Higgs-boson contributions to gauge-boson mass shifts in extended electroweak models

Stephen Richard Moore
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HIGGS-BOSON CONTRIBUTIONS TO GAUGE-BOSON MASS SHIFTS IN EXTENDED ELECTROWEAK MODELS

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Higgs-boson contributions to gauge-boson mass shifts in extended electroweak models

by

Stephen Richard Moore

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

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Major: High Energy Physics

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I. INTRODUCTION

The weak interactions,\(^1\) which are responsible for β decay, μ decay, most hyperon decays, etc., are characterized as being very weak and very short ranged. The interactions do not conserve such quantum numbers as strangeness, charm, and isospin, as do the strong and electromagnetic interactions. In addition, the weak interactions do not conserve parity and charge conjugation separately, at least in the low-energy domain, meaning they distinguish between left- and right-helicity particles. Only in the last decade have we started to gain a fundamental understanding of the weak force.

The first picture of weak interactions was the Fermi current-current theory.\(^2\) Weakly-interacting fermions coupled via a local interaction involving four particles, as shown in Figure 1.1. The interaction was characterized by a

![Figure 1.1. Muon decay in the Fermi current-current theory](image-url)
dimensional coupling constant, $G_F = 1.17 \times 10^{-5}$ GeV$^{-2}$. The theory gave an adequate description of $\mu$ and $\beta$ decays. However, it had several fatal flaws. At high energies, the theory violated unitarity. That is, cross-sections grew larger than bounds set by the conservation of probability. If we consider some cross-section $\sigma$, then on dimensional grounds we expect the high energy behavior to be

$$\sigma \approx \text{constant} \times G_F^2 s,$$  \hspace{1cm} (1.1)

where $s$ is the total center of mass energy square. But in every partial wave, the unitarity limit reads

$$\sigma \approx \text{constant} / s.$$  \hspace{1cm} (1.2)

Unitarity violation occurs when energies are of the order $\sqrt{s} = G_F^{-1/2}$. This meant that the theory was at best a low-energy effective theory. In addition, the theory was not renormalizable. Physical quantities were approximated using a perturbation expansion; higher order terms of the expansion (quantum fluctuations) of the Fermi theory gave divergent expressions. The two difficulties of the Fermi theory are related to each other. Violation of unitarity makes the divergences appearing in the higher order terms uncontrollable. It is this unrenormalizability feature which sustained a continued search for an improvement of the Fermi theory and eventually led the electroweak theory to its present form.
Physical processes can be calculated from a theory by use of the framework known as quantum field theory. This was vividly demonstrated with the success of Quantum Electrodynamics (QED) in the 1950s. Unfortunately, we can not solve a theory such as QED exactly. Rather, we make a perturbation expansion about some small parameter, such as the fine structure constant $\alpha$. This procedure can be represented by the Feynman diagram notation. The first term in the expansion (the lowest-order, or tree-level, diagrams) can be used in a straightforward manner. The higher-order diagrams, representing the presence of quantum fluctuations, are expected to give corrections of order of powers of $\alpha$. Instead, these diagrams lead to divergent integrals. The procedure by which these divergences are consistently removed is known as renormalization. The systematic application of renormalization in QED is due to Dyson, Tomonoga, Feynman, and Schwinger. The success of QED with renormalization is well-documented: its predictions have been verified to an incredible accuracy (approximately one part in $10^9$).

On the other hand, the Fermi current-current theory is non-renormalizable. This is due to the coupling constant $G_F$ not being dimensionless. At high enough energy, we can not be content with the tree-level approximation. However, when we include higher-order diagrams, we are plagued by divergences.
These divergences can be removed, but at the expense of a growing number of arbitrary parameters. Thus, we have a lack of predictive power. In practice, nonrenormalizability makes computations impossible beyond the tree level.

We would therefore like to transform the Fermi theory into a renormalizable field theory. A reasonable attempt, parallel to the formation of QED, would be to postulate an intermediate vector boson \( W \) which mediates the weak interaction. Then the four-fermion interaction would be a low-energy approximation to a finite-range interaction (Figure 1.2). Our theory now resembles QED in that they are both mediated by the exchange of a vector particle. Unlike the photon, however, the \( W \) is
electrically charged, massive, and couples only to left-handed particles and right-handed antiparticles. The coupling of the $W$ to fermions has a strength $g$. We assume the $W$ to be very massive; the theory reduces to the Fermi theory at low energies provided

$$\frac{g^2}{M_W^2} = \frac{G_F}{\sqrt{2}}$$

(1.3)

where $M_W$ is the $W$-boson mass.

Unfortunately, the theory is still not renormalizable.

The tree-level approximation to a scattering amplitude such as $\nu \bar{\nu} \rightarrow \nu \bar{\nu}$ would be reduced by a factor of $M_W^2/s$ with respect to the Fermi amplitude, and thus would no longer be a problem. However, other amplitudes, such as $\nu \bar{\nu} \rightarrow W^+W^-$, still have a bad behavior. Because the gauge bosons are massive, their propagators have a longitudinal component, which gives rise to the divergences. The problem is that, unlike QED, this is not a gauge theory. In QED, we could add a photon mass, which did not spoil the renormalizability. This results from the noninteraction of the longitudinal and transverse components of the propagator. But in this theory, the longitudinal and transverse parts do interact, and the theory is not renormalizable for an arbitrary $W$-coupling to matter fields. Thus, our theory is still incomplete.
Through the work of Yang and Mills, it seemed obvious that the solution was to use a non-abelian group for the local gauge symmetry. The non-abelian case differs from the abelian case in that the gauge fields carry the "charges" associated with the generators of the group. That is, while the photon is neutral (electric charge), the W will carry weak "charge" - called weak isospin. A result of this is that there will be self-interactions among the gauge bosons. Such a theory will be renormalizable, with divergences being removed by the redefinition of a finite number of masses and coupling constants. Unfortunately, the addition of vector-meson mass terms to the Lagrangian would break the gauge invariance and lead to a nonrenormalizable theory. While this is fine for the photon and QED, the weak bosons are not massless. Thus, it would seem to not fit into the gauge theory properly.

The problem was solved in 1967-68 independently by Weinberg and Salam, culminating a long line of work due to many people. In 1957, Schwinger proposed a model with a triplet $W^{\pm,0}$ of vector bosons, with the $W^0$ being identified with the photon. Bludman started with a gauge group $SU(2)$, and identified the neutral member as a new gauge boson, giving rise to a neutral weak current. The unification of the electromagnetic and weak forces was proposed in 1961 by Glashow based on a $SU(2) \times U(1)$ gauge group. As yet the
bosons were massless; it was still unknown how to insert mass terms. This problem was solved when Weinberg\textsuperscript{12} and Salam\textsuperscript{13} suggested the use of the Higgs mechanism.\textsuperscript{14} They proposed starting with a massless theory involving a SU(2)×U(1) symmetry. The local gauge symmetry is then broken spontaneously, and the gauge bosons gain a mass. Through the work of \textquoteleft\textquoteleft t Hooft,\textsuperscript{15} \textquoteleft\textquoteleft t Hooft and Veltman,\textsuperscript{16} and Lee and Zinn-Justin,\textsuperscript{17} it was proven that the Higgs mechanism maintains enough of the essence of the unbroken theory to render the broken theory renormalizable also. Quarks were included through the use of the GIM mechanism.\textsuperscript{18} Thus, we have a complete theory to describe weak and electromagnetic interactions - the Glashow-Weinberg-Salam Model, or Standard Model (SM).

