1985

The impact of farmland price changes on farm size, financial structure enterprise choice

James M. Lowenberg-DeBoer
Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Agricultural and Resource Economics Commons, and the Agricultural Economics Commons

Recommended Citation
Lowenberg-DeBoer, James M., "The impact of farmland price changes on farm size, financial structure enterprise choice " (1985). Retrospective Theses and Dissertations. 8723.
https://lib.dr.iastate.edu/rtd/8723

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
INFORMATION TO USERS

This reproduction was made from a copy of a manuscript sent to us for publication and microfilming. While the most advanced technology has been used to photograph and reproduce this manuscript, the quality of the reproduction is heavily dependent upon the quality of the material submitted. Pages in any manuscript may have indistinct print. In all cases the best available copy has been filmed.

The following explanation of techniques is provided to help clarify notations which may appear on this reproduction.

1. Manuscripts may not always be complete. When it is not possible to obtain missing pages, a note appears to indicate this.

2. When copyrighted materials are removed from the manuscript, a note appears to indicate this.

3. Oversize materials (maps, drawings, and charts) are photographed by sectioning the original, beginning at the upper left hand corner and continuing from left to right in equal sections with small overlaps. Each oversize page is also filmed as one exposure and is available, for an additional charge, as a standard 35mm slide or in black and white paper format.*

4. Most photographs reproduce acceptably on positive microfilm or microfiche but lack clarity on xerographic copies made from the microfilm. For an additional charge, all photographs are available in black and white standard 35mm slide format.*

*For more information about black and white slides or enlarged paper reproductions, please contact the Dissertations Customer Services Department.
Lowenberg-DeBoer, James M.

THE IMPACT OF FARMLAND PRICE CHANGES ON FARM SIZE, FINANCIAL STRUCTURE AND ENTERPRISE CHOICE

Iowa State University

University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106

Ph.D. 1985
The impact of farmland price changes on farm size, financial structure and enterprise choice

by

James M. Lowenberg-DeBoer

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

Department: Economics
Major: Agricultural Economics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1985
## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>GENERAL INTRODUCTION</strong></td>
<td>1</td>
</tr>
<tr>
<td>Explanation of Dissertation Format</td>
<td>4</td>
</tr>
<tr>
<td><strong>PART I  BASIC THEORETICAL MODEL</strong></td>
<td>6</td>
</tr>
<tr>
<td>Previous Research</td>
<td>9</td>
</tr>
<tr>
<td>Theoretical Framework</td>
<td>11</td>
</tr>
<tr>
<td>Wealth Approach</td>
<td>16</td>
</tr>
<tr>
<td>Finite Planning Horizon</td>
<td>30</td>
</tr>
<tr>
<td>Income Approach</td>
<td>42</td>
</tr>
<tr>
<td>Two Output Static Model</td>
<td>51</td>
</tr>
<tr>
<td>Dynamic Model, Wealth Approach</td>
<td>54</td>
</tr>
<tr>
<td>Dynamic Model, Income Approach</td>
<td>89</td>
</tr>
<tr>
<td>Summary and Conclusions</td>
<td>99</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>104</td>
</tr>
<tr>
<td><strong>PART II  EXTENSIONS OF THE THEORETICAL MODEL</strong></td>
<td>107</td>
</tr>
<tr>
<td>Previous Research</td>
<td>108</td>
</tr>
<tr>
<td>The Basic Model</td>
<td>112</td>
</tr>
<tr>
<td>Limited Availability of Land</td>
<td>116</td>
</tr>
<tr>
<td>Adjustment Costs</td>
<td>136</td>
</tr>
<tr>
<td>Equity Investment Limits</td>
<td>143</td>
</tr>
<tr>
<td>Land Price Risk</td>
<td>156</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>193</td>
</tr>
<tr>
<td><strong>PART III  EXPLORATORY EMPIRICAL WORK</strong></td>
<td>195</td>
</tr>
<tr>
<td>Previous Empirical Research</td>
<td>199</td>
</tr>
<tr>
<td>Conceptual Framework</td>
<td>201</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>The Numerical Model</td>
<td>211</td>
</tr>
<tr>
<td>Input Data and Parameter Estimation</td>
<td>230</td>
</tr>
<tr>
<td>Numerical Results</td>
<td>258</td>
</tr>
<tr>
<td>Higher Leverage Solutions</td>
<td>272</td>
</tr>
<tr>
<td>Conclusions and Implications</td>
<td>291</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>312</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>317</td>
</tr>
<tr>
<td>APPENDIX B</td>
<td>366</td>
</tr>
<tr>
<td>GENERAL SUMMARY AND CONCLUSIONS</td>
<td>377</td>
</tr>
<tr>
<td>REFERENCES FOR INTRODUCTION AND SUMMARY</td>
<td>380</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>381</td>
</tr>
</tbody>
</table>
GENERAL INTRODUCTION

This volume is a collection of three studies which develops and tests a theory of how capital gains and losses affect production and finance decisions. That this work was carried on within the discipline of agricultural economics is largely an historical accident. The organization of U.S. agriculture in small closely held businesses which combine, under one management investment, production and finance decisions and the farmland capital gains and losses of recent years have forced farmers, the makers of farm policy, and agricultural economists to be aware of the potential link between land price changes and management choices. When capital gains and losses are a small part of the returns, the usual microeconomic theory which assumes that all inputs are consumed in the production process and that the firm is only interested in current income from production may be an adequate approximation. But, when capital gains are as large as current returns, as they were in U.S. agriculture during the 1970s (Melichar, 1979, p. 1086), that approximation can be called into question. It seems reasonable that producers take into account the fact that some inputs are not consumed in the production process and may be sold for more than their purchase price after use.

The concern about the impact of farmland capital gains and losses is not limited to the U.S. in the twentieth century. For instance, Brannon has hypothesized that the low rate of investment in agricultural equipment and inputs in Uruguayan agriculture in the 1950s and 1960s was due to land price increases which made it more profitable for landowners
to invest in more land than in more intensive use of their existing land. Work by agricultural historians provides evidence that farmland price changes and expected capital gains had an effect on the rate and the form of development of land resources on the U.S. frontier (A. Bogue, 1985; M. Bogue, 1959; Swerenga, 1968; Gates, 1973).

The potential for capital gain and loss effects is, however, not limited to agriculture. It can happen wherever there are inputs that are not consumed in the production process or are consumed relatively slowly. For instance, does the pattern of price change on urban real estate affect how land is developed? Capital gain and loss effects may also occur in a consumption context. For instance, antiques or art works may be thought of as producing esthetic services, but they may also appreciate in value. Will individuals hold a greater quantity of antiques or art if their price is rising than if they held these items solely for esthetic reasons? Was the boom in the demand for antiques in the U.S. in the 1970s related to increasing esthetic awareness or was it a function of the price appreciation potential? Similarly, residential housing can be thought of as providing housing services — shelter, location, comfortable environment, etc. Will the demand for residential housing change if home prices are rising? Can the desire to earn additional capital gains explain some of the real estate market behavior in the U.S. in the 1970s? Are such diverse phenomena as condominium conversions and the entry of unmarried, childless individuals into the housing market related to capital gains? Will some people who would ordinarily be content to rent housing desire to buy if appreciation potential is large?
These studies do not answer all these questions, but they provide a first step toward a better understanding of the role of capital gains and losses in economic decisions.

The capital gain and loss effect identified here differ from pure speculation activity in that the assets acquired have some legitimate use in the firm or the household. Farmers normally use land in their production process. Individuals need residential housing. The question here is how price changes affect the choice to hold and use the appreciating or depreciating asset.

Some researchers have argued that capital gains and losses are discounted changes in future income and cannot be analyzed as a separate form of returns (Fischer, 1930, p. 25). In this view, a model which includes both the capital gain or loss and the income stream from the asset earning the capital gain or loss is potentially double counting some income. The identification of capital gains and losses with changes in future income depends, however, on perfect capital markets. With heterogeneous expectations, imperfect information, and transactions cost, the direct link between these two forms of return may be severed. For instance, the farmer who anticipated the low farm income and declining land prices of the mid-1980s could have sold land in 1981, the high point in land prices, for the same price as a farmer who anticipated a future of strong farm income and continued rise in land value. The capital gain realized would be independent of the farmer's own view of the future.

Similar arguments can be advanced about the value of unrealized capital gains. The value of unrealized capital gain as collateral will
depend on the lender's view of the future, but it does not necessarily depend on the borrower's. The individual who anticipated price declines can borrow against accrued capital gains to the same extent as a person who expects increases, though the use of the loans may be different. If the consumption decision depends on wealth, then consumption may be affected by a capital gain even if the individual believes that the future income increase anticipated by the market will never occur. For example, land owners in the 1970s may have changed their consumption patterns to reflect increased wealth in the form of farmland, even if they did not believe that income from land would continue to rise.

Using the common microeconomic assumption that the decisionmaker cannot affect the prices of inputs or outputs, current income and capital gains can be treated as two separate sources of returns. Some future research may completely specify the relationship of income and capital value under varying degrees of expectational heterogeneity, imperfect information, and transactions cost, but these studies adhere to the simplifying assumption that the decisionmaker is a small part of the input and output markets and does not affect prices.

Explanation of Dissertation Format

The organization structure of this dissertation is three separate studies which build upon each other. The original research for the writing of these papers was done solely by the author. In the first study, a theoretical model of how capital gains and losses affect production and finance is constructed. This is developed in the framework of a
modified Vickers model; first with a static approximation and then with a more complete dynamic model using optimal control theory techniques. The second study extends the theoretical model to an environment which includes limitations on land availability, adjustment cost, equity investment constrained to retained earnings, and land price risk. The third study is an exploratory empirical test of the theoretical framework developed in the first two studies. It was a discrete, deterministic dynamic programming model based on data for central Iowa during the period 1970-1984. The policy implications of the three papers are discussed in the last section of the third study. A brief statement of summary and conclusions follows the third paper.
Statements in the media and the professional economic journals indicate a widespread belief that farmland price changes have important impacts for farm production and finance. Media accounts of the current financial stress of some producers frequently attribute a large part of the problem to falling land prices which reduce the farmer's net worth and curtail the borrowing power of the farm business. Agricultural economists have suggested that land price changes may have a wide variety of impacts. Castle and Hoch (1982) said that farmland capital gains may reduce incentives to adopt "land saving practices and technologies" (pp. 16-17). They indicate that farmland price increases may help explain the increase in acreage per farm, the use of larger machinery and the relatively low growth of land productivity in the 1970s. Economists have often hypothesized that farm expansion and capital gains are related because the appreciated value of land holdings provides a base for additional purchases (Lins and Duncan, 1980, p. 1025; Plaxico, 1979, p. 1089; Raup, 1978, p. 303; and Robinson, 1979, p. 906). It is suggested that the preferential tax treatment of capital gains has exacerbated the farm expansion trend by offering high income land owners additional incentives for purchasing more land (Davenport, Boehlje and Martin, 1982, p. 17-18; Raup, 1978, p. 305). Lins and Duncan indicate that the impact of farmland capital gains on farm financing goes beyond increasing the borrowing capacity of the farmer. They suggest that rising prices encourage greater reliance on debt financing. In an attempt to reap the
benefits of capital gains, farmers buy sooner and go into debt further than they would in a stable price environment (p. 1051).

In spite of the many hypotheses, little research has been done on the impact of capital gains and losses on production and finance decisions. Current firm level microeconomic theory does not provide a well-developed framework for such research since it usually assumes that the firm is only interested in current income and that inputs are consumed in the production process. In most industries, these assumptions may be adequate, but in U.S. agriculture capital gains and losses have been very important concerns. Melichar (1979) indicates that in the 1970s real capital gains were roughly equal to current income from agricultural production (pp. 1085-1086). In the 1950s and 1960s, capital gains were about one-third as large as current income. In the 1920s and early 1930s, and again in the early 1980s, capital losses were large in some areas. Because of the dominant position of land in the farm asset structure, most of the capital gains and losses stem from farmland price changes. On the individual asset or project level, the impact of capital gains and losses is well-known, for instance through the salvage values incorporated in net present value or internal rate of return calculations, but the asset or project level analysis does not capture the firm level impacts that have to be hypothesized. Similarly, the impact of capital gains and losses on purely investment decisions is well-known; investment models frequently assume that return can be either in the form of current income or capital gain. But the pure investment approach does
not provide an adequate framework to analyze choices which also include output levels, enterprise choice and financing production.

A better understanding of the interrelationship of capital gains, farm size and financial structure would be useful on the microeconomic level in analyzing farm business decisions and on the macroeconomic level in considering the effect of tax and price support policies. As an initial step in understanding this relationship, this study develops a simple model of the farm firm which includes land price changes. The model uses the common microeconomic assumption that the firm is small enough so that it has no impact on input or output prices. The interaction between producer decisions and farmland prices is an important subject, but it is not the primary focus of this research. The financing of the farm firm is considered as a constraint in the manner outlined by Vickers.

Both a static and dynamic version of the model are developed. The following section reviews previous research on farmland capital gain and loss impacts. Subsequent sections detail the static model, which assumes that variables are constant throughout the planning horizon and that the land price path is linear, and the dynamic model allows wider variety of land price paths and varying input and financial asset levels. The dynamic model is examined under several hypotheses about the value of unrealized capital gain and loss, and under various tax regimes. A two generation model is also developed. The final sections examine the enterprise choice decision and provide a brief look at the implications of the model.
Previous Research

Research which has explicitly examined the impact of capital gains on production and finance choices is rare. The research on breeding livestock capital gains does not generalize to the case of farmland capital gains because breeding livestock capital gains are primarily a result of tax rules which allow the costs of raising these animals to be written off as a current expense, but considers the proceeds from the sale of raised breeding stock to be capital gain. Ordinarily, the capital gain calculated for tax purposes on breeding stock includes little price appreciation. A study by Vandeputte and Baker (1970) which did consider the impact of farmland capital gains on the input mix and enterprise choice found that the gains and how they are taxed had important effects. In a linear programming example, they show intensive hog production in the optimal solution when capital gains are fully taxed, but land ownership and crop production when the partial exemption of capital gains from income taxes was included in the model (pp. 526-527). The analysis did not include financial activities and, the authors note, "were credit specifications also to be included, the differences might be still greater because of the credit consequences of assets accumulated under the capital gains provisions of the tax laws" (p. 526).

An important precedent for this research is the debate in agricultural economics literature over the last three decades concerning the value of unrealized capital gains. In the traditional methods of analyzing the individual purchase decision, capital gains have not been
recognized until they are realized. Because land is often held for long periods before it is sold, the net present value of the gain at the time it occurs may be small even if the gain is substantial. Some researchers have, however, argued that unrealized gains have value because they add to the wealth position of the owner (Boyne, 1964; Crowley, 1974; Grove, 1950). Bhatia (1972) argued that in a world of perfect capital markets and equal taxes on all types of income, unrealized capital gains would be a perfect substitute for current income in the wealth of the individual, and hence there is justification for including at least a part of the capital gain in the current income stream of an individual. Plaxico and Kletke (1979) formalize this approach by recognizing a fraction of capital gain as income, while deferring the remaining gain and the taxes on the gain to the end of the holding period. They emphasize that the fraction of the gain which can be considered a perfect substitute for current income depends on the cash flow problems of the owner and on whether the gains are viewed as permanent. The recognition of a portion of unrealized capital gains or loss as a substitute for cash income is denoted as the wealth approach (Plaxico and Kletke, 1980, p. 264; Bhatia, 1972, p. 868) because unrealized capital gain substitutes for income in determining wealth.

A second approach to the value of unrealized gain is to argue that the unrealized gain is a substitute for equity in the financial negotiation (Plaxico and Kletke, 1979, p. 238). Unrealized gain increases the financial base for acquiring credit and reduces risk for borrower and lender. In this approach, the value of unrealized gain is the extra
income earned because of the added borrowing capacity and increased financial flexibility. The fraction of unrealized gain that would be a perfect substitute for equity in the financial negotiation is primarily a function of the lender's view of those gains as security for the loan. The second method of valuing unrealized capital gains and losses is denoted as the income approach (Plaxico and Kletke, 1980), p. 264) because the focus is on the income or earning capacity generated by accumulated capital gains and losses through their effect on borrowing power. Because of the long period of generally rising real land prices from 1940 to 1980, much of the research on farmland price changes is formulated exclusively in terms of capital gains. In most cases, the extension of the arguments to capital losses is straightforward, involving mainly changes in the tax impact because capital gains and losses are not treated symmetrically in the U.S. tax systems.

Theoretical Framework

The basic theoretical framework of this study was developed by Vickers (1968). The approach is to determine the optimal input mix and financial structure that maximizes the present value of the firm's income subject to a money capital constraint. The Vickers' model is concerned primarily with long run planning of investment, finance and production, hence all asset levels and even enterprise choice is considered variable (p. 43). Wealth maximization is chosen as the appropriate objective function in the Vickers model because it allows consideration of the timing of the income stream, and through the risk premium in the discount
rate, it allows the owner's risk preferences to be expressed (p. 7). The net present value maximization also approximates utility maximization for cases in which utility is primarily a function of the level of money income and the owners may borrow against or save cash returns to achieve the desired consumption pattern. In its simplest form, the Vickers model maximizes:

$$V = \frac{\pi}{\rho} - K$$ (1)

where:

- $V$ = the present value of the stream of income from the firm,
- $\pi$ = the expected income per period,
- $\rho$ = the discount rate, and
- $K$ = invested equity.

This simple form of the present value of the firm can be derived using either a discrete or continuous time model assuming a stream of returns over an infinite planning horizon:

$$V = \frac{\pi}{\rho} - K = \int_0^\infty e^{-\rho t} \pi \, dt - K \tag{2}$$

or

$$V = \frac{\pi}{\rho} - K = \sum_{t=0}^\infty \frac{\pi}{(1+\rho)^t} - K$$

Convergences to the simple static form (2) requires that the income term ($\pi$) be independent of time. The difference between the two models is in
the definition of \( \pi \) and \( \rho \), but with appropriately defined variables both should yield the same solutions. The discount rate used in this study is assumed to be based on the yields of alternative investment opportunities with risk levels similar to those encountered in agriculture. The simple static form of the objective function in equation (1) assumes that input levels, debt use and equity investment are set initially and remain constant throughout the planning horizon.

In the agricultural context where most firms are sole proprietorships, the infinite life firm can be reinterpreted to be a family farm on which the farm family intends to go on farming indefinitely passing the farm business from generation to generation. The idea of an infinite life family farm fits well into the traditional view of the family farm that underlies much of the public debate on farm policy, but it abstracts away from the distinct entry and exit pattern of U.S. farming.

The \( \pi \) in the Vickers' model is specified to be price times quantity minus the cost of inputs used in the production process and the interest on debt:

\[
\pi = P \cdot f(X, L) - \gamma_1 X - \gamma_2 L - r(\frac{D}{K}) D
\]

where:

\( P = \) the price of output,
\( L = \) farmland,
\( X = \) composite input which includes everything but land,
\( D = \) debt,
\( \gamma_1, \gamma_2 \) = the current cost of inputs used,

\[ f(X, L) = \text{a strictly concave production function with:} \]
\[ f_X, f_L > 0, f_{XX}, f_{LL} < 0 \text{ and} \]

\[ r(\frac{D}{K}) = \text{the debt supply function with:} \]
\[ r' > 0, r'' > 0. \]

In Vickers' (1968) general model price is a function of the quantity of output, but for much of agriculture the assumption that the firm is a price taker is realistic. The gammas are the costs of inputs actually consumed in the production process. For example, in the case of depreciable property, the gammas define the value of wear and tear, regular maintenance, and replacement costs (p. 128). In the case of farm real estate, the gamma coefficient would be primarily composed of property taxes, insurance and the depreciation on buildings, fences, tile lines and soil conservation structures. In this simple model, the current costs \( (\gamma_1, \gamma_2) \) are assumed to be constant throughout the planning horizon. Vickers suggests that the supply of debt capital and the cost of credit can be modeled as a function of the leverage ratio (pp. 67-68). The argument is that the interest rate will rise as the leverage ratio increases because the lender must assume more risk in the highly leveraged firm. The income term will be independent of time and the objective function will converge to the simple static farm (2) if input, debt and equity levels are set at the initial time and remain unchanged.

The money capital constraint on the maximization is of the form:

\[ (K + D - \alpha X - \beta L) = 0 \]

where: \( \alpha, \beta \) = the amount of capital absorbed by each input.
The financial constraint (4) must be satisfied at all points in time, but in the static case it is enough to satisfy the constraint at the initial time because input, debt and equity levels are not changed during the planning horizon. The capital absorption coefficients \((\alpha, \beta)\) differ from the current cost terms \((\gamma_1, \gamma_2)\). For example, if a tractor is purchased with cash, the capital absorbed is the cash outlay. The current costs of using that tractor are the fuel, maintenance, repairs and depreciation. Alpha and beta may reflect special financial arrangements that are only available with a specific input, but for simplicity this analysis will assume that \(\alpha\) and \(\beta\) are equal to the prices of the input.

The basic Vickers' model is the maximization of the present value of the income stream (1) subject to the financial constraint (4). The basic model was modified in several ways to fit the special characteristics of farm firms. The wealth approach to the value of unrealized gain was modeled by including a fraction of the capital gain or loss in the current income expression (3). The income approach was examined by including a fraction of the unrealized capital gain in the denominator of the argument of the debt supply function. Because of the distinct entry and exit pattern of much of U.S. agriculture, a terminal horizon model is examined. Analysis of the static model is primarily through examination of the first order conditions for maximization and through formal comparative statics (Henderson and Quandt, 1980, pp. 25-27, pp. 80-81). The dynamic models relax the assumption that input, debt and equity levels remain constant over the planning horizon. Analysis of the dynamic
models is through examination of the necessary conditions for maximization and through explicit control equations for the change in input use and financial structure.

**Wealth Approach**

The wealth approach model combines (2), (3), (4) a capital gains term and taxes in a Lagrangian expression:

\[
\text{Max } Z = \frac{1}{(1-\tau)} \left[ \left( \frac{P_{f}(X, L)}{K} - \gamma_{1}X - \gamma_{2}L - \tau \left( \frac{D}{K} \right) D \right) (1-\tau) \right] \\
+ \phi \theta L - K + \lambda \left( X + D - \alpha X - \beta L \right)
\]

where:  
\( \phi = \) the portion of unrealized gain that is a perfect substitute for income,  
\( \theta = \) the change in the land price per acre per rod,  
\( \tau = \) the average tax rate, and  
\( \lambda = \) the Lagrangian multiplier.

The capital gain term \((\phi \theta L)\) represents the capital gain or loss that is recognized as a substitute for cash income or loss in determining wealth. The convergence of the objective function to the simple static form (2) is maintained because the land price increase \((\theta)\) is independent of time. The land price path is assumed to be a linear function of time:

\[
\beta_{t} = \beta + \theta t
\]

where: \( \beta_{t} = \) the land price at time \( t \).
Only the land price at the initial time (β) is relevant for the financial constraint in the static model because land is acquired only at the beginning of the planning horizon. Thus, the capital absorbed by land does not change though the land price changes. For simplicity, the model uses a constant tax adjustment factor (1-τ), however, it should be remembered that this ignores the impact of tax rate progressivity. The average tax rate is used because the whole farm business is the subject of analysis here, not just the marginal asset or project. The tax adjustment of the discount rate assumes that the alternative investment opportunities on which the discount rate calculation is based are fully taxed. The model does not include the effects of inflation; it is assumed that all prices and costs are in real terms. The price and other parameter values may be thought of as expected values used in long range planning.

Optimization of the Lagrangian expression (5) is a straightforward calculus problem. The decision variables are land (L), nonland inputs (X), debt (D) and equity capital (K). The first order conditions for the static model are:

\[\frac{\partial Z}{\partial X} = \frac{1}{(1-\tau)\rho} \left( \alpha \tau L - \gamma_1 \right) (1-\tau) - \alpha \lambda = 0\]

\[\frac{\partial Z}{\partial L} = \frac{1}{(1-\tau)\rho} \left[ (\alpha L - \gamma_2 \tau) (1-\tau) + \theta \phi \right] - \beta \lambda = 0\]

\[\frac{\partial Z}{\partial D} = - \frac{1}{(1-\tau)\rho} \left[ \tau + \tau' \left( \frac{D}{K} \right) \right] (1-\tau) + \lambda = 0\]
The second order conditions (SOC) are satisfied if the Hessian matrix of second derivatives of the objective function is negative for all changes which satisfy the constraint; hence, the second order conditions will certainly be satisfied if the Hessian is negative definite for all possible changes. The Hessian is a block diagonal matrix of the form:

\[
H = \begin{bmatrix}
A & 0 \\
0 & C
\end{bmatrix}
\]

where \( A \) is the 2x2 matrix of second derivatives of the production function multiplied by the constant \( P/K \). \( 0 \) is a conformable null matrix and \( C \) is:

\[
C = -\frac{1}{\rho K} \left[ 2r^r + r^r - K \right] \begin{bmatrix}
1 & -\frac{D}{K} \\
-\frac{D}{K} & \left(\frac{D}{K}\right)^2
\end{bmatrix}
\]

For \( H \) to be negative definite the condition is:

\[
h'Hh = \left( h_1 \ h_2 \right) \begin{bmatrix}
A_{h_2} & h_1 \\
h_2 & C_{h_4}
\end{bmatrix} \begin{bmatrix}
h_1 \\
h_2
\end{bmatrix} < 0
\]

\[\frac{\partial Z}{\partial \lambda} = K + D - \alpha X - BL = 0\]
for all $h \neq 0$. The term in $A$ is negative definite for all $h$ by the assumption of a strictly concave production function. The term in $C$ can be expanded and rewritten as:

\begin{equation}
\begin{aligned}
(7.3) \quad \left( h_3 h_4 \right)^h C_{h_4} &= -\frac{1}{\rho^K} \left( 2r + r' \frac{D}{K} \right) \left( h_3 - h_4 \frac{D}{K} \right)^2
\end{aligned}
\end{equation}

Hence, the term in $C$ is negative semidefinite for all $h$ under the assumptions of a convex debt cost function ($r'^{>0}$, $r''^{>0}$) and the second order conditions are satisfied.

The financial structure of the optimal solution can be characterized by solving equation (6.4) for $\lambda$, substituting the $\lambda$ value into equation (6.3) and simplifying, yielding:

\begin{equation}
\begin{aligned}
(8) \quad \rho - r' \left( \frac{D}{K} \right)^2 &= r + r \frac{D}{K}
\end{aligned}
\end{equation}

The left hand side of equation (8) is the marginal cost of equity capital and the right hand side is the marginal cost of debt capital. Equation (8) indicates the common financial result that in the optimal financial structure the marginal cost of all sources of capital is equal. In theory, equation (8) could be solved for the optimal leverage ratio as a function of the discount rate. Capital gains do not affect the financial structure of the model if equity is variable and the wealth approach to the value of unrealized gain is used. Taxes also do not affect the
financial structure in this case. This occurs because of the assumption that the investment alternatives which are used to develop the discount rate are fully taxed. Tax terms would appear in equation (8) if alternative investments were partially or totally tax exempt.

The marginal rate of substitution (MRS) of land for other inputs can be found by substituting (6.3) into (6.1) and (6.2), and solving:

\[
\frac{f_x}{f_L} = \frac{\left[ Y_1 + \alpha (r + r_D D) \right] (1-\tau)}{\left[ Y_2 + \beta (r + r_D D) \right] (1-\tau) - \phi \theta}
\]

As is typical of the Vickers' formulation, the MRS is equal to a factor cost ratio which includes the cost of capital absorbed by the input. Unlike the usual Vickers' model, the denominator of the factor cost ratio includes a land price change term \( \phi \theta \). If capital gains are occurring \( \theta > 0 \) and part of those gains are substitutable for current income \( \phi > 0 \), then the capital gains will tend to offset the current cost of owning land. Under those conditions the factor cost ratio, and hence the optimal MRS, will be larger than in the usual microeconomic model which ignores capital gains. The increased MRS indicates that when capital gains are taken into account the optimal solution includes more land in the input mix than it otherwise would (Figure 1). Capital losses have the opposite effect; they tend to increase the cost of land ownership, reduce the factor cost ratio and hence reduce the farm size in the optimal solution. In an environment of capital losses, the decision
Figure 1. If capital gains are not considered in the model the input mix would that described by point like A. However, if capital gains are considered the MRS is higher and the optimal input mix is described by some point like B. In addition to shifting along the isoquant, capital gain and loss effects may also force changes in the output level.
maker approximated by this model would tend to economize on land use to avoid the capital losses. The impact of the land price change on input use is increased if the capital gains or losses are larger or if a larger percentage of the gains or losses can be substituted for current income. It is important to separate the effects of the land price level and the rate of change of land prices. If land price is at a constant high price, the financing cost will be large, and less land is used in the input mix than at a lower land price. The land price change can, however, either offset or add to the cost of land ownership, depending on whether the price is rising or falling.

The model suggests that at least part of the increase in farm size since World War II may be a result of the almost continuous capital gains that occurred during that period. It also indicates that, all other things being equal, if capital gains during the period had been smaller or if those unrealized gains had been less substitutable for wealth, farmers would have invested in more nonland inputs, such as labor, fertilizer, pesticides, improved seed, irrigation or improvements to land they already own. Of course, the "ceteris paribus" condition of this simple static model does not hold. Many forces work simultaneously changing agriculture. These forces may reinforce or counteract each other. For example, the presence of large capital gains may reinforce the trend toward larger farming units brought about by technological change. It should not be inferred from this study that land price change is the only force changing farm decision making, but rather that land
price changes have wide ranging impact on farm decisions and merit further attention.

Equation (9.1) suggests that farm size and use of nonland inputs will be affected if government policy changes the expected rate of land price change or the substitutability of capital gains for current income. For instance, a major land price formation hypothesis suggests that land prices are the capitalized value of expected future income from land ownership. If this expected future income is rising, capital gains are likely to occur. If a price support program increases the rate at which future income from land is expected to rise, the model suggests that there will be a tendency for farm size to increase and for land use to become more extensive. Conversely, a weakening of government price support commitment which reduced income expectations and resulted in a lower rate of land price change would tend to reduce the optimal farm size and encourage more intensive farming.

Similarly, policy which affects the substitutability of capital gains for current income can affect farm size and input use. Plaxico and Kletke (1980) argue that the proportion of capital gain which is a substitute for current income is affected by cash flow problems. Unrealized gain cannot be used to meet cash requirements, hence if cash flow is a problem the proportion of unrealized gain which substitutes for current income will be reduced. If a government program reduces output price fluctuations or otherwise reduces the possibility of cash flow problems, the proportion of capital gain which is substitutable for current income will be increased. The model suggests that an indirect
effect of the government stabilization program would be larger farm size
and more extensive land use. The model suggests that the weakening of
government programs which tend to decrease the substitutability of
capital gains for current income, would, all other things being equal,
tend to reduce farm size and encourage more intensive farming.

In cases where the parameters $\theta$ and $\phi$ are large, the model indicates
that the optimal MRS might even be negative, implying that the producer
would acquire land to the point where extra land reduced current income,
but was still profitable in terms of capital gains. A more complete
model would include the option of renting out land so that the marginal
current revenue of land would be bounded below by the rental rate.
Renting out land becomes a viable option when the output price times the
marginal product of land is less than the rental rate. If large capital
gains substitutable for current income were occurring and a family had
access to enough equity to acquire more land than they could effectively
work, they could rent that land to a lower equity producer with a lower
MRS, and still enjoy the capital gains accruing to the land. Before the
mechanization of U.S. agriculture, it was relatively common for prosper­
ous farmers to own rental land, while in the 1980s it is relatively
uncommon for producers to rent out land. This may be an indication of
how mechanization changed the production function. In a horse drawn
agriculture, the marginal product of land probably declined sharply
beyond some relatively modest farm size, while research indicates that on
modern farms the marginal product of land is relatively constant over a
wide range (Miller, Rodewald and McElroy, 1981, pp. 15-17).
Fixed equity

For many farmers the amount of equity capital available for investment in the farm business is limited; adequate equity capital may not be available to achieve the optimal land and nonland input mix. In a static model this limitation can be approximated by assuming that equity is fixed. This approximation probably overstates the impact of the equity limits, because additions to equity from retained farm and nonfarm earnings are often possible, but it allows a relatively simple static model to be defined which shows the impacts of the equity constraint.

In the fixed equity model, the equation (6.4) is eliminated and the optimal debt load is dependent on the production function relationships, output prices and costs. A constant optimal leverage ratio cannot be defined. The optimal land and nonland input mix can still be characterized by the MRS equation (9.1), but analysis is difficult because changes in land and nonland input use affect the optimal amount of debt, and hence the marginal cost of debt in the factor cost ratio.

The fixed equity model satisfies the second order conditions. In the Hessian, the submatrix C is reduced to a scalar.

\[ 7.1' \quad c = \frac{-1}{k_D} \left( 2r'' + \frac{r' D}{K} \right) \]

The fixed equity objective function will be negative definite if:

\[ 7.2' \quad (h_1 h_2) \left[ A_{h_2}^h \right] + h_3^2 c < 0 \]
Inequality (7.2') holds for all \( h \) under the assumption of a strictly concave production function and the negativity of \( c \). Hence, the second order conditions are satisfied.

Changes in input and debt use for changes in the capital gain or loss, or changes in the substitutability of capital gains and losses for current income can be examined by comparative static analysis of the system:

\[
\begin{bmatrix}
A & 0 & (\frac{a}{\beta}) & \frac{3X}{3L} \\
0 & c & 1 & \frac{3L}{3D} \\
(\infty, \beta) & 1 & 0 & \frac{3D}{3\lambda}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \alpha} \\
\frac{\partial}{\partial \beta} \\
\frac{\partial}{\partial \lambda}
\end{bmatrix} =
\begin{bmatrix}
0 \\
\frac{-\theta}{\rho(1-\tau)} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \theta} \\
0
\end{bmatrix}
\]

By Cramer's rule, the change in land use for a change in \( \theta \) is:

\[
\frac{3L}{3\theta} = \frac{\phi}{(1-\tau)\rho^2} \left[ Pf_{XX} - \frac{a^2(2r' + r''D)}{K} \right] > 0
\]

which is always positive under the assumptions of the model and the negativity of the determinant of the bordered Hessian, denoted \(|BH|\).

The determinant \(|BH|\) can be written:

\[
|BH| = -\frac{\rho^2}{\rho} \left[ Pf_{XX}f_{LL} - f_{LX}^2 \right] - c\left( \frac{\rho}{\rho} \left[ a^2f_{XX} - 2a\beta f_{XL} + \beta f_{LL} \right] \right)
\]
and must be negative under the assumptions of the model because the first term in brackets is the determinate of the Hessian (H) of the strictly concave production function which is always positive and the second term in brackets is a quadratic form in H which must be negative under the assumed concavity. Equation (10.1) suggests that the tendency identified in the variable equity model for farm acreage to change in the same direction as the land price is carried over into the fixed equity version. Similarly, the effect on farm size of changes in the substitutability of capital gains for current income is the same as identified in the variable equity model. By Cramer's rule:

\[ \frac{\partial L}{\partial \phi} = \frac{-\theta}{(1-\tau)\rho^2} \left[ \frac{PF_{XX} - \alpha^2(2\tau' + \tau D/K)}{|BH|} \right] \]

Equation (10.2) is always the same sign as the land price change, suggesting that if land prices are increasing (decreasing) farm size will be enlarged (reduced) even more if the substitutability of capital gains (losses) for current income is increased.

When equity is fixed, the change in nonland input use for a change in the rate of capital gain (loss) or the substitutability of gains (losses) for current income is ambiguous. By Cramer's rule the changes in nonland inputs are:

\[ \frac{\partial X}{\partial \theta} = \frac{-\theta}{(1-\tau)\rho} \left[ \frac{PF_{XL} - \alpha^2(2\tau' + \tau D/K)}{|BH|} \right] \]
The sign of both (10.3) and (10.4) is ambiguous. It depends on the relative magnitudes of the change in productivity of nonland inputs as acreage is changed and the change in the marginal cost of debt as debt use changes. Over the range of operation of commercial agriculture the change in productivity of nonland inputs for a change in acreage \( f_{XL} \) will probably be positive. Hence, as capital gains increase (decrease) or the substitutability of capital gains for income increases (decreases) the productivity effect will tend to increase (decrease) nonland inputs along with acreage:

\[
\begin{align*}
\frac{-\frac{\phi pf_{XL}}{1-\tau} \, (2r' + r^\frac{D}{K})}{(1-\tau)\rho^2 BH} & \geq 0 \quad \text{for} \quad \theta \geq 0 \\
\frac{-\frac{\theta pf_{XL}}{1-\tau} \, (2r' + r^\frac{D}{K})}{(1-\tau)\rho^2 BH} & \leq 0 \quad \text{for} \quad \theta \geq 0
\end{align*}
\]

But the productivity effect is offset by the increasing cost of debt:

\[
\begin{align*}
\frac{\phi \alpha \beta}{(1-\tau)\rho^2 K BH} (2r' + r^\frac{D}{K}) & \geq 0 \quad \text{for} \quad \theta \geq 0 \\
\frac{\theta \alpha \beta}{(1-\tau)\rho^2 K BH} (2r' + r^\frac{D}{K}) & \leq 0 \quad \text{for} \quad \theta \geq 0
\end{align*}
\]

When capital losses occur, the productivity effect will tend to pull nonland input use down along with acreage, but this may be offset by the reduced debt cost as the farm business uses less capital and reduces leverage.
The change in debt use for changes in land appreciation rate is always positive if the productivity of land and nonland inputs is positively related \((f_{XL} > 0)\). The change in debt use for a change in the substitutability of capital gains (losses) for current income is of the same sign as the land price change for \(f_{XL}\) positive. By Cramer's rule:

\[
\frac{\delta P}{\delta f} \left( \frac{[\beta f_{XX} - \alpha f_{LX}]}{(1-\tau)\delta^2 \left| BH \right|} \right) > 0, \text{ for } f_{XL} > 0
\]

Equations (10.5) and (10.6) suggest that if equity is limited debt use will tend to increase (decrease) as farmers attempt to acquire (reduce use of) the appreciating (depreciating) asset. The fixed equity model suggests that the increased debt use by U.S. farmers in the 1970s may have been in part a response to the large capital gains. It also suggests that the financial problems of some producers in the early 1980s may be in part problems of adjusting their debt use to the declining land prices of that period. The financial structure which was optimal in the period of rising land prices, may no longer be optimal in a period of stable or falling land prices. Government policies may affect debt use through the land price change \((\delta)\) and substitutability of capital gains or losses for current income \((\phi)\) parameters. Price support or other programs which increase (decrease) capital gains or the substitutability of those gains for current income will tend to increase (decrease) debt use.
The Lagrangian multiplier ($\lambda$) may be interpreted as the value of the marginal unit of equity. The impact of a change in $\theta$ on the value of additional equity is positive for $f_{LX} > 0$. The change in the marginal value of equity for a change in the substitutability of capital gain or loss for current income is of the same sign as the land price change.

$$\frac{3\lambda}{3\theta} = \frac{P_\phi}{\rho^3(1-\tau)K} \left[ 2r^t+r^n D_K \right] \left[ f_{XX}B - \alpha f_{LX} \right] / |BH| > 0$$

$$\frac{3\lambda}{3\phi} = \frac{P_\phi}{\rho^3(1-\tau)K} \left[ 2r^t+r^n D_K \right] \left[ f_{XX}B - \alpha f_{LX} \right] / |BH|$$

Equations (10.7) and (10.8) suggest that as capital gains increase more equity capital would be attracted to agriculture. If total returns in agriculture are high relative to the rest of the economy, it is reasonable to suppose that some of that equity would come from outside the farm sector. Hence, it should come as no surprise that during the 1970s some nonfarm corporations, pension funds and other nonfarm investors found agricultural investments attractive.

Finite Planning Horizon

For most U.S. farm families, the infinite life assumption is probably not very realistic. At a minimum, an explicit choice must be made by each generation as to whether or not to continue agricultural production. Even if younger family members decide to pursue farming careers, it is common for them to establish separate operations from
their parents. Hence, a finite horizon model in which the land is sold at the end of the planning period may be a more adequate description. A more complex model might assume that at least part of the farm assets are passed on to the next generation, but if it is assumed that the next generation would carry on conventional, commercial agriculture the farm asset market value at the terminal date is probably at least a rough measure of their value to the next generation. The finite horizon model may also be used to model the situation of the investor who wishes to sell the land and realize the gain or loss after a fixed period of time.

In the finite horizon case, the objective must be to maximize the sum of the net present value of current income and the terminal or salvage value:

\[
V = \int_{0}^{T} e^{-\rho t} \pi dt - K + S
\]

\[
= \frac{\pi}{\rho} (1 - e^{-\rho T}) - K + S
\]

where:

- \( S \) = salvage value and
- \( T \) = terminal time.

If it can be assumed that the farm assets are sold without liquidity losses the salvage value may be written as the present value of the equity capital returned to the owner when the farm business is dissolved and that of the capital gain or loss. In the wealth approach, a proportion (\( \phi \)) of the capital gain or loss is recognized as current income,
hence a proportion \((1-\phi)\) is left to be recognized upon sale. Under U.S. law, realized long term capital gain is partially taxed and long term capital losses may be partially deducted. Because land is normally held for long periods, it will be assumed that all capital gains and losses are long term. For simplicity, it is assumed that no depreciation has been taken on the real estate and that the tax treatment of soil conservation expense does not change the tax status of the capital gain or loss. In the finite horizon case, capital gains and losses must be modeled separately because capital gains and losses are not treated symmetrically in the U.S. tax system. In the case of capital gain, the salvage value can be written:

\[
S = e^{-\rho(1-\tau)T}[K + (1-\psi-\psi_T)\delta TL]
\]

where:

- \(\tau_T\) = the average tax rate at the terminal time and
- \(\psi\) = the proportion of the capital gain that is taxable.

In the case of capital losses, the limitations on use of capital loss deductions must be acknowledged: only one-half of the loss is deductible, if there is no short term loss the maximum annual deduction is the lesser of $3,000 or taxable income over the zero bracket amount and there is no carryback provision for capital losses on farmland. If it is assumed that taxable income is at least $3,000 more than the zero bracket amount throughout the period after the farm business is dissolved and that the individual lives long enough to use all the deductions, value of
the capital loss deduction can be modeled in continuous time as the net present value of the stream of deductions. The upper bound on this present value calculation is the proportion of total capital loss deductible divided by the annual deduction. Hence, the salvage value would be:

\[
S = e^{-\rho(1-\tau)T} [K + T0L(1-\phi) + \int_0^{-T0L\delta/\varepsilon} e^{-\rho(1-\tau)t}T_\varepsilon ds] \\
= e^{-\rho(1-\tau)T} [K + T0L(1-\phi) + \frac{T_\varepsilon}{\rho(1-\tau)} (1-e^{-\rho(1-\tau)T0L\delta/\varepsilon})]
\]

where:
\[
\varepsilon = \text{the annual deduction limit and} \\
\delta = \text{proportion of the loss deductible.}
\]

In the case of capital gains, definition (11.1) can be substituted into equation \(2'\). The objective function is then maximized subject to the financial constraint \(4\). The first order condition for land becomes:

\[
(6.2') \quad \frac{\delta Z}{\delta L} = \frac{(1-e^{-\rho(1-\tau)T})}{(1-\tau\rho)} \left[ (P_L - Y_2)(1-\tau) + \theta \phi \right] \\
+ e^{-\rho(1-\tau)T}(1-\phi - \psi T) \theta T - \beta \lambda = 0
\]

and the other first order conditions \(6.1\)-\(6.5\) are unchanged. The second order conditions are unchanged, except for the discounting term because land enters the terminal value linearly.
The MRS between land and nonland inputs becomes:

\[
\frac{f_X}{f_L} = \frac{\{\gamma_1 + \alpha [r + r_t D] (1 - \tau)\}}{\{\gamma_2 + \beta [r + r_t D] (1 - \tau) - \theta (1 - \tau) e^{\rho (1 - \tau)^T - 1} - \theta T (1 - \phi - \psi T)\}}
\]

In the denominator of the MRS (9.2), the exponential term converts the after tax value of capital gain realization at time \( T \) into a stream of payments. The exponential term is always positive. For an environment of land price increase (\( \theta > 0 \)), the sign of the realization term in the denominator depends on the tax parameters and the substitutability of unrealized gain for current income. As the proportion of unrealized gain that can be substituted for current income approaches unity, the primary impact of the sale of land will be the tax liability, which would increase the cost of owning land and tend to offset the capital gain. The impact of the tax term will depend on the taxable proportion of capital gain (\( \psi \)) and the tax rate at the terminal date. If the exemption level is higher, taxes will be smaller and capital gains will tend to have a greater impact on land use. If the proportion of unrealized gain which can be substituted for current income is small, the realization term tends to be positive for most parameter values and the impact of capital gains on land use is larger than it would be in the infinite horizon case. The impact of realization is always larger when the horizon is smaller. There may be important planning horizon differences between various classes of land owners. For instance, it is probable that for the decisions modeled here, nonfarm investors would have a
shorter planning horizon than a family farmer. This would be true because of the investor's institutional structure and liquidity needs. In that case, the tax parameters will have a larger impact on the investor than on the family farmer. It should be noted that the overall capital gains impact is always nonnegative and is only zero if no unrealized gain is substitutable for income (\(\phi=0\)) and the taxes are confiscatory (\(\psi_T=1\)).

The model suggests that the tendency of capital gains to encourage extensive land use exists under a wide variety of tax regimes. The effect is reduced if capital gains are fully taxed (\(\psi=1\)), but not eliminated. The effect also persists for a wide range of parameter values if capital gains are taxed on an accrual basis. Accrual taxation of gain can be modeled by dropping the tax term in the terminal value and subtracting from current income the accrual tax liability, the current capital gain multiplied by the average tax rate, (\(\theta L\tau\)). In the accrual case, the MRS between land and nonland inputs is:

\[
\frac{f_X}{f_L} = \frac{[\gamma_2 + \sigma(r+\tau')^\text{D}] (1-\tau)}{[\gamma_1 + \sigma(r+\tau')^\text{D}] (1-\tau) - \theta (\phi-\tau) - \rho (1-\tau) [e^{-\rho (1-\tau) T} - 1]^{1-\theta T}}
\]

The tendency of capital gains to encourage extensive land use persists as long as the combined value of unrealized gains substituted for current income and realization of gain is larger than the taxes:

\[
\phi + \rho T (1-\phi) (1-\tau) [e^{\rho (1-\tau) T} - 1]^{-1} > \tau
\]
Capital gains can reduce farm acreage with accrual taxation of gain if the substitutability of unrealized gain for current income is small or the tax rate is high.

A fixed equity version of the finite margin model can also be defined. The conditions for maximization are the same as for the variable equity model, except first order condition (6.4) is omitted and the second order conditions are as shown in expressions (7.1') and (7.2'). For the fixed equity model, the effect of the capital gains rate ($\theta$), the substitutability of unrealized gain for current income ($\phi$) and the taxability of capital gains ($\psi$) can be examined with formal comparative statics. The system is of the same form as (10), but the second element on the right hand side becomes:

$$
\begin{align*}
13) & \quad -\theta \left[ \left( 1 - e^{-\rho(1-\tau)T} \right) / \rho (1-\tau) - Te^{-\rho(1-\tau)T} \right] d\phi \\
& \quad - \left[ \phi \left( 1 - e^{-\rho(1-\tau)T} \right) / \rho (1-\tau) + e^{-\rho(1-\tau)T(1-\phi-\psi)T} \right] d\theta \\
& \quad + e^{-\rho(1-\tau)T} \theta T d\psi
\end{align*}
$$

The changes in the decision variables for changes in these parameters are of the same form as equations (10.1-10.8); the first term in each expression is merely replaced with the appropriate term from (13).

The signs of the changes with respect to changes in the capital gains rate and the substitutability of unrealized gain for current income are the same as in the infinite horizon case. The square bracketed term in the change in the rate of capital gain expression is the sum of two
noting that the term approaches a minimum of zero as the planning horizon becomes very short. For longer planning horizons, the term is always positive because the exponential term becomes small at a faster rate than \( T \) increases.

For each of the decision variables, the sign of its change for a change in the taxability of capital gains (\( \psi \)) will be the opposite of the change in the tax parameter. If the exemption is increased, hence taxability decreased (\( d\psi < 0 \)), more debt and land will be used, fewer nonland inputs will be applied and the value of the marginal unit of equity will be higher.

The case of capital losses can be examined substituting equation (12.2) into the objective function (2') and maximizing given the financial constraint. The first order condition for land becomes:

\[
(6.2'') \quad \frac{\partial Z}{\partial L} = \frac{1-e}{(1-\tau)\rho} \left[ \left( PF_L - \gamma \right) (1-\tau) + \theta \phi \right] \\
+ e^{-\phi(1-\tau)T} \left( 1-\phi - \delta T e^{\rho(1-\tau)T + \phi T} \right) \lambda - \delta \lambda = 0
\]

The other first order conditions are unchanged for (6.1-6.5). The second order conditions are satisfied. This can be seen by noting that the Hessian matrix will of the same form as (7), except that the second derivative of the land equation (6.2'') will be the sum of the second derivative of the production function with respect to land and a nonpositive term of the form:
The objective function will be negative definite if:

\[ h'Hh = (h^h^) a(h^E + h^C) < 0 \]

Inequality (7.2') is always satisfied under the assumptions of the model and hence the second order conditions are satisfied.

The basic implications of the capital loss case can be seen by forming the MRS between land and nonland inputs:

\[
\frac{f_X}{f_L} = \frac{[\gamma_2 + \alpha(r + r')^D \frac{D}{K} ] (1-\tau)}{\gamma_1 + \phi(r + r')^D K(1-\tau) - \phi - \phi^{-1} - \phi^{-1} e^{\phi(1-\tau)T/L}}
\]

As in the infinite horizon case, the total factor cost of land is increased and the MRS is decreased by capital losses but the capital loss exemption tends to offset some of the impact of capital losses. When capital loss deductions are allowed the tendency of land use to become more intensive during periods of capital loss is reduced. The impact of the deduction provision depends on the annual deduction limit, the proportion of the loss that is deductible and the tax rate after the farm business is sold. If the family farm operator retires and sells his land, the tax rate in the retirement period is likely to be small unless other income producing property is owned, hence the impact of the
deduction provisions on the family farmer is likely to be small. The
investor, on the other hand, may have a relatively high tax rate after
selling the farm because of subsequent investment on other projects. For
the investor, the deduction provisions are likely to be more important.
For all decision makers, an increase in the annual deduction limit or
the proportion of the loss that is deductible increases the value of the
deduction and hence tends to further offset the tendency of capital
losses to encourage intensive farming.

For the fixed equity case, formal comparative statics can be used to
analyze the impact of the tax and capital gain parameters on the
decisions variables. The system is of the same form as matrix equation
(10), except that the second element in the second row of the Hessian
contains term E and on the right hand side the second element is:

\[
13' \quad -\delta \left[ (1 - e^{-\rho(1 - \tau)T})/\rho(1 - \tau) - Te^{-\rho(1 - \tau)T} \right] d\phi
\]

\[
-\left[ \delta (1 - e^{-\rho(1 - \tau)T})/\rho(1 - \tau) \right]
\]

\[
+e^{-\rho(1 - \tau)T} (1 - \phi - \delta T) \left( 1 + \frac{\rho(1 - \tau)T^2}{\epsilon} \right) e^{-\rho(1 - \tau)T^2}\phi^{2}/\epsilon \left[ 1 + \frac{\rho(1 - \tau)T^2}{\epsilon} \right] d\phi
\]

\[
+e^{-\rho(1 - \tau)T} \theta T e^{-\rho(1 - \tau)T^2}/\epsilon (1 + \phi(1 - \tau)T^2\phi^{2}/\epsilon) \theta \delta
\]

\[
+e^{-\rho(1 - \tau)T} \theta T e^{-\rho(1 - \tau)T^2}/\epsilon (\rho(1 - \tau)T^2\phi^{2}/\epsilon)^2 \theta \epsilon
\]

The signs of the changes in the decision variable depend on the signs of
the multipliers of the differential terms in (13'). The multiplier of \( d\phi \)
is the same as in the increasing land price finite horizon case. The
multiplier of $d^\theta$ differs from the increasing land price finite horizon case because of the difference in the tax impact. Nonetheless, the multiplier of $d^\theta$ is positive for all plausible parameter specifications. This can be seen by rewriting the $d^\theta$ multiplier as:

\[ 14) \quad \frac{\phi[1-e^{-\rho(1-\tau)T(1-\rho(1-\tau)T)}]}{\rho(1-\tau)} + e^{-\rho(1-\tau)T(1-\delta_{T}T_w)} \]

where: \( W = \left[ 1 + \frac{\rho(1-\tau)\delta_{T}T_T}{\epsilon} + \rho(1-\tau)TL\delta_{T}/\epsilon \right] \theta \).

Because the land price change coefficient ($\theta$) is negative, $W$ is always less than or equal to one. The first term in (14) is always nonnegative because:

\[ e^{-\rho(1-\tau)T(1-\rho(1-\tau)T)} < 1 \text{ for } T \leq \frac{1}{\rho(1-\tau)} \]

and

\[ -e^{-\rho(1-\tau)T(1-\rho(1-\tau)T)} > 0 \text{ for } T \geq \frac{1}{\rho(1-\tau)} \]

The second term is always positive under the assumption that the tax rate is less than one and at most the capital loss is 100 percent deductible. Hence, the impact of change in the substitutability of capital losses for current income and the change in the rate of land price change have the same implications as in previously derived models.

The sign of the multiplier for $d^\theta$ depends on the magnitude of the capital loss, the deductibility of the loss and the limit on annual deductions in the term:
If the deductible capital loss is large compared to the annual deduction, the multiplier term can be negative. This occurs because when the proportion of the capital loss which is deductible is increased, not only is the total tax deduction increased but, given a limit on annual deductions, the stream of deductions is stretched further into the future. Therefore, the deduction generated by the marginal unit of land or using the marginal unit of debt is further in the future and less valuable. The objective function is always increased by increasing the deductibility of losses, but the use of land or debt is not always increased because the optimal solution depends on the value of the marginal deduction and that may be reduced by a change in the deductibility proportion.

The multiplier of $dE$ is unambiguously positive, because the only negative parameter, the land price change, is squared. This implies that all other things equal, the optimal farm acreage and debt use will be higher when the annual loss deduction limit is larger. The use of nonland inputs relative to land is reduced when the limit is raised. The farm becomes a more attractive equity investment when the limit is increased. It should be noted that in the limit, when no annual constraint on deductions is imposed, the multiplier of $d\delta$ is also unambiguously negative because the second term in goes to zero, indicating that an increase in deductibility has the same effect as an
increase in the level of exemption for capital gains \((1-\psi)\). Because of discounting, the overall capital loss deduction effect is likely to be small; not only is the loss realized at the end of the period, but realization of the tax benefits of the loss is pushed even further into the future by the annual limit on deductions.

