix's spectrum. Additionally, understanding the linear algebra techniques of cospectral constructions helps us understand the scope of the technique. Some construction methods are very specialized, while others apply to many matrices. These ideas can help us as we continue to explore new graph matrices and build on previous work in this field.

Further, the pattern and technique is sometimes dependent on the matrix structure. Some matrices, such as the adjacency matrix, are easy to work with because they are $(0, 1)$-symmetric matrices. The following are the graph matrices we will discuss:

- The *adjacency matrix* A has a 1 in the i, j entry if there is an edge between vertices i, j, and 0 otherwise.

- The *combinatorial Laplacian* (or Laplacian when the context is clear) is defined as $L = D - A$, where D is the diagonal matrix with the degrees of the vertices down the diagonal.

- The *signless Laplacian* is defined as $|L| = D + A$, where D is the diagonal matrix with the degrees of the vertices down the diagonal.

- The *normalized Laplacian*, for graphs without isolated vertices, is defined as $\mathcal{L} = D^{-1/2}LD^{-1/2}$, where D is the diagonal matrix with the degrees of the vertices down the diagonal.

- The $(1,x)-$*adjacency matrix* is defined as $A_x = (x-1)A + J$. It can be thought of as x in the i,j entry if $i \sim j$ and a 1 otherwise as a generalization of the adjacency matrix for some variable x. This matrix has been narrowly examined compared to other matrices discussed here.

- The *distance matrix* $\mathcal{D}$ has entries $\mathcal{D}_{i,j} = \text{dist}(i,j)$ where the distance is the length of the shortest path between vertex i and vertex j. The graph is assumed to be connected.

- The *distance Laplacian matrix* $\mathcal{D}^L = \mathcal{T} - \mathcal{D}$ where $\mathcal{T}$ is the diagonal matrix with the transmission of the vertices (sum of the distances to a particular vertex) down the diagonal and $\mathcal{D}$ is the distance matrix. The graph is assumed to be connected.

Each matrix has a unique set of structural properties that it does not preserve in the spectrum. Figure 3.2 shows several pairs of graphs that are cospectral with respect to different matrices.

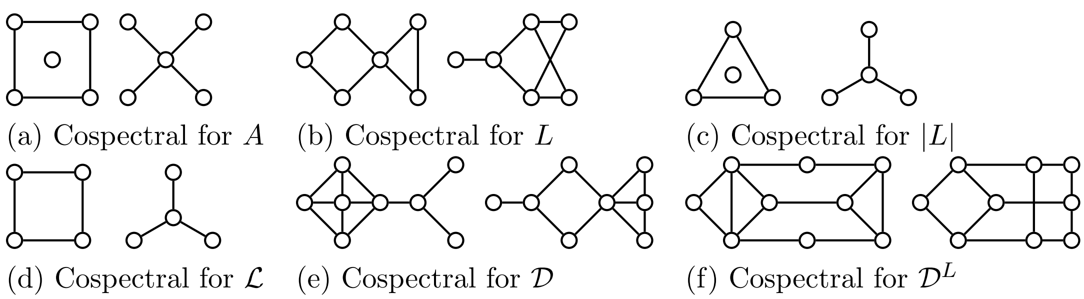

Figure 3.2 Examples of cospectral graphs for A, L, $|L|$, $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{D}^L$

In addition to surveying cospectral construction methods, we look to give insight into why certain methods work for particular matrices. This will give the reader a deeper understanding of cospectral constructions and patterns of cospectrality across many graph matrices. We will start with cospectral constructions that use similarity matrices in their proofs. Then we will discuss using the characteristic polynomial and cycle decompositions to prove cospectrality. Finally, we end on using the eigenvectors to show two graphs have the same spectrum.

3.3 Similarity Matrices

Finding a similarity matrix between two matrices representing a pair of cospectral graphs is a common way to construct cospectral graphs. In general, the constructions give structural properties of graphs such that a special similarity matrix can be applied.

3.3.1 Seidel Switching

The idea of *switching* on a graph is credited to J.J. Seidel. Let G be a graph with a sub-vertex set S. Let the graph G' be the graph on the vertex set of G with edges in G with both or neither endpoint in S, and the edges not in G with exactly one endpoint in S. Then G' is the result of switching G with respect to S. An example of Seidel Switching is shown in Figure 3.3 where $S = \{v\}$. This operation forms an equivalence relation on graphs.

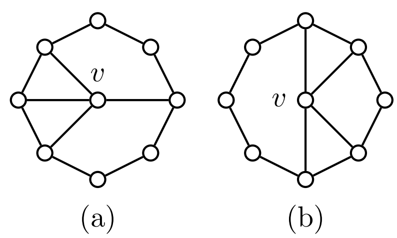

Figure 3.3 Graph (b) is formed from Seidel switching (a) with switching set $S = \{v\}$. These graphs are cospectral using the Godsil-McKay Construction

This graph operation is interesting for spectral graph theorists as it preserves the spectrum of the Seidel matrix (the adjacency matrix of the complement of G minus the adjacency matrix of G). In addition, Seidel [49] showed that two k-regular switching equivalent graphs are cospectral for the adjacency matrix. Additionally, Doob [21] furthers this result by classifying regular graphs that have a non-isomorphic mate obtained by switching.

Others studied Seidel switching on variations of the adjacency matrix. One variation is the $(1, x)$ adjacency matrix studied by Johnson and Newman [37] where $A_x = (x - 1)A + J$. Despite having a variable, the authors showed this matrix still produced pairs of cospectral graphs since if the similarity matrix for A is regular (all rows and columns sum to 1), it will also be a similarity matrix for A_x. Further spectral properties and cospectrality of the $(1, x)$ adjacency matrix are explored in Liu and Wang [40].

Investigating a particular value of x, x_0, we get more interesting results than the general case. For example, van Dam et al. [16] showed that all graphs have a cospectral mate for A_x when $x = 1/2$ by Seidel Switching. Looking at x_0, when $0 < x_0 < 1$, allowed the authors to find examples of A_{x_0} cospectral graphs where one is regular, and the other is not. This result is fascinating because the adjacency spectrum determines regularity. This result is also discussed and expanded by Haemers [26]. Haemers demonstrated how distance regular graphs have a cospectral mate that is not distance regular by constructing an adjacency matrix cospectral to a distance regular graph and directly showing it is not necessarily distance regular.

This graph operation has connections to cospectral graphs for the adjacency matrix, but is limited to regular graphs. For variations, we also see it producing cospectrality for regular graphs or all graphs depending on the variation. Seidel switching's larger legacy is a modification known as Godsil-McKay switching.

3.3.2 Godsil-McKay Switching

Godsil and McKay [25] showed a modification of Seidel switching produced a cospectral mate for the adjacency matrix.

Theorem 3.3.1. *[25] Let G be a graph with equitable partition and switching set S: $\{C_1, \ldots, C_k, S\}$. Let m_i be the size of each C_i. If each vertex of S is connected to all, half, or none of the vertices in each C_i, then G is cospectral with respect to the adjacency matrix to the graph formed from switching between S and each C_i for which it is connected to half the vertices.*

Proof. Let $Q = diag(\frac{1}{m_1} - I, \ldots, \frac{1}{m_k} - I, I)$, then $Q^T A Q = \hat{A}$. □

An example of this construction is shown in Figure 3.3. This construction method has also been proved using partial joins of a graph by Deo et al. [20].

This cospectral construction method is the most well known because of its prevalence in explaining cospectrality, as discussed by Haemers and Spence [30]. Further, it has been used in

special cases to find cospectral graphs with certain graph structure:
[5, 13, 15, 19, 22, 24, 27, 28, 50, 51].

In addition to being a pervasive cospectral construction for the adjacency matrix, with some modifications, Godsil-McKay switching produces cospectral mates for matrices of the form $M = \alpha A + \beta D + \gamma I$, which include the combinatorial Laplacian and signless Laplacian. This result relies on the following linear algebra result.

Theorem 3.3.2. *[30] Let N be a $(0,1)$-matrix of size $b \times c$ whose column sums are $0, b,$ or $b/2$. Define N^* to be the matrix obtained from N by replacing each column with $b/2$ ones with its complement. Let B be a symmtric square matrix with constant row and column sums. Put* $M = \begin{bmatrix} B & N \\ N^T & C \end{bmatrix}$ *and* $M^* = \begin{bmatrix} B & N^* \\ (N^*)^T & C \end{bmatrix}$. *Then M and M^* are cospectral.*

Proof. Define $Q = \begin{bmatrix} \frac{2}{b}J & O \\ O & I \end{bmatrix}$. Then $QMQ^{-1} = M^*$. □

This result extends the Godsil-McKay construction to many variations of the adjacency matrix as well as provides a way to classify graphs. For the adjacency matrix, Theorem 3.3.2 applies when the induced graph on the vertices corresponding to B is regular, and every vertex in the switching set C is connected to all, half, or none of the vertices in the partition. This graph condition is called the GM-property.

If we take our matrix such that the vertices corresponding to B have the same degree in B, we say our graph has the GM*-property. This graph condition is a sufficient condition for all matrices of the form $M = \alpha A + \beta D + \gamma I$, including the combinatorial and signless Laplacian matrices. In the case of the combinatorial Laplacian, we can relax this a bit. Since the submatrix B will have constant row sums, even if it is not regular. As long as the vertices corresponding to B have the same number of neighbors of the vertices corresponding to C, we can apply Theorem 3.3.2 to obtain Godsil-McKay switching for the Laplacian matrix.

Using these ideas, Wang et al. [54] gave graph conditions to be simultaneously cospectral for the adjacency matrix, Laplacian matrix, signless Laplacian matrix, and normalized Laplacian matrix.

There have been analogous developments to Godsil-McKay switching by determining what graph conditions are necessary to use a different similarity matrix. The focus has been on regular rational orthogonal matrices. A matrix is called regular rational orthogonal if $Q^TQ = I$ and $Q\mathbf{1} = \mathbf{1}$. The level of the matrix is the smallest integer such that kQ is integral.

Abiad and Haemers [2] classified all graphs that are cospectral with a regular rational orthogonal matrix of level 2 with the partition parts of size 2-8 for the adjacency matrix. Wang et al. [55] furthered this idea and classified all pairs of cospectral graphs via a regular rational orthogonal matrix of level p where p is an odd prime with one set of switched vertices for the adjacency matrix.

Theorem 3.3.3. *[55] Let G be a graph with partition A_1, A_2 where $|A_1| = 2p$. Let B_1, B_2 be a partition of A_1 into equal parts. If for every vertex v in B_i, $d_v(B_i) - d_v(V \backslash B_i)$ is constant and $v \in A_2$ is adjacent to all the vertices in B_i or the same number of vertices in B_1, B_2, then the graph formed by switching between A_1, A_2 is cospectral with respect to the adjacency matrix.*

Proof. Let $U = I + \frac{1}{p}\begin{bmatrix} -J & J \\ J & -J \end{bmatrix}$. Let $Q = diag(U, I)$. Then $Q^TAQ = \hat{A}$. □

Qiu et al. [46] extended this construction to graphs with multiple partition parts for the adjacency matrix. These results further generalize Godsil-McKay switching.

Godsil-McKay switching is a local edge switching so it does not produce cospectral mates with large structural differences. This construction method produces most known cospectral graphs on a small number of vertices for the adjacency matrix and combinatorial Laplacian matrix. Having a $0, 1$ off diagonal pattern allows the similarity matrices to replicate the switching. In general, this method does not work for the distance matrix or its variations. Except when the diameter of the graph is two, then this construction does produce cospectral graphs for the distance matrix and its variations.

3.3.3 Local Subgraph Swapping

Subgraph swapping and edge switchings are local operations, and through strategic labeling, some cospectral mates fall into multiple constructions. However, other cospectral mates can only be explained with one method. Initially, this looks identical to switching and uses many similar tools to finding switching but is a distinct graph operation.

Dutta and Adhikari [23] describe a subgraph swapping method that uses similarity matrices to show two graphs are cospectral mates for the adjacency matrix. The authors partition the graph into clusters of equal size, creating a pseudo-bipartite graph. They also define a method to swap edges between the clusters.

Definition 3.3.4. *[23] Let G be a graph with clusters $C_i = \{v_{i,1}, \ldots, v_{i,m}\}$. Then G^τ is obtained from G by removing $(v_{i,k}, v_{j,l})$ and adding the edges $(v_{i,l}, v_{j,k})$ for all $k \neq l$, $i \neq j$. This operation is called the graph theoretical partial transpose (GTPT).*

An example of a graph and its GTPT is given in Figure 3.4.

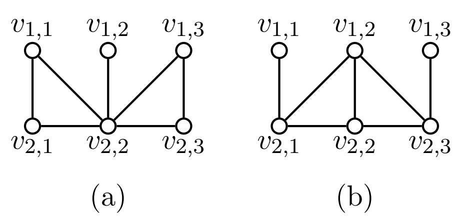

Figure 3.4 (a) A graph G with two clusters (b) GTPT of G, G^τ with swapped edges. G and G^τ are non-isomorphic and cospectral with respect to the adjacency matrix.

This operation is shown to produced cospectral graphs when the original graph G satisfies a commuting and normality condition relating to neighborhoods of each vertex. The proof stems from an observation that the blocks of matrices between the clusters create a family of commuting normal matrices. It is well known that for $\{A_i\}$ of normal commuting matrices, there exists an invertible matrix X such that $A_i^t = X^{-1} A_i X$. This result assembles a larger similarity matrix.

Dutta [22] extends clustering and partial transposes extend to the signless Laplacian. This construction has empirically classified a majority of sparse examples of $|L|$-cospectral graphs.

Another subgraph swapping method using similarity matrices comes from Liu et al. [41] developing a method for swapping out subgraphs that are $k-$cospectrally rooted, which is a generalization of Schwenk's work [47]. Two graphs G_1, G_2 are $k-$cospectrally rooted if they both contain a set of k vertices such that for any graph H with a set of k vertices, identifying G_1 to H at the k vertices is cospectral to identifying G_2 to H at the k vertices. The authors observe that when two graphs with such a property occur, the set of eigenvectors can be chosen in a way such that constructing a similarity matrix is simple. In addition, this allows Liu et al. to construct cospectral graphs that have Hamiltonian cycles.

Thus far, we described constructions involving similarity matrices only for the adjacency matrix, combinatorial Laplacian matrix, signless Laplacian matrix, generalized adjacency matrix, and to a lesser extent, the normalized Laplacian matrix. This circumstance is due to the zero-nonzero pattern being slightly easier to work with. We will now describe a similarity matrix construction for the distance Laplacian, which is the only known construction for the distance Laplacian matrix (with diameter greater than two).

Brimkov et al. [6] defined a graph structure called a vertex cousin set on four vertices (which is a relaxation of vertex twins). They swapped a subgraph within the cousin set to create a non-isomorphic cospectral graph for the distance Laplacian.

Definition 3.3.5. *In a graph G, the pair of sets of vertices U and W are called* cousins *if*

1. $|U| = |W| = m$;

2. $dist(u_i, v) = dist(u_j, v)$ *and* $dist(w_i, v) = dist(w_j, v)$ *for all* $u_i, u_j \in U$, *all* $w_i, w_j \in W$, *and all* $v \in V(G)\backslash\{U \cup W\}$.

   Additionally, we would call a pair co-transmission cousins *if*

3. $\sum_{v \in V(G)\backslash\{U \cup W\}} dist(u_i, v) = \sum_{v \in V(G)\backslash\{U \cup W\}} dist(w_j, v)$ *for all* $u_i \in U$ *and all* $w_j \in W$.

Lorenzen [42] (a version is Chapter 2) extended this result to vertex cousin sets on $2m$ vertices and all other well studied matrices (distance matrix, adjacency matrix, combinatorial Laplacian, signless Laplacian, and normalized Laplacian). The proof relies on a linear algebra result.

Theorem 3.3.6. *Let* $M = \begin{bmatrix} N & Q \\ Q^T & B \end{bmatrix}$ *be a symmetric* $m \times m$ *matrix with* $2k \times 2k$ *submatrix* N *having constant row sums and every column of* Q *is of the form* $[p, \ldots, p, r, \ldots, r]^T$ *where* p, r *occur* k *times each. Then,* M *is similar to* $M' = \begin{bmatrix} {}^T N & Q \\ Q^T & B \end{bmatrix}$.

Proof. Consider the matrix

$$\mathcal{S} = \begin{bmatrix} \frac{1}{k} J_{2k} - \widehat{I_{2k}} & 0 \\ 0 & I_{m-2k} \end{bmatrix}$$

where J is the all ones matrix and $\widehat{I}$ is the matrix with ones along the anti-diagonal and zeros elsewhere. Then $\mathcal{S}^{-1} M \mathcal{S} = M'$. □

Depending on the matrix, certain graph conditions are sufficient to apply this result. For the adjacency matrix, combinatorial Laplacian matrix, signless Laplacian matrix, and normalized Laplacian matrix, the definition of vertex cousins is relaxed since the property is to ensure the columns of Q are in the appropriate configuration.