The SM has a SU(2)×U(1) gauge structure. We start with four massless gauge bosons. A scalar-boson weak-isospin doublet, consisting of two complex fields, is introduced. Spontaneous symmetry breaking (SSB) is activated by giving the neutral real field a non-zero vacuum expectation value (vev). (Only the neutral scalar field receives a vev to preserve charge conservation.) After SSB, we have three massive gauge bosons, one massless gauge boson, and one massive scalar boson. Three of the real scalar fields have become Goldstone bosons. They were "eaten" by the gauge bosons, giving rise to
the gauge-boson masses. The total number of degrees of freedom, four, is preserved by the SSB. The three massive and one massless gauge bosons are of course identified as the $W^+$, the $Z$, and the photon $A$, respectively. The remaining physical scalar boson is called the Higgs boson. The theory remains renormalizable; we say the SSB was "soft". We note that the doublet is the simplest scalar field which will insure a proper mass structure for all particles in the theory. The $SU(2) \times U(1)$ gauge structure, together with this simplest Higgs sector, is known as the Minimal Standard Model (MSM).

Up to the present time, all experimental evidence points to the SM as being the correct theory for electroweak interactions. The SM predicts the existence of neutral current weak interactions mediated by the $Z$ boson; the discovery of neutral currents in 1974 was a major success for the SM. Since that time many neutral current experiments have been performed. The results of these experiments have been in good agreement with each other, providing strong evidence for the SM. In 1983, the UA1 and UA2 experiments at the CERN SPS collider announced the discovery of the $W$ and $Z$ bosons, with masses close to those predicted. Thus, the SM is generally regarded as being the proper description of electroweak interactions.
Yet, there are still many unanswered questions. The non-abelian nature of the gauge group gives rise to couplings between the gauge bosons, which have not been observed. The Higgs-boson mass is not determined by the theory, and the Higgs boson has not been discovered. The structure of the Higgs sector is not constrained by the gauge interaction; and it is generally felt that the dynamics of the SSB are not understood. The parity violation observed in the weak interactions is inserted by hand. Experimental measurements are not of sufficient accuracy to probe the higher-order corrections and renormalization structure. It is uncertain just how the SM will fit into a Grand Unified Theory (GUT) of strong, weak, and electromagnetic interactions, and whether it is an adequate description of electroweak interactions for energies above a few hundred GeV. Until these and other problems are solved, we cannot give full acceptance to the SM, and it has to be challenged continually.

Therefore, we can ask if any other theories, while agreeing with the SM at low (present) energies, give different and presumably richer predictions for higher-energy phenomena. In particular, we would like to make an extension of the SM, keeping the basic idea yet allowing for more structure which will appear at higher energies. We find that the SM is not unique; there are many such variations. Possible methods
include:

1) extensions of the Higgs sector,\textsuperscript{24}

2) extensions of the gauge sector,\textsuperscript{25}

3) introduction of fermion-number-violating Majorana mass terms for neutrinos,\textsuperscript{26}

4) horizontal symmetries.\textsuperscript{27}

This list is by no means exhaustive.

The most straightforward way to extend the SM is to increase the number of Higgs bosons. The model described earlier had one scalar doublet, giving rise to one physical Higgs. The only reason to have just one doublet was economy; it is sufficient to activate the SSB. We can, however, consider a different Higgs structure. We can activate the SSB with a different Higgs multiplet, or we can add more Higgs multiplets (more doublets or other multiplets). There are some valid reasons for adding more Higgs representations. The nonzero vev of the neutral Higgs field gives rise to the gauge-boson masses. This vev also gives rise to the fermion masses, with the masses proportional to the vev times the Higgs-fermion coupling constant. Yet, the fermions vary in mass over a wide range of values (by about five orders of magnitude), while the W and Z bosons are of an even heavier mass scale. This can be accounted for by allowing the coupling constants to range widely. This is generally felt to
be unsatisfying. An alternative would be to add more Higgs multiplets with different vevs. Then, different vevs could give rise to different mass scales. The details of which vev gives rise to which mass scale depend on the details of the Higgs potential and imposed symmetries.

The simplest extension of the SM would be to add a second Higgs doublet. This 2-doublet model has five physical Higgs bosons - three neutral ones and a charged pair. We may expect the 2-doublet model to differ from the MSM in some non-trivial ways. A generalization of this model would be to have $n$ doublets. With the $n$-doublet model there are $2n-1$ neutral Higgs bosons and $n-1$ charged pairs.

We are somewhat constrained when we try to add other multiplets. We can define the rho parameter as

$$\rho = \frac{\frac{M_W^2}{M_Z^2 \cos^2 \theta_W}}, \tag{1.4}$$

where $M_W$ and $M_Z$ are the $W$ and $Z$ masses and $\theta_W$ is the Weinberg angle, a free parameter of the theory. The $\rho$ parameter is determined by the weak-isospin structure of the Higgs multiplets. Given $n$ multiplets, we find that

$$\rho = \frac{\sum_{i=1}^{n} \left[ t_i(t_i+1)-t_{3i}^2 \right] v_i^2}{\sum_{i=1}^{2n} t_{3i}^2 v_i^2} \tag{1.5}.$$
where $t^i_1$ is the total weak isospin of the $i^{th}$ Higgs multiplet, $t^i_3$ is the third component of isospin for the neutral field of that multiplet, $v^i$ is the vev for that neutral, and the sum is over all multiplets. Thus, for a doublet, $\rho = 1$. One of the phenomenological successes of the MSM is that neutral current experiments have determined $\rho \neq 1$. This provides a constraint on the allowed Higgs sectors. For instance, a triplet ($t = 1$) with $t^3 = \pm 1$ will give $\rho = 1/2$. To maintain $\rho = 1$ as an exact identity we can only allow doublets (ignoring exotic cases with large isospin). However, the experimental constraint of $\rho \neq 1$ can still be satisfied if the vevs of the $t \neq 1/2$ multiplets are small compared to the vevs of the doublets, or if the number of doublets is much larger than the number of other multiplets. For instance, a model with a doublet and a $t^3 = \pm 1$ triplet will have

$$\rho = \frac{1 + 2k^2/v^2}{1 + 4k^2/v^2}, \quad (1.6)$$

where $k$ and $v$ are the vevs of the triplet and doublet, respectively. This expression is approximately one for $k \ll v$. We will require this to preserve the $\rho \neq 1$ relation.

Another alternative to the SM is to extend the gauge boson sector. This is done by starting with a larger gauge group structure. Possibilities include the $SU(2) \times U(1) \times G$ natural
models.\textsuperscript{28} SU(2)\texttimes U(1)\texttimes U(1) models,\textsuperscript{29} and SU(2)\textsubscript{L}\texttimes SU(2)\textsubscript{R}\texttimes U(1)\textsubscript{B-L} left-right models.\textsuperscript{30} These models are characterized by a richer gauge-boson structure, with a set of one or more additional gauge bosons. These new bosons are generally more massive than the SM gauge bosons. Thus, the extended models reduce to the SM in the low-energy regime.