**Income Approach**

The income approach to the value of unrealized capital gains is essentially a dynamic concept; it is hypothesized that the debt cost and the borrowing power of the farm firm are dependent on the accumulated unrealized gain at any point in time. The financial condition of the firm changes as capital gains accumulate. A static approximation to the income approach model can, however, be defined by assuming that the financial negotiation takes into account the gain to be realized over the entire period, that is the sum of accumulated and anticipated gains at any point. If the land price path is known, and asset levels are chosen initially and maintained throughout the period, the total capital gain to be realized in the period is known. Hence, if the debt cost function depends on the total unrealized gain, the integral in equation (2) can converge to a compact static form dependent on the initial asset levels. This approximation probably overstates the impact of capital gains because the impact of anticipated unrealized gains is likely to be small. Nonetheless, the model offers insight into the income approach implications without the mathematical complications of control theory.
In general, the income approach can be modeled by redefining the argument of the debt cost function to be debt divided by the sum of equity and some proportion of the unrealized gain. In terms of the textbook discussion of agricultural finance, the equity variable (K) can be thought of as net worth when assets are valued using book values, while the sum of equity (K) and a proportion of the unrealized gain can be viewed as net worth when conservative market values are used (see for instance Lee et al., 1980, pp. 143-144). The proportion of unrealized gain which is recognized in the market value net worth estimate will depend on selling cost, capital gains taxes and uncertainty about whether or not the gain will actually be realized. The views of lenders and their expectations about the future will have an important impact on the proportion of unrealized gains recognized as a substitute for invested equity. If the proportion of unrealized gain that is recognized is denoted by ω, then the debt cost function can be written:

\[
(15) \quad r = r\left(\frac{D}{K + \omega TL}\right)
\]

The argument of the modified debt cost function will be referred to as the market value leverage ratio. A pure income approach model can be defined by substituting debt cost (15) into the finite horizon objective function (2') and setting the proportion of unrealized gain substitutable for current income to zero (ϕ=0). Mixed income and wealth approach models can be defined and are probably a better approximation of reality.
than the pure versions, but for expository convenience the pure income approach will be examined here.

In the income approach static model, the first order condition for land becomes:

\[
(6.2') \quad \frac{\partial Z}{\partial L} = \left[ \frac{1 - e^{-\varphi(1-\tau)T}}{\rho(1-\tau)} \right] \left[ PfL - \gamma_2 + \omega T r'\left( \frac{D_{KT}}{K+\omega TL} \right)^2 \right](1-\tau) \\
- \beta \lambda - e^{-\varphi(1-\tau)T} \theta T(1-\psi T) = 0
\]

The other first order conditions are unchanged except for the substitution of the modified debt cost function into equation (6.3) and (6.4).

The second order conditions for the income approach problem may be examined by noting that the Hessian matrix may be partitioned and written as the sum of a matrix of production function derivatives and a matrix of debt cost derivatives:

\[
(7') \quad H = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} R & Q \\ Q' & C \end{bmatrix}
\]

where:

\[
R = - \left[ \frac{1 - e^{-\varphi(1-\tau)T}}{\rho(K+\omega TL)} \right] \left[ 2r' + r'' \frac{D_{KT}}{K+\omega TL} \right]\begin{bmatrix} 0 & 0 \\ 0 & (D_{KT})^2 \end{bmatrix}
\]

\[
Q = - \left[ \frac{1 - e^{-\varphi(1-\tau)T}}{\rho(K+\omega TL)} \right] \left[ 2r' + r'' \frac{D_{KT}}{K+\omega TL} \right]\begin{bmatrix} 0 & 0 \\ - (D_{KT})^2 & (D_{KT})(D_{KT}) (\frac{D}{K+\omega TL}) \end{bmatrix}
\]
The matrices A and C are as previously defined for the finite horizon variable equity case, except that the modified debt function argument is substituted into C. The matrix of production function derivative is negative definite under the assumption of strict concavity. The debt cost function matrix will be negative semidefinite if:

\[(7.2') \quad \begin{bmatrix} h_1 & h_2 \\ h_1 & h_2 \end{bmatrix} R(h_1) + 2(h_1 h_2) Q(h_3) + (h_3 h_4) C(h_4) \leq 0 \]

for all \( h \) is not equal to zero.

The inequality \((7.2')\) can be expanded and written as:

\[(7.3') \quad - \left[ \frac{1-e^{-\rho(1-\tau)t}}{\rho(K+\omega^2t)} \right] [2r' + r'' D K + \omega^2 t] \left[ h_2 \frac{D \omega T}{K + \omega^2 t} - h_3 \right. \]
\[\left. + h_4 \frac{D}{K + \omega^2 t} \right] \leq 0 \]

Inequality \((7.3')\) always holds under the assumption of a convex debt cost function. Thus, the second order conditions are satisfied.

The optimal financial structure in the variable equity case can be characterized by using the first order conditions for debt and equity to derive an expression similar to \((7)\):

\[(8.1) \quad \rho - r' \left( \frac{D}{K + \omega^2 t} \right)^2 = r + r' \frac{D}{K + \omega^2 t} \]

In theory, equation \((7')\) could be solved for the optimal modified leverage ratio as a function of the discount rate. This optimal leverage
ratio would not change with changes in the rate of land price change ($\theta$) or in the proportion of unrealized gain that is recognized as a permanent addition to net worth ($\omega$), through there would be changes in the optimal allocation of net worth between invested equity ($K$) and unrealized capital gain. These changes in the net worth term cannot be further characterized without a more complete specification of the functional forms and parameter values. The complexity of the bordered Hessian, with interactions between the production function and debt cost function submatrices, makes sign determination in formal comparative statics at this level of generality very difficult. It can be shown that land and debt use rise with increases in the rate of land price appreciation or the recognition of unrealized gain as a permanent addition to net worth, but the sign of the change in equity is ambiguous. Hence, the question of the changes in the allocation of net worth between invested equity and unrealized gain in response to parameter changes cannot be explored without a more complex model.

The optimal input use can be characterized by solving equations (6.1), (6.2") and (6.3) for the MRS between land and nonland inputs:

\[
\rho \left(1 - \tau \right) \left[ e^{\rho \left(1 - \tau \right) T} - 1 \right]^{-1} \theta T \left(1 - \psi T\right)
\]

\[
\frac{f_x}{f_L} = \frac{\gamma_1 + \alpha \left(1 + r^e - \frac{D}{K + \omega^e T L} \right) \left(1 - \tau\right)}{\gamma_2 + \beta \left(1 + r^e - \frac{D}{K + \omega^e T L} \right) - \omega^e T r^e \left(1 - \frac{D}{K + \omega^e T L} \right)^2 \left(1 - \tau\right)}
\]
The reduction in debt cost term:

\[ \omega \theta \text{Tr}'(\frac{D}{K+\omega \theta TL})^2 \]

plays the same role in equation (8.4) as the substitutability of unrealized capital gain for current income term (\( \Phi^0 \)) played in previously derived MRS expressions. The reduction debt cost due to the recognition of unrealized capital gain in net worth offsets the costs of land ownership and hence the optimal MRS is increased, indicating that more land tends to be used in the input mix. The income tax rate plays a more important role in the income approach model than in the wealth approach case, because the extra earnings due to reduced debt cost are taxed at the full rate, while unrealized gains that are substitutable for current income are not taxed until realization. The impact of the debt cost reduction due to unrealized gain is reduced by taxation. The impact of deferral and partial exemption of capital gains from taxation is unchanged from the wealth approach model.

The fixed equity income approach model can also be defined. It can be analyzed with formal comparative statics on the system:

\[
\begin{bmatrix}
A + R \quad Q^* \quad \alpha \\
Q^* \\
(-\alpha - \beta) \\
0
\end{bmatrix}
\begin{bmatrix}
\partial X \\
\partial L \\
\partial D \\
\partial \lambda
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
= 
- \left( \frac{D^T}{K+\omega TL} \right)^2 \left( J + \omega LC^* \right) \partial \omega
\]

\[
\begin{align*}
\frac{D^T}{K+\omega TL} LC^* \partial \omega
\end{align*}
\]
where:

\[ C^* = - \left[ \frac{1-e^{-\rho(1-\tau)T}}{\rho(K+\omega TL)} \right] \left[ 2r' + r'' \left( \frac{D}{K+\omega TL} \right) \right] \]

\[ Q^* = \begin{bmatrix} 0 \\ - C^* \left( \frac{D \omega T}{K+\omega TL} \right) \end{bmatrix} \]

\[ J = \left[ \frac{1-e^{-\rho(1-\tau)}}{\rho} \right] r'/\theta T \]

For relatively modest amounts of total unrealized gain, unambiguous signs can be determined for the changes in land and debt use in response to a change in the amount of unrealized gain recognized as a permanent addition to net worth. By Cramer's rule:

\[ (10.1') \frac{\partial L}{\partial \omega} = \left[ J \left( \frac{D \omega T}{K+\omega TL} \right)^2 \left[ \frac{1-e^{-\rho(1-\tau)}}{\rho} \right] Pf_{XX} \right] \alpha^2 C^* \]

\[ - Pf^* \left[ \frac{1-e^{-\rho(1-\tau)T}}{\rho} \right] \left( \frac{D \omega TL}{K+\omega TL} \right) \left[ f_{XX} \left( \beta - \omega TL \frac{D}{K+\omega TL} \right) - \alpha f_{LX} \right] \left[ \left( \frac{\omega TL}{K+\omega TL} \right)^2 \right] \]

The change in land for a change in the proportion of unrealized gain recognized as an addition to net worth is always positive under the assumptions of the model when the original land price is greater than the capital gain multiplied by the leverage ratio:

\[ (16) \quad \beta > \omega TL \frac{D}{K+\omega TL} \]
Because agricultural lenders are often reluctant to lend more money to a farm business than the owner has invested in it (Barry, Hopkin and Baker, 1983) it is unlikely that the planned leverage rates would be greater than one. Thus, inequality (16) implies that equation (10.1') is usually positive for cases in which the land price does not more than double during the planning period. Equation (10.1') may be positive for larger amounts of capital gain. Only the multiplier of $F_{XX}$ in the second term can become negative with large amounts of gain. But clear sign determination is not possible for large amounts of unrealized gain without additional specification of functional forms and parameter magnitudes.

The determinate of the bordered Hessian can be written:

$$|BH| = -p^2 \left[ \frac{1 - e^{-\rho(1-\tau)}}{\rho} \right]^2 \left( f_{XX} f_{LL} - f_{LX}^2 \right) - c^* \left[ \frac{1 - e^{-\rho(1-\tau)}}{\rho} \right]^T \left[ \frac{D\delta{T}}{K + w^0 T} \right]^2 - 2f_{XL} \alpha (B - \frac{D\delta{T}}{K + w^0 T}) + \alpha^2 f_{LL}^2 \right] < 0$$

and is known to be negative for all parameter values under the assumption of a strictly concave production function. This is because the first term in second derivatives is the Hessian determinate for the production function and the second term in production production function second derivatives is a quadratic expansion of the same Hessian; both terms in the production function derivatives must be negative under the assumed concavity.
The change in debt use for a change in the proportion of unrealized gain recognized as net worth is also positive for modest amounts of gain. By Cramer's rule:

\[ 10.5' \] \[
\frac{\partial D}{\partial \omega} = \frac{1-e^{-\rho(1-t)T}}{\rho} \{ J[\frac{D^T}{K+\omega TL}]^2 (f_{xx} - \alpha f_{XL} + \alpha^2 \frac{D^T \omega}{K+\omega TL}) \}
\]

Equations (10.1') and (10.5') suggest that in many ways the impacts of recognizing a portion of unrealized gains as additions to net worth are like those of substituting a portion of the unrealized gain for current income. In both cases, the effective cost of land ownership is reduced, more land is used and in the fixed equity case the firm becomes more highly leveraged to acquire that land. The signs of changes in nonland input use and the marginal value of equity (\( \lambda \)) cannot be clearly determined without further functional form and parameter magnitude specifications, but for a wide range of parameter values the tendency of capital gains to encourage extensive farming and to increase the value of equity capital is maintained. Because \( \theta \) and \( \omega \) always appear together in the comparative statics of changes in the rate of land price appreciation are of exactly the same form as expression (7'), (10.1') and (10.5'), except that equations in (10.1'), (10.5') and the right hand side of (7') are multiplied through by \( \omega/\theta \). An income approach model for capital loss environment could also be defined in which the unrealized losses offset
the investment equity capital and reduce the denominator of the debt cost function argument. In that case, the loss would increase the cost of land ownership by increasing the cost of debt. The capital losses would tend to result in smaller optimal farm size and less debt use.

Two Output Static Model

A yet unanswered question is does the presence of capital gains have an impact on the choice of outputs. In its simplest form, this question may be examined by modifying the Lagrangian expression 5) to include a second concave production function \( g(X, L) \) representing an alternative enterprise. Let \( P_1, X_1, L_1 \) be variables related to the original production function and \( P_2, X_2, L_2 \) be variables related to this new production function. It is assumed that input use for the two enterprises is independent; the same input cannot be used for both enterprises. The Lagrangian expression becomes:

\[
(5') \quad \text{Max } Z = \frac{1}{1-\tau} \left\{ \left[ P_1 f(X_1, L_1) + P_2 g(X_2, L_2) \right] - \gamma_1 (X_1 + X_2) \right. \\
- \left. \gamma_2 (L_1 + L_2) - \left( \frac{D}{K} \right) D \right\} (1-\tau) \\
+ \phi \theta \left( L_1 + L_2 \right) + \lambda [K + D - \varepsilon (X_1 + X_2) - g(L_1 + L_2)]
\]

The first order conditions for maximization of this problem are of the same form as before, with two added equations because of the allocation of inputs between outputs. The second order conditions are also
satisfied by this model. The Hessian is a block diagonal matrix with a third negative definite submatrix along the diagonal.

The significant implication of this two output model can be seen in the marginal rates of substitution of the inputs for both outputs. Because the input costs are the same for both outputs, the MRS must both be equal to the factor cost ratio:

\[
\frac{f_X}{f_L} = \frac{g_X}{g_L} = \frac{\{\gamma_1 + \alpha[r + r' \frac{D}{K}]\} (1-r)}{\{\gamma_2 + B[r + r' \frac{D}{K}]\} (1-r) - \phi \theta}
\]

Both the marginal rates of substitution for the f and g production functions, MRSf and MRSg, will be larger than is traditionally the case because of the presence of capital gains. If one production function has a lower marginal product of land, the output and use of land in the production of that commodity will be curtailed relative to the production and acreage of the other commodity, when capital gains are recognized in the decision making process. For instance, assume g describes the production of fruit and vegetables such that at some relatively small amount of land \(g_L\) becomes small compared to \(g_X\), that is the marginal product of land becomes small compared to the marginal product of other inputs such as equipment, labor, pesticides. Assume f describes the production of grain; the marginal product of land in grain production can remain relatively high even if substantial amounts of land are already in use. Under these conditions, MRSg would be equal to the factor cost ratio at
some low level of land input, but a much larger level of land input would be required to equate MRSf and the factor cost ratio.

As capital gains increase, the factor cost ratio increases and the output which lends itself to land extensive production assumes a larger share of the output mix. It may be the case for some levels of capital gain and some production functions, that the land input for g must be made so small to achieve equality (9.6) that for practical purposes the production of output 2 drops out of the model. It should be noted that the model does not suggest that the most land extensive output is always favored. It does not suggest that high capital gains would lead cornbelt farmers to seed their fields to grass and graze sheep. Rather it indicates that the favored output in the presence of capital gains is one in which the production process is relatively land extensive and the marginal product of land remains relatively high even when the firm uses large amounts of land. For capital loss, the opposite effect occurs and enterprise choice tends toward land intensive options.

The two output models may partially explain the movement away from land intensive crops in Midwest farming. For example, at the turn of the century Iowa was a major producer of fruits and vegetables, but today the amount of these products grown in the state is miniscule, even though economic research indicates that the current returns from producing such products exceeds returns from common grain crops (Calkins and Weimar, 1984, pp. 29-30; Kirschling and Sullivan, 1979). The explanation in terms of the model presented here is that fruit and vegetable production is not part of the optimal solution for most Iowa farmers because its
land intensiveness reduces the possibility for earning capital gains. For instance, if a Midwestern producer with a given amount of equity and labor resources could choose between a 300 acre grain farm and a 40 acre fruit and vegetable operation, and both options had the same current net income from production, the grain farm would be favored because it has the greater potential for capital gains. As a result of the capital gain pressure for extensive farming, fruit and vegetable production occurs primarily in those locations, such as California, where climate, soil type, transportation cost and other factors allow productivity large enough to overcome the capital gains advantage of extensive farming. The model suggests that the interest shown in intensive crops by some Midwestern farmers and farm groups or agencies may be related to the falling farmland prices.

Dynamic Model, Wealth Approach

If the level of input use and financial structure is allowed to change over time, the integral equation (2) no longer converges to a compact static form and optimization requires use of dynamic mathematical methods, such as optimal control theory. In addition, the mechanism of input and financial adjustment become important in a dynamic environment. In particular, the model must be modified to show new investment and retained earnings. For most U.S. farmers, the equity invested in the farm firm will change primarily because of retained earnings, further investment by the farmer from nonfarm sources or withdrawals by the farmer. This route for changing equity can be incorporated into the
model by subtracting an investment term from the current after-tax return. Let this term be denoted $u_{lt}$. When $u_{lt}$ is positive and less than current income, it indicates retained earnings which reduce income flow to the owner for the period. When $u_{lt}$ is greater than current income, the net stream to the owners is negative indicating net investment in the business by the owners from nonfarm wealth. When $u_{lt}$ is negative, dissaving is occurring. The simplest model does not place restraint on $u_{lt}$; the rate of investment or dissaving may be any real number. A more complex model would allow the owners to maximize their net worth by attracting outside partners or investors, but since this would involve issues of timing the outside investment and the return to those investors, it is omitted from this simple model.

The land purchase and sale transactions could be modeled in several ways. The most realistic model would allow land to be sold at any time and a major part of the optimization problem in that case would be the optimal selling time. Such a model would pose a major recordkeeping problem. The purchase price and area acquired in each increment would have to be recorded separately to calculate the capital gain or loss. Part of the selling problem might include the order of tract sales; for instance, for tax purposes it might be best to sell the land with the least accumulated capital gain first. Such a model would be very complex and unwieldy for analytical purposes. To simplify the situation, the static model assumption that land is sold only at the end of the planning period is used in the dynamic model. This reduces the recordkeeping problem because only the cumulative capital gain has to be known at the
end of the period, not the gain at each increment. This assumption implies that no land is sold during the planning period, but additional land may be acquired. In a deterministic model with no unexpected financial reversals, the exclusion of land sales is probably not an unrealistic approximation of farmland ownership behavior. The basic dynamic model can be thought of as the situation of the family farmer who buys land with the intention of holding it at least until retirement. This model is a traditional view of farmland ownership motives, in that it excludes purely speculative land buying. The basic model will concern itself with only one generation; all the land will be sold at the end of the period. A later section will show how the model can be extended to the multigenerational case and to include more speculative activity. This formulation avoids the question of optimal selling time and selling order of tracts, but still captures many important aspects of U.S. farmland markets, in which farm entry and exit transactions are a major part.

Because land prices are changing the financial constant must also be modified. The amount of capital absorbed by land is no longer the market price multiplied by the area ($aBL$), but rather the sum of the purchase prices. The difference between the current market value of the land and the sum of the purchase prices is the unrealized capital gain; hence, the capital absorbed by land can be expressed as the difference between the current market price and the unrealized gain.

The financial constraint then can be written:

$\begin{align*}
(4') & \quad x_t + d_t - aX_t - \beta L_t + g_t = 0 
\end{align*}$
where: \( G_t \) = accumulated unrealized capital gain or loss.

The \( t \) subscript is used on variables and parameters that may change over time. The same constraint can be expressed as the time derivative of equation (4') and the initial conditions on the levels of the input and financial variables.

\[
(4'') \quad \dot{K}_t + \dot{D}_t - \alpha \dot{X}_t - \dot{\beta}_t L_t - \dot{L}_t \hat{\beta}_t + \dot{G}_t = 0
\]

\[
K_0 = K(0), \ D_0 = D(0), \ X_0 = X(0), \ L_0 = L(0), \ G_0 = L(0)
\]

The dot notation is used to indicate the derivative with respect to time. In this section, the land price path is assumed to be smooth and non-decreasing with continuous first and second derivatives. The generalization of the model to other land price paths will be discussed in subsequent sections. For expositional convenience, it is assumed that the only parameter which changes over time is the land price. Models in which additional parameters change are notationally more complex but conceptually identical to the model developed here. Under the assumption that no land is sold during the planning period, the change in unrealized gain is:

\[
(17) \quad \dot{G}_t = L_t \hat{\beta}_t
\]

By substituting term (17) into equation (4'') and solving for \( \dot{D}_t \), the constraint can be expressed in a form convenient for the control theory problem:
The control problem for the wealth approach in the case of increasing land prices can be expressed by subtracting the investment control variable \( u_{1t} \) from the current income term and defining control variables for the change in land use \( u_{2t} \) and the change in nonland inputs \( u_{3t} \):

\[
\text{Max } Z = \int_0^T e^{-(1-\tau)T} \left[ \left( \frac{D_t}{K_t} \right) D_t \right] (1-\tau) - u_{1t} - \hat{\beta}_t L_t \phi \right] dt \\
+ e^{-(1-\tau)T} \left[ K_T + (1-\phi-\psi)G_T \right] - K_0
\]

s.t.

\[\begin{align*}
\dot{K}_t &= u_{1t} \\
L_t &= u_{2t}, \quad u_{2t} > 0 \\
X_t &= u_{3t} \\
D_t &= \beta_t u_{2t} + \alpha u_{3t} - u_{1t} \\
C_t &= L_t \hat{\beta}_t \\
\end{align*}\]

\( K_0 = K(0), \quad D_0 = D(0), \quad L_0 = L(0), \quad X_0 = X(0), \quad G_0 = G(0) \)
The Hamiltonian for this problem is:

\[ H = e^{-\phi(1-t)t} \left\{ \left[ P_t \left( X_t, L_t \right) - \gamma_1 X_t - \gamma_2 L_t - t \left( \frac{D_t}{K_t} \right) D_t \right] (1-t) \right\} \]

\[ - u_{1t} + \beta_{t^2} \lambda_{2t} \lambda_{3t} + \lambda_{1t} u_{1t} + \lambda_{2t} u_{2t} + \lambda_{3t} u_{3t} \]

\[ + \lambda_{4t} \left( \beta_{t^2} u_{2t} + a_{u_{3t}} - u_{1t} \right) + \lambda_{5t} L_t \hat{\lambda}_t \]

where \( \lambda_{it} \) is the adjoint variables

\[ i = 1, 2, 3, 4, 5 \]

Because the control variables enter linearly, the solution to this problem will entail some mix of bang-bang and singular control. When the control variables enter linearly, the usual Pontryagin necessary conditions do not provide information about the optimal control because no control variables appear in the optimality conditions and hence it is not possible to solve for the optimal control in the usual way. The problem is called singular because the matrix of second derivatives of the Hamiltonian with respect to the control variables is a singular; null matrix. To cope with this problem, several researchers have examined the second variation of the singular problem for further necessary and sufficient conditions. This study will make use of the Generalized Legendre-Clebsch necessary condition developed by such researchers as Kelley, Kop and Moyer (1967), and Goh (1966). Several sufficiency theorems have been developed for specific subsets of the singular control problem (Jacobson and Speyer, 1971 or McDanell and Powers, 1971), but general sufficiency theorems which include the case of mixed bang-bang
singular control are not well-developed. Arguments for the optimality of the controls suggested by the necessary conditions are presented below.

Nonnegativity constraints on the state variables \( K_t, L_t, X_t, D_t, G_t \) have not been explicitly imposed because of the complexity of optimality conditions with pure state variable constraints, and because the singular control results used in this study have not been extended to the case of pure state variable path constraints. For many control problems, not only singular problems, it is often more efficient to initially solve the problem without the nonnegativity constraints and to implement them explicitly only if they are violated in the optimal solutions (Kamien and Schwartz, p. 215).

The usual necessary conditions are (Kamien and Schwartz, p. 133):

\[
\begin{align*}
\frac{\partial H}{\partial K_t} &= -\lambda_{1t} = e^{-\rho(1-\tau)t} \left( \frac{D_t}{L_t} \right)^2 (1-\tau) \\
\frac{\partial H}{\partial L_t} &= -\lambda_{2t} = e^{-\rho(1-\tau)t} \left( \left( PF_L - Y_2 \right)(1-\tau) + \beta_t \right) + \lambda_{5t} \theta_t \\
\frac{\partial H}{\partial X_t} &= -\lambda_{3t} = e^{-\rho(1-\tau)t} \left( PF_X - Y_1 \right)(1-\tau) \\
\frac{\partial H}{\partial D_t} &= -\lambda_{4t} = -e^{-\rho(1-\tau)t} \left( r + r' \frac{D_t}{K_t} \right)(1-\tau) \\
\frac{\partial H}{\partial G_t} &= -\lambda_{5t} = 0 \\
\frac{\partial H}{\partial u_{lt}} &= e^{-\rho(1-\tau)t} + \lambda_{1t} - \lambda_{4t} = 0
\end{align*}
\]
with the transversality conditions that as time approaches the terminal time the following conditions must hold:

\[ (20.1) \quad \lambda_{1T} = e^{-\rho(1-\tau)T} \]
\[ (20.2) \quad \lambda_{2T} = 0 \]
\[ (20.3) \quad \lambda_{3T} = 0 \]
\[ (20.4) \quad \lambda_{4T} = 0 \]
\[ (20.5) \quad \lambda_{5T} = (1-\phi-\psi T)e^{-\rho(1-\tau)T} \]

The optimal control scheme implied by these necessary conditions is bang-bang and singular:

\[ (12.1) \quad u_{1T} = \begin{cases} 
-\infty & \text{if } -e^{-\rho(1-\tau)t} + \lambda_{1T} - \lambda_{4T} < 0 \\
\text{singular} & \text{if } -e^{-\rho(1-\tau)t} + \lambda_{1T} - \lambda_{4T} = 0 \\
+\infty & \text{if } -e^{-\rho(1-\tau)t} + \lambda_{1T} - \lambda_{4T} > 0 
\end{cases} \]
The adjoint variables can be interpreted as the value of the marginal unit of the state variable (Kamien and Schwartz, 1981, p. 125). They can be characterized by integrating equations (19.1-19.5) and evaluating the constants with the transversality conditions. The adjoint variable for equity capital \((\lambda_{1t})\) can be written as:

\[
\lambda_{1t} = -\int_0^T e^{-\rho(1-\tau)s} r'(\frac{D_s}{K})^2 (1-\tau) ds + C
\]

where: \( s = \) variable of integration.

Using the transversality condition (20.1), the constant can be evaluated:

\[
C = \lambda_{1T} + \int_0^T e^{-\rho(1-\tau)s} r'(\frac{D_t}{K})^2 (1-\tau) ds
\]

\[
= e^{-\rho(1-\tau)T} + \int_0^T e^{-\rho(1-\tau)s} r'(\frac{D_s}{K})^2 ds
\]
and hence the adjoint variable can be characterized as the discounted value over the remaining time horizon of the marginal reduction in interest payments due to an additional unit of equity, plus the discounted value of the capital withdrawn from the farm business at the end of the planning period:

\[
(22.1) \quad \lambda_{1t} = \int_t^T e^{-\rho(1-\tau)s} \left[ r'(\frac{s}{K})^2 (1-\tau)ds + e^{-\rho(1-\tau)T} \right]
\]

Using the same procedure the adjoint variables for land, nonland input, and debt can be characterized:

\[
(22.2) \quad \lambda_{2t} = \int_t^T e^{-\rho(1-\tau)s} \left[ \left( pf_L - \gamma_2 \right)(1-\tau) + \beta_s \right] ds
\]

\[
+ e^{-\rho(1-\tau)T} \left[ \beta_T - \beta_L \right] (1-\phi-\psi_T)
\]

\[
(22.3) \quad \lambda_{3t} = \int_t^T e^{-\rho(1-\tau)s} \left( pf_X - \gamma_1 \right)(1-\tau) ds
\]

\[
(22.4) \quad \lambda_{4t} = -\int_t^T e^{-\rho(1-\tau)s} \left( r + r_s \frac{D}{K} \right) (1-\tau) ds
\]

The adjoint variable for land can be characterized as the discounted value of the current marginal value product of land over the remaining horizon plus the after tax value of selling the land at the end of the horizon. Under the assumptions of the wealth approach, the current
marginal value product of land includes both income from production and a portion of the capital gain. The adjoint variable for nonland inputs can be characterized as the discounted value of the marginal value product of nonland inputs over the remaining horizon. The adjoint variable for debt can be characterized as the discounted value of the marginal cost of debt over the remaining horizon. The adjoint variable for unrealized capital gain can be characterized by noting that equation (19.5) indicates that it is a constant and the transversality condition shows that the constant is

$$\lambda_{5t} = \lambda_{5T} = e^{-\rho(1-\tau)T} (1-\Phi-\Phi t)$$

Equation (22.5) shows that the marginal value of unrealized capital gain in the model is simply its addition to the terminal value.

By substituting equations (22.1) and (22.4) into (21.1), it can be seen that the investment decision depends on whether or not the marginal cost of added equity is greater than or less than the marginal cost of debt over the remaining horizon for the candidate investment path:

$$e^{-\rho(1-\tau)t} - e^{-\rho(1-\tau)T} - \int_t^T e^{-\rho(1-\tau)s} \left( r + r' \frac{D}{K_s} \right)^2 (1-\tau) ds$$

$$\leq \mathcal{N} \int_t^T e^{-\rho(1-\tau)t} \left( r + r' \frac{D}{K_s} \right) (1-\tau) ds$$
If either inequality holds at any point the rate of investment or dis-investment approaches infinity and instantaneously adjusts the level of equity to the point where the quality can hold. The inequalities will be relevant only in the initial instant of the problem; if the starting value is not at the optimal levels they will be instantaneously adjusted. After that initial adjustment, the change in investment and all other changes in the model are driven by the change in land prices. In this model with a smooth land price path, the path of investment will also be smooth and singular after the initial adjustment. The singular control paths for all the state variables are derived below.

By substituting equations (22.2) and (22.4) into (21.2), it can be seen that the land purchase decision depends on whether or not the marginal return from land over the remaining horizon in terms of income from production and capital gain is greater or less than the financial cost of owning that marginal unit of land:

$$\frac{D}{s} \beta \int_{t}^{T} e^{-\rho(1-\tau)s} \left( [P_{t} - \gamma_{2}(1-\tau) + \beta_{s} \phi] \right) ds + e^{-\rho(1-\tau)T} (\beta_{t} - \beta_{c})(1-\phi(1-\tau)^{2})$$

If the marginal return from land is greater than the financial cost of ownership, the rate of land acquisition will approach infinity and the level of land ownership will instantaneously be brought up to the level which enables the marginal returns over the remaining horizon to exactly match the marginal financial cost. In determining the initial adjustment
marginal returns along optimal singular and bang-bang arcs are used. Under the assumptions of the model land purchases will be zero if the marginal returns from land are less than the financial cost. As in the case of investment, the instantaneous adjustment will apply only at the initial instant. After the initial adjustment, changes in land use will be a smooth singular control path driven by the land price change.

By substituting equations (22.3) and (22.4) into (21.3) it can be seen that the nonland input acquisition decision depends on whether or not the marginal return from nonland inputs is greater or less than the financial cost of using those inputs over the remaining horizon:

$$
\int_t^T e^{-(1-\tau)s} \left( Pf_k - \gamma \right)(1-\tau)ds \geq \int_t^T e^{-(1-\tau)s} \left( r + \frac{D_s}{K_s} \right)(1-\tau)ds
$$

Like the investment decision, the nonland input acquisition will be characterized by an initial instantaneous adjustment to the optimal level, with a smooth singular path thereafter.

Expressions for the singular optimal control paths can be derived by repeatedly differentiating with respect to time the derivatives of the Hamiltonian with respect to the controls (Bryson and Ho, 1975, Chapter 8). The differentiation with respect to time is repeated until the control variables appear explicitly. These time derivatives can also be used to derive expressions showing the financial and input structure of the optimal solutions. The financial structure can be characterized by differentiating equation (19.6) with respect to time and substituting in equation (19.1) and (19.4):
This condition implies that along the optimum singular path equation (8) holds and the marginal cost of equity equal the marginal costs of debt. If the debt cost function were specified, equation (8.2) could be solved for the leverage ratio as a function of the discount rate.

**Optimal singular controls**

The fixed leverage ratio implies that along the singular arc equity investment and acquisition of new debt stand in a constant relationship. This relationship can be shown by differentiating the leverage ratio with respect to time:

\[
(23.1) \quad \frac{\partial \left( \frac{D_t}{K_t} \right)}{\partial t} = \frac{D_t}{K_t} - \frac{D_t}{K_t} \frac{K_t}{K_t} = 0
\]

The derivative of the leverage ratio with respect to time must be zero because the leverage ratio is a constant. By solving equation (23.1) simultaneously with equations (2.1) and (2.4), it can be shown that the capital requirements of new land and nonland input acquisitions are divided between debt and equity according to the optimal debt and equity proportions:

\[
(23.2) \quad u_{1t} = \frac{K_t}{K_t + D_t} \left[ \beta u_{2t} + au_{3t} \right]
\]
The marginal value product of land along the singular arc can be characterized by differentiating equation (19.7) with respect to time, and substituting in equations (19.2), (19.4) and (22.5):

\[
(24.1) \quad P_f(L(1-\tau)) = \left[ Y + \beta_t \left( 1 - e^{-\rho(T-t)} \right) / \rho \right] (1-\tau)
\]

\[
= \beta_t \left[ \phi + e^{-\rho(T-t)} (1-\phi-\psi) \right]
\]

In defining the marginal value product, the integral in the expression for the cost of debt adjoint variable (20.3) has been solved as:

\[
\lambda_{4t} = e^{-\rho(T-t)} \left[ \frac{r+\rho K_t}{\rho} \right] [1-e^{-\rho(T-t)}]
\]

This solution is possible because of the constant leverage ratio.

The right hand side of equation (24.1) is simply the cost of using land and thus this expression is a version of the conventional economic result that at the optimum the after tax marginal value product should equal the after tax cost of using the input. The cost of using land is affected by the capital gains through the reduction in finance costs by buying now instead of later:
the capital gain that is substitutable for current income \( \hat{\beta}_t \phi \) and the realized gain term:

\[
\hat{\beta}_t e^{-\rho(1-\tau)(T-t)} (1-\phi-\psi \tau)
\]

Under model assumptions in an environment of capital gain these terms reduce the costs of land ownership. The impact of capital gain on the cost of using land is enhanced if a higher proportion of gain is substitutable for current income \( \phi \), the capital gains tax exemption \( 1-\psi \), is greater, or the decision maker is nearer to the terminal date.

The marginal value product of nonland inputs along the singular arc can be characterized by differentiating equation (19.8) with respect to time and substituting in equations (19.3) and (19.4):

\[
(24.2) \quad Pf_x(1-\tau) = [\gamma_1 + \alpha(r + r' \frac{D_t}{K_t})](1-\tau)
\]

By dividing (24.2) by (24.1) an expression for the marginal rate of substitution of land for nonland inputs at each point on the singular arc can be derived:
The MRS is of the usual Vickers' form including the financial costs of ownership. The MRS of the dynamic wealth approach model differs from the static model MRS (equation 9.1) in that it includes the reduction in land cost achieved by buying now instead of later. This strengthens the impact of land price increases on the input mix. As in the static model, the MRS will be larger when capital gains are larger, indicating that more land will be used and farming will be more extensive when capital gains occur.

The dynamic model, like the static version, is sensitive to the size of the proportion of capital gain that can be substituted for current income (\(\beta\)). If that proportion is small, the capital gain effect will be smaller. The realized gain effect can occur, however, even without the substitutability of capital gains for income. If the decision maker is relatively close to the terminal time or if the discount rate is small, the realization of gain at the end of the period can have a substantial effect. If the model is viewed as representing the career of a single farmer, it suggests that age may have a significant impact on the input choices. As the terminal time grows nearer, the impact of capital gain realization grows larger and more land is incorporated in the input mix.
This is consistent with the observation that a high proportion of farmland is in the hands of older, established farmers. The age effect will be examined further in the multigenerational model, but is should be noted that if the problem (2") is seen as a representation of several generations of a farm family and the terminal date is far in the future, then the MRS approaches its static form and gain realization has no effect. The time to the terminal date affects the impact of the capital gain exemption. If the realization gain is relatively close, the exemption can substantially increase returns to land and increase the optional land use. As T becomes large the exemption becomes less important. The impact of government programs and other factors that may affect the proportion of gain that is substitutable for income is the same as in the static model.

By differentiating (24.1) and (24.2) with respect to time and substituting in equation (2.2) and (2.3), two equations in the land and nonland input control variables are derived:

\[
(25.1) \quad -P[f_{LL}^2 u_{2t} + f_{LX} u_{3t}] (1-\tau) - \tilde{\beta}_t[\phi + e^{-\rho(1-\tau)(T-t)}(1-\phi-\psi)] \\
- \left[ \frac{D_{LX} + D_{L}}{\rho \left(t + e^{-\rho(1-\tau)(T-t)} \right)} \right] \\
+ \tilde{\beta}_t \left[ \frac{D_{LX} + D_{L}}{\rho \left(t + e^{-\rho(1-\tau)(T-t)} \right)} \left(1 + e^{-\rho(1-\tau)(T-t)} - \rho(1-\tau) e^{-\rho(1-\tau)(T-t)} \right) \right] = 0
\]

\[
(25.2) \quad -P[f_{XX}^2 u_{3t} + f_{XL} u_{2t}] (1-\tau) = 0
\]
Solving equations 25.1) and (25.2) simultaneously for the control yields:

\[(17) \quad u_{2t} = F[\hat{\beta}_t M - \hat{\beta}_t N]\]

where:

\[F = \frac{\frac{f_{XX}}{p(1-\tau)(f_{LL} f_{XX} - f_{LX})^2}}{p}\]

\[M = \left[\left(\frac{r}{K_t} + \frac{1}{\rho}\right)(1 + e^{-\rho(1-\tau)(T-t)}) - \rho e^{-\rho(1-\tau)(T-t)}(1-\phi-\psi\tau)\right](1-\tau)\]

\[N = \phi + e^{-\rho(1-\tau)(T-t)}(1-\phi-\psi\tau) + \frac{D_t}{\rho} \left[1 - e^{-\rho(1-\tau)(T-t)}\right]\]

(26.2) \( u_{3t} = \frac{f_{XL}}{f_{LL}} u_{2t} \)

Equation (26.1) indicates that land will be purchased only if the change in the return to land ownership, composed of the part of the capital gain that is substitutable for current income, the gain that is recognized upon realization and the reduction in financial cost, is greater than the change in the financial cost. The second term in M and all of term N represent the benefits of capital gains. The first term in M reflects the increased financial cost of and ownership when the land price rises. The multiplier (F) of the land use cost and return term is always
negative under the assumptions of the model because the denominator is simply the determinant of the matrix of second derivatives of the production function multiplied by the after tax return price of output. Under the assumption of strict concavity this determinant must be positive and the second derivative of the production function with respect to nonland inputs must be negative. The after tax price of output will usually be positive. Land acquisition requires the difference in brackets in (26.1) to be negative; clearly this is more likely if the land price is increasing at an increasing rate ($\delta_t > 0$).

Like the decision on the level of land use, the land purchase decision is sensitive to the proportion of capital gain which can be substituted for current income. If the parameter ($\phi$) is larger, the change in returns to land ownership are larger and purchase will be larger. However, even if the parameter ($\phi$) is zero, capital gains can have an effect. Likewise, if the capital gain exemption ($1-\psi$) is larger, the value of realized gain will be larger and the rate of land acquisition will be larger for a finite terminal date. Equation (26.2) indicates that the nonland input purchase decisions depends on how the productivity of those inputs changes with increased land area per farm. If the marginal product of nonland inputs increases rapidly as land increases, acquisition of nonland inputs will be greater. It should be noted that because all the costs and returns of nonland inputs are taxed equally, the tax rate does not directly affect the nonland input purchase but only through the land purchase decision. Taxes are, however, an important factor in the land purchase decision primarily because of the
capital gain exemption and the fact that unrealized gain is not taxed. The effect of the tax rate is complex because it appears in every term in the land purchase equation and in some terms more than once, but it is clear that a higher tax rate increases the importance of the capital gain exemption in the land purchase decision.

**Junction conditions**

According to the work on junctions between singular and nonsingular arcs by McDanell and Powers (1970), the junction between the initial adjustments and the singular control arc should be discontinuous for equity debt and nonland input controls, and for the case in which the nonsingular land control is positive. Their theorem 3 states that if the second time derivative of $\tilde{H}_u$ (the derivative of the Hamiltonian with respect to the control variable) is not equal to zero on the nonsingular side of the junction, then control must be discontinuous (p. 166). Because it is likely that equations (25.1), (25.2) and the time derivative of financial structure equation (8.2) will be nonzero during the initial adjustment when state variables are changing rapidly, the conditions of this theorem will be met in most cases. This theorem does not require the control on the nonsingular side of the junctions to have well-defined derivatives and therefore applies to the case of instantaneous adjustment on the nonsingular side. Hence, the discontinuous control path described earlier, in which the control is at one instant approaching negative or positive infinity and at the next instant is part of a smooth, finite singular control path, fits the theoretical requirements for optimality.
In the case of the land control at its lower bound either at the initial instant or otherwise, McDanell and Powers (1970) suggest that the junction may be either a jump discontinuity or continuous with a continuous first derivative (p. 165). This result is based on their theorem 1 which requires the control to be differential around the junction. The continuous junction is likely for most plausible parameter specifications. This is true because the MRS in equation (9.7) characterizes the optimal level of land and nonland inputs and it changes smoothly with land price changes. The control equations (26.1) and (26.2) are simply the relationships which will maintain the optimal MRS. If the MRS cannot reach the optimal level, because the initial level of land holding was above the optimal level or for some other reason, the MRS will change smoothly because the land price, which is the only parameter which changes in the MRS equation, changes smoothly. If the MRS is not optimal, but is approaching optimality it will approach it smoothly; at the junction point the time derivative of the denominator of the MRS will be zero and the singular control equation will take over smoothly. This scenario might occur if the capital gain in the early part of the planning period was small and the initial land holding was relatively large, but later in the planning period large capital gains occur. In this case the MRS (9.7) would indicate a relatively modest land holding during the first part of the planning period, perhaps smaller than the initial holding, so no land acquisitions would be made at first. Later, when the capital gains increase, a larger land holding may be justified and land purchases would be made.
It should be noted that after the initial adjustment when no land purchases are being made the MRS between land and nonland inputs is a constant defined at the relationship which allows equation (24.2) to hold for the fixed land level. Essentially, the model is optimizing the use of nonland inputs for a given acreage in this case. Because the parameters of equation (24.2) are constant, the chosen nonland input level is maintained until the acreage changes. In addition, because input levels are not changing equations (23.2) and (23.3) suggest that debt and equity levels are unchanged while no land purchases are being made.

The ordinary Legendre-Clebsch necessary condition which requires the matrix of second derivatives of the Hamiltonian with respect to the controls to be negative semidefinite for maximization is satisfied trivially throughout the problem. The second order matrix is everywhere a null matrix because of the linear entry of the controls. The generalized Legendre-Clebsch condition is meaningfully satisfied. This condition requires that the matrix of the derivatives of equations (25.1) and (25.2) and the time derivative of equation (8.3) with respect to the control be positive semidefinite (Bryson and Ho, p. 258):

\[ h' \begin{bmatrix} \frac{\partial}{\partial u} & \frac{\partial^2 H}{\partial u \partial t} \\ \frac{\partial}{\partial t} & 2 \end{bmatrix} h > 0, \text{ for all } h \neq 0. \]

In this problem, equation (27.1) take simple form because terms including the control variables cancel out of the time derivative of equation (8.3) along the singular path. For this problem, the generalized
Legendre-Clebsch condition is:

\[
(26.2) \quad h' \left[ \partial^2 H \over \partial u \partial t^2 \right] h = h' \begin{bmatrix} 0 & 0 \\ 0 & -A \end{bmatrix} h \geq 0
\]

The matrix A is as defined in the static model except that the discount term \((\rho/p)\) is replaced by \((\rho(1-t)e^{-\rho(1-t)t})\). A must be negative semi-definite under the assumption of strict concavity. Hence, the generalized Legendre-Clebsch condition is satisfied everywhere along the singular arc, including the endpoints.

Argument for optimality of the necessary conditions

Because formal mathematical sufficiency arguments for the mixed singular and bang-bang control problem are not well-developed, an informal argument for the optimality of the control scheme suggested by the necessary conditions will be presented. It will be based on the subject matter reasonableness of the control solution. A common formulation of the control theory sufficiency condition states that if the Hamiltonian is concave when evaluated at the optimal adjoint and control variables, then the necessary conditions are also sufficient. Concavity of the Hamiltonian requires the matrix of second derivatives with respect to the state variables to be negative definite. Except for the discount terms the matrix of second derivatives can be partitioned and written in the same form as the static model Hessian (7.1). Thus, the concavity of the Hamiltonian can be shown by the arguments of the same form as those used for sufficiency of the simple static model.
The problem is then reduced to questioning the optimality of the singular control scheme. The bang-bang controls fit well into usual sufficiency conditions. If the control variables were more tightly bounded and the parameters and functional specification was such that the entire control path was bang-bang, such a control scheme would be considered to satisfy the sufficient conditions for optimality. For the singular control scheme to be optimal, it must at least maintain the equalities in equations (27.1-3). Given the optimal adjoint variables there is only one singular control solution that satisfies this requirement, the solution given by equations (23.2, 26.1-2).

The alternative to the singular control would be some sort of solution that allowed the controls to "chatter" between the upper and lower bounds. Based on knowledge of the farm operations and economic reasoning this "chattering" solution is an unlikely candidate for optimality. For instance, in the case of nonland inputs a chattering solution would mean that large quantities of inputs would be acquired in one instant and disposed of the next instant, and this pattern would continue through extended periods. Though in the absence of adjustment cost a chattering solution cannot be entirely ruled out, it is hard to see any economic rationale for such a solution. The overall control solution is made more plausible by the fact that it brings the state variables to levels at which the marginal costs of all sources of capital are equal and the marginal return from inputs is equal to their marginal cost, and then maintains those relationships inasmuch as that is possible with the assumption of no land sales. The fact that the costs and
returns include capital gains and effects over time goes beyond the usual static results, but is not inconsistent with them. Hence, the control scheme defined here is a promising candidate for optimality. A numerical solution of the problem is being conducted with dynamic programming that will further test the optimality of the proposed control solution.

The multigenerational problem

If it is assumed that each generation solves a problem like (2") given some initial endowment passed from the previous generation and the objective is to maximize the net present value of income, then an overall solution can be analyzed using dynamic programming arguments. The dynamic programming principle of optimality states:

The best path from A to B has the property that, whatever the initial decision at A, the remaining path to B starting from the next point after A, must be the best path from that point to B (Dreyfus, 1965, p. 3).

The principle of optimality is implemented by maximizing the sum of the present periods returns and the present value over the remaining horizon given the choices in the present period. In this context, the recurrence relation would be problem (2") plus the optimal solution for all the generations after the one being considered given the level of assets passed from the generation being considered to the next one. Essentially, this means that the terminal value term must be modified to account for the value of the terminal assets to future generations. For expository purposes, a two generational model will be considered. More generations would simply mean a more complicated terminal value term added to problem (2"). With a positive discount rate and relatively long generational period, the two generation model will capture almost all of the net present value information.
Under the assumptions of problem (2") all assets were sold at the end of the generation's planning period and equity capital was returned to the owner of the farm business. With these assumptions the generations are independent; no assets are passed on to the next generation and hence the value of terminal assets have no importance for the decisions of the future generations. If this assumption is changed so that only a portion of the terminal assets are sold after the first generation, then the terminal value function could be written:

\[ e^{-(1-\tau)T}UK_T + uG_T(1-\psi_T) + V[(1-u)K_T, (1-u)D_T, (1-u)X_T, (1-u)I_T, (1-u)G_T] \]

where

- \( u \) = the proportion of assets retained by the first generation and
- \( V(\cdot) \) = the present value of the assets passed on to the next generation.

The proportion \( u \) could differ between asset types, but for simplicity the same proportion is used for all assets. The proportion is assumed to be a constant that depends on the desire of the decision maker to make a bequest to the next generation. The exact functional form of \( V(\cdot) \) will depend on the forms specified for the production and debt cost functions, but the first derivatives of \( V(\cdot) \) can be defined. These derivatives simply express the marginal value of each asset at the initial point of the second generation and are equal to the adjoint variables for the second generation evaluated at that initial point. If the terminal value function (28) is used in problem (2") the necessary conditions remain unchanged and the transversality conditions become:
\begin{align}
(20.1') \quad \lambda_{1T} &= \hat{u} e^{-(1-\tau)T_1} + (1-u) \left\{ \int_{T_1}^{T_2} e^{-(1-\tau)s} \frac{D_s}{K_s} (1-\tau)ds \right. \\
&\quad \left. + e^{-(1-\tau)T_2} \right\} \\
(20.2') \quad \lambda_{2T} &= (1-u) \left\{ \int_{T_1}^{T_2} e^{-(1-\tau)s} \left[ (\beta_2 - \gamma_2)(1-\tau) + \hat{\beta}_s \right] ds \right. \\
&\quad \left. + e^{-(1-\tau)T_2} (\beta_{T_2} - \beta_{T_1})(1-\phi-\psi_\tau) \right\} \\
(20.3') \quad \lambda_{3T} &= (1-u) \int_{T_1}^{T_2} e^{-(1-\tau)s} (\beta_1 - \gamma_1)(1-\tau)ds \\
(20.4') \quad \lambda_{4T} &= - (1-u) \int_{T_1}^{T_2} e^{-(1-\tau)s} \left[ \gamma + \frac{D_s}{K_s} \right] (1-\tau)ds \\
(20.5') \quad \lambda_{5T} &= \hat{u}(1-\phi-\psi_\tau) e^{-(1-\tau)T_1} + (1-u)(1-\phi-\psi_\tau)e^{-(1-\tau)T_2}
\end{align}

where:

- \( T_1 \) = the terminal time for the first generation and
- \( T_2 \) = the terminal time for the second generation.

This formulation assumes that the land is sold in such a way that the amount of the capital gain realized is the same proportion of total unrealized gain as the land sold is a proportion of the total land owned at the terminal date. In addition, it is assumed that the first generation's income tax basis for the land passed on to the heirs becomes the
basis for the second generation. This model omits the question of estate or gift taxes and their effect on the income tax basis. The transversality conditions could then be used in defining the optimal adjoint variables for the multigenerational problem. For instance, the adjoint variable for land becomes:

\[
(22.2') \quad \lambda_{2t} = \int_{t}^{T_1} e^{\rho(t)} s \left[ (p_{fL} - y_2)(1-\tau) + \beta \phi \right] ds \\
+ (1-u) \int_{T_1}^{T_2} e^{\rho(t)} s \left[ (p_{fL} - y_2)(1-\tau) + \phi \beta \right] ds \\
+ u(1-\phi-\psi)(\beta \beta_{T_1} - \beta_T) e^{-\rho(1-\tau)T_1} \\
+ (1-u)(1-\phi-\psi)(\beta \beta_{T_1} - \beta_T) e^{-\rho(1-\tau)T_2} \\
+ (l-\phi-\psi)(\beta \beta_{T_1} - \beta_T) e^{-\rho(1-\tau)T_2}
\]

In the multigenerational model, both current income and capital gains from the second generation affect the input choices in the first. When the adjoint variables defined with equations \((22.1'-22.5')\) are substituted into the control expression \((21.1-3)\), the optimal combination of bang-bang and singular control is affected.

The impact of succeeding generations is not great along the singular arc. The MRS and land purchase equations \((9.7)\) and \((26.1)\) take almost the same form in the multigenerational model as they do in the single generation problem. The only change is in the realized capital gain
term. In the multigenerational case part of the capital gain is discounted back from the end of the second generation, so the realized gain is likely to have a smaller impact. The primary impact of events in the second generation is on the optimal combination of bang-bang and singular arcs. For example, consider the case of land prices rising at an increasing rate throughout the first generation and stable land prices in the second generation. The model suggests that the optimal farm size is larger for the first generation than for the second. If all of the assets of the first generation are passed on to the second \( (v=0) \), then the acreage passed to the second generation would be larger than the optimal farm size if the first generation followed the single generation optimal solution for period 0 to \( T_1 \), and returns would be reduced in the second generation by larger than optimal farm size. In the multigenerational model, the marginal losses in the second generation would be felt in the first generation and the land acquisition control variable may be set to its lower bound where it would be positive in a single generation model.

The impact of capital gain realization is likely to be reduced in the multigenerational model. Because of discounting, the first generation gain that is not realized until after the second generation is less valuable than it would be in a single generation model. The larger the discount rate, the time period, or the percentage of property passed to heirs, the larger the reduction in capital gain impact will be.

The impact of the proportion of capital gain that is substitutable for current income has the same impact in the multigenerational model.
that it has in the static and single generation dynamic models. When the proportion goes up the impact of capital gains on input choice increases and more land is included in the input mix.

A version of the multigenerational model could also be used to examine activity that is more speculative. Instead of viewing the time periods as generations, they might be viewed as holding periods. For instance, the investor may wish to realize some capital gain every five years by selling some proportion of the land. The same necessary conditions hold as in the multigenerational model. Again, the exact impacts depend on how the parameters change between selling times, but the impact of the proportion of capital gain substitutable for current income would be the same as in the static model. A larger value for \( \phi \) would tend to increase land holding. Capital gain realization would have a larger impact in the investor model than in the family farm model because realization occurs earlier. Consequently, the capital gain exemption would be more important. A tax exemption that is only a few years in the future in the investor problem has a greater impact than the exemption that may be 30 or more years in the future in the traditional family farm optimization problem.

To modify the single generational problem for declining land values, the capital gain terminal value should be replaced with the capital loss terminal value (11.2). The necessary conditions will be unchanged and the transversality conditions (20.5) becomes:

\[
20.5') \quad \lambda_{5T} = e^{-\rho(1-\gamma)T} [1-\phi-\tau T \delta^\rho(1-\gamma) / \epsilon T]
\]
The optimal control is bang-bang and singular, as it was for the increasing land price case. The optimal control is described by equations (21.1-3) with equation (20.5") used in defining the adjoint variables. The optimal financial structure is unchanged. Using equation (20.5") the MRS along the singular arc:

\[
\frac{\frac{\partial X}{\partial L}}{\frac{\partial L}{\partial L}} = \frac{\frac{D}{\gamma_1 - \alpha \left( r + \frac{r}{R_L} \right)(1-\tau)}}{\{\gamma_2 - \frac{B}{R_L} (1-e^{-\rho(1-\tau)(1-\tau)}) / \rho(1-\tau)\}\left( r + \frac{r}{R_L} \right)(1-\tau)}
\]

\[
= \frac{D}{\gamma_2 - \frac{B}{R_L} (1-e^{-\rho(1-\tau)(1-\tau)}) / \rho(1-\tau)\left( r + \frac{r}{R_L} \right)(1-\tau)}
\]

In an environment of declining land price, the marginal cost of land use is increased by the capital losses. The optimal solution is now to economize on land holdings and use more of the nonland input in the farming operations to avoid capital losses. When capital losses are larger, the denominator of the MRS will be larger indicating a larger optimal marginal product of land and/or a smaller optimal marginal product of nonland inputs at any time. The impact of capital losses on the input mix is sensitive to the same variables as the impact of capital gains. If a higher proportion of losses are recognized as current earnings, the impact is greater. If the planning horizon is longer, the impact tends to be reduced because the discounted value of the realized loss is small. Increasing the capital loss deduction limit or the proportion of losses deductible would tend to reduce the impact of the
losses by increasing the value of the tax benefits, but the effect is likely to be small because the tax benefit occurs only at the end of the horizon.

Along the singular arc land purchases are described by an expression similar to (26.1) except that the capital gain realization term \((1-\phi-\psi_T)\) is replaced by a capital loss realization term. Land purchases can occur in the declining land price case if lower financial cost of the new acquisition is enough to outweigh the losses due to additional land ownership. For instance, this might occur if larger capital losses are occurring during the initial part of the planning period, but the land price stabilized later in the period. Initially, the optimal input mix as defined by (9.8) would include a relatively small amount of land. Later, in the period a larger land input can be justified with a smaller capital loss rate.