In general, we have seen methods utilizing a similarity matrix for local graph structure changes. In addition, these methods appear to classify most cospectral graphs. Further, by applying tools from linear algebra we can see the necessary graph conditions in order to apply a method to a particular matrix. This observation makes many of these methods adaptable to multiple matrices.

3.4 Direct Computation of Characteristic Polynomial

For graphs with a high symmetry or structure, the characteristic polynomial is easy to compute directly. Graph properties such as cut vertices allow the characteristic polynomial to be expressed as combinations of the characteristic polynomials of subgraphs. This breaks the characteristic polynomial into its building blocks. By swapping a cospectral mate of one of the building blocks we create larger cospectral graphs.

The characteristic polynomial is found through cycle decomposition (which can be arranged into co-factor expansion). For the adjacency matrix and its variants, the $0, 1$ pattern means that cycles and matchings in the graph correspond to non-zero cycles in the matrix. Harary et al. [31] derived a formula for the characteristic polynomial of the adjacency matrix of a graph in terms of a subgraph with one leaf vertex removed. This was extended by Schwenk [47] to any vertex removed.

Theorem 3.4.1. *[47] Let G be a graph with vertex v and $\mathcal{C}(v)$ be the collection of cycles containing v. Let $\phi(G)$ be the characteristic polynomial of the adjacency matrix of G. Then*

$$\phi(G) = x\phi(G - v) - \sum_{u \sim v} \phi(G - v - u) - 2 \sum_{Z \in \mathcal{C}(v)} \phi(G - Z)$$

The single leaf vertex version of this result (v in included in no cycles and has only one neighbor) exists for almost all well-studied matrices. An interesting application of this to cut vertices gives a significant cospectral construction. Recall that for rooted graphs G, r and H, s, that $G \circ H$ is the graph formed by identifying the two roots r and s.

Corollary 3.4.2. *[47] Let G, H be two rooted graphs with roots v, u. Let $\phi(G)$ $(\phi(H))$ be the characteristic polynomial of the adjacency matrix of G (H). Then*

$$\phi(G \circ H) = \phi(G)\phi(H - u) + \phi(G - v)\phi(H) - \phi(G - v)\phi(H - u)$$

This result leads to natural cospectral construction.

Theorem 3.4.3. *[47] Let G_1, G_2 be a pair of cospectral graphs with roots v_1, v_2 such that $G_1 - v_1$ is also cospectral to $G_2 - v_2$. Then $G_1 \circ H$ is cospectral to $G_2 \circ H$ for any rooted graph H.*

Applying this theorem to trees with the rooted trees in Figure 3.5, Schwenk showed that almost all trees had a cospectral mate for the adjacency matrix. Notice that the two trees are isomorphic and have two different roots. Herndon and Ellzey [33] showed a method for finding two cospectral roots of the same graph using eigenvectors in addition to creating large pairs of cospectrally rooted graphs from smaller ones.

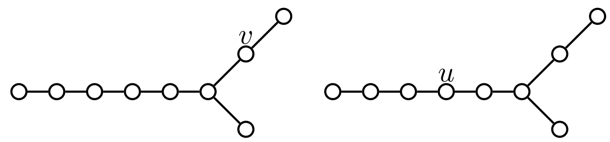

Figure 3.5 A cospectrally rooted tree with roots u, v. For any graph G with root w, the graph resulting from w identified with u is cospectral to the graph resulting from w identified with v for the adjacency matrix.

For the adjacency matrix, the idea of cospectrally rooted graphs expands to *removal-cospectral sets*.

Definition 3.4.4. *[48] Let G be a graph with subset of vertices $S = \{u_1, \dots, u_k\}$ and H be a graph with subset of vertices $T = \{v_1, \dots, v_k\}$ with a correspondence $\theta : S \to T$ by $\theta(u_i) = v_i$. Then S and T are removal-cospectral sets if for every $A \subseteq S$, $\phi(G - A) = \phi(H - \theta(A))$.*

Furthermore, after showing the conditions for three consecutive sizes of A, the rest follows. The proof of this result is by showing, inductively, the number of walks in G of length p is the same as the number of walks in H of length p. This result allows for multiple identifications to occur simultaneously which constructs larger cospectral graphs without cut vertices.

Theorem 3.4.5. *[48] If G_1 and G_2 have removal-cospectral sets of size k, then any graph H with at least k vertices may be attached to G_1 by identifying k of its vertices to S_1 and may be similarly attached to G_2 by identifying the same k vertices to S_2 to form cospectral graphs.*

The proof follows from the number of walks of length p between the vertices in the removal-cospectral sets from G_1 to G_2 is the same. Since if $\mathrm{tr}(A^p_{G_1}) = \mathrm{tr}(A^p_{G_2})$ for all p, then the graphs are cospectral. This technique is quite common for the adjacency matrix when toggling different graphs adjoined to many vertices on and off and is related to computing the adjacency matrix's characteristic polynomial coefficients.

McKay [43] extended the idea of forming larger cospectral graphs using identification to the Laplacian matrix and distance matrix for trees. In addition, the graphs shown in Figure 3.6 are cospectrally rooted for the Laplacian and distance matrix. Therefore, almost all trees have a cospectral mate for most well-studied matrices.

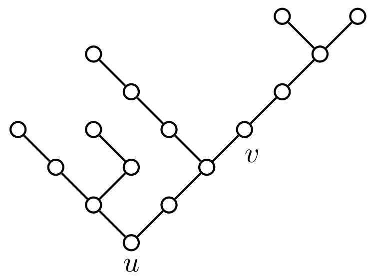

Figure 3.6 Special tree T with vertices v, u. For any graph G with vertex w, the graph resulting from w identified with v is cospectral to the graph resulting from w identified with u for the distance matrix, combinatorial Laplacian matrix, and adjacency matrix.

Using removable-cospectral sets of vertices with certain conditions can create many specialized constructions, some with specific graph properties as shown in [32, 39, 57, 58, 59].

Identification is one graph operation to form cospectral graphs. Other graph products (especially when done with regular graphs) create a high amount of symmetry within the matrix, making it easy to calculate the characteristic polynomial directly (see [3, 4, 17, 38, 16, 53, 56]).

For example, Abiad et al. [1] constructed cospectral graphs for the distance matrix using a graph product. The q-coclique of a graph G, denoted G_q, is the graph with vertex set $V \times \{1, \dots, q\}$ where $(x, i) \sim (y, j)$ if and only if $x \sim y$ in G. The q-clique of a graph G, denoted G_q^+, is the graph with vertex set $V \times \{1, \dots, q\}$ with $(x, i) \sim (y, j)$ if and only if $x \sim y$ in G or $x = y$ and $i \neq j$. Using these graph products, Abiad et al. gave the following construction.

Theorem 3.4.6. *[1] Let G and H be cospectral graphs for the distance matrix. Then G_q and H_q are cospectral graphs for the distance matrix, and G_q^+ and H_q^+ are cospectral for the distance matrix.*

The proof follows from the distance matrix of q-coclique and q-clique graphs are sums, differences, and Kronecker products of the identity matrix, the all ones matrix, and the distance matrices of G and H. The matrix resulting from these operations has a spectrum that is determined by the spectrum of the identity matrix, all ones matrix, and the distance matrix. Therefore, swapping a cospectral mate does not change the spectrum.

These construction methods fall into the larger category of toggling. We can think of toggling as swapping certain subgraphs in a more global sense. For example, for the signless Laplacian, Carvalho et al. [12] used a recursive formula to find the characteristic polynomial of a Cartesian product of a pair of graphs in terms of their original characteristic polynomial. Therefore substituting, or toggling, a cospectral mate for one of the graphs created a cospectral mate to the graph resulting from the Cartesian product. Notice that this swap does not happen on a local set of vertices but in many different parts of the graph.

These constructions show that the characteristic polynomial coefficients remain unchanged using walk counting for the adjacency matrix and other techniques of cycle decompositions. For example, Butler and Heysse [11] showed a method of cospectrality for the normalized Laplacian by toggling small subgraphs or modules chained in a cycle. The three modules are the path on four vertices, the cycle on four vertices, and the edge on two vertices, pictured in Figure 3.7. These modules are pieced together by identifying unlike vertices (v to u) in a cycle.

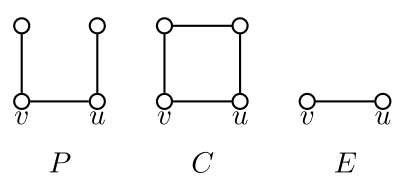

Figure 3.7 Three graph modules: P is the path on four vertices, C is the cycle on four vertices, and E is an edge on two vertices. Let G be a graph formed by identifying unlike vertices of the modules. We can toggle P and C (replace instances of P with C and C with P) to create a cospectral graph for the normalized Laplacian.

Theorem 3.4.7. *[11] Let G be a graph formed by identifying the modules in Figure 3.7 in a cyclic manner. Let H be the graph formed from G be replacing every instance of P with C and C with P. Then G and H are cospectral for the normalized Laplacian.*

The proof looks at decompositions of the graph into edges and cycles (which are the non-zero terms of the permutation definition of the $\det(xI - \mathcal{L})$) and shows that the toggled graph produce equivalent results. Note that this construction can produce cospectral pairs that have a different number of edges.

These constructions allow for large changes in the graph structure. In addition, they almost exclusively work for the adjacency matrix and its variant. For the distance matrix and its variants, almost all permutations produce a non-zero term. Therefore, when translating to the linear algebra we lose graph properties. In trees, the characteristic polynomial of the distance matrix can be manipulated to create a more desirable situation. This construction method by McKay [43] for the distance matrix, does not apply to the distance Laplacian because the submatrix does not correspond to the subgraph (different diagonal entries).

3.5 Manipulation of Eigenvectors

For some graph families with certain structures, their eigenvalues are relatively easy to compute from the eigenvectors. When this occurs we can identify cospectral matrices. For example, matrices of circulant graphs are circulant matrices with easily computed eigenvalues. Mönius [44] used results from number theory to show when two circulant graphs were cospectral with respect to the adjacency matrix.

One graph structure that produces a specific structure of eigenvectors is twin vertices.

Definition 3.5.1. *Let v, u be vertices of graph G such that $N(v) = N(u)$. We say that v, u are vertex twins. If in addition, $v \sim u$, we say that v, u are adjacent twins.*

Vertex twins and extensions to twin subgraphs create easy ways to compute the spectrum with eigenvectors and construct cospectral graphs.

The existence of vertex twins in a graph leads to at least one known eigenvalue. Since twin vertices have the same neighborhood, then for any vertex $w \in V(G)\backslash\{v, u\}$, $\text{dist}(w, v) = \text{dist}(w, u)$. This fact means that in most well-studied matrices, the rows and columns corresponding to v, u have the same entries and therefore have $[1, -1, 0, \ldots, 0]^T$ as an eigenvector where the 1 and -1 entry corresponds to v, u. The corresponding eigenvalue is then in terms of the adjacency or distance between v, u. If the matrix is real symmetric (thus, the eigenvectors are simultaneously orthogonal), then the remaining eigenvectors can be chosen such that the entries corresponding to u, v are the same.

If the matrix is real symmetric, changing the adjacency between twin vertices does not change the eigenvectors. In fact, it only changes the eigenvalue corresponding to the eigenvector $[1, -1, 0, \ldots, 0]^T$. This result naturally extends to (at least) two sets of simultaneously pairwise twin vertices resulting in a cospectral construction. For example, for the combinatorial Laplacian matrix, Das [18] shows if we have two sets of isolated twin vertices of the same size and have the same degree, adding k edges into one set is cospectral to adding k edges into the other set. In other words, these k edges can be *swapped* from one twin set to another. An example of this construction is shown in Figure 3.8. This construction method produces cospectral graphs for the combinatorial Laplacian matrix, the normalized Laplacian matrix, and the distance Laplacian matrix.

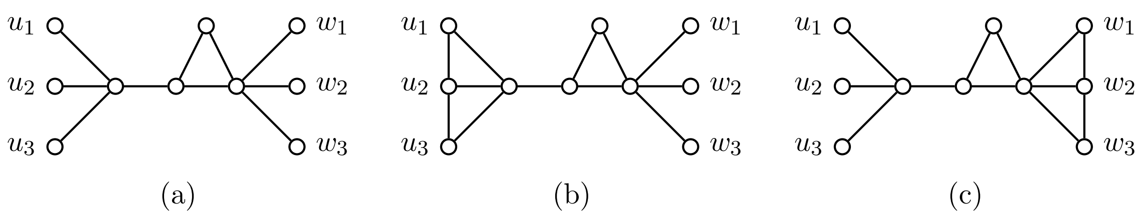

Figure 3.8 (a) A graph with isolated twin set $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$ that are the same size and with degree 1. By adding 2 edges into each set, we produced cospectral graphs ((b) and (c)) for the Laplacian matrix.

For the normalized Laplacian, this notion of twins extends a step further. Butler [9] defined *twin subgraphs* which are isomorphic subgraphs of G connected to the rest of the graph in the same way. Butler showed that most of the normalized Laplacian eigenvectors could be chosen in a particular way so that the entries corresponding to the subgraphs are the same.

For example, consider the graph G in Figure 3.9(a) which has H_1, H_2, H_3 are twin subgraphs. Therefore, the normalized Laplacian matrix has the following structure

$$\begin{bmatrix} M & 0 & 0 & C \\ 0 & M & 0 & C \\ 0 & 0 & M & C \\ C^T & C^T & C^T & N \end{bmatrix}$$

where M corresponds to H_i and N corresponds to the rest of the matrix. The spectrum of this matrix can be found from the *collapsed matrix*.

Theorem 3.5.2. *[9] The eigenvalues (with multiplicities) of graph G with $H_1, \ldots, H_k$ twin subgraphs are*

- *the eigenvalues of $[M]$ with multiplicity $k-1$ (this correspond to the eigenvalues of H_i with weighted endpoints as shown in Figure 3.9(b)).*
- *the eigenvalues of $\begin{bmatrix} kM & kC \\ kC & N \end{bmatrix}$ (this correspond to the eigenvalues of the collapsed graph as shown in Figure 3.9(c)).*

To prove this Butler used *harmonic eigenvectors*. Let $\mathcal{L}x = \lambda x$, then the harmonic eigenvector corresponding to λ is $y = D^{-1/2}x$. This allows us to write $Ay = (1-\lambda)Dy$ and more importantly the following is satisfied for every $u \in V(G)$

$$\sum_{v \sim u} y_v = (1-\lambda)d(u)y_u.$$

Starting with a harmonic eigenvector for a reduced graph, Butler then lifts them to harmonic eigenvectors of a larger graph.

Proof. Let $\vec{x}$ be an eigenvector of M, then the vector y^i as defined

$$y^i_u = \begin{cases} x_u & \text{if } u \in H_1 \\ -x_u & \text{if } u \in H_i \\ 0 & \text{else.} \end{cases}$$

Using computation of harmonic eigenvectors, this is an eigenvector for $i = 1, \ldots, k-1$. Let $\vec{x}$ be an eigenvector of $\begin{bmatrix} kM & kC \\ kC & N \end{bmatrix}$, then the vector $\vec{y}$ with entries corresponding to H_i are the entries of $\vec{x}$ to the principal submatrix of size $|H_i|$. The remaining entries of $\vec{y}$ are the remaining entries of $\vec{x}$. Again, through harmonic eigenvector computation, $\vec{y}$ is an eigenvector for G. □

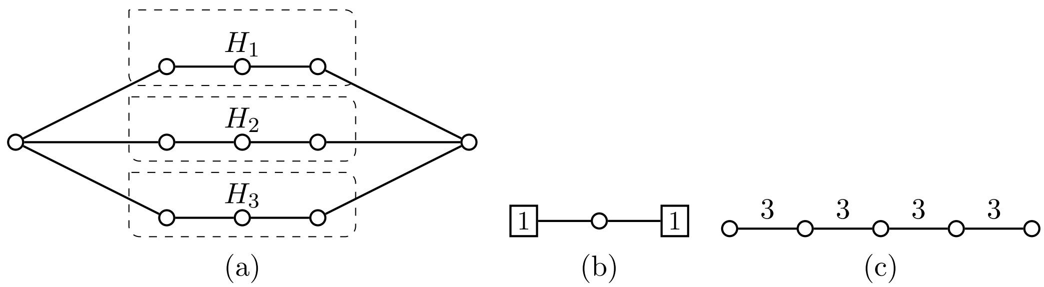

Figure 3.9 (a) Graph G with three twin subgraphs H_1, H_2, H_3. (b) The weighted graph corresponding to the matrix M, the eigenvalues of this graph are eigenvalues of G with multiplicity 2. (c) The weighted collapsed graph, the eigenvalues of this graph are the eigenvalues of G.