An illustration of the richer structure of extended models is the anomalous magnetic moment $a_\mu = (\alpha - 2)/2$ of the muon.\textsuperscript{31} The calculation and measurement of $a_\mu$ has been a cornerstone in the experimental verification of QED, and has the potential to provide an excellent testing ground for electroweak gauge theories. Experiments under consideration should allow the probing of the weak contributions to $a_\mu$ in the near future. A calculation of these weak effects in the SM\textsuperscript{32} and in extended models\textsuperscript{33} reveals that the extended models, while constrained to agree with the SM at low energies, differ significantly from the SM in their $a_\mu$ predictions. For instance, the gauge-boson contributions to $a_\mu$ may vary by as much as 50% from the SM value. More interesting, but less certain, are contributions from the Higgs bosons. In certain scenarios, the Higgs-boson contribution can dominate the moment. Thus, a detailed analysis of Higgs-boson effects in models with a non-minimal Higgs sector should be undertaken.
Such an extended Higgs sector will not upset the renormalizability of the models. If we ignore the Higgs-fermion couplings (which are proportional to $m_f^2/M_W^2$), then at the tree level the Higgs particles are not involved in low-energy processes such as muon decay and $\nu e$ scattering. These models are presently indistinguishable from the SM as long as $\rho \approx 1$. This situation will be changing, however, with the next generation of accelerators. The SLC $e^+e^-$ collider at Stanford, the TRISTAN collider in Japan, and the LEP collider in Europe, coupled with more precise calibration and analysis at CERN, should give us a very good determination of the $W$ and $Z$ masses within the next few years. We can use this to check the validity of radiative corrections in the SM and extended models, and to differentiate between extended models.

In the SM, we can calculate different numerical quantities. Comparison of these with low-energy experiments gives us predictions for the $W$ and $Z$ masses. When we include first-order corrections, however, we get different predictions for the masses. This difference is known as the mass shift. We expect this shift to be of order $\alpha (1\%)$, the fine structure constant, as it is our perturbation-expansion parameter. A detailed calculation shows that the mass shift is actually of order $5\%$, due to terms of the form $\ln(m_f^2/M_W^2)$. It is this
first-order-corrected value, or shifted value, which we expect to be the physical mass (except for terms of higher powers of $\alpha$).

Thus we would like to ask: what are the mass shifts in models with a more complex Higgs structure? There is hope they may be significantly different from the MSM values. For instance, in the MSM there are terms of the form

$$\frac{M_H^2}{M_W^2} \ln(M_H^2/M_W^2) \quad (\text{for } M_H^2 \gg M_W^2), \quad (1.7)$$

where $M_H$ is the Higgs-boson mass, in the percentage mass shift. Since the Higgs mass is not set in the theory and is only loosely bounded by other considerations, these terms may potentially be large. However, these terms are cancelled in the renormalization process. The leading Higgs-dependent term turns out to be

$$\ln(M_H^2/M_W^2) \quad (\text{for } M_H^2 \gg M_W^2), \quad (1.8)$$

which is not an important part of the shift. As we add more multiplets, we may speculate that some potentially large Higgs-dependent terms may survive. Thus, our goal is to calculate the Higgs-dependent portion of the $W$ and $Z$ mass shifts, looking for terms which may become dominant in the limit of large Higgs mass. Comparison to experimental values of the gauge-boson masses may enable us to differentiate
between models with different Higgs sectors, and possibly even offer limits on Higgs-boson masses.

The plan of this paper is as follows. In Chapter II, we will review the SM in more detail, concentrating on the Higgs sector. We will also present the 2-doublet, n-doublet, and doublet-triplet models. In Chapter III, we will review the renormalization procedure. In Chapter IV, we show the renormalization of the MSM and compare renormalization schemes. We will calculate the Higgs-dependent parts of the mass shifts for the MSM and models with more complex Higgs structures, finding the potentially dominant terms, in Chapter V. We present our conclusions in Chapter VI. Some Feynman diagram integrals are calculated in Appendix A, and Appendix B has some needed large-mass expansions.
II. STANDARD AND EXTENDED ELECTROWEAK MODELS

A. Introduction

In this chapter, we will review in detail the minimal version of the Standard Model. We will examine the Lagrangian and derive the Feynman rules which enable us to calculate various quantities relating to the weak interactions. To do this, we assume the basic tools of quantum field theory and the formalism of gauge theories. In addition, we will derive the Lagrangian and Feynman rules for standard models with extended Higgs sectors. We will consider the case of two doublets of Higgs fields, and generalize this to \( n \) doublets, where \( n \) is arbitrary. We will also consider the case of the triplet Higgs, with and without doublets.

B. Minimal Standard Model Lagrangian

The SM is based on the gauge group \( \text{SU}(2) \times \text{U}(1) \). \( \text{SU}(2) \) is the group of \( 2 \times 2 \) unitary matrices with determinant one. It has three generators, \( T^i \), \( i = 1, 2, 3 \). \( \text{U}(1) \) is the one-dimensional unitary group, with generator \( Y \). The generators of \( \text{SU}(2) \times \text{U}(1) \) obey the commutation relations

\[
[T^i, T^j] = i\epsilon^{ijk} T^k
\] (2.1)
\[ \varepsilon^{ijk} = 0 , \]

where \( \varepsilon^{ijk} \) is the anti-symmetric Levi-Civita tensor in three dimensions, which forms the structure constants of the SU(2) group. The gauge coupling constants for SU(2) and U(1) are denoted by \( g \) and \( g' \), respectively. The group generators have eigenvalues \( t, t_3, \) and \( y \), usually referred to as the weak isospin and weak hypercharge. We will denote multiplets by their quantum numbers \((t,y)\). The third component of weak isospin and the weak hypercharge are related through the Gell-Mann-Nishijima formula, which in operator form is given by

\[ Q = T^3 + \frac{Y}{2} , \quad (2.2) \]

where \( Q \) is the charge operator. We introduce the gauge fields \( W^i_\mu, i = 1, 2, 3 \), and \( B_\mu \). The covariant derivative is then

\[ D_\mu = \partial_\mu - ig T^i_\mu W^i_\mu - ig' \frac{Y}{2} B_\mu . \quad (2.3) \]

Here, we use \( T^i_j \) to denote the weak isospin operators and their matrix representations.

The weak interaction has been observed to violate parity (at least in the low-energy domain). That is, it differentiates between particles of left- and right-handed helicities. We can incorporate this parity violation into a gauge theory by putting the left- and right-handed fermions
into different representations. We define the helicity states by

\[ \Psi_L = \frac{1 + \gamma_5}{2} \psi_R. \]  

We assign the left-handed electrons and neutrinos to doublets of the group SU(2), and right-handed electrons to singlets:

\[ L \equiv \left( \begin{array}{c} \nu_e \\ e \end{array} \right), \]

\[ R \equiv e_R, \]  

where \( e_L \equiv \psi_L \), and so on. The left-handed doublets have \( y_1 = -1 \), and the right-handed singlets have \( y_2 = -2 \). The inclusion of a right-handed neutrino singlet will give rise to neutrino masses, if desired. For the moment, the only fermions we will consider are the electron and its associated neutrino.