The multigenerational model does not need to be changed substantially for the declining land price case. Realized loss terms replace realized capital gain terms in equations (20.1'-20.5') for periods in which the land price declines. Because capital loss deductions are not passed on to succeeding generations, the capital loss situation does not add greatly to the complexity of the model. If land price declines continue over more than one generation, the tendency of the model to whom smaller farm size and more intensive farming is likely to be increased because losses in the second generation will tend to further reduce land holdings in the first.
The situation of the investor requires a somewhat more complicated model because capital losses in one period can be used to offset farm income during the following period in the same investor's career. This might be modeled by defining a capital loss deductions account, denoted CLD, which is carried into a period and a control variable to allocate the deductions. The change in the capital loss deductions account would be the negative of the control variable. The average tax rate multiplied by the control would be added to current income in the objective function. The control variable would be bounded by zero and $3000, the current maximum annual deduction. With a constant average tax rate, control would be completely bang-bang for the capital loss deductions; the maximum deduction of $3,000 would be taken as long as the account is positive. The implications of declining land prices for the investor would be the same as for the other models; land holding would be smaller and farming more intensive with larger price declines. The tax implications are likely to be greater for the investor. Because losses are realized earlier, the present value of the capital loss deduction is greater. The size of the maximum deduction would tend to be more important for the investor than for the other models. In the investor model, the deductions will tend to offset income that is taxed at a higher rate than the retirement income assumed in the other models.

The model might also be used to examine mixed increasing and decreasing land price patterns. In the model, one period may use the increasing land price formulation while the next period uses the decreasing price definitions. Because the length of the periods is
arbitrary and they do not have to be of equal length, the periods can be chosen to fit the periods of increasing and decreasing land price. For instance, sharply rising land prices might characterize the first part of a farmer's career, while land price declines occur in the second part. If the single generation model assumption of no land sales until the end of the career is maintained, knowledge of future land price declines will tend to decrease land holding during the initial period of capital gain. If midcareer sales are allowed, the impact on initial period decisions would be minimized because land holdings could be reduced at the end of the first period to the level adapted to the conditions of declining land prices. The model suggests that contraction of the farm business by selling land and other assets upon entering a period of declining land prices can be a rational, wealth maximizing move that should not be viewed with alarm by farmers or lenders. In the absence of adjustment costs, the model suggests contraction of the farm business upon entering a period of declining prices may be rational even if the declining price period is short and followed by another period of rising prices. For example, the land price pattern might be characterized by an initial period of rising prices, a second short period in which prices fall and a final period in which prices rise and exceed the terminal price of the first period. If midcareer sales are allowed, the model indicates that the optimal input mix will include more land at the end of the first period than at the beginning of the second, and that optimal land holdings will rise again upon entering the third period. Of course, adjustment costs, uncertainty about parameter values in the future and
other factors complicate actual economic decisions, but the point remains that farm business contraction or perhaps ceasing production altogether may be the wealth maximizing course of action when coming out of a period of capital gains into a period of capital losses.

Dynamic Model, Income Approach

In the dynamic model, a more realistic version of the income approach model can be considered. The total capital gain term in the modified debt cost function (15) is replaced by the accumulated capital gain \( G_t \). A pure income approach dynamic model can be defined by setting the proportion of capital gain substitutable for current income to zero \((\phi=0)\) and replacing the debt cost function in problem (2") with equation (16). The necessary conditions (19.1-4 and 16.6-9) are unchanged. Equation (19.5) becomes:

\[
(19.5") \quad \lambda_t = \frac{3H}{3G_t} = e^{-\rho(1-\tau)t} \omega r'(\frac{D_t}{K + \omega G_t})^2(1-\tau)ds
\]

The transversality conditions are not changed and the optimal control scheme is still described by expression (21.1-3). In the pure income approach model, the adjoint variables become:

\[
(22.1') \quad \lambda_{1t} = \int_t^T e^{-\rho(1-\tau)s} r'(\frac{D_s}{K + \omega G_s})^2(1-\tau)ds + e^{-\rho(1-\tau)T}(1-\psi_t)
\]
(22.2') \[ \lambda_{2t} = \int_t^T e^{-\rho(1-\tau)} s \left( Pf_L - \gamma_2 \right) (1-\tau) \]

\[ + \beta \int_T^t e^{-\rho(1-\tau)} \omega \left( \frac{D \nu}{K_v + \omega G_v} \right)^2 (1-\tau) dv \] ds

\[ + e^{-\rho(1-\tau)} (\beta_T - \beta_t) (1-\psi_T) \]

(22.3') \[ \lambda_{3t} = \int_t^T e^{-\rho(1-\tau)} s \left( Pf_X - \gamma_1 \right) (1-\tau) ds \]

(22.4') \[ \lambda_{4t} = \int_t^T e^{-\rho(1-\tau)} s \left( r + \tau \left( \frac{D}{K_s + \omega G_s} \right) \right) (1-\tau) ds \]

(22.5') \[ \lambda_{5t} = \int_t^T e^{-\rho(1-\tau)} \omega r \left( \frac{D}{K_v + \omega G_v} \right)^2 (1-\tau) ds + e^{-\rho(1-\tau)} T (1-\psi_T) \]

where: \( v \) = variable of integration.

In the adjoint variable for land, the value of unrealized gain substitutable for current income \( (\phi \beta_t) \) has been replaced by the debt cost reductions due to the marginal unrealized gain:

\[ \beta_t \int_t^T e^{-\rho(1-\tau)} \omega r \left( \frac{D \nu}{K_v + \omega G_v} \right)^2 (1-\tau) dv \]

The adjoint variable for unrealized gain now includes both the value of gain realization at the end of the period and the effect of unrealized gain on interest costs.
By substituting the above equations into expression (21.1-3), it can be seen that the optimal control is still characterized by an initial adjustment of asset levels and largely singular control thereafter except for the case of no land purchases. The optimal singular arc can be characterized by the method of differentiating conditions (19.6-8) with respect to time that was used for the wealth approach. The financial structure under the income approach can be characterized by differentiating equation (19.6) with respect to time and substituting in equations (19.1' and 19.4):

\[ e^{-p(1-r)t}(1-\tau)[r-r'(\frac{D_t}{K_t+\omega G_t})^2 - r - r'(\frac{D_t}{K_t+\omega G_t})] = 0. \]

Equation (8.3) implies that along the singular path the marginal cost of debt will equal the marginal cost of equity, as in the wealth approach model. For a constant discount rate, equation (8.3) can in theory be solved for the optimal leverage ratio, which is constant over the planning horizon. The constant optimal leverage ratio implies that along the singular arc equity investment, new debt acquisition and the change in unrealized capital gains stand in a fixed relationship. This relationship can be shown by differentiating the leverage ratio with respect to time:

\[ (23.1') \quad \frac{\delta(\frac{D_t}{K_t+\omega G_t})}{\delta t} = \frac{\dot{D}_t}{K_t+\omega G_t} - \frac{D_t}{(K_t+\omega G_t)^2} \left( \dot{K}_t + \omega \dot{G}_t \right) \]
By solving equation (23.1') simultaneously with constraints (2.1, 2.4-5)
it can be shown that equity investment and new debt depend on land and
nonland input purchases and on appreciation:

\[
(23.2') \quad u_{1t} = \frac{K_t + \omega G_t}{D_t + K_t + \omega G_t} [\beta u_{2t} + \alpha u_{3t} - \left(\frac{D_t}{K_t + \omega G_t}\right) \omega \beta L_t]
\]

\[
(23.3') \quad D_t = \frac{D_t}{D_t + K_t + \omega G_t} [\beta u_{2t} + \alpha u_{3t} + \omega \beta L_t]
\]

The optimal accumulation of unrealized capital gains is defined by the
optimal land holding through constraint (2.5). Equation (23.2') suggests
that at any point along the singular arc equity investment will be
reduced if the rate of capital gains is increased because of the negative
term in acreage and land price change. Equation (23.3') indicates that
at any point along the singular arc the acquisition of new debt will rise
as the capital gains rate increases because of the positive term in
acreage and land price change. The behavior suggested by the dynamic
income approach model is similar to that identified in the static model.
It suggests that the increased debt use by U.S. farmers during the 1960s
and 1970s may have been at least partially in response to the large
capital gains. The magnitude of this impact depends largely on the
equity substitution parameter (\(\omega\)). If lenders are willing to accept a
large proportion of appreciation equity, the equity investment will be
reduced and debt use increased. This is apparently what happened in the
1970s. After the 1973-74 commodity boom, the mood of farmers and
agricultural lenders was optimistic. There was an expectation of continued high demand for farm products and of continued land price increases. Real estate appreciation was viewed as a relatively solid asset. Under such conditions, the equity substitution parameter would tend to be large. The model suggests that new debt acquisitions would be large and savings would stagnate.

Unlike the wealth approach model, equations (23.2') and (23.3') indicate that the financial structure can be changing even when no change is occurring in the input levels. For example, if the rate of land price change is constant and input levels have adjusted to that rate, the model suggests that land and other input acquisitions would be zero. But equations (23.2') and (23.3') indicate that as the land price increase continued additional debt would be acquired and equity would be withdrawn from the business; dissaving would occur as unrealized gain substituted for equity in the financial process.

An income approach MRS can be defined by differentiating equation (19.2) and (19.3) with respect to time, substituting in the adjoint variables (22.1") (22.2") and (22.3") and simplifying:

\[
9.9) \quad \frac{f_X}{f_L} = \frac{[\gamma_1 - \alpha(\tau + \tau \frac{D_t}{K_t})](1-\tau)}{\{\gamma_2 + [\beta_t - \frac{\beta_t}{\rho(1-\tau)}(1-e^{-\rho(1-\tau)(T-t)})[r+\tau \frac{D_t}{K_t+\omega G_t}])(1-\tau) \\
- \beta t \int_t^T e^{-\rho(1-\tau)(s-t)w(\frac{D_s}{K_s+\omega G_s})^2}(1-\tau)ds \\
- e^{-\rho(1-\tau)(T-t)(1-\psi T)}
\]
As in the wealth approach model, increasing land prices tend to encourage the use of more land in the income approach model input mix. The denominator of the factor cost ratio is decreased by the impacts of capital gains: the financial benefits of buying now instead of later in the finance cost term, the reduced interest cost because of the substitutability of unrealized capital gains for equity, and the after tax value of realized gain at the end of the period. The capital gain exemption has the same effect in the income approach model as the wealth approach model. A larger exemption enhances the capital gain impact. In the income approach, the equity substitution factor \( (\omega) \) has an impact similar to that of the proportion of gain substitutable for income \( (\phi) \) in the wealth approach model; a larger adjustment factor will tend to increase the benefits from unrealized gain and encourage larger farm size and more extensive farming.

As in the wealth approach model, equations (19.7-8) can be differentiated twice with respect to time and explicit equations for the control variables can be derived. The nonland output control is unchanged. The land purchase control variable is described by a system similar to (26.1) but with the substitution of unrealized capital gain term \( (\delta) \) replaced by a term reflecting the debt cost reduction due to additional unrealized capital gain. Again, after the initial adjustment land will be acquired only if the change in the returns to land ownership is larger than the change in the financial costs of ownership. The rate of land acquisition will tend to be greater if more unrealized gain is substitutable for
equity, if the capital gains tax exemption is larger, and if the decision maker is closer to the terminal date at which gains will be realized.

The income approach model can be extended to the multigenerational case, to allow more speculative activity, to permit declining land prices and to the multiple enterprise problem in the same manner as the wealth approach model and with the same results. In addition, the income approach does not have to be used in its pure state. The two approaches can be combined and this is perhaps the most realistic model.

Sufficiency arguments for the income approach

The arguments of the optimality of the control scheme defined for the income approach are very similar to those advanced for the wealth model. All the necessary conditions are satisfied by the income approach. The Legendre-Clebsch is satisfied trivially everywhere. The generalized Legendre-Clebsch condition is unchanged from the wealth approach model and is satisfied. The form of the matrix of second derivatives of the Hamiltonian is affected by the income approach modifications because the unrealized gain variable enters the Hamiltonian. The matrix may be partitioned and written:

\[
\begin{bmatrix}
A & 0 & 0 \\
0 & C & E^* \\
0 & E^* & I \\
\end{bmatrix}
\]
where matrices $A$ and $C$ are as defined under the wealth approach with the market value leverage ratio as argument of the debt function and $E$ and $I$ are:

$$E^* = e^{-\rho(1-\tau)\tau(1-\tau)[\frac{r_nD_t}{(K_t+\omega G_t)^2} + \frac{2r^\prime}{K_t+\omega G_t}]} \begin{bmatrix} D_t \omega(\frac{1}{K_t+\omega G_t}) \\ -\omega(\frac{1}{K_t+\omega G_t})^2 \end{bmatrix}$$

$$I = -e^{-\rho(1-\tau)\tau(1-\tau)[\frac{r_nD_t}{(K_t+\omega G_t)^2} + \frac{2r^\prime}{K_t+\omega G_t}]} \begin{bmatrix} D_t \omega(\frac{1}{K_t+\omega G_t}) \\ -\omega(\frac{1}{K_t+\omega G_t})^2 \end{bmatrix}$$

Concavity requires for all vectors $h$:

$$\begin{bmatrix} C & E^* \\ E^* & I \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} \begin{bmatrix} h_3 \\ h_5 \end{bmatrix} \leq 0$$

$A$ is negative definite by assumption. The terms in $C$, $E^*$, and $I$, can be factored and written:

$$-e^{-\rho(1-\tau)\tau(1-\tau)[\frac{r_nD_t}{(K_t+\omega G_t)^2} + \frac{2r^\prime}{K_t+\omega G_t}]} [-h_3$$

$$+ h_4 \frac{D_t}{K_t+\omega G_t} + h_5 \frac{\omega D_t}{(K_t+\omega G_t)^2}].$$
Which is negative definite for all $h=0$ under the assumptions of the model. Hence, the Hamiltonian is concave under the income approach modifications and the informal optimality arguments advanced for the wealth approach model can be applied to the income approach case. The optimality of the proposed control plan is not proven, but all available information indicates that it is the most likely candidate for optimality.

**Limitations of the model**

In an effort to simplify the model, numerous assumptions have been made and complications ignored. The primary limitation of the model is that it ignores the optimal selling time for land. In modeling the traditional idea of a family farm, this assumption does not cause major problems. In this traditional view, the farmer does not intend to sell the land, at least not before retirement. The assumption is, however, a limitation in examining activity in which a speculative motive is present. Another important limitation is that uncertainty does not enter into the model. Several important constraints on farmer behavior have been neglected. Ordinarily, the farmer does not face an unlimited supply of land that will fit into the farming operation, is within a reasonable distance, and is for sale at the market price. Only a small percentage of the land in the U.S. is offered for sale each year. Usually, farmers do not have unlimited supplies of nonfarm equity capital to draw on. Their equity investments are often limited to retained earnings.

The assumption of a fixed equity discount rate was made to improve the mathematical tractability of the model and allow clear determinations
of signs in the comparative statics of the static model. Variable
discount rate models seem to have implications similar to the fixed
discount rate models presented with the variable discount rate
accentuating the financial influences already present in the model
through the debt supply function. The implications are not the same,
however. For example, the separation of financial and production
decisions in the dynamic wealth approach model does not occur if the
discount rate is a function of the leverage ratio. In a more complete
model, the fixed discount rate assumption might be relaxed, but only if
the production and debt supply functions are more completely specified so
that all second order conditions can be determined.

An additional limitation of the analysis is that it focuses on the
microeconomic impact of land price appreciation. The ties between the
individual buyers who make up the land market and market behavior are
left unexplored. The model certainly has implications in that direction
that could be the subject of further research. For example, in the case
of the two period model in which land price rises in the first period and
falls during the second period, the model suggests that the optimal
choice is to sell land at the end of the first period to reduce holdings
to the level justified in an environment of capital losses. If many land
owners followed this plan, large amounts of land would be dumped on the
market at the end of the period in which land prices were rising and
prices in the second period would decline further and faster than they
otherwise would. Further research might show that a part of land price
decline in the mid-1980s was caused by a glut of farmland marketings as
land owners adjusted their holding to an environment of smaller capital gains.

Another question is the impact of the capital gains effect on overall efficiency. Do capital gains tend to push the decision maker toward greater overall efficiency? Macroeconomic analysis of land appreciation should also consider the social costs and benefits associated with farm size and intensity that do not appear in the calculation of the individual producer.

Summary and Conclusions

This study developed an optimization model of the impact of farmland price changes on the production and finance choices of a wealth maximizing decisionmaker. The model allows for both capital gains and losses, and the possibilities that some proportion of the unrealized capital gain or loss may be substituted for current income or recognized in the financial negotiation as collateral. The model offers a rigorous theoretical foundation for the hypotheses on the impact of capital gains that have been informally advanced by agricultural economists and it allows preliminary analysis of the impact of individual parameters under broad assumptions on their signs and magnitudes. In general, the model suggests that in an environment of large capital gains farmers will tend to enlarge farm acreage and incur higher debt loads in their attempts to take advantage of the farmland appreciation. These effects are found in both dynamic and static versions of the model. If equity is a constraining factor, the model suggests that farmers will tend to use more highly
leveraged financial structures when land prices are rising. Over a wide range of parameter values, the model suggests that capital gains encourage extensive farming. Capital losses have the opposite effect of capital gains. The magnitude of these effects is an empirical question, but it should be noted that the model is broadly consistent with the farm size expansion and greater debt use that occurred during the period of farmland capital gains in the 1950s, 1960s and 1970s. The model also offers an explanation for the virtual disappearance of fruit and vegetable production from the Midwest in the same period and the renewed interest in horticultural crops that has been shown by some Iowa farmers during the recent period of declining farmland prices.

An important insight offered by the model is that the impact of capital gains on farm decisionmaking is not purely a result of tax rules. The effects of land price changes on finance and production choices are likely to show up under a wide variety of tax planning. The model suggests that the present tax rules tend to enhance the impact of capital gains by deferring the taxation until realization of the gain and exempting 60 percent of the capital gains from taxes. Accrual taxation of the gain would reduce the impact of the capital gain on decisionmaking and for some parameter values it may even reverse the effects by making the tax liability larger than the gain. With the current tax system, the impact of capital gains on farm size and debt use is positive for all plausible parameter values. In the case of capital losses, the deductibility of the loss tends to offset the negative impact of the loss, but the impact of these tax deductions is sharply limited by the tax rules.
which only allow half of the loss to be deducted, limit the annual
maximum deduction and only allow losses to be deducted when they are
realized.

The model may be used to describe either the traditional farm family
or the long term planning decision of an investor. If the planning
horizon of these two decision makers differ the model suggests that the
tax impacts of on farm size and intensiveness will differ. For instance,
if the investor's institutional structure and need for liquidity require
more frequent realization of gain than the farm family, then the capital
gain tax rules will have a larger impact on the investor than the farm
family. In the limiting case of the farm family which intends to pass
its land from generation to generation, the income tax rules have no
effect. Hence, the model suggests that advocates of the moderate size
family farm should push for elimination of the capital gains tax exemp-
tion for farmland. The exemption encourages larger farm size and more
extensive farming; the opposite of the family farm ideal. In addition,
if it is true that investors tend to realize the gain more frequently
than family farmers, then elimination of the exemption would tend to have
a far greater impact on investor choices than on family farm decisions.

The model also suggests that government price support and other
programs may have indirect impacts on farm size and debt use through
their effect on capital gains or the substitutability of unrealized gains
for income. If a price support program raises expectations about future
income to land or increases the certainty of that income, the value of
capital gains to the decisionmaker is likely to increase. With higher
capital gains a higher level of unrealized gains that are substitutable for current income, the model suggests that farm size is likely to expand and the debt load is likely to increase. Weakening an existing program which increases capital gains or the certainty of those gains, is likely to reduce farm acreage expansion and reduce debt use. The implication of the model is that price support and other programs which are ostensibly developed to protect the moderately sized family farm may have indirect effects through capital gains that are the opposite of the desired protection.

Decisions by public and private lenders also can affect the impact of capital gains. The income approach model suggest that by increasing the proportion of unrealized gains that they are willing to accept as net worth in the financial negotiation, lenders tend to increase total returns to land and offer incentives to increase farm size and debt load.

If the model describes a long term planning process in which parameter magnitudes are expected values, then it is useful to think of revisions of the plan that would occur if new information were acquired. For instance, in going from a period in which capital gains were expected to continue indefinitely to a period of capital losses, the model suggests that there may be large adjustments in debt use, farm size and production decisions. The financial structure that is appropriate for a period of rapid capital gains may be too highly leveraged for a period of losses. Because of the adjustment costs that are not treated in this model the reality of these changes in asset use and financing is likely
to be a wrenching experience for many farmers. Because of the lumpiness of many farm assets and liquidity losses changing the scale of production may be difficult. The model suggests that the economic distress of many farmers in the early 1980s can be at least partially explained by difficulty in making the adjustments required to adapt to a period of declining, or at least more stable land prices.

In general, the model indicates that the capital gains which made farming an attractive career for farm families in the 1970s also helped create the conditions which undermine the family farm structure. Capital gains on land create economic pressures which force farmers away from the family farm ideal of the small, intensively managed farm. The additional debt that is likely to be acquired in periods of capital gains makes farmers vulnerable in recessionary periods. Certainly, capital gains are not the only force changing agriculture. Technological change, the development of export markets and other factors have played major roles, but this model allows the identification and understanding of another force behind the historic changes in farming.
REFERENCES


PART II
EXTENSIONS OF THE THEORETICAL MODEL

A recently developed model of farm production and finance decisions suggests that farmland capital gains tend to encourage larger farm acreage, greater debt use and the choice of land extensive enterprises (Lowenberg-DeBoer, herein). The model indicates that capital losses tend to have the opposite effect. These results were derived assuming that the decision maker could purchase an unlimited supply of farmland at the current market price, that there were no adjustment costs in expanding farm sizes, that the producer had access to enough equity capital to achieve the optimal asset levels with the optimal financial structure and that the farmland price path was known with certainty. These assumptions allow a relatively simple model to be developed, but they leave something to be desired in terms of realism. The supply of farmland that would fit into a producer's operation is usually limited. Not only is the total supply of farmland relatively inflexible, but producers generally desire additional land to be within a certain distance of the existing farm and they may require specific types of land. The simple model assumes that there is nonfarm wealth available for investment in the farm business, but the amount of equity available to U.S. farmers is often limited to retained earnings and savings from off-farm earnings. This occurs because in the structure of U.S. agriculture most farm equity is supplied by the operators and on-farm equity sources are generally undeveloped. Hence, for some enterprise choices the availability of equity may be a binding constraint. Land prices are not known with
certainty. The unexpected land price boom of the 1970s and the largely unforeseen land price drop of the early 1980s have emphasized land price risk. Plans made in the 1970s based on continued farmland capital gains resulted in financial problems in the 1980s for some producers when the expected land price path was not realized. Hence, it is reasonable to ask whether land price uncertainty affects the impact of farmland capital gains and losses on production and finance plans.

The goal of this study was to extend the previously developed model of farm production and finance decisions to cases in which the supply of land is limited, adjustment costs exist, the sources of equity are constrained to retain earnings, and the land price is uncertain. This research utilizes the modified Vickers model for dynamic conditions specified by Lowenberg-DeBoer. The problems considered are inherently dynamic and no static approximations are defined. The optimization in the presence of land market limits, adjustment costs, and equity constraints is a straightforward extension of the optimal control theory used in the derivation of the basic model. The examination of land price uncertainty utilizes the stochastic optimal control methods outlined by Dreyfus and solves the resulting partial differential equations to derive analytical expressions for the "first best choices" assuming that the decision maker maximizes expected net worth.

Previous Research

Lowenberg-DeBoer reviewed the previous research on the impacts of farmland capital gains on production and finance decisions. He found
that while agricultural economists have suggested many hypotheses about the impacts of farmland price changes at the firm level, rigorous models and empirical tests of the effects were scarce. Research on breeding livestock capital gains does not generalize to the farmland case because livestock capital gains occur primarily because of the tax rules which allow the costs of raising breeding stock to be written off as current expenses, but consider the proceeds from the sale of that livestock to be capital gain. In contrast, farmland capital gains and loss are not a simple product of tax rules. The effects of capital gains and losses at the individual asset level are well-known through the salvage value in the net present value or internal rate of return analysis. The impact of capital gains and losses is commonly included in analysis of pure investment decisions. But the individual asset and pure investment models do not capture important parts of the agricultural decision maker's problem. For example, the financial structure reflected in the weighted average cost of capital discount rate may be sensitive to capital gains and losses. Or the ability to borrow against unrealized capital gains may affect both the supply and the cost of debt capital. In an environment of large farmland capital gains, it may be more profitable to farm a large acreage haphazardly, than a smaller acreage carefully, because the added capital gains with the larger acreage could more than offset the losses from such less intensive management as practices less timely field operations and use of lower quality seed. The usual present value decision framework would not encompass such choices. In addition, the
farm decision maker must often make both investment and management
decisions. The farm decision maker has greater flexibility in reorganiz­
ing production and finance to maximize capital gains or minimize losses
than is usually assumed in analysis of investment in common stock. For
agriculture, the capital gains and losses have not been just a minor part
of the asset purchase decision, as is the case for most nonfarm
industries, but a major planning choice.

The stochastic control method used in this study has not been
frequently used in agricultural economics and has only occasionally been
utilized in other areas of economics. The primary reason for this is the
difficulty of solving the partial differential equation that is derived
in the optimization process. There are many other techniques for dealing
with stochastic problems that have been more widely used. Numerical
techniques have often been used by agricultural economists. Those
methods include: Monte Carlo simulation (see for instance: Held and
Helmers, 1984; Richardson and Condra, 1981; Skees and Reid, 1984),
stochastic linear programming (see for instance: Rae 1971 or Johnson et
al., 1967), quadratic or other programming methods in connection with
specific assumptions about the nonlinear form of the utility function
(see for instance: How and Hazel, 1968), stochastic dynamic programming
(see for instance: Burt, et al. 1980), approximate solutions to
stochastic control problems (see for instance: Rausser and Hochman,
1979, Chapter 5) and approximate solutions to adaptive control problems
(see for instance: Rausser and Hochman, 1979, Chapter 9). The impact
of farmland price variability on farm firm survival was recently examined by Reid and Skees with a Monte Carlo simulation model of Illinois corn and soybean farms. They found that the farmland price risk substantially increased the total risk of failure and that the risk was increased more for larger farms than for smaller farms. Numerical techniques have offered useful insights into the impact of uncertainty in specific situations, but they may be difficult to generalize because in a model of any complexity it is usually impossible to consider all of the possible parameter permutations that may be encountered in other situations than the one being studied.

Ideally, a model should offer both analytical insights and a direct route to empirical tests. In practice, the inherent characteristics of the system being modeled and the ultimate use of the research results play an important part in the choice of technique. While many methods have been used to derive analytical results for dynamic stochastic problems, they tend to fall into two groups: either the objective function is multiplied by a probability density function and the expected value is optimized with deterministic control techniques (see for instance: Rausser and Hochman, 1979) or the equations of motion are specified in terms of stochastic difference or differential equations and stochastic control techniques used (Merton, 1969). Direct empirical application of the first methods generally requires estimation of probability density functions. Because the second method employs dynamic programming arguments, a dynamic programming approximation is probably
the most direct route to empirical application in most cases. Again, the choice of techniques depends on the system studied and research needs.

The Basic Model

The basic model proposed by Lowenberg-DeBoer maximizes the net cash flow from agricultural production plus the capital gain effects over a finite horizon. This study will consider only one planning period; the model can be adapted to multigenerational or multiperiod problems with a dynamic programming arguments (Lowenberg-DeBoer, 1985, p. 74). The capital gain or loss effects may enter the model in three forms: the terminal value of selling the land at the end of the horizon, the substitutability of at least a portion of the unrealized capital gain for current income in the wealth of the individual (wealth approach) and the substitutability of some portion of unrealized gain for invested equity in the debt supply function (income approach). The last two capital gain or loss effects are based on arguments advanced by Plaxico and Klekte (1979). Most of the work in this study is with the pure wealth approach model because it is analytically simpler; all of the results derived for the wealth approach model hold for the income approach model though the mathematical details become more cumbersome. The simplifying assumption that capital gains and losses are treated symmetrically is used in this study. Hence, the proportion of capital gain that is taxable will be the same as the proportion of capital loss that is deductible and no annual limit is imposed on capital loss deductions. For economy of exposition, analysis of the model will be primarily in terms of an environment of
rising land prices. It should be remembered the impact of capital losses is a mirror image of the capital gain effect. The objective function for the pure wealth approach model can be written as:

$$z = \int_0^T e^{-\rho(1-\tau)t} \left\{ \left[ Pf \left( X_t, L_t \right) - \gamma_1 X_t - \gamma_2 L_t \right] - \frac{D_t}{K_t} (1-\tau) - u_{1t} + \beta_t L_t \phi \right\} dt + e^{-\rho(1-\tau)T} [K_T + (1-\phi t) G_T]$$

where:

- $T$ = terminal time,
- $\rho$ = discount rate,
- $\tau$ = average tax rate,
- $P$ = product price,
- $X_t$ = nonland inputs,
- $L_t$ = land,
- $f(\cdot)$ = strictly concave production function,
- $\gamma_i$ = current cost of using an input,
- $D_t$ = debt,
- $K_t$ = invested equity,
- $r(\cdot)$ = strictly convex debt cost function,
- $u_{1t}$ = investment,
- $\phi$ = the proportion of unrealized gain that is substitutable for current income,
- $\beta_t$ = land price,
\( \psi \) = the proportion of capital gain that is taxable or of capital loss that is tax deductible, and

\( G_t \) = accumulated unrealized capital gain.

Only the land price parameter, state variables and the rate of change of state variables are assumed to vary over time; they are designated with \( t \) subscripts. Straightforward extensions of the model could allow other parameters to vary. The land price path is assumed to be smooth with continuous first and second derivatives. The dot notation is used to denote the time derivative of a variable or parameter. It is important to note that the gamma terms are defined following Vickers as the current cost of using the input in production. For items used up in production, such as fertilizer or seed, this is their whole cost. For durable nonland inputs, the gamma parameter includes the repairs, maintenance and depreciation. In the case of land, the gamma parameter is primarily composed of property taxes, insurance and repairs, maintenance and depreciation on real estate improvements.

The constraints on the maximization of the objective function (1) are:

\[
\begin{align*}
(2.1) & \quad \dot{K}_t = u_{1t} \\
(2.2) & \quad \dot{L}_t = u_{2t} \\
(2.3) & \quad \dot{X}_t = u_{3t} \\
(2.4) & \quad \dot{D}_t = \beta_t u_{2t} + \alpha u_{3t} - u_{1t}
\end{align*}
\]
(2.5) \[ \dot{G}_t = \beta_t \dot{L}_t \]

(2.6) \[ u_{2t} \geq 0 \]

where:

\( a \) = the capital absorbed by nonland inputs,

\( u_{2t} \) = land purchases and

\( u_{3t} \) = nonland input purchases.

and the initial conditions on the state variables \( X_t, L_t, D_t, K_t, \) and \( G_t \).

Constraint (23.4) and the initial conditions impose the financial constraint of the Vickers' model that equity plus debt must be equal to the capital absorbed by the inputs. It is derived by differentiating the usual Vickers' constraint with respect to time. It is assumed here that the capital absorbed by land is its price, though a more complex model could examine the effect of special financing arrangements and other factors that would make the capital absorbed different than the price.

The land purchase constraint (2.6) is imposed in an effort to define a relatively simple model. If land sales during the planning period are permitted the model must deal with the questions of the timing of individual sales and the calculation of capital gain on specific tracts, which would require a more complex model. The model may be viewed as covering the career of a family farmer who may sell the farmland at retirement, but does not intend to sell before that, or it may be seen as the situation of a more investment oriented decision maker who wishes to hold the land for a fixed period before realizing the capital gains or
losses. Equation (1) and constraints (2.1-2.6) define an optimal control problem with the controls being: \( u_{1t} \), \( u_{2t} \), and \( u_{3t} \).

**Limited Availability of Land**

The simplest method of analyzing the impact of the limited availability of land is to impose an upper limit on land purchases at anytime:

\[
(27) \quad u_{2t} \leq \zeta
\]

An upper limit on land purchases might be generated in various ways; through legal limits, social pressure or lender concern about rapid expansion. The limit might vary over time depending on the amount of land on the market and other economic conditions, but this study will consider only the simplest case of the constant limit.

The Hamiltonian for the control problem formed by the maximization of equation (1) subject to constraints (2.1-2.7) is unchanged from Lowenberg-DeBoer's basic wealth approach model:

\[
(3) \quad H = e^{-\sigma(1-\tau)t} \left[ \left\{ \frac{Pf(X_t, L_t)}{K_t} \right\} - \gamma_1 X_t - \gamma_2 L_t - \tau(X_t) \right] (1-\tau)
\]

\[
- u_{1t} + \beta_t L_t \dot{\beta}_t + \lambda_2 u_{2t} + \lambda_3 u_{3t} + \lambda_4 (2u_{2t} + a u_{3t} - u_{1t}) + \lambda_5 L_t \dot{\beta}_t
\]

where \( \lambda_t \) are the adjoint variables, \( i = 1, \ldots, 5 \).
Because the control variables enter linearly, the optimal control will be bang-bang and singular:

\[
(4.1) \quad u_{1t} = \begin{cases} 
-\infty & \text{if } -e^{-\rho(1-\tau)t} + \lambda_{1t} - \lambda_{4t} < 0 \\
\text{singular} & \text{if } -e^{-\rho(1-\tau)t} + \lambda_{1t} - \lambda_{4t} = 0 \\
+\infty & \text{if } -e^{-\rho(1-\tau)t} + \lambda_{1t} - \lambda_{4t} > 0 
\end{cases}
\]

\[
(4.2) \quad u_{2t} = \begin{cases} 
0 & \text{if } \lambda_{2t} + \beta \lambda_{4t} < 0 \\
\text{singular} & \text{if } \lambda_{2t} + \beta \lambda_{4t} = 0 \\
\zeta_t & \text{if } \lambda_{2t} + \beta \lambda_{4t} > 0 
\end{cases}
\]

\[
(4.3) \quad u_{3t} = \begin{cases} 
-\infty & \text{if } \lambda_{3t} + \alpha \lambda_{4t} < 0 \\
\text{singular} & \text{if } \lambda_{3t} + \alpha \lambda_{4t} = 0 \\
+\infty & \text{if } \lambda_{3t} + \alpha \lambda_{4t} > 0 
\end{cases}
\]

This control scheme differs from the basic wealth approach solution only in the upper bound on the land purchase control. The adjoint variables are defined by the necessary conditions:

\[
(5.1) \quad -\lambda_{1t} = \frac{\partial H}{\partial K_t} = e^{-\rho(1-\tau)t} \frac{D_t}{2} (1-\tau)
\]

\[
(5.2) \quad -\lambda_{2t} = \frac{\partial H}{\partial L_t} = e^{-\rho(1-\tau)t} \left[ (P_{L_t} - \gamma_2)(1-\tau) + \hat{\beta_t} \phi \right] + \lambda_{5t} \hat{\beta_t}
\]
\[ \lambda_{3t} = \frac{\partial H}{\partial x_t} = e^{\rho(1-\tau)t}(P_f x - \gamma_1)(1-\tau) \]

\[ \lambda_{4t} = \frac{\partial H}{\partial D_t} = -e^{\rho(1-\tau)t}(\tau + \tau')(\frac{D}{K_t})(1-\tau) \]

\[ \lambda_{5t} = \frac{\partial H}{\partial G_t} = 0 \]

and the transversality conditions:

\[ \lambda_{1T} = e^{-\rho(1-\tau)T} \]

\[ \lambda_{2T} = 0 \]

\[ \lambda_{3T} = 0 \]

\[ \lambda_{4T} = 0 \]

\[ \lambda_{5T} = (1-\phi-\psi) e^{-\rho(1-\tau)T} \]

Sufficiency conditions for totally singular optimal control problems are not well-developed, but the conditions (4.1-4.3), (5.1-5.5) and (6.1-6.5) form the most promising set of candidate optimal controls for this problem. The problem satisfies known conditions for optimality including the concavity of the Hamiltonian in the state variables and the
generalized Legendre-Clebsch condition. In addition, the candidate controls are highly plausible. The primary alternative set of controls would entail a "chattering" solution, in which the controls would continuously alternate between the upper and lower boundaries. Investment and purchasing behavior of this type is not observed and there are no apparent economic forces which would generate such behavior. The mathematic details of the sufficiency argument are found in Lowenberg-DeBoer. An introduction to the solution of singular optimal control problems can be found in Bryson and Ho, 1975, Chapter 8.

Equations 5.1-5.5 and 6.1-6.5 can be solved for the optimal adjoint variables:

\begin{align*}
(7.1) \quad \lambda_{1t} &= \int_t^T e^{-\rho (1-\tau)} s \frac{D}{K_s} (1-\tau) ds + e^{-\rho (1-\tau) T} \\
(7.2) \quad \lambda_{2t} &= \int_t^T e^{-\rho (1-\tau)} s \left[p_f \gamma_2 (1-\tau) + \beta_s \phi \right] ds \\
&\quad + e^{-\rho (1-\tau) T} (\beta_T - \beta_c) (1-\psi - \phi T) \\
(7.3) \quad \lambda_{3t} &= \int_t^T e^{-\rho (1-\tau) s} \left[p_f - \gamma_1 \right] (1-\tau) ds \\
(7.4) \quad \lambda_{4t} &= -\int_t^T e^{-\rho (1-\tau) s} \left[\tau + \tau \frac{D}{K_s} (1-\tau) ds \\
(7.5) \quad \lambda_{5t} &= e^{-\rho (1-\tau) T} (1-\psi - \phi T)
\end{align*}
The adjoint variables (7.1-7.5) represent the marginal cost or return from each of the state variables over the remaining horizon. By substituting (7.1) and (7.2) into expression (4.1) it can be seen that if the marginal benefit of an extra unit of investment is less than the marginal cost of debt along a candidate path then investment will instantaneously drop until the marginal benefit of investment is equal to the marginal cost. Likewise, if the marginal benefit of investment along the candidate arc is greater than the marginal cost of debt then investment will instantaneously increase until the singular equality holds.

Financial structure

The financial structure along the singular arc can be characterized by noting that if the singular equality:

\[ -e^{-\varphi(1-\tau)t} + \lambda_{lt} - \lambda_{4t} = 0 \]

holds over time then the time derivative must also be zero. Equations (5.1) and (5.4) can be substituted into that time derivative and the resulting expression simplified to yield:

\[ \rho + r'\left(\frac{\mathcal{E}}{K_t}\right)^2 = r + r'\frac{\mathcal{E}}{K_t} \]

The left hand side of equation (9) is the marginal cost of equity investment and the right hand side is the marginal cost of debt. This condition must hold at every point along the singular arc. Given a
constant discount rate equation (9) could in theory be solved for a constant optimal leverage ratio that is not affected by capital gains or by the limit on land availability. The constant optimal leverage ratio is a result of the effectively unlimited supply of equity capital which allows an optimal ratio of debt and equity to be used in financing every asset level. It is assumed that nonfarm wealth can be pulled into the far business if needed.

The optimal investment and debt acquisition rates can be characterized by differentiating equation (9) twice with respect to time and solving for:

\[ u_{1t} = \frac{K_t}{K_t + D_t} \left[ \beta \cdot u_{2t} + \alpha u_{3t} \right] \]

\[ \dot{D}_t = \frac{D_t}{K_t + D_t} \left[ \beta \cdot u_{2t} + \alpha u_{3t} \right] \]

The term in brackets in (10.1) and (10.2) is the capital required to finance new land and nonland purchases. The changes in optimal debt and equity over time given by (10.1) and (10.2) indicate that the capital required for new purchases will be financed to maintain the optimal leverage. Because the change in equity investment is unconstrained the singular path can be maintained indefinitely. New debt and equity merely respond to the capital requirements of the optimal assets levels. If the land and nonland acquisition controls are continuous, the debt and equity time paths will also be continuous.
The land and nonland input use can be characterized by similar methods. By substituting adjoint variable equations (7.3) and (7.4) into expression (4.3) it can be seen that if the return from the marginal unit of nonland inputs along the candidate path exceeds the current and financial cost then nonland inputs are instantaneously acquired until the singular equality can hold. Similarly, if the return from the marginal unit of nonland input along the candidate path is less than the current costs and financial cost, nonland inputs will be decreased until the singular equality can hold. If the singular equality:

\[(11) \quad \lambda_{3t} + \alpha \lambda_{4t} = 0\]

is to be maintained through time, then the time derivative of (11) must also be equal to zero. By substituting conditions (5.3) and (5.4) into the time derivative of (11) and simplifying, it can be seen that along the singular arc the marginal value product of nonland inputs must be equal to its marginal cost:

\[(12) \quad P_{\lambda X} = \gamma_1 + \alpha (r + \frac{r}{K_t})\]

The marginal cost of the nonland inputs follows the usual Vickers formulation of including both the current cost of using the input and the financial cost, which is the capital absorbed multiplied by the marginal cost of capital.
The optimal control for nonland inputs along the singular path can be found by differentiating the singular equality (11) twice and solving for $u_{3t}$:

\begin{equation}
(12) \quad u_{3t} = -\frac{f_{XL}}{f_{LL}} u_{2t}
\end{equation}

The optimal nonland input purchases are some proportion of the land acquisitions, where the proportion is always positive under model assumptions and depends on the current level of land and nonland input use. Like the debt and equity time paths, the time path of nonland input use will be characterized by an initial adjustment to a level which allows the equality of the returns from the marginal unit of nonland input and its costs and singular adjustments thereafter which allow that equality to be maintained.

The optimal controls for land use are complicated by the existence of the upper and lower bounds. By substituting adjoint variables (7.2) and (7.4) into expression (4.2), it can be seen that if the returns to the marginal unit of land along the candidate path is greater than the marginal current and financial cost of land ownership then land acquisitions will be at the upper bound until the singular equality is reached. Similarly, if the return to the marginal unit of land along the candidate path is less than the marginal current and financial cost of land ownership the land acquisition will be zero until the singular equality
can hold. Along the singular path land use can be characterized by
noting that if the singular equality:

\[(14) \quad \lambda_{2t} + \beta \lambda_{4t} = 0\]

is to hold over time then its time derivative must be zero. Because it
is necessary to frequently refer to the value of the left hand side of
equation (14), it will be denoted as \(Q_t\). By substituting conditions
(5.2) and (5.4) into the time derivative of (14) and simplifying it can
be seen that along the singular arc the marginal cost of land ownership
must be equal to the marginal value product of land:

\[(15) \quad Pf_L(1-\tau) = [\gamma_2 + \beta_t^{*} (1-e^{-\rho(1-\tau)(T-t)})/\rho (1-\tau)](r+r')/K_t^{*} (1-\tau)\]

\[-\beta_t^{*} [\phi + e^{-\rho(1-\tau)(T-t)}(1-\phi-\psi)]\]

The marginal cost of land ownership in (15) is reduced by the capital
gains effects the substitution of unrealized gain for current income
(\(\beta_t^{*} \phi\), the realization of gain:

\[\beta_t^{*} e^{-\rho(1-\tau)(T-t)}(1-\phi-\psi)\]

and the capital cost saving of buying now rather than later:

\[\beta_t^{*} (1-e^{-\rho(1-\tau)(T-t)})/(r+r')D_t^{*} /\rho\]
Because the capital gains tend to offset the cost of land ownership, the optimal marginal value product of land will be lower than it would be if capital gains were ignored, and the farm acreage will be greater.

When both land and nonland inputs are on the singular arc, the optimal relationship between the inputs can be characterized by dividing equation (11) by equation (14) to yield the marginal rate of substitution (MRS) between land and nonland inputs:

\[
\frac{f_X}{f_L} = \frac{\frac{D}{\gamma_1 + \alpha (r + r' \frac{T}{K_e}) (1-\tau)}}{\frac{D}{\gamma_2 + \beta_T - \beta_T (1-e^{-\rho (1-\tau)(T-t)}/\rho (1-\tau))(r + r' \frac{T}{K_e}) (1-\tau)}} - \beta_T \phi e^{-\rho (1-\tau)(T-t) (1-\phi-\psi_t)}
\]

Because the factor cost ratio on the right hand side of (16) is increased by the capital gains term in the denominator which offsets the ownership costs of land, the optimal MRS will be larger than it would otherwise be if capital gains were ignored and the input mix will tend to include more land. This suggests that capital gains provide incentive for extensive land use. Lowenberg-DeBoer shows that enterprise choice in the presence of capital gains tends toward land extensive production. When the land use control path is at one of its bounds, the choice of nonland inputs becomes optimization of the nonland input given a certain acreage. In this case the tendency toward extensive land use is not necessarily maintained.
The optimal control along the singular path can be characterized by differentiating (14) twice with respect to time and solving for the control:

\[(17) \quad u_{2t} = F[\dot{\beta}_tM - \ddot{\beta}_tN]\]

where:

\[F = \frac{f_{XX}}{p(1-\tau)[f_{XX}f_{XX} - f_{XX}^2]}\]

\[M = \left[(r+\tau)^{D_t} \left[1 + e^{-\rho(1-\tau)(T-t)} - e^{-\rho(1-\tau)(T-t)}(1-\phi - \psi\tau)\right]\right](1-\tau)\]

\[N = \phi + e^{-\rho(1-\tau)(T-t)}(1-\phi - \psi\tau) + \frac{(r+\tau)^{D_t} \left[1 + e^{-\rho(1-\tau)(T-t)}\right]}{\rho}\]

The variable \(F\) is always negative under the assumption of strict concavity. The first term in \(M\) reflects the increasing financial cost of land ownership when the land price is rising. The second term in \(M\) and all of \(N\) represent the benefits of capital gains: the substitutability of unrealized capital gains for current income, the realization of gain and the capital cost reduction due to buying now rather than later. Land will be acquired if the capital gains effects outweigh the increasing capital cost. The singular control is defined for purchases between zero and \(\zeta\). The combination of singular and bang-bang controls is determined by expression (4.2) and it may be difficult to identify because the
return to the marginal unit of land \( \lambda_{2t} \) depends on the control path that is being determined. Regardless of the optimal control for land the previously defined debt, equity and nonland input controls hold when the appropriate land purchase control is defined. If land purchases are zero the other controls are also zero and debt, equity and nonland use are constant. If land purchases are at the upper limit, nonland input will change to maintain the optimal land, nonland input relationship and debt and equity changes will allocate the financial cost of increased land and nonland use to maintain the optimal financial structure. If the land control has a jump discontinuity, the other controls will also show jump discontinuities.

**Two special cases**

Though the exact combination of bang-bang and singular controls is difficult to identify for the general case, it is useful to note that the land availability constraint may either dampen or exaggerate the impact of capital gains. This can been seen by examining the characteristics of the land use path for two special cases: the producer who begins the period with a small farm acreage in an economic environment in which a large farm acreage is optimal, and the situation in which land prices are relatively stable during the initial part of the planning horizon, but large capital gains occur during the final portion of the period.

The first case might be seen as the situation of the beginning farmer or of the small farmer in a community where land is tightly held. In the model, large optimal farm size may occur for several reasons; high capital gains, low land ownership costs and production function
characteristics. In the basic model in which land and equity availability are not constrained, the land use path would be generally characterized by an initial adjustment from the small farm acreage up to the optimal acreage and a singular or no acquisition path there. In the basic model, the initial adjustment would be instantaneous, because the rate of land acquisition may approach infinity. The land use path for the model with constrained land availability also is generally composed of an initial adjustment and a singular or no acquisition path thereafter, but the initial adjustment period may occupy a major portion of the planning period. If the constraint is tight relative to the difference between the initial and optimum acreage, the adjustment period may extend through the whole planning period and the full impact of capital gains on farm size and the tendency toward extensive farming operation may not be observed. This may be the case, for instance, in much of Western Europe where relatively little land is bought or sold and land transactions are often subject to strict legal limits.

The examination of the second case will concentrate on the possibility that the land availability limit may encourage "buying ahead" to avoid the constraint and hence exaggerate the impact of capital gains in the form of larger farm size in some periods. To clearly distinguish this situation from the first case, it is assumed that the initial acreage is at the stable land price optimum in the sense that equation (14) holds at the initial farm size. To simplify the analysis, it is assumed that equation (17) holds throughout the planning horizon for the unlimited land availability model (ζ→∞). For most parameter
specifications, this requires the land price to be increasing at an increasing rate in the second part of the planning horizon (Figure 1a). Along the singular path in the basic model $Q_c$ and $\dot{Q}_c$ are zero everywhere reflecting the fact that the farm is earning the maximum net return at each point along the path. When land availability is limited, the large land acquisitions determined by the singular path during the period of the land price increase may no longer be feasible and the optimal control will contain at least one segment of limit level land acquisition (Figure 1b, dashed and dotted lines). One approach to determining the optimal control might suggest following the singular path until the constraint becomes binding and then following the boundary until the singular path can be reattained (Figure 1b, dotted line). This control path is non-optimal because it does not satisfy expression (4.2). Evaluating the return to the marginal unit of land $[\lambda_{2t}]$ along the boundary following control path leads to a control that is at limit level for most of the planning horizon because $\dot{Q}$ is positive throughout the segment of limit level land acquisition and hence $\dot{Q}$ is positive from time zero up to the point at which the singular control path is reattained during the period of rising land prices ($t_\phi$). To satisfy expression (4.2) the control path indicated must be the same as the control path used to evaluate $\lambda_{2t}$. Starting the limit level optimal control later than the point at which the constraint becomes binding in the singular path ($t_5$) will be non-optimal because that would mean delaying acquisition of the optimal acreage and increasing the period of losses from below optimal farm size. Hence, all optimal paths must include some limit level land acquisition
Figure 1. Part (a) shows the assumed land price path increasing at an increasing rate after $t_1$. Part (b) shows the land use path. Throughout the figure the solid line shows the no land availability limit optimal solution, the dotted line shows the boundary following solution and the dashed line shows the most promising candidate for the optimal solution in the limited land availability case. Part (c) shows the return to the marginal unit of land along each land use path. Part (d) shows the return to the marginal unit of land at each point along the land use path. In the no land availability limit case, both $Q_L$ and $Q_C$ follow the $t$ axis.
before land prices start rising (Figure Ib, dashed line). Actual deter-
mination of the optimal path requires specification of parameter values
and functional forms, but the trade-off involved in finding the junction
points between the singular and limit level controls is simple. The
optimal control balances the marginal losses from acreage above the
optimal farm size (between $t_3$ and $t_2$) and the positive marginal returns
from the segment in which acreage is below the optimal level ($t_2$ to $t_4$).
In Figure 1, the points $t_2$, $t_3$ and $t_4$ must satisfy:

$$Q_{t_3} = Q_{t_3} = Q_{t_4} = Q_{t_4} = Q_{t_2} = 0$$

where the land use path is known to be:

$$L_t = L_{t_3} + \tau(t-t_3), \text{ for } t_3 \leq t \leq t_4$$

If the efficiency penalty for farm size above the optimum is small,
"buying ahead" will tend to begin earlier. As the efficiency penalty for
above optimum farm size becomes larger, the beginning point of the limit
level acquisitions will approach the point at which the constraint
affects the singular path. This control path will usually involve jump
discontinuities at the beginning and end points. The presence of
controls with jump discontinuities is within the necessary conditions for
junctions between the singular and bang-bang control paths outlined by
McConnel and Powers and discussed by Lowenberg-DeBoer for the context of
this problem. Hence, it is likely that for at least some land price
paths and parameter values, "buying ahead" is a promising strategy.
Based on the analysis of these two special cases, the overall impact of the land available constraint can be seen to depend on the land price path, the parameter values, and the functional forms. Under some conditions which provide a plausible approximation to observed situations, the land availability constraint will tend to dampen the effect of capital gains by constraining the expansion of farm size. Under other conditions, it is possible for the land availability limit to lead to larger than optimal farm acreage in some periods as decision makers "buy ahead" to mitigate the impact of the constraint.

**Limited availability of land, the income approach**

In the income approach to the value of unrealized gain, the optimal leverage ratio is not independent of land use and capital gains. Hence, the limited availability of land will have an impact on the financial structure for that model. The income approach can be modeled by redefining the argument of the debt cost function to be debt divided by the sum of invested equity and some portion of unrealized gain. This term is the leverage ratio when net worth is calculated based on market values and will be labeled the market value leverage ratio. The portion of unrealized gain recognized in the debt function argument is an approximation of the impact of farmland price increases on conservative market value. The debt cost function can then be written as:

\[
(18) \quad r = r\left(\frac{D_t}{K_t + \omega G_t}\right)
\]
where:

\[ \omega = \text{the proportion of unrealized gain recognized in the financial negotiation.} \]

If (18) is substituted into the Hamiltonian (3) and the optimization process repeated, a financial structure equation similar to (9) can be derived:

\[ (9') \quad \rho - r\left(\frac{D_t}{K_t + \omega G_t}\right)^2 = r + r \left(\frac{D_t}{K_t + \omega G_t}\right) \]

which in theory can be solved for the optimal market value leverage ratio as a function of the discount rate and debt cost function coefficients. It can be shown that (9') implies debt and equity investment paths of:

\[ (19.1) \quad u_{1t} = \frac{K_t + \omega G_t}{D_t + K_t + \omega G_t} \left[ \beta u_{2t} + \alpha u_{3t} - \left(\frac{D_t}{K_t + \omega G_t}\right) \omega \beta L_t \right] \]

\[ (19.2) \quad \dot{D}_t = \frac{D_t}{D_t + K_t + \omega G_t} \left[ \beta u_{2t} + \alpha u_{3t} + \omega \beta \dot{L}_t \right] \]

The negative term in farm acreage in equation (19.1) indicates equity investment is reduced if the level of land ownership is increased. The positive term in farm acreage in equation (19.2) suggests that debt is an increasing part of the financial structure when land use is increased. This is reasonable because it is less costly to meet the collateral
requirement in the financial negotiation with unrealized capital gain than with invested equity. The model suggests that the increasing debt use among U.S. farmers in the 1970s may have been in part a response to the substitutability of unrealized gain for invested equity in the financial negotiation.

If the land availability constraint is binding and farm acreage is less than the optimal level identified in the unconstrained model, then equation (19.2) indicates that a smaller part of the new capital needs will be financed by debt. More invested equity will be needed, because with a smaller farm less unrealized capital gain is accumulated. Conversely, if land use is above the level defined along the singular arc in a "buy ahead" case, a larger portion of new capital needs will be satisfied by debt use because more unrealized capital gain will be available for collateral. Hence, the direction of the impact of the land availability constraint on debt use is the same as it is for land use. For instance, in the case of the small initial acreage and large optimal acreage, the land use is below that which would be expected in the unconstrained case for at least part of the horizon and debt use would also tend to be less. In the "buying ahead" situation, the land availability constraint would tend to expand debt use over a portion of the planning horizon.

The impact of capital gains in the income approach to land use is similar to that identified in the wealth approach case; the ownership costs of land are offset by capital gains effects. The optimal marginal value product of land can be expressed as:
\[ Pf_L (1-\tau) = \gamma_2 + \beta_t \left( 1 - e^{-\rho(1-\tau)(T-t)}/\rho(1-\tau) \right) \left( 1 + \frac{D_t}{K + \omega G} \right) (1-\tau) \]

\[ - \beta_t \int_T^T e^{-\rho(1-\tau)s} \omega_r \left( \frac{D_s}{K + \omega G} \right)^2 (1-\tau) ds \]

\[ + e^{-\rho(1-\tau)(T-t)} (1-\psi_t) \]

in which the proportion of unrealized capital gains substitutability for current income term \( \phi \beta_t \) has been replaced by:

\[ \beta_t \int_T^T e^{-\rho(1-\tau)s} \omega_r \left( \frac{D_s}{K + \omega G} \right)^2 (1-\tau) ds \]

which is the debt cost reduction from the marginal unit of land over the remaining horizon. The larger the proportion of unrealized gain recognized as a permanent addition to net worth \( (\omega) \), the larger debt cost reduction \( (20) \) will tend to be larger and land use will be expanded. The pattern of bang-bang and singular land purchase controls for the income approach will be similar to that identified in the wealth approach case, but the junction points may differ because the value of debt cost reduction due to the marginal unit of land \( (20) \) may differ from the value of substituting unrealized gain for current income.