This technique can build cospectral graphs by combining twin subgraphs. Osborne [45] showed a special case of this construction. Butler and Grout [10] also used harmonic eigenvectors to show that swapping out a cospectral bipartite subgraph created a cospectral mate.

Butler [8] used the technique of collapsing eigenvectors to unfold a bipartite graph in two different ways to create a pair of cospectral graphs for the adjacency matrix and normalized Laplacian. An example of this construction is shown in Figure 3.10. This result was generalized by Ji et al. [36] to unfolding a bipartite graph k times.

Harmonic eigenvectors translate eigenvectors for the normalized Laplacian matrix into matrix equations involving the adjacency matrix. This results in relating the graph structure to eigenvalues in more natural ways. The normalized Laplacian can easily have irrational entries and trying to replicate a graph structural change within the matrix is challenging. Harmonic eigenvectors are an excellent tool to help calculate the eigenvalues.

Collapsing eigenvectors and twin subgraphs are especially useful tools for constructing cospectral graphs for the normalized Laplacian matrix. This is due to the spectrum of the matrix

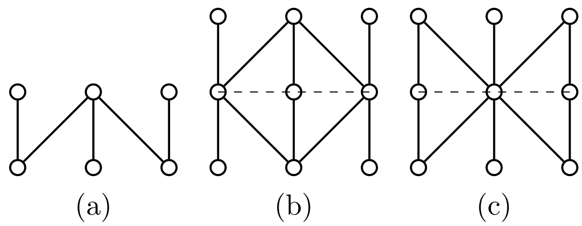

Figure 3.10 (a) A bipartite graph G. (b) Graph G unfolded one way (if we fold along dashed line we get G). (c) Graph G unfolded a second way (if we fold along the dashed line we get G). Using collapsing eigenvectors, graph (b) and (c) are cospectral for the normalized Laplacian.

not changing under scaling which is equivalent to a proportional increase into edge weight. By being able to collapse the spectrum and eigenvectors into basic building blocks, we can unfold it in different ways to create cospectral graphs.

Another way to create cospectral graphs is to perturb the eigenvectors of one graph to create a set of eigenvectors of a second graph.

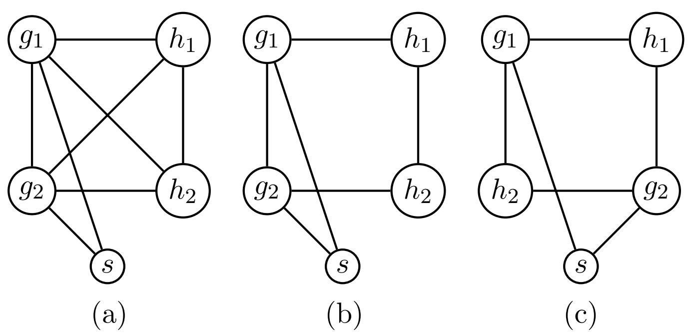

Figure 3.11 Subgraph switching candidates for a cospectral construction for the distance matrix. In the switch, vertex s switches from being adjacent to g_1, g_2 to h_1, h_2.

Heysse [35] used a perturbation of eigenvectors of the distance matrix to showing a subgraph swapping method and a local edge switching. The subgraph swapping is structurally similar to cospectral constructions using cospectral roots. However, the proposed subgraphs have a different number of edges (instead of being isomorphic graphs with two different roots). Heysse showed that any graph identified with one special subgraph at a special vertex is cospectral to the graph identified with the other special subgraph at a special vertex. The proof for this cospectral construction for the distance matrix is by computation. Since the proposed subgraphs have different number of edges, through multiple swaps this construction gives cospectral graphs with an arbitrary different number of edges.

The edge switching Heysse presents in [35] is like Godsil-McKay switching, but it only occurs on a subset of the vertices. The subset the switch occurs on is shown in Figure 3.11 with vertex s switching its adjacency from g_1, g_2 to h_1, h_2 to create a cospectral mate.

Theorem 3.5.3. *[35] Let G be a graph with one of the induced subgraphs show in in Figure 3.11 with an appropriate partition of the vertices. Let H be the graph constructed from G by switching adjacency of s from g_1, g_2 to h_1, h_2. If this switch changes the distances in a prescribed way, then G and H are cospectral for the distance matrix.*

Proof. Let (λ, x) be an eigenpair for the distance matrix of G. Define $y = x + \Delta$ where

$$\Delta_i = \begin{cases} 0 & \text{if } i \notin \{g_1, g_2, h_1, h_2\} \\ \alpha & \text{if } i \in \{g_1, g_2\} \\ -\alpha & \text{if } i \in \{h_1, h_2\} \end{cases}$$

for some prescribed α. Then (λ, y) is an eigenpair for the distance matrix of H by computation. □

Although this might seem like a restrictive construction, it explains all distance matrix cospectral graphs on seven vertices.

The distance matrix has many non-zero entries which can be challenging when proving cospectral constructions. Perturbations in the eigenvectors help focus the spectrum into a slight change in a linear equation. As we have seen, these constructions can create structural differences (such as number of edges) that are challenging with other methods.

3.6 General Conclusions

We discussed cospectral constructions for many different matrices with a heavy emphasis on the adjacency matrix. Structurally, this is an easy matrix to work with since all of the entries are either 0 or 1. Variations such as the combinatorial Laplacian and normalized Laplacian are interesting matrices, and their cospectral constructions develop more intricate linear algebra tools, some of which still apply (with modification) to the adjacency matrix. The distance matrix and its variants are more difficult to work with because they are not sparse matrices.

Some cospectral constructions we discussed apply (with modification) to both adjacency and distance matrices. Most notably, McKay's construction of cospectral trees [43]. The only matrix we discussed for which there is not some version of this is the distance Laplacian. This development is because even at a cut vertex, a subgraph does not correspond to a submatrix. There is a conjecture that the distance Laplacian spectrum determines trees. In the future, we hope to see more interest in this matrix and new linear algebra techniques that further explain cospectrality for graph matrices.

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CHAPTER 4. SPECTRAL PROPERTIES OF VARIANTS OF THE DISTANCE MATRIX

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4.1 Abstract

Let the distance matrix of a graph be the matrix the distance between two vertices as entries. The spectrum of the distance matrix has been studied, including finding inertia of graph families, cospectral constructions, and the coefficients of the characteristic polynomial. In this paper, we look at graph matrices that are variations of the distance matrix which are not spectrally well understood. First, we examine the exponential distance matrix. We find the inertia of several graph families and a cospectral construction for a pair of graphs with a different number of components. Second, we examine the distance Laplacian matrix and add theoretical and empirical evidence to the conjecture that the distance Laplacian spectrum determines all trees.

4.2 General Introductions

A graph G is a collection of objects called the vertex set, $V(G)$, and the connections between them called the edges, $E(G)$. Two vertices are adjacent, denoted $v \sim u$, if there is an edge between them. A *path* of length k is a sequence of distinct vertices $(v_1, \ldots, v_{k+1})$ where $v_i \sim v_{i+1}$. Notice that we count the length of a path by the number of edges. A *cycle* is a path that starts and ends at the same vertex. The *distance* between two vertices $v, u \in V(G)$, denoted $\text{dist}_G(v, u)$, is the length of the shortest path between the vertices. The *distance matrix* of a graph is the matrix with entries $\mathcal{D}_{i,j} = \text{dist}(v_i, v_j)$. The *spectrum*, denoted $\text{spec}(M)$, of a square matrix M is the multiset of eigenvalues of M. The *inertia* of a matrix with a real spectrum, denoted inertia$(M) = (n_-, n_0, n_+)$, is the ordered tuple giving the number of negative eigenvalues, the number of zero eigenvalues, and the number of positive eigenvalues respectively.

The distance matrix was originally studied to solve the addressing problem on networks by Graham and Pollak [12] by using graph decomposition. The addressing problem looked to find an efficient labeling system of the nodes of a network such that messages arrive to the correct recipient and in a timely manner. Graham and Pollak found, that for the desired addressing system, the length of an address needs to be at least $\max(n_+, n_-)$ where n_+ (n_-) is the number of positive (negative) eigenvalues of $\mathcal{D}$. Winkler [17] later showed an address needs to be at most $n-1$ in length.

Graham and Pollak also established that for any tree T, a graph with no cycles, on n vertices $\det(\mathcal{D}(T)) = (-1)^{n-1}(n-1)2^{n-2}$ as well as results on the inertia of trees. Bapat et al. [5] extended these results to weighted trees and also established results on the inertia of unicyclic graphs. Graham and Lovász [13] found the the coefficients of the characteristic polynomial in terms of subtrees for the distance matrix. Since then, many spectral proprieties have been studied (for a survey of results see [3]).

Two graphs are called *cospectral with respect to a matrix M* if their M spectrum is the same. For the distance matrix, there exists constructions of cospectral graphs. McKay [16] used matrix properties to show that cospectrally rooted trees T_1, T_2 could be identified with any rooted tree to create a pair of larger cospectral trees. This resulted in showing almost all trees have a cospectral mate for the distance matrix. Heysse [14] gave two more cospectral constructions for the distance matrix both using a perturbation of the eigenvalues. One constructed graphs with a different number of edges and the other constructed all cospectral graphs on seven vertices. Abiad et al. [1] gave a cospectral construction for the distance matrix by creating large well structured graphs. This construction resulted in cospectral pairs with different diameters.

We will discuss two variants of the distance matrix. The first variant is the exponential distance matrix. For any $q \in \mathbb{R}$, the *exponential distance matrix* is the matrix with entries

$$\mathcal{D}^q_{i,j} = q^{\text{dist}(i,j)}.$$

In this paper, we will restrict $0 \leq q \leq 1$. Therefore the diagonal entries are ones and that this matrix is well-defined for disconnected graphs. This variation's spectral properties have been briefly studied before by Bapat et al. [6] (where the exponential distance matrix was introduced) and Butler et al. [8]. The previous work includes finding the explicit spectrum for several well-known families of graphs such as the complete graph, cycle graph, hypercubes, the characteristic polynomial of the path graph, and the spectrum resulting from graph operations such as the join and Cartesian product. In addition, Bapat et al. [6] the determinant for all trees T on n vertices is $\det(\mathcal{D}^q(T)) = (1-q^2)^n$.

Butler et al. [8] used a technique of classifying factors of the characteristic polynomial using pendant vertices. This led to a cospectral construction for a family of unicyclic graphs.

We will find the inertia of trees, cycles, unicylic, and multipartite graphs for all values of q (with $0 \leq q \leq 1$). In addition, we will show a cospectral construction for two families of graphs that are cospectral for exactly one value of q. We will also discuss open problems related to cospectrality for the exponential distance matrix.

The second variant is the distance Laplacian matrix first introduced by Aouchiche and Hansen [2]. The *distance Laplacian matrix* is the matrix with entries

$$\mathcal{D}^L_{i,j} = \begin{cases} -\operatorname{dist}(i,j) & i \neq j \\ \sum_{k \in V} \operatorname{dist}(i,k) & i = j. \end{cases}$$

Spectral properties and cospectral graphs have been explored in [2, 4, 7]. Brimkov et al. [7] gave a construction of cospectral graphs using a relaxation of twin vertices.

Thus far there are no known examples of cospectral trees. Aouchie and Hansen [2] showed this is true through 20 vertices leading to conjecture the following.

Conjecture 4.2.1. *[2] Let T be a graph tree. For the distance Laplacian matrix, T is spectrally determined.*

This is surprising because, for many other well studied matrices including the distance matrix, almost all trees have a cospectral mate as shown by McKay [16].

In addition to the empirical results, we know $K_{1,n-1}$ and P_n are spectrally determined. We show, through studying the trace of distance Laplacian of trees, there are two more families of trees that are spectrally determined among trees. Furthermore, we investigate the reduced determinant of the distance Laplacian which gives an easy computation of a coefficient of the characteristic polynomial. We will give an improved algorithm technique to increase the known number of vertices with no example of cospectral trees from 20 to 23 vertices.

The spectrum of the distance matrix gives a more detailed picture of the whole graph. Similarly, these variations store a broader picture of the network than variations of the adjacency matrix. Through understanding their relatively unstudied spectrum, we gain deep insights into interactions on the network. This paper is divided into two parts: the first part covering the results of the exponential distance matrix, and the second part covering the results of the distance Laplacian matrix.

4.3 Exponential Distance Matrix

4.3.1 Introduction

Bapat, Lal, and Pait [6] introduced a variation of the distance matrix called the exponential distance matrix, $\mathcal{D}^q(G)$ whose entries are $(\mathcal{D}^q)_{i,j} = q^{\text{dist}(i,j)}$ if i, j are in the same connected component and 0 otherwise. When we restrict $q \in (-1, 1)$, then we see the definition can simplify to $(\mathcal{D}^q)_{i,j} = q^{\text{dist}(i,j)}$ since $q^\infty = 0$. This variation is able to work with disconnected graphs contrary to conventions of the distance matrix.

This matrix has variable entries so the spectrum is reliant on the variable. For some graphs such as the complete graph and hypercube graphs, Butler et al. [8] computed the spectrum directly in terms of q.

Proposition 4.3.1. *[8] Consider the complete graph on n vertices, K_n. Then*

$$spec(\mathcal{D}^q(K_n)) = \{q(n-1)+1, (1-q)^{(n-1)}\}.$$

Proposition 4.3.2. *[8] Consider the hypercube graph on 2^n vertices, Q_n. Then the spectrum of Q_n for $\mathcal{D}^q$ is $(1-q)^k(1+q)^{n-k}$ with multiplicity $\binom{n}{k}$ for $0 \le k \le n$.*

Finding the spectrum explicitly for a variable q is a challenging task for most graphs. Instead, we can look at the inertia of the graph (number of negative, zero, and positive eigenvalues) as a way of understanding the spectrum.

This section will classify several graph families' inertia for all values of $0 < q < 1$. For all graphs when $q = 0$, then $\mathcal{D}^q(G)$ is the identify matrix and when $q = 1$, then $\mathcal{D}^q(G)$ is the all-ones matrix. Both of these matrices have a well-known spectrum, and we are interested in the spectrum for q between these values. We will show that for trees and unicyclic graphs, the inertia is not dependent on q, but for the complete multipartite graph, it is dependent.

Butler et al. [8] discussed cospectrality for the exponential distance matrix and gave a construction when the two graphs are cospectral for all values of q. Here, we look at two families of cospectral graphs for exactly one value of $q = 1/2$. These examples create pairs of cospectral graphs with a different number of components. There are no other known values of q where there is a pair of cospectral graphs for exactly that value of q. After our construction, we present a conjecture questioning the necessity of leaving q as a variable when studying cospectrality.

4.3.2 Inertia of Multipartite Graphs

Let $G = K_{n_1,\ldots,n_k}$ be the multipartite graph with k independent sets each with size n_i and with all possible edges between vertices in two different parts. Let $\{\bar{n}_i : 1 \le i \le t\}$ be the set of distinct part sizes and p_i be the number of times each $\bar{n}_i$ occurs in $\{n_1, \ldots, n_k\}$. In this section we will prove the following result.

Theorem 4.3.3. *For the multipartite graph G, there exists $0 = q_0 < q_1 < q_2 < \cdots < q_t < q_{t+1} = 1$ determined by $n_1, \ldots, n_k$ such that the inertia of G is as follows for a value of q. Let $0 < j < t$ be an even index and recall that p_i is the number of times a* *part size occurs (we assume these are ordered from largest $\bar{n}_i$ to smallest).*

$$
inertia(G) = \begin{cases}
(0,0,n) & 0 \leq q < q_1 \\
(0, p_1 - 1, n - p_1 - 1) & q = q_1 \\
(p_1 - 1, 0, n - p_1 - 1) & q_1 < q < q_2 \\
\left(\sum_{i \leq \frac{j}{2}} (p_i - 1) + \frac{j}{2} - 1, 1, n - \sum_{i \leq \frac{j}{2}} (p_i - 1) - \frac{j}{2} \right) & q = q_j \\
\left(\sum_{i \leq \frac{j}{2}} (p_i - 1) + \frac{j}{2}, 0, n - \sum_{i \leq \frac{j}{2}} (p_i - 1) - \frac{j}{2} \right) & q_j < q < q_{j+1} \\
\left(\sum_{i \leq \frac{j}{2}} (p_i - 1) + \frac{j}{2}, p_{\frac{j}{2}+1} - 1, n - \sum_{i \leq \frac{j}{2}+1} (p_i - 1) - \frac{j}{2} \right) & q = q_{j+1} \\
\left(\sum_{i \leq \frac{j}{2}+1} (p_i - 1) + \frac{j}{2}, 0, n - \sum_{i \leq \frac{j}{2}+1} (p_i - 1) - \frac{j}{2} \right) & q_{j+1} < q < q_{j+2} \\
(k-1, 0, n-k+1) & q_t < q < 1 \\
(0, n-1, 1) & q = 1
\end{cases}
$$

To prove this, we will follow the techniques demonstrated by Esser and Harary [11] where they calculated the inertia of the multipartite graph for the adjacency matrix.