The Lagrangian may now be written down. One part contains the fermions and their interactions with the gauge bosons and the other contains gauge fields only. They are separately gauge invariant.

\[ L_F = \bar{L} \left( i \gamma^\mu (\partial_\mu - igT^a W^a - i \frac{g'}{2} B_\mu \right) L \]

\[ + \bar{R} i \gamma^\mu (\partial_\mu - i \frac{g'}{2} y_2 B_\mu) R, \]  

and
where
\[ F_{\mu\nu}^i \equiv \partial_{\mu} A_{\nu}^i - \partial_{\nu} A_{\mu}^i + g \epsilon^{ijk} A_{\mu}^j A_{\nu}^k , \]
\[ B_{\mu\nu} \equiv \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} . \]

Summation over repeated group indices is implied. Note that there are no bare mass terms. The possible fermion mass terms \( \bar{L}L \) and \( \bar{R}R \) vanish identically, while \( \bar{L}R \) is gauge non-invariant. Gauge invariance also prohibits gauge-boson mass terms. At this stage we have a massless theory.

We are now ready to activate the spontaneous symmetry breaking. We introduce a scalar doublet
\[ \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \]
with hypercharge \( y = +1 \). This is denoted as a \((1/2,1)\) multiplet. The most general renormalizable, gauge invariant potential for \( \Phi \) is
\[ V(\Phi^\dagger \Phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 . \]

Since the energy of a physical system is bounded below, this requires that \( \lambda > 0 \). However, one has a choice of the sign of \( \mu^2 \). For \( \mu^2 > 0 \), the quadratic term of the potential is just a mass term and \( \mu \) is the bare mass of \( \phi \). The potential has a unique minimum at \( |\phi|^2 = 0 \). We say that the system is in the
symmetry mode. For $\mu^2 < 0$, the quadratic term does not
describe a mass term, and there is a continuum of minima at
$|\phi|^2 = -\mu^2/2\lambda$. The vacuum is not unique, and therefore the
breakdown of the gauge invariance of the vacuum is possible.
In this circumstance, we say the system is in the spontaneous
symmetry breakdown mode. This allows $\phi$ to acquire a non-zero
vacuum expectation value for the neutral component:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix},$$

(2.10)

where

$$v \equiv \left( -\frac{\mu^2}{\lambda} \right)^{1/2}.$$

(2.11)

Note that $\langle \phi \rangle$ is one of the minima of the potential (2.9). We
can require $v$ to be real without loss of generality; a complex
vev introduces a common phase to all components of the scalar
fields, which does not enter the Lagrangian due to its
hermiticity.

We can see the physical consequences of this non-zero vev
if we make the transformation

$$\phi \rightarrow \begin{pmatrix} \phi^+ \\ (\phi + i\psi + v)/\sqrt{2} \end{pmatrix},$$

(2.12)

where $\phi$ and $\psi$ are real neutral fields with vanishing vevs.
Then, the Higgs-boson term of the Lagrangian becomes
\[ L_H = \left| (\partial - i\sigma \cdot \vec{A} - i\sigma' B) \psi \right|^2 - V(\psi^\dagger \psi) \] 
\[ = L_{\text{Mass}} + L_{\text{KE}} + L_{\text{Int}} + L_V. \] 

Note that we have suppressed a \( \mu \) index on \( \sigma \) and the gauge fields. We will use this notation throughout. There is now a mass term:

\[ L_{\text{Mass}} = \frac{1}{8} v^2 [g^2 W_1 W_1 + g^2 W_2 W_2 + g^2 W_3 W_3 \] 
\[ + g'^2 B B - 2gg' W_3 B] . \] 

We introduce the charged fields

\[ W^\pm = (W_1 \mp iW_2)/\sqrt{2}. \] 

As a field operator, \( W^+(W^-) \) annihilates a positive (negative) \( W \) boson and creates a negative (positive) \( W \) boson. The charged sector is in diagonal form; the neutral sector is not.

The neutral sector mass matrix is proportional to

\[ \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}, \] 

which has a zero determinant. This tells us that after diagonalization the neutral sector will contain a massless particle. We write

\[ Z^0 = (g' B - g W_3)/\sqrt{g^2 + g'^2} , \]
\[ A = \frac{(g'W_3 + gB)}{\sqrt{g^2 + g'^2}}. \]

The mass terms of the Lagrangian become

\[ L_{\text{Mass}} = M_W^2 W^+ W^- + \frac{1}{2} M_Z^2 Z Z; \]

\[ M_W = \frac{1}{2} g v, \]

\[ M_Z = \frac{1}{2} g_Z v, \quad (2.18) \]

\[ M_A = 0, \]

where

\[ g_Z = \sqrt{g^2 + g'^2}. \quad (2.19) \]

We introduce the Weinberg angle \( \theta_W \) as the angle of rotation between \( W_3 \) and \( B \). That is,

\[ \tan \theta_W = \frac{g'}{g}, \]

\[ e \equiv \frac{gg'}{\sqrt{g^2 + g'^2}} \quad (2.20) \]

\[ e = g' \cos \theta_W = g \sin \theta_W \]

\[ x_W \equiv \sin^2 \theta_W. \]

Note that

\[ g_Z^2 = g^2 / \cos^2 \theta_W. \quad (2.21) \]
The $W^\pm$, $Z$, and $A$ represent the charged weak bosons, the neutral weak boson, and the photon respectively. The 
$SU(2) \times U(1)$ symmetry has been broken, leaving a residual $U(1)$ symmetry describing the electromagnetic interaction. There is no mass term corresponding to the photon because it couples to the charge of the remaining symmetry, $Q$. The other terms in (2.13) are:

\[ L_{\text{KE}} = \frac{1}{2} |\phi|^{2} + |\phi^+|^2 + \frac{1}{2} |\phi^+_\psi|^2, \]

\[ L_{\text{Int}} = \frac{1}{2} g \delta W^{-}(\phi^+) + c.c. + \frac{1}{2} g Z v Z(\phi) \]

\[ + \frac{1}{2} g^2 \delta W^{+} W^{-} \phi + \frac{1}{2} g^2 Z v Z \phi \]

\[ + \frac{1}{2} g \delta W^{-} e^{-1} (\phi^+ - \phi^+) \phi + \frac{1}{2} g Z v Z + c.c. \]

\[ + \frac{1}{2} g Z e^{-1} (\phi^+ - \phi^+) \phi + c.c. \]

\[ + \frac{1}{2} g Z (\phi^+ - \phi^+) \phi + \frac{1}{2} \delta v Z (\phi^+ - \phi^+) \phi \]

\[ + \frac{1}{2} g^2 W^{+} W^{-} (\phi^+ + \phi^+) + \frac{1}{2} g^2 Z Z (\phi^+ + \phi^+) \phi \]

\[ + \delta e^{-1} A A + e g Z (1-2x_{W}) A Z + \frac{1}{4} g Z^2 Z Z + \frac{1}{2} g^2 W^{+} W^{-} (\phi^+ \phi^-) \]

\[ L_{\nu} = - V(\phi^+ \phi) \]
\[ = - 2\mu^2 \phi \phi \\
- 2\nu\lambda(\phi^+\phi^-) - \nu\lambda(\phi\psi\psi) - \nu\lambda(\phi\phi\phi) \]
(2.24)

\[- \lambda(\phi^- \phi^+ \phi^+) - \lambda(\phi^- \phi^+ \psi\psi) - \lambda(\phi\phi\phi^+ \phi^-) \]

\[- \frac{1}{4}\lambda(\psi\psi\phi\phi) - \frac{1}{4}\lambda(\phi\phi\phi) - \frac{1}{2}\lambda(\phi\phi\psi) , \]

where c.c. denotes the complex conjugate of an expression.