**Adjustment Costs**

Adjustment cost may be significant in farm size expansion.

Purchasing real estate is not costless; there may be realtor's fees,
search costs and other charges. Expansion demands additional management skill and time. The costs of changing the equipment and facilities to match the larger scale operation might also be considered adjustment cost. It is possible to consider separate adjustment costs for land, nonland inputs and the financial structure, but this study will focus on the adjustment cost of land because in the model farmland capital gains and losses drive changes in all state variables and changes in nonland inputs and financial structure can be seen in response to land change. Hence, the costs of changing nonland inputs and financial structure can be seen as part of the land adjustment cost. In addition, limiting adjustment costs to one input reduces the mathematical complexity. In some situations, land price sensitivity to producer purchases could also be seen as an adjustment cost. For example, in a local land market large purchases by a single decision maker may drive up local price temporarily without affecting the long run land price, which is probably determined in light of long run regional and national investment opportunities that are not affected by local conditions. Hence, the difference between the usual market price and the price paid by the decision maker intent on expansion can be seen as a premium for enlarging his or her farm acreage at a certain time, an adjustment cost.

The adjustment cost can be modeled by subtracting an adjustment cost term under the integral in the objective function (1). The adjustment cost term may take a variety of function forms, but a quadratic form is commonly used because it recognizes that adjustment cost may increase as
the size of the adjustment increases and it maintains the mathematical tractability of the model. The adjustment cost can be written as:

\[ \frac{1}{2}a \left( u_{2t} \right)^2 \]  

where: \( a \) = a constant adjustment cost coefficient. \( a > 0 \)
The one-half is added to simplify subsequent notation. When the derivative is taken, the numerical coefficients will cancel leaving only the term \( (au_{2t}) \). The constant \( a \) is twice the constant in an adjustment cost term defined without the one-half. If the optimization process is repeated assuming that there is no fixed limit on land purchases \( (\zeta \to \infty) \), the optimal control for equity investment and nonland inputs is unchanged from expressions (4.1) and (4.3). Because the Hamiltonian is no longer linear in the land purchase control, an interior solution holds whenever the nonnegativity constraint on acreage change is satisfied. The quadratic adjustment cost allows an explicit solution of the interior land acquisition control. The optimal control for land is no longer singular and becomes:

\[ u_{2t} = \begin{cases} 
\frac{1}{a}\left( \lambda_2 + \lambda_{4t} \right), & \lambda_2 + \lambda_{4t} \geq 0 \\
0, & \lambda_2 + \lambda_{4t} < 0
\end{cases} \]  

The adjoint variables are still described by equations (7.1-7.5). The argument for the optimality of these conditions is similar to that
previously advanced in the totally singular case. Goh (1966) indicates that the necessary conditions for a partially singular model in which the first derivative of the Hamiltonian with respect to the nonsingular controls is not a function of the state variables, include the ordinary Legendre-Clebsch condition for the nonsingular variables and the generalized Legendre-Clebsch condition (pp. 725-726). The ordinary Legendre-Clebsch condition for the land purchase control is:

\[(22) \quad H_{u_2} = -\alpha \leq 0,\]

Hence, the ordinary Legendre-Clebsch condition is satisfied. The generalized Legendre-Clebsch condition is unchanged from the basic model. The argument for sufficiency of these conditions is based on the plausibility of the solution and the lack of economic incentive for the alternative solutions involving chattering.

**Optimal control paths**

Optimal changes in debt, equity and nonland inputs can be characterized by the singular solutions (10.1), (10.2) and (13). The optimal land purchases can be complex for the general land price path, but the important implications of the adjustment cost for this study can be seen by examining the special cases used in the land availability limit analysis. Land purchases can be characterized by noting that if the interior solution:

\[(23.1) \quad u_{2t} = \frac{1}{}\lambda_2t + \beta t \lambda_4t\]
is to hold over time, then its first derivative with respect to time must also hold:

\[
\dot{u}_{2t} = \frac{1}{a} (\lambda_t x_{2t} + \beta_t \lambda_{4t} + \beta_t \lambda_t^4) \\
\dot{u}_{2t} = -\frac{1}{a} PF_L \gamma_2 \left[ e^{-\rho(1-\tau)(T-t)} / \rho(1-\tau) \right] \left[ r + \gamma \frac{D_L}{K_t} \right] (1-\tau) \\
- \frac{\beta}{a} \left[ \phi + e^{-\rho(1-\tau)(T-t)} (1-\phi-\rho) \right]
\]

The optimal change in the rate of land purchase at any time is known if the current farm acreage is known. If land use is above the optimum and land is being acquired, the land purchases will be increased because the right hand side of equation (23.2) is the negative of the net marginal product of land. If land is below the optimum, the rate of change in land purchases will be decreasing. This may seem to be a paradoxical relationships, but the intuitive sense behind this plan can be seen by observing the connection between the current level of net marginal product of land and land acquisitions. If the net marginal product of land is high indicating that land use is below the optimum, acreage are also likely to be large, unless the conditions encouraging large farm size are only temporary. Hence, condition (23.2) indicates that when the net marginal value product of land is large, land acquisition is increasing, but at a decreasing rate. This is logical because as land is purchased, the land use level approaches the optimum and land
acquisitions must decrease. If the net marginal value product of land is negative, land acquisitions will be made only if there is some demand for a larger farm size in the future so that the return to the marginal unit of land over the remaining horizon is positive \( Q>0 \). Hence, land acquisitions are building up to some future larger optimal farm size. With adjustment cost it is likely to be optimal to spread the acquisitions out over a long period to reduce purchases and adjustment cost in each period. As the time the large farm size will be optimal approaches, acquisitions are likely to increase because the present value of the gain from optimal farm size becomes larger. Equation (23.2) suggests precisely this pattern of land purchases increasing rate.

For the beginning farmer problem, the adjustment cost model suggests and land acquisition path that begins with large land purchases that decrease as the optimal farm size is approached in contrast to the linear land acquisition path imposed by the constant land availability limit. If the difference between the initial farm size and the optimal size is large, then farm size is likely to be below the optimum for some time and the return to the marginal unit of land over the remaining horizon is likely to be large at time zero. This suggests a large initial level of land purchases. This is reasonable because initial land purchases will produce income over the whole planning period and, hence, a larger adjustment cost can be justified in that case. As in the limited land availability model, the adjustment cost will tend to obscure the impact of capital gain on farm size. Farm acreage will be below the optimum for a substantial part of the planning horizon and if adjustment costs are
large or the difference between the initial and optimum acreage is large the optimum farm size may never be reached.

As in the land availability limit case, the second special case shows the "buying ahead" phenomenon. This case assumes that the land price is stable in the initial part of the planning horizon, but large capital gains occur in the later part of the horizon. The singular control path of the basic model showed no land acquisitions until land prices start rising and then large purchases are made. The large purchases are unlikely to be optimal with adjustment costs so there will be incentive to spread buying over a larger period. If it is assumed that the producer starts with a farm size at the stable price optimum at time zero, condition (23.2) suggests that the optimal acquisition strategy demands positive land purchases at time zero. If the plan of waiting until land prices start to rise to begin buying land is used the net marginal value product of land will be positive at the point when land prices begin to rise and capital gains start to offset other ownership costs. This implies that the rate of land purchases should decrease at that point. But the rate of purchase up to that point is zero so no purchases would ever be made. Condition (23.2) does not allow "buying ahead" to start anywhere in the period between time zero and the land price rise. In this period, if no initial purchases are made, the net marginal value product of land is zero because initial farm size is the stable price optimum, implying that the rate of change of land purchases is zero. If land purchases are zero and the rate of change of land purchases are zero then purchasing can not begin. If an initial
land purchase is made, the net marginal value product of land will become negative as the stable price optimal farm size is exceeded. Using condition (23.2), this implies land purchases increasing at an increasing rate, building up farm acreage to take advantage of the later capital gains without incurring large adjustment costs. The crucial choice in this strategy is the initial purchase. The initial purchase and condition (23.2) imply the land purchase strategy. The initial purchases must be chosen so that equation (23.1) is satisfied when the return to the marginal unit of land over the remaining horizon is evaluated along that strategy. This problem can be expressed as a second order differential equation with boundary value conditions. The problem can be solved, but does not add analytical insights.

An adjustment cost problem could also be constructed for the income approach model. The results would be similar to those in the land availability limits case. Debt usage would be restricted in part of the beginning farmer problem and exaggerated in part of the "buying ahead" problem.

**Equity Investment Limits**

Retained earnings is the major source of investment equity on U.S. farms. The model can be manipulated to limit investment to retained earnings by defining a new control variable for operator withdrawals from current earning. Let this control be noted $u_{4t}$. The portion of unrealized gain that is a substitute for current income must be treated as a withdrawal to avoid double counting the unrealized gain in meeting the
financial constraint. Unrealized gain is deducted from the capital requirement in the financial constraint (Lowenberg-DeBoer, 1985, p. 51); if the fraction of unrealized that is a substitute for current income could be saved and become part of invested equity \( K_e \) then that part of unrealized gain would be subtracted twice. The unrealized capital gain could be saved and added to equity, and included in the accumulated unrealized capital gain term \( G_e \). With the control variable for operation or withdrawal from current earnings equity investment can then be expressed as current income minus the withdrawal variables. Hence, constraint (2.1) becomes for the wealth approach case:

\[
\dot{K}_e = u_{1t} = \left[ PF(X_t, L_t) - \gamma_1 X_t - \gamma_2 L_t - r \left( \frac{D}{K_t} \right) \left( 1 - \tau \right) \right] - u_{4t}
\]

The objective function in this model is the maximization of the discounted value of withdrawals:

\[
\text{(1') } \max Z = \int_0^T e^{-\rho(1-\tau)t} \left[ u_{4t} + \phi \dot{\phi} L_t \right] \, dt + e^{-\rho(1-\tau)T} \left( u_T + (1 - \phi - \psi) G_T \right)
\]

It is assumed that the entire terminal value is withdrawn from the business. If the control variable for withdrawals is limited to positive values, equity investment is limited to current income. This formulation is a rearrangement of equations (1) and (2.1) which allows the equity investment to be constrained to retained earnings by limiting the range of a control variable within exogenous bounds. Most of the body of
Theorem on singular control deals with controls limited in this way. The alternative would be to place an inequality constraint on the basic model limiting equity investment to less than or equal to the current income. Singular control problems in the presence of such constraints have not been widely studied. Savings from off-farm earnings could be included in the model, by adding an off-farm income term to the current income expression. In a simple model, this term could be a function of time to reflect earning trends over time.

The relationship of this model to previous formulations can be seen by substituting constraint (2.1') into the basic objective function (1). The Hamiltonian becomes:

\[
H = e^{-\rho(1-\tau)t} \left[ u_{4t} + \phi \frac{\partial}{\partial \tau} \int_{t}^{T} \right] + \lambda_{1t} \left[ Pf\left(X_t, L_t\right) - \gamma_1 X_t - \gamma_2 L_t - \tau \left( \frac{D}{K_t} \right) \right] (1-\tau) \\
- \lambda_{2t} u_{2t} + \lambda_{3t} u_{3t} \\
+ \lambda_{4t} \left( au_{3t} + 2u_{2t} - Pf\left(X_t, L_t\right) - 2X_t - \gamma^2 L_t - \tau \left( \frac{D}{K_t} \right) \right] (1-\tau) \\
+ u_{4t} + \lambda_{5t} L_t \frac{\partial}{\partial \tau} 
\]

The adjoint variables are defined by equations similar to (5.1-5.5):

\[
\frac{\partial H}{\partial K_t} = -\lambda_{1t} = (\lambda_{1t} - \lambda_{4t}) \left( \frac{D}{K_t} \right)^2 (1-\tau) \\
\frac{\partial H}{\partial L_t} = -\lambda_{2t} = (\lambda_{1t} - \lambda_{4t}) \left[ Pf\left( X_t, L_t \right) - \gamma_2 \right] (1-\tau) + e^{-\rho(1-\tau)t} \frac{\partial \phi + \lambda_{5t}}{\partial \tau} 
\]
and the transversality conditions (6.1-6.5). The optimal control is singular and bang-bang defined by (4.2), (4.3) and:

\[
(4.4) \quad u_{4t} = \begin{cases} 
0 & \text{if } e^{-\rho(1-\tau)t} - \lambda_{1t} + \lambda_{4t} < 0 \\
\text{singular if } e^{-\rho(1-\tau)t} - \lambda_{1t} + \lambda_{4t} = 0 \\
\infty & \text{if } e^{-\rho(1-\tau)t} - \lambda_{1t} + \lambda_{4t} > 0
\end{cases}
\]

The sufficiency arguments are the same as for previous models. The Hamiltonian can be shown to be concave in the state variables. The form of the Hessian is the same as for the basic model with the exponential discounting term \(e^{-\rho(1-\tau)t}\) replaced by the adjoint variable term \((\lambda_{1t} - \lambda_{4t})\), which is shown below to take an exponential form in the optimal solution. Along the singular arc, the generalized Legendre-Clebsch condition is the same as for the basic model.

**Alternative discounting term**

The optimal pattern of withdrawals can be characterized by solving equations (5.1') and (5.4') simultaneously and substituting the resulting
expressions into (4.4). The sum \(-\lambda_{1t} + \lambda_{4t}\) in (4.4) can be seen as an alternate discounting term that captures the impact of equity shortages. Expression (4.4) indicates that if the absolute value of the alternative discounting term is greater than the value of the standard discounting term \(e^{-\rho(1-\tau)t}\) then all current income should be saved. Solving the first order differential equation system (5.1') and (5.4') for the alternative discounting term can be accomplished by subtracting (5.4') from (5.1') yielding:

\[
(24) - (\dot{\lambda}_{1t} - \dot{\lambda}_{4t}) = (\lambda_{1t} - \lambda_{4t})(r'(\frac{D}{K_t})^2 + (r + r'(\frac{D}{K_t}))(1-\tau)
\]

and using the transformation:

\[
\begin{align*}
\dot{Y}_t &= \lambda_{1t} - \lambda_{4t} \\
\ddot{Y}_t &= \dot{\lambda}_{1t} - \dot{\lambda}_{4t}
\end{align*}
\]

Hence, the system of differential equations can be expressed as a single first order equation in \(Y\) with the solution:

\[
(25) \quad \dot{Y}_t = e^{-\rho(1-\tau)T}\int_T^T \eta_t \, dr
\]

where:

\[
\eta_t = (r'(\frac{D}{K_t})^2 + (r + r'(\frac{D}{K_t}))(1-\tau)
\]
where the constant has been evaluated using the transversality conditions. The term \( Y_t \) reflects the return to the marginal unit of equity over the remaining horizon. By substituting \( Y_t \) into (4.4) it can be seen that along the singular arc:

\[
(26.1) \quad e^{-\rho(1-t)t} - e^{-\rho(1-t)T} \int_t^T h_r dr = 0
\]

which implies:

\[
(26.2) \quad \rho(1-t)t - \rho(1-t)T + \int_t^T h_r dr = 0
\]

If equation (26.2) is to hold over time, its first derivative with respect to time must also be zero. But the first derivative of equation (26.2) is the financial structure equation (9) developed in the case of unlimited access to equity. Hence, along the totally singular arc the previously defined optimal capital structure and controls hold and the alternative discounting term \( Y_t \) reduces to the simple discounting term because:

\[
-\rho(1-t)T + \int_t^T h_r dr = -\rho(1-t)T + \int_t^T \rho(1-t) dr
\]

\[
= -\rho(1-t)T + \rho(1-t)T - \rho(1-t)t = -\rho(1-t)t
\]
The adjoint variables needed to evaluate expressions (4.2-4.4) can be found by substituting \( Y_t \) into equations (5.1'-5.5') and integrating, using the transversality conditions to evaluate the constants:

\[
(7.1') \quad \lambda_{1t} = \int_t^T e^{-\rho(1-\tau)T} [r'\left(\frac{DS}{K_s}\right)(1-\tau)] ds + e^{-\rho(1-\tau)T} \\
+ \int_t^T e^{-\rho(1-\tau)(1-\frac{\phi_s^*}{\phi_s})} ds \\
+ e^{-\rho(1-\tau)(1-\phi-\psi)} \\
(7.2') \quad \lambda_{2t} = \int_t^T e^{-\rho(1-\tau)T} [\left(P_L^f - \gamma_2\right)(1-\tau)] ds \\
+ \int_t^T e^{-\rho(1-\tau)T} \left(\frac{\beta_t - \theta_t}{\phi_s}\right)(1-\phi-\psi) \\
(7.3') \quad \lambda_{3t} = \int_t^T e^{-\rho(1-\tau)T} \left[(P_L^f - \gamma_1)(1-\tau)\right] ds \\
(7.4') \quad \lambda_{4t} = -\int_t^T e^{-\rho(1-\tau)T} \left[r + r'\left(\frac{DS}{K_s}\right)\right](1-\tau) ds \\
(7.5') \quad \lambda_{5t} = e^{-\rho(1-\tau)T}(1-\phi-\psi) \\
\]

From equation (7.1') and (7.4') it can be seen that an alternate expression for \( Y_t \) is the present value of the marginal cost of debt and the marginal return to equity over the remaining horizon, discounted at a rate that is less than or equal to the discounting in the freely variable.
equity case. When equity is a binding constraint at some point in the planning horizon, the input and financial variables must be determined simultaneously. Equity will be constraining only as production or land ownership returns create a demand for expansion. The problem is essentially to decide whether current withdrawals or savings, with the promise of higher withdrawals in the future, will maximize wealth.

**Control paths in two special cases**

In general, the exact combination of singular and bang-bang controls depends on the functional forms and the parameter values. It is, however, useful to re-examine the two special cases of the beginning farmer and the established farmer faced with a stable land price early in the planning period, but sharply rising land price later. If the beginning farmer starts with no land or debt and a pool of equity far less than would be needed to finance the optimum farm size identified in the perfectly variable equity case, then the initial adjustment would entail acquiring land, nonland inputs and debt until the singular equalities in expressions (4.2) and (4.3) hold when evaluated with the adjoint variables (7.1'-7.5'). Withdrawals would be zero and all current production income would be reinvested if the leverage is higher than the optimal level for the perfectly variable equity case at any point along the candidate path without offsetting low leverage points. This is because when leverage is higher than the variable equity optimum, the term \( h_t \) is larger than the discount rate, the absolute value of the exponent in \( \gamma_t \) is smaller than the after-tax discount rate, \( \gamma_t \) is larger than the discounting term \( e^{-p(1-\tau)t} \) and decision rule (4.4) indicates
zero withdrawals. In nonmathematical terms, this means that the returns to equity in the future are sufficiently large that retaining current earnings increases wealth.

Offsetting low leverage points can occur in several situations. One is the "saving ahead" phenomena, analogous to the "buying ahead" identified in previous models. "Saving ahead" is discussed below for the established farmer special case. In the beginning farmer case low leverage ratios may occur if conditions are such that expansion is slowed or temporarily stopped at some point in the horizon, thereby reducing total capital demand and debt use at those points. This would reduce the present value of the marginal capital cost and signal a smaller incentive to current savings. For instance, this could happen if there is a period of farmland price declines in this general period of capital gains. Expansion just prior to and during the decline could be slowed, but savings may continue so that the firm is in a position to expand when capital gains resume. Thus, equity may be relatively plentiful in this period.

That leverage will be higher than the variable equity optimum early in the horizon of the beginning farmer problems can be seen by examining the singular land purchase equation:

\[
\int_T^e \rho(1-t)T^3 \int_s^T \frac{DR}{s} \left[ \left( \frac{P_L - \gamma_L}{2} \right) - \beta_L \frac{r + r' \left( \frac{D_s}{K_s} \right)}{1-t} \right] (1-t) ds
\]

\[
+ \int_T^e \rho(1-\tau) \phi \beta D_s ds + e^{-\rho(1-\tau)T} \left( \beta_L \phi - \phi \right) (1-\phi-\psi) = 0
\]
In the initial instantaneous adjustment, the equity can be treated as fixed; the adjustment can be thought of as taking no time, so there is no time to generate current production income to be saved. If the leverage ratio is set at the variable equity optimum given the relatively small and fixed amount of equity, the amount of land that can be financed will be small and the marginal product of land higher than the variable equity optimum. The marginal value product of land minus current and financial costs would be positive until the variable equity optimum farm size could be acquired and the singular equality would not hold. By taking on a debt load beyond that specified by the unconstrained model, more land could be acquired, increasing financial cost and lowering the marginal value product of land. In this way, the singular equality could be achieved.

If the decision maker starts out with some land and debt, it is possible that the debt cost may be high enough so that the left hand side of equation (27) is negative and no land acquisitions are made until the debt load is reduced. Given that the beginning farm size is far below the variable equity optimum and the marginal productivity of land is relatively high, this would still indicate a leverage ratio higher than those indicated in the unconstrained problem over at least part of the horizon.

If the land price path was such that equation (14) held throughout the horizon in the unconstrained equity case, reinvestment of all production income would continue until the variable equity optimum leverage ratio could be achieved or the end of the planning horizon. Retaining
earnings is the wealth maximizing decision because the marginal dollar invested when the farm size is below the optimum and the leverage ratio above the variable equity optimum earns a return larger than the discount rate, thereby increasing the net present value. Because there are no liquidity loss in the terminal value, limit level saving can occur right up to the end of the planning horizon. As farm size grows the leverage ratio needed to maintain the equality (27) would drop, with the leverage ratio approaching the variable equity optimum as the farm size approached the optimum. Explicit singular control variables for land and nonland inputs can be derived by solving simultaneously the second derivatives with respect to time of the singular land and nonland input equations and the change in debt constraint (2.4). These controls contain terms similar to control equations (13) and (17) with adjustments for limited equity which reduce acquisitions, plus terms depending on equity investment. Because leverage is changing along the singular path these control equations are much more complicated than (13) and (17). They are not derived here because they do not add substantially to the analytic power of the model.

Farmland capital gains have an effect on the farm size and leverage ratio path needed to achieve equality (27). All other things equal, the larger the capital gain the higher the leverage ratio and the larger the farm size needed to establish the equality. Hence, when equity investment is limited to retained earnings for the beginning farmer problem, farm size is smaller than it would be in the unconstrained case, but the effects of capital gains are still present. Farm size is still larger
than it would be if the capital gains were ignored in the decision process. When equity is constrained, capital gains increase the wealth maximizing leverage, even for the wealth approach model. Similar results could be derived for the income approach case, with leverage being increased both by the substitutability of unrealized capital gain for net worth and by the equity constraint.

In the second problem, if prices are stable in the early part of the period and a farmer begins the planning process with farm size, nonland input use and leverage at the stable price, variable equity optimum, then both "saving ahead" and "buying ahead" will tend to occur. In the unconstrained problem, equity investment would not start until land prices started to rise in the later part of the planning horizon. If the rate at which equity investment is demanded in the unconstrained model exceeds current income, and the land price path is as shown in Figure 1a, there will be a segment of limit level retained earnings. As in the case of the land availability constraint, the problem is in deciding where that limit level segment begins. If expression (4.4) is evaluated along the equity investment path suggested by the unconstrained problem, a logical contradiction occurs. The candidate path shows no equity investment until late in the planning horizon, but expression (4.4) shows limit level investment at the initial time because equity will be a binding constraint at some point and the debt use indicated by the simultaneous solution of the singular land and nonland input equation and the financial constraint will be above the variable equity optimum, with the result that $Y^*_t$ is greater than the discounting term $e^{-\delta (1-\tau) t}$. It is
not reasonable to start the limit level investment later than the time at which it occurs in the unconstrained model, because this would increase the losses from suboptimal farm size and high leverage. By beginning limit level savings before the land price started to rise, the decision maker could find a feasible control path that would balance the losses from larger than usual equity levels in the period before the land price rise with gains from a leverage level closer to the optimum after land price starts to increase. Losses from equity levels above the variable equity optimum would occur because as equity is increased, the marginal cost of debt drops and in the decision rules for land and nonland input use this means lower financial cost and greater input use. But the extra inputs beyond the variable equity optimum earn returns below the cost of equity and hence the objective function value will be reduced because the present value of returns will not exceed the present value of immediate withdrawal. The interval between the beginning of limit level savings and of the land price rise would depend primarily on how sharp the land price rise is and on the size of current production income. If the land price rise is modest or the production income is close to the equity demand, then the interval will be small. If the capital gains are very large, so that the demand for equity is large, or if the production income is small, the interval may be large. In some cases, limit level saving may start at the beginning of the planning horizon.

The "buying ahead" of farmland in this model occurs at least partially because there are no other investment opportunities within the decision framework. Even if the return on this alternative investment
were lower than the discount rate, so that it would not be a relevant option in the unconstrained problem, it may increase returns to store equity needed in the future in the nonfarm investments, compared to investments in farmland and nonland inputs beyond the optimal levels. This example should not be taken to indicate that "buying ahead" is a common result of anticipated equity shortages by farmers. The dampening effect on land purchase by equity constraints suggested in the beginning farmer problem is a more likely scenario. The buying ahead in the second problem does indicate that constraining equity does not always dampen the effects of capital gains.

Land Price Risk

The future path of land prices is not known with certainty and this lack of certain knowledge may have major consequences for the decision maker. If farm decisions are based on the expected level and rate of change of land prices and the expected path is not realized, then farm size, financial structure and enterprise choice decision may be non-optimal. In some cases, the nonoptimal choice may be disastrous. For instance, if a producer expects large land price increases and chooses a large farm size with a heavy debt load, but capital losses are realized, then he or she may be forced out of agricultural production when lenders observe the large debt compared to the reduced value of assets and decide that continued lending to this business is too risky. This part of the study sought to model and identify important ways in which land price risk affects the input levels, financial structure and enterprise choice
of the expected wealth maximizing producer. This study concentrates on the risk neutral case because it is mathematically more tractable and because risk neutral results are often used as a benchmark for further analysis. It was assumed that the state variables at the initial time are known and the paths of parameters other than the land price is deterministic.

The use of expected wealth as the appropriate objective function is based on von Neumann-Morgenstern utility theory (Hey, pp. 27-37) and requires the assumption that the lack of certainty about land prices is risk in the classic sense, that is probabilities can be attached to future land prices. Obviously, the exact probability distributions of future land prices are not known, but it seems reasonable that a decision maker could take objective information on past land price realizations and current economic trends, plus subjective judgments, and form some probability estimates.

Land purchases by the decision maker are unlikely to contribute to the reliability of the land price path estimates, hence an adaptive control formulation is not required. If the area were isolated from larger markets for durable assets and if the producer had potential market power in that area's farmland market, it might be possible to learn more about the extent of that market power by varying land purchase, but this seems to be an implausible scenario for U.S. conditions. For similar reasons, the usual control theory formulation in which the objective function includes a probability function as a state variable or as a function of state variables is inappropriate. For most
U.S. producers, the probability density of farmland prices is not controllable and does not depend on their input and financial decisions individually.

For some versions of the model, however, the evolution of the state variables is affected by land price path realizations because the equations of motion (2.4 and 2.5) include stochastic land price terms. Hence, the stochastic control formulation which utilizes the stochastic calculus of Ito and dynamic programming arguments is most appropriate. A more detailed explanation of this methodology may be found in Dreyfus (1965, pp. 214-219), Kamien and Schwartz (1981, Section 21) or Arnold, (1973, pp. 220-224). This formulation derives the optimal feedback control in which new information is taken into account at every point in the planning horizon. Hence, the focus of the analysis must be on the rules for the "first best choice" of input levels and financial structure. The time paths of land, nonland inputs, debt and equity cannot be meaningfully derived because the decisions will depend on land price realizations at every point.

Stochastic land price path

The stochastic control formulation depends on the assumption that the change in the land price path over time can be modeled as the sum of a deterministic term which may depend on time the current land price realization, and a white noise (or random walk) term:

\[
(28.1) \quad \dot{\beta}_t = \theta(t, \beta_t) + \sigma \xi_t
\]
where:

\( \theta(\cdot) \) = any real function of time and the current land price,

\( \xi \) = a white noise process and

\( \sigma \) = a constant which is related to the variance of the land price.

The white noise process is defined to be normal and independently distributed with a zero mean at every point. There is no known physical process which can exactly be called white noise because the assumption of independence demands extremely irregular behavior and because the assumption of constant spectral density requires an infinite variance. The white noise process is a useful mathematical concept because it can be linked to ordinary calculus methods by the stochastic calculus of Ito (Arnold, Chapters 4 and 5). Using the theory of generalized functions white noise can be shown to be the first derivative of a Weiner process (Arnold, 1973, p. 68):

\[
(29) \quad \dot{W}_t = \xi_t \quad \text{or} \quad W_t = \int_0^t \xi_s \, ds
\]

where: \( W \) = a Weiner process.

A Weiner process is assumed to be such that \( \{W(t_i) - W(t_j)\} \) are normally and independently distributed with zero mean and variance \( (t_i - t_j) \) for all \( t_i > t_j \). A Weiner process is continuous, but nowhere differentiable in the usual sense of the word. This leads to the differential representation of the land price path:

\[
(28.2) \quad dB_t = \theta_t \, dt + \sigma \, dW
\]
or the integral representation:

\[(28.3) \quad \beta_t = \beta_0 + \int_0^t \theta_s \, ds + \int_0^t \sigma dW \]

The expected value of the land price level is then:

\[(29.1) \quad E[\beta_t] = \beta_0 + \int_0^t \theta_s \, ds \]

and the land price change expected value is:

\[(29.2) \quad E[\delta_t] = \theta_t \]

**Impact of land price risk, wealth approach**

The wealth approach problem for the case of stochastic land price can be represented by substituting the land price path (28.1) into equation (1) and taking the expected value:

\[(1') \quad Z = E\left[ e^{-(1-\tau)t} \left\{ \pi_t + L_t(\theta_t + \xi) - u_{lt} \right\} \right] \]

where:

\[\pi_t = \left[ pf(X_t, L_t) - \gamma_1 X_t - \gamma_2 L_t - r \left( \frac{D_t}{K_t} \right) D_t \right] (1-\tau)\]

The form of constraints (2.1-2.4) and (2.6) is unchanged. The debt acquisition constraint (2.4) contains the land price level, which is
stochastic, but the constraint is not a function of the change in land price. Hence, representation as a stochastic differential equation is not required. The accumulated unrealized gain constraint becomes a stochastic differential equation:

\[(2.5') \quad \frac{dG_t}{dt} = L_t \theta_t d\xi_t + \sigma L_t dW\]

when the land price change (28.1) and the Weiner process representation (28.2) are substituted into (2.5)

Because of the stochastic land price, the usual control theory techniques can not be used in the maximization of equation (1'). The optimization can, however, be carried out by converting the problem to a discrete time dynamic programming problem and allowing the increments of time to become very small. The optimal value function is:

\[(30.1) \quad J(I, t) = \text{the maximum expected discounted value of current income and capital gains and losses over the remaining horizon from time } t \text{ to } T, \text{ with the vector of initial values } I = (K_t, L_t, X_t, D_t, G_t) \text{ and assuming an optimal feedback policy is followed.}\]

The recurrence relation with current income and capital gains and losses discounted to time zero is:

\[(30.2) \quad J(I, t) = \max E\left\{ e^{-\rho (1-T)t} \left[ \pi_t + L_t (\theta_t + \xi_t) - u_{lt} \right] \Delta t + J(I + \Delta I, t + \Delta t) \right\} \]

where: \( \Delta t = \) a small increment of time.
The maximization in recurrence relation (30.2) is subject to constraints (2.1-2.4), (2.5') and (2.6). The boundary condition is:

$$J(I, T) = e^{-\rho(1-t)T} [K_T + G_T (1-\phi-\psi T)]$$

The final term on the right hand side of the recurrence relation can be evaluated by expanding it in a Taylor series and substituting in the control constraints:

$$J(I+\Delta I, t+\Delta t) \equiv J(I, t) + J_\Delta t + J_\Delta I + h.o.t.$$

$$= J(I, t) + J_\Delta t + J_K_1 t + J_L 2t + J_K u_3t$$

$$+ J_D (b u_2t + au_3t - u_1 t) \Delta t$$

$$+ J_G (\theta L \Delta + J_L G \Delta W)$$

$$+ \frac{1}{2} J_{GG} \sigma^2 L_2 \Delta W^2 + h.o.t.$$

where:

- $\Delta W$ = a small increment in the Weiner process,
- $\Delta I$ = a small increment in each element of the vector $I$ and
- $h.o.t$ = higher order terms.

Expression (30.4) is a stochastic differential equation and it may be simplified using Ito's Theorem (Arnold, 1973, pp. 101-105). In
particular, this theorem shows that $\Delta W^2 = \Delta t$. This can be intuitively understood by noting that the Weiner process is defined so that:

$$E[\Delta W] = E[W_{t2} - W_{t1}] = 0$$

and

$$E[\Delta W^2] = E[(W_{t2} - W_{t1})^2] = t_2 - t_1 = \Delta t.$$ 

By substituting expression (30.4) into (30.2), taking the expectation, subtracting $J(I, t)$ from both sides, dividing by $t$ and allowing $\Delta t$ to go to zero, the recurrence relation can be characterized by the deterministic differential equation:

\begin{equation}
0 = \max u_{1t}, u_{2t}, u_{3t} \{e^{-p(1-t)}t[\pi_t + L_t \theta_t - u_{1t}] + J_t
\end{equation}

+ \int K_t u_{1t} + \int L_t u_{2t} + \int X_t u_{3t} +

+ \int D_t \{E[\theta_t]u_{2t} + \alpha u_{3t} - u_{1t}\}

+ \int D_t \theta_t + \frac{1}{2} \int G_t \sigma_t^2 L_t^2

The expected value of the land price in (30.5) is defined by (19.1). The expectation of the product $[J_D \theta_t u_{2t}]$ equals the product of the expected land price multiplied by the land purchase and the derivative of the recurrence relation with respect to debt because the land purchase, like
the other controls, is deterministic and because the recurrence relation is already in expected value terms.

The conditions for an interior maximum are:

31.1) \[ -e^{-\rho(1-\tau)t} + J_K - J_D = 0 \]

31.2) \[ J_L + E(\beta_j)J_D = 0 \]

31.3) \[ J_X + \alpha J_D = 0 \]

Based on the arguments present for the deterministic model with freely variable equity the control path for equity and nonland inputs will be characterized by an initial adjustment followed by a singular arc which maintains the interior maximum conditions (31.1) and (31.3). Hence, equations (31.1) and (31.3) will hold everywhere except possibly at the first instant while the initial adjustment occurs. If there is unlimited land availability the land purchase path after the first instant will be characterized by equation (31.2) or by \( u_{2t} = 0 \). Hence, along the maximizing path after the first instant the terms containing control variables will vanish and the recurrence relation can be characterized by:

30.6) \[ -J_t = e^{-\rho(1-\tau)t}[\pi_t + L_t \theta_L \phi] + J_G L_t \theta_L + \frac{1}{2} u_{2t} \sigma^2 L_t^2 \]
If the initial adjustment occurs instantaneously and is thought of as taking no time, equation (30.6) will hold over the entire remaining planning horizon after t, though it may not hold as \( t_1 \) approaches t from below. Because the stochastic process is assumed to be continuous, planned discontinuous adjustments in the control variables will occur only in the initial adjustment at time zero. The "first best choice" at time zero will be the initial adjustment that allows conditions (31.1, 31.3 and 31.2 or \( u_2c=0 \)) to hold over the remaining horizon. After time zero, the first best choice will be the control that maintains those conditions in the next instant. As in the case of deterministic, singular control, the argument for optimality of this solution is based on satisfaction of known necessary conditions and the plausibility of the solution.

**Solving the differential equation**

Analysis of the first best choice decision rules implied by conditions (31.1-31.3) requires an explicit solution of the partial differential equation (30.6). Expression (30.6) takes the form of a non-homogeneous diffusion equation and analytic solutions to this type of problem are well known. The approach here will be to guess a solution to the problem based on solutions constructed formally for a more limited set of conditions. It will be verified that the solution satisfies equation (30.6) and the conditions under which the solution is bounded will be specified. Proof of the uniqueness of this solution is desirable, but is not attempted in this study. The focus will be on the applicability of the decision rules generated.
It can be shown (Cannon, 1984, p. 338) that the solution to the problem:

\[ dR = R_{GG} \, dt + h(G, t) \, dt \]

\[ R(G, 0) = g(G) \]

is of the form:

\[ R(G, t) = \int_{-\infty}^{\infty} K(G-s, t)g(s)\, ds + \int_{0}^{t} \int_{-\infty}^{\infty} K(G-s, t-v)h(s, v)\, ds\, dv \]

where:

\[ K(G-s, t) = -\frac{(G-s)^2}{4t} \frac{1}{(4\pi t)^2} \cdot \frac{1}{\text{e}^{(G-s)^2}} \]

\( s \) is a variable of integration.

In constructing this solution the functions \( F \) and \( g \) are assumed to be bounded. The equation (30.6) can be put in the form of problem (32) by employing the following transformations (Cannon, pp. 15-16):

33.1) \( G^*_t = G_t + \int_{t}^{T} \theta_s \, ds \)

33.2) \( t^* = T-t \)

33.3) \( t^{**} = -\int_{0}^{t} \int_{0}^{T} \frac{1}{2} \sigma^2 \, L_s \, ds = \int_{0}^{T} \frac{1}{2} \sigma^2 \, L_s \, ds \)
It is also useful to express the inverse of the relationship defined in equation (33.3):

\[ t^* = A(t^{**}) \text{ or } t = T - A(t^{**}) \]

The transformation (33.1) eliminates the term in (30.6) which involves the first derivative of the recurrence relation with respect to the accumulated capital gain \( J_0 \). The transformation (33.2) reverses the time index so that the terminal value (30.3) becomes the initial value. Transformation (33.3) incorporates the time varying coefficient of the second derivative term into the time index variable.

The recurrence relation is expressed as a function of the original levels of the state variables. In order for solution (32) to be of that form, the deterministic state variables must be expressed as a function of their original level and of time. The stochastic accumulated capital gain variable will be expressed in this way because of transformation (33.1). Assuming that the appropriate singular controls can be defined as functions of time, land, nonland inputs, debt and equity can be expressed as the sum of their original level and the integral over the dynamic constraints (2.1-2.4). For example, the equity variable becomes:

\[ K_t = K_0 + \int_0^t U^{*}_t \lambda^v dv \]

where: \( U^{*}_{1t} \) = the optimal equity investment control
If the transformed function is denoted \( \tilde{J} \) then its first derivative with respect to time is:

\[
\tilde{J}_t = J_t + U_{1t}^* J_K + U_{2t}^* J_L + U_{3t}^* J_X + J_D(\theta_{2t}^* U_{2t}^* + \alpha U_{3t}^* - U_{1t}^*) - J_{GL}^t \theta_t
\]

where: \( U_{2t}, U_{3t} \) = optimal land and nonland input controls which can be simplified to

\[
\tilde{J}_t = J_t + e^{-\rho(1-\tau)t} U_{1t} - J_{GL}^t \theta_t
\]

using the interior maximum conditions (31.1-31.3). Thus, the problem (30.6) can be written:

\[
(30.6') \quad \tilde{J}^{**} = \tilde{J} (G^*, t^{**}) + e^{-\rho(1-\tau)[T-A(t^{**})]}[\pi_{T-A(t^{**})} + L_{T-A(t^{**})} T_A(t^{**}) - U_{1, T_A(t^{**})}]^T - 2 L_{T-A(t^{**})}^2
\]

where: \( \tilde{J} = \tilde{J} (G^*, t^{**}) \)

with initial value:

\[
(30.3') \quad \tilde{J}(G^*, 0) = e^{-\rho(1-\tau)T} [K_T + G^* (1-\phi-\psi T)]
\]

Even with the transformation, however, the partial differential equation problem defined by equations (30.6') and (30.3') does not
exactly fit the form of problem (32), which requires the functions f and g to be bounded. As the accumulated capital gain goes to infinity, the initial value (30.3') becomes unbounded so that this condition can not be met. But it is often true in solving partial differential equations that the solutions apply to a broader range of case than the formal conditions necessary for constructing those solutions. Hence, the guess is made that the solution to the problem defined by (30.6') and (30.3') is of the same form as the solution to problem (32). In terms of the original variables, the candidate solution is:

\[
J(I, t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(G+T_s \theta_s - S)^2}{2\sigma_s^2}} ds \\
\times \exp\left[-(1-\tau)T_s \left[ \psi_T + S(1-\phi-\psi_T) \right] ds \right] \\
+ \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(G+T_s \theta_s - S)^2}{2\sigma_s^2}} ds \\
\times \exp\left[-(1-\tau)T_s \left[ \psi_T + S(1-\phi-\psi_T) \right] ds \right] \\
\times \exp\left[-(1-\tau)T_s \left[ \psi_T + S(1-\phi-\psi_T) \right] ds \right]
\]

where: s, S = variables of integration.

The solution is the expected value of current income and capital gain over the remaining horizon, given a normal distribution of accumulated unrealized capital gains with a mean at any time of:

\[
\mu_v = G_t + \int_T^\infty L_s \theta_s ds, \quad v \geq +
\]
The form of the distribution function is a result of the assumption of normal white noise in the land price change. The mean of the distribution is simply the expected value of capital gains given a specified land ownership path. Because of the time varying coefficient of the second derivative term in equation (30.6), the variance takes an integral form and is the sum over time of the total variance in capital gain for all land owned. In the simple case in which land ownership is constant and the decision maker stands at point zero, the variance takes the form:

\[
\frac{1}{2} \int_{0}^{t} \sigma^2 \mathcal{L}^2 ds = \frac{1}{2} \sigma^2 \mathcal{L}^2 t
\]

In solving the partial differential equation problem, the state variables other than \( G \) can be treated as parameters of the \( J \) function, because they are controllable and their derivatives do not appear in (30.6). The form of the solution for the wealth approach can be simplified by noting that the state variables other than \( G \) are not affected by the integration over \( S \). Hence, the terms in those state variables can be brought through the integral. It is also true that:

\[
\int_{-\infty}^{\infty} \frac{1}{\left(2\pi\int_{0}^{T} \sigma L^2 ds\right)^{\frac{1}{2}}} e^{-\left(\frac{1}{2} \left(\mathcal{G} + \int_{0}^{T} \mathcal{L} \theta ds - S\right)^2}{2\int_{0}^{T} \sigma^2 \mathcal{L}^2 ds} ds = 1
\]
because the integrant of (35) is a normal density function. Hence, the solution can be written:

\[
(33.1) \quad J(t) = e^{-\rho (1-\tau)T} [K_T + (1-\phi-\psi)] \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \frac{1}{(2\sigma L_s ds)^T} e^{-(C+\int_{t}^{T} L_s \theta_s ds - s)^2/2} \int_{t}^{T} L_s^2 ds \, ds \]

\[
+ \int_{t}^{T} \frac{\rho (1-\tau)\nu}{[\pi_1 + L_s^2 \phi - U_{1s}^*]} \, ds \]

\[
= e^{-\rho (1-\tau)T} [K_T + E(G_T)(1-\phi-\psi)]
\int_{t}^{T} \frac{\rho (1-\tau)\nu}{[\pi_1 + L_s^2 \phi - U_{1s}^*]} \, ds
\]

Along the optimal path, the recurrence relation is simply the expected net present value of current return and capital gain. The land purchases, acquisition of nonland inputs, equity investment and changes in debt used are determined by decision rules (31.1-31.3).

The solution can be verified by differentiating the candidate solution and forming the equation (30.6). In taking the derivatives, it should be remembered that the state variables are initial levels of assets and can be represented in the form:

\[
K_t = K_0 + \int_0^t u_{1s} ds
\]
The initial level of the asset does not vary with time and hence taking the derivative of the recurrence relation with respect to the state variables can be treated as taking the derivative under the integral with respect to a parameter. By substituting the derivatives of the candidate solution into equation (30.6) and using the interior maximum conditions (31.1-31.3) to simplify the following identity is formed:

\[
- \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma^2} e^{-\left[\frac{(u-S)^2}{2\sigma^2}\right]} \left[ \frac{1}{2\pi \sigma^2} \right] \left[ K_T + S(1-\psi_T) \right] \frac{\partial}{\partial \pi_T} \left[ \rho (1-T)T \right] \left[ K_T + S(1-\psi_T) \right] \left[ K_T + S(1-\psi_T) \right] \left[ K_T + S(1-\psi_T) \right] ds
\]

\[
\frac{1}{2\sigma^2} \left[ \rho \left( \frac{\partial}{\partial \pi_T} \right) \left[ K_T + S(1-\psi_T) \right] \right] ds
\]

\[
+ e^{-\rho (1-T)} T_t \left[ \pi_t + L_t \theta \right]
\]

\[
e^{-\rho (1-T)} T_t \left[ \pi_t + L_t \theta \right]
\]

\[
+ \left[ e^{-\rho (1-T)} T_t \left[ K_T + S(1-\psi_T) \right] \right] ds
\]

\[
+ \frac{1}{2} \left[ e^{-\rho (1-T)} T_t \left[ \pi_t + L_t \theta \right] \right] ds
\]

\[
+ \frac{1}{2} \left[ e^{-\rho (1-T)} T_t \left[ \pi_t + L_t \theta \right] \right] ds
\]
The solution (33) will be bounded even though the terminal value term is unbounded as the accumulated unrealized capital gain approaches infinity because the exponential term in $S$ will dominate the linear term in $S$ and drive the integrand to zero as $S$ goes to $+\infty$ or $-\infty$.

**Decision rules examined**

The decision rules can be better understood by taking the derivative of the recurrence relation with respect to the state variables and substituting those derivatives into equations (31.1-31.3). The decision rules for the financial structure and the nonland inputs take the same form as in the deterministic case. The derivatives of the recurrence relation with respect to equity, debt and nonland inputs are identical to the adjoint variables previously defined (7.1) (7.3) and (7.4). At first glance, the derivative with respect to land differs from its deterministic counterpart because of the way land affects the expected value of unrealized gain. Land occurs in both the mean and variance terms of the accumulated unrealized gain density function. The change in the expected value of accumulated unrealized gain with respect to land is:

\[
\frac{\partial \mathbb{E}(G^T)}{\partial L_o} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{G+\int_{t_s}^{T_t} \theta_s ds - S}{\int_{t_s}^{T_t} \sigma_s^2 ds}\right)^2}{2\int_{t_s}^{T_t} \sigma_s^2 ds}} \left\{ \int_{t_s}^{T_t} \sigma_s^2 ds \right\}^{-2}\left\{ [(G+\int_{t_s}^{T_t} \theta_s ds - S)^2 - \int_{t_s}^{T_t} \sigma_s^2 ds] \left\{ \int_{t_s}^{T_t} \sigma_s^2 ds \right\} \right\} \left\{ \int_{t_s}^{T_t} \sigma_s^2 ds \right\} ds
\]
By expanding the mean minus the random variable term, the derivative can be written as a sum of the moments and products of moments of $G_t$:

\[ \frac{\partial E(G_t)}{\partial L_0} = \frac{1}{\sigma^2} \left[ \left( \mu^3 - 2\mu(\sigma^2 + \mu^2) + (\mu^3 + 3\mu\sigma^2 - \mu\sigma^2) \right) \right. \\
\left. \left. \int_0^T \sigma^2 L_s ds \right] - \left[ \mu^2 - (\sigma^2 + \mu^2) \right] \int_0^T \theta_s ds \right] \]

where:

\[
\mu = G + \int_0^T L_s \theta_s ds
\]

\[
\sigma^2 = \int_0^T \sigma^2 L_s ds .
\]

In deriving equation (36.2) it is useful to note that the third moment of a nominal distribution can be expressed as $3\mu^2 + \mu^3$. Equation (36.2) can be simplified to:

\[ \frac{\partial E(G_t)}{\partial L_0} = \int_0^T \theta_s ds = E(\theta_t) - E(\delta_t) \]

Hence, the decision rule for land purchases is of the same form as the control developed in the deterministic case, with the land price path terms replaced by their expected values. This certainty equivalence result occurs because the land price and capital gain terms enter the model linearly. Land price risk has a more complex effect in the income approach model because the accumulated unrealized capital gain term occurs in the nonlinear debt cost function.
Impact of land price risk, income approach

Up to the simplification of the recurrence relation in equation (33.1) the development of the income approach model is identical to the wealth approach model except that the current income term \( \pi_t \) is replaced with:

\[
\pi_* = [PF(X_{1t}, L_t) - \gamma_1 X_t - \gamma_2 L_t - r\left(\frac{D}{K + \omega G_t}\right) D_t] (1-\tau)
\]

Because \( G \) occurs in the current income term of the income approach model, the simplification includes an expected debt cost term:

\[
(33.1') \quad J(I, t) = e^{-\rho(1-\tau)T} F[K_T + (1-\phi-\psi) T] \int_0^\infty \frac{1}{2\pi\sigma^2} e^{-\frac{(u-s)^2}{2\sigma^2}} \int_0^\infty e^{-\frac{(u-s)^2}{2\sigma^2}} \left[ PF(X_v, L_v) - \gamma_1 X_v - \gamma_2 L_v \right] e^{r\left(\frac{D_v}{K_v + \omega S_v}\right) D_v} (1-\tau) dS + L_v \theta_v \phi - U_{1v} \] dv

\[
= e^{-\rho(1-\tau)T} F[K_T + (1-\phi-\psi) E(G_T)]
\]

\[
+ \int_0^T e^{-\rho(1-\tau)v} \left[ PF(X_v, L_v) - \gamma_1 X_v - \gamma_2 L_v \right] e^{r\left(\frac{D_v}{K_v + \omega S_v}\right) D_v} (1-\tau) dS + L_v \theta_v \phi - U_{1v} \] dv

\[
- E\left[r\left(\frac{D_v}{K_v + \omega G_v}\right) D_v\right] (1-\tau) + L_v \theta_v \phi - U_{1v} \] dv
That this is indeed a solution can be verified by differentiating and substituting into equation (30.6). The solution will be bounded if the debt cost function is bounded everywhere except perhaps as the leverage ratio goes to negative infinity and if the exponential term in the normal density function dominates the debt cost function as capital losses become infinite. The boundedness of the solution revolves around the behavior of debt cost as net worth becomes small or negative. Usually, the credit available to a decision maker will be limited and it is likely that for some value of the leverage ratio no more credit will be available at any interest rate. Certainly, for negative leverage ratios, commercial credit sources will dry up and even family members or friends will be reluctant to lend. Taken literally, this means that the cost of debt is infinite beyond the credit cut-off point. But this literal interpretation ignores the financial and legal responses open to the borrower and to the lender. The first response to leverage position beyond the cut-off point is likely to be debt and asset restructuring. Assets would be sold to retire debt and reestablish a viable financial structure. This liquidation may be voluntary or forced through foreclosure. The cost of the restructuring to the borrower is the liquidity loss incurred due to rapid sale of assets. The liquidity loss may be large, but it will be finite. If the liquidity losses are very large or if the asset prices have dropped substantially, the decision maker may not be able to reestablish a viable financial structure and is forced out of business. In that case another cost is incurred, the loss of future income from the firm. Again, this may be large, but it will be
finite in most cases. If some debt remains after the liquidation of assets, the decision maker has the legal option of bankruptcy which limits the liability for debt cost as the net worth becomes very small or negative. Without bankruptcy, the remaining debt would continue to generate interest costs which could never be repaid because there is no income after firm liquidation. The present value of this cost of remaining debt would tend toward infinity because the debt cost would be high in a negative net worth situation and because the interest charges would accumulate over an infinite horizon. Without bankruptcy or similar legal procedure, the decision maker always remains liable for the debt. With the availability of the bankruptcy option the cost of the remaining debt after liquidation is also finite. Hence, the cost of having a debt load such that the leverage ratio is greater than the cut-off point is likely to be large, but finite under the existing financial and legal system, because all the costs incurred in such a situation are likely to be finite.

If the bankruptcy option did not exist, the optimal financial structure would be all equity when land price are risky and the leverage is measured on a market value basis. When the debt cost function becomes infinite at some finite leverage ratio, the expected cost of debt and the objective function (1') go to negative infinity for any positive level of debt. This occurs because if a leverage ratio beyond the cut-off point occurs with some positive probability, no matter how small, the expected debt cost become infinitely negative and cancels out all past and future income. The marginal cost of debt would then be the discount rate and
the certainty equivalence results for land and nonland inputs derived above would hold.

If the debt cost function is bounded everywhere except perhaps as capital losses approach infinity, and in the case that it is not bounded for infinite capital losses, it is dominated by the exponential term of the normal density function, then the decision rules (31.1-31.3) become relevant. Taking the derivative of (33.1') with respect to the debt and equity and substituting into rule (31.1) the financial rule becomes:

\[
(31.1') - e^{-p(1-\tau)t} + \int_t^T e^{-p(1-\tau)v}E[r'(\frac{D}{K_v+G_v})^2]dv + e^{-p(1-\tau)T} \\
+ \int_t^T e^{-p(1-\tau)v}E[r+r'(\frac{D}{K_v+G_v})^2]dv = 0
\]

Expression (31.1') indicates that along the optimal path the discounted marginal cost of investment at time t is equal to the expected present value of the discounted marginal cost of debt over the remaining horizon. It should be noted that the expected values in expression (31.1') would be defined even if the first derivative of the debt cost function become unbounded at a finite number of points. For proof of the integrability of a function with a finite number of discontinuities, see Ferrar, p. 70. An example of a point at which the debt cost function may become unbounded is the credit cut-off value; the debt cost function may become vertical at this point before leveling off at some high finite value over the remaining domain.
The optimal leverage ratio at the initial point can be found by noting the equality:

\[ e^{\rho(1-\tau)t} = \int_0^T e^{\rho(1-\tau)v} \rho(1-\tau)dv + e^{\rho(1-\tau)T} \]

and substituting expression (37) into (31.1), yielding:

\[ 31.1''') \int_T^t e^{\rho(1-\tau)}\{E[r'(\frac{D_v}{K_v+\omega G_v})^2] + E[r+r'\frac{D_v}{K_v+\omega G_v}] - \rho\}(1-\tau)dv = 0 \]

Equation (31.1'') is satisfied if the integrand is always equal to zero along the optimal arc:

\[ 9''') \ E[r'(\frac{D_t}{K_t+\omega G_t})^2] + E[r+r'\frac{D_t}{K_t+\omega G_t}] - \rho = 0 \]

At time \( t \), the amount of unrealized capital gain at that moment is known with certainty, so equation (9''') is identical to equation (9) at time \( t \). The first best choice of financial structure is the same as under the deterministic decision rule, but the financial plan on which this first best choice is based differs substantially from the deterministic paths of equity and debt. The financial plan referred to here is not a set of optimal control paths. Such paths can not be meaningfully derived in an optimal feedback problem. Rather the financial plan is the set of expected control paths based on expected land price paths. These expected paths are important in calculating the first best choice, but
random events may force the actual control path to deviate from the expected path.

Because of the nonlinear debt cost function, the expected marginal cost of debt and the expected marginal cost of equity for a constant leverage ratio will rise rapidly as time passes. To maintain the equality \(9'''\), the financial plan must specify constantly decreasing leverage. In the basic deterministic case, the leverage ratio was constant. In most cases in which the planning horizon is relatively long, the decision maker must effectively plan to achieve an all equity financial structure in the future because the variability of the expected land price path in the distant future is likely to become so large that equation \(9'''\) will hold only for minute amounts of debt or there will be no positive leverage ratio that satisfies it.