We will be using results about twin vertices to obtain certain eigenvalues.

Definition 4.3.4. *Two vertices v_1, v_2 are called* twin vertices *if $N(v_1) = N(v_2)$.*

Proposition 4.3.5. *[8] Let G be a graph with a pair of twin vertices v_1, v_2. Then its exponential distance matrix $\mathcal{D}^q$ has $(1-q^2)$ as an eigenvalue if $v_1 \not\sim v_2$ and $(1-q)$ as an eigenvalue if $v_1 \sim v_2$.*

The proof of Proposition 4.3.5 uses that the vector $[1, -1, 0, \ldots, 0]^T$ is an eigenvector where the non-zero entries correspond to the twin vertices. This naturally extends to a set of vertices that are simultaneously pairwise twin vertices: $N(v_1) = N(v_2) = \cdots = N(v_m)$. Therefore, if G has m simultaneously pairwise twin vertices, then the exponential distance matrix $\mathcal{D}^q$ has $(1-q^2)$

as an eigenvalue with multiplicity $m-1$. This is because we can form $m-1$ pairwise orthogonal eigenvectors with $(1-q^2)$ as an eigenvalue.

In order to prove Theorem 4.3.3, we will establish nice factors of the characteristic polynomial. We will use this to find all but t eigenvalues explicitly and bound the remaining t eigenvalues.

Proposition 4.3.6. *Let $G = K_{n_1,\ldots,n_k}$ be a complete multipartite graph as defined above. Then the eigenvalues are*

(I) $(1-q^2)$ *with multiplicity* $n-k$

(II) $(1-\bar{n}_i q + q^2(\bar{n}_i - 1))$ *with multiplicity* $p_i - 1$ *for* $1 \leq i \leq t$

and the roots of

(III) $$1 - \sum_{i=1}^{t} \frac{p_i \bar{n}_i q}{(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)}.$$

Proof. First, we will show the eigenvalues (I) using twin vertices.

By Proposition 4.3.5, $(x-1+q^2)^{n-k}$ is a factor of our characteristic polynomial of G since there are $n_i - 1$ simultaneous twin vertices in each part.

To find the remaining eigenvalues, we will use the multipartite partitions which are an equitable partition of the graph. Therefore, we can collapse our matrix to find the remaining eigenvalues. Thus the remaining k eigenvalues are the eigenvalues of

$$M = \begin{bmatrix} q^2(n_1-1)+1 & n_2 q & \cdots & n_k q \\ n_1 q & q^2(n_2-1)+1 & \cdots & n_k q \\ \vdots & & \ddots & \vdots \\ n_1 q & n_2 q & & q^2(n_k-1)+1 \end{bmatrix}.$$

We will find the characteristic polynomial of M by expanding $\det(xI - M)$. This addition will allow us to make row and column reductions on M

$$\det(xI-M)=\det\begin{bmatrix} x-q^2(n_1-1)-1 & -n_2q & \cdots & -n_kq \\ -n_1q & x-q^2(n_2-1)-1 & \cdots & -n_kq \\ \vdots & & \ddots & \vdots \\ -n_1q & -n_2q & & x-q^2(n_k-1)-1 \end{bmatrix}$$

First, we will add a special row and column that do not change the determinant.

$$\det(xI-M)=\det\begin{bmatrix} 1 & n_1q & n_2q & \cdots & n_kq \\ 0 & x-q^2(n_1-1)-1 & -n_2q & \cdots & -n_kq \\ 0 & -n_1q & x-q^2(n_2-1)-1 & \cdots & -n_kq \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & -n_1q & -n_2q & & x-q^2(n_k-1)-1 \end{bmatrix}$$

We will now add the first row to the other rows. This does not change the determinant but will make it easier to expand.

$$\det(xI-M)=\det\begin{bmatrix} 1 & n_1q & n_2q & \cdots & n_kq \\ 1 & x-q^2(n_1-1)-1+n_1q & 0 & \cdots & 0 \\ 1 & 0 & x-q^2(n_2-1)-1+n_2q & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & & x-q^2(n_k-1)-1+n_kq \end{bmatrix}$$

Now we expand along the top row to get the following formula for the characteristic polynomial:

$$\begin{aligned}\det(xI-M) &= \prod_{i=1}^{k}(x-q^2(n_i-1)-1+n_iq)-\sum_{i=1}^{k}n_iq\prod_{j=1,j\neq i}^{k}(x-q^2(n_j-1)-1+n_jq)\\ &= \prod_{i=1}^{k}(x-q^2(n_i-1)-1+n_iq)-\sum_{i=1}^{k}\frac{n_iq}{(x-q^2(n_i-1)-1+n_iq)}\prod_{j=1}^{k}(x-q^2(n_j-1)-1+n_jq)\end{aligned}$$

We will now substitute in the variables that represent the set of unique sizes of parts $\bar{n}_i$ with multiplicity of p_i for $1 \leq i \leq t$.

$$\prod_{i=1}^{k}(x - q^2(n_i - 1) - 1 + n_i q) - \sum_{i=1}^{k} \frac{n_i q}{(x - q^2(n_i - 1) - 1 + n_i q)} \prod_{j=1}^{k}(x - q^2(n_j - 1) - 1 + n_j q)$$

$$= \prod_{i=1}^{t}(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)^{p_i} - \sum_{i=1}^{t} \frac{p_i \bar{n}_i q}{(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)} \prod_{j=1}^{t}(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)^{p_i}$$

$$= \left(\prod_{i=1}^{t}(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)^{p_i - 1}\right) \tag{4.1}$$

$$\left(\prod_{i=1}^{t}(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q) - \sum_{i=1}^{t} \frac{p_i \bar{n}_i q}{(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)} \prod_{j=1}^{t}(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)\right) \tag{4.2}$$

Thus, the roots of (4.1) are the desired eigenvalues of the form (II). Furthermore, our remaining t eigenvalues are the roots of the equation in (4.2). We know that $(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)$ are not roots for $1 \le i \le t$ (we factored these out), so we can divide through without changing the roots.

Thus, the last eigenvalues are the roots of

$$1 - \sum_{i=1}^{t} \frac{p_i \bar{n}_i q}{(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)}.$$

Therefore, our eigenvalues are

(i) $(1 - q^2)$ with multiplicity $n - k$

(ii) $(1 - \bar{n}_i q + q^2(\bar{n}_i - 1))$ with multiplicity $p_i - 1$ for $1 \le i \le t$

and the roots of

(iii) $1 - \sum_{i=1}^{t} \frac{p_i \bar{n}_i q}{(x - q^2(\bar{n}_i - 1) - 1 + \bar{n}_i q)}$.

as desired. □

We will now prove Theorem 4.3.3 about the inertia of the complete multipartite graph for $0 < q < 1$.

Proof. The eigenvalues of form (I) are positive for all $0 < q < 1$.

Next consider the eigenvalues of form (II): $\lambda_j = 1 - \bar{n}_i q + q^2(\bar{n}_i - 1) = (1 - q)(1 - q(\bar{n}_i - 1))$. Therefore,

- $\lambda_j > 0$ when $0 < q < \frac{1}{\bar{n_i}-1}$
- $\lambda_j = 0$ when $q = \frac{1}{\bar{n_i}-1}$
- $\lambda_j < 0$ when $\frac{1}{\bar{n_i}-1} < q < 1$.

These eigenvalues have multiplicity $p_i - 1$ and the special value of q where they shift from being positive to zero to negative is $q_j = \frac{1}{\bar{n_i}-1}$.

Finally, consider the eigenvalues of form (III) which are the roots of

$$1 - \sum_{i=1}^{t} \frac{p_i \bar{n_i} q}{(x - q^2(\bar{n_i} - 1) - 1 + \bar{n_i} q)}.$$

Let $a_i = p_i \bar{n_i} q$ and $b_i = q^2(\bar{n_i} - 1) + 1 - \bar{n_i} q$. Therefore, the eigenvalues are the roots of $1 - \sum \frac{a_i}{x - b_i}$ and we know that all of our b_i are distinct. The function $\sum \frac{a_i}{x-b_i}$ is a well known function with asymptotes at $x = b_i$ and exactly one root between each asymptote. A graph of the function $1 - \sum \frac{a_i}{x-b_i}$ with asymptotes at $x = b_i$ is shown in Figure 4.1.

Therefore, the last t eigenvalues are interlaced with the values of b_i as follows (let $b_1, \ldots, b_t$ be ordered from smallest to largest value):

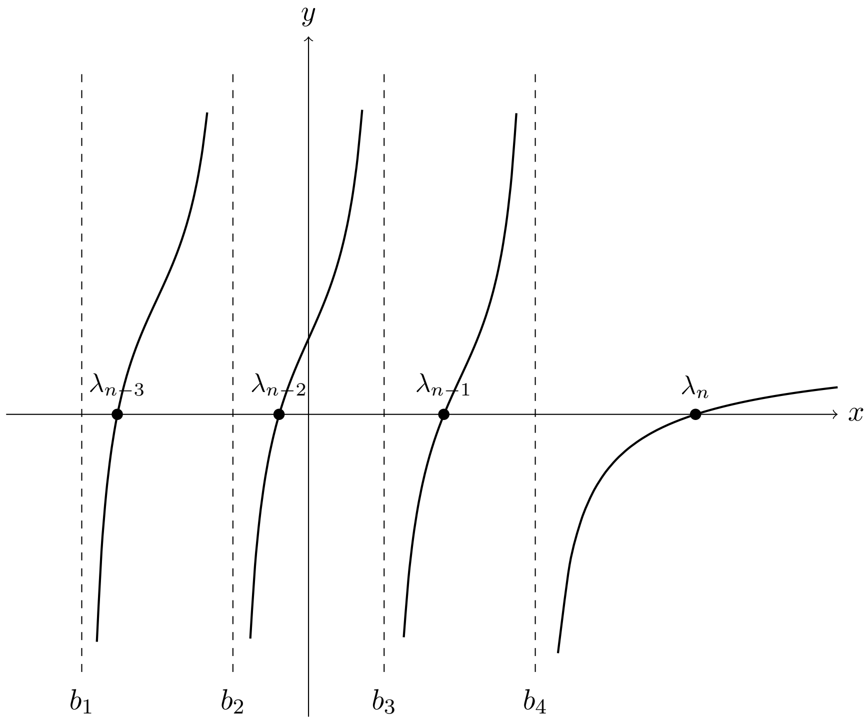
$$b_1 < \lambda_{n-t+1} < b_2 < \cdots < b_t < \lambda_n.$$

Recall that b_i is quadratic in q with $q = 1, \frac{1}{\bar{n_i}-1}$ as roots. Therefore, $b_i > 0$ when $q \in \left[0, \frac{1}{\bar{n_i}-1}\right)$ and $b_i < 0$ when $q \in \left(\frac{1}{\bar{n_i}-1}, 1\right)$. So, there exists some q_j with $1 \leq j \leq t-1$ such that $\frac{1}{\bar{n_i}-1} < q_j < \frac{1}{\bar{n_{i+1}}-1}$, if $q = q_j$ then $\lambda_{n-t+i} = 0$. Moreover, $\lambda_{n-t+i} > 0$ if $q < q_j$ and $\lambda_{n-t+i} < 0$ if $q > q_j$.

Note that q_j for these $t-1$ eigenvalues are distinct where as q_j for eigenvalues of form (II) are distinct up to multiplicity of λ_j.

Lastly, let us classify the sign of λ_n.

Our eigenvalues of the form (II) are exactly the b_is which bound our eigenvalues of the form (III). Therefore, $k-1$ or our eigenvalues are upper bounded by $1 - q^2$ for $0 < q < 1$. In other



[ht]

Figure 4.1 Graph of $1 - \sum \frac{a_i}{x-b_i}$ with asympototes at $x = b_i$. The roots of this function make up the remaining eigenvalues of the complete multipartite graph. The asympototes allow us to strictly bound the roots (and eigenvalues).

words

$$\begin{aligned} 1 - nq + nq^2 - q^2 &= 1 - q^2 - nq(1-q) \\ &\leq 1 - q^2. \end{aligned}$$

Recall that $n-k$ of our eigenvalues of the form (I) are $1-q^2$. We also know that the trace of $\mathcal{D}^q$ is n, therefore

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i &= n\\ \lambda_n+\sum_{i=1}^{n-1}\lambda_i &= n\\ \lambda_n+(n-1)(1-q^2) &\geq n\\ \lambda_n &\geq n-(n-nq^2-1+q^2)\\ &\geq nq^2-q^2+1\\ &\geq q^2(n-1)+1\\ &\geq 0\end{aligned}$$

for all q.

We end by noting when $q=0$, $\mathcal{D}^q=I$, the identity matrix which has inertia$(I)=(0,0,n)$. When $q=1$, $\mathcal{D}^q=J$, the all ones matrix which has inertia$(J)=(0,n-1,1)$. This gives us the desired result. □

Let us now explicitly explore examples of Theorem 4.3.3.

Example 4.3.7. *Let $G=K_{n,m}$, the complete bipartite graph. Therefore, the spectrum of G is*

- $(1-q)^2$ *with multiplicity* $n-2$
- *roots of* $((m-1)q^2+1-x)((n-1)q^2+1-x)-nmq^2$.

These last two eigenvalues work out to be

$$\phi_i=\frac{1}{2}\left((m+n-2)q^2\pm q\sqrt{(m^2-2mn+n^2)q^2+4mn}+1\right)$$

which when taking (+) is always positive. Therefore, the only possible eigenvalue which is not always positive for $0<q<1$ is

$$\phi_2=\frac{1}{2}\left((m+n-2)q^2-q\sqrt{(m-n)^2q^2+4mn}+1\right).$$

By letting $\phi_2 = 0$ and solving for q, we find that $\phi_2 = 0$ when $q = (mn - m - n + 1)^{-1/2}$. Further, $\phi_2 > 0$ when $q < (mn - m - n + 1)^{-1/2}$ and $\phi_2 < 0$ when $q > (mn - m - n + 1)^{-1/2}$.

If $m = n$, then $\phi_1 = (m - 1)q^2 - mq + 1$ which is exactly of the form of (II).

4.3.3 Inertia of Unicyclic and Noncyclic Graphs

In this section we will prove that unicyclic and noncyclic graphs have positive eigenvalues for the exponential distance matrix for $0 < q < 1$. Many of the techniques shown demonstrate an interesting multiplicative property to manipulate the matrix. This technique is unique to this matrix and makes some spectral results easier to compute when compared to other graph matrices. We will be showing that series of matrices are similar by row and column operations. We say matrix M is similar to matrix N by row and column operations, denoted $M \sim N$.

A cycle in a graph G is a sequence of unique vertices $(v_1, \ldots, v_{k+1})$ such that $v_i \sim v_{i+1}$ and $v_1 = v_{k=1}$. If G contains exactly one cycle (up to permutation), then G is *unicyclic*.

If a vertex v is contained in no cycle of G, then we say it is in a branch of G.

Lemma 4.3.8. *Let G be a connected graph with m vertices in branches, with H being a subgraph of G with all vertices in branches removed. Then* $\det(G) = (1 - q)^m \det(H)$.

Proof. We will prove this by induction. Let $m = 1$, so G has one leaf vertex v. Let v have u as its neighbor. Notice that $\mathcal{D}^q(G)$ can be written in the following block form with the last column and row corresponding to vertex v.

$$\begin{bmatrix} \mathcal{D}^q(G - u - v) & \vec{x} & q\vec{x} \\ \vec{x}^T & 1 & q \\ q\vec{x}^T & q & 1 \end{bmatrix} \sim \begin{bmatrix} \mathcal{D}^q(G - u - v) & \vec{x} & \vec{0} \\ \vec{x}^T & 1 & 0 \\ \vec{0}^T & 0 & 1 - q^2 \end{bmatrix}$$

Therefore, $\det(\mathcal{D}^q(G)) = (1 - q^2) \det(\mathcal{D}^q(G - v))$.