The fields \( \phi^\dagger \) and \( \psi \) are the Nambu-Goldstone bosons\(^{38}\) of the \( \tilde{W}^+ \) and \( Z \) bosons. The Goldstone bosons are massless. They are unphysical particles and appear only with zero norm in the Hilbert space. They have been "eaten" to give masses to the gauge bosons, and are now associated with the longitudinal parts of the \( \tilde{W} \) and \( Z \) bosons. The remaining scalar field \( \phi \) is the physical Higgs scalar. It has mass

\[ M_H = \sqrt{-2\mu^2} . \]
(2.25)

However, the theory does not predict what the Higgs potential parameters \( \mu \) and \( \lambda \) are. Thus, the theory gives no prediction for what the Higgs mass should be. This reflects our lack of knowledge about the dynamics of SSB. We will consider constraints on the Higgs mass later.

The scalars and leptons interact:

\[ L_{\text{Yuk}} = - \frac{2}{\sqrt{2}} \lambda R(\phi^+ L) + (\bar{L}\phi)R \]
(2.26)
where $G_e$ is known as the Yukawa coupling constant. This gives rise to fermion mass terms. Equation (2.26) becomes:

$$L_{Yuk} = -\frac{G_e}{\sqrt{2}} \bar{e} e \bar{e} e \phi - \frac{G_e}{\sqrt{2}} \bar{e} e \phi + \frac{G_e}{\sqrt{2}} \bar{e} \gamma_5 e \phi$$

(2.27)

$$- G_e \bar{\nu}_L e_R \phi^+ - G_e \bar{e}_R \nu_L \phi^- .$$

The electron has mass $m_e = G_e v/\sqrt{2}$. The electron couples to the Higgs particle with a strength of $G_e/\sqrt{2}$. The constant $G_e$ is not determined by the theory. However, we know the fermion masses from other considerations. Then, by use of (2.19), we find the Higgs-fermion coupling to be of strength

$$\frac{g m_f}{2 M_W} .$$

(2.28)

As the known fermions all have $m_f \ll M_W$, the Higgs-boson contributions to rates of fermion processes will be suppressed by a factor of $(M_f/M_W)^n$, $n = 2$ or larger. For the purposes of this work, we will ignore the Higgs-fermion couplings.

We can also rewrite the Lagrangian pieces containing the fermion-gauge-boson interactions (2.6) and gauge-interaction terms (2.7) with the physical vector bosons given in (2.15) and (2.17). Explicitly, the $L_F$ piece becomes:

$$L_F = \frac{g}{2\sqrt{2}} \left[ \bar{\nu}_Y^\mu (1-\gamma^5) e_W^\mu + c.c. \right]$$

$$- \frac{g}{2} \tan \theta_W \left( 2 \bar{e}_R \gamma^\mu e_R + \bar{\nu}_Y^\mu \nu + \bar{e}_L \gamma^\mu e_L \right)$$
The last term in equation (2.29) is the usual electromagnetic coupling to the field $A_\mu$. We identify $|e|$ with the electric charge of the electron (defined so $e > 0$). The first term in (2.29) is the charged weak current. We can relate this to the Fermi coupling of Chapter I provided:

$$
\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}.
$$

This interaction has a vector minus axial-vector (V-A) form.

We also have a new interaction - a coupling of $Z$ to a neutral, parity-violating weak current. This is the second term in equation (2.29).

The extension to include the $\mu$ and $\tau$ leptons and the quarks is straightforward. We put each charged lepton and its neutrino in a left-handed $(1/2,-1)$ doublet and the right-handed charged lepton in a $(0,-2)$ singlet. The left-handed quarks go into $(1/2,1/3)$ doublets, while the right-handed quarks go into $(0,4/3)$ (for $u$, $c$, $t$) and $(0,-2/3)$ (for $d$, $s$, $b$) singlets. In detail, these are

$$
\begin{pmatrix}
\nu_e \\
\text{e}_L
\end{pmatrix},
\begin{pmatrix}
\nu_\mu \\
\mu_L
\end{pmatrix},
\begin{pmatrix}
\nu_\tau \\
\tau_L
\end{pmatrix},
$$
for the leptons, and
\[
\left( \begin{array}{c} u' \\ d' \\ c' \\ s' \\ t' \\ b' \end{array} \right)_L,
\left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \right)_L,
\left( \begin{array}{c} t' \\ b' \end{array} \right)_L,
\]
(2.31)

for the quarks, where \( d' \), \( s' \), and \( b' \) are the Kobayashi-Maskawa mixed states. We introduce the K-M matrix \( U_{ii'} \), where the index \( I \) runs over the \( t_3 = +1/2 \) states \( (\Psi_I = \nu_L, \ldots, u_L, \ldots) \) and the index \( i \) runs over the \( t_3 = -1/2 \) states \( (\Psi_i = e_L, d_L, \ldots) \). The index \( n \) runs over all states \( (\Psi_n = e, \nu, u, d, \ldots) \). Then, equation (2.29) becomes

\[
L_F = i\bar{\psi}_n \gamma_\mu \psi_n \\
+ \frac{g}{\sqrt{2}} (\bar{\psi}_I U_{iI} e^\mu \frac{1-\gamma_5}{2} \psi_i W^+ + \bar{\psi}_I U_{iI} \nu^\mu \frac{1-\gamma_5}{2} \psi_i W^-) \\
+ \frac{g'}{g_Z} Q_n \bar{\psi}_n \gamma_\mu \psi_i A^\mu \\
+ \frac{g_Z}{2} \left( \bar{\psi}_I \gamma_\mu \frac{1-\gamma_5}{2} - 2Q_I \left( \frac{g'}{g_Z} \right)^2 \right) \psi_i \\
- \bar{\psi}_I \gamma_\mu \frac{1-\gamma_5}{2} + 2Q_I \left( \frac{g'}{g_Z} \right)^2 \psi_i \right) Z_\mu .
\]
(2.32)

Note that each fermion has its own coupling to the Higgs...
particle, \( G^e \), \( G^\mu \), ...), and hence each is generated a different mass.

The final step in developing the Lagrangian is to perform the canonical quantization. We denote the variable canonically conjugate to \( W^i_\mu \) as \( \Pi^i_\mu \). However, we find that \( \Pi^i_0 \) vanishes. This means that the propagators of the gauge fields have no inverse and hence are singular. Thus, it is impossible to make the covariant quantization consistently. This difficulty is removed if we break the local gauge invariance by adding a gauge-fixing term. The singularity disappears, and we can perform the quantization. It is convenient to work in a special class of gauges known as the \( R_\xi \)-gauges.\(^{40}\) Here, \( \xi \) parameterizes the gauge, running from 0 to \( \infty \). The quantization has two effects on the Lagrangian. First, we have added gauge-fixing terms. These terms will be dependent upon the gauge-parameter \( \xi \). Second, four pairs (one for each gauge boson) of scalar fields which follow anticommutation relations are introduced. These are known as the Fadeev-Popov (FP) ghosts.\(^{41}\) The FP ghosts are unphysical and play only an algebraic role. In the general \( R_\xi \)-gauge, we must consider contributions from the gauge-fixing terms and the FP ghosts.