In equation \(9'''\), the leverage ratio is a random variable. The distribution of this random variable at any point will depend on the expected value of capital gain \(\theta\), the land price variance parameter \(\sigma\), the fraction of the unrealized capital gain that is recognized as a substitute for invested equity \(\omega\) and on the farm size, because the land area appears in the variance term \(34.2\). All other things equal the larger the expected capital gain, the smaller the expected marginal cost of debt, because with large values of unrealized capital gains for the same variance the leverage ratio is less likely to stray into the high leverage ratio, high debt cost area. With larger values of the land price variance parameter or farm size, the marginal cost of debt will tend to be higher because the leverage ratio will have a higher
probability of taking the value in the high debt cost area. The relationship of the expected debt cost and the leverage ratio distribution is illustrated in Figure 2. With a larger fraction of the unrealized capital gain recognized as a substitute for invested equity, more of the land price variability is transferred to the debt cost and the expected marginal cost of debt will be higher. Because the expected values in equation (9'') depend on the farm size, the leverage ratio for planning purposes beyond the initial choice must be determined simultaneously with planned land and nonland inputs.

If equity is perfectly variable, the financial plan on which first best choice of financial structure is based is not important. In the variable equity case, any financial structure adjustment required by future realizations can be met. The financial plan is important only if there is some reason to believe the plan infeasible. For instance, if equity investment is limited to retained earnings there may be many cases in which not enough equity is available to satisfy equation (9'''') and the land and nonland input decisions everywhere even if they can be satisfied initially. If equity investment is at a positive limit level somewhere during the planning horizon, the conditions under which the differential equation (30.3') was developed no longer hold. The differential equation was solved assuming that interior maximum decision rules (31.1-31.3) held everywhere except perhaps as \( t_1 \) approaches \( t_0 \) from below. The new problem would have to be solved in segments defined by where the controls were at limit levels and where the interior solution holds. An added transformation would be needed in the limit level.
Figure 2. The figure shows the market value leverage ratio on the horizontal axis, and the debt cost and the probability of leverage ratio values on the vertical axis. The solid line is the debt cost as a function of the leverage ratio. It is the convex debt cost function used throughout this research. The dotted and dashed lines show leverage ratio distributions. The tight distribution illustrated by the dashed line would have a relatively low expected cost of debt because the chances of a high leverage ratio realization are small. With the fat tailed distribution illustrated by the dotted line, the expected debt cost would be higher because there is a substantial probability of high debt cost realizations beyond \( \frac{D_t}{K_t + \omega G_t} \).
segments because the first order terms in the limit level control would not disappear. Solutions are possible, but complicated because the combination of limit level and interior solution control depend on function forms and parameter values. General, analytical solutions can not be easily derived and other numerical techniques, such as dynamic programming, appear to offer better empirical tests. Hence, analytical solutions to the constrained control case will not be derived here. From the analytical framework developed thus far it can, however, be seen that reduction in the initial choice of farm size may be an important consequence of equity constraint in this model. If land can be sold only at the end of the planning horizon, equity available is less than the interior solution level at some points, and the expected marginal cost of debt becomes very high at the debt level needed to finance the interior solution level of land and nonland inputs at those points, then the optimal initial choice of farm size may be less than that specified by the interior solution. With less land, less debt will be needed in the future, expected debt cost will be reduced and greater expected returns may be earned.

The decision rules for land and nonland inputs take the form of expression (4.2) and (4.3) with adjoint variables replaced by the derivatives of the recurrence relation along the optimal path. The recurrence relation derivatives differ from the deterministic adjoint variables only in that the marginal cost of capital is in expected value form. The singular path is defined by expressions like (11) and (14) with the expected marginal debt cost substituting for its deterministic counter.
part. As in the deterministic case the planned path of the inputs will be characterized by an initial adjustment followed by a singular path for nonland inputs and a singular or zero purchase path for land. The planned path of input use and financial structure is important for the first best choice because of the constraint on land sales. If land were perfectly variable during the planning horizon, the initial adjustment would always be such that the decision rule equalities (31.2-31.3) would hold over the remaining horizon and the first choice in the stochastic case would be the same as in the deterministic problem. This occurs because at the initial point in the financial structure equation (9''') is identical to the deterministic financial structure equation (9) and the initial point of the singular path is identical to that defined by land and nonland input equations (11) and (14). But if land sales are disallowed, the initial land purchase may not be such that the singular land purchase path (14) holds. It may be optimal in some cases to make no initial land purchases to avoid a farm size larger than optimal in the future. In that case, the expected cost of debt and planned land and nonland input use matter, but these variables must be determined simultaneously and hence require more information about functional forms and parameter values. If the land price path and other information is such that the initial adjustment allows the singular land purchase path to hold, then the first best choice in the risky land price case is identical to the deterministic path.

Arguments for the sufficiency of the optimality conditions for the risky land price case are similar to those advanced for the deterministic
Because the recurrence relation (33) is concave in the initial levels of the state variables, decision rules (31.1-31.3) describe a maximum. The control variables which maintain the state variables at the maximizing level are obvious candidates for the optimal control. Chattering controls which alternate between the control limits are unappealing for practical and empirical reasons; such behavior does not appear to serve any economic purpose and is not observed. As in the deterministic case, such arguments do not prove the optimality of the conditions, but they do suggest that they are good candidates for a maximizing solution.

Limitations of the model

The models presented in this study overcome several of the limitations of the simple model developed by Lowenberg-DeBoer. The limited availability of land, adjustment cost, equity investment constraints and risky land prices are incorporated here, through the interaction of the constraints and risk effect was not examined. The model, however, still has many of the limitations previously identified (Lowenberg-DeBoer, 1985, p. 93). A fixed discount rate is used through it can be argued that the discount rate might vary with firm size and risk. The focus of the model is on the microeconomic impact of land price changes on buyers. Macroeconomic impacts and the supply side of the land market are not taken into account.

Though land price risk has been incorporated, the model is still only a very rough approximation of decision making in a risky environment. Only land price risk has been considered, but the producer faces
many other risks some of which are correlated with land price risk. For instance, commonly used land price models suggest that input and output price risk is related to land price risk because the land price is seen as a function of returns to land and if input price is higher or output price lower, the returns to land will be reduced and hence land price will decline. This problem can be dealt with in the framework developed here. If more than one equation of motion was stochastic, the differential equation (30.6) would be a higher dimension diffusion equation. Analytic solutions to these higher dimension problems are possible (Widder, 1975, Chapter 12) and may offer some useful insights. Another limitation of the model is the assumption of normal distributed land prices which is used to make the problem mathematically tractable. This limitation can not be overcome with currently available mathematical techniques in the framework used here. Nonnormal distribution can be used in numerical dynamic programming solutions. In addition, the model does not include the cash flow problem or the difference between long and short term debt, that may be especially important in making decisions in a risky environment.

Summary and conclusions

This research considered the impact of limited land availability, adjustment costs, equity investment constraint and land price risk on the farm decision making in an environment of farmland capital gains. A dynamic model farm firm is used which allows acreage expansion and change over time in nonland inputs, debt and equity. Optimal control theory techniques are used to maximize the present value of returns from farm
investments. The limited land availability condition was imposed on the model by placing an upper limit on the land acquisition control variable. Adjustment costs were considered by subtracting a quadratic function of the land purchase control from the income stream; as more land is purchased the quadratic adjustment cost term is larger, dampening the incentive for rapid changes. Equity investment was limited to retained earnings by rewriting the problem so that the objective function is maximization of withdrawals and then limiting the withdrawal control variable to nonnegative values. Land price risk was considered by assuming the land price change is the sum of a deterministic trend term and a normal white noise process. In this situation, the equation of motion for the accumulation of capital gains and losses is a stochastic differential equation. The problem of maximizing expected wealth given the stochastic accumulation of capital gains and losses is solved using the stochastic calculus of Ito and dynamic programming arguments.

These extensions of the basic model developed by Lowenberg-DeBoer suggest that the impact of land price changes identified in the earlier work can persist under a wide variety of conditions, though the additional constraints may dampen or exaggerate the effects. The earlier work indicated that capital gains create incentives to enlarge farm size, increase debt use, reduce use of nonland inputs, and choose land extensive enterprises. Capital losses have the opposite effect.

Because the combination of singular and bang-bang controls in the constraint problems depends on the function forms and parameter values, that part of the study focused on an analysis of two special cases: the
beginning or small farmer problem and the case of an established farmer faced with a land price path that is stable for the early part of the planning horizon and rises sharply later. In the limited land availability, adjustment cost and equity constraint problems, the impact of capital gains on farm size, and debt use is dampened for the beginning farmer case. If the initial farm size is below the optimal acreage, the land purchase limitation can create an initial period of limit level expansion. The simple model used in previous research assumed the farm size could be instantaneously adjusted upward to the optimal level. During the period of limit level expansion, the full effect of capital gains may not be apparent, because enough land can not be acquired to achieve the optimal level of the sample model. Some capital gain effects may still be seen though. For instance, if the initial farm size were such that it was optimal for a problem without recognition of capital gains and large capital gains occurred in the planning period, then a model which did not recognize capital gains would show no acreage expansion, but the model developed here would show a period of limit level expansion. The expansion in the model with the land purchase constraint may not be as great as in the simple model, but it will be greater than in a model which does not recognize the capital gains impact.

In the beginning farmer case, adjustment costs may reduce the initial land purchases. It becomes too expensive to expand rapidly, so acquisition are spread over a longer period. During the acquisition period, farm acreage is less than it would be in the simple model.
Capital gains will affect the speed of adjustment. With larger capital gains it will be profitable to expand more rapidly, because the additional capital gains offset the higher adjustment cost.

As in the simple model, nonland input use, debt levels and equity investment in the limited land availability and adjustment cost models follow the land purchase pattern. With limited land availability nonland input use will tend to be larger. Debt increase and equity investment will be lower if borrowing against unrealized capital gains is not permitted. When accumulated unrealized capital gain can not be substituted for equity, retained earnings will be increased and debt use decreased compared to the simple model because less unrealized capital gain will be available to replace equity investment.

When the initial acreage is small relative to the optimum size, limiting equity investment to retained earnings reduces the expansion that can be financed and provides incentive to incur a higher debt load. The higher debt use can be justified because with the smaller than optimal farm size the marginal product of inputs is larger than it would be in the simple model, so a higher marginal debt cost can be tolerated before marginal cost equals marginal revenue. The incentive to use larger amounts of land relative to nonland inputs persists in the limited equity investment case because capital gain still offsets the cost of using land in production.

In the established farm case, the land availability constraint adjustment cost and equity limits exaggerate the capital gains impact by
incentives for producers to "buy ahead", thereby enlarging farm size during the stable price period. In the limited land availability case, the prospect of future capital gains makes it profitable to buy land beyond the optimal farm size when it is available early in the planning horizon, rather than to lose potential capital gains because land cannot be acquired fast enough later on. Adjustment costs create incentives to start acquiring land earlier and spread out the purchases over a longer period. In both the limited land availability and adjustment cost cases, the problem is to find the optimal trade-off between lower current income from too large a farm size in the early part of the planning horizon and gains from being closer to the optimum size after the capital gains period starts.

Both saving ahead and buying ahead can occur in the limited equity case. If the lack of equity is a binding constraint in the higher capital gain period at the end of the planning horizon, then it may be profitable to increase retained earnings early in the planning horizon to reduce later equity shortages. The added equity in the initial stable land price period can be used to reduce debt and increase land and nonland input use.

The beginning or small farmer problem probably describes situations of many farmers in the U.S. and elsewhere. Hence, the impact of capital gains is probably less than would be expected from the unconstrained version. The importance of the second special case is not as a description of farm conditions. The ability to forecast sharp land price rises in the distant future is unreliable enough that relatively few decision
makers ever face such a problem. The value of the second problem is as a counter example that shows that the constraints do not always dampen the capital gain effects.

Analysis of the optimality conditions for the stochastic land price model shows that in the wealth approach case for the expected wealth maximizing producer the deterministic decision rules hold. Because the land price and unrealized capital gain enter the wealth approach model linearally, the deterministic rules evaluated at the expected land price and unrealized gain amounts given the first best choice. Hence, for the wealth approach the impacts of land price changes identified in the deterministic model persist. In the income approach model, risky land price have a more complex effect. If equity is freely variable the first best choice of financial structure is identical to the deterministic case, but if equity investment is a binding constraint at some point in the planning horizon land and debt use may be reduced. This occurs because the first best choice of financial structure is based on a financial plan that requires decreasing leverage as time passes. If available equity is inadequate to decrease leverage fast enough, land and debt use are likely to be reduced. The decreasing leverage financial plan is a response to the fact that land price risk tends to increase when planning for periods in the distant future.

Forecasting land prices 5 or 10 years from the initial date is simply less reliable than forecasting land prices a month or a year from the initial date. With a higher land price variance the probability increases that land prices will fall enough below the expected value to
drive the market value leverage ratio to dangerously high levels. In practical terms, extremely high leverage ratios can mean that further credit is cut-off and foreclosure may result when financial obligations can not be met. Effectively, the cost of debt becomes very high; it may mean the loss of the farm and future farm earnings.

The income approach model suggests that the traditional desire of farmers to be debt free may have been part of a wealth maximizing strategy to cope with land price risk. The model indicates that the plan to reduce or eliminate debt use is an important part of the financial structure decision regardless of whether or not a review of debt use over a farmer's career shows progress toward that goal. The actual series of first best choices of debt use may differ from the financial plan on which those choices are made. For instance, if the model were solved at annual intervals over a 40-year horizon with freely variable equity, the series of first best choices would show debt use at the level identified in the deterministic model, though that choice is based on a financial plan which requires decreasing debt use. In a practical decision making sense, the impact of the decreasing debt use plan is likely to be an increased emphasis on financial flexibility; the producer plans in the ability to reduce debt use instead of planning for continuing current debt use.
REFERENCES


The recent financial problems of U.S. farmers have accentuated the need to understand the impact of capital gains and losses on farm production and finance decisions. Were the farmers who expanded their operations and incurred large debt loads in the farmland capital gains period of the 1970s speculators or were they rational entrepreneurs reorganizing their businesses to take advantage of profit opportunities? Are the farmers experiencing financial stress merely individuals who gambled on further land price increases and lost? Or did they respond to incentives created by lending practices and government policy which increased expected returns, but also made them more vulnerable to cash flow crises, collateral risk and other financial problems? Can the asset restructuring of the 1970s toward greater investments in land and less in liquid forms (Boehlje and Eidman, 1983, p. 937) be explained by the large capital gains of the period?

Many economists have suggested a link between farmland price changes and production and finance decisions. Farmland capital gains have been connected to incentives to increase farm size (Lins and Duncan, 1980), slowing growth of land productivity (Castle and Hoch, 1983) and increasing debt use (Lins and Duncan, 1980). Lowenberg-DeBoer (1985a, b) has developed a theoretical model of the farm firm which incorporates the impact of capital gains and losses on production and finance decisions. The analytical results from this model suggest that if enough equity is
available to achieve the optimal capital structure and if the firm may acquire as much land as desired then:

1) land ownership tends to be greater when capital gains are larger,

2) the use of nonland inputs tends to be reduced when capital gains rise, and

3) debt use tends to be greater when capital gains are larger.

Capital losses have the opposite effect. Lowenberg-DeBoer (1985b) shows that under plausible circumstances the capital gains effects may be dampened or disappear entirely if the solution is constrained by the availability of equity or land.

Casual observation suggests that land and equity may be binding constraints for many farmers. Because most U.S. farms are relatively small, closely held businesses it is difficult for them to attract outside equity investment. Hence, their equity supply is often limited to family savings and retained earnings. Only about three percent of U.S. farmland changes hands each year and in most areas the amount of land for sale that will fit into a given farmer's business is only a fraction of the land put on the market. A cash grain farmer may not be interested in buying rough pasture land that comes up for sale nearby. Similarly, a farmer specializing in a beef cow calf operation may not be in the market for tillable land. Travel time and costs usually circumscribe the area in which farmland purchases are considered.

Therefore, the impact of capital gains and losses on farm production and finance decisions is an empirical issue. Were the parameter values
in the U.S. in the 1970s and early 1980s such that the capital gains of
that period could have led to the expansion of farm acreage and debt use,
and the reduction in the use of nonland inputs? If the capital gain
effects are observed in the model under the parameter values of the 1970s
and 1980s, then the view of the expanding, debt incurring farmer making
rational, economic decisions given available information is supported.
If, however, the capital gains effects disappear under the equity and
land availability constraints of the period, then speculative motives
become a more plausible alternative explanation.

The primary objective of this research is to provide a preliminary
test of whether the capital gains effects on acreage, nonland inputs and
debt use are plausible under recent economic conditions. The methodology
involves a deterministic dynamic programming (DP) model of a farm firm
based on central Iowa conditions. Solutions to the model under various
land price paths are compared. Sensitivity testing is used to examine
the impact of the unobservable parameters, the proportion of unrealized
capital gain or loss substitutable for income and the proportion of
unrealized capital gain or loss substitutable for equity in the financial
negotiation (ϕ and ω respectively in Lowenberg-DeBoer's (1985) notation).
Because the tax treatment of capital gains and losses plays an intimate
role in the impact of land price changes, solutions under several tax
parameter levels are also compared.

A secondary objective of the research is to verify the sufficiency
arguments for the theoretical model. Because sufficiency conditions are
not well-developed for singular control models of the type used by
Lowenberg-DeBoer (1985a, b), the optimality arguments are based on satisfying known necessary conditions and the plausibility of the solutions when compared to other candidate solutions. Another form of evidence for the optimality of the candidate solution is to solve the problem numerically via dynamic programming (Bryson and Ho, 1975, p. 268). If the DP solution is consistent with the proposed singular control solution, the optimality of the candidate is supported.

This analysis is an exploratory first step. It does not consider enterprise choice, adjustment cost and other problems examined in the theoretical model. In particular, it does not include the impact of land price risk, which Lowenberg-DeBoer (1985b) suggests may substantially alter the optimal decisions in some cases. Though simple and preliminary, it is hoped that this research provides an empirical test of the explanatory power of the model and indicates fruitful areas for further work.

The general organization of this paper is to briefly review previous research, explain the empirical model, discuss the numerical results, and consider the implications of the work. The next section reviews previous empirical research on the impact of land price change on production and finance decisions. The theoretical background of the research is covered in the third section. A basic model of farm firm decisions in an environment of capital gains and losses is outlined and the primary analytic results are reviewed in the third section. The implementation of the model in a deterministic DP model is explained in the fourth section. Scenarios to be analyzed with the DP algorithm are developed in the fifth section. The DP solutions are analyzed and compared in the
sixth section. The final section discusses the conclusions and implications of the empirical research and earlier theoretical work.

Previous Empirical Research

Though economists have hypothesized many impacts of capital gains and losses on farm production and finance decisions, empirical tests of these hypotheses are rare. The body of research on livestock capital gains does not generalize to the case of farmland price changes, because livestock capital gains are almost purely a result of tax rules, but farmland capital gains are primarily price appreciation.

Researchers have often included changing land prices in programming models of the farm firm, but the full range of capital gain and loss impacts have not been treated. For example, Van Arsdall and Elder (1969) found that increasing the land value in a single period linear programming model (LP) of an Illinois farm firm reduced farm size and led to more land intensive activities in the optimal solution. Their model, however, did not include realization of the capital gain or the substitution of unrealized capital gains for income or equity. Vandeputte and Baker (1970) assumed that capital gains affect farm production decisions through the substitutability of capital gains for current income in the consumption decisions and through realization of the gain at the end of the period. They develop a multiperiod linear programming example in which intensive hog production is in the optimal solution when capital gains are fully taxed, but production switches to crops when the capital gains tax deduction is incorporated in the model. They did not include
financial activities in their model, but they note that, "were credit specifications also to be included, the differences might be still greater because of the credit consequences of assets accumulated under the capital gains provisions of the tax laws" (p. 526).

Farm firm simulation models have often included changing land prices. For instance, in their model of a Nebraska wheat farm Held and Helmers assume a constant annual appreciation rate and that all land value increase was an addition to equity. They found that a higher rate of land price appreciation tended to increase the chance of firm survival and enhance growth by helping to maintain the minimum equity level needed for farm existence and by increasing the borrowing power of the firm. Skees and Reid assume that the land price is a function of returns to land and that these returns are stochastic. Hence, the land price is stochastic in their model. They show that the correlation of land returns and land prices tends to substantially increase farm risk levels and that the impact of land price risk tends to be greater for larger farms. Because production and finance relationships are assumed in simulation models, capital gain and loss impacts on these relationships could not easily be tested in these studies.

The research on land price formation approaches the impact of land prices on production and finance decisions from an aggregate perspective. What is the equilibrium price level given that farmers have certain avenues of production and financial response available when the land price changes? Econometric models have shown that expected capital gains, government land diversion programs, technological change,
conservation payments, farm enlargement and the rate of return on alternative investments can all be used to explain farmland price changes (Reynolds and Timmons, 1969; Tweeten and Martin, 1968; Herdt and Cochrane, 1966; Klinefelter, 1973; and Brown and Brown, 1984). Shalit and Schmitz (1982) demonstrate that the debt carrying capacity of farmland can be important in determining farmland prices; if more money can be borrowed against land, the demand for land increases and prices are forced up. They suggest that lenders are crucial in land purchasing behavior because they determine the supply of credit. The capital asset pricing model was used by Barry in comparing the risk of farmland investments with that of nonfarm financial instruments. He found that the systematic risk associated with farmland was relatively low; thus, farmland was "a promising candidate for risk reduction in a well diversified portfolio."

Conceptual Framework

The conceptual framework of this research is based on a modified Vickers' model of firm level decision making. The model assumes that the firm's owners seek to maximize the net present value of returns to the business subject to the constraint that equity plus debt must equal the capital absorbed in acquiring inputs. The maximization of net present value allows considerations of the timing of cash flows. Through the discount rate information on the owner's risk preferences and returns on alternative assets enter the problem. The net present value maximization also approximates utility maximization for cases in which utility is
primarily a function of money income and the owners may borrow against or save cash returns to achieve the desired consumption pattern.

In continuous time, the objective function can be written:

\[
\max Z = \int_0^T e^{-\rho (1-\tau) t} \pi_t \, dt + S - K_0
\]

where:
- \( T \) = total number of periods,
- \( \pi_t \) = the income in period \( t \),
- \( \rho_t \) = the discount rate,
- \( \tau \) = the average tax rate,
- \( S \) = the salvage value of the firm's assets,
- \( K_t \) = equity capital invested in the firm, and
- \( t \) = variable of integration, time.

The financial constraint is:

\[
K_t + D_t - \alpha_t X_t - \beta_t L_t + G_t = 0
\]

where:
- \( D_t \) = debt,
- \( X_t \) = nonland inputs,
- \( L_t \) = land area,
- \( G_t \) = unrealized capital gains, and
- \( \alpha_t, \beta_t \) = capital absorbed by nonland and land inputs.

The capital absorption parameters (\( \alpha, \beta \)) may differ from the price of the input. For instance, special financing agreements may reduce the capital.
required. For simplicity, the following analysis assumes that the
capital absorbed by land is equal to its price. The income term can be
more fully specified as the sum of after tax current income from
production and the proportion of the capital gain in period t that is
substitutable for current income:

\[ \pi_t = \left[ P_t f(X_t, L_t, t) - \gamma_1 X_t - \gamma_2 L_t - r\left(\frac{D_t}{K_t + \omega G_t}, t\right)D_t \right] (1-\tau) \\
+ \phi \delta_t L_t \]

where: \( P_t \) = the output price,
\( \gamma_1, \gamma_2 \) = the current costs of using nonland and land inputs,
\( f(*) \) = production function,
\( r(*) \) = a convex credit supply function with \( r' > 0 \) and \( r'' > 0 \),
\( \omega \) = the proportion of unrealized capital gain that can be
substituted for equity in the financial negotiation,
\( \phi \) = the proportion of unrealized capital gain that can be
substituted for current income, and
\( \delta_t \) = the change in the land price at time t.

As in a conventional Vickers' model, the current cost terms \( (\gamma_1, \gamma_2) \) are
the costs of inputs actually consumed in the production process. For
nondurable inputs, like fertilizer, the current cost is the full purchase
price. For durable nonland inputs, the current cost includes maintenance
and depreciation. Property taxes are a primary component of the current
cost of real estate ownership. The cost of debt is assumed to rise with
increasing leverage. The argument is that when debt is larger relative
to equity, the lender incurs more risk and this risk cost is passed
through in the form of higher interest rates (Vickers, 1968, pp. 67-68).
With time subscripts on the discount rate, the nonland capital absorption
parameter the current cost coefficients and the output price, and a time
variable in the production and debt cost functions, this specification is
slightly more general than that used by Lowenberg-DeBoer. The time
varying coefficients and the changes in the functions over time do not
present any conceptual problem for the theoretical model, but they do
make derivation of the explicit control expressions more complicated and
hence they were omitted in the original discussion. The time variable in
the production function is assumed to capture technological change. In
the debt cost function, time is assumed to reflect changing financial
conditions which may shift interest rates for all borrowers and may
affect the premium charged for risk.

Unlike the usual Vickers' model, the income term includes a propor-
tion of unrealized capital gains or losses and some unrealized capital
gain or loss can be substituted for equity in the debt cost function.
The argument for the substitutability of unrealized capital gains for
current income is based on the idea that with perfect capital markets
unrealized capital gains or losses would be a perfect substitute for
current income because one could at any time cash in the unrealized gains
or losses without penalty (Bhatia, 1972). In an imperfect capital market
with differing expectations among agents and transactions cost,
unrealized capital gains are no longer a perfect substitute for cash, but
it is reasonable to assume that for at least some agents the unrealized capital gains are an imperfect substitute for cash. The proportion \( \phi \) reflects the degree to which the farm decision maker is willing to substitute accrued capital gains or losses for cash gains or losses.

The substitutability of unrealized capital gains or losses for equity is based on the common practice in agricultural lending of valuing collateral assets at a "conservative market value." This is a market value adjusted for selling costs, taxes and the uncertainty about whether the gain or loss will ever be realized. The views of lenders and their expectations about the future influence the equity substitution parameter \( (\omega) \). If lenders are willing to recognize a larger proportion of unrealized price appreciation as an addition to equity, the effective leverage will be reduced and debt costs lowered, leading in many cases to expanded debt use. If lenders believe that the price appreciation is ephemeral then they will be unlikely to recognize it in the financial negotiation as an addition to equity and debt costs will be unchanged by the land price change, regardless of the farmer's view on the permanence of the land price change.

Because the tax treatment of capital gains and losses is not symmetric in the U.S., the salvage value term will differ depending on the land price path and the land acquisition strategy. If capital gains are earned, the salvage value term can be written:

\[
S = e^{-\tau T} \left[ K_T + (1 - \phi - \psi) G_T \right]
\]
where: $\psi =$ the proportion of capital gains that is taxable and

$\tau_T =$ equals the average tax rate after the business is terminated.

This specification assumes that all the assets are sold for their full market value at the end of the horizon and all debt is paid off. Hence, the terminal cash flow is the value of invested equity plus the after tax capital gain income $\left( (1-\psi T_T)G_T \right)$. To prevent double counting of capital gain, the proportion of the unrealized capital gain that has already been recognized $(\psi G_T)$ is subtracted. For simplicity, the salvage value term (4) assumes that there are no liquidity losses. Such losses could be incorporated by recognizing only a portion of the terminal value of assets. Under current U.S. law only, 40 percent of capital gains are taxable.

In the capital loss case, the salvage value must account for the partial deductibility of capital losses from taxable income. The value of capital loss deductions is limited because only a proportion of the loss can be deducted (currently 50 percent of long-term capital losses) and because the annual capital loss deduction is constrained (to the lower of the taxable income over the zero bracket amount or $3,000 under current law). The salvage value term is then the sum of the invested equity, the capital loss which has not yet been recognized, and the value of the capital loss deduction. If the decision maker lives long enough to use the entire deduction, if income is at least high enough to allow the maximum deduction each period, and the discount rate and average tax rate are constant, then the salvage value term may be written (Lowenberg-DeBoer, 1985a, p. 28):
207

(5) \[ S = e^{-\rho(1-\tau)}T[K_T + (1-\phi)G_T + \frac{\tau e}{\rho(1-\tau)}(1-e^{-\rho(1-\tau)\delta G_T/e})] \]

where: \( \delta = \) the proportion of the capital loss that is tax deductible and 
\( e = \) the annual limit on capital loss deductions.

The last term in the capital loss salvage value term (5) is the present value of the stream of tax benefits from the capital loss deduction. The period during which these deductions occur is calculated as the total deductible loss, \((\delta G_T)\) divided by the annual deduction \((e)\).

A crucial simplifying assumption used in solving the theoretical model is that land is sold only at the end of the planning horizon. This models the situation of the farmer who may wish to sell the land upon retirement, but does not plan to sell before then. Or it may model the case of an investor who plans to hold the land for some specified period. Without this assumption it would be necessary to record the capital gain for each tract separately so that if it were necessary to sell it during the planning horizon, the exact capital gain or loss could be calculated. If land is sold at the terminal date, only the aggregate capital gain or loss is needed and this can be recorded as a running total of all capital gains and losses. The assumption also simplifies the maximization problem. If land could be sold at any time, the order of tract sales might be important. For instance, even if land is homogeneous, it may be advantageous for tax reasons to sell the most recently acquired tract first. The selling price of the recently acquired tract would be the same as for other tracts, but in a rising land price market the taxable capital gain would be smaller. In a deterministic model of an average
farm firm, the assumption of sales only at the end of the period is reasonable. Farmers do not usually buy land with the intention of selling it while they are still in business, though they may be forced to sell by unforeseen circumstances.

The decision variables in this model are the farm acreage \((L_t)\), nonland inputs \((X_t)\), debt \((D_t)\), and equity \((K_t)\). If the land price path is linear in time and the levels of the decision variables are set initially and remain unchanged over the horizon, this is a static constrained maximum problem in ordinary calculus. For other land price paths or if the decisions variables change over time, optimization requires control theory techniques. The key analytic insight of both the static or dynamic problems can be understood by examining the marginal rate of substitution between land and nonland inputs. For the dynamic case of rising land price in the absence of land or equity constraints, the MRS equation takes the form (Lowenberg-DeBoer, 1985a, p. 65):

\[
\frac{f_x}{f_L} = \frac{[\gamma_1 + \alpha(\tau+r_t' \frac{D_t}{K_t + \omega G_t})](1-\tau)}{\{\gamma_2 + [\beta_t - \frac{\beta_t}{\rho(1-\tau)}(1-e^{-\rho(1-\tau)(T-t)})](\tau+r_t' \frac{D_t}{K_t + \omega G_t})

- \frac{\beta_t \omega r_t'}{\rho(1-\tau)}(\frac{D_t}{K_t + \omega G_t})2(1-e^{-\rho(1-\tau)(T-t)})\}(1-\tau)

- \frac{\beta_t}{\rho(1-\tau)}[\phi e^{-\phi(1-\tau)(T-t)}(1-\phi-\psi T)]
\]

where \(\frac{f_x}{f_L}\) is the first derivative of the production function with respect to nonland inputs and
\( f_1 = \) the first derivative of the production function with respect to land.

Like the usual Vickers model, the factor cost ratio in the MRS equation contains both the current cost of using the input \( (\gamma_1, \gamma_2) \) and the financial cost. Unlike the conventional Vickers model, the cost of using land (assuming capital gains) is reduced by the savings from buying now rather than later at a higher price:

\[
(6.1) \quad \frac{\hat{B}_t}{\rho(1-\tau)} \left( 1-e^{-\rho(1-\tau)(T-t)} \right) \left( r+\tau \frac{D_t}{K_t}(1-\tau) \right)
\]

the value of interest savings due to recognition of unrealized capital gains as a substitute for equity:

\[
(6.2) \quad \frac{\hat{B}_t \omega_{t'}}{\rho(1-\tau)} \left( \frac{D_t}{K_{t}+\omega_{t}} \right)^2 \left( 1-e^{-\rho(1-\tau)(T-t)} \right) (1-\tau)
\]

the value of unrealized gains that are recognized as current income \( (\hat{\phi}_t) \) and the realization of the gains at the end of the period \( \left( e^{-\rho(1-\tau)(T-t)}(1-\hat{\phi}_t) \right) \).

Because the cost of land use is reduced by capital gains, the optimal input mix tends to include more land and fewer nonland inputs. Taxes affect the magnitude of the impact because the unrealized capital gain that is substituted for current income is not taxed and the realization of capital gains is only partially taxed at the end of the horizon, while the benefits of buying now rather than later and the debt
cost reduction due to added equity in the form of unrealized gain are fully taxed as they occur. The greater the substitutability of capital gains for income (φ) and for equity (ω) and the smaller taxable proportion of capital gain (ψ), the greater the incentive to expand land use and decrease use of other inputs. It should be noted that it is not inconsistent for accrued capital gain to be both recognized as current income and used as an equity substitute in the financial negotiation, though for analytic purposes dealing with a pure model of one effect or the other simplifies analysis by reducing the number of channels for capital gain and loss effects. In a capital loss case, the MRS equation has a similar form with the value of capital loss deduction tax benefits replacing the taxation of capital gain term \( e^{\rho(1-\tau)(T-\tau)} \psi I \). When land prices are declining (\( \Delta_t < 0 \)), the land price change terms in the cost ratio, except for the value of capital loss tax deductions, increase the cost of owning land. This creates incentives to economize on land use and farm more intensively.

If there is adequate equity to achieve the optimal financial structure in which the marginal cost of equity equals the marginal cost of debt, then the leverage ratio \( \frac{D_t}{(K_t+G_t)} \) is a constant (Lowenberg-DeBoer 1985a, p. 86) and the factor cost ratio is completely determined at any point in the horizon independent of the input choice. If the optimal financial structure cannot be achieved, the debt and equity levels must be solved for simultaneously with the input levels. With adequate equity, the absolute level of debt use may rise with farmland capital gains because land tends to be a capital intensive
input, but the level of debt use relative to effective equity \( (K_t + \omega G_t) \) is constant. With the substitutability of unrealized capital gain for equity \( (\omega > 0) \), the level of debt relative to invested equity \( (K_t) \) rises with capital gains, because it is less expensive to meet the equity requirement in the financial negotiation with unrealized capital gains than with investment of retained earnings or outside capital (Lowenberg-DeBoer, 1985a, p. 87).

With constraints on the availability of land and capital, the optimal capital structure or the optimal MRS may not be achievable during at least some part of the horizon. If equity capital is a binding constraint, the leverage ratio is no longer constant and the factor cost ratio depends on the debt level. Lowenberg-DeBoer (1985b) shows that for some plausible circumstances this results in a dampening of the land price change effects, but exaggerated effects are also possible. A more complete analysis of the model is available in Lowenberg-DeBoer (1985a) and Lowenberg-DeBoer (1985b).

The Numerical Model

Dynamic programming was chosen for the numerical solution to the problem because it requires a minimum of additional assumptions and because it facilitates extending the model to a stochastic environment. The primary alternative to DP would have been a multiperiod LP or quadratic programming (QP) algorithm, which would have required strict assumptions about the functional forms involved. With DP, no functional form assumptions are required. This exploratory research utilizes only
one production function form, a power function, and one debt cost function form, a step function in which the interest is at the average market rate up to some specified maximum leverage and very high above this leverage. Testing the sensitivity of the solutions to the functional forms is a relatively simple matter of estimating new forms and substituting them into the algorithm.

With discrete control techniques, a wide choice of functional forms would also have been available, but extension of the model to an environment of sequential decisions in a stochastic environment would have been difficult. DP allows the solution of sequential stochastic problems with simple extensions of the basic algorithm (Dreyfus and Law, 1977, Chapter 9). Sequential, stochastic problems can also be solved by discrete stochastic programming (DSP) (Cocks, 1969; and Rae, 1971) in an LP or QP framework. It is not clear, however, that DSP has an advantage over DP because DSP requires a separate block in the LP or QP matrix for each possible outcome. Hence, the matrix becomes very large and the advantage of lower problem solving cost with LP or QP software and the ability to use a larger number of decision variables is diminished.

In a stochastic environment, the sequential nature of land and debt use decisions may have important consequences for the optimal choices. For example, if an open loop solution, which specifies levels of the control variables over the entire horizon, is used in a risky environment in which large land price declines are possible, then low levels of land and debt use may be chosen to avoid the large losses that could be associated with large farm size and a highly leveraged position in a
period of declining prices. An optimal feedback solution, which assumes that the decision maker observes the environment at each decision point before making a decision, may indicate a higher initial land and debt use level because there is more flexibility in adjusting to the conditions that may occur. Because stochastic DP is a simple extension of deterministic methods, the deterministic DP algorithm constructed for this exploratory research can serve as the foundation for software to solve the stochastic problem.

The primary problems with DP are the "curse of dimensionality" and the lack of commercially available software. Because DP is essentially an efficient enumeration technique, the number of possible solutions over which it is necessary to enumerate and problem solving costs rise rapidly as the number of variables increase. Most DP problems have three or fewer decision variables. Larger problems can be solved, but the cost becomes prohibitive given the current state of computer technology. In this case, working with a small number of variables is not a severe distortion of the decision situation. In a long-term planning context, it is unlikely that the farmer will plan to raise a specific number of livestock or to plant a specific number of acres to a given crop at some point in the distant future. It is more likely that the decision maker would seek to plan major variables, like farm size and financial structure, and then construct more detailed short-run plans within the context of the major long-run decisions.

The lack of readily available DP software is not a major barrier for anyone with rudimentary programming skills. DP programs are relatively
simple, consisting primarily of a set of nested do loops, with a mechanism for accepting input and producing output. Kennedy reviews a wide range of successful DP applications in agriculture, forestry, and fisheries. He suggests that the interest in long-term planning and control, the increased efficiency of computer equipment, and the diffusion of programming skills will make DP a more common research tool in the future (Kennedy, 1981, pp. 141-142).

**DP algorithm**

The central idea of DP is stated in the principle of optimality: "The best path from A to B has the property that, whatever the initial decision at A, the remaining path to B, starting from the next point after A, must be the best path from that point to B" (Dreyfus and Law, 1977). This principle is put into practice by enumerating backwards from the end of the planning horizon. In the last period, the problem is a simple one-period optimization. This optimization is carried out for all relevant values of the state variables at the beginning of the last period. In the second to the last period, one must choose the control variables which optimize the sum of that period's return and the optimal objective function value for the subsequent period given initial state variable levels generated by the control choice in the second to the last period. This process is repeated in each period with an optimization over the stage at hand and the rest of the process. By enumerating only over the optimal choice in each period given the state variables at the beginning of the period, DP substantially reduces the size of the
enumeration problem compared to enumerating over all possible paths. Dreyfus and Law provide a good introduction to the techniques of DP.

DP problems are commonly formulated in three statements: an optimal value function, which is a statement of the problem in words; a recurrence relation, which is mathematical expression of the relationship which is optimized in each period; and the boundary condition, which is the value at the end of the planning horizon. The boundary condition serves as a starting point for the backward enumeration. The optimal value function for the above problem is:

optimal value function = the maximum present value of returns for the periods t through T given an initial farm acreage L, equity level K, and unrealized capital gain G.

Debt is eliminated from the problem by substituting in the financial constraint (2). This substitution eliminates one state variable and substantially reduces problem solving cost. Typically, DP constraints simplify the problem by reducing the number of independent state variables or by reducing the range over which enumeration must occur.

The recurrence relation is a function of the initial land, equity, and unrealized capital gain levels. It can be expressed as the sum of the optimal choice for the period at hand and the recurrence relation over the remaining horizon:
\begin{equation}
S_t(L, K, G) = \max \left\{ \sum_{i=1}^{n} \left( \frac{\pi_{t+i-1} - u_2}{(1+\rho (1-\tau)^{i})} \right), \right.
\left. \frac{1}{(1+\rho (1-\tau)^{n}} S_{t+1}(L+u_1, K+u_2, G+(L+u_1) \right) }\right. \\
(B_t - B_{t-1}) \right) \right. 
\end{equation}

where: 
- $u_1 = \text{land purchases}$,
- $u_2 = \text{equity investments}$, and
- $n = \text{number of years for decision period}$.

The maximization in equation (7) is carried out over land purchases, equity investment, and the level of nonland inputs. New investment ($u_2$) is subtracted from the current income stream because it represents a cash outflow to the decision maker. If the income term ($\pi_u - u_2$) is constrained to be nonnegative, new equity investment is limited to retained earnings.

Years are grouped into decision periods to simplify the problem, with the number of years per decision period being denoted by $n$. Grouping years reduces the number of possible sets over which the enumeration must occur and reduces problem solving cost. Because land purchase is not usually an every year decision, this is not a totally unrealistic assumption, though it is possible the choice of decision years may affect the outcome. It should be noted that if $n$ is one and the time units are allowed to become very small, the DP problem becomes identical to the continuous time optimal control problem solved by Lowenberg-DeBoer (1985). In fact, DP could be used as an alternative
solution technique for the optimal control problem (Kamien and Schwarz, 1981, section 20) or it could be used to provide analytic solutions to the analogous discrete time problem (Dreyfus and Law, 1977, Chapter 7). With appropriate defined variables, discrete and continuous time formulation suggest identical optimal solution approaches. Discrete time is used for the empirical work because the data are in discrete time form.

The income term is a discrete version of equation (3):

\[
(3.1) \quad \pi_t = \left[ P_t f(L+u_2, X, t) - \gamma_{1t}X - \gamma_{2t}L - t\left(\frac{B_{t-1}L + \alpha X - K - G}{K + \omega G}, t\right)(B_{t-1}L + \alpha_t X - K - G)\right](1-t) + \phi(B_t - B_{t-1})
\]

The subscripting in equation (3.1) reflects the assumed timing of decisions, sales, and purchases. The input and financing decision is assumed to be made at the beginning of the year. Land may be purchased only at the beginning of the n year period. Debt and nonland inputs are assumed to be adjusted at the beginning of each year. It is assumed that input costs are paid and output sold at the end of each year. Because land must be purchased before the production period, it is acquired at the price established at the end of the previous year \(B_{t-1}\). Though nonland input levels are set at the beginning of the year, it is assumed that the nonland inputs are actually acquired during the production season as needed. Hence, the current cost of nonland inputs \(\gamma_{1t}\) is based on the price during the year. The current costs of land ownership
and the capital absorbed by nonland inputs \((a_t^*)\) is also assumed to be based on values during the year. The capital gain \((\beta_t - \beta_{t-1})\) is the difference between the price that land could be purchased for at the beginning of the year and the price which is established at the end of the year. For this exploratory research, a power function:

\[
(8) \quad f(X_t, L_t, t) = A_t^{a_1} X_t^{a_2} L_t^{a_3}
\]

was used for the production relationship and a step function:

\[
(9) \quad r = \begin{cases} 
    i & \text{for } \frac{D_t}{K_t + \omega G_t} < L \\
    \text{otherwise} & \text{otherwise}
\end{cases}
\]

where \(i\) = the average market rate of interest, 
\(L\) = the maximum allowable leverage, and 
\(M\) = a very large number,

for the debt real estate and nonreal estate cost. The power function has been widely used for production function approximation. It requires the factors of production to be limiting; an advantage in this case since it is unlikely that any significant production could occur with no land or with no nonland inputs. It is relatively easy to estimate in logarithmic form. Heady and Dillon (1961) suggest that the unit elasticity of substitution which is inherent in the functional form may be a satisfactory approximation for a firm level relationship, though it is unrealistic for
per acre or per animal situations (p. 84). Attempts were made to estimate coefficients for a translog function, which would avoid the unit elasticity of substitution assumption. In the Central Iowa data for 1957-1969, this resulted in many statistically insignificant coefficients and some implausible signs. The simple power function model estimation resulted in statistically significant and plausible coefficient estimates for all variables. Predicted output levels appear reasonable. The power function seems to be an adequate approximation for exploratory research, however, further work in the area should not be tied to continued use of this assumption.

A separate debt cost function of the form (9) was specified for real estate and nonreal estate debt. It was assumed that real estate debt, which is usually lower cost, is incurred first, up to a maximum percentage of the market value of real estate. If the debt level permitted under the maximum leverage ($A$) is greater than the maximum amount that can be borrowed on real estate, nonreal estate debt may be contracted up to the maximum leverage. This ordering of debt acquisition assumes that the decision maker chooses the debt source with the lowest nominal cost. In the context of a model which abstracts away from the maturity structure of debt and interest rate risk, this is a reasonable approximation of the debt acquisition decision, though in a more detailed model that order may sometimes be violated.

A step function was used for the debt cost because it appears to be a reasonable approximation in the financial situation of U.S. agriculture where lenders tend to respond to risk by nonprice methods such as capital
rationing instead of adjusting the interest rate (Barry et al., 1981, pp. 220 and 224; Baker, 1968, p. 519), and because available data are inadequate to estimate the hypothesized smooth credit supply curve. Barry, Hopkin, and Baker (1983) suggest a debt cost function which is flat for initial borrowing and then slopes upward as the farmer is forced to seek credit from higher cost lenders (p. 141). They state, "Available evidence suggests that a given lender is not likely to respond to a higher interest rate with more loan funds for a given borrower" (p. 141). When this observation is combined with the fact that split lines of credit are discouraged in agricultural lending, a credit profile is derived that consists of a series of steps. As more credit is demanded by the decision maker, all of the debt is shifted up to a higher cost lender. In the long-term planning context, it is not clear that there are very many steps. The Farm Credit System lenders have usually had the lowest stated interest rates among commercial agricultural lenders, yet there is little evidence that they are willing to provide less credit to a given borrower than other lenders. In fact, they have a reputation in many areas of being willing to provide more credit than their chief rivals, commercial banks. The relative advantage of the FCS lenders in effective debt cost may not be as great as the differential in stated rates, because of the FCS stock requirements and because FCS lenders often require greater documentation, thereby increasing transaction cost. FCS lenders frequently require additional income and cash flow information, while the traditional once a year balance sheet update may be adequate for small commercial banks.
The relevance of the Farmers Home Administration (FmHA) finance companies and merchant and dealer credit to the long-term credit profile in a deterministic decision framework is questionable. Would the representative farmer modeled here consider these lenders as long-term sources for all business credit needs? It is true that some farmers remain FmHA borrowers throughout their careers, but it is unlikely that this is a relevant alternative in planning. Certainly, it is the FmHA policy to graduate borrowers to commercial sources as soon as possible. Would the representative farmer plan to forego cash discounts and pay the service charges on merchant and dealer credit as a long-term credit source? In a stochastic environment, merchant and dealer credit, the FmHA, and finance companies may be very important in adding flexibility to the system, but for this deterministic model a first approximation of the debt cost function without them will probably not be a major source of distortion. This leaves a debt cost function with one step, which can be specified as the average interest rate on FCS borrowing or on borrowing from one of its commercial competitors, depending on data availability, up to some critical leverage level and no credit available beyond that leverage. The lack of data on lender imposed borrowing limits, beyond statutory maximums, reinforces the decision to use a step function. The available data are inadequate to specify a more sophisticated function and many arbitrary decisions would be required to implement such a choice.

In the context of the model, the critical leverage level may be the result of either internal or external capital rationing. With internal
rationing, the decisionmaker stops credit use before the lender's cutoff leverage. Internal capital rationing is commonly observed among U.S. farmers and Barry and Baker have demonstrated that it can be seen as a rational response to the value of liquidity. Unused credit is a source of liquidity and one could argue that the value of unused credit rises as more debt is incurred and the liquidity of the firm falls (Barry and Baker, 1983, p. 223). This would result in a smooth upward sloping debt cost function instead of the step function assumed here. Unfortunately, there is little empirical evidence on the magnitudes of liquidity premiums, so specifying the rising debt cost due to liquidity costs is impractical. Hence, for both internal and external capital rationing, a step function is used to approximate the firm's debt cost, leverage relationship.

The cost of debt beyond the credit cutoff is specified as some very large but finite number because the existence of bankruptcy laws imposes a limit on the decisionmaker's liability. Lowenberg-DeBoer (1985b) argues that the debt cost does not become infinite because the decisionmaker cannot lose more than the equity in the firm and the present value of future earnings (p. 170). The magnitude of $M$ is such that credit use beyond the credit cutoff will never be an optimal choice.

The boundary conditions are discrete versions of the salvage value terms of the theoretical model (4, 5). The boundary condition for capital gains is:

\[(4.1) \quad S_1(L, K, G) = K + \left(1-\phi-\psi T \right)G \]

for $G \geq 0$
and for capital losses it is:

\[ S_T(L, K, G) = K + (1-\phi)G + \tau \int \frac{1 - (l - \rho(1-\tau) - \delta G/\epsilon)}{\rho(1-\tau)} \]

for \( G < 0 \).

It is assumed that the firm is liquidated at the beginning of the terminal year \( T \), so no production takes place during that period.

**Implementation of algorithm**

The algorithm is implemented in a computer program written in PL/1. The program consists of five subroutines: Main, which accepts the input data and sets up the decision problem; Recurr, which calculates the recurrence relation values for each initial land, equity, and unrealized capital gain combination; Bounds, which checks if the state variable values generated by the recurrence relation are consistent with the user specified range for the subsequent period; Trans, which saves recurrence relation values from one period's calculation for use in the next period; and Output, which prints out the optimal control path and recurrence relation values for each possible initial state variable combination in each decision period. Flow charts of the algorithm are found in Figures 1 and 2.

The recurrence relation subroutine consists of five nested do loops. The first three loops are indexed over the land, equity, and unrealized capital gain ranges that are specified in the input. For each combination of state variables, an optimum land purchase, savings choice, and nonland input level is calculated and recorded. The control variable
Main: Accept Input
Calculate Frequently Used Values
Calculate Boundary Values
Do for Each Year Starting with the Last Year:

- Call Recurr,
- Call Output,
- Call Trans,

End.

Trans: Save optimal recurrence relation value to be used in the next iteration as value over the remaining horizon.

Output: Print optimal recurrence relation values and control choices.

Recurr: Find the control choices which maximized the present value of returns over the remaining horizon for each combination of user specified initial land, equity and unrealized capital gain levels. (See Figure 2 for more detail).

Bounds: Check if control choices result in land, equity or unrealized capital gain choices exceeding the user specified range in the subsequent period.

Figure 1. Flow chart of the dynamic programming algorithm
Do for each user specified level of initial land, equity and unrealized capital gain:

Calculate bounds for land purchase;

Do for all land purchase levels:

Calculate bounds for retained earnings;

Do for all retained earnings levels:

Do for each year within the period:

Check calculus interior and corner solutions for nonland inputs;

Calculate present value of production before interest given land purchase and nonland input choice;

Calculate present value of interest payments, property taxes, retained earnings and unrealized capital gain substitutable for income.

End;

End;

End;

Check if the present value current income minus interest, taxes and retained earnings over the period is nonnegative; Call bounds;

Check if the present value of current income in the period, minus interest on debt, property taxes and retained earnings, and plus unrealized capital gain substitutable for income and the present value of returns over the remaining horizon is greater than the previous maximum;

If the returns exceed the previous maximum, record the present value of returns as a potential optimal recurrence relation value and record the control path.

If returns exactly match those under some previous control choice, the record the nonunique control path

End;

Figure 2. Flow chart of the recurrence relation subroutine
values which maximize wealth are chosen by iterating over feasible land purchase and savings choices, with the nonland input level chosen by calculus arguments for each land purchase and savings combination. The bounds of the land purchase and savings iteration are partially internally calculated. The land purchase is varied from zero to the maximum amount that can be financed with the existing level of equity and unrealized capital gain or the maximum land available for purchase. The model allows both equity investment and disinvestment. The savings choices range from the maximum amount that can be dissaved and still allow the existing farm acreage to be financed the next period to the savings that would produce the largest equity amount specified by the user for the beginning of the next period. Dissaving implies that equity is withdrawn from the farm firm for consumption or investment elsewhere.

The land purchase and savings choice is made by period. The land purchase is assumed to be made at the beginning of the period and the savings decision is made for the period as a whole, with a constant amount saved each year. The nonland input choice is made separately each year to allow equity accumulation and unrealized capital gain to have an impact within the period. The alternative would be to assume savings were put in a bank account until the beginning of the next period, clearly an unrealistic assumption in a situation dominated by a lack of capital. Retained earnings are most likely to be reinvested in the farm business as soon as possible.

The calculus arguments for the nonland input optimality are designed to check interior and corner solutions for maximizing one year net farm
income given the farm acreage, equity level, and savings plan. The nonland input is assumed to be perfectly variable; the level in one year does not constrain the choice in the next year. This overstates the flexibility of nonland input use because acquiring or selling specialized equipment and facilities may be difficult, but it is a simplifying assumption that is consistent with the theoretical model and reduces the number of state variables in the model. The program checks five possible nonland input levels:

1) the calculus interior solution given farm acreage assuming no nonreal estate debt is used;

2) if choice (1) results in a capital requirement exceeding available equity and real estate debt, then the calculus interior solution assuming that the marginal nonland input is financed with borrowed capital is tried;

3) if choice (2) can be financed without nonreal estate debt, then the optimal nonland input level lies in between the levels of (1) and (2) at the maximum nonland input that can be purchased without using nonreal estate debt;

4) if choice (2) results in a capital requirement exceeding available equity and debt, then the nonland input level is set at the maximum amount that can be financed;

5) if choice (4) results in a negative nonland capital amount, then the farm acreage is unfeasible because it cannot be financed with available debt and equity.

The debt level is determined simultaneously with the nonland input
because the financial constraint (2) has been solved for debt and substituted into the nonland input choice. arguments.

In developing the algorithm, an alternative method of optimizing the nonland input was initially attempted, which iterated over discrete levels of nonland input. The range of the nonland input index was user specified and assumed to be based on observed input use. Even if only one nonland input level is chosen per period, this results in high problem solving cost because of the many iterations that must be performed or in inaccurate solutions because the nonland increment must be made large.

After the nonland input path is determined, the constraint limiting equity investment to retained earnings is checked. The present value of the period's current income from production minus expenses, including interest, and minus retained earnings must be nonnegative. This formulation of the constraint assumes that interyear transfers within the period are possible so long as they are made up with interest within the period. For example, if the current income before savings in a two-year period is $6,000 the first year and $10,200 the second year with a discount rate of ten percent, then savings of up to $8,000 annually is permitted because the present value of the deficit in the first year ($1,818) is offset by the present value of the surplus in the second year ($1,818). The interyear transfers might be accomplished by delaying consumption or drawing on personal nonfarm wealth to augment farm earnings temporarily. Imposing the constraint in this way reduces the lumpiness problem in savings. The alternative would be to require that current income after
retained earnings be nonnegative in each year, which is a more stringent requirement. Under the alternative specification, there is no allowance for income increasing with savings and technological change; the savings sum must be available in the lowest income year.

The maximization of wealth for a given land, equity, and unrealized capital gain level involves comparing wealth under the current plan with the highest wealth amount calculated up to that point for the state variable combination. The wealth is defined as the present value over the period of current income, minus retained earnings, plus any unrealized capital gain that is substitutable for income, plus the present value of wealth over the remaining horizon given the current land purchase and savings choices. It is possible for nonunique optima to occur if differing land purchase and savings plans result in the same wealth. The program will record one nonunique plan. This tells the user that the solution is not unique, but does not indicate all possible nonunique plans. Because the possible number of nonunique plans is very large, recording them all was not considered feasible. Because the discrete increments are used for land and saving, the chance of nonunique solutions is small and none have been encountered in multiperiod models using this algorithm.

The input consists of annual time series over the relevant period for interest rates, input and output prices, land prices, and property taxes, plus the parameters which are assumed to be constant over the planning horizon such as production function coefficients, the average
tax rate, the capital gains exemption, and the maximum leverage. Debugging was handled by running the algorithm for representative input data and manually verifying the results. The source code for the computer program is reproduced in Appendix A.

Input Data and Parameter Estimation

The input data for the problem are assumed to be in some sense expected values. The focus of this research was not expectations formation, so sophisticated models of this process were not constructed. Rather, plausible scenarios were constructed employing two simple expectation assumptions:

1) the expectations are the "best" statistical forecast using data up to the first year of the planning horizon and

2) perfect foresight, in which the expectations are the observed values for the period.