Now consider the case when G has m vertices in branches. Let the vertices in branches of G be labeled $v_1, \ldots, v_m$ such that the branches of G can be constructed by iteratively adding vertex v_i as a leaf. Let v_{m-1} have vertex v_m as a neighbor.

Notice that $\mathcal{D}^q(G)$ can be written in the following block form with the last column and row corresponding to vertex v_m

$$\begin{bmatrix} \mathcal{D}^q(G - v_{m-1} - v_m) & \vec{x} & q\vec{x} \\ \vec{x}^T & 1 & q \\ q\vec{x}^T & q & 1 \end{bmatrix} \sim \begin{bmatrix} \mathcal{D}^q(G - v_{m-1} - v_m) & \vec{x} & \vec{0} \\ \vec{x}^T & 1 & 0 \\ \vec{0}^T & 0 & 1 - q^2 \end{bmatrix}$$

Therefore, together with the induction hypothesis,

$$\begin{aligned} \det(\mathcal{D}^q(G)) &= (1 - q^2)\det(\mathcal{D}^q(G - v_m)) \\ &= \qquad \vdots \\ &= (1 - q^2)^m \det(\mathcal{D}^q(G - \{v_1, \ldots, v_m\})) \end{aligned}$$

□

Theorem 4.3.9. *Let H be a graph where $\mathcal{D}^q(H)$ is a positive definite matrix on n vertices. Let G be the graph constructed by adding m vertices as branches to H. Then $\mathcal{D}^q(G)$ is a positive definite matrix.*

Proof. We will prove this with induction. Let $m = 1$. By Lemma 4.3.8, we know $\det(\mathcal{D}^q(G)) = (1 - q^2)\det(\mathcal{D}^q(H))$. By interlacing theorem, we have

$$\lambda_{i-1}(\mathcal{D}^q(G)) \leq \lambda_i(\mathcal{D}^q(H)) \leq \lambda_i(\mathcal{D}^q(G))$$

for $i \in \{1, \ldots, n\}$.

Therefore, $\mathcal{D}^q(G)$ has at least n positive eigenvalues. We also know that $\det(\mathcal{D}^q(H))$ is a positive number and $(1 - q^2)$ is a positive number. Therefore, $\det(\mathcal{D}^q(G))$ is a positive number and the remaining eigenvalue must also be positive for all n.

Consider the graph G, constructed from H by adding m vertices as branches iteratively. Let v be the last vertex added to construct G. By Lemma 4.3.8, we know $\det(\mathcal{D}^q(G)) = (1 - q^2)^m \det(\mathcal{D}^q(H))$. By interlacing theorem and the induction hypothesis, we have

$$\lambda_{i-1}(\mathcal{D}^q(G)) \leq \lambda_i(\mathcal{D}^q(G - v)) \leq \lambda_i(\mathcal{D}^q(G))$$

for $i \in \{1, \dots, n+m-1\}$.

Therefore, $\mathcal{D}^q(G)$ has at least $n+m-1$ positive eigenvalues. We also know that $\det(\mathcal{D}^q(H))$ is a positive number and $(1-q^2)^m$ is a positive number. Therefore, $\det(\mathcal{D}^q(G))$ is a positive number and the remaining eigenvalues must also be positive. □

Corollary 4.3.10. *The matrix $\mathcal{D}^q$ of tree T is positive definite for $0 < q < 1$, i.e. inertia$(\mathcal{D}^q(T)) = (0, 0, n)$.*

Proof. Consider K_2, which is the tree on two vertices whose exponential distance matrix is

$$\mathcal{D}^q(K_2) = \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}$$

and has eigenvalues $\{1+q, 1-q\}$. For $0 < q < 1$, both of these eigenvalues are positive. Any tree T on n vertices can be thought of as adding $n-2$ vertices in branches to K_2. Therefore, by Theorem 4.3.9, $\mathcal{D}^q(T)$ is a positive definite matrix. □

Proposition 4.3.11. *The matrix $\mathcal{D}^q$ of C_n is positive definite for $0 < q < 1$, i.e. inertia$(\mathcal{D}^q(C_n)) = (0, 0, n)$.*

Proof. Let C_n be the cycle graph on n vertices. In [8], the explicit form of the eigenvalues which are polynomial in q are

$$\left\{ 1 + 2 \sum_{k=1}^{(n-1)/2} q^k \cos(\tfrac{2\pi k j}{n}) \mid 1 \le j \le n \right\}$$

when n is odd and

$$\left\{ 1 + (-1)^j q^{n/2} + 2 \sum_{k=1}^{(n-2)/2} q^k \cos(\tfrac{2\pi k j}{n}) \mid 1 \le j \le n \right\}$$

when n is even.

This result follows from $\mathcal{D}^q(C_n)$ being a circulant matrix with representation $circ(c_0, c_1, c_1, \dots, c_{n-1}) = circ(1, q, q^2, \dots, q^2, q)$. Therefore, we also know the eigenvectors corresponding to these eigenvalues are $\vec{v_j} = [1, \omega^j, \omega^{2j} \dots, \omega^{n-1} j]^T$. When $j = n$ we have a positive eigenvector for a symmetric positive matrix, by Perron-Frobinus, $\lambda_1 > 0$ and is the dominant eigenvalue.

We will now examine the remaining eigenvalues which are $\lambda_j = 0$ when $q = 1$ and $\lambda_j = 1$ when $q = 0$. We can also note that our eigenvalues are polynomial in q.

Let ζ_q be the multiplicity of the eigenvalue 0 of $\mathcal{D}^q(C_n)$ for some value q, n. Let $P(x, q)$ be the polynomial representer of $\mathcal{D}^q(C_n)$. Therefore,
$P(x, q) = \sum_{i=0}^{n-1} c_i x^i = 1 + q^{\text{dist}(1,2)}x + \cdots + q^{\text{dist}(1,n)}x^{n-1}$. Corollary 10 in [15] states

$$\zeta_q = deg(gcd(P(x, q), x^n - 1)).$$

We will show that for all $0 < q < 1$, $\zeta_q = 0$ and since our eigenvalues are polynomial in q, at $q = 0$ are $\lambda_j = 1$, and at $q = 1$ are $\lambda_j = 0$, it follows that they must be positive for all $0 < q < 1$ by the intermediate value theorem.

We know that $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$. Let
$a(x) = x^{n-1} + x^{n-2} + \cdots + x + 1$.

Clearly $a(x)$ and $P(x, q)$ are relatively prime when $0 < q < 1$ since they are the same degree, $a(x)$ is irreducible, and they are not equal.

Additionally, $(x - 1)$ and $P(x, q)$ are relativily prime. Suppose otherwise, then $P(x, q)$ would have $x = 1$ as a root. But $P(1, q) = 1 + q + q^2 + \cdots + q > 0$.

So neither $x - 1$ nor $a(x)$ have a common factor with $P(x, q)$. Therefore, when $0 < q < 1$ $x^n - 1$ and $P(x, q)$ are relatively prime. So $\zeta_q = 0$. Thus, we have that cycles are positive definite. □

This result combined with the result about the inertia of complete multipartite graphs in Theorem 4.3.3 might seem like contradicting results for the graph C_4, which is both a cycle and a complete bipartite graph. In Example 4.3.7, we found the value of q where exactly one eigenvalue of a complete bipartite graph change from being positive to zero, and then negative for q larger than the value. This value is $q = (mn - m - n + 1)^{-1/2}$.

When $m = n = 2$ (which is the graph C_4), then this special value of $q = (1)^{-1/2} = 1$. So Proposition 4.3.11 holds.

Using the previous results, we can now state our main theorem.

Theorem 4.3.12. *Let G be a unicyclic graph. Then inertia$(D_q(G)) = (0, 0, n)$ for all $0 < q < 1$.*

Proof. This result immediately follows from Proposition 4.3.11 and Theorem 4.3.9. □

4.3.4 Infinite Cospectral Families for Exactly One Value of q

Next, we will describe two pairs of graphs that are cospectral only for $q = 1/2$. For graphs on seven vertices, these are the only known pairs that are neither cospectral for all values of q nor cospectral for no values of q.

We will be using the recursive definition of the characteristic polynomial of a path graph for the exponential distance matrix proved by Butler et al. [8].

Proposition 4.3.13. *[8] Let P_n be the path graph on n vertices. Then the characteristic polynomial $P_{\mathcal{D}^q,P_n}(x)$ has the following recurrence with initial conditions*

$$P_{\mathcal{D}^q,P_0}(x) = 1$$

$$P_{\mathcal{D}^q,P_1}(x) = x - 1$$

$$P_{\mathcal{D}^q,P_n}(x) = \big((q^2+1)x - 1 + q^2\big)P_{\mathcal{D}^q,P_{n-1}}(x) - \big(q^2x^2\big)P_{\mathcal{D}^q,P_{n-2}}(x)$$

We say matrix M is similar to matrix N by row and column operations, denoted $M \sim N$.

Theorem 4.3.14. *Let G_k and H_k be the graphs depicted in Figure 4.2 on $k + 6$ vertices. Then G_k and H_k are cospectral for the exponential distance matrix exactly when $q = 1/2$.*

Proof. Let $P_{\mathcal{D}^q,P_k}(x)$ be the characteristic polynomial of the path graph on k vertices. By Proposition 2.1 in [8], we know that the characteristic polynomial of G_k can be written as $P_{\mathcal{D}^q,P_k}(x)f_q(x)$ for some polynomial $f_q(x)$. We will show that H_k has the same characteristic polynomial for the exponential distance matrix exactly when $q = 1/2$.

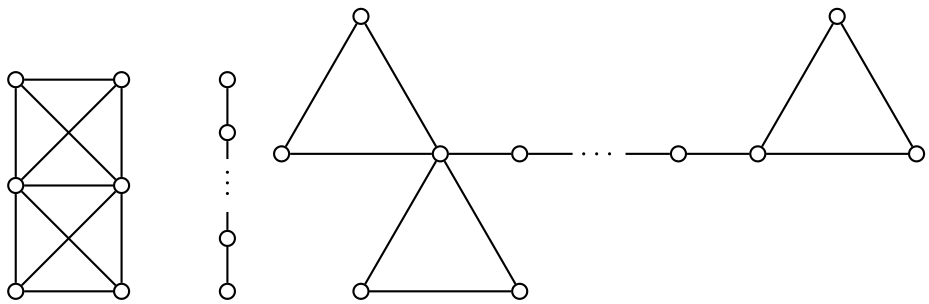


Figure 4.2 (a) The graph G_k (on $k+6$ vertices) which includes a component which is P_k. (b) The graph H_k (on $k+6$ vertices) which has P_k as an induced subgraph. H_1 is the friendship graph on six vertices. G_k is cospectral to H_k for the exponential distance matrix only when $q = 1/2$.

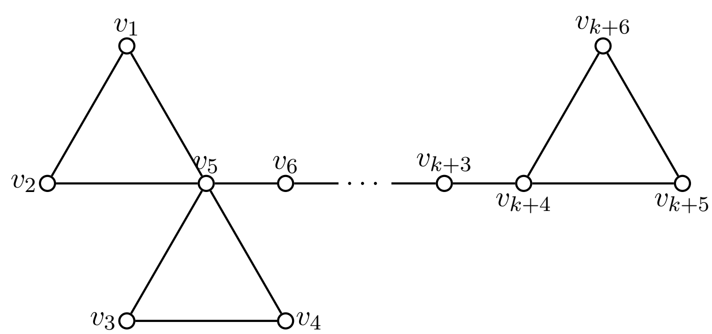


Figure 4.3 A labeling of the vertices of H_k.

Consider the order of the vertices shown in Figure 4.3 for the graph H_k. Notice that the matrix $\mathcal{D}^q(H_k)$ is as follows

$$\mathcal{D}^q(H_k) = \left[\begin{array}{cccc|ccccc|cc}
1 & q & q^2 & q^2 & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & 1 & q^2 & q^2 & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q^2 & q^2 & 1 & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q^2 & q^2 & q & 1 & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
\hline
q & q & q & q & & & & & & q^k & q^k \\
q^2 & q^2 & q^2 & q^2 & & & & & & q^{k-1} & q^{k-1} \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k) & & & \vdots & \vdots \\
q^{k-1} & q^{k-1} & q^{k-1} & q^{k-1} & & & & & & q^2 & q^2 \\
q^k & q^k & q^k & q^k & & & & & & q & q \\
\hline
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & 1 & q \\
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & q & 1
\end{array}\right]$$

We will find the characteristic polynomial of the exponential distance matrix for H_k by expanding the $\det(\mathcal{D}^q - xI)$.

$$\det(\mathcal{D}^q(H_k) - xI) = \left|\begin{array}{cccc|ccccc|cc}
1-x & q & q^2 & q^2 & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & 1-x & q^2 & q^2 & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q^2 & q^2 & 1-x & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q^2 & q^2 & q & 1-x & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
\hline
q & q & q & q & & & & & & q^k & q^k \\
q^2 & q^2 & q^2 & q^2 & & & & & & q^{k-1} & q^{k-1} \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k) - xI & & & \vdots & \vdots \\
q^{k-1} & q^{k-1} & q^{k-1} & q^{k-1} & & & & & & q^2 & q^2 \\
q^k & q^k & q^k & q^k & & & & & & q & q \\
\hline
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & 1-x & q \\
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & q & 1-x
\end{array}\right|$$

$$\sim \left|\begin{array}{cccc|ccccc|cc}
 & & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & M_1 & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
\hline
q & q & q & q & & & & & & q^k & q^k \\
q^2 & q^2 & q^2 & q^2 & & & & & & q^{k-1} & q^{k-1} \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k) - xI & & & \vdots & \vdots \\
q^{k-1} & q^{k-1} & q^{k-1} & q^{k-1} & & & & & & q^2 & q^2 \\
q^k & q^k & q^k & q^k & & & & & & q & q \\
\hline
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & M_2 & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & &
\end{array}\right|$$

where

$$M_1 = \begin{bmatrix} 1-x-q^2 & q-q^2 & q^2-q & q^2+x-1 \\ q-q^2 & 1-x-q^2 & q^2-q & q^2+x-1 \\ 0 & 0 & 1-x-q & q+x-1 \\ q^2 & q^2 & q & 1-x \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1-x & q \\ q+x-1 & 1-x-q \end{bmatrix}.$$

In our first row operations step, we have subtracted row 4 from rows $1, 2, 3$ and subtracted row $k+5$ from row $k+6$. Next, we will do a similar operation to the columns.

$$
\sim \left|\begin{array}{cccc|ccccc|cc}
2(1-x-q^2) & 1-x+q-2q^2q^2 & 1-x-q & q^2+x-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1-x+q-2q^2 & 1(1-x-q^2) & 1-x-q & q^2+x-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1-x-q & 1-x-q & 2(1-x-q) & q+x-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
q^2+x-1 & q^2+x-1 & q+x-1 & 1-x & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & 0 \\
\hline
0 & 0 & 0 & q & & & & & & q^k & 0 \\
0 & 0 & 0 & q^2 & & & & & & q^{k-1} & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k)-xI & & & \vdots & \vdots \\
0 & 0 & 0 & q^{k-1} & & & & & & q^2 & 0 \\
0 & 0 & 0 & q^k & & & & & & q & 0 \\
\hline
0 & 0 & 0 & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & 1-x & q+x-1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & q+x-1 & 2(1-x-q)
\end{array}\right|
$$

In our second row operations step, we have subtracted column 4 from column $1, 2, 3$ and subtracted column $k+5$ from column $k+5$. Now we will leverage the fact that we have the matrix of a path in our middle block.

$$
\sim \left|\begin{array}{cccc|ccccc|cc}
 & & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & M_1 & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & & -qx & 0 & \cdots & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -qx & & & & & & 0 & 0 \\
0 & 0 & 0 & 0 & & & & & & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k)-xI & & & \vdots & \vdots \\
0 & 0 & 0 & 0 & & & & & & & 0 \\
0 & 0 & 0 & 0 & & & & & & -qx & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -qx & M_2 & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & &
\end{array}\right| \tag{4.3}
$$

where

$$
M_1 = \begin{bmatrix}
2(1-x-q^2) & 1-x+q-2q^2 & 1-x-q & q^2+x-1 \\
1-x+q-2q^2 & 2(1-x-q^2) & 1-x-q & q^2+x-1 \\
1-x-q & 1-x-q & 2(1-x-q) & q+x-1 \\
q^2+x-1 & q^2+x-1 & q+x-1 & 1-x-q^2(1+x)
\end{bmatrix}
$$

$$
M_2 = \begin{bmatrix}
1-x-q^2(1+x) & q+x-1 \\
q+x-1 & 2(1-x-q)
\end{bmatrix}.
$$

We have just subtracted q times row (and column) 5 from row (and column) 4. Then we have subtracted q times row (and column) $k+4$ from row (and column) $k+5$. All of these operations do not change the determinant and, therefore, the characteristic polynomial.