Physical quantities will be independent of the gauge we choose. Particular choices for \( \xi \) include: the unitary gauge
(U-gauge), $\xi \to 0$; the Feynman gauge, $\xi \to 1$; and the renormalization gauge, $\xi \to \infty$. The U-gauge is particularly meaningful, as it is the physical gauge. In the U-gauge, the unphysical particles - the Goldstone bosons $\phi^\pm$ and $\psi$ and the FP ghosts - decouple from the physical particle. This property of the U-gauge can be seen from the following argument. The U-gauge can be obtained by quantization after the SSB. Then, the $W$ and $Z$ are massive particles. The difficulties associated with the vanishing of the canonical momenta of the massless vector particles are absent, and the Goldstone bosons and the FP ghosts disappear. The only massless particle is the photon, and its ghost decouples from all physical particles due to the $U(1)$ nature of its interactions with other particles. Thus, we only need to consider the physical particles, and can disregard the unphysical ones. The trade-off is a more complicated form for the gauge-boson propagators. We will work exclusively in the U-gauge.

We summarize the Lagrangian as follows (in the U-gauge):

$$L = L_G + L_F + L_H + L_{Yuk};$$

$$L_G = -\frac{1}{2} F_{\mu \nu} F^{\mu \nu} - \frac{1}{4} \rho Z F_{\mu \nu} F^{\mu \nu} - \frac{1}{4} A F^A F_{\mu \nu}$$

$$+ i \frac{g}{2 \rho Z} (g^{\alpha \gamma} g^{\beta \delta} - g^{\alpha \delta} g^{\beta \gamma}) \epsilon_{\alpha \beta \gamma \delta} \partial \chi (J^\alpha W^\beta)^+ W^\gamma Z_S \tag{2.33}$$
\[
\begin{align*}
&+ (\alpha \beta) Z Y S + (\alpha \beta)^+ Y S) \\
&+ g^2 (\alpha \beta) Z Y S + (\alpha \beta)^+ Y S + (\alpha \beta)^+ Y S) \\
&\left[ \left( g \overline{g} g^+ g^+ Y S - g^+ g^+ Y S \right) \right] \not\rightharpoonup Y S (g^2 Z Y S + g^2 Z Y S) \\
&+ g g^+ (2g^+ g^+ Y S - g^+ g^+ Y S - g^+ g^+ Y S) \not\rightharpoonup Y S \\
&+ g^2 (g^+ g^+ Y S - g^+ g^+ Y S) \not\rightharpoonup Y S \\
\end{align*}
\]

where

\[
F_{\mu \nu} \equiv \alpha \mu \nu - \alpha \mu \nu , \text{ etc.}
\]

\[
L_F = i \bar{\psi}_n Y^\mu \gamma^\mu \psi_n \\
+ \frac{g}{\sqrt{2}} \left( \bar{\psi}_u \gamma^\mu \psi_\nu \right) + \frac{g}{\sqrt{2}} \left( \bar{\psi}_u \gamma^\mu \psi_\nu \right) \\
+ \frac{g g^+}{\sqrt{2}} \bar{\psi}_n Y^\mu , \psi_n + \frac{g g^+}{\sqrt{2}} \bar{\psi}_n Y^\mu , \psi_n \\
+ \frac{g^2}{\sqrt{2}} \left( \bar{\psi}_u Y^\mu \right) \\
- \left( \bar{\psi}_u Y^\mu \right) + \frac{g^2}{\sqrt{2}} \left( \bar{\psi}_u Y^\mu \right) \\
L_H = \frac{1}{2} |\lambda_{\mu} \phi|^2 + M_{\phi}^2 \phi
\]
From this Lagrangian, using quantum field theory techniques, the Feynman rules can be found. We summarize the Feynman rules in Figure 2.1.

C. Phenomenology of the Standard Model

As mentioned in Chapter I, the SM has been in good agreement with experimental data. The theory has three fundamental parameters (aside from fermion and Higgs-boson masses) which we can take as $G_F$, $\epsilon$, and $\theta_W$. Two of these are well-known: $G_F = 1.16637 \times 10^{-5}$ GeV$^{-2}$ and $\epsilon^2 = 4\pi\alpha = 4\pi/137.036$. Thus, a measurement of $\theta_W$ will determine all the predictions of our theory.

The Weinberg angle has been measured in many neutral current experiments. The experiments have all been in good agreement and thus provide strong evidence for the SM.
Figure 2.1. Feynman rules for the minimal standard model: (a) propagators, (b) fermion-gauge-boson vertices, (c) fermion-Higgs-boson vertex, (d) gauge-boson trilinear vertices, (e) gauge-boson quartic vertices, (f) gauge-boson-Higgs-boson trilinear vertices, (g) gauge-boson-Higgs-boson quartic vertices, and (h) Higgs-boson trilinear and quartic vertices.
Figure 2.1. (Continued)
Figure 2.1. (Continued)
Figure 2.1. (Continued)
Figure 2.1. (Continued)
Among the experiments are:

1. Neutrino and antineutrino deep inelastic scattering from isosinglet targets.

2. Neutrino and antineutrino deep inelastic scattering by protons.

3. Elastic $\nu_\mu p$ and $\bar{\nu}_\mu p$ scattering.

4. Exclusive and inclusive $\pi$ production in neutral-current events.

5. Neutrino disintegration of the deuteron: $\bar{\nu}_e d \rightarrow \nu_e np$.

6. Polarized-electron deuteron deep inelastic scattering.

7. Forward-backward asymmetry in $e^+ e^- \rightarrow \mu^+ \mu^-$.

8. Elastic $\nu e$ scattering.

9. Production of $W^+$ and $Z$ bosons.

In addition, the $\rho$ parameter can be measured as the ratio of charged to neutral weak currents. A detailed analysis finds

$$\sin^2 \theta_W = 0.234 \pm 0.013 \pm (0.009) ,$$

$$\rho = 1.002 \pm 0.015 \pm (0.011) .$$

From equations (2.18) and (2.30), the gauge-boson masses are

$$M_W = \frac{37.3 \text{ GeV}}{\sin \theta_W} = 77 \text{ GeV} ,$$

$$M_Z = \frac{M_W}{\cos \theta_W} = 88 \text{ GeV} .$$
These quantities are obtained from the lowest order in the perturbation expansion; they will be modified by radiative corrections. We will consider this further in the next chapters.