Which forecasting methodology is "best" depends on the characteristics of the data, but in most cases univariate time series models were used. Parameters which could not directly be observed are estimated when data is available, as in the case of the production function, or their impact on model solutions is examined by means of sensitivity testing, as in the case of the substitutability parameters ($\phi, \omega$). The simple step function used in the algorithm did not allow the full range of financial choices assumed in the theoretical model, thus sensitivity testing is done on the leverage ratio and the maximum percentage of real estate debt to
determine their impact on solutions. Several solutions are also calculated with alternative tax parameter values.

Parameter values other than land prices differ substantially under the two expectation assumptions. This can make it difficult to identify pure land price path effects. To more closely analyze land price path effects, two additional scenarios were defined based on the forecasts, but with the land price altered. A higher capital gain scenario was created by multiplying the dollar value of forecast capital gains by ten. The resulting land price path more closely approximates the nominal changes in land prices observed in the 1970s than the forecasts. The factor of ten is, however, arbitrary. It could be eight or 12. A factor of ten is used because it is large enough so that if land prices can significantly affect production and finance decisions, that effect should be visible. Yet it is small enough so that the expectation is within the realm of possibility. The other scenario uses the forecast land price increases multiplied by ten up to 1981 and then drops to the forecast path. This creates a land price bubble similar to that found in the observed values. Hence, the basic scenarios are:

1) forecast values,
2) perfect foresight,
3) higher capital gains, and
4) land price bubble.

The treatment of parameter values in each basic scenario is summarized in Table 1.
Table 1. The treatment of parameter values in expectation scenarios

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Observed values</th>
<th>Statistically estimated values</th>
<th>Sensitivity testing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Land price ($\beta_t$)</td>
<td>Perfect foresight</td>
<td>Forecast</td>
<td>Higher capital gain</td>
</tr>
<tr>
<td>Input and output prices, interest rates, discount rate, and nonland input capital requirement ($\gamma_1$, $\gamma_2$, $p_t$, $i$, $\rho$, $\sigma_t$)</td>
<td>Perfect foresight</td>
<td>Forecast</td>
<td>None</td>
</tr>
<tr>
<td>Tax parameters ($\tau$, $\psi$, $\tau_T$, $\delta$, $\epsilon$)</td>
<td>All basic(^1) scenarios</td>
<td>None</td>
<td>Higher capital gain</td>
</tr>
<tr>
<td>Substitutability parameters ($\phi$, $\omega$)</td>
<td>None</td>
<td>None</td>
<td>All scenarios</td>
</tr>
<tr>
<td>Production function ($a_1$, $a_2$, $a_3$)</td>
<td>All basic(^1) scenarios</td>
<td>All scenarios</td>
<td>None</td>
</tr>
<tr>
<td>Debt supply function parameters, minimum leverage ratio ($L$), and percentage real estate debt</td>
<td>None</td>
<td>None</td>
<td>Higher capital gain</td>
</tr>
</tbody>
</table>

\(^1\)The basic scenarios are: forecast, perfect foresight, higher capital gain and land price bubble, with a maximum leverage of 0.2 and a maximum real estate debt of 50 percent of market value.
Wherever possible, univariate Box-Jenkins models were used to construct the statistical forecasts. Univariate Box-Jenkins models offer numerous advantages over multivariable econometric methods in terms of the logical approach to forecasting and theoretical consistency (Jenkins, 1979, pp. 88-94). For the purposes of this research, the univariate models offered a feasible method of generating predicted values for a relatively large number of variables. In many forecasting circumstances, univariate models perform at least as well as more complicated multiequation econometric models (Naylor et al., 1972). For this project, the land price forecast is crucial and previous research indicates that a univariate model predicts short-term land price changes at least as well as econometric methods (Pope et al., 1979).

The problem in using univariate models in this context is that the forecast period is relatively long, while the greatest strength of Box-Jenkins model forecasting is in the short term. Because the models usually rely on stochastic trend, forecasts for the distant future are usually flat. Stochastic disturbances in the most recent observations work themselves out in the short term and no new disturbances are introduced. Given the uncertainty about the distant future, however, a forecast of "more of the same" is not necessarily an unrealistic expectation.

The time period to be considered is the 15-year span, 1970-1984. The period was chosen because it encompasses a time of large capital gains and losses, when land price changes were a potentially major force in agricultural decisions. It comes at the end of a 40-year period of
almost continuously rising land prices, so that expectations of capital gains were reasonable almost irrespective of how that expectation is formed, and the capital losses of the 1980s were not widely anticipated. In general, U.S. farmers responded to the conditions of this period by increasing their holdings of intermediate and long-term assets and incurring additional debt (Boehlje and Eidman, 1983, p. 935; USDA, Balance Sheet of the Farming Sector, 1978, pp. 98-99). This reduced their liquidity and increasing debt service requirements, leaving them vulnerable to cash flow difficulties and other financial problems. Insight into why farmers restructured their balance sheets during this period would be useful to policymakers in developing programs to mitigate the effects of financial stress and avoid its resurgence, and to decisionmakers and those who advise them in identifying the consequences of certain approaches to valuing capital gains and losses. Other periods in U.S. history would offer an equally interesting test of the explanatory power of this model — the land price boom and crash of the World War I period for instance — but no other period offers such immediate and compelling need for study and readily available data.

Central Iowa was used as the focus of this exploratory research because it is in a state that showed among of the largest farmland capital gains from 1970-1981 and among the largest capital losses after 1981, because adequate data are available for estimating production relationships from the Iowa Farm Business Association (IFBA) annual summaries, and because the central region of the state contains a mix of crop and livestock production that may reduce the impact of enterprise
choice decisions on the research results. The choice of a state with land price boom and bust extremes is logical in research that hopes to test the potential for capital gain and loss impacts. If they will show up anywhere, then they are most likely to show up when price changes are greater.

The IFBA is an independent farm management and recordkeeping organization. IFBA members are not a random sample of Iowa farmers. They participate voluntarily and it can be argued that their interest in detailed recordkeeping indicates management skills that are above those of the average farmer. The advantage of IFBA data is that it offers a relatively detailed look at a substantial group of commercial farmers. Membership in the central region has varied over the years, but recently it has ranged around 400. Unfortunately, individual records are not available, but the Iowa State University Extension Service does publish an annual summary of average assets, costs, and income for various farm acreage classes to give other farmers some standard for comparison. The format of these publications has been relatively consistent since 1957, so 13 years of data are available before 1970. The role of these summaries as a public standard for comparison strengthens the argument for using this data as a proxy for the information used by farmers and their advisors in formulating expectations. It is not known how influential the summaries are in the formation of expectations, but they are intended to be part of that process.
Production function estimation

The production function (8) was estimated from the cross-sectional time series data of central Iowa IFBA farmers using the error components method (Judge et al., 1982, p. 499). This technique assumes that in addition to the general error term (e_{it}) there is some year specific error (u_t) that is the same across farm size classes. It is assumed that the trend term (t) captures systematic productivity changes, such as those generated by technological innovation. The year error term captures random effects from such sources as weather, disease, and pest problems common to all the IFBA farmers in a given year. The general error term is assumed to capture all other sources of error: omitted variables, the collection of individuals who happen to belong to the IFBA in a certain size class, and events on individual farms. In logarithmic form, the model can be written:

\[(10) \quad \ln Y_{it} = A + a_1 \ln t + a_2 \ln X_{it} + a_3 \ln L_{it} + u_t + e_{it}\]

where: \(i\) = the farm size class.

The general and year specific errors are assumed to be independent, with zero mean and constant variance of \(\sigma_e^2\) and \(\sigma_u^2\), respectively. The general and year specific errors are assumed to be uncorrelated. The covariance matrix has the sum of the two variances on the diagonal \(\sigma_u^2 + \sigma_e^2\) and the year specific variance everywhere else (Judge et al., 1982, p. 489). Because this variance structure deviates from the constant variance multiplied by an identity matrix assumed for ordinary
least squares estimation, generalized least squares (GLS) is required. The GLS transformation involves expressing the original observations as a deviation from a fraction of their means. The fraction is:

\[
b = 1 - \frac{\sigma^2_e}{N\sigma^2_u + \sigma^2_e}
\]

where: \( N \) = the number of farm size classes.

The other primary method for dealing with cross-sectional time series data employs dummy variables for the year effect. Instead of assuming that there is a random term for each year, it is assumed that there is a separate intercept term for each year. The dummy variable is adaptable to a wider range of situations because the assumptions about the distribution of the year specific error (\( u \)) are not required (Judge et al., p. 497). It is, however, unsuitable for capturing expectations about the future because the model provides no information about future intercept values.

The problem of correlation between the year specific error and the independent variable raised by Mundlak is a possibility here. For instance, if pesticides are a large part of nonland input, then a year in which insect infestations are particularly heavy might be characterized by a large nonland expenditure for insecticide and a large negative year specific residual for all four sizes. Similarly, when the infestation is unusually light, the nonland input level may be small and the year specific residual a large positive. If there is correlation between the
year error and the independent variables, the error component model is misspecified and coefficient estimates will be biased. The formal test for correlation of the year error and the independent variable suggested by Hausmann (1978, pp. 1261-1264) does not appear to apply here because the dummy variable and the error component model contain a differing number of slope coefficients. This occurs because the trend term must be dropped from the dummy variable estimation in order for the model to be full rank. The argument used by Hausmann, however, does apply. If the model is properly specified, the slope coefficient estimates for the dummy variable and error component models will be similar because the specification error would not bias the dummy variable estimate, and if the error component model is correct it will similarly be unbiased (Hausmann, 1978, p. 1263). The two models differ only in how the year effects are treated. The slope parameters being estimated should be the same, though the estimates may differ in efficiency. For the central Iowa data, the dummy variable and error component model estimates for the land and nonland input coefficients were almost identical.

The need for year effect terms can be examined with an F-test comparing the residual sum of squares for the dummy variable model and for a restricted model in which it is assumed that a common intercept exists for all years (Judge et al., 1982, p. 495).

\[
F_{12, 43} = \frac{(SSE \text{ restricted model} - SSE \text{ unrestricted model})/(T-1)}{\text{MSE unrestricted model}}
\]

\[
= \frac{(0.5521 - 0.0386)/12}{0.00089831}
\]

\[
= 47.6.
\]
where: \( \text{SSE} = \text{sum of squares error} \),
\( \text{MSE} = \text{mean square error} \), and
\( T = \text{number of years} \).

The F-test is statistically significant at the 0.01 level, indicating that the hypothesis of a common intercept is not accepted.

Because the variance terms \( (\sigma^2_e, \sigma^2_u) \) are not known, they must be estimated before the GLS transformation can be made. The variance of the general error can be estimated from the residuals of the dummy variable model, and the year error variance can be estimated from a regression of the mean output in each year on the mean land and nonland input levels. The estimation and transformation procedure suggested by Judge et al. requires the same number of farm size classes in each year, but the published IFBA data had four farm size classes before 1964 and five after that date. A modified procedure for handling the differing number of classes is outlined in Appendix B.

The variables used in the estimations were: \( Y \), the real gross income measured in 1969 dollars; \( X \), all expenses except for interest and property taxes measured in 1969 dollars; and \( L \), farm area measured in acres. The units for output and nonland input where chosen to allow aggregation. A central Iowa farm may produce corn, soybeans, hogs, and many other products; no one physical unit is suitable for this diverse collection of outputs. The gross income did not include income from off-farm employment, nonfarm investment, or sales of equipment and real estate. The convention of measuring farm income after deducting purchased feed and livestock was followed. It is argued that this yields
a better estimate of actual farm production because the purchased feed and livestock were not produced on the farm in question (Lee et al., 1980, p. 154).

In addition to the operating expenses, such as fuel, fertilizer, and hired labor, the nonland input category includes the imputed value of family and operator labor, all depreciation, building repairs, and insurance. The family and operator labor is taken from estimates in the published summaries that are based on the wages for hired farm labor. With this specification, the objective function is the returns to capital and management. The problem of insuring some income for family consumption is thereby reduced because some family income is always included in the nonland input. The model does not require that operator and family labor be fully employed. It is assumed that off-farm employment is available for operator and family labor not used on the farm, at least to the extent that minimum consumption requirements are met.

For the purposes of the study, building and other real estate improvements were taken to be nonland inputs. With land measured in acres, the impact of building and other improvements would not be reflected in the estimation if real estate depreciation and repairs were excluded from the nonland input category. Property taxes are the only cost strictly attributable to the land and they do not contribute directly to production, so it is reasonable to exclude them from the measure of productive inputs. This definition of inputs also provides additional flexibility in the decision environment. There may be cases in which large capital gains occur and it is optimal to neglect building
repair and replacement to invest in more land. It should be noted that improvements affect the price of a particular piece of property, but general capital gains on U.S. farmland cannot be attributed to improvements. Major farmland price formation hypotheses suggest that capital gains and losses depend on technological change which makes more farm income attributable to land and less to labor (Melichar, 1979), or the interaction of inflation and tax rules (Feldstein, 1980). Consistent with this observation, it is assumed that farmland capital gains are independent of the real estate improvement repair and maintenance decision.

The estimated equation for the 1957-1969 data was:

\[
(10.1) \quad Y_{it} = -3.1995 + 1.0312 \cdot 1nX_{it} + 0.7734 \cdot 1nL_{it} + 0.2924 \cdot 1nL_{it}
\]

\[
(1.5986) (0.4088) (0.0595) (0.0408)
\]

where: \( Y_{it} \) = gross income in 1969 dollars,
\[ t = 57, 58 \ldots 69, \]
\[ i = 1 \ldots 4 \text{ for } 1957-1963 \text{ and } 1 \ldots 5 \text{ for } 1964-1969 \]
\[ X_{it} \] = nonland inputs in 1969 dollars, and
\[ L_{it} \] = land in acres.

Standard errors are listed under the coefficient estimates in equation (10.1). All the coefficient estimates are significantly different from zero at the 0.05 level. The overall F-test is significant at the 0.01 level with \( F = 53,871 \). The coefficient estimates are plausible. The trend coefficient is positive, consistent with observed changes in productivity. If the intercept and the trend term are considered to be
parts of a year specific intercept, the model can be seen as a two input power function in which the sum of the input coefficients indicates the returns to scale. The estimated coefficients sum to 1.07, indicating a slight degree of increasing returns to scale. This is reasonable for the range of farm sizes considered. Economies of size research suggests the cost curve continues to decline over a wide range of farm sizes (Miller, Rodewald, and McElroy, 1981; Madden, 1967). Decreasing returns to scale may set in for very large farms, but no observations are available in the very large size range and it is unlikely that conditions on the very large farms are relevant to the representative farm modeled here. The increasing returns to scale do not create a problem for the solution algorithm because capital constraints will force a finite firm size, even if production relationships do not.

In the context of the theoretical model, sufficient conditions for optimality under increasing returns to scale can be found in the models with equity restrictions (Lowenberg-DeBoer 1985a, p. 13). When the negative debt cost terms in the bordered Hessian or matrix of second derivatives of the Hamiltonian outweigh the production function terms, which may be positive with increasing returns to scale, the matrix can be negative definite for the parameter combination considered. Explosive solutions can occur in the theoretical model with increasing returns to scale, but they do not necessarily occur.

The dummy variable and error component model estimates for the land and nonland input coefficients are almost identical, supporting the argument that the model is correctly specified. Using data for 1957—
1969, the dummy variable coefficient estimates were 0.2902 for land and 0.7761 for the nonland inputs for the dummy variable. Simple t-tests fail to reject the hypothesis that the estimates from the two models are equal. It is reasonable that at this level of aggregation the year and input correlation are negligible. The year effect may be correlated with use of some pesticide or other individual input, but when one aggregates nonland inputs across enterprises and subgroups of inputs (fertilizer, pesticides, equipment, etc.), then it is likely that under Iowa conditions the correlation of the year residual with the specific input is swamped by other variation.

**Time series forecasts**

Interest rates, input and output prices, indexes, and land prices were forecast with the Box-Jenkins methodology of identification, estimation, and diagnostic checking (Pankratz, 1983, p. 17). Sample autocorrelation functions were calculated for the time series to be forecast and compared to the theoretical autocorrelation functions for autoregressive and moving average models. Models that fit the patterns in the sample autocorrelation were estimated. Criteria for the "best" model were the minimum mean square error, plausibility of the forecast, and the randomness of the residuals. The residuals were checked visually by plotting them against time and against the original series, and with the Box-Pierce chi-square approximation (Malbert, 1975, p. 21).

Because most of the time series were nonstationary, differencing was used to induce stationarity. This is standard procedure in Box-Jenkins models (Pankratz, 1983, p. 157; Malbert, 1975, p. 12; Jenkins, 1979,
Explicit tests of the appropriateness of differencing, such as those suggested by Fuller (1976, pp. 366-382), were not employed because of their complexity and because the problems associated with inappropriate differencing were not encountered. For instance, inappropriate differencing may cause forecasts to wander off to plus or minus infinity.

The land price forecast was based on the USDA land price index for Iowa as reported in Agricultural Statistics. The rule used in constructing all the data series is that the most recently reported estimate is used. This assumes that the most recent revision contains the most up to date and accurate information. The USDA index was used in preference to the one estimated by Iowa State University economists, because the ISU series is too short for effective use of the Box-Jenkins methodology.

The estimated model for the 1912-1969 period is a first-order autoregressive in the first differences of the land price index:

\[ \text{LP}_t - \text{LP}_{t-1} = 0.8099 + 0.4505(\text{LP}_{t-1} - \text{LP}_{t-2}). \]

This model utilized a constant term, implying deterministic trend in land prices. As a general rule, Box-Jenkins models do not use deterministic trend unless there is some compelling reason to believe that there is some systematic change in the process (Pankratz, 1983, p. 189). A stochastic trend which follows recent disturbances in the data is preferred because it is more flexible and does not require the time series to follow exactly the same pattern that it has in the past. Pankratz suggests fitting models with and without a constant term in the
preliminary stages of estimation to check the possibility of deterministic trend. The constant term is used here because it substantially reduces the mean square error of the model compared to estimates with no constant, and because it generates a more plausible long-term forecast. The Iowa land price forecast rises steadily with increases of $6 to $8 annually during the early 1970s, dropping off to a regular $5 annual increase through the remainder of the period (Table 2). Given the continuous, modest rise in land price through the 1950s and 1960s, this is a more plausible forecast than the small initial increase and then no change prediction generated by models without deterministic trend. An argument for deterministic trend in land prices is that technological change is altering the factor shares in agriculture so that more income is attributable to land.

The forecasted prices are substantially below the observed land price increases of the 1970s (Table 3), but are consistent with the experience up to 1970. It should be noted that this forecast is based on all available data, including the land price declines of the 1920s and 1930s. If one thought that the structure of the land market had changed with the advent of government farm programs and that the only relevant data were after 1933 or after World War II, then the estimate would probably show much larger increases. A model based on the shorter data period may also overestimate the land prices for the 1980s because of the optimistic trend estimate based on the continuously rising prices of the data period. In real terms, the forecasted values appear close to the observed values for the 1980s. It is not clear which price index is most appropriate for deflating land prices, but if the observed land price for
Table 2. Price, interest rate, and tax expectations for 1969-1984 based on statistical forecasts

<table>
<thead>
<tr>
<th>Year</th>
<th>Discount rate</th>
<th>Real estate interest</th>
<th>Nonreal estate interest</th>
<th>Output price index</th>
<th>Input price index</th>
<th>Land price</th>
<th>Property tax per acre</th>
</tr>
</thead>
<tbody>
<tr>
<td>1969</td>
<td>0.070</td>
<td>0.058</td>
<td>0.078</td>
<td>1.000</td>
<td>1.000</td>
<td>382</td>
<td>6.3</td>
</tr>
<tr>
<td>1970</td>
<td>0.070</td>
<td>0.058</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>394</td>
<td>6.5</td>
</tr>
<tr>
<td>1971</td>
<td>0.070</td>
<td>0.059</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>402</td>
<td>6.8</td>
</tr>
<tr>
<td>1972</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>409</td>
<td>7.1</td>
</tr>
<tr>
<td>1973</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>415</td>
<td>7.3</td>
</tr>
<tr>
<td>1974</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>420</td>
<td>7.5</td>
</tr>
<tr>
<td>1975</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>425</td>
<td>7.8</td>
</tr>
<tr>
<td>1976</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>430</td>
<td>8.0</td>
</tr>
<tr>
<td>1977</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>435</td>
<td>8.3</td>
</tr>
<tr>
<td>1978</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>440</td>
<td>8.5</td>
</tr>
<tr>
<td>1979</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>445</td>
<td>8.7</td>
</tr>
<tr>
<td>1980</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>450</td>
<td>9.0</td>
</tr>
<tr>
<td>1981</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>456</td>
<td>9.2</td>
</tr>
<tr>
<td>1982</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>461</td>
<td>9.5</td>
</tr>
<tr>
<td>1983</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>466</td>
<td>9.7</td>
</tr>
<tr>
<td>1984</td>
<td>0.070</td>
<td>0.060</td>
<td>0.078</td>
<td>1.013</td>
<td>1.016</td>
<td>481</td>
<td>9.9</td>
</tr>
</tbody>
</table>

--nominal dollars--
Table 3. Observed price, interest rate, and tax values for 1969-1980

<table>
<thead>
<tr>
<th>Year</th>
<th>Discount rate^a</th>
<th>Real estate interest^b</th>
<th>Nonreal estate interest^b</th>
<th>Output price index^c</th>
<th>Input price index^c</th>
<th>Land price^d</th>
<th>Property tax per acre^d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1969</td>
<td>0.020</td>
<td>0.057</td>
<td>0.078</td>
<td>1.000</td>
<td>1.000</td>
<td>382</td>
<td>6.12</td>
</tr>
<tr>
<td>1970</td>
<td>-0.107</td>
<td>0.058</td>
<td>0.090</td>
<td>1.017</td>
<td>1.038</td>
<td>392</td>
<td>8.67</td>
</tr>
<tr>
<td>1971</td>
<td>0.198</td>
<td>0.060</td>
<td>0.073</td>
<td>1.051</td>
<td>1.096</td>
<td>392</td>
<td>8.91</td>
</tr>
<tr>
<td>1972</td>
<td>0.129</td>
<td>0.061</td>
<td>0.070</td>
<td>1.169</td>
<td>1.173</td>
<td>414</td>
<td>8.48</td>
</tr>
<tr>
<td>1973</td>
<td>0.011</td>
<td>0.063</td>
<td>0.081</td>
<td>1.661</td>
<td>1.404</td>
<td>466</td>
<td>8.60</td>
</tr>
<tr>
<td>1974</td>
<td>-0.176</td>
<td>0.066</td>
<td>0.094</td>
<td>1.780</td>
<td>1.596</td>
<td>597</td>
<td>8.59</td>
</tr>
<tr>
<td>1975</td>
<td>0.076</td>
<td>0.069</td>
<td>0.089</td>
<td>1.712</td>
<td>1.750</td>
<td>719</td>
<td>9.39</td>
</tr>
<tr>
<td>1976</td>
<td>0.206</td>
<td>0.072</td>
<td>0.082</td>
<td>1.729</td>
<td>1.865</td>
<td>920</td>
<td>9.78</td>
</tr>
<tr>
<td>1977</td>
<td>0.006</td>
<td>0.073</td>
<td>0.079</td>
<td>1.695</td>
<td>1.923</td>
<td>1259</td>
<td>11.05</td>
</tr>
<tr>
<td>1978</td>
<td>0.026</td>
<td>0.073</td>
<td>0.088</td>
<td>1.949</td>
<td>2.077</td>
<td>1331</td>
<td>11.00</td>
</tr>
<tr>
<td>1979</td>
<td>0.117</td>
<td>0.077</td>
<td>0.107</td>
<td>2.237</td>
<td>2.404</td>
<td>1550</td>
<td>11.19</td>
</tr>
<tr>
<td>1980</td>
<td>0.192</td>
<td>0.081</td>
<td>0.129</td>
<td>2.271</td>
<td>2.654</td>
<td>1811</td>
<td>12.35</td>
</tr>
<tr>
<td>1981</td>
<td>0.120</td>
<td>0.086</td>
<td>0.149</td>
<td>2.356</td>
<td>2.846</td>
<td>1941</td>
<td>12.31</td>
</tr>
<tr>
<td>1982</td>
<td>-0.011</td>
<td>0.096</td>
<td>0.143</td>
<td>2.254</td>
<td>2.865</td>
<td>1802</td>
<td>14.32</td>
</tr>
<tr>
<td>1983</td>
<td>0.362</td>
<td>0.099</td>
<td>0.119</td>
<td>2.271</td>
<td>2.942</td>
<td>1568</td>
<td>12.85</td>
</tr>
<tr>
<td>1984</td>
<td>0.041</td>
<td>0.099</td>
<td>0.119</td>
<td>2.407</td>
<td>2.981</td>
<td>1396</td>
<td>12.85</td>
</tr>
</tbody>
</table>

^aAfter tax return on common stock, calculated from dividend yields and annual percentage price changes on the Standard and Poor's composite of 500 stocks, Standard and Poor's Statistical Service, Security Price Index Record, 1984.


1984 is deflated by the observed index of prices paid by farmers for production items, the real value is $468 (constant 1969 dollars), and if the 1984 forecast land price is deflated by the forecast prices paid index, then the real value is $465. The fact that the forecast misses the land price boom of the 1970s is not surprising because it was not generally anticipated by the experts of the time and it is unlikely that it would be anticipated by a simple, univariate model.

The forecast of prices paid by farmers was based on the USDA index of prices paid by farmers for all production commodities, including fertilizer, seed, and building and fencing material as reported in Agricultural Statistics. It excluded interest, taxes, and wages. For the aggregate nonland input defined for this study, an index including wages would have been better, but the available index including wages also had interest and taxes, which are explicitly handled elsewhere in the model. Iowa data were not used for the prices and interest rates because these variables are based on national markets, not local markets like land. Also, data series for Iowa prices and interest were either nonexistent or too short for effective use of the Box-Jenkins methodology. The estimated model is a first-order moving average on the first differences.

\[ \text{PAID}_t - \text{PAID}_{t-1} = -0.4588e_{t-1} \]

where: \( \text{PAID}_t \) = index of prices paid in year \( t \) (1969=100) and \( e_{t-1} \) = prediction error in period \( t-1 \).

The forecast shows a price increase to 1.0163 in 1970 and constant
thereafter (Table 2). Again, this forecast is substantially lower than
the observed values (Table 3), but consistent with the relatively low
inflation experience in the 1950s and 1960s. The index was either
dropping or constant through most of the 1950s and rising slowly in the
1960s. Model estimates with a constant term will generate continuously
rising prices, but that constant term is not statistically significant in
the models estimated and, more important, there is no reason to believe
that prices should be systematically rising.

With the nonland inputs defined in real terms, the prices paid index
plays the role of the current input cost \( (\gamma_1) \) in the theoretical model
(3). If prices are unchanged, the price of nonland inputs is one.

The forecast of prices received by farmers is based on the USDA
index of prices received for all farm products as reported in
Agricultural Statistics. Because central Iowa farmers produce grain, oil
seeds, livestock, and other products, no one commodity group index was
appropriate for measuring changes in prices received. The estimated
model was a first-order moving average on the first differences:

\[
\text{RECEIVED}_t - \text{RECEIVED}_{t-1} = -0.4297e_{t-1}
\]

where: \( \text{RECEIVED}_t \) = index of prices received in period \( t \) and
\( e_t \) = prediction error in period \( t-1 \).

The forecasts show a modest rise in 1970 to 1.0130 and constant
thereafter (Table 2). The observed index shows a much larger rise
(Table 3), but the forecast is plausible given the roughly constant index
of the 1950s and 1960s. The 1969 index value is only slightly above the 1950 value, and it is below the Korean War commodity boom values in 1951 and 1952.

The average interest rate on outstanding real estate debt was used to forecast real estate debt cost. The 1910-1969 data was found in the Ag Finance Databook. Rates on outstanding debt were used in preference to rates on new debt because of the longer data series available and because they capture some of the effect of the relatively long maturities of real estate credit. The model does not explicitly include debt maturity, but it is unrealistic to base decisions on the expected cost of new debt when the debt cost of the average farmer during the period was based on the interest rates from 5, 10, or 15 years ago because of long-term fixed interest loans. Currently, the difference between the average interest rate on outstanding debt and on new debt is decreasing because of the widespread use of variable interest loans, but there is no reason to think that the average farmer in 1970 could anticipate that change.

The estimated model is a second order autoregressive on the first differences:

\[ R_I_t - R_I_{t-1} = 0.5286 (R_I_{t-1} - R_I_{t-2}) + 0.1702 (R_I_{t-2} - R_I_{t-3}) \]

where: \( R_I_t \) = real estate interest

The forecasts show the real estate interest rate rising to 5.8 percent in 1970, 5.9 in 1971 and 6.0 for 1972 and later (Table 2). Observed interest rates rose substantially more in the 1970s and 1980s, reaching
almost 10 percent on outstanding real estate debt in 1983 (Table 3). During the 1950s and 1960s interest rates rose modestly from 4.5 percent in 1950 to 5.7 percent in 1969. Models with constant terms can generate forecasts of continuously rising rates that more closely mimic the observed values from the 1970s and 1980s, but the constant term is statistically nonsignificant and the mean square error rises compared to models without the constant. Though interest rates had risen steadily in the decades immediately preceding 1969, there was no compelling reason to believe that they would continue to rise systematically in the 1970s and 1980s.

Average interest rates on outstanding Production Credit Association debt were used in forecasting the nonreal estate debt cost. The PCA series was used because that lender was an increasingly important source of nonreal estate credit through the period being considered and because the data series is slightly longer than the commercial bank series (Melichar, 1979, p. 28). With only 29 observations the PCA series is almost too short for the application of the Box and Jenkins methodology. Longer series exist (USDA, 1957, p. 30), but they are plagued by unreliable data and changing definitions. The random walk model:

\[
NRI_t - NRI_{t-1} = e_t
\]

where: \( NRI_t \) = the nonreal estate interest rate in period t.

was used because the sample autocorrelation and partial autocorrelation functions showed no statistically significant estimates. Also, in simple
moving average and autoregressive models estimated on the data, the
coefficient estimates were invariably not significant. Plotting the,
first difference of the series and the Box-Pierce statistic suggest that
the hypothesis that the first differences are random can not be rejected.
PCA loans are generally short term, often a year or less, so it is not
surprising that in annual data lagged effects are weak. With a random
walk model the best estimate of the next period price is this period's
price. Hence, the forecast is constant at the 1969 value, 7.8 (Table 2).
The forecast is lower and more regular than the observed values
(Table 3), but there is no information in the 1940-1969 PCA series or
other statistical sources which would cause one to anticipate the high
rates of the late 1970s and early 1980s. In a market which is character­
ized by efficient information flow and rapid adjustment, the forecast of
"more of the same" is reasonable in the absence of knowledge concerning
factors which will shock the market.

The discount rate was based on return to common stocks as measured
by the dividend yield and annual percentage price change for Standard and
Poor's composite of 500 stocks. For farm firms organized as sole propri­
etorships, the most straight forward discount rate is the after-tax
return to nonfarm investments of comparable risk (Alpin, Casler and
Francis, 1977, p. 50; Lee et al., 1980, p. 77). For comparability of
risk equity investments are more suitable than the fixed and relatively
sure return of debt securities. Common stock is the most widely held
equity investment in the U.S. and its returns are well-documented. The
argument for using the common stock return does not require that the farm
motivated people in the financial industry. In the absence of an appropriate model, the mean value 4.4 percent was used as the best available forecast. Though the stock price level has risen substantially over the data period, the annual percentage change shows no apparent trends. This forecast misses the cyclical variation in returns that continue in the 1970s and 1980s, but it provides information about the expected long term gains that could be expected in the stock market.

The discount rate should be measured net of taxes. Little empirical information exists on average farm tax rates. The Internal Revenue Service Data (IRS) studied by Sisson is of little use to this study because it is difficult to identify farmers from tax data alone. The taxpayer with farm income may be a commercial farmer, a nonfarm investor in agricultural production; or an urban worker with a rural residence that produces some farm products. The best available data appears to be a survey of Indiana farmers conducted in 1980. The data relate to tax returns filed in 1978 and 1979. The survey found a mean tax rate of about 12 percent (Baker and Lapp, 1981, p. 16) and showed an average tax rate that is roughly constant over all farm sales classes. This supports the argument for using a constant average tax rate though the farm business size may vary substantially over the planning horizon. The 12 percent mean tax rate includes income and social security self-employment taxes. The rate is after deductions and exemptions. Because Iowa and Indiana farmers engage in the same enterprises and operate under similar conditions, the average tax rate among central Iowa farmers is likely to be similar to that used in Indiana. State income taxes were not included.
Under the law in effect in 1969, the capital gains deduction was 50 percent. This was changed in October, 1978 to 60 percent (Harl, 1983, Section 27.06), but here is no reason to believe that the average farmer could anticipate that change. Hence, the capital gain deduction for the farmland sold at the end of the planning horizon is specified at 50 percent throughout the planning horizon for the forecast solutions.

With a 12 percent average tax rate, the after-tax dividend returns is 2.82 percent. Assuming a 50 percent capital gain deduction and annual realization of capital gain, the after-tax capital gain rate on common stock was 4.14 percent. The after-tax discount rate for the forecast model was 7.0 percent.

The remaining parameters for the forecast solutions are the capital absorbed by nonland inputs \( (a_t) \), the maximum leverage \( (\Lambda) \) and the maximum percentage of real estate capital that may be borrowed. In the central Iowa IFBA data, there are about $2.14 invested in inventories of feed, stored grain, livestock equipment and miscellaneous supplies for every dollar of the nonland input. For example, if expenditures on crop and livestock inputs, labor, insurance and other nonland costs is about $20,000 then the incentives carried by that farm in utilizing those inputs will be around $42,800. The proportion is roughly constant over the data period. For example, the proportion is 2.14 for the 1957-1969 period and 2.10 for the entire data set, 1957-1983. The IFBA data are based on year-end inventories so it does not include items that are purchased and used up during the year, such as seed, fertilizer, and fuel. This working capital requirement was estimated at 50 percent of
the nominal value of nonland inputs, based on a six month production cycle. Hence, the capital absorbed by nonland inputs in the forecast solutions is estimated as 2.64 multiplied by the prices paid index, because the investment was calculated on a nominal basis.

Solutions were calculated with several leverage levels to test the sensitivity of the model to the availability of debt. Results with leverage ratios of 0.2 and 0.5 are reported in detail. These modest leverage levels are consistent with the general conservative financial behavior of U.S. farmers. The 0.2 leverage level is close to the observed average level of the 1960s and 1970s. The 0.5 leverage is representative of the greater debt us of some farmers observed in the 1970s. Modest leverage reduced problem solving costs because it constrains firm growth and shrinks the number of potential paths over which the algorithm must enumerate.

Property taxes were forecast based on a linear trend model estimated from the IFBA data:

\[ \text{PT}_t = -10.25 + 0.2404t \]

where: \( \text{PT}_t \) = property tax per acre.

The Box-Jenkins methodology was not appropriate for the short time series and cross sectional nature of the IFBA data. Because property taxes tend to be a localized phenomena, longer state and national series were not used. The forecast shows property taxes rising at about $0.20 to 0.30 per acre annually.
The maximum percentage of real estate capital that can be borrowed was set at 50 percent of market value in the base solutions. This level is consistent with the 65 percent of normal value that the Federal Land Bank could lend in 1969. Normal value was based on a standard set of input and output prices. It was often below market values because of lags in adjusting the reference set of normal prices for inflation. In the early 1970s the Federal Land Bank statutory maximum was changed to 85 percent of market value, but the average farmer in 1969 would probably not have anticipated that change.

For the perfect foresight solutions, the observed values of the time series variables and other parameters were used when available. The most recent values of the debt cost variables were not available when the research was being conducted. The 1983 value for PCA interest was taken to be the average rate on new loans (Amols and Kaiser, 1984, p. 53) and the 1984 rates for both debt types were assumed to be constant from 1983. The simple algorithm used in their exploratory research did not allow a changing tax rate or maximum real estate debt. Hence, the 12 percent average tax rate was used for both expectation assumptions and the 50 percent maximum real estate debt was maintained.

The production function for the perfect foresight solution was reestimated following the same procedure as was used for the forecast solution, except that all available data, 1957-1983 was used. Presumably, in the perfect foresight case there is knowledge of how productivity will change during the planning horizon. The estimated equation with standard errors listed under the coefficient estimates is:
\[ \ln Y_{it} = -2.4582 + 0.6875 \ln t + 0.2257 \ln X + 0.881 \ln L \]
\[ (0.7453) (0.1919) (0.0375) (0.0548) \]

All the coefficient estimates are statistically significant at the 0.01 level. The overall F test is significant at the 0.01 level with F=76,109. Consistent with the observation of slower technological change during the 1970s, the trend coefficient is smaller than in the earlier estimate. The land and nonland input coefficients are similar to the earlier estimates, though the land coefficient is slightly smaller and the nonland input coefficient is larger.

Numerical Results

The model was solved for the forecast and perfect foresight scenarios assuming an initial equity of $120,000, approximately the average equity of an Iowa farmer in 1970 if all unrealized capital gain is considered an addition to equity (USDA, 1978, p. 27). This initial equity level assumes that the decision maker can cash in all assets at market value and make a fresh allocation decision. In the first period of the basic models, no initial land holding is assumed. Solutions calculated for variations on the basic scenarios, such as higher leverage or alternative increment values, are started at the lowest initial land purchase identified for the basic model of that expectation scenario. For instance, if the smallest first year farm size for the scenario is 240 acres, then variations of the scenario use an initial farm size of 240 acres. This reduces problem solving cost for sensitivity testing and
is unlikely to affect the solutions, given the continuous functions employed and the importance of the capital constraint. For instance, with increasing returns to scale it is likely that if any production is profitable all available capital will be used. Increasing capital availability by allowing higher leverage ratios is unlikely to reduce the initial acreage chosen. The choice in the higher leverage model is between the optimal path identified for the lower leverage case and those additional paths that are made possible by greater debt use. The impact of assuming an initial landholding in the sensitivity testing was spot checked by solving a few cases for the whole range of land values; in no case did this solution differ from that calculated for the more restricted land purchase assumption.

The 15-year planning horizon was divided into three 5-year periods (n=5). The increment values were set at: land, 80 acres; annual saving, $8,000; and capital gains, $10,000 for the forecast scenario and $100,000 for higher capital gain, land price bubble and the perfect foresight scenarios. The choice of increment size was made to balance the need for small increments to aid in finding accurate solutions and larger increments to reduce problem solving cost. It must be emphasized that the numerical magnitudes in the solutions are conditional on the increment values. For instance, allowing 40 acre land increments may speed growth and increase the present value of returns if acquiring capital for the larger land unit is a major impediment to expansion. Smaller increments would be preferred for savings and capital gains variables, because they are essentially continuous in actual farm
operations. Lumpiness in land sales is realistic, though the size of the tracts may differ from area to area. Eighty acre tracts are not unreasonable for central Iowa conditions. In the limited sensitivity testing done increment sizes did not change the general impact of land prices on the solutions.

The range of state variable values over which the solutions were sought was set to allow all feasible paths. Because the capital constraint dominated the solutions, this meant in practice input combinations that could be financed and savings levels that could be achieved with inputs at the maximum level that can be financed. The capital gains range was determined by the minimum and maximum landholding. If a solution specified the extreme value of a state variable range as optimal, the model was rerun with a broader range. The adequacy of the ranges was spotchecked by rerunning a few models with broadened ranges; in no case did this result in a different solution when the previous solution was not at an extreme value of a range.

Land purchases were limited to a maximum of 400 acres per 5-year period. This is a relatively low level of land availability. In the traditional 36 section township, this implies that the decision maker places successful bids on only 2 percent of the land every 5-year period. Usually about 3 to 4 percent of U.S. farmland changes hands in any given year. Many farmers would be willing to search for land over a wider area than a single township. At maximum leverage ratios of 0.5 and less, however, the land availability is not a binding constraint; more land could not be financed even if it was available.
Solutions were calculated with the proportions of unrealized capital gain or loss that could be substituted for current income or equity set at zero and one ($\phi=1, 0; \omega=1, 0$). These values were chosen to test the sensitivity of the solutions to extreme values of the substitutability parameters. To facilitate the discussion the substitutability parameter combinations will be referred to as:

1. the no flow value case when both substitutability parameters are set to zero ($\phi=0, \omega=0$) and capital gain or loss is recognized only at the end of the horizon;
2. the income substitution case when capital gains and loss are substitutable for income alone ($\phi=1, \omega=0$);
3. the equity substitution case when capital gains and losses are substitutable for equity alone ($\phi=0, \omega=1$); and
4. the maximum flow value case when both substitutability parameters are set to one ($\phi=1, \omega=1$).

**Forecast scenario**

When expectations are based on statistical forecasts the optimal solution for 1970 is to purchase 240 acres of land and use $19,500 on non-land inputs (Table 4). The non-land input per acre is $81. This initial period choice is the same for all values of the unrealized capital gain or loss substitutability parameters. Debt use is set at $24,000, the maximum allowable with a 0.2 leverage limit. All the solutions discussed here use the maximum available debt given the effective leverage. This is reasonable given the increasing returns to scale and the generally low, after-tax cost of debt when compared to the
Table 4. Land purchases, saving, nonland inputs, and debt use in the forecast and higher capital gain scenarios, solutions using a maximum leverage of 0.2

<table>
<thead>
<tr>
<th>Period</th>
<th>$\phi^c$</th>
<th>$\omega^d$</th>
<th>Land purchases</th>
<th>Annual saving</th>
<th>Total$^a$</th>
<th>Nonland$^b,a$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Forecast scenario:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970-1974</td>
<td>0</td>
<td>0</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td>1975-1979</td>
<td>0</td>
<td>0</td>
<td>80</td>
<td>16,000</td>
<td>32,000</td>
<td>25,200</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>80</td>
<td>16,000</td>
<td>32,000</td>
<td>25,200</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>80</td>
<td>16,000</td>
<td>34,000</td>
<td>25,900</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>80</td>
<td>16,000</td>
<td>34,000</td>
<td>25,900</td>
</tr>
<tr>
<td>1980-1984</td>
<td>0</td>
<td>0</td>
<td>160</td>
<td>40,000</td>
<td>48,000</td>
<td>35,200</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>160</td>
<td>40,000</td>
<td>48,000</td>
<td>35,200</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>160</td>
<td>40,000</td>
<td>52,000</td>
<td>36,700</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>160</td>
<td>40,000</td>
<td>52,000</td>
<td>36,700</td>
</tr>
<tr>
<td><strong>Higher capital gain scenario:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970-1974</td>
<td>0</td>
<td>0</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>320</td>
<td>8,000</td>
<td>24,000</td>
<td>8,100</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>320</td>
<td>8,000</td>
<td>24,000</td>
<td>8,100</td>
</tr>
<tr>
<td>1975-1979</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>16,000</td>
<td>32,000</td>
<td>40,700</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>16,000</td>
<td>32,000</td>
<td>18,000</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>80</td>
<td>16,000</td>
<td>52,000</td>
<td>25,400</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>16,000</td>
<td>52,000</td>
<td>25,400</td>
</tr>
<tr>
<td>1980-1984</td>
<td>0</td>
<td>0</td>
<td>80</td>
<td>32,000</td>
<td>48,000</td>
<td>61,200</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>80</td>
<td>32,000</td>
<td>48,000</td>
<td>31,000</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>80</td>
<td>40,000</td>
<td>88,000</td>
<td>45,900</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>80</td>
<td>40,000</td>
<td>88,000</td>
<td>45,900</td>
</tr>
</tbody>
</table>

$^a$During the first year of the period.

$^b$Nonland inputs are measured in real 1969 dollars. Savings and debt are in nominal dollars.

$^c$The proportion of unrealized capital gain or loss that is substitutable for current income.

$^d$The proportion of unrealized capital gain or loss that is substitutable for equity in the financial negotiation.
discount rate. Annual savings are set at $8000 for the first period, the maximum earnings that can be retained given the constraint that the present value of current income in the period must be nonnegative.

The 1970 solution is close to the observed average values for Iowa. In 1970 the average Iowa farm was 237 acres (Iowa Crop and Livestock Reporting Service, 1982, p. 4). This average includes tenant and part owner farms. Full owner operations probably had smaller acreages. The 1969 Census of Agriculture indicates an average farm size of 196 acres for full owners with gross sales over $2,500. In the 1970 IFBA data, the average nonland input use for farms in the 180-259 acre category was $18,800 when deflated by the index of prices paid for all production inputs. An estimate of farm retained earnings for Iowa is not available, but a rough idea of the savings can be reached by subtracting the average capital gain per farm from the average change in equity (USDA, 1978, p. 27). The residual change in equity varies widely between years, but the average value for the five year period 1970-1974 is $10,174. In this context, the $8,000 annual savings appears reasonable. The use of $24,000 of debt in the optimal solutions is largely a result of the maximum leverage assumption and is not useful for model validation. It should be noted, however, that the model correctly identifies moderate debt use as a wealth increasing strategy.

In general, the 1970 values in the solution for the expectations based on forecast values appear to fit the observed mean values. The optimal control paths for the second and third periods are interesting from a planning perspective, but they do not offer much information for
validating the model. For example, the observed values for 1975 should not be compared to the second year solution of this scenario, but to another solution with its initial year in 1975 and based on forecasts utilizing information through 1974.

The optimal solution shows the farm firm planning to expand its land base through the second and third periods. Eighty acres are to be purchased in 1975 and 160 acres in 1980 for all substitutability parameter values. As farm size is enlarged the use of nonland inputs increases and the savings capacity is greater. Debt use follows the maximum amount allowed given the equity level and the substitutability of unrealized capital gain for equity. Debt use for models in which some unrealized gain is substitutable for equity exceeds debt use for other models slightly. This additional credit is used to purchase a higher level of nonland inputs. Because of the lumpiness of land, the modest amount of additional credit derived from borrowing against unrealized capital gain is not enough to finance more land. The expected wealth differs only slightly depending on the unrealized gain and loss substitutability parameters. In the no flow value case, the expected wealth is $216,600. In the equity substitution case the wealth is only slightly higher: $218,200. In the income substitution and maximum flow value case, the expected wealth is somewhat greater: 226,900 and 228,500, respectively.

Solutions which assume the land price is constant at the 1969 level are close to those calculated with forecast land prices. With a maximum leverage of 0.2, the land purchases in the constant land price scenario are: 1970, 240 acres; 1975, 80 acres and 1980, 240 acres. With constant
prices, a larger acreage can be financed with a given amount of equity, so a larger purchase is planned for 1980, than in solutions with the land price increasing at forecasted rates. Nonland input levels exceed those in the forecast value solutions for period two and three slightly. In 1975, the nonland input amounts is: $26,000; compared to $25,200 in the no flow value and income substitution cases and $25,900 in the solutions that permit borrowing against unrealized capital gains. Forecast and constant land price solutions both have 320 acres in the second period. In the third period the constant land price solution shows $45,000 of nonland inputs on 560 acres, or $80 per acre. The forecast solution indicates a nonland input level on 480 acres of $35,200 for the no flow value and income substitution cases and $36,700 for the equity substitution and maximum flow value cases. On a per acre basis the forecast solution nonland inputs are $73 and $76 respectively. Slightly higher nonland input levels are possible because less capital is tied up in land. The higher nonland input level in turn generates a larger current income than in the forecast value solutions and thus greater savings capacity. The annual average savings is set at: 1970-1974, $8,000; 1975-1979, $24,000; and 1980-1984, $56,000, compared to the forecast solution savings of: 1970, $8,000; 1975, $16,000 and 1979, $40,000.

Debt use in the final period of the constant land price plan exceeds that of the forecast value solutions because greater savings has increased equity and expanded borrowing capacity. The constant land price solution shows $56,000 of debt in 1980 compared to forecast scenario solution values of $48,000 for the no flow value and income
substitution cases and $52,000 for the maximum flow value and equity substitution cases. The expected wealth is $226,200, which is higher than wealth in the forecast solution when no unrealized capital gain is substitutable for income. This suggests that even when borrowing against capital gain is permitted, modest land price increases do not necessarily increase wealth. Problems in financing land at higher prices may more than offset the capital gains.

**Higher capital gains scenario**

When solutions are calculated with expectations based on statistical forecasts, except that the land price increase is multiplied by 10, the larger capital gains provide incentive to purchase land earlier and reduce initial nonland input levels. In the basic scenario, the optimal first period choice when capital gain is substitutable for current income is 320 acres of land and $8,100 of nonland inputs (Table 4). The nonland input use per acre is $250. This is a larger farm acreage and lower nonland input level than is found in solutions with the land price increasing as forecast. The smaller farm size and greater nonland input in solutions which do not allow substitution of unrealized capital gain for income is largely the result of the capital gain increment ($100,000). If the no flow value and equity substitution cases are solved with a $30,000 capital gain increment, the solutions are closer to those identified for the income substitution and maximum flow value case. The first period solutions are identical with 320 acres purchased, $8,100 of nonland input and $8,000 annual saving.
In the no flow value and equity substitution cases, the size of the capital gain increment can have impact on the solution. Solutions with a large capital gains increment may differ from solution for exactly the same input data except that the capital gain increment is smaller. In these cases, the value of the capital gain that is recognized depends on rounding errors in the system. In the first period, the amount of unrealized capital gains is $91,200 on a 240 acre farm and $121,600 on a 320 acre unit. With a $100,000 capital gain increment, the amount of capital gain that is recognized in the next period is the same for both farm sizes. Thus, there is little incentive to buy the additional 80 acres and reduce current income by cutting nonland inputs. When it is possible to borrow against unrealized capital gains, the land price increase enlarges the credit available with the period, but this is not a large enough incentive to justify purchasing an extra unit of land. In contrast, the capital gain increment has little effect on the solution when unrealized capital gains are 100 percent substitutable for current income. The initial land purchase is 320 acres for both $100,000 and $30,000 capital gain increment sizes in the income substitution and maximum flow value cases. All the unrealized capital gains are recognized within the period in the income substitution and maximum flow value cases. The only capital gain effect outside the period is the tax in the terminal value.

At first glance, the realism of an $8,100 nonland input level on 320 acres can be questioned. It is far below the $24,900 average level observed for the IFBA 260-359 acre category in 1970. It must however, be
noted that in the short term nonland input use can vary widely from farm to farm. Regular repair and maintenance of equipment and facilities can be delayed. Fertilizer applications may be reduced, especially maintenance applications that may have only a small impact on current yield, but are intended to maintain soil fertility. Maximum participation in government acreage set aside programs may substantially cut input use. In an environment of lumpy land purchases a decision maker may be content in the short run with less timely field operations, lower yields and less current income if acquisition of another tract is possible. It would be possible to place a lower limit on the amount of nonland input that must be used, but there is little information in the IFBA data or elsewhere that would indicate this minimum level.

A comparison of initial period choices in the forecast scenario and higher capital gain solutions suggests that the modest capital gains that might have been expected in the 1970s based on previous experience may not have had a major impact on production and finance choices. But when the land price boom of the mid-1970s occurred and capital gain expectations were revised upward, the parameter values indicate that incentives existed for change in acreage and nonland input use.

In the second and third periods of the higher capital gain scenario solutions land purchases tend to be smaller than when forecast land price values are used. For the no flow values, income substitution and maximum flow value cases no land is purchased in the second period and only 80 acres are acquired in the third period. This can be compared to the 80 acre purchased in the second period and 160 acres acquired in the third
period in the forecast scenario solutions. For the equity substitution case, the second period purchase is the same for both forecast and higher capital gain scenarios, but the third period purchased in the higher capital gain scenario is 80 acres less than the 160 acres acquired in the forecast scenario solution. The larger first period purchases strain available equity, and with higher land prices more capital is required to purchase a given amount of land. The 80 acre second period purchase in the equity substitution case occurs because the first period purchase is not increased to 320 acres (Table 4). In solutions with a $30,000 capital gain increment, this second period purchase does not occur for the equity substitution case.

The increased use of nonland inputs for some parameter combinations in the second and third periods of the higher capital gains scenario solutions suggests that at low leverage levels the capital gains effects are quickly disipated. For instance, in the forecast scenario solutions the nonland input for the no flow value case is $25,200 in the second period and $35,200 in the third period or $79 and $73 per acre respectively. In the higher capital gain scenario, the nonland input levels are $40,700 and $61,200 for the second and third period respectively. In per acre terms, this is $170 in the second period and $191 in the third period. When borrowing against unrealized capital gains is allowed, the second period nonland input level is about the same in both the forecast and higher capital gain scenarios, but the third period nonland input level is $11,200 higher in the higher capital gain scenario than in the forecast land price situation. This is $39 per acre higher in the higher
capital gain case than in the forecast value scenarios. Given the capital constraint and the high cost of owning land, nonland inputs become a better investment. The relatively low nonland input use in the income substitution case, $56 per acre in the second period and $78 per acre in the third period, can be traced to the financial strain created by a larger first period land purchase and the inability to borrow against accrued capital gains to finance nonland inputs. The total capital available in this case is the same as for the no flow value case, $192,000 in 1975 and $288,000 in 1979 but the farm acreage is 80 acres larger throughout the horizon. The ability to recognize unrealized capital gain as income creates the incentive to own more land, without generating the financial means to maintain nonland input levels.

Savings in the higher capital gain scenario solutions are almost equal to those in the forecast value scenario. Annual retained earnings are $8,000 in the first period and $16,000 in the second period for all substitutability parameter values in both the forecast and higher capital gain scenario solutions. In the third period, annual saving is $40,000 for all substitutability parameter values in the forecast value scenario and for the equity substitution and maximum flow value cases in the higher capital gain scenario. The lower, $32,000 savings in the third period in the no flow value and income substitution cases of the higher capital gain scenario can be linked to the land purchase choice. In the no flow value case, the 320 acre land holding is substantially below the levels identified for the forecast scenario and the nonland input amount in 1980 is close to the optimum identified by calculus agreements. In
most solutions, the optimal nonland input level is set at the amount that can be financed, choice (4) in the algorithm, and this nonland input level is far below the optimum amount identified by calculus arguments. With less land, more nonland inputs can be financed with existing equity and there is less incentive to save to increase nonland inputs. In the income substitution case, the lower savings level occurs because the 400 acre farm size strains available financing and nonland input level is relatively lower. This reduces current income and savings capacity.

When unrealized capital gains are substitutable for equity, the use of debt is substantially increased in the second and third period of the higher capital gain scenario even with a maximum leverage ratio of 0.2. For the equity substitution and maximum flow value cases, debt in the second period is $34,000 in the forecast scenario solutions, but $52,000 in the higher capital gain scenario solutions. Similarly, the third period debt for those cases is $52,000 for the forecast scenario and $88,000 for the higher capital gain scenario. This increase is a result of being able to borrow against large accumulated capital gains. In the third period, debt use is almost twice as large in the equity substitution and maximum flow value cases as it is in the no flow value and income substitution cases.

The expected wealth is substantially increased by the higher land price increase, even if expansion in firm size is dampened. The expected wealth values are: no flow value case, $293,500; income substitution, $398,800; equity substitution, $300,400; and maximum flow value, $418,600. Even though the substitutability of unrealized capital gain
for equity can induce changes in the production and finance plans, the benefit in both forecast and higher capital gain scenarios is small when leverage is limited to 0.2. This can be seen in the higher capital gain scenario by noting that the difference between the expected wealth for the no flow value case and the equity substitution case is only $6,900. In contrast, the expected wealth for the income substitution and maximum flow value cases are $105,300 and $125,100 greater than the no flow value wealth, respectively. In the forecast scenario, the difference between the no flow value and equity substitution cases is only $1,600. The income substitution and maximum flow wealth figures are $10,300 and $11,900 higher than the no flow value wealth, respectively.

Higher Leverage Solutions

When a maximum leverage ratio of 0.5 is used in the forecast scenario, farm size, nonland input use and debt rise proportionately (Table 5). The initial land purchase is 320 acres. Nonland inputs are set at $21,500, which is close to the 1970 level in the IFBA data for 260-359 acre farms of $24,900. The first period annual saving is $16,000 and initial debt is $60,000. The 0.5 leverage ratio is twice the Iowa average leverage ratio in 1970 of 0.25. Thus, larger than average farm size and greater than average retained earnings are consistent with the larger amount of capital used in the firm. Choices in 1970 are the same for all values of the substitutability parameters.