We will now expand the determinate. Define the following sub-matrices of the matrix in (4.3) as

$$S_1 = \begin{bmatrix} 2(1-x-q^2) & 1-x+q-2q^2q^2 & 1-x-q & q^2+x-1 \\ 1-x+q-2q^2 & 1(1-x-q^2) & 1-x-q & q^2+x-1 \\ 1-x-q & 1-x-q & 2(1-x-q) & q+x-1 \\ q^2+x-1 & q^2+x-1 & q+x-1 & 1-x-q^2(1+x) \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1-x-q^2(1+x) & q+x-1 \\ q+x-1 & 2(1-x-q) \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 2(1-x-q^2) & 1-x+q-2q^2q^2 & 1-x-q \\ 1-x+q-2q^2 & 1(1-x-q^2) & 1-x-q \\ 1-x-q & 1-x-q & 2(1-x-q) \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 2(1-x-q) \end{bmatrix}.$$

Thus,

$$\begin{aligned} \det(\mathcal{D}^q(H_k)-xI) = {} & \det(S_1)\det(S_2)\det(\mathcal{D}^q(P_k)-xI) \\ & -(qx)^2\det(T_1)\det(S_2)\det(\mathcal{D}^q(P_{k-1})-xI) \\ & -(qx)^2\det(T_2)\det(S_1)\det(\mathcal{D}^q(P_{k-1})-xI) \\ & -(qx)^4\det(T_1)\det(T_2)\det(\mathcal{D}^q(P_{k-2})-xI). \end{aligned}$$

We will now show computationally, that

$$\begin{aligned} \det(\mathcal{D}^q(H_k)-xI) = {} & s(x,q)\det(\mathcal{D}^q(P_k)-xI) \\ & -r(x,q)[(8q^2x+6q^2-3q+3x-3)\det(\mathcal{D}^q(P_{k-1})-xI) \\ & -x^2q^2\det(\mathcal{D}^q(P_{k-2})-xI)] \end{aligned}$$

for some polynomials s, r of x, q. Then, with the aid of `Sage`, a mathematical computational software, we will show that this is equal to the characteristic polynomial of the exponential distance matrix of G_k exactly when $q = 1/2$.

First, let us examine the expanded determinants of our submatrices S_1, S_2, T_1, T_2.

$$\det(S_1) = (4q^2x + 2q^2 - q + x - 1)(2q^2 - q + x - 1)(q + x - 1)^2$$
$$\det(S_2) = (2q^2x + 2q^2 - q + x - 1)(q + x - 1)$$
$$\det(T_1) = -4(2q^2 - q + x - 1)(q + x - 1)^2$$
$$\det(T_2) = -2(q + x - 1)$$

Thus,

$$\begin{aligned}\det(\mathcal{D}^q(H_k) - xI) = {} & \det(S_1)\det(S_2)\det(\mathcal{D}^q(P_k) - xI) \\ & - 8(qx)^2(2q^2 - q + x - 1)(q + x - 1)^3 \\ & \left[\frac{1}{4}(8q^2x + 6q^2 - 3q + 3x - 3)\det(\mathcal{D}^q(P_{k-1}) - xI) - (qx)^2\det(\mathcal{D}^q(P_{k-2}) - xI)\right]\end{aligned}$$

In order to use Proposition 4.3.13 so that we can factor $\det(\mathcal{D}^q(P_k) - xI)$ out of the characteristic polynomial (since we know $\det(\mathcal{D}^q(P_k) - xI)$ if a factor of the characteristic polynomial of the exponential distance matrix of G_k), we need to find values of q where

$$\frac{1}{4}(8q^2x + 6q^2 - 3q + 3x - 3) = (q^2 + 1)x - 1 + q^2$$

for all x. We can solve this equation in `Sage` and get the result of exactly $q = 1/2$.

Therefore, when $q = 1/2$ our characteristic polynomial of $\mathcal{D}^q$ for H_k is

$$\begin{aligned}\det(\mathcal{D}^q(H_k) - xI) = {} & \det(\mathcal{D}^q(P_k) - xI)\left[\det(S_1)\det(S_2)\right. \\ & \left.-8(qx)^2(2q^2 - q + x - 1)(q + x - 1)^3\right].\end{aligned}$$

Lastly, recall we defined $P_{\mathcal{D}^q,G_k}(x) = P_{\mathcal{D}^q,P_k}(x)f_q(x)$ where

$$\begin{aligned}
f_q(x) = x^6 &- 6x^5 + (-4q^4 - 11q^2 + 15)x^4 \\
&+ (-8q^5 - 16q^3 + 44q^2 - 20)x^3 \\
&+ (-24q^5 + 27q^4 + 48q^3 - 66q^2 + 15)x^2 \\
&+ (8q^7 - 48q^6 + 88q^5 - 38q^4 - 48q^3 + 44q^2 - 6)x \\
&+ (4q^8 - 24q^7 + 55q^6 - 56q^5 + 15q^4 + 16q^3 - 11q^2 + 1).
\end{aligned}$$

Therefore, H_k and G_k have the same characteristic polynomial when

$$\det(S_1)\det(S_2) + 8(qx)^2(2q^2 - q + x - 1)(q + x - 1)^3 = f_q(x).$$

With the aid of `Sage`, we can compute that these two polynomials are equal for exactly $q = 1/2$ and $q = 0$ (which we disregard since our previous computations solved only when $q = 1/2$).

Therefore, G_k and H_k have the same characteristic polynomial exactly when $q = 1/2$.

□

This proof highlights how the determinant expansion definition of the characteristic polynomial is an effective proof technique. However, it is not too insightful why this construction is valid only when $q = 1/2$. Surprisingly, this is not the only family of matrices that are cospectral for exactly $q = 1/2$. Figure 4.4 shows G'_k and H'_k which are also cospectral for $\mathcal{D}^q$ exactly when $q = 1/2$ which we will prove below. It resembles the proof above; thus, we will provide a shortened version.

Theorem 4.3.15. *Let G'_k and H'_k be the graphs depicted in Figure 4.4 on $k + 6$ vertices. Then G'_k and H'_k are cospectral for the exponential distance matrix exactly when $q = 1/2$.*

Proof. Let $P_{\mathcal{D}^q,P_k}(x)$ be the characteristic polynomial of the path graph on k vertices. By Proposition 2.1 in [8], we know that the characteristic polynomial of G'_k can be written as $P_{\mathcal{D}^q,P_k}(x)f_q(x)$ for some polynomial $f_q(x)$. We will show that H'_k has the same characteristic polynomial for the exponential distance matrix exactly when $q = 1/2$.

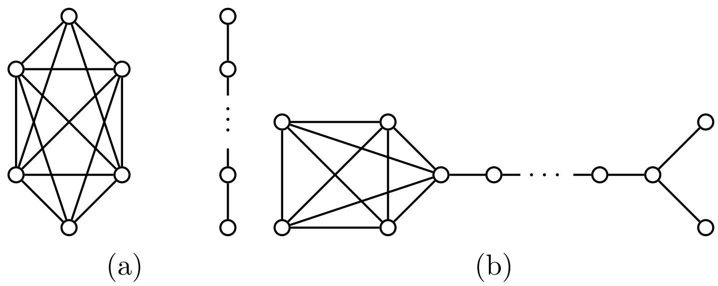


Figure 4.4 (a) The graph G'_k (on $k+6$ vertices) which includes a component which is P_k. (b) The graph H'_k (on $k+6$ vertices) which has P_k as an induced subgraph. G'_k is cospectral to H'_k for the exponential distance matrix only when $q = 1/2$.

[ht]

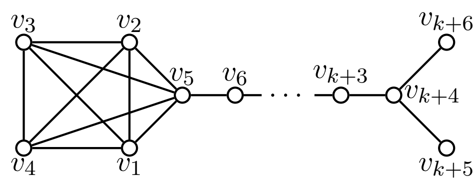


Figure 4.5 A labeling of the vertices of H'_k.

Consider the order of the vertices shown in Figure 4.5 for the graph H'_k. Notice that the matrix $\mathcal{D}^q(H'_k)$ is as follows

$$\mathcal{D}^q(H_k) = \left[\begin{array}{cccc|ccccc|cc}
1 & q & q & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & 1 & q & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & q & 1 & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & q & q & 1 & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
\hline
q & q & q & q & & & & & & q^k & q^k \\
q^2 & q^2 & q^2 & q^2 & & & & & & q^{k-1} & q^{k-1} \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k) & & & \vdots & \vdots \\
q^{k-1} & q^{k-1} & q^{k-1} & q^{k-1} & & & & & & q^2 & q^2 \\
q^k & q^k & q^k & q^k & & & & & & q & q \\
\hline
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & 1 & q^2 \\
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & q^2 & 1
\end{array}\right]$$

We will find the characteristic polynomial of the exponential distance matrix for H_k by expanding the $\det(\mathcal{D}^q - xI)$.

$$
\det(\mathcal{D}^q(H_k) - xI) = \left|\begin{array}{cccc|ccccc|cc}
1-x & q & q & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & 1-x & q & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & q & 1-x & q & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
q & q & q & 1-x & q & q^2 & \cdots & q^{k-1} & q^k & q^{k+1} & q^{k+1} \\
\hline
q & q & q & q & & & & & & q^k & q^k \\
q^2 & q^2 & q^2 & q^2 & & & & & & q^{k-1} & q^{k-1} \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k) - xI & & & \vdots & \vdots \\
q^{k-1} & q^{k-1} & q^{k-1} & q^{k-1} & & & & & & q^2 & q^2 \\
q^k & q^k & q^k & q^k & & & & & & q & q \\
\hline
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & 1-x & q^2 \\
q^{k+1} & q^{k+1} & q^{k+1} & q^{k+1} & q^k & q^{k-1} & \cdots & q^2 & q & q^2 & 1-x
\end{array}\right|
$$

$$
\sim \left|\begin{array}{cccc|ccccc|cc}
 & & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & M_1 & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & & -qx & 0 & \cdots & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -qx & & & & & & 0 & 0 \\
0 & 0 & 0 & 0 & & & & & & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \mathcal{D}^q(P_k) - xI & & & \vdots & \vdots \\
0 & 0 & 0 & 0 & & & & & & & 0 \\
0 & 0 & 0 & 0 & & & & & & -qx & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -qx & M_2 & \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & &
\end{array}\right| \tag{4.4}
$$

where

$$M_1 = \begin{bmatrix} 2(1-x-q) & 1-x-q & 1-x-q & q+x-1 \\ 1-x-q & 2(1-x-q) & 1-x-q & q+x-1 \\ 1-x-q & 1-x-q & 2(1-x-q) & q+x-1 \\ q+x-1 & q+x-1 & q+x-1 & 1-x-q^2(1+x) \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1-x-q^2(1+x) & q^2+x-1 \\ q^2+x-1 & 2(1-x-q) \end{bmatrix}.$$

We followed the same row and column operations in the proof of Theorem 4.3.14 which do not change the determinant and, therefore, the characteristic polynomial.

We will now expand the determinate. Define the following sub-matrices of (4.4) as

$$S_1 = \begin{bmatrix} 2(1-x-q) & 1-x-q & 1-x-q & q+x-1 \\ 1-x-q & 2(1-x-q) & 1-x-q & q+x-1 \\ 1-x-q & 1-x-q & 2(1-x-q) & q+x-1 \\ q+x-1 & q+x-1 & q+x-1 & 1-x-q^2(1+x) \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 1-x-q^2(1+x) & q^2+x-1 \\ q^2+x-1 & 2(1-x-q) \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 2(1-x-q) & 1-x-q & 1-x-q \\ 1-x-q & 2(1-x-q) & 1-x-q \\ 1-x-q & 1-x-q & 2(1-x-q) \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 2(1-x-q) \end{bmatrix}$$

Thus,

$$\begin{aligned} \det(\mathcal{D}^q(H_k') - xI) &= \det(S_1)\det(S_2)\det(\mathcal{D}^q(P_k) - xI) \\ &\quad - (qx)^2 \det(T_1)\det(S_2)\det(\mathcal{D}^q(P_{k-1}) - xI) \\ &\quad - (qx)^2 \det(T_2)\det(S_1)\det(\mathcal{D}^q(P_{k-1}) - xI) \\ &\quad - (qx)^4 \det(T_1)\det(T_2)\det(\mathcal{D}^q(P_{k-2}) - xI). \end{aligned}$$

We will now show computationally, that

$$\begin{aligned}\det(\mathcal{D}^q(H_k') - xI) &= s(x,q)\det(\mathcal{D}^q(P_k) - xI)\\ &\quad - r(x,q)\left(\frac{1}{4}(8q^2x + 6q^2 - 3q + 3x - 3)\det(\mathcal{D}^q(P_{k-1}) - xI)\right.\\ &\quad \left. - x^2q^2\det(\mathcal{D}^q(P_{k-2}) - xI)\right)\end{aligned}$$

for some polynomials s, r of x, q. Then, with the aid of `Sage`, a mathematical computational software, we will show that this is equal to the characteristic polynomial of the exponential distance matrix of G_k exactly when $q = 1/2$.

First, let us examine the expanded determinants of our submatrices S_1, S_2, T_1, T_2.

$$\begin{aligned}\det(S_1) &= (4q^2x + 4q^2 - 3q + x - 1)(q + x - 1)^3\\ \det(S_2) &= (2q^2x + q^2 + x - 1)(q^2 + x - 1)\\ \det(T_1) &= -4(q + x - 1)^3\\ \det(T_2) &= -2(q^2 + x - 1)\end{aligned}$$

Thus,

$$\begin{aligned}\det(\mathcal{D}^q(H_k') - xI) &= \det(S_1)\det(S_2)\det(\mathcal{D}^q(P_k) - xI)\\ &\quad - 8(qx)^2(q^2 + x - 1)(q + x - 1)^3\\ &\quad \left[\frac{1}{4}(8q^2x + 6q^2 - 3q + 3x - 3)\det(\mathcal{D}^q(P_{k-1}) - xI) - (qx)^2\det(\mathcal{D}^q(P_{k-2}) - xI)\right].\end{aligned} \tag{4.5}$$

Note that the term (4.5) is the same as in the proof of Theorem 4.3.14. Hence, we will be eliminating solving the equation to factor out a $\det(\mathcal{D}^q(P_k) - xI)$ since it is the same as in the previous proof.

Therefore, when $q = 1/2$ our characteristic polynomial of $\mathcal{D}^q$ for H_k' is

$$\det(\mathcal{D}^q(H_k') - xI) = \det(\mathcal{D}^q(P_k) - xI)\left[\det(S_1)\det(S_2) - 8(qx)^2(q^2 + x - 1)(q + x - 1)^3\right].$$

Lastly, recall we defined $P_{\mathcal{D}^q,G'_k}(x) = P_{\mathcal{D}^q,P_k}(x)f_q(x)$ where

$$\begin{aligned} f_q(x) =& x^6 - 6x^5 + (-q^4 - 14q^2 + 15)x^4 \\ & (-4q^4 - 32q^3 + 56q^2 - 20)x^3 \\ & + (6q^6 - 24q^5 - 9q^4 + 96q^3 - 84q^2 + 15)x^2 \\ & + (8q^7 - 36q^6 + 40q^5 + 34q^4 - 96q^3 + 56q^2 - 6)x \\ & (3q^8 - 16q^7 + 30q^6 - 16q^5 - 20q^4 + 32q^3 - 14q^2 + 1). \end{aligned}$$

Therefore, H'_k and G'_k have the same characteristic polynomial when

$$\det(S_1)\det(S_2) - 8(qx)^2(q^2 + x - 1)(q + x - 1)^3 = f_q(x).$$

With the aid of `Sage`, we can compute that these two polynomials are equal for exactly $q = 1/2$ and $q = 0$ (which we disregard since our previous computations solved only when $q = 1/2$).

Therefore, G'_k and H'_k have the same characteristic polynomial exactly when $q = 1/2$. □

Both pairs, G_k, H_k and G'_k, H'_k, are the only graphs on seven vertices that are cospectral for only $q = 1/2$. These are also the only known cospectral graphs for any value of q that have a different number of components on seven vertices.

The lack of other examples of graphs that are cospectral for only some values of q (even by algebraic accident) indicates that there might be something special about $q = 1/2$. This observation leads us to state the following conjecture about the spectrum of the exponential distance matrix.