We saw before that the Higgs mass depends on the scalar-potential parameter $\mu$. Low-energy experiments, which involve fermions, give no predictions for the parameters of the potential. We can, however, find some bounds from theoretical considerations. Linde and Weinberg\textsuperscript{43} found that the stability of the vacuum sets a lower bound of order 10 GeV on the physical Higgs boson mass. More recently,\textsuperscript{44} it has been shown that in models with additional multiplets, this limit applies to only the heaviest Higgs boson. An upper bound may be established by noting that the trilinear and quartic Higgs couplings are proportional to the Higgs mass. As the Higgs mass reaches a certain threshold, the couplings become large enough that perturbation theory breaks down and the Higgs sector becomes strongly interacting. This threshold has been evaluated by a variety of means.\textsuperscript{45} Various estimates range from 125 GeV to 3 TeV. We will take about 1 TeV as the upper limit for the Higgs mass. Note that breakdown of perturbation theory above this limit does not necessarily invalidate the SM. While one might prefer a theory that is perturbatively
calculable, a strongly-interacting Higgs sector should give rise to some interesting phenomena.\textsuperscript{46}

D. The 2-Doublet and n-Doublet Models

In the MSM described above, one scalar doublet was introduced. The introduction of only one Higgs multiplet was for economical considerations; it was sufficient to activate the SSB and give masses to the gauge bosons. In view of the diverse mass scales of the leptons and quarks, there may well be more than one Higgs multiplet to achieve the SSB. The different vevs can give rise to different mass scales. In addition, other interesting phenomena, such as CP violation, can arise. Therefore, we would like to consider adding more and/or different Higgs multiplets to our theory.

The most straightforward extension of the MSM would be to add a second doublet. We define

\[
\Phi_1 = \begin{pmatrix} \phi_1^+ \\ \phi_1^0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}.
\]  

(2.40)

The scalar contribution to the Lagrangian is then

\[
L_H = |D_\mu \phi_1|^2 + |D_\mu \phi_2|^2 + L_V,
\]

(2.41)

where $D_\mu$ is the covariant derivative defined in eq. (2.3). Here $L_V$ is the general gauge-invariant potential.
corresponding to (2.24). In addition, we may need to impose symmetries on $L_V$ to prevent flavor-changing neutral currents. As the potential will involve unknown parameters, we will not specify $L_V$.

We now activate the SSB. The doublets $\Phi_1$ and $\Phi_2$ are given vevs $v_1$ and $v_2$, respectively. Making the transformation (2.12) and rotating to the physical gauge bosons, we find (corresponding to eqs. (2.18) and (2.23)):

$$L_{\text{Mass}} = \frac{1}{4} g^2 (v_1^2 + v_2^2) W^+ W^- + \frac{1}{8} g_Z^2 (v_1^2 + v_2^2) ZZ,$$

and

$$L_{\text{Int}} = \frac{1}{2} g_W (v_1^a \Phi_1^+ + v_2^a \Phi_2^+) + \text{c.c.} + \frac{1}{4} g_Z (v_1^a \Phi_1 + v_2^a \Phi_2)$$

$$+ \frac{1}{2} g_W (\delta A - g_Z Z) (v_1^a \Phi_1 + v_2^a \Phi_2) + \text{c.c.}$$

$$+ \frac{1}{2} g_W [i (\psi_1^+ \Phi_1^+ - \Phi_1^+ \psi_1^+) + i (\psi_2^+ \Phi_2^+ - \Phi_2^+ \psi_2^+)] + \text{c.c.}$$

$$+ \frac{1}{2} g_Z [\epsilon (\Phi_1^+ \phi_1^+ + \phi_1^+ \Phi_1^+) + (\Phi_2^+ \phi_2^+ - \phi_2^+ \Phi_2^+)] + \text{c.c.}$$

$$+ \frac{1}{2} g_Z [\epsilon (\phi_1^+ \Phi_1^+ - \Phi_1^+ \phi_1^+) + (\phi_2^+ \Phi_2^+ - \Phi_2^+ \phi_2^+)]$$

$$+ [\epsilon A + \frac{1}{2} g_Z (1 - 2 x_W) Z] [\epsilon (\phi_1^+ \Phi_1^+ - \Phi_1^+ \phi_1^+) + (\phi_2^+ \Phi_2^+ - \Phi_2^+ \phi_2^+)]$$

$$+ \frac{1}{4} g^2 W^+ W^- + \frac{1}{8} g_Z^2 ZZ (\phi_1^+ \phi_1^+ + \phi_1^+ \phi_1^-) + (\phi_2^+ \phi_2^+ - \phi_2^+ \phi_2^-)]$$

$$+ \frac{1}{4} g^2 AA + \epsilon g_Z (1 - 2 x_W) A \phi_1^+ \phi_1^+ + \frac{1}{4} g_Z^2 ZZ + [\frac{1}{2} g^2 W^+ W^- J (\phi_1^+ \phi_1^+ + \phi_2^+ \phi_2^-).$$
We see that
\[ M_W = \frac{1}{2} g v, \quad (2.44) \]
\[ M_Z = \frac{1}{2} g_Z v, \]
where
\[ v^2 = v_1^2 + v_2^2. \]

The combinations
\begin{align*}
\xi^\pm &\propto v_1 \phi_1^\pm + v_2 \phi_2^\pm \\
\xi^0 &\propto v_1 \psi_1 + v_2 \psi_2 \quad (2.45)
\end{align*}
correspond to the Goldstone bosons that are eaten. The gauge-boson–scalar-boson direct coupling terms
\[ \frac{i g_W}{2} (v_1 \partial_\phi_1^+ + v_2 \partial_\phi_2^+ + \text{c.c.} + \frac{1}{4} g_Z Z (v_1 \partial_\psi_1 + v_2 \partial_\psi_2) \quad (2.46) \]
will then exist only for the Goldstone bosons. Thus, we make the rotations
\[ \begin{pmatrix} \xi^\pm, 0 \\ H^\pm, 0 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \phi_1^+ \psi_1 \\ \phi_2^+ \psi_2 \end{pmatrix}, \quad (2.47) \]
where
\[ c = \frac{v_1}{v}, \quad s = \frac{v_2}{v}. \]

The Goldstone bosons \( \xi^\pm, \xi^0 \) are eaten, while the \( H^\pm, H^0 \) are
physical Higgs particles. In addition, we have neutral Higgs particles $\phi_1$ and $\phi_2$. In general, these will not be mass eigenstates; we have to diagonalize them to obtain the physical particle states, similar to (2.47). However, the rotation arising from the diagonalization will have no consequence for our purposes (it merely redefines the angles $c$ and $s$, which are unknown anyway), and hence we will not bother with it. We will treat $\phi_1$ and $\phi_2$ as if they were the actual mass eigenstates.

Our scalar-boson–gauge-boson interaction Lagrangian for the physical particles is then

\[
L_{\text{Int}} = g_W W^+ W^- (c \phi_1 + s \phi_2) + \frac{1}{2} g_Z Z Z (c \phi_1 + s \phi_2)
\]

\[
+ \frac{i}{2} g_W [s (H^+ a \phi_1 - \phi_1 a H^+) - c (H^+ a \phi_2 - \phi_2 a H^+)]
\]

\[
+ i (H^+ a H^0 - H^0 a H^+)] + c.c.
\]

\[
+ \frac{1}{2} g_Z Z [s (H^0 a \phi_1 - \phi_1 a H^0) - c (H^0 a \phi_1 - \phi_1 a H^0)]
\]

\[
+ i (1 - 2 x_W) (H^- a H^+ - H^+ a H^-)]
\]

\[
+ i e A (H^- a H^+ - H^+ a H^-)
\]

\[
+ (\frac{1}{4} g^2 W^+ W^- + \frac{1}{8} g^2 Z Z Z) (\phi_1 \phi_1 + \phi_2 \phi_2 + H^0 H^0)
\]

\[
+ i e A + e g_Z (1 - 2 x_W A Z) + \frac{1}{2} g^2 Z Z Z + \frac{1}{2} g^2 W^+ W^- J H^+ H^-.
\]
The gauge-boson—scalar-boson vertices and coupling strengths are given in Figure 2.2. We have used a condensed notation; for instance, the ZZ$\phi^2$ coupling has strength $1/2g_Z^2$, while the ZZ$H^+H^-$ coupling has strength $1/2g_Z^2(1-2x_W)^2$. Note that the coupling strengths, not the vertex factors, are given. That is, we have left off Lorentz and momentum factors, and factors of $i$ which come from expanding the Lagrangian.