The second and third period choices differ only slightly with the substitutability parameter values and these differences are primarily due
Table 5. Land purchases, saving, nonland inputs, and debt use in the forecast and higher capital gain scenario solutions using a maximum leverage of 0.5

<table>
<thead>
<tr>
<th>Period</th>
<th>φ&lt;sup&gt;c&lt;/sup&gt;</th>
<th>ω&lt;sup&gt;d&lt;/sup&gt;</th>
<th>Land purchases</th>
<th>Annual saving</th>
<th>Total&lt;sup&gt;a&lt;/sup&gt; debt</th>
<th>Nonland&lt;sup&gt;b,a&lt;/sup&gt; inputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forecast scenario:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970-1974</td>
<td>0 0</td>
<td>320</td>
<td>16,000</td>
<td>60,000</td>
<td>21,500</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0</td>
<td>320</td>
<td>16,000</td>
<td>60,000</td>
<td>21,500</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1</td>
<td>320</td>
<td>16,000</td>
<td>60,000</td>
<td>21,500</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 1</td>
<td>320</td>
<td>16,000</td>
<td>60,000</td>
<td>21,500</td>
<td></td>
</tr>
<tr>
<td>1975-1979</td>
<td>0 0</td>
<td>320</td>
<td>32,000</td>
<td>100,000</td>
<td>15,400</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0</td>
<td>240</td>
<td>32,000</td>
<td>100,000</td>
<td>27,900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1</td>
<td>320</td>
<td>32,000</td>
<td>105,000</td>
<td>17,200</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 1</td>
<td>240</td>
<td>32,000</td>
<td>105,000</td>
<td>29,700</td>
<td></td>
</tr>
<tr>
<td>1980-1984</td>
<td>0 0</td>
<td>320</td>
<td>96,000</td>
<td>180,000</td>
<td>53,200</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0</td>
<td>400</td>
<td>88,000</td>
<td>180,000</td>
<td>49,500</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1</td>
<td>400</td>
<td>96,000</td>
<td>195,000</td>
<td>45,500</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 1</td>
<td>400</td>
<td>96,000</td>
<td>195,000</td>
<td>53,200</td>
<td></td>
</tr>
<tr>
<td>Higher capital gain scenario:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970-1974</td>
<td>0 0</td>
<td>400</td>
<td>8,000</td>
<td>60,000</td>
<td>10,100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0</td>
<td>400</td>
<td>8,000</td>
<td>60,000</td>
<td>10,100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1</td>
<td>400</td>
<td>16,000</td>
<td>60,000</td>
<td>10,100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 1</td>
<td>400</td>
<td>16,000</td>
<td>60,000</td>
<td>10,100</td>
<td></td>
</tr>
<tr>
<td>1975-1979</td>
<td>0 0</td>
<td>0</td>
<td>32,000</td>
<td>80,000</td>
<td>50,400</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0</td>
<td>80</td>
<td>32,000</td>
<td>80,000</td>
<td>27,700</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1</td>
<td>240</td>
<td>40,000</td>
<td>200,000</td>
<td>41,900</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 1</td>
<td>240</td>
<td>40,000</td>
<td>200,000</td>
<td>41,900</td>
<td></td>
</tr>
<tr>
<td>1980-1984</td>
<td>0 0</td>
<td>240</td>
<td>64,000</td>
<td>160,000</td>
<td>49,300</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 0</td>
<td>160</td>
<td>64,000</td>
<td>160,000</td>
<td>49,300</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1</td>
<td>400</td>
<td>96,000</td>
<td>400,000</td>
<td>55,000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 1</td>
<td>320</td>
<td>104,000</td>
<td>400,000</td>
<td>85,200</td>
<td></td>
</tr>
</tbody>
</table>

<sup>a</sup>Debt and nonland inputs during the first year of the period.
<sup>b</sup>Nonland inputs are measured in real 1969 dollars. Savings and debt are in nominal dollars.
<sup>c</sup>The proportion of unrealized capital gain or loss that is substitutable for current income.
<sup>d</sup>The proportion of unrealized capital gain or loss that is substitutable for equity in the financial negotiation.
to the lumpiness of the capital gain increment. The differences in the later periods tend to disappear if a smaller increment is used. For example, the solutions with a $100,000 capital gain increment show a 320 acre land purchase in the second period in the no flow value and equity substitution cases, but a 240 acre acquisition for the income substitution and maximum flow value situations. If a $30,000 capital gain increment is used, 240 acres are purchased in the second period for solutions for all substitutability parameter values. Because of the higher leverage ratio the ability to borrow against unrealized capital gain increases debt use more than it did in the 0.2 leverage case, but the effect is still small. In the second period, the ability to borrow against unrealized capital gains results in a debt level for the equity substitution and maximum flow value cases that is $5,000 higher than debt use for the other two substitutability parameter combinations. In the third period, the substitution of unrealized capital gain for equity results in a debt level that is $15,000 higher.

The forecast scenario solutions with a leverage ratio of 0.5 are similar to a solution which assumes the land price is constant at the 1969 level, but uses forecast values for other variables. The land, savings, and debt use are identical to those identified for the model with both substitutability parameters set to one. Nonland input use is greater in the second and third periods because less capital is needed to finance land. Nonland input use in the constant land price solutions is $32,100 in the second period and $76,000 in the third period or $57 and $79 per acre respectively. This can be compared to the highest nonland
input use in the forecast value solution: $29,700 in the second period and $53,200 in the third period for the maximum flow value case. On a per acre basis this is $53 in the second period and $55 in the third period of the higher leverage forecast scenario solution.

In the higher capital gain scenario, the initial land purchase jumps to 400 acres in solutions for all substitutability parameter values when the maximum leverage ratio is 0.5. This is the maximum purchase under the 400 acre per period constraint and also the maximum acreage that can be financed. Even if the decisionmaker were willing to eliminate all nonland inputs and concentrate all efforts on earning capital gains, another unit of land could not be financed. As it is, the solution trades current income for capital gain potential to the maximum extent possible given the tract size. The strain of financing the larger land purchase shows up in the savings capacity. Because nonland inputs and current income are reduced when borrowing against unrealized capital gain is not allowed, the annual savings is cut to $8,000 annually for the no flow value and income substitution cases. When unrealized gains can be substituted for equity, the additional borrowing power allows current income to expand more rapidly within the period and thus permits the $16,000 annual saving of the 0.5 leverage forecast scenario to be retained.

In the second period, land purchases are reduced when borrowing against unrealized capital gain is not permitted in the higher capital gain scenario. In the maximum flow value and equity substitution cases 240 acres are acquired, but in the income substitution case only 80 acres
is purchased, and in the no flow case farm size is constant. This occurs because equity is smaller than at the same point in the 0.5 leverage forecast scenario solutions due to reduced saving, and at the same time capital requirements are greater because of higher land prices. Recognizing the large accumulated capital gains as equity permits land purchases comparable to those in the forecast scenario, but with greater nonland input levels. For both the equity substitution and maximum flow value cases, the nonland input level in the second period of the higher capital gains scenario solutions is $41,900 or $65 per acre. In the forecast scenario, the second period nonland input level is $17,200, or $27 per acre, for the equity substitution case and $29,700, or $53 per acre for the maximum flow value situation. This generates greater current income and savings capacity in the 0.5 leverage higher capital gain scenario cases which permit borrowing against unrealized capital gain than in the no flow value and income substitution cases. The higher nonland input level can be attributed to the lumpiness of real estate purchases. Capital gains generate more than enough borrowing capacity to maintain firm expansion plans, but not enough to finance another unit of land and accompanying nonland inputs.

In the third period, land purchases, and retained earnings are lower than in the forecast solutions when unrealized capital gain is not substitutable for equity. For example, in the no flow value case the third period control variables in the higher capital gain case are: land, 240 acres; annual savings, $64,000; and nonland inputs, $49,300 or $77 per acre. In the forecast value scenario, the third period control
variable values are: land, 320 acres; annual savings, $96,000; and nonland inputs, $53,000 or $55 per acre. Again, the higher land price and reduced savings dampen expansion when unrealized capital gain is not substitutable for equity. The higher nonland input on a per acre basis suggests that without the ability to borrow against unrealized capital gains the incentive to hold larger amounts of land is quickly exhausted.

When borrowing against equity is permitted, the third period land purchases and retained earnings are similar to those in the forecast solutions. Nonland input levels are somewhat higher. In the equity substitution case, 400 acres of land are purchased and annual saving is $96,000 for both the forecast and higher capital gain scenarios in the third period. In that case, nonland inputs are $55,000, or $53 per acre, in the third period of the higher capital gain scenario, but $9,500, or $9 per acre less in the same period of the forecast scenario solution. In the maximum flow value case, the land purchase in the third period is 400 acres in the forecast scenario and 320 acres in the higher capital gain scenario. Third period savings for the maximum flow value case are $96,000 in the forecast scenario and $104,000 in the higher capital gain situation. The nonland input level in the third period for the maximum flow value case is $53,200, or $55 per acre for the forecast scenario and $85,000, or $89 per acre for the higher capital gain situation. The ability to substitute unrealized capital gains permits farm firm expansion to continue unabated in the higher leverage case.

In the second and third periods, the level of debt use is 2.5 times greater when borrowing against unrealized capital gains is permitted than
in the income substitution and no flow value cases. In the second period of the higher capital gain scenario, debt is $80,000 when borrowing against unrealized capital gain is not allowed and $200,000 in the equity substitution and maximum flow value cases. In the third period, debt is $160,000 for the no flow value and income substitution case and $400,000 for the cases which allow unrealized capital gain to be substituted for equity. The larger debt load leaves the firm in a vulnerable position if unexpected land price declines occur. The decision which is optimal, given expectations of continued land price increases, may not even be feasible if the land price declines and erodes away that accumulated unrealized capital gain.

Expected wealth increases substantially in the higher capital gain solutions. In the no flow value case, the present value of income is $325,900 in the 0.5 leverage forecast solution, but $426,200 in the 0.5 leverage higher capital gain solution. In the income substitution case with 0.5 leverage, the present value is $344,900 in the forecast scenario, but $539,000 in the higher capital gain case. With higher leverage the increase in expected wealth due to borrowing against unrealized capital gains is greater than the increase from recognizing unrealized gain as current income. In the equity substitution case with 0.5 leverage, the present value is $334,900 forecast scenario but $577,200 in the higher capital gain case. In the maximum flow value 0.5 leverage case, the present value is $353,300 for the forecast scenario, but $732,500 in the higher capital gain scenario.
At the relatively low leverage levels of 0.2 and 0.5, some degree of internal capital rationing is probably involved. In deciding on the degree of internal rationing, it is plausible that the decisionmaker tries to assess the tradeoff between the higher return and the risk in using more debt. The same decisionmaker may use a low leverage level in one environment and a higher leverage if the returns to taking the added financial risk are sufficient. The numerical solutions suggest that the return to using higher leverage is substantially larger when capital gains are larger. This difference occurs for all parameter values, but especially when unrealized gain can be substituted for equity. When borrowing against unrealized capital gain is not permitted, the high land price dampens expansion plans, thus reducing the increase in expected wealth. In the forecast scenario and the equity substitution case, the difference in present values between the 0.2 and 0.5 leverage solutions is $108,000, but in the higher capital gain scenario it is $276,800. Similarly, in the maximum flow value case the difference is $124,800 for the forecast scenario, but $313,900 when the higher capital gain scenario is used. When capital gains are small, the higher return from additional debt use may not be adequate to compensate for the risk, but in a high capital gains environment the higher leverage may be seen as worthwhile. The ability to substitute unrealized capital gain for income or equity enhances returns in the high capital gains environment and increases the returns to higher leverage.

Several cases were solved for higher leverage specifications. The solutions suggest with higher leverage the land availability limit
becomes more important. For instance, even with a constant land price, the solution with other variables at forecast values show land purchases at the 400 acre limit in all three periods. With the land at maximum levels, the impact of greater capital gains on the control path is limited to the greater use of nonland inputs when borrowing against unrealized capital gains is allowed. These solutions indicate that limited land availability can eliminate the impact of capital gains on farm acreage. With greater land availability, the incentive to hold land could be expected to increase with higher leverage. As previously noted, the 400 acre per five-year period land availability constraint is probably a more stringent limit than is experienced by most Iowa farmers, though it may be realistic for areas in which land is closely held, such as some ethnic communities.

Solutions were also calculated for the maximum borrowing on real estate set at 65 and 85 percent of market value. The basic solutions assume that 50 percent of the market value of land may be borrowed. With the specified leverage levels and land constraints, the 65 and 85 percent solutions are identical to the 50 percent solutions. This occurs because the maximum leverage on the land availability becomes a binding constraint before the maximum real estate debt percentage.

**Perfect foresight scenario**

The initial input decision in the perfect foresight scenario solutions with a maximum leverage of 0.2 are virtually identical to those in solutions using forecast values (Table 6). Savings are higher in the perfect foresight solutions in the first period because current income is
Table 6. Land purchases, saving, nonland inputs, and debt use in the perfect foresight and land price bubble scenario solutions using a maximum leverage of 0.02

<table>
<thead>
<tr>
<th>Period</th>
<th>$c$</th>
<th>$d$</th>
<th>Land purchases</th>
<th>Annual saving</th>
<th>Total debt</th>
<th>Nonland inputs</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Perfect foresight scenario:</strong> --acres--</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970-1974</td>
<td>0</td>
<td>0</td>
<td>240</td>
<td>16,000</td>
<td>24,000</td>
<td>19,400</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>240</td>
<td>16,000</td>
<td>24,000</td>
<td>19,400</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>240</td>
<td>16,000</td>
<td>24,000</td>
<td>19,400</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>240</td>
<td>16,000</td>
<td>24,000</td>
<td>19,400</td>
</tr>
<tr>
<td>1975-1979</td>
<td>0</td>
<td>0</td>
<td>160</td>
<td>40,000</td>
<td>40,000</td>
<td>22,200</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>240</td>
<td>24,000</td>
<td>40,000</td>
<td>11,700</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>240</td>
<td>40,000</td>
<td>60,000</td>
<td>16,100</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>240</td>
<td>40,000</td>
<td>60,000</td>
<td>16,100</td>
</tr>
<tr>
<td>1980-1984</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>80,000</td>
<td>52,200</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>56,000</td>
<td>64,000</td>
<td>34,800</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>8,000</td>
<td>200,000</td>
<td>66,100</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>8,000</td>
<td>200,000</td>
<td>66,100</td>
</tr>
<tr>
<td><strong>Land price bubble scenario:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1970-1974</td>
<td>0</td>
<td>0</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>320</td>
<td>8,000</td>
<td>24,000</td>
<td>8,100</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>240</td>
<td>8,000</td>
<td>24,000</td>
<td>19,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>320</td>
<td>8,000</td>
<td>24,000</td>
<td>8,100</td>
</tr>
<tr>
<td>1975-1979</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8,000</td>
<td>32,000</td>
<td>40,700</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>16,000</td>
<td>32,000</td>
<td>18,000</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>52,000</td>
<td>48,100</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>16,000</td>
<td>52,000</td>
<td>25,400</td>
</tr>
<tr>
<td>1980-1984</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-8,000</td>
<td>40,000</td>
<td>73,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>24,000</td>
<td>48,000</td>
<td>61,200</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>8,000</td>
<td>72,000</td>
<td>70,500</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>24,000</td>
<td>88,000</td>
<td>76,100</td>
</tr>
</tbody>
</table>

*Debt and nonland inputs during the first year of the period.

*aNonland inputs are measured in real 1969 dollars. Savings and debt are in nominal dollars.

The proportion of unrealized capital gain or loss that is substitutable for current income.

The proportion of unrealized capital gain or loss that is substitutable for equity in the financial negotiation.
higher than in the forecast solutions. First period annual saving is $16,000 in the perfect foresight solution compared to $8,000 in the forecast scenario. The forecast values do not anticipate the commodity price boom of the mid-1970s and hence do not include this period of relatively favorable prices.

In the second period of the perfect foresight solutions, the land purchases are much larger than those in the forecast scenario solutions. The second period land acquisition in the perfect foresight scenario is 240 acres for the income substitution, equity substitution, and maximum flow value cases, and 160 acres for the no flow value cases. This can be compared to the 80 acre purchase for all substitutability parameter values in the second period of the forecast scenario with a maximum leverage of 0.2. It can also be compared to the no purchase decision in the second period of the higher capital gain scenario with a maximum leverage of 0.2 in the no flow value, income substitution, and maximum flow value cases. Or to the 80 acre purchase for the equity substitution case that higher capital gain situation. In the perfect foresight scenario, the second period contains most of the capital gains and it pays to own as much land as possible during that period. Land prices increase $215 in the first period, $953 in the second period, and drop $154 in the third period.

The third period capital losses are not large enough to wipe out the previous capital gains, nor are they large enough to discourage second period purchases. The third period capital loss, however, provides an incentive to eliminate third period purchases. The total farm acreage in
the third period is about the same as in the forecast solution, but the purchases are made earlier to take advantage of the large capital gain in the second period. The 1979 total farm acreage is 480 acres for all substitutability parameter values in the forecast scenario with a maximum leverage of 0.2. In the perfect foresight scenario, the 1979 total farm acreage is 480 acres for the income substitution, equity substitution, and maximum flow value cases, and 400 acres for the no flow value cases. Total farm acreage in 1979 in the perfect foresight scenario is larger than in the solution with ten times the forecast capital gain. In the higher capital gain scenario, the 1979 total farm acreage is 400 acres for the income substitution, equity substitution, and maximum flow value cases. It is 320 acres for the no flow value case. This is 80 acres smaller than the 1979 perfect foresight solution acreage for all substitutability parameter combinations. This occurs because the relatively more favorable output prices in the first and second period allow greater savings, and because the observed land value in 1974 is lower than the land price at ten times the forecast capital gain. This allows a larger acreage to be purchased for a given amount of capital.

With the capital gain concentrated in the second period, the optimal solution is to focus on generating current income in the first period. In the second period, the focus shifts to earning capital gains. The level of nonland inputs and hence current income is low during the second period because almost all available capital is invested in land. Second period nonland input use ranges from $24 to $56 per acre in the perfect foresight scenario, compared to nonland input use of about $80 per acre.
in the forecast scenario. The third period can be characterized as a holding pattern. No land is purchased. Little current income is saved, except for the income solution case. Third period saving is $56,000 in the income substitution case, $8,000 for the cases which allow borrowing against unrealized capital gains, and no saving in the no flow value case. In the income substitution case, the third period solution includes a relatively higher level of saving to compensate for the low saving in the second period because of an extremely low nonland input use. Annual saving in the second period is $24,000 for the income substitution case, but $40,000 for the other substitutability parameter combinations. Third period saving is also possible for the other three parameter combinations, but not profitable. The capital losses discourage further investment in land. With existing debt and equity, a relatively high level of nonland inputs can be maintained without additional saving. Further increases in the nonland input use do not offer large enough returns to justify retaining earnings. The relatively high level of nonland inputs used in models where unrealized gains are substitutable for equity provides the financial flexibility to deal with the anticipated capital losses. When the land price declines and effective equity decreases, nonland input use can be reduced to maintain the feasibility of the solution. Under model assumptions, investments in land are locked in for the duration of the planning horizon and could not be liquidated to cope with the decreased borrowing power.

The differences in input and output prices, interest rates, the discount rate, and other parameters makes detailed comparison between the
perfect foresight and forecast scenario solutions difficult. A better comparison might be achieved by using a land price bubble scenario with forecast values except for the land price path, which is at ten times the forecast capital gain through 1981, the observed high point in land prices, and then drops to the forecast value. In this scenario, the capital loss is much larger ($541 per acre) than in the observed values ($154 per acre).

The solutions show the same initial period decisions as the higher capital gains scenario. The second and third period choices differ substantially, however. In the equity substitution case, the second period plan shows no land purchase and nonland input levels similar to the higher capital gain scenario. The second period nonland input level is exactly the same as the higher capital gain scenario choice for the no flow value, income substitution, and maximum flow value cases. In the equity substitution case, the nonland input level is raised to $48,100, or $200 per acre compared to the $25,400 level or $79 per acre in the higher capital gain scenario because the 80 acre second period land purchase is eliminated in the perfect foresight solution and more capital is available for nonland inputs.

Saving is reduced for the equity substitution and no flow value cases because the nonland input is already relatively high and the returns on savings in the third period are not favorable. In the no flow case, second period annual saving is $8,000. No saving occurs in the equity substitution solution in the second period. The models with unrealized gain substitutable for income continue relatively high levels
of saving in the second and third periods because their larger land purchase in the initial period provides greater investment opportunities. The third period annual saving is $24,000 for the income substitution and maximum flow value cases, compared to a $8,000 saving in the equity substitution case on $8,000 annual withdrawal in the no flow value case. For no flow value and equity substitution cases, the nonland input level is $73,500 and $70,500 respectively in the third period. Nonland inputs approach $300 per acre in those cases. In the income substitution case, the third period nonland input is $61,200 or $191 per acre and in the maximum flow value case it is $76,100 or $239 per acre. All the solutions show third period nonland inputs at higher levels per acre than all previous solutions. Debt use is at maximum levels in the land price bubble solutions, with higher debt levels in the models that allow borrowing against unrealized capital gain. Debt is still a relatively low cost source of capital.

The solutions for scenarios with high capital gains in at least some periods suggest that the observed tendency of U.S. farmers to increase land investments and reduce holdings of liquid assets in the 1970s and early 1980s can be explained as a rational response to capital gains given certain price expectations. If capital gains are expected to continue, it is reasonable to hold more of the asset which earns both through production and appreciation. This tendency occurs in models with all substitutability parameter combinations, though substitutability of unrealized capital gain for income or equity increases the incentives for this kind of decision. The increase in land holding for a given time and
the decrease in nonland inputs, the liquid assets, can even be observed when anticipated capital losses occur in the last period of the model. In the perfect foresight solutions, the second period farm acreage is larger than that of the forecast scenario solutions and nonland inputs are lower. In the land price bubble scenario, the initial land purchase for models in which unrealized capital gains are substitutable for income is larger than those in the forecast solutions. Anticipated losses change the solutions in these cases by altering the land purchase pattern in the second and third periods and increasing the use of nonland inputs during the period of declining prices. In the perfect foresight case, land purchases are made earlier than they would be otherwise and third period purchases are eliminated. In the land price bubble solution, no land is purchased in the second and third periods. The behavior of the average U.S. farmer appears to correspond to the model of continued capital gains. In general, there was no move in the late 1970s and the early 1980s toward more liquid investments among U.S. farmers, though individual farmers may have made this adjustment.

**Tax parameter impacts**

Solutions for an average tax rate of 0.5 and for capital gains taxed as ordinary income at the end of the horizon (ψ=1) were calculated for the higher capital gain scenario in the equity substitution case. The higher capital gains scenario was used in this exploratory research because the theoretical model suggests that tax impacts are most likely to exist when the land price increase is large. The equity substitution case is used because it best approximates known practices. The values of...
both substitutability parameters are largely unobservable, but it is known that market valuation is commonly used in agricultural lending in the U.S. There is less evidence that unrealized gain is substituted for income. A maximum leverage of 0.2 was used in the tax parameter solutions. The capital increment was reduced to $20,000; with the $40,000 capital increment, saving was difficult in the high tax rate solutions where savable income was sharply reduced. The discount rate was adjusted to reflect the varying tax parameters. With the 0.12 average tax rate and full taxation of capital gain income, the after tax discount rate is 6.7 percent. With the 0.5 average tax rate, the after tax discount rate is 4.9 percent with the 60 percent deduction for capital gain income, and 3.8 percent if capital gain income is fully taxed. The adjustment of the discount rate reflects the assumption that the tax parameters affect income from alternative assets in the same way as they affect agricultural income.

For comparison purposes, a new solution to the 0.12 average tax rate, 60 percent capital gain deduction scenario was calculated with the smaller capital increment. The land purchase path in the new solution was identical to the previously identified one. The smaller capital increment allowed greater saving in the first two periods, and this additional equity was used to purchase extra nonland inputs. Hence, income was greater and the expected wealth was increased to $308,400, compared to $300,400 with the larger capital increment. Savings was $12,000 annually in the first period and $24,000 annually in the second. The nonland inputs are: 1970, $19,500; 1975, $34,300; and 1980, $72,800,
compared to $19,500, $25,400 and $45,900 in the higher capital gain solution for the equity substitution case. When a solution is calculated for the identical parameter values, except that the capital gains is fully taxed at the end of the horizon, the control paths are virtually the same. The present value of expected income declines only slightly, to $306,100, with full taxation of capital gains.

The major impact of the higher tax rate solutions is to reduce firm growth by cutting retained earnings. Whether or not the capital gain is fully taxed does not affect the solutions. With the higher tax rate, land purchases are 240 acres in 1970 and 80 acres in 1980. Annual savings are: 1970-1974, $4,000; 1975-1979, $8,000; and 1980-1984, $16,000. Nonland input use in the second and third periods is reduced by the lack of capital to: 1975, $39,200, and 1980, $49,300. First year nonland input use is identical to the low tax rate solution, $19,500.

The solutions with varying tax parameters indicate that the tax treatment of capital gains does not have a major impact under the average farm conditions modeled in this research. Several solutions with varying tax parameters were calculated for other values of the substitutability parameters in testing the model. Though not exactly comparable to the solutions reported here, they suggest that the tax treatment of capital gains do not have a greater impact for other substitutability parameter combinations. This finding of no tax effect is conditional on the length of the planning horizon, the savings, acreage and capital gain increment used, and the constraint which limits equity growth to retained farm earnings. There may be some combination of short planning horizon and
tax rate for which the present value of the capital gains deduction tax benefit is large enough to affect land holding, saving, and nonland inputs. The capital gains deduction may provide some incentive to hold more land, but it may not be large enough to justify owning 80 acres more.

The argument about the capital gains deduction in agriculture usually centers on investments by nonfarmers who seek to shelter income from taxes. Equity growth for these investors is not limited to retained farm earnings. It is precisely because they have other sources of funds which they wish to shelter from taxation that they are interested in farm assets. As it stands, the DP algorithm does not handle equity investment from nonfarm sources, so the impact of the capital gains deduction on the tax shelter investor is not well tested.

Sufficiency of the theoretical solutions

The numerical solutions are generally consistent with those derived in the theoretical model, though the capital constraint, limited land availability, and the lumpiness problem sometimes dampen or eliminate the theoretical response to various land price paths. The solutions show a tendency toward larger land holdings and reduced nonland input use when capital gains are larger. The simple step function approximation to the debt supply function eliminates some flexibility in credit used, but nonetheless debt use increases in the model when capital gains are substitutable for equity, and the comparison of models with higher and lower maximum leverage levels shows that the incentive to use higher leverage increases with capital gains. The consistency between the numerical and
theoretical solutions suggests that the proposed theoretical decisions are good candidates for optimality.

Conclusions and Implications

The primary conclusion of this research is that the responses of the wealth maximizing decision maker to capital gains and losses identified in the theoretical model can be observed under the parameter values prevailing in the U.S. in the 1970s and 1980s. The tendency toward larger land holdings in an environment of capital gains is dampened by the capital constraint, especially when land values are high, but it is often observable in decisions to buy land earlier when land prices are rising more rapidly. This allows purchase at a lower price and a greater capital gain. Nonland input use is reduced when land prices rise more rapidly because it is impossible to finance both the earlier land purchases and nonland inputs.

In low leverage situations and when unrealized capital gain is not substitutable for equity, the incentive for greater land holdings and a lower nonland input level is rapidly dissipated. The high cost of buying land and the financial constraint combine to dampen acreage expansion and increase the incentive to use nonland inputs. With higher leverage ratios and the ability to borrow against unrealized capital gains, firm expansion in the high capital gains case is comparable to that found in the constant land price models. Because expansion plans can be maintained in the face of rising land prices, the incentive to use higher leverage is much greater when land prices are increasing rapidly.
In scenarios which involve capital losses at the end of the horizon, planned land purchases are rescheduled or eliminated and higher nonland input levels are specified. The nonland inputs provide a cushion in models where borrowing against unrealized capital gain is permitted. When land prices fall and the borrowing capacity of the firm is reduced, nonland inputs can be liquidated to allow the financial constraint to be met.

The empirical model solutions do not suggest that the ability to borrow against unrealized capital gain or to substitute that capital gain for current income is necessary for land price increases to affect farm decisions. The additional cash flow from realization of the capital gain at the end of the horizon can be enough to increase land holding and reduce nonland asset levels. The substitutability of unrealized capital gain for income increases incentives and at times is the key element in the choice between land and nonland inputs.

For the land price scenarios considered, the substitution parameters do not have a major impact on the first period choices. The impact on the second and third period decisions suggests, however, that for some land price paths and for some sets of initial endowments the ability to borrow against unrealized capital gain or to recognize unrealized capital gain as current income will make the difference between an input choice focused on earning current income with a relatively high nonland input proportion and an input decision to concentrate on earning capital gains. For instance, there is some multiple of the forecast land price for which the capital gain effect occurs for positive substitutability parameter
values, but not when both parameters are set to zero. Similarly, decisions in models with varying levels of initial unrealized capital gain levels may vary widely.

This exploratory research suggests that the asset and financial restructuring that occurred in U.S. agriculture in the 1970s and early 1980s can be explained as a rational response to an environment of large farmland capital gains. More land is acquired to earn more capital gains. More debt is incurred to purchase the additional land and to maintain nonland input levels. The comparison of high and low leverage solutions suggests that the ability of farmers to borrow against unrealized capital gains may have been crucial in allowing the restructuring to proceed. Without the ability to substitute unrealized capital gain for equity, the incentive for acquiring a farm acreage larger than that indicated by production relationships is short lived.

Although this research leaves many questions unanswered, especially in the area of response to land price risk, it does shed some light on the decisions that led to the farm financial stress of the mid-1980s. The farm operator's decision process can be viewed as repeated runs of the model. Few farm operators explicitly perform the complex calculations like those of the computer algorithm, but the correspondence between the solutions and observed values suggests that the model captures many of the important decision criteria that are implicitly used by farmers to evaluate production and finance choices. The model is based on expectations. When new information becomes available, expectations are revised and plans adjusted to fit the new environment. Farm
financial stress can be seen as a situation in which that adjustment becomes very difficult or in some cases impossible. For instance, consider the case in which there is a large amount of accumulated unrealized capital gain and it is anticipated that land prices will continue to rise rapidly. The exact optimal solution will depend on output prices, interest rates, and other variables, but it may be to acquire as much land as possible by reducing the holding of liquid assets and incurring additional debt. If borrowing against unrealized gain is permitted, the debt may rise substantially. If land prices were then to fall unexpectedly, there may be no feasible solution to the model. There may not be enough nonland assets to absorb the loss in borrowing power. The amount of debt required to finance just the land may exceed the credit available under the lower land price. Selling land may be the only alternative. Even if a feasible solution exists, the adjustment may be difficult. Selling nonland inputs at the same time when everyone else is making the same adjustment may mean enormous liquidity losses.

The parameters of the model may be changed in the new environment of farmland capital losses. The lender who formerly accepted unrealized capital gain as an addition to equity may now view the much reduced amount of accumulated capital gain with skepticism, thereby further reducing borrowing power. In short, the farm financial stress of the mid-1980s can be seen as an adjustment from a period of rapid land price increases to a period of more stable or declining values. Among the substantial proportion of farmers that are not undergoing financial stress, the model leads to the hypothesis that there are at least some
who had land price expectations that differed from their peers and who made the adjustments suggested by the perfect foresight and land price bubble models toward smaller land holdings and greater investment in liquid assets.

**Expectation formation**

If this view of farm financial problems is accepted, then the formation of the parameter values in the agricultural decision framework is involves a crucial policy question. If the important parameters are formed primarily on the basis of the decisionmaker's own observations and utility function, then the public responsibility for those actions is small. Public policy may focus on minimizing the loss to the economy in general, but there is little obligation to resecure failing firms. Such a view of the situation implicitly underlies much of the questioning of government responsibility in refinancing debt that has been incurred by "consenting adults" (Anthan, 1985, p. 6s). If, on the other hand, important parameters are exogenously formed by lenders or government, for instance, then in fairness these external institutions might be expected to bear some of the adjustment burden. In addition, the role of those who advise farmers, such as the extension personnel and researchers of land grant universities, in the formation of expectations must be questioned. Do the advisors bear any responsibility for decisions directly based on expectations that they helped to create?

The results of this research suggest that the land price expectations, the substitutability parameters, and the maximum leverage ratio can have important consequences for production and finance decisions.
The formation of these expectations and parameter values have not been extensively studied, but it seems clear that the personal observations and characteristics of the individual decisionmaker are crucial in this process. Farmland markets tend to be localized. State or regional average land prices may be available only annually and may not be relevant to the price structure in a particular community. When prices are changing rapidly, the decisionmaker is almost forced to depend on personal observation to form expectations. Similarly, the substitutability parameters appear to depend heavily on personal characteristics and observation. The risk attitude of the individual is likely to be important. If the decisionmaker perceives the capital gain as relatively certain and is willing to take the risk of future losses, then the substitutability of unrealized capital gains for both income and equity is likely to be increased. When the maximum leverage ratio is a result of internal capital rationing, the individual's risk aversion is probably a primary determinant of the level. The fact that not all farmers are in financial trouble, though all were subject to very similar exogenous influences, suggests that individual factors are important in the decision making framework.

The individual does not operate in a vacuum, however. The lender, for instance, determines the supply of credit and influences the role of unrealized capital gain in the financial negotiation. When external credit rationing is a binding constraint, the lender determines the maximum leverage level. Even internal capital rationing may not be independent of the lender's influence. For instance, the internally
imposed limit may be set as some proportion of the external limit. If the lender is willing to accept more unrealized gain as equity, then the internal debt limited will also be higher. To guard against future capital losses, the risk averse decisionmaker may assume that a smaller proportion of unrealized capital gain is substitutable for equity than would be acceptable to the lender. The decisionmaker cannot, however, assume a larger value than the lender does. Plans based on greater borrowing against unrealized gain than is allowed by the lender would be unfeasible because they would tend to call for greater debt use than is available. Many agricultural lenders are trusted financial advisors, in addition to being a source of credit. In this advisory role, the lender may create a climate of increased certainty about treating unrealized capital gains as an addition to equity, or may reinforce overly optimistic land price path expectations.

Government action may affect virtually all the parameters. Government regulations affect the supply of debt. For the Federal Land Bank, the maximum percentage of the land value which may be lent is specified by law. Raising the percentage, as occurred in the Farm Credit Act of 1971, effectively increases the supply of credit to farmers and may increase the impact of capital gains on farm decisions. Bank examiners' view of farm assets as collateral affect the lenders' willingness to extend credit to agriculture. For instance, if bank examiners had insisted that farm balance sheets use the valuation rule used in most nonfarm businesses, that is the value is the lower of cost or market value, then agricultural lending and hence farm decisions in the 1970s
would probably have been different. Similarly, if examiners in the mid-
1980s are more skeptical of unrealized capital gain as collateral than
they were in the 1970s, then farm adjustment problems will be increased.
In addition to the decline in real estate market value, the proportion of
that market value which could serve as a base for borrowing would
decline.

Government price support programs may affect the land price expecta-
tions and the reliability of cash flows. Incorporating the land value
into inflation indexed price supports, as was discussed in the late
1970s, would tend to create expectations of continued capital gains
(Harris, 1977). The interaction of inflation and tax effects may create
farmland capital gains (Feldstein, 1980). Price supports in a period of
technological change may create land price increases as profits from
increased productivity are bid into land.

Plaxico and Kletke (1980) argue that the substitutability of
unrealized capital gain for current income depends on the reliability of
cash flows (p. 263). Unrealized capital gain cannot be used to meet cash
obligations, so if cash flow is a problem, the weight given capital gains
in the decision process will be reduced. By increasing the reliability
of cash flows, price support programs may tend to encourage decision-
makers to view a larger proportion of capital gains as a substitute for
current income. Similarly, the growing unwillingness of Congress and the
executive branch to support farm prices in the mid-1980s may cause
downward revision in the income substitutability parameter because farm
income is likely to be less reliable in the future. Such a revision
would tend to increase the size of the downward adjustment required to meet the new situation.

Land grant universities are in the business of expectation formation. This role is most explicit in the marketing of grains, oilseeds, and meat animals. Extension economists regularly publish price forecasts and discuss the factors influencing these markets. In land prices, this role is not quite so obvious. Some universities regularly survey and publish state and area land price averages, but explicit forecasting is not usually part of this activity. Some agricultural economists have been active in informing farmers about methods which could justify the land prices of the 1970s, for instance, by including future capital gains in the calculations of land value or by averaging recently purchased higher priced land with low cost land to achieve some reasonable average land cost. Unfortunately, the variance of land prices has not been widely treated. The possibility of downside land price risk was not emphasized in extension communications. It is possible that the actions of land grant university extension personnel and researchers reinforced the expectations of continued capital gains.

Policy implications

Documentation of farmer expectation formation is beyond the scope of this research, but casual observation suggests that there is ample evidence to show that while decisionmaker characteristics and observations are important in expectation formation, other institutions and individuals may play an instrumental role. What does this recognition imply for those other individuals and institutions? How may it affect
their role in the current financial problems and their policy in the future?

This research suggests that by permitting borrowing against unrealized capital gains and expanding the supply of credit, lenders enabled farmers to restructure their balance sheets in ways that left them financially vulnerable. The model suggests that without the ability to borrow against unrealized capital gains, long-term restructuring may not have been possible during the 1970s. In this context, farm bankruptcies and other financial problems of the 1980s are signs of the inadequacy of past lender policy, as well as indications of poor farm level decisions. Lenders should, therefore, expect to bear part of the adjustment cost. The principal write down in the Farmers Home Administration debt restructuring and guarantee plan for commercial lenders can be seen as the lender's share of the cost of bad decisions. This study does not quantify the magnitude of the lender's share or suggest the principal write down is too large or too small, but supports the general principle of lender responsibility in the financial crisis.

A clear temptation to lenders and their regulators in the mid-1980s is to immediately discourage all borrowing against unrealized capital gains. If the ability to substitute unrealized capital gain for equity tends to make farm firms financially unstable, one can argue that lenders should eliminate the practice. Rapid transitions from lending on market values to cost based valuation would, however, exacerbate the adjustment problems because it would increase the required adjustment in farm plans. In the longer term, a policy of no borrowing against unrealized capital
gains would force farmers to sell land to realize any benefit from land price increases. This would increase transactions cost and would require repeated adjustments in the farm business as land holdings were sold and new tracts acquired to monetize capital gains. This may tend to discourage long-term investments in land productivity, such as soil conservation improvements. Instead of instinctively reacting against borrowing on market values, an improved lender policy is likely to include some recognition of the possibility of long-term real capital gains in agriculture and of the role of these price increases in building equity, but with a recognition that market prices have variances and that the variability of land prices over the long term can be particularly wide. This recognition of land price variability might take a variety of forms, including looking at the borrower's balance sheet under several land price path assumptions or requiring borrowers to have a contingency plan for dealing with capital losses.

The argument for government responsibility in the farm financial problems of the 1980s is similar to that for lender responsibility. Government actions expanded credit supply, helped create an expectation of rising land prices, and stabilized cash flows, thus encouraging the substitution of unrealized capital gains for current income in making farm decisions. This research suggests farmers reacted to these stimuli in an economically rational manner by increasing land holding, incurring debt, and cutting investment in liquid assets. Farm financial problems are not simply contracts between "consenting adults" gone awry, but the result of decisions made under certain limitations and incentives created
in part by government. The magnitude of government responsibility is a more difficult question. To what degree is there an implied guarantee with government programs that protect the participant and the sector in general against adverse side effects of the policy? Should farmers have been more wary of government programs and the expectations they created? Increased skepticism of government programs will probably be a result of the 1980s financial crisis in agriculture if government is seen as having abandoned farmers in a mess that it helped create. This distrust of government would make future farm programs more difficult to administer and probably less effective.

This research adds to the evidence that suggests that the price support and credit programs carried out by the U.S. government in the name of preserving the family farm may have been counter productive (see U.S.D.A., 1981, for other evidence). The government programs appear to have destructive side effects that leave family farmers vulnerable to cash flow crises and insolvency. There is no standard definition of the family farm, but often mentioned characteristics include: a relatively small acreage, depending primarily on family labor and capital, owner operation, and intensive management. This type of operation does not appear to be well-adapted to the economic environment created by government price supports and credit programs. In particular, it is not likely to be successful in a situation involving large farmland capital gains. As long as land prices rise, the larger more heavily leveraged operations will tend to be more successful. The short-term price supports and credit programs may make life more comfortable for the family farmer, but
this research suggests that in the longer run these policies tend to undermine the system that they are created to preserve.

The development and testing of alternative policies is beyond the scope of this study, but the research does suggest some questions that should be asked about newly-proposed programs and program modifications:

1) can the program benefits be capitalized into the value of land or other assets?

2) does the policy create incentives to increase farm business size beyond that indicated by production relationships, market prices, and conservative financial management?

3) does the program encourage asset or debt restructuring in ways that increase financial risk?

It is often assumed that higher prices or farm incomes will strengthen the family farm system. If benefits can be capitalized into land values, the program will be an inefficient means for achieving this goal. The ultimate beneficiary will be the landowners, many of who are not farmers. Who actually receives the government payment is irrelevant because the benefits can be bid directly into land or they can be transmitted to the real estate purchase market through rents. Entry for new farmers may be more difficult because of increased capital requirements. In addition, this study shows that capital gains can provide incentives for agriculture to move away from the family farm toward larger farms, more extensive land use, and higher leverage.

Almost all the price and income support programs that have been tried in the U.S. can potentially create capital gains because they are
tied to the volume of production or to the acreage itself. The type of program can affect the pattern of the capital gains. For instance, if acreage allotments are tied to certain farms, as they have been for peanuts and tobacco, then capital gains will be concentrated on the farms with the large allotments. In contrast, if commodity prices are supported by a loan and storage program, government purchase of commodities, acreage set asides, export subsidies, or deficiency payments, the capital gain is likely to be spread over all land capable of producing the product. Allowing the allotment to be sold separate from the farm does not solve the problem, but merely creates another asset which can earn capital gains or losses.

Theoretically, it may be possible to design a program which eliminates or at least minimizes the capital gains or loss side effects, but such policies have major administrative and political drawbacks. Farmland capital gains and losses occur because there is in the U.S. a free market in land. If the land market were rigidly controlled, as it is in some parts of Europe, then land price change could also be controlled. This option is unlikely to be politically feasible in the U.S. where government intervention in the real estate market is not well accepted. Strict payment limitations could reduce the capital gain and loss problem. For instance, if benefits for Midwestern grain producers were limited to the average yield on 160 acres, then the incentive to bid up the price would be reduced. The program benefits could be bid into a farmer's first 160 acres or it could be averaged out over all the land farmed. Payment limitations have, however, not been particularly
successful in the past. The political will required to create and enforce these limits is great and the benefits are visible primarily in the long term.

U.S. price and income support policies can generate capital gains and losses because they are tied to the volume of production or the acreage. It would be more difficult to capitalize benefits that were tied to labor and management. For instance, instead of deficiency payments calculated on some average yield per acre, the benefit could be based on average yield per operator. To sever the connection between the payment and farm size, the average yield could not be based on the operator's personal production record, but must be some state or county figure. There would be administrative problems in defining who was eligible for the program. Should the definition concentrate on production of some minimum amount of the commodity? Will resource misallocations result if farmers must produce minimum amounts of some commodities to qualify? There may also be maximum production limits established on the argument that large-scale operations can be expected to fend for themselves, but there is a public interest in aiding moderate size family farms.

Linking the loan based deficiency payment to a bundle of commodities may provide a better indication of farm earning power and produce better resource allocation. Eligibility could be defined by producing any one of the alternative commodities. For instance, in Iowa the payment might be based on a weighted average of corn, soybean, hog, and cattle prices. Production of some minimum amount of any agricultural commodity could be
used to establish eligibility using the argument that alternative or specialty enterprises must be competitive with the four basic commodities. With a deficiency payment based on a bundle of commodities, the farmer could choose the most profitable alternative without jeopardizing eligibility.

Because it would increase farm income, a labor based income support could still be bid into land prices, but the problem would be greatly reduced. The value of additional land beyond the acreage required to produce the amounts on which the payment is based would have to be based primarily on expected market prices. Extra program benefits could not be captured by acquiring additional land.

The second question concerning incentives to increase farm business size is closely tied to the first because capital gains can encourage expansion. By helping to generate land price increases, price and income support programs can provide incentives to expand. Credit programs reinforce this tendency by creating a greater supply of funds to use in expansion and by expanding the ability to borrow against unrealized capital gains. In addition, price and income policies may tend to increase the value of capital gains by stabilizing cash flows. Plaxico and Kletke have argued that cash flow uncertainty reduces the ability to recognize capital gains as current income. Price and income stabilization may also have broader effects. The theoretical work on land price risk suggests that uncertain land values can provide incentives to keep farm acreage relatively small, limit debt use, and encourage intensive farming. Does uncertainty about yield and other prices have a similar
effect? Traditionally, the strength of the family farm has been its flexibility. It can cope with low income periods by reducing consumption expenditures. A more industrialized operation with employees on a fixed wage schedule would have a much harder time making labor cost adjustments. The family farm can change production plans or adopt innovations without clearing the decision through a management hierarchy. But, when risk is reduced, the flexibility becomes less valuable. In the more certain environment, economies of scale or the ability to raise large amounts of capital may outweigh flexibility in the competition for firm survival. It is not immediately clear that society benefits by public programs which absorb farm risks and allow these larger-scale units to develop. Even with significant economies of scale in production or finance, large-scale units may reduce overall efficiency compared to smaller, more flexible units that can potentially handle risk at a lower cost. Disaster payments, income insurance, and other stabilization measures may make life more pleasant for family farmers in the short run, but reduce their ability to compete in the long run.

The impact of government policies on asset and financial structure is the concern of the third question. This research suggests that some government policies may have encouraged restructuring toward more risky combinations than would have been chosen on the basis of market conditions alone. Farmland capital gains provided incentives to purchase more land and hold smaller amounts of liquid, nonland capital. Expansion of the credit supply and allowing greater borrowing against unrealized capital gains facilitated the restructuring and created heavier debt
loads. In addition, there has often been the implicit assumption in
government credit and tax policies that the more capital intensive farm
is the more efficient unit and, therefore, capital use should be encour-
aged. This assumption does not allow for the increasing financial risk
that often accompanies increased capital use, because of the greater debt
load. The high capital costs of the 1980s suggest that the capital labor
substitution arguments underlying the assumption that capital intensive
farms are efficient should be reexamined. Unfortunately, the incentives
embedded in government programs do not change quickly with market
conditions. The subsidy inherent in FmHA lending and in the FCS implied
agency status, the expansion of credit supplies by the operation of
government sponsored or associated lenders, and the investment incentives
in the tax code may be creating and preserving farms that are over-
capitalized, too large, and vulnerable to price and financial risk. At
the same time, market conditions may indicate that the future belongs to
relatively small firms that can adapt quickly to changing conditions and
which can supply their relatively small capital requirements through low
leverage financial structures.

In short, the three questions help identify areas in which long-term
destructive side effects of government involvement in agriculture may
appear. The questions help answer the more general query of whether or
not the proposed policy will do more harm than good. It should be noted
that though the ideal policy would allow one to answer no to all three
questions, the best practical course may allow other responses. For
instance, on the basis of this and other research one might decide that
the best government policy for agriculture is no special policy. Government should provide infrastructure and facilitate communications as it does for all parts of the economy, but prices and credit allocation should be left to the market. The adjustment from the current situation of heavy government involvement to that minimal government situation may, however, require an extended period of second best policies, which may generate capital gains or encourage expansion. In that case, it would still be useful to identify the situation which minimizes the long-term cost.

This study also has implications for the activities of land grant universities. Research suggests that the expectations held in the 1970s may go a long way in explaining why some farm firms in the mid-1980s are financially strong and others are on the verge of collapse. Land grant university extension and research personnel had a role in creating these expectations. Was this financial crisis at least partially a result of poor or misunderstood information and advice? If it was poor advice, could these recommendations be traced to a good system that simply could not foresee all possible contingencies? Or are there flaws in the system that led agricultural economists to overlook certain possibilities?

**Land price formation**

Though the model used here is microeconomic in focus, it does suggest how a land price boom and bust cycle might be generated. If the model captures the important decision criteria for the bulk of participants in some farmland market, then land price increase expectations, once formed, would tend to become self-fulfilling. As capital
gains increase, farmers and other buyers desire to own more land. But, land is in limited supply so this drives the price up even more. This cycle of capital gains generating demand and that demand creating land price increases is eventually dampened by the inability to finance purchases at high prices. When the capital gains decrease, the demand for land also drops and the land price weakens. This can form a downward price spiral. As the price drops, buyers attempt to economize on land use to avoid capital losses and this drives the price even lower. The downward price movement may be halted when the current income from land ownership becomes favorable relative to the potential capital loss. This turning point at the bottom of the cycle may be far below the equilibrium suggested by conventional models of land price formation that rely on the present value of current income from land because the expected capital loss offsets the current income.

The model suggests that with an expanded credit supply and the ability to borrow against unrealized capital gains, the boom would tend to last longer and the price would go higher. With greater borrowing against unrealized capital gain, the land price bust would tend to be more disastrous because farmers would have incurred debt further beyond the level suggested by their invested equity than they would incur without the ability to substitute unrealized capital gain for equity.

This research suggests that modest capital gains may not be enough to set off a boom and bust cycle. Small capital gains increase the incentive to own land, but the incentive is not great enough to overcome the lumpiness of acquiring real estate tracts. This suggests that during
the long period of modest capital gain on farmland from the 1940s through the early 1970s, the price increases had only a small impact on production and finance decisions. If some exogenous event increases capital gains expectations, a boom may be triggered. The increased export demand for U.S. farm products in the mid-1970s can be identified as a possible triggering event. Earlier in this century, the commodity demand generated by rising U.S. population and World War I is probably the crucial expectation altering event.

Would a better understanding of land price cycles and their causes and effects, eliminate them? It probably would not. The hog cycle is well-known and reasonably well-understood and yet it continues unabated. One can hope, however, that if farmers understood the downside risk in land prices that they would not leave themselves financially vulnerable as they did in the 1980s.
REFERENCES


Anthan, George. "Stockman Squeals 'Blackmail' at Farm Representatives." Register. (Des Moines, Iowa) February 6, 1985, p. 6s.


Malbert, Vincent A. *An Introduction to Short Term Forecasting Using the Box-Jenkins Methodology.* Norcross, Georgia: Production Planning Division, American Institute of Industrial Engineers, 1975.