Conjecture 4.3.16. *Every pair of graphs is cospectral for* $0, 1,$ *or all values of* q*. In addition, a pair of graphs are cospectral for exactly* 1 *value of* q *if and only if they have a different number of components and are cospectral for* $q = 1/2$*.*

In this chapter, we have only considered q to be in the interval $(0, 1)$. There are examples of cospectral graphs for $q > 1$ and $q \in \mathbb{C}$ (see [8]). It would be interesting how the results represented here vary for different values of q.

Further, there appear to be other families of graphs that have the same determinant like trees and unicyclic graphs. One such family is clique-path graphs (Cheng and Lin [9] showed a similar result for the distance matrix). One idea would be to prove this with induction and use the multiplicative property to co-factor expand the determinant wisely.

4.4 Distance Laplacian Matrix

4.4.1 Introduction

The distance Laplacian matrix differs from most other well studied matrices by having no examples of cospectral trees. Almost all trees have a cospectral mate for the adjacency matrix, Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix, distance matrix, and exponential distance matrix. Therefore, this spectral property is a considerable departure from previously studied matrices.

For the adjacency matrix, combinatorial Laplacian matrix, and distance matrix, it was shown by McKay [16] that by identifying a vertex w of a tree to vertex v of a special tree T shown in Figure 4.6 is cospectral to identifying to vertex u of T. Therefore, an exponentially large number of trees have a cospectral mate. The proof utilizes that in a graph G, an induced subgraph (attached to the rest of the graph via a bridge) corresponds to a principal submatrix.

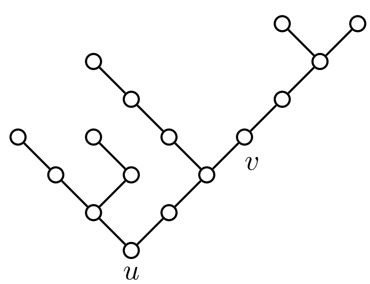


Figure 4.6 Rooted tree T with roots v, u. For any rooted tree G with root w, the graph resulting from w identified with v is cospectral to the graph resulting from w identified with u for the distance matrix.

The proof for the combinatorial Laplacian matrix case is slightly more intricate because one diagonal entry (corresponding the the degree of a vertex) changes as we take induced subgraphs of G. This problem foreshadows why this construction method does not work for the distance Laplacian matrix. Recall the definition of the distance Laplacian matrix as the diagonal matrix with each vertex's transmission in the diagonal entry, denoted $\mathcal{T}$, with the distance matrix subtracted,

$$\mathcal{D}^L = \mathcal{T} - \mathcal{D}.$$

Therefore, when we consider an induced subgraph, none of the diagonal entries of the submatrix align.

This section adds to the theoretical and empirical evidence that the distance Laplacian matrix spectrally determines trees. First, we will investigate coefficients of the characteristic polynomial.

In our work presented here, we will add two more families that are spectrally determined by showing that a special tree on n has a unique coefficient of the characteristic polynomial. We will also discuss another coefficient and its relation to the weighted spanning tree number of a complete graph.

Finally, we improve empirical results by showing that no trees share a coefficient up to 23 vertices (in addition to being spectrally determined). This result improves the known number of vertices where tress are spectrally determined previously found by Aouchie and Hansen [4].

4.4.2 Coefficients of the Characteristic Polynomial of Trees

Let

$$P_{\mathcal{D}^L,T}(x) = x^n + d_{n-1}x^{n-1} + \cdots + d_1 x + d_0$$

be the characteristic polynomial of the distance Laplacian matrix of a tree T. We know that $\mathcal{D}^L$ of any graph has at least one eigenvalue equal to zero, therefore $d_0 = 0$ for all graphs. Brimkov et al. [7] showed that the sequence $(d_{n-1}, d_{n-2}, \cdots, d_2, d_1)$ is an alternating increasing sequence with d_{n-1} the trace of the matrix $\mathcal{D}^L$ (which is the sum of all the transmissions of the vertices). In addition, Aouchie and Hansen [2] used extreme values of the trace of $\mathcal{D}^L$ (which is $-d_{n-1}$) to classify families of trees that are spectrally determined.

This section is organized into two parts. First, we will present theoretical results about d_{n-1} (trace) for trees. Second, we will present theoretical results about d_1 for trees.

To learn information about these coefficients, we will be using *vertex slide.*

Definition 4.4.1. *Let G be a graph with vertex v with* $\deg(v) = 1$*. Let $e = uv$ be the edge that contains the vertex v and w be a neighbor of u. Therefore, dist$(v, w) = 2$. A* vertex slide *on (G, v) to vertex w results in a new graph $G' = (V(G), E(G) - e + vw)$.*

Figure 4.7 contains an example of a slide-move on a tree T. When discussing slide moves on trees, we will talk about subgraphs not involved in the slide. Let T_u (T_w) denote the induced connected subgraph of $T - vu - uw$ containg the vertex u (w). For clarity, this is noted in Figure 4.7

Proposition 4.4.2. *Every graph T that is a tree is a finite number of vertex slide moves from the star graph and the path graph.*

This is a well known result.

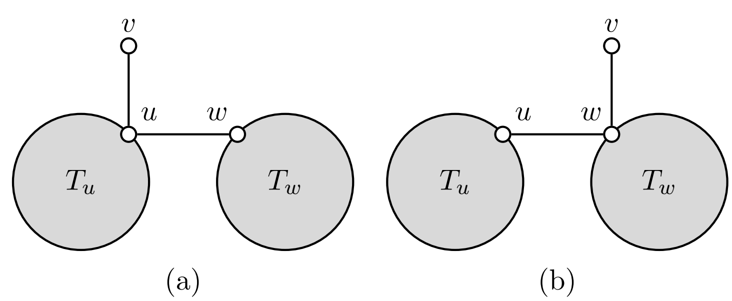


Figure 4.7 (a) A tree T with leaf vertex v with neighbor u. (b) Graph T' as result of vertex slide (T, v) to w.

4.4.2.1 Results about d_{n-1}

The transmission of a vertex is the sum of all distances between a vertex and the rest of the vertices

$$\mathrm{tr}(v) = \sum_{u \in V} \mathrm{dist}(u, v).$$

Let us define a partial ordering of trees by the number of increasing vertex slide moves a tree is away from a star graph. An increasing vertex slide move is a vertex slide move of v from u to w when $|T_w| < |T_u|$. The number of vertex slide moves a graph is away from a star creates a level of trees. We will now prove an interesting observation about the transmission of vertices of trees on an even number of vertices.

Lemma 4.4.3. *Let v be a leaf vertex of tree T (adjacent to u and $w \in N(u)$) with odd (even) transmission. Let tree T' be a vertex slide of (T, v) to w. Then in T', tr(v) is even (odd).*

Proof. Let v be a leaf vertex of tree T (adjacent to u and $w \in N(u)$). Let tree T' be a vertex slide of (T, v) to w. Define T_u to be the subtree of $T - uw - uv$ that contains vertex u and define T_w to be analogously defined for vertex w. We can write the transmission of v for T and T' using the distance from the other vertices in T_u, T_w to u, w with an added adjustment. Therefore,

$$\begin{aligned}
\mathrm{tr}_T(v) &= \mathrm{dist}(u,v) + \mathrm{dist}(w,v) + \sum_{x \in T_u} (\mathrm{dist}(u,x) + 1) + \sum_{x \in T_w} (\mathrm{dist}(w,x) + 2) \\
&= 1 + 2 + \sum_{x \in T_u} (\mathrm{dist}(u,x) + 1) + \sum_{x \in T_w} (\mathrm{dist}(w,x) + 2) \\
\mathrm{tr}_{T'}(v) &= \mathrm{dist}(u,v) + \mathrm{dist}(w,v) + \sum_{x \in T_u} (\mathrm{dist}(u,x) + 2) + \sum_{x \in T_w} (\mathrm{dist}(w,x) + 1) \\
&= 2 + 1 + \sum_{x \in T_u} (\mathrm{dist}(u,x) + 2) + \sum_{x \in T_w} (\mathrm{dist}(w,x) + 1)
\end{aligned}$$

Thus,

$$\mathrm{tr}_T(v) - \mathrm{tr}_{T'}(v) = |T_w| - |T_u| \tag{4.6}$$

We know that $|T_w| + |T_u| + 1$ is an even number (total number of vertices in our tree), so $|T_w| + |T_u|$ is an odd number. Since we have the addition of two numbers resulting in an odd number, then one number must be odd and the other must be even. Moreover, the difference of an odd number and an even number is an odd number. Therefore, $\mathrm{tr}_T(v)$ and $\mathrm{tr}_{T'}(v)$ have different parity. □

Proposition 4.4.4. *Let T be a tree on an even number of vertices. Then for all $v \in V$, either all $tr(v)$ are even or odd. Moreover, if T is an odd (even) levels of slide moves from a star graph, each $tr(v)$ is even (odd).*

Proof. Let vertex v have an even transmission. For some $u \in N(v)$, define T_v to be the connected component of $T - vu$ that contains the vertex v and define T_u analogously. Therefore, we can write the transmission of u in terms of the distance from the other vertices to v with some adjustment.

$$\begin{aligned}
\text{tr}(u) &= 1 + \sum_{w \in T_v, w \neq v} (\text{dist}(w,v) + 1) + \sum_{w \in T_u, w \neq v,u} (\text{dist}(w,v) - 1) \\
&= \text{tr}(v) + \sum_{w \in T_v, w \neq u} 1 - \sum_{w \in T_u, w \neq v,u} 1 \\
&= \text{tr}(v) + |T_v| - |T_u|
\end{aligned}$$

We know $|T_v| + |T_u|$ is our number of vertices in T, which is even. Since we have the addition of two numbers resulting in an even number, then both numbers must be odd or even. We know the subtraction of two even or odd numbers is always even. Therefore $\text{tr}(u)$ must have the same parity of $\text{tr}(v)$.

Therefore, in a tree T, the parity of a vertex's transmission must match their neighbors.

We will now show the graphs alternate parity by the level of vertex slide move. Consider the star graph on n vertices where n is even. This graph is also known as $K_{1,2k+1}$ for some integer k. The center vertex has transmission $2k + 1$, which is an odd number. The leaf vertices have transmission $1 + (2k)(2)$, which is an odd number. So all the vertices of a star graph have odd transmission.

Applying Lemma 4.4.3, we know that at least one vertex changes parity after each slide move. Therefore, all transmissions of vertices in a tree on an even number of vertices have the same parity. □

This interesting observation helps us understand the sum of the diagonal entries of the distance Laplacian matrix. The trace denoted $\text{trace}(M)$ is the sum of the diagonal entries in a matrix M. It is also the sum of the eigenvalues of M. For the distance Laplacian, the trace is also the coefficient (up to sign) of x^{n-1} in the characteristic polynomial.

Lemma 4.4.5. *Let v be a leaf vertex of tree T (adjacent to u and $w \in N(u)$) with odd (even) transmission. Let tree T' be a vertex slide of (T, v) to w. Define T_u be the subtree of $T - uw - v$ that contains vertex u and define T_w be analogously defined for vertex w. Then*

$$\textit{trace}(T') = \textit{trace}(T) + 2(|T_u| - |T_w|).$$

Proof. Let us examine the difference of the transmission of each vertex from T to T'. This difference for vertex v is given in (4.6) as $|T_u| - |T_w|$.

The vertices in T_u have their transmissions increased by 1 since all the distances that do not include v do not change. Analogously, vertices in T_w have their transmissions decreased by 1.

Therefore,

$$\begin{aligned}
\text{trace}(T) - \text{trace}(T') &= \sum_{x \in V} (\text{tr}_T(x) - \text{tr}_{T'}(x)) \\
&= |T_u| - |T_w| + \sum_{x \in T_u} (1) + \sum_{x \in T_w} (-1) \\
&= 2(|T_u| - |T_w|)
\end{aligned}$$

as desired. □

Proposition 4.4.6. *For trees on an even number of vertices with all tr(v) even for $v \in V$, then 4 is a divisor of the trace of the distance Laplacian. If tr(v) is odd for all $v \in V$, then 4 is not a divisor of the trace of the distance Laplacian.*

Note that the sum of an even number of odd numbers not being divisible by 4 is not always true. For example $1 + 1 + 1 + 1 = 4$ which is divisible by 4.

Proof. We will show the result by induction. Consider a star graph or a $K_{1,n-1}$.

$$\begin{aligned} trace(K_{1,n-1}) &= (n-1) + (n-1)(2(n-2)+1) \\ &= (n-1) + (n-1)(2n-3) \\ &= (n-1)(1+2n-3) \\ &= 2(n-1)^2 \end{aligned}$$

Since n is an even number, it follows that the trace of a star is divisible by two but not four.

Now let us perform one slide move on $K_{1,n-1}$. Since $|T_u| = n-2$ and $|T_w| = 1$, it follows that our trace is

$$\begin{aligned} 2(n-1)^2 + 2(n-2) - 2 &= 2(n^2 - 2n + 1) + 2n - 6 \\ &= 2n^2 - 2n - 4 \end{aligned}$$

which is divisible by 4 since n is even and each term of the sum is divisible by 4.

We will show the two induction steps resulting in the trace oscillates between being divisible by 4 and divisible only by 2.

Let v be a leaf vertex of tree T (adjacent to u and $w \in N(u)$) with odd (even) transmission. Let tree T' be a vertex slide of (T, v) to w. Define T_u be the subtree of $T - uw - v$ that contains vertex u and define T_w be analogously defined for vertex w. Let a be the trace of T that is divisible by 4. Therefore, the trace of T' is $a + 2(|T_u| - |T_w|)$ where $|T_u| + |T_w| = n - 1$ by Lemma 4.4.5.

Therefore

$$\begin{aligned} a + 2(|T_u| - |T_w|) &= a + 2(|T_u| - (n - 1 - |T_u|)) \\ &= a + 2(2|T_u| - n + 1) \\ &= a + 4|T_u| - 2n + 2 \end{aligned}$$

since n is even, all the terms are divisible by 4 except for the last one. Thus the new trace is not divisible by 4 but divisible by 2.

Let b be the trace of T that is divisible by 2 but not by 4. So $b = 2k$ for some odd number k. Therefore, the trace of T' is $b + 2(|T_u| - |T_w|)$ where $|T_u| + |T_w| = n - 1$ by Lemma 4.4.5.

Therefore

$$\begin{aligned} b + 2(|T_u| - |T_w|) &= b + 2(|T_u| - (n - 1 - |T_u|)) \\ &= b + 2(2|T_u| - n + 1) \\ &= 2k + 4|T_u| - 2n + 2 \\ &= 2(k + 1) + 4|T_u| - 2n \end{aligned}$$

which is divisible by 4 since $k + 1$ and n are even. □

We can learn from Proposition 4.4.6 that if cospectrality were to occur for the distance Laplacian between two trees with an even number of vertices, it must occur within the same parity of slide moves away from a star.

We also know the trace of the matrix increases if $|T_w| < |T_u|$ when we perform our slide move by Lemma 4.4.5.

We know that the trace of the star is unique. By making one slide move, we get a graph depicted in Figure 4.8(a). From this graph, any vertex slide move increases the trace or returns the star. Therefore, the trace is unique; moreover, spectrally determined among trees.

We know that the trace of the path is unique. By doing one vertex slide move we get a graph depicted in Figure 4.8(b) (comet graph). From this graph, any vertex slide move decreases the trace or returns us to the path. Therefore the trace is unique; moreover, spectrally determined.

Families of trees with a unique trace is atypical. There are at least three graph operations that we can do to a graph that preserves the trace, shown in Figures 4.9, 4.10, 4.11.

Proposition 4.4.7. *The graph operations shown in Figures 4.9, 4.10, 4.11 preserve the trace for the distance Laplacian.*

Proof. First, let us begin with the graph operation in Figure 4.9. By applying Lemma 4.4.5 twice to the two pendant vertices, we get our result.

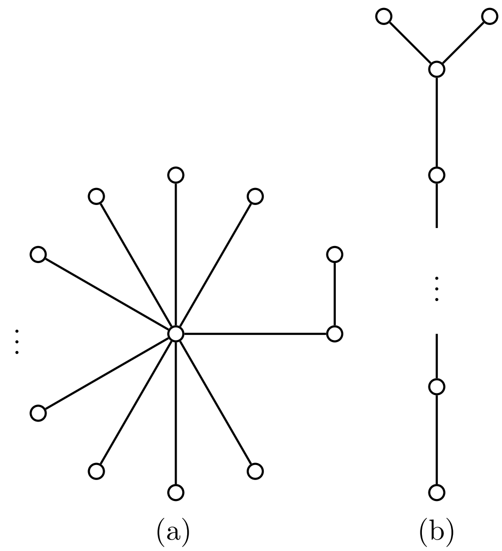


Figure 4.8 (a) A family of graphs one vertex slide moves away from a star which has a unique trace. (b) A family of graphs one vertex slide moves away from a path that has a unique trace. Therefore, both (a) and (b) are spectrally determined among all trees.