APPENDIX A
MAIN:PROCEDURE OPTIONS (MAIN);

DECLARE FORE(0:50,9) FLOAT EXTERNAL;

/* COLUMN 1=AFTER TAX DISCOUNT RATE
  2=INTEREST ON DEBT
  3=OUTPUT PRICE
  4=INPUT PRICE INDEX
  5=LAND PRICE PER ACRE
  6=PROPERTY TAX
  7=NONREAL ESTATE INTEREST
  8=CUMULATIVE CAPITAL GAIN WITHIN PERIOD
  9=CAPITAL GAIN PER PERIOD */

/* ROWS ARE PERIODS 0-50 */

DECLARE MAXM(5,10) FLOAT EXTERNAL;

/* ROW 1=MAX LAND IN EACH PERIOD
  2=MIN EQUITY IN EACH PERIOD
  3=MAX EQUITY IN EACH PERIOD
  4=MIN ACCUMULATED CAPITAL GAIN EACH PERIOD
  5=MAX ACCUMULATED CAPITAL GAIN EACH PERIOD
  6=UNUSED

COLUMNS ARE FOR DECISION PERIODS 1-10 */

DECLARE IN(4) FLOAT EXTERNAL;

/* COLUMN 1= INITIAL LAND INCREMENTS
  2= INITIAL EQUITY INCREMENTS
  3= INITIAL UNREALIZED CAPITAL GAIN INCREMENTS
  4= UNUSED */
DECLARE DECIS(8,5) FIXED(15,4) DEC EXTERNAL;

/* THE MATRIX OF PARAMETERS IS:

<table>
<thead>
<tr>
<th>Land</th>
<th>Capital</th>
<th>Gain</th>
<th>Unused</th>
<th>Unused</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increment</td>
<td>Increment</td>
<td>Increment</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Beginning</th>
<th>Number</th>
<th>Years Per</th>
<th>Max Land</th>
<th>Adjust</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>Decision</td>
<td>Period</td>
<td>Purchase</td>
<td></td>
</tr>
<tr>
<td>Periods</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Used</th>
<th>Unused</th>
<th>Unused</th>
<th>Debugging</th>
<th>Debug2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Output=0</td>
<td>Else&gt;0</td>
<td>Output=0</td>
<td>Else&gt;0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rent Land=1</th>
<th>Rent Max</th>
<th>Max Land</th>
<th>Unused</th>
<th>Debug3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Else=0</td>
<td>Else=0</td>
<td>Debt Percent</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X Exponent</th>
<th>Unused</th>
<th>X Land</th>
<th>X Year</th>
<th>Y Quad</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Interact</td>
<td>Interact</td>
<td>Term</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Intercept</th>
<th>Land 1</th>
<th>Land Quad</th>
<th>Land Year</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>Term</td>
<td>Interact</td>
<td>Term</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tau</th>
<th>Terminal Tau</th>
<th>Psi</th>
<th>Delta</th>
<th>Epsilon</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>PHI</th>
<th>OMEGA</th>
<th>LEVERAGE</th>
<th>NONLAND</th>
<th>RENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>CUTOFF</td>
<td>CAP RATIO</td>
<td>PERCENT%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

DECLARE (DISCOUT(*,*)*, DEBTCST(*,*,*), GAIN(*), REVENUE(*,*,*), ANNUITY(*,*)*, LENDING(*,*))

FLOAT DEC (10) EXTERNAL CONTROLLED;

DECLARE P(*,*,*,*) FLOAT DEC(10) EXTERNAL CONTROLLED;

DECLARE (S(*,*,*), REMAIN(*,*,*)) FLOAT DEC (10) EXTERNAL CONTROLLED;

DECLARE (TAX, CAPTAX, HORIZON, PERIODS,
          INLAND, UNIQUE, OMEGA, PHI,
          BUYLAND, CAPINC, LANDINC, GAININC, ANNUAL) EXTERNAL

FLOAT DEC(10);

DECLARE (RANGE, NUMBER, ZMAX, ZMIN, YEAR) FLOAT DEC(10);

DECLARE (LINDEX, EXP, EXP2, EXP3, XINDEX, TEMP) FLOAT DEC(10);

DECLARE (U, I, J, K, L, M, N, T, W, V) FIXED DEC(6,0);

DECLARE (LEVER, DEBUG1, DEBUG2, RENTIN, RENTOUT, NONLCAP,
          DEBUG3, RENTPER, ADJUST, BEGIN) FIXED DEC(6,2) EXTERNAL;

DECLARE (ALPHA6, ALPHA8, ALPHA9, LR) FIXED DEC(10,4) EXTERNAL;

DECLARE X FLOAT DEC(10) INIT(0);

DECLARE RECURR ENTRY;

DECLARE OUTPUT ENTRY;

DECLARE TRANS ENTRY;

DECLARE BOUNDS ENTRY;

DECLARE AA FILE OUTPUT;
OPEN FILE (AA) OUTPUT STREAM PRINT LINESIZE(132);

DO I= 0 TO 50;

   GET EDIT ((FORE(I,J) DO J= 1 TO 9))((8)F(9,2),F(8,2));

END;

PUT FILE (AA) EDIT ('**MAXM**') (SKIP,X(5),A);

DO I=1 TO 6;

   GET EDIT ((MAXM(I,J) DO J=1 TO 10))((10)F(8,0));
   PUT FILE (AA) EDIT ((MAXM(I,J) DO J=1 TO 10))
       (SKIP,X(2),(10)F(6,0));

END;

PUT FILE (AA) EDIT ('**INITIAL**')(SKIP,X(5),A);

GET EDIT ((IN(I) DO I=1 TO 4))((4)F(20,0));
PUT FILE (AA) EDIT ((IN(I) DO I=1 TO 4))(SKIP,X(2),(4)F(10,0));
PUT FILE (AA) EDIT ('***DECISION***')(SKIP,X(5),A);

DO I=1 TO 8;

GET EDIT ((DECIS(I,J) DO J=1 TO 5))((5)F(16,4));

PUT FILE (AA) EDIT ((DECIS(I,J) DO J=1 TO 5))
(SKIP,X(2),(5)F(16,4));
END;

/* CALCULATE COMMONLY USE VALUES */

PHI=DECIS(8,1);

PUT FILE (AA) EDIT ('PHI=',PHI)(SKIP,A(4),F(8,2));

TAX= 1-DECIS(7,1);

PUT FILE (AA) EDIT ('TAX=',TAX)(SKIP,A(4),F(8,2));

CAPTAX=1-PHI-DECIS(7,2)*DECIS(7,3);

PUT FILE (AA) EDIT ('CAPTAX=',CAPTAX)(SKIP,A(7),F(8,2));

HORIZON=DECIS(2,2)*DECIS(2,3);

PUT FILE (AA) EDIT ('HORIZON=',HORIZON)(SKIP,A(8),F(8,2));

CAPLOSS=DECIS(7,2)*DECIS(7,5)/FORE(HORIZON,1);

PUT FILE (AA) EDIT ('CAPLOSS=',CAPLOSS)(SKIP,A(8),F(8,2));

/* CAPLOSS=TERMINAL TAU * ANNUAL DEDUCTION LIMIT
   DIVIDED BY THE TERMINAL DISCOUNT RATE */
LANDINC = DECIS(1,1);
PUT FILE (AA) EDIT ('LANDINC=' , LANDINC )(SKIP, A(8), F(8,2));
CAPINC = DECIS(1,2);
PUT FILE (AA) EDIT ('CAPINC =', CAPINC )(SKIP, A(8), F(8,2));
GAININC = DECIS(1,3);
PUT FILE (AA) EDIT ('GAININC=' , GAININC )(SKIP, A(8), F(8,2));
NONLINC = DECIS(1,4);
PUT FILE (AA) EDIT ('NONLINC=' , NONLINC )(SKIP, A(8), F(8,2));
INLAND = IN(1);
OMEGA = DECIS(8,2);
PUT FILE (AA) EDIT ('OMEGA =', OMEGA )(SKIP, A(8), F(8,2));
BUYLAND = DECIS(2,4);
PUT FILE (AA) EDIT ('BUYLAND=' , BUYLAND )(SKIP, A(8), F(8,2));
UNIQUE = 0;

PERIODS = DECIS(2,2);
PUT FILE (AA) EDIT ('PERIODS=' , PERIODS )(SKIP, A(8), F(8,2));
ANNUAL = DECIS(2,3);
PUT FILE (AA) EDIT ('ANNUAL =', ANNUAL )(SKIP, A(8), F(8,2));
LEVER = DECIS(8,3);
DEBUG1 = DECIS(3,4);
DEBUG2 = DECIS(3,5);
DEBUG3 = DECIS(4,5);
NONLCAP = DECIS(8,4);
RENTOUT=DECIS(4,1);
RENTIN=DECIS(4,2);
RENTPER=DECIS(8,5);
ADJUST=DECIS(2,5);
BEGIN=DECIS(2,1);
ALPHA6=DECIS(5,1);
ALPHA8=DECIS(5,3);
ALPHA9=DECIS(5,4);
LR=DECIS(4,3);

/* CALCULATE CUMULATIVE UNREALIZED CAPITAL GAIN WITHIN THE
   DECISION PERIOD */

DO I= 1 TO PERIODS;

    DO U= 1 TO ANNUAL;

        FORE((I-1)*ANNUAL+U,8)=FORE((I-1)*ANNUAL+U,5)
              - FORE((I-1)*ANNUAL,5);

    END;

END;
/* CALCULATE ANNUAL CAPITAL GAIN PER ACRE */

DO I = 1 TO 50;

FORE(I,9)= FORE(I,5)-FORE(I-1,5);

END;

PUT FILE (AA) EDIT ('***FORECASTS***')(SKIP,X(5),A);

DO I= 0 TO 50;

PUT FILE (AA) EDIT ((FORE(I,J) DO J= 1 TO 9))
(SKIP,X(2),(9)F(8,2));

END;

/* CREATE ARRAYS OF DISCOUNT AND INTEREST TERMS */

ALLOCATE DISCOUNT(PERIODS,ANNUAL) INIT(((PERIODS)*ANNUAL)1),
DEBT CST(PERIODS,ANNUAL,2) INIT(((PERIODS)*2*ANNUAL)0),
ANNUITY(PERIODS,2) INIT((PERIODS)0),LENDING(PERIODS,
ANNUAL) INIT((PERIODS*ANNUAL)0);

DO T= 1 TO PERIODS;
DISCOUT(T,1)=1/(1+FORE((T-1)*ANNUAL+1,1));

IF ANNUAL>1 THEN DO I=2 TO ANNUAL;

    DISCOUT(T,I)=DISCOUT(T,I-1)/(1+FORE((T-1)*ANNUAL+1,1));

END;

DO I= 1 TO ANNUAL;

    DEBTCST(T,I,1)=FORE((T-1)*ANNUAL+1,2)*TAX*DISCOUT(T,I);
    DEBTCST(T,I,2)=FORE((T-1)*ANNUAL+1,7)*TAX*DISCOUT(T,I);
    LENDING(T,I)=FORE((T-1)*ANNUAL+1,1)*DISCOUT(T,I);

END;

END;

DO T= 1 TO PERIODS;

    ANNUITY(T,2)=1;

DO I= 1 TO ANNUAL;

\text{ANNUITY}(T,1) = \text{ANNUITY}(T,1) + \text{DISCOUT}(T,I);

\text{END;}

\text{END;}

/* CREATE AND ARRAY OF UNREALIZED CAPITAL GAIN TERMS */

\text{ALLOCATE GAIN(PERIODS) INIT((PERIODS)0);

DO T = 1 TO PERIODS;

\text{DO I = 1 TO ANNUAL;}

\text{IF PHI} > 0 \text{ THEN DO;}

\text{GAIN}(T) = \text{FORE((T-1)\*ANNUAL+I,9)} * \text{DISCOUT}(T,I) + \text{GAIN}(T);

\text{END;}

\text{END;}

\text{END;}

/* PRINT OUT DISCOUT, DEBTCST, ANNUITY AND GAIN ARRAYS FOR CHECKING */

PUT FILE (AA) EDIT ('***DISCOUT***')(SKIP,X(5),A);

DO I=1 TO PERIODS;

    PUT FILE (AA) EDIT ((DISCOUT(I,J) DO J=1 TO ANNUAL))
           (SKIP,X(2),(ANNUAL)F(9,4));

END;

PUT FILE (AA) EDIT ('***LENDING***')(SKIP,X(5),A);

DO I=1 TO PERIODS;

    PUT FILE (AA) EDIT ((LENDING(I,J) DO J=1 TO ANNUAL))
           (SKIP,X(2),(ANNUAL)F(9,4));

END;

PUT FILE (AA) EDIT ('***ANNUITY***')(SKIP,X(5),A);

PUT FILE (AA) EDIT ((ANNUITY(I,1) DO I=1 TO PERIODS))
           (SKIP,X(2),(PERIODS)F(9,4));
PUT FILE (AA) EDIT ('***GAIN***')(SKIP,X(5),A);

PUT FILE (AA) EDIT ((GAIN(I) DO I=1 TO PERIODS))
    (SKIP,X(2),(PERIODS)F(9,4));

PUT FILE (AA) EDIT ('***REAL ESTATE DEBTCOST***')(SKIP,X(5),A);

DO I=1 TO PERIODS;

    PUT FILE (AA) EDIT ((DEBTCST(I,J,1) DO J=1 TO ANNUAL))
        (SKIP,X(2),(ANNUAL)F(9,4));

END;

PUT FILE (AA) EDIT ('***NONREAL ESTATE DEBTCOST***')(SKIP,X(5),A);

DO I=1 TO PERIODS;

    PUT FILE (AA) EDIT ((DEBTCST(I,J,2) DO J=1 TO ANNUAL))
        (SKIP,X(2),(ANNUAL)F(9,4));

END;

/* CREATE AN ARRAY OF DISCOUNTED REVENUE BEFORE INTEREST FOR EACH
POSSIBLE LAND LEVEL IN EACH PERIOD

ALLOCATE REVENUE(HORIZON, INLAND: (INLAND+PERIODS*BUYLAND*(1+RENTIN)), 3)

INIT((HORIZON*3*(1+PERIODS*BUYLAND*(RENTIN+1))));

U=0;
W=1;

DO T=1 TO HORIZON;

U=U+1;
IF U=ANNUAL+1 THEN DO;
U=1;
W=W+1;
END;

DO I=MAX(INLAND,0) TO INLAND+W*BUYLAND*(1+RENTIN);

IF I>0 THEN DO;

YEAR=BEGIN-1+T;

EXP=DECIS(6,2)+DECIS(6,3)*LOG(I*LANDINC);
EXP=EXP+DECIS(6,4)*LOG(YEAR);
\[ \text{EXP3} = \text{DECIS}(5,1) + \text{DECIS}(5,3) \times \text{LOG}(\text{LANDINC}^*I) + \text{DECIS}(5,4) \times \text{LOG}(\text{YEAR}); \]

\[ \text{EXP2} = \text{DECIS}(6,5) + \text{DECIS}(5,5) \times \text{LOG}(\text{YEAR}); \]

\[ \text{TEMP} = \text{DECIS}(6,1) \times (I \times \text{LANDINC})^* \times \text{EXP* YEAR}^* \times \text{EXP2}; \]

\[ \text{REVENUE}(T, I, 1) = \text{FORE}(T, 3) \times \text{TEMP}; \]

\[ \text{REVENUE}(T, I, 2) = (\text{FORE}(T, 4) / (\text{REVENUE}(T, I, 1) \times \text{EXP3}))^* \times (1/(\text{EXP3}-1)); \]

\[ \text{XINDEX} = \text{FORE}(T, 4) \times (1 + \text{FORE}(T, 7) \times \text{NONLCAP}); \]

\[ \text{REVENUE}(T, I, 3) = (\text{XINDEX} / (\text{REVENUE}(T, I, 1) \times \text{EXP3}))^* \times (1/(\text{EXP3}-1)); \]

\[ \text{IF DEBUG1=0 THEN DO; } \]

\[ \text{PUT FILE } (\text{AA}) \text{ EDIT ('TEMP=',TEMP,'LAND=',I,'TIME=',T,'EXP=',EXP,'EXP2=',EXP2,'EXP3=',EXP3)(SKIP,(2)(A(6),F(8,4)),A(6),F(4),A(4),F(9,4),(2)(A(5),F(9,4))); } \]

\[ \text{END; } \]

\[ \text{END; } \]
ELSE DO X=1 TO 3;

    REVENUE(T,I,X)=0;

    END;

END;

ENDIF=0 THEN DO X=1 TO 3;

    PUT FILE (AA) EDIT ('***REVENUE',X,'***')
                        (SKIP,X(5),A(10),F(1),A);

    RANGE=PERIODS*BUYLAND;
    NUMBER=TRUNC(RANGE/10);

    DO U=0 TO NUMBER;

        ZMIN=INLAND+U*10;
        ZMAX=MIN(INLAND+U*10+9,INLAND+PERIODS*BUYLAND*(1+RENTIN));

        PUT FILE(AA) EDIT ('LAND EQUALS',ZMIN,'TO',
ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2),
A(2),X(2),F(5,0));

DO T =1 TO HORIZON;

PUT FILE (AA) EDIT ((REVENUE(T,J,X) DO J=
ZMIN TO ZMAX))
(SKIP,X(2),(ZMAX-ZMIN+1)F(8,0));

END;

END;

END;

END;

/* BOUNDARY CONDITION */

ALLOCATE REMAIN(INLAND:MAXM(1,PERIODS+1),
MAXM(2,PERIODS+1):MAXM(3,PERIODS+1),
MAXM(4,PERIODS+1):MAXM(5,PERIODS+1));

DO I= INLAND TO MAXM(1,PERIODS+1);

DO J= MAXM(2,PERIODS+1) TO MAXM(3,PERIODS +1);
IF MAXM(5,PERIODS+1)>0 THEN DO;

    DO K=MAX(1,MAXM(4,PERIODS+1)) TO MAXM(5,PERIODS+1);

    REMAIN(I,J,K)=CAPINC*J+
    K*GAININC*CAPTAX;

    END;

END;

IF MAXM(4,PERIODS+1)<=0 THEN DO K=
    MAXM(4,PERIODS+1) TO MIN(0,MAXM(5,PERIODS+1));

    IF DECIS(7,5)>0 THEN EXP=
        DECIS(7,4)*K*GAININC/DECIS(7,5);
    ELSE EXP=0;

    REMAIN(I,J,K)=CAPINC*J+
    (1-PHI)*K*GAININC+CAPLOSS*(1-
    (1+FORE(HORIZON,1))**EXP);

/* THE CAPITAL LOSS TERM ASSUMES THE MAXIMUM
   ANNUAL DEDUCTION WILL BE TAKEN UNTIL THE
   ALLOWABLE CAPITAL LOSS DEDUCTION IS EXHAUSTED.
   IT IS ASSUMED THAT THE FIRST DEDUCTION OCCURS
ONE YEAR AFTER THE SALE, WHICH OCCURS AT THE 
BEGINNING OF THE TERMINAL YEAR. THE DISCOUNT 
IS ASSUMED TO STAY CONSTANT AT TERMINAL LEVEL.*/

END;

END;

END;

PUT FILE (AA) EDIT ('**BOUNDARY CONDITION IS THE SAME', 
' FOR ALL LAND LEVELS**') (SKIP, X(5), A(33), A);

RANGE = ABS(MAXM(5, PERIODS+1) - MAXM(4, PERIODS+1));
NUMBER = TRUNC(RANGE/10);

DO U= 0 TO NUMBER;

ZMIN = MAXM(4, PERIODS+1) + U*10;
ZMAX = MIN(MAXM(4, PERIODS+1) + U*10 + 9,
MAXM(5, PERIODS+1));

PUT FILE (AA) EDIT ('GAIN EQUALS', ZMIN, 'TO', ZMAX) 
(SKIP, X(2), A(13), X(2), F(5, 0), X(2), 
A(2), X(2), F(5, 0));
DO J = MAXM(2, PERIODS+1) TO MAXM(3, PERIODS+1);

PUT FILE(AA) EDIT ('EQUITY=', J*CAPINC, 
(REMAIN(INLAND, J, K)
DO K = ZMIN TO ZMAX)(SKIP, A(7), F(8, 0), X(5), 
(ZMAX-ZMIN+1)F(10, 0));

END;

END;

/* RECURRENCE RELATION */

DO T = -PERIODS TO -1;

CALL RECURR(-T);

FREE REMAIN;

CALL OUTPUT(-T);

FREE P;

CALL TRANS(-T, INLAND);
FREE S;

END; /* T LOOP */

CLOSE FILE (AA);
END MAIN;
*PROCESS('S');
RECURR:PROCEDURE(TIME);
DECLARE FORE(0:50,9) FLOAT EXTERNAL;
DECLARE MAXM(5,10) FLOAT EXTERNAL;
DECLARE IN(4) FLOAT EXTERNAL;
DECLARE DECIS(8,5) FIXED(15,4) DEC EXTERNAL;
DECLARE (DISCOUT(*,*) ,REVENUE(*,*,*))
  FLOAT EXTERNAL DEC (10) CONTROLLED;
DECLARE P(*,* ,* ,*) FLOAT DEC(10) EXTERNAL CONTROLLED;
DECLARE (S(* ,* ,* ,*) ,REMAIN(* ,* ,*)) FLOAT DEC (10) EXTERNAL CONTROLLED;
DECLARE (CONTROL(* ,* ,* ,*) ,LEVEL(* ,* ,* ,*)) FLOAT DEC (10) CONTROLLED;
DECLARE (LAND_B,GAIN_B, CAP_B) FLOAT DEC(10) INIT(0);
DECLARE (LAND_LB,GAIN_LB, CAP_LB) FLOAT DEC(10) INIT(0);
DECLARE (MAXL, MINM, MANNUAL,EQUITY,
  DEBT) FLOAT DEC(10);
DECLARE (ZMAX,ZMIN,NEXTLAND,NEXTCAP,NEXTGAIN) FIXED DEC(6,0);
DECLARE (TAX,UNIQUE,OMEGA,PHI,INLAND,
  BUYLAND,CAPINC, LANDING, GAININC,ANNUAL) EXTERNAL
FLOAT DEC(10);
DECLARE (LEVER,DEBUG1,DEBUG2,RENTIN, RENTOUT,NONLCAP,
        DEBUG3,RENTPER,ADJUST,BEGIN) FIXED DEC(6,2) EXTERNAL;
DECLARE (ALPHA6,ALPHA8,ALPHA9,LR) FIXED DEC(10,4) EXTERNAL;
DECLARE (RENT,TAXES,LAND,RATIO,ASSETS,XINDEX) FLOAT DEC (10) INIT(0);
DECLARE (APPR,INTEREST,CURRENT,TEST,SAVING) FLOAT DEC (10) INIT(0);
DECLARE (GAINING,EXP,YEAR,CAPGAIN,EQUITY1,XCAP,DR,DNR,INCOME)
        FLOAT DEC(10) INIT(0);
DECLARE (TEMP,FLOW,RR,RNR) FLOAT DEC(10);
DECLARE (U,I,J,K,L,M,N,T,W,V) FIXED DEC(6,0);
DECLARE X FLOAT DEC(10) INIT(0);
DECLARE TIME FIXED DEC(6,0);
DECLARE BOUNDS ENTRY;

ALLOCATE S(INLAND:MAXM(1,TIME),MAXM(2,TIME):MAXM(3,TIME),
          MAXM(4,TIME):MAXM(5,TIME)),P(INLAND:MAXM(1,TIME),MAXM(2,TIME):
          MAXM(3,TIME),MAXM(4,TIME):MAXM(5,TIME),8);

DO I=INLAND TO MAXM(1,TIME);

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

DO K=MAXM(4,TIME) TO MAXM(5,TIME);

S(I,J,K)=-999999999;
P(I,J,K,1)=-999999999;
P(I,J,K,2)=-999999999;
P(I,J,K,3)=-999999999;
P(I,J,K,4)=-999999999;
P(I,J,K,5)=-999999999;
P(I,J,K,6)=-999999999;
P(I,J,K,7)=-999999999;
P(I,J,K,8)=-999999999;

/*DETERMINE THE MAXIMUM LAND PURCHASE THAT CAN BE FINANCED IF ALL AVAILABLE CAPITAL IN LAND*/

DR=LEVER*(J*CAPINC+K*OMEGA*GAININC);
MAXL=J*CAPINC+K*GAININC+DR;
DR=MIN(DR,LR*MAXL);
MAXL=J*CAPINC+K*GAININC+DR;
MAXL=MAXL/FORE((TIME-1)*ANNUAL,5);
MAXL=TRUNC(MAXL/LANDINC)-I;
MAXL=MIN(BUYLAND,MAXL,(MAXM(1,TIME+1)-I));

IF MAXL<0 THEN MAXL=0;

DO L=0 TO MAXL;

DO W=-(I+L)*RENTOUT TO
REN Tin* BUY LAND* TIME;

IF DEBUG1=0 THEN DO;

PUT FILE (AA) EDIT ('MAXL',MAXL,'LAND',I,' CAP',J,
    'GAIN',K)(SKIP,A(4),F(8,0),A(4),F(8,0),A(4),
    F(8,0),A(4),F(8,0));

END;

/* MAXL CALCULATED FROM:

D/K+OMEGA*G=LEVERAGE CUTOFF WHEN ALL CREDIT USED=L

IMPLIES D=L*(K+OMEGA*G)

ALSO CAPITAL ABSORBED MUST EQUAL DEBT PLUS EQUITY

THAT IS: BETA*LAND=D+K

IMPLIES LAND=(K+G+L*(K+OMEGA*G))/BETA

DETERMINE MINIMUM SAVING OR MAXIMUM DISAVING.
EQUITY LEVEL MUST NOT GO BELOW THE MAXIMUM AMOUNT
NEEDED IN THE COMING PERIOD USING THE SAVINGS PLAN*/
MINM=-999999999;
YEAR=(TIME)*ANNUAL;
LAND=(I+L)*LANDINC*FORE(YEAR,5);
TEMP=(LAND-
(1+LEVER*OMEGA)*K*GAININC)/(1+LEVER);
TEMP=MAX(TEMP,LAND*(1-LR));
MINM=MAX(MINM,TEMP);

/* MINIMUM CAPITAL WHEN ALL CAPITAL DEVOTED
 TO LAND CALCULATED AS:

K=(LAND*BETA-L*OMEGA*G)/(1+L)

SINCE D/(K+OMEGA*G)=L AND

D+K+G=BETA*LAND

IMPLIES D=BETA*LAND-K-G

IMPLIES BETA*LAND-K-G=L*(K+OMEGA*G)

IMPLIES BETA*LAND-G-L*OMEGA*G=(1+L)*K */
MINM=TRUNC(MINM/CAPINC)-J;
MINM=MIN(MINM,(MAXM(3,TIME+1)-J));
MINM=MAX(MINM,(MAXM(2,TIME+1)-J));

IF DEBUG1=0 THEN DO;

    PUT FILE (AA) EDIT ('MINM',MINM,'LAND',I,'CAP',J,'GAIN',K,'RENTED',W)
             (SKIP,A(4),F(8,0),A(4),F(8,0),A(4),
             F(8,0),A(4),F(8,0),A(6),F(8));

END;

DO M=MINM TO (MAXM(3,TIME+1)-J);

    INTEREST=0;
    TAXES=0;
    RENT=0;
    INCOME=0;
    MANNUAL=M/ANNUAL;
    SAVING=0;
    GAINING=0;

    DO U=0 TO ANNUAL-1;
YEAR = (TIME - 1) * ANNUAL + U;
LAND = (I + L) * LANDINC * FORE(YEAR, 5);
IF U = 0 THEN CAPGAIN = K * GAININC;
ELSE CAPGAIN = K * GAININC + (L + I) *
   LANDINC * FORE(YEAR, 8);
EQUITY1 = (J + U * MANNUAL) * CAPINC
   + CAPGAIN;
EQUITY = EQUITY1 - (1 - OMEGA) * CAPGAIN;
IF (I + L + W) > 0 THEN DO;
   EXP = ALPHA6 + ALPHA8 * LOG(I + L + W) +
       ALPHA9 * LOG(YEAR + BEGIN - 1);
END;
X = REVENUE(YEAR + 1, I + L + W, 2);
XINDEX = X * FORE(YEAR + 1, 4);
XCAP = XINDEX * NONLCAP;
DR = LAND + XCAP - EQUITY1;
IF DR < 0 THEN DNR = 0;

/*FIRST TRY ONLY REAL ESTATE DEBT*/

IF DR > 0 THEN DO;

IF DR <= LR * LAND &
   DR <= LEVER * EQUITY THEN
DNR=0;

/*IF THIS LOOP IS ENTERED THEN REAL ESTATE DEBT ALONE IS ADEQUATE. ALL X FINANCED WITH EQUITY. */

ELSE DO;

IF DR>LEVER*EQUITY THEN
  DR=LEVER*EQUITY;
IF DR>LR*LAND THEN
  DR=LR*LAND;

IF EQUITY1+DR<LAND THEN
  DO;

X=0;
XINDEX=0;
XCAP=0;
DNR=0;
INTEREST=999999999;

/* IF THIS DO GROUP ENTERED, LAND CAN
NOT BE FINANCED, EVEN WITH X=0 */

END;

ELSE DO;

/*/ RECALCULATE X ASSUMING NONREAL
ESTATE DEBT USE */

X=REVENUE(YEAR+1, I+L+W, 3);
XINDEX=X*FORE(YEAR+1, 4);
XCAP=XINDEX*NONLCAP;
DNR=LAND+XCAP-EQUITY1-DR;

IF DNR<0 THEN DO;

/*/ INTERIOR OPTIMUM WITH NONREAL ESTATE
DEBT USE IS LOW ENOUGH THAT NON REAL
ESTATE DEBT IS NOT REQUIRED. THEN USE
ALL X THAT CAN BE EQUITY FINANCED */

DNR=0;
XCAP=EQUITY1+DR-LAND;
XINDEX=XCAP/NONLCAP;
X=XINDEX/FORE(YEAR+1, 4);
IF (DNR+DR)>LEVER*EQUITY
    THEN DO;

    /* X IS AT MAXIMUM THAT CAN BE FINANCED */

    DNR=LEVER*EQUITY-DR;
    XCAP=EQUITY1+DNR+DR-LAND;
    XINDEX=XCAP/NONLCAP;
    X=XINDEX/FORE(YEAR+1,4);
    END;

    END;

    END;

    END;

    END;

    END;

    IF DR>=0 & (EQUITY1+DR)>=LAND THEN
    DO;

    RR=TAX*DISCOUT(TIME,U+1,4)*
FORE(YEAR+1,2);
RNR=TAX*DISCOUT(TIME,U+1)*
FORE(YEAR+1,7);
TEMP=RR*DR+RNR*DNR;
INTEREST=TEMP+INTEREST;

END;

IF DR<0 THEN INTEREST=
DR*FORE(YEAR+1,1)*
DISCOUT(TIME,U+1)+INTEREST;

/* NEGATIVE DEBT EARN AT THE DIS
RATE      */

IF X>0 THEN
TEMP=X^EXP;
ELSE TEMP=0;
TEMP=REVENUE(YEAR+1,L+L+W,1)*
TEMP-XINDEX;
INCOME=TEMP*TAX*
DISCOUT(TIME,U+1)+INCOME;
RENT=RENT+DISCOUT(TIME,U+1)*
W*LANDINC*RENTPER*TAX*
FORE((TIME-1)*ANNUAL+U,5);
TAXES = TAXES + DISCOUNT(TIME, U+1) * 
     (I+L)*LANDINC*TAX 
        *FORE((TIME-1)*ANNUAL+U+1,6);
SAVING = MANNUAL*CAPINC*DISCOUNT(TIME, U+1) 
        + SAVING;
GAINING = (L+I)*LANDINC*FORE(YEAR+1,9) 
        *PHI+GAINING;

IF U=0 THEN X1=X;

IF DEBUG1=0 & L<DEBUG3 THEN DO;

    PUT FILE (AA) EDIT 
    (\'RENT=',RENT,\'TAXES=',TAXES, 
     \'DR   =',DR,\'LAND=',LAND, 
     \'DNR   =',DNR,\'XINDEX=', 
     XINDEX,\'TEMP=',TEMP, 
     \'SAVING=',SAVING) 
    (SKIP,A(5),F(8,0),A(6), 
     F(8,0),A(6),F(8),A(5),F(8), 
     A(7),F(8),A(7),F(8),A(5),F(8), 
     A(7),F(8));

END;
CURRENT = INCOME - INTEREST
- SAVINGS - RENT - TAXES - ADJUSTMENTS;

NEXT LAND = I + L;
NEXT CAP = J + M;
APPR = (I + L) X LAND INC
FORE ((TIME) X ANNUAL, 8) / GAIN INC;
NEXT GAIN = TRUNC (APPR);
IF (APPR - NEXT GAIN) >= 0.5 THEN
   NEXT GAIN = NEXT GAIN + 1;
IF (APPR - NEXT GAIN) <= -0.5 THEN
   NEXT GAIN = NEXT GAIN - 1;
   NEXT GAIN = NEXT GAIN + K;

CALL BOUNDS (NEXT LAND, NEXT CAP,
NEXT GAIN, CURRENT, LAND_B, GAIN_B, CAP_B,
LAND LB, GAIN LB, CAP LB, TIME);
IF CURRENT >= 0 THEN DO;

    TEST = CURRENT + GAINING;
    TEMP = REMAIN(NEXTLAND, NEXTCAP, NEXTGAIN);
    TEMP = TEMP * DISCOUT(TIME, ANNUAL);
    TEST = TEST + TEMP;

END;

ELSE TEST = -999999999;

IF DEBUG1 = 0 THEN DO;

    PUT FILE (AA) EDIT ('DEBT', DR+DNR, 'INTEREST', INTEREST, 'CURRENT', CURRENT, 'TEST', TEST, 'APPR', APPR, 'NG', NEXTGAIN, 'R', REMAIN(NEXTLAND, NEXTCAP, NEXTGAIN))
        (SKIP, A(4), F(10,0), A(8), F(10,0), A(7), F(10,0), A(4), F(10,0), A(4), F(6,2), A(2), F(6,2), A(8), F(10,0));

END;
/* CURRENT IS THE VALUE OF CURRENT
PRODUCTION INCOME AFTER INTEREST
AND TAXES. BECAUSE EQUITY INVESTMENT
IS LIMITED TO RETAINED EARNINGS,
CURRENT MUST BE POSITIVE FOR RELEVANT
SOLUTIONS. */

IF TEST>S(I,J,K) THEN DO;

S(I,J,K)=TEST;

P(I,J,K,1)=L;
P(I,J,K,2)=M;
P(I,J,K,3)=W;
P(I,J,K,4)=X1;

END;

/* RECORD NONUNIQUE CONTROL PATHS.
ONLY ONE RECORDED BECAUSE THAT SHOWS
THE CONTROL PATH IS NOT UNIQUE, BUT
DOES NOT REQUIRE MUCH STORAGE. */
ELSE IF TEST=S(I,J,K) & TEST>-999999999 THEN DO;

IF L>P(I,J,K,1) | L<P(I,J,K,1) |
X>P(I,J,K,4) | X<P(I,J,K,4) |
THEN DO;

P(I,J,K,5)=L;
P(I,J,K,6)=M;
P(I,J,K,7)=W;
P(I,J,K,8)=X1;
UNIQUE=UNIQUE+1;

END;

END;

END; /* M LOOP */

END; /* W LOOP */

END; /* L LOOP */
END; /* K LOOP */

END; /* J LOOP */

END; /* I LOOP */

PUT FILE (AA) EDIT ('PERIOD IS', TIME)(SKIP,A(9),F(8,0));

PUT FILE (AA) EDIT ('LAND AT BOUNDS', LAND_B,' TIMES')
     (SKIP,X(5),A(14),F(8,0),A(7));

PUT FILE (AA) EDIT (' CAP AT BOUNDS', CAP_B,' TIMES')
     (SKIP,X(5),A(14),F(8,0),A(7));

PUT FILE (AA) EDIT ('GAIN AT BOUNDS', GAIN_B,* TIMES')
     (SKIP,X(5),A(14),F(8,0),A(7));

PUT FILE (AA) EDIT ('LAND AT LOWER BOUNDS', LAND_LB,' TIMES')
     (SKIP,X(5),A(20),F(8,0),A(7));

PUT FILE (AA) EDIT (' CAP AT LOWER BOUNDS', CAP_LB,' TIMES')
     (SKIP,X(5),A(20),F(8,0),A(7));

PUT FILE (AA) EDIT ('GAIN AT LOWER BOUNDS', GAIN_LB,' TIMES')
(SKIP,X(5),A(20),F(8,0),A(7));

END RECURR;

*PROCESS('S');

OUTPUT:PROCEDURE(TIME);

DECLARE MAXM(5,10) FLOAT EXTERNAL;
DECLARE P(*,*,*,:) FLOAT DEC(10) EXTERNAL CONTROLLED;
DECLARE (S(*,*,:),REMAIN(*,*,:)) FLOAT DEC (10) EXTERNAL CONTROLLED;
DECLARE (RANGE, NUMBER, ZMAX, ZMIN) FLOAT DEC(10);
DECLARE (UNIQUE,INLAND,CAPINC) FLOAT EXTERNAL DEC(10);
DECLARE TIME FIXED DEC(6,0);
DECLARE (U,I,J,K,L,M,N,T,W,V) FIXED DEC(6,0);
DECLARE X FLOAT DEC(10) INIT(0);
PUT FILE (AA) EDIT ('***S MATRIX PERIOD ',TIME,'***')
     (SKIP,X(5),A(19),F(6,0),A);

RANGE=ABS(MAXM(5,TIME)-MAXM(4,TIME));
NUMBER=TRUNC(RANGE/10);
DO I=INLAND TO MAXM(1,TIME);

PUT FILE (AA) EDIT ('LAND EQUALS',I)
     (SKIP,X(2),A(12),X(2),F(6,0));

DO U= 0 TO NUMBER;

ZMIN=MAXM(4,TIME)+U*10;
ZMAX=MIN(MAXM(4,TIME)+U*10+9,
         MAXM(5,TIME));

PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,'TO',
         ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2),
         A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

PUT FILE(AA) EDIT ('EQUITY=',J*CAPINC,
         (S(I,J,K) DO K=ZMIN TO ZMAX))(SKIP,
         A(7),F(8,0),X(5),(ZMAX-ZMIN+1)F(10,0));

END;

END;

END;

PUT FILE(AA) EDIT ('***P MATRIX PERIOD ',TIME, '***')
         (SKIP,X(5),A(19),F(6,0),A);

DO I=INLAND TO MAXM(1,TIME);

PUT FILE(AA) EDIT ('LAND EQUALS ',I)(SKIP,X(2),A(12),
DO U=0 TO NUMBER;

ZMIN=MAXM(4,TIME)+U*10;
ZMAX=MIN(MAXM(4,TIME)+U*10+9,
   MAXM(5,TIME));

PUT FILE (AA) EDIT ('LAND PURCHASES')(SKIP,X(2),A);
PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,'TO',
   ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2),
   A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

PUT FILE(AA) EDIT ('EQUITY=',J*CAPINC,(P(I,J,K,1)
   DO K=ZMIN TO ZMAX))
   (SKIP,A(7),F(8,0),X(5),(ZMAX-ZMIN+1)(F(10,0)));

END;

PUT FILE (AA) EDIT ('SAVINGS')(SKIP,X(2),A);

PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,'TO',
   ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2),
A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

PUT FILE(AA) EDIT ('EQUITY=", J*CAPINC, (P(I,J,K,2)
DO K=ZMIN TO ZMAX))
(SKIP,A(7),F(8,0),X(5), (ZMAX-ZMIN+1)(F(10,0)));

END;

PUT FILE (AA) EDIT ('LAND RENTED') (SKIP,X(2),A);
PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,’TO’,
ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2),
A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

PUT FILE(AA) EDIT ('EQUITY=", J*CAPINC, (P(I,J,K,3)
DO K=ZMIN TO ZMAX))
(SKIP,A(7),F(8,0),X(5),(ZMAX-ZMIN+1)(F(10,0)));

END;

PUT FILE (AA) EDIT ('NONLAND INPUTS') (SKIP,X(2),A);
PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,'TO',
ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2),
A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

PUT FILE(AA) EDIT ('EQUITY=',J*CAPINC,(P(I,J,K,4)
DO K=ZMIN TO ZMAX)
(SKIP,A(7),F(8,0),X(5),(ZMAX-ZMIN+1)(F(10,0)));

END;

END;

END;

/*NONUNIQUE CONTROL PATHS PRINTED OUT */

IF UNIQUE>0 THEN DO;

PUT FILE (AA) EDIT ('***NONUNIQUE CONTROL PATHS IN PERIOD',TIME,'***')
(SKIP,X(5),A(36),F(6,0),A);

PUT FILE (AA) EDIT ('UNIQUE EQUALS',UNIQUE)(SKIP,A(13),F(6,0));

UNIQUE=0;
PUT FILE(AA) EDIT ('***P MATRIX PERIOD ', TIME, '***')
(SKIP, X(5), A(19), F(6,0), A);

DO I=INLAND TO MAXM(1, TIME);

PUT FILE (AA) EDIT ('LAND EQUALS ', I)(SKIP, X(2), A(12), X(2), F(6,0));

PUT FILE (AA) EDIT ('LAND PURCHASES')(SKIP, X(2), A);

DO U=0 TO NUMBER;

ZMIN=MAXM(4, TIME)+U*10;
ZMAX=MIN(MAXM(4, TIME)+U*10+9, MAXM(5, TIME));

PUT FILE(AA) EDIT ('GAIN EQUALS', ZMIN, 'TO', ZMAX)(SKIP, X(2), A(13), X(2), F(5,0), X(2), A(2), X(2), F(5,0));

DO J=MAXM(2, TIME) TO MAXM(3, TIME);

PUT FILE(AA) EDIT ('EQUITY=', J*CAPINC, (P(I,J,K,5)
DO K=ZMIN TO ZMAX))
(SKIP,A(7),F(8,0),X(5),(ZMAX-ZMIN+1)(F(10,0))); END;

PUT FILE (AA) EDIT ('SAVINGS')(SKIP,X(2),A);

PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,'TO', ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2), A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

PUT FILE(AA) EDIT ('EQUITY=',J*CAPINC,(P(I,J,K,6) DO K=ZMIN TO ZMAX)) (SKIP,A(7),F(8,0),X(5),(ZMAX-ZMIN+1)(F(10,0))); END;

PUT FILE (AA) EDIT ('LAND RENTED')(SKIP,X(2),A);

PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,'TO', ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2), A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);
PUT FILE(AA) EDIT ('EQUITY=', J*CAPINC, (P(I,J,K,7)
  DO K=ZMIN TO ZMAX))
  (SKIP, A(7), F(8,0), X(5), (ZMAX-ZMIN+1)(F(10,0)));

END;

PUT FILE(AA) EDIT ('NONLAND INPUTS')(SKIP,X(2),A);

PUT FILE(AA) EDIT ('GAIN EQUALS',ZMIN,'TO',
  ZMAX)(SKIP,X(2),A(13),X(2),F(5,0),X(2),
  A(2),X(2),F(5,0));

DO J=MAXM(2,TIME) TO MAXM(3,TIME);

PUT FILE(AA) EDIT ('EQUITY=', J*CAPINC, (P(I,J,K,8)
  DO K=ZMIN TO ZMAX))
  (SKIP, A(7), F(8,0), X(5), (ZMAX-ZMIN+1)(F(10,0)));

END;

END;

END;

END OUTPUT;
*PROCESS('S');

TRANS: PROCEDURE (TIME, LAND);

DECLARE MAXM(5, 10) FLOAT EXTERNAL;

DECLARE (S(*, *, *), REMAIN(*, *, *)) FLOAT DEC (10) EXTERNAL CONTROLLED;

DECLARE LAND FLOAT DEC(10);

DECLARE TIME FIXED DEC(6, 0);

DECLARE (I, J, K, U) FIXED DEC(6, 0);

ALLOCATE REMAIN(LAND: MAXM(1, TIME), MAXM(2, TIME): MAXM(3, TIME),

MAXM(4, TIME): MAXM(5, TIME));

DO I = LAND TO MAXM(1, TIME);

DO J = MAXM(2, TIME) TO MAXM(3, TIME);

DO K = MAXM(4, TIME) TO MAXM(5, TIME);

REMAIN(I, J, K) = S(I, J, K);

END;

END;
END TRANS;

*PROCESS('S');

BOUNDS: PROCEDURE (NEXTLAND, NEXTCAP, NEXTGAIN, CURRENT,
  LAND_B, GAIN_B, CAP_B,
  LAND_LB, GAIN_LB, CAP_LB, TIME);

DECLARE MAXM(5,10) FLOAT EXTERNAL;
DECLARE (LAND_B, GAIN_B, CAP_B) FLOAT DEC(10);
DECLARE (LAND_LB, GAIN_LB, CAP_LB) FLOAT DEC(10);
DECLARE (NEXTLAND, NEXTCAP, NEXTGAIN) FIXED DEC(6,0);
DECLARE (CURRENT) FLOAT DEC(10);
DECLARE DEBUG2 FIXED DEC(6,2) EXTERNAL;
DECLARE TIME FIXED DEC(6,0);
DECLARE INLAND FLOAT DEC(10) EXTERNAL;

IF NEXTLAND>MAXM(1,TIME+1) THEN DO;

  NEXTLAND=MAXM(1,TIME+1);
  LAND_B=LAND_B+1;

END;

IF NEXTCAP>MAXM(3,TIME+1) THEN DO;

  NEXTCAP=MAXM(3,TIME+1);
  CAP_B=CAP_B+1;

IF NEXTGAIN > MAXM(5, TIME+1) THEN DO;

NEXTGAIN = MAXM(5, TIME+1);
GAIN_B = GAIN_B + 1;

END;

IF NEXTLAND < INLAND THEN DO;

CURRENT = -999999999;
NEXTLAND = INLAND;
LAND_LB = LAND_LB + 1;

END;

IF NEXTCAP < MAXM(2, TIME+1) THEN DO;

NEXTCAP = MAXM(2, TIME+1);
CURRENT = -999999999;
CAP_LB = CAP_LB + 1;

END;
IF NEXTGAIN<MAXM(4,TIME+1) THEN DO;

NEXTGAIN=MAXM(4,TIME+1);
CURRENT=-999999999;
GAIN_LB=GAIN_LB+1;

END;

END BOUNDS;

/
Because the number of farm size classes in the 1957-1963 period differs from that in later years, the error component estimation method outlined by Judge et al. (1982), does not exactly fit the situation. This appendix outlines the adaptation of the error component method to a case in which the number of classes differ. This exposition will generally use the notation and framework of Judge et al.

The complete set of observations can be written in matrix notation as:

$$
\begin{align*}
Y_t & = X_t \beta + u_t J_{N_1} + e_t \\
\end{align*}
$$

where:

- $Y_t$ = the vector of dependent variable observations for the $t$th year,
- $X_t$ = the matrix of independent variables for the $t$th year,
\( u_t \) = the scalar error term specific to the \( t \)th year,
\( e_t \) = the vector of general error terms for the \( t \)th year,
\( N_1 \) = the number of classes within a year in the first \( T_1 \) years,
\( N \) = the number of classes in the years \( T_1+1 \) to \( T \),
\( T \) = the total number of years,
\( T_1 \) = the number of year with \( N_1 \) classes,
\( T_2 \) = the number of years with \( N_2 \) classes
\( J_N \) = a vector of 1's, where the subscript denotes the size of the vector and
\( \beta \) = the vector of regression coefficients.

In all there are \( T_1 \cdot N_1 + T_2 \cdot N_2 = NT \) observations. The dimensions of the \( Y, X \) and \( e \) vectors are assumed to be comfortable to the \( J \) vectors, hence not all \( Y, X \) and \( e \) vectors have the same dimensions. Expression B1 can be more compactly written as:

\[
(Y)_{(Bl.1)} = X \beta + \begin{bmatrix}
U_1 & J_{N_1} \\
U_2 & J_{N_2}
\end{bmatrix} + e
\]

where:

\( Y \) = a vector of the \( Y_t \) vectors,
\( X \) = a vector of the \( X_t \) matrices,
\( e \) = a vector of the \( e_t \) vectors
\( U_1 \) = a vector of the year errors \( 1-T_1 \), and
\( U_2 \) = a vector of the year errors \( T_1+1-T_2 \).
The symbol \( \times \) denotes the Kronecker product, which is the product of each element of the \( U \) vector with the \( J \) vector. Assumptions used in finding the covariance matrix of the error in expression (B1.1) are:

(B2.1) \( E[U_1 U_1'] = \sigma_u^2 I_{N_1} \)

(B2.2) \( E[U_2 U_2'] = \sigma_u^2 I_{N_2} \)

(B2.3) \( E[ee'] = \sigma_e^2 I_{NT} \)

Equality (B2.4) states that the general and year errors are independent. Then following Judge et al. (1982) the covariance matrix will be:

\[
\Phi = E \begin{bmatrix} U_1 \otimes J_{N_1} & U_1 \otimes J_{N_1} \\ U_2 \otimes J_{N_2} & U_2 \otimes J_{N_2} \end{bmatrix} + e' \begin{bmatrix} I_{N_1 T_1} & 0 \\ 0 & I_{N_2 T_2} \end{bmatrix}
\]

\[
= \sigma_u^2 \begin{bmatrix} I_{N_1 T_1} & 0 \\ 0 & I_{N_2 T_2} \end{bmatrix} + \sigma_e^2 \begin{bmatrix} I_{N_1 T_1} & 0 \\ 0 & I_{N_2 T_2} \end{bmatrix}
\]
The block diagonal property of the disturbances vector arises because the errors between years are uncorrelated, but errors within years are related by the common year error.

The GLS estimator is then:

$$\beta = (X\phi^{-1}X)^{-1} X\phi^{-1}Y$$

By the rules for finding the inverse of Kronecker products and partitioned matrices, the inverse of the covariance matrix can be written as:

$$\phi^{-1} = \begin{bmatrix}
I_{T_1} \otimes V_1^{-1} & 0 \\
0 & I_{T_2} \otimes V_2^{-1}
\end{bmatrix}$$

where:

$$V_1^{-1} = \frac{J_{N_1}J_{N_1}'}{N_1(N_1\sigma^2 + \sigma e^2)} + \frac{I_{N_1}}{\sigma e^2} - \frac{J_{N_1}J_{N_1}'}{N_1\sigma e^2}$$

$$V_2^{-1} = \frac{J_{N_2}J_{N_2}'}{N_2(N_2\sigma^2 + \sigma e^2)} + \frac{I_{N_2}}{\sigma e^2} - \frac{J_{N_2}J_{N_2}'}{N_1\sigma e^2}$$
The transformed variables are:

\[ Y^* = PY = \begin{bmatrix}
I_{T_1} & x \left( I_{N_1} - \frac{b_1}{N_1} J_{N_1} J_{N_1}' \right) \\
0 & I_{T_2} \times \left( I_{N_2} - \frac{b_2}{N_2} J_{N_2} J_{N_2}' \right)
\end{bmatrix} \]

where:

- \( P \) = the transformation matrix such that \( P'P = \Phi^{-1} \),
- \( b_1 = \left( 1 - \frac{\sigma^2}{N_1 \sigma^2 + \sigma_e^2} \right) \) and
- \( b_2 = \left( 1 - \frac{\sigma^2}{N_2 \sigma^2 + \sigma_e^2} \right) \).

As a result of the differing class number the weights in the transformation matrix differ between the early and later observations.

The remaining problem caused by differing class sizes concerns the estimation for the term under the radical in the \( b \) weights \( \{N_1 \sigma^2 + \sigma_e^2 \) or \( N_2 \sigma^2 + \sigma_e^2 \)\}. This term is estimated from the mean square error of regression of the mean output over farm sizes on the mean inputs over farm sizes (Judge et al., 1982, p. 483). If there were data for a large number of years relative to the number of independent variables in each group, then the observations could be split into two groups by the farm size class number and separate regression run. Unfortunately, the number of years with each class number are not adequate for such a strategy. In
the 1957-1969 data, the first six years have four classes and the last seven years have five classes. With four independent variables, that leaves the mean square error estimate with only two or three degrees of freedom. The problem with using the mean square error term from data with both sets of farm size classes is that it is theoretically some average of the value which would be derived using the class number from each group. This introduces an additional level of approximation into the estimation.

To avoid the unreliable estimates from the separate regressions on an inadequate number of observations and the added approximation in using the unadjusted estimate from the combined data set, the exact expected value of the mean square error from the combined regression was derived and the bias identified. The expected value expression was then used to find an improved estimate of the year variance (\(\sigma_u^2\)), which was in turn used in calculating the \(b\) weights. The model for the regression on the mean values can be stated.

$$\bar{Y} = \bar{X} + W$$

where:

$$W = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \begin{bmatrix} e_1 'J_{N_1} \\ N_1 ' \\ e_1 'J_{N_2} \\ N_2 \end{bmatrix}$$

\(e_1\) = the vector of errors from the years 1-T, and
\(e_2\) = the vector of errors from the years T_{1+1}-T.
Under the assumptions (B2.1-2.4) the covariance matrix of the error term \((W)\) is:

\[
B4) \quad E[WW'] = \begin{bmatrix}
I_{T_1} \left( \sigma_u^2 \frac{\sigma_e^2}{N_1} \right) & 0 \\
0 & I_{T_2} \left( \sigma_u^2 \frac{\sigma_e^2}{N_2} \right)
\end{bmatrix}
\]

The regression residual from model (B3) may be written (Johnston, 1972, p. 128):

\[
Z = \bar{Y} - \bar{X} \beta
\]

\[
= \left[ I_T - \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}' \right] W
\]

The residual sum of squares can then be written:

\[
B5) \quad Z'Z = W' \left[ I_T - \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}' \right] W
\]

because the matrix \((I-\bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}')\) is symmetric and idempotent.

The expected value of the sum of squared residuals can be expressed as the expected value of the trace of the right hand side of equation (B5) because the sum of squares is a scalar and the trace of a scalar is equal to itself:

\[
E[Z'Z] = E[TrW' \left[ I_T - \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}' \right] W]
\]
By the circular cumulative property of traces and the usual regression assumption that the Xs are known without error:

\[
E[Z'Z] = E[WW'(I_T - \bar{x}(\bar{x}'\bar{x})^{-1}\bar{x}')] \\
= E[WW](I_T - \bar{x}(\bar{x}'\bar{x})^{-1}\bar{x}') \\
= \begin{bmatrix}
I_{T_1}(\sigma_u^2 + \frac{\sigma_e^2}{N_1}) & 0 \\
0 & I_{T_2}(\sigma_u^2 + \frac{\sigma_e^2}{N_2})
\end{bmatrix} (I_T - \bar{x}(\bar{x}'\bar{x})^{-1}\bar{x}')
\]

By noting that the expected value of the error term (W) can be written as a sum of the year variance multiplied by an identity matrix of dimension T, and the general variance multiplied by a matrix with reciprocals of the farm class number along the diagonal, and that the trace of a sum is the sum of the traces, the expected value of the square residuals becomes:

B6) \[
E[Z'Z] = tr\sigma_u^2I_T(I_T - \bar{x}(\bar{x}'\bar{x})^{-1}\bar{x}) \\
+ tr\sigma_e^2 \begin{bmatrix}
\frac{1}{N_1} I_{T_1} & 0 \\
0 & \frac{1}{N_2} I_{T_2}
\end{bmatrix} (I_T - \bar{x}(\bar{x}'\bar{x})^{-1}\bar{x}')
\]
The first term in expression (B6) follows the classic OLS pattern and collapses to the variance multiplied by the degrees of freedom:

$$\text{tr} \sigma^2 \mathbf{I}_T \left( \mathbf{I}_T - \bar{X}' \overline{\mathbf{X}}^{-1} \bar{X}' \right) = \sigma^2 (T-K)$$

where:

$$K = \text{the number of independent variables including the intercept.}$$

The second term can be expanded to:

$$\text{tr} \sigma^2 \begin{bmatrix} \frac{1}{N_1} \mathbf{I}_{T_1} & 0 \\ 0 & \frac{1}{N_2} \mathbf{I}_{T_2} \end{bmatrix} - \text{tr} \sigma^2 \begin{bmatrix} \frac{1}{N_1} \mathbf{I}_{T_1} & 0 \\ 0 & \frac{1}{N_2} \mathbf{I}_{T_2} \end{bmatrix}$$

$$\cdot \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}$$

where the trace of the product of the general variance and the diagonal matrix is:

$$\text{tr} \sigma^2 \begin{bmatrix} \frac{1}{N_1} \mathbf{I}_{T_1} & 0 \\ 0 & \frac{1}{N_2} \mathbf{I}_{T_2} \end{bmatrix} = \sigma^2 \left( \frac{T_1}{N_1} + \frac{T_2}{N_2} \right)$$

but the trace of the term in the $X$ matrices is not easily evaluated in general. With an estimate of the general variance ($\sigma^2$), the $X$ matrix term can be calculated for specific cases from the data. An estimate of
the general variance term can be derived from the dummy variable model.

The expected value of the sum of square residuals is then:

$$E[Z'Z] = \sigma^2 u^2 (T-K) + \sigma^2 e^2 \left[ \frac{T_1}{N_1} + \frac{T_2}{N_2} - \text{tr} \begin{bmatrix} \frac{1}{N_1} I_{T_1} & 0 \\ 0 & \frac{1}{N_2} I_{T_2} \end{bmatrix} \right]$$

This expected value can be solved to yield an improved estimate of the year variance:

$$\sigma^2 u^2 = \frac{Z'Z}{T-K} - \frac{\sigma^2 e^2}{T-K} \left[ \frac{T_1}{N_1} + \frac{T_2}{N_2} - \text{tr} \begin{bmatrix} \frac{1}{N_1} I_{T_1} & 0 \\ 0 & \frac{1}{N_2} I_{T_2} \end{bmatrix} \right]$$

which is in turn used to calculate the \(b\) weights.
GENERAL SUMMARY AND CONCLUSIONS

These three studies suggest that capital gains and losses may have a major impact on production and finance decisions. Capital gains provide an incentive to increase use of the appreciating asset, incur greater debt to buy the asset, and decrease holdings of items that are consumed in the production process. Capital losses tend to have the opposite effects as decision makers attempt to economize on the use of the deprecating asset. Various constraints may dampen or eliminate the capital gains and loss effects from observable decisions. These constraints include limited availability of the appreciating resource and the inability to raise equity capital from outside sources, beyond the decisionmaker's own funds.

The theoretical model indicates that if borrowing against unrealized capital gains is permitted, the optimal farm decisions may be affected by land price risk. When uncertain unrealized capital gains can be substituted for equity, the leverage ratio becomes a random variable. Solutions to the stochastic optimal control problem suggest that the main result of land price uncertainty is to force added financial flexibility. More liquid assets may be held and less debt incurred to allow for the possibility that borrowing power may be eroded by unanticipated capital losses.

The exploratory empirical work suggests that the capital gains and loss effects may be observable under the economic conditions of the 1970s and 1980s. The model has the potential of explaining much of the asset
and financial restructuring that occurred in U.S. farm firms during the 1970s. This restructuring involved increasing investment in long-term assets, such as land, and decreasing holdings of liquid assets, such as financial assets, grain inventories, and livestock (Boehlje and Eidman, 1983, p. 937). Debt use increased substantially during the period.

Within the framework of this model, the asset and financial restructuring is consistent with the decisions that would be undertaken in an environment of large capital gains. The larger land investment and smaller nonland holdings allow more capital gain to accrue, but it also leaves the farm firm financially vulnerable in times of lower output prices and unexpected farmland price declines. The model indicates that if farmland price decreases are anticipated, the growth in farm acreage will tend to be smaller and more nonland inputs will be used. Debt use is not decreased in the simple model used for the exploratory research when land prices are expected to decline, but the increased holdings of liquid nonland assets allows greater flexibility in coping with the financial effects of capital losses.

The empirical work in the third study has only begun to explore and test the implications of the theoretical model outlined in the first two sections. Research on the impact of capital gains and losses on enterprise choice would be a straightforward extension of the existing model. Adjustment cost and availability of rental land would make the empirical work more realistic. The issue of land price risk is important in both farm management decision making and in the formulation of public
policy. Extension of the model to a stochastic framework would help in examining land price uncertainty effects. In a more general framework, the theoretical framework could be extended to other situations in which capital gains and losses occur, such as urban real estate, art, or antiques.
REFERENCES FOR INTRODUCTION AND SUMMARY


I gratefully acknowledge the support and guidance of my major professor, Dr. Michael Boehlje, which made this dissertation possible. He planted the idea which grew into this dissertation and he faithfully pruned, edited and criticized it as it developed. I also acknowledge the assistance of my other committee members: Dr. J. Arne Hallam, Dr. Roy Hickman, Dr. Bob Jolly and Dr. John Schroeter. Finally, I would like to thank my typist, Carolyn Millage, for her untiring efforts.