Second, for the graph operation in Figure 4.10. By applying Lemma 4.4.5 twice to the pendant vertex, we get our result.

Finally, for the graph operation in Figure 4.11. By applying Lemma 4.4.5 to a pendent vertex adjacent to v and then to the pendant vertex adjacent to $N(u)$ we get our result. □

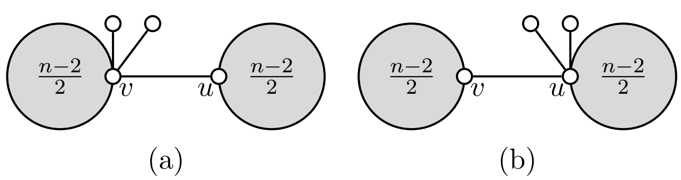


Figure 4.9 Two trees with the same trace for the distance Laplacian

4.4.2.2 Results about d_1

The spectrum of the combinatorial Laplacian determines the number spanning trees of a graph. This comes from a result know as the 'Matrix-Tree Theorem' which says that for any i, j, $\det(L_G(i|j))$ is a constant and the number of spanning trees of G.

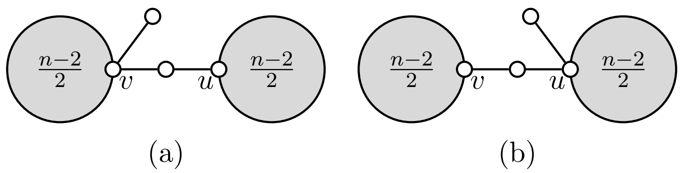


Figure 4.10 Two trees with the same trace for the distance Laplacian

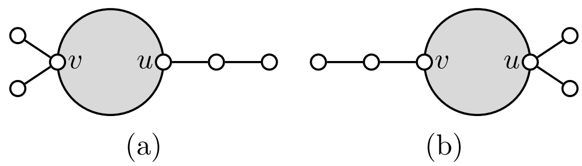


Figure 4.11 Two trees with the same trace for the distance Laplacian

Theorem 4.4.8. *[10] Let L_G be the combinatorial Laplacian matrix of a graph G. Then each cofactor of L_G $\det(L_G(i|j))$ is equal to the number of spanning trees of G.*

This result is sometimes called Maxwell's rule or Kirchhoff's rule and generalizes Cayley's formula for counting the number of spanning trees of a complete graph.

If G is a weighted graph with weighting function w, then cofactor of L_G represents the *weighted spanning tree number* denoted by τ which is the sum of the weights of a graph's spanning trees. The weight of a tree is the multiplication of the weights of the edges. Therefore,

$$\begin{aligned}\tau &= \det(L_{G,w}(i|j)) \\ &= \sum_{T \in SP(G)} \prod_{e \in T} w(e).\end{aligned}$$

We can consider the distance Laplacian matrix as a special case of the weighted combinatorial Laplacian of a complete graph. This view allows us to define τ as a special case of the weighted spanning tree number, calculated using the distance Laplacian's reduced determinant.

We will explicitly calculate τ for the star graph in two different ways. In addition, for trees, we will show a general formula for calculating τ for a graph T' resulting from a slide move of T. Since all trees are a finite number of vertex slide moves from the star Proposition 4.4.2, it follows that this gives a process to calculate τ for all trees.

From the weighted spanning tree number, we can directly computed the characteristic polynomial coefficient of $\mathcal{D}^L$ d_1 as $n\tau = d_1$. Therefore, by learning information about τ we are learning information about a coefficient of the characteristic polynomial of the distance Laplacian matrix.

Theorem 4.4.9. *Let S_n be the star graph on n vertices. So $S_{n+1} = K_{1,n}$. Then*

$$\tau(S_{n+1}) = \sum_{k=1}^{n} \binom{n}{k} 2^{n-k} \sum_{m_1+\cdots+m_k=n} \binom{n-k}{m_1-1,\ldots,m_k-1} m_1^{m_1-2}\ldots m_k^{m_k-2} \tag{4.7}$$

$$= (2n+1)^{n-1} \tag{4.8}$$

where $m_1, \ldots, m_k$ are an order partition of n.

Proof. First, we will prove (4.7). Let $G = K_n$ be a weighted graph such that all edges incident to vertex v have weight 1 and the remaining edges have weight 2. Therefore, $L_G = \mathcal{D}^L(S_n)$.

Let us consider $\prod w(e)$ of the spanning trees of G. There is precisely one tree with $\prod w(e) = 1$, and the remaining trees are powers of 2.

Therefore, each spanning tree's weight is 2^m if m edges of weight 2 are selected. Since there are n vertices, it follows that a spanning will have $n-1$ edges.

Let x be the central vertex of $K_{1,n}$ (the vertex with degree n). If we choose k edges emitting from the vertex to be in our spanning tree, then the tree will have weight 2^{n-k} where $1 \le k \le n$. For each value of k, there are $\binom{n}{k}$ ways to do this.

After this initial selection into the tree T, we need to count how to add the remaining edges. There is a set of vertices U which are an endpoint of an edge already in the tree. Since we want to create a tree, each of these vertices needs to be in its own branch. Each of these k branches has size $m_1, \ldots, m_k$ where $m_1 + \cdots + m_k = n$ and each branch is a spanning tree of the complete graph of the branch size. We know that the number of spanning trees of the complete graph on m vertices is m^{m-2}. Furthermore, we want this $m_1, \ldots, m_k$ partition to be an ordered partition.

Therefore, after the initial selection of edges, there are

$$\sum_{m_1+\cdots+m_k=n} \binom{n-k}{m_1-1,\ldots,m_k-1} m_1^{m_1-2}\ldots m_k^{m_k-2}$$

spanning trees.

Putting this all together, we get

$$sp(K_{1,n}) = \sum_{k=1}^{n} \binom{n}{k} 2^{n-k} \sum_{m_1+\cdots+m_k=n} \binom{n-k}{m_1-1,\ldots,m_k-1} m_1^{m_1-2} \ldots m_k^{m_k-2}.$$

To prove (4.8), we will use the reduced determinant of $\mathcal{D}^L$. If we reduce the matrix by the row and column corresponding to x, then we have a circulant matrix with eigenvalues $\lambda_0 = 1$ and

$$\begin{aligned}\lambda_k &= \sum_{j=0}^{n-1} c_j \omega^{jk} \\ &= (2n-1)e^{2\pi i(0)k/n} - 2\sum_{j=1}^{n-1} e^{2\pi ijk/n} \\ &= (2n-1) - 2(-1) \\ &= 2n+1.\end{aligned}$$

Therefore,

$$\begin{aligned}\tau(K_{1,n}) &= \det(\mathcal{D}^L_{K_{1,n}}(0|0)) \\ &= \prod_{k=0}^{n-1} \lambda_k \\ &= (2n+1)^{n-1}.\end{aligned}$$

□

Note that by using combinatorics identities, these two expressions are also equal.

It is clear that for all trees, the star graph S_n is the smallest τ since it is the only tree with a diameter of 2. We will now prove a formula for $\tau(T')$ computable from $\mathcal{D}^L(T)$ where T' is a vertex slide move of T.

We will use the following proposition which can be derived from the Laplace expansion theorem.

Proposition 4.4.10. *Let A, B be square matrices.*

$$\det(A+B) = \sum_{r=0}^{n} \sum_{\alpha,\beta} (-1)^{s(\alpha)+s(\beta)} \det(A[\alpha|\beta]) \det(B(\alpha|\beta))$$

where α, β are subsets of n of length r and $s(\alpha)$ as the sum of elements of α.

Theorem 4.4.11. *Let T' be a graph obtained by performing a single slide move on (T, v) from u to w. Let T_w be the connected component containing the vertex w of $T - vw$. Then*

$$\tau(T') = \sum_{r=0}^{n} \sum_{\alpha} (-1)^{\sigma(\alpha)} \det(\mathcal{D}^L(T)(\alpha|\alpha))$$

where α is the subsets of n of length r and $\sigma(\alpha)$ is the parity of $\alpha \cap V(T_w)$.

Proof. Let us index our matrix's vertices such that the row and column corresponding to v is the last row and column. The only distances that change from T to T' are the ones that include v, which either increase or decrease by 1. So the transmissions change by 1.

Therefore we can write the weighted spanning tree number of T' as

$$\begin{aligned}
\tau(T') &= \det(\mathcal{D}^L(T')([n-1]|[n-1])) \\
&= \det(\mathcal{D}^L(T) + \operatorname{diag}(\pm 1, \ldots, \pm 1))
\end{aligned}$$

because the only entries of $\mathcal{D}^L(T')$ that change are the diagonal entries.

Since the $\det(\operatorname{diag}(\pm 1, \ldots, \pm 1)[\alpha|\beta]) = 0$ unless $\alpha = \beta$ and is ± 1 for $\alpha = \beta$. The sign depends on the number of vertices that correspond to T_w (whose transmission decreases) who have index in α.

Therefore by applying Proposition 4.4.10,

$$\begin{aligned}
\tau(T') &= \det(\mathcal{D}^L(T) + \operatorname{diag}(\pm 1, \ldots, \pm 1)) \\
&= \sum_{r=0}^{n} \sum_{\alpha} (-1)^{\sigma(\alpha_w)} \det(\mathcal{D}^L(T)(\alpha|\alpha)).
\end{aligned}$$

Note that $\det(\mathcal{D}^L(T)(\alpha|\alpha))$ are the square principle subdeterminants of the matrix $\mathcal{D}^L(T)$. □

This parameter for trees is not dependent on diameter. For example consider the graphs given in Figure 4.12.

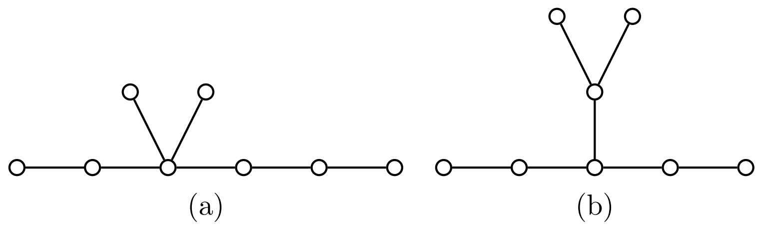


Figure 4.12 (a) A tree with $\text{diam}(T) = 5$ and $\tau = 92,855,071$ (b) A tree with $\text{diam}(T) = 4$ and $\tau = 93,794,169$. Therefore, τ does not always increase as diameter increases.

Since the weighted spanning tree number grows large quickly, it is unpromising to determine that some pair of trees must have the same value for some large value of n.

There is no known example of two trees having the same value of τ up through 23 vertices as the algorithm to determine potential cospectral trees used only τ. This parameter would be interesting to consider for future studies into spectrally determined trees for the distance Laplacian matrix.

4.4.3 Search Algorithms for Cospectral Trees

Aouchie and Hansen [4] empirically found no evidence of cospectral trees through 20 vertices. A new search technique, presented in this section, using the weighted spanning tree number, verified this result; and increased the known data through 23 vertices. The development of this technique came from an observation that no trees share the last coefficient of their characteristic polynomial on small numbers of vertices. Since the weighted spanning tree number is a smaller set of data to store than the entire characteristic polynomial, we can reduce our search to look at trees that share this coefficient.

Since our search space increases rapidly, we use a dictionary data type to store our graphs. In python (and Sage), a dictionary is a hashtable. A hashtable is a data type that stores information using a key. Hashtables have been implemented for large data sets to find repetition.

First, we will create a dictionary of all our graphs on n vertices using the graph's `graph6_string` as the key (these are unique to graphs) and the determinant of the matrix with one row and column deleted (τ) as the value. Then, we create a reverse dictionary where the value is the set of all graphs with a particular τ. We can then do a quick search on all these sets and pull the ones out with more than one graph attached to the key. Pseudo-code of this process is given in Algorithm 3.

This algorithm runs in polynomial time (τ is computed in polynomial time), making it efficient for a large number of vertices. The longest-running part is creating the initial dictionary. For $n = 19$, this took about 20 minutes to run on the Sage server.

Algorithm 3: `cospecTreeFinder`(n)

Input: number of vertices n

Output: List of distance Laplacian cospectral trees

treedict = {g.graph6_string():$\mathcal{D}^L$(g)(1—1) for g in graphs.trees(n)}
rev_dict = {}
foreach *key,value in treedict* **do**
  rev_dict.setdefault(value, set()).add(key)
result=filter(lambda x: len(x) > 1,rev_dict.values())
return result

This program was run through 23 vertices and found no graphs that share this coefficient.

4.5 General Conclusions

Information from both of these matrices adds to what we know about graphs by looking at their spectrum. The added evidence to Conjecture 4.2.1 is exciting and, if true, would be a significant result in the field. This area needs more theoretical work as the limits of computational power will eventually be reached.

The exponential distance matrix is new, and its behavior of being multiplicative rather than additive creates a unique flavor of the matrix. There is still plenty to learn about the spectrum. In particular, there is evidence that all clique-paths of the same generating sequence have the same determinant mirroring a result for the distance matrix by Cheng and Lin [9]. Additionally, there is an open problem about cospectrality for only some values of q. Here we provided a construction for exactly $q = 1/2$. Currently, there is no evidence that another value of q has the same property. Both of these variations have interesting spectral properties, many of which remain unexplored.

4.6 References

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CHAPTER 5. GENERAL CONCLUSIONS

Spectral graph theory tries to understand what information is preserved or lost about a graph from its spectrum. We presented many cospectral constructions to show predictable ways that information can be lost. In Chapter 2, we gave a new construction method that swaps subgraphs. The proof of this construction technique is similar to Godsil-McKay switching, one of the more famous cospectral construction methods. However, our new method is valid on both adjacency and distance matrices. We enumerated this method for graphs on a small number of vertices and observed that the method occurs more frequently on some matrices than others.

In Chapter 3, we looked at cospectral constructions. Understanding cospectral constructions gives us insight on how they can be used and improved upon. For small, local graph changes, we observed many authors using similarity matrices. When constructing graphs with specific structure like a cut vertex, using the co-factor expansion definition of the characteristic polynomial helps break it into smaller parts which can be swapped out. These constructions methods apply to several matrices and showcase the relationship of cycle decompositions between graphs and matrices. This allows us to toggle certain subgraphs on and off in graphs. We observed many authors used this method when large graph changes occur between graphs. For certain matrices, finding a relationship between eigenvectors of different graphs finds cospectral pairs. From all of these methods, we learn more about the relationship between the spectrum and the graph structure. Additionally, depending on the pattern of cospectrality, a certain construction method will be more fruitful.

The spectrum of variations of the distance matrix behave in new and interesting ways. In Chapter 4, we considered the exponential distance matrix and found the inertia for several graph families. This helps us compare the matrix to other well studied matrices. We also gave a cospectral construction for exactly one value of q. The proof highlights the multiplicative property of this matrix showing how the co-factor expansion definition of the characteristic polynomial is a powerful way to connect structure and spectrum.

Additionally in Chapter 4, we considered the distance Laplacian matrix and developed results about the coefficients of the characteristic polynomial for trees. Trees are an interesting case for this matrix as there are currently no known cospectral trees. We developed techniques for more efficient algorithms to find cospectral trees. Both of these new matrices differ from previously studied matrices and offer new ways to connect spectrum to graph structure.

5.1 Future Work

The new cospectral construction in Chapter 2 uses a proof technique that is similar to Godsil and McKay's famous cospectral construction in [2]. Both of these proofs show that $\mathrm{diag}(\frac{1}{k}J - P, I)$ is a similarity matrix where $P = I$ in Godsil-McKay and $P = \hat{I}$ in Chapter 2. Both $I, \hat{I}$ are symmetric permutation matrices. We are curious if for every symmetric permutation matrix P, does there exist a cospectral construction.

Additionally, this new cospectral construction method applies to almost all well-studied matrices. Are there other known copsectral construction methods that can be modified to apply to other matrices? In Chapter 3, many constructions had similar proof techniques. We are curious if there are other novel ways to show two graphs have the same spectrum.

New variations of the distance matrix, such as the exponential distance matrix, provide new ways that graph structure connects to the spectrum. We are curious about the determinant and inertia for other families of graphs, such as co-clique graphs. Like trees, they can be constructed by iteratively adding a vertex. Therefore, similar arguments may apply. We conjectured in Chapter 4 that every pair of graphs is cospectral for $0, 1$, or all values of q. We are curious about results relating to this.

Finally, Aouchiche and Hansen [1] conjectured that all trees are determined by the distance Laplacian spectrum. We are curious about this problem and look for future development.

5.2 References

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