We may extend this process for $n$ doublets in the same manner. The Lagrangian is

$$L_H = \sum_{j=1}^{n} |D_\mu \phi_j|^2 + L_V ,$$

(2.49)

with obvious notation. Again $L_V$ is the potential term. The mass terms are

$$L_{\text{Mass}} = \frac{1}{4} g^2 (\sum v_j^2) W^+ W^- + \frac{1}{8} g_Z^2 (\sum v_j^2) Z Z$$

(2.50)

giving

$$M_W = \frac{1}{2} g v ,$$

(2.51)

$$M_Z = \frac{1}{2} g_Z v ,$$

where

$$v = \sum v_j^2 .$$

The summations go from 1 to $n$. The gauge-boson—scalar-boson direct terms are
Figure 2.2. Couplings and strengths for the 2-doublet model
Figure 2.2. (Continued)
\[ \frac{g^2}{2} \operatorname{tr} (\Sigma v_j \psi_j) + \frac{i g}{2} \operatorname{tr} (\Sigma v_j \phi_j^+) + \text{c.c.} . \]  

Thus, we define the Goldstone bosons as

\[ \xi^0 = \frac{1}{v_j} \Sigma v_j \psi_j , \]  

\[ \xi^\pm = \frac{1}{v_j} \Sigma v_j \phi_j^\pm . \]

We make a general unitary rotation on the charged and imaginary neutral components:

\[ \phi_j^\pm = \Sigma U_{jk} H_k^\pm , \]  

\[ \psi_j = \Sigma V_{jk} H_k^0 . \]

or

\[ H_k^\pm = \Sigma U_{jk}^* \phi_j^\pm , \]  

\[ H_k^0 = \Sigma V_{jk}^* \psi_j . \]

The physical particles are \( H_j^\pm, H_j^0, j = 1 \) to \( n-1 \), while the Goldstone bosons are \( H_n^\pm = \xi^\pm, H_n^0 = \xi^0 \). There are also \( n \) physical particles from the real components, which we can rotate to the mass eigenstates:

\[ \phi_j = \Sigma T_{jk} \phi_k^0 . \]
Note that since all the vevs are assumed real to preserve CP, the unitary matrices $U$, $V$, and $T$ are real.

With these rotations, the interaction Lagrangian becomes

$$L_{\text{Int}} = g M_{WW}^{-1} (\Sigma V_{jk} T_{jk} \phi_0^0) + \frac{1}{2} g_{Z} Z_{ZZ} (\Sigma V_{jk} T_{jk} \phi_0^0)$$

$$+ \frac{1}{2} W^+ \Sigma (\Sigma T_{jk} U_{jk}^T) (\phi_k^0 \partial H^+_1 - H^+_1 \partial \phi_k^0) + \text{c.c.}$$

$$+ \frac{1}{2} W^+ \Sigma (\Sigma V_{jk} U_{jk}^T) (H^0 \partial H^+_1 - H^+_1 \partial H^0_1) + \text{c.c.}$$

$$+ \frac{1}{2} Z^+ \Sigma (\Sigma T_{jk} V_{jk}^T) (\phi_k^0 \partial H^0_1 - H^0_1 \partial \phi_k^0)$$

$$+ i e A - \frac{1}{2} g_{Z} (1 - 2 x_w) \Sigma \Sigma (\Sigma H^+_j - H^+ - H^+_j)$$

$$+ e^2 A + \text{c.c.} + \frac{1}{4} g_{Z}^2 (1 - 2 x_w) \Sigma \Sigma (\Sigma H^+_j - H^+ - H^+_j).$$

The summations are from 1 to $n-1$ for $H^+$ and $H^0$, and from 1 to $n$ for $\phi^0$. The relevant vertices and coupling strengths are given in Figure 2.3.

E. Triplet and Doublet-Triplet Models

The next extension of the Higgs sector would be to consider a multiplet other than a doublet. Thus, we consider models with a scalar triplet. We have an additional motivation that a triplet may generate Majorana masses for the
Figure 2.3. Couplings and strengths for the n-doublet model
Figure 2.3. (Continued)
neutrinos. As pointed out in Chapter I, these models will have $\rho \neq 1$; the experimental value $\rho \leq 1$ puts strong constraints on the triplet representations. Nonetheless, they are informative both as unrealistic and realistic models.

The first triplet we consider is the $\{1,2\}$-triplet:

$$\Delta(1,2) = \begin{pmatrix} \xi^{++} \\ \xi^+ \\ \xi^0 \end{pmatrix}.$$  \hfill (2.58)

The Higgs contribution to the Lagrangian is:

$$L_H = |D_\mu \Delta|^2 + L_V.$$ \hfill (2.59)

Inserting (2.40) and making the transformation

$$\xi^0 \to (\xi + i\chi + \kappa) / \sqrt{2},$$ \hfill (2.60)

where $\kappa$ is the vev of $\xi^0$ and $\xi$ and $\chi$ are real fields with zero vev, we find:

$$L_{\text{Mass}} = \frac{1}{2} g^2 \xi^2 \bar{W}^+ W^- + \frac{1}{2} g^2 \kappa^2 ZZ,$$ \hfill (2.61)

giving

$$M_W = g\kappa / \sqrt{2},$$

$$M_Z = g_Z \kappa,$$ \hfill (2.62)

$$\rho = \frac{1}{2}.$$  

The fields $\xi^\pm$, $\chi$ are eaten to give the $W^\pm$, $Z$ masses. The
remaining fields $\phi^{++}$, $\phi$ are physical Higgs particles. The interaction Lagrangian is

$$L_{\text{Int}} = \sqrt{2g_M M_W^{-} \phi^{++} + g_Z M_Z^{++} \phi \phi^{++} + c.c.]$$

$$+ i[2eA + g_Z(1-2x_W)Z](-\phi^{--} \phi^{++} - \phi^{++} \phi^{--})$$

$$+ \frac{1}{2} g^2 W^+ W^- \phi \phi + \frac{1}{2} g_Z^2 Z \phi \phi + \left[ \frac{1}{4} g^2 W^+ W^- \phi \phi + c.c. \right]$$

$$+ [4e^2 A A + 4eg_Z^2(1-2x_W)A Z + g_Z^2 (1-2x_W)^2 Z Z + g^2 W^+ W^- \phi \phi + c.c.]$$

The couplings for the (1,2)-triplet model are summarized in Figure 2.4. We emphasize that this triplet is not a realistic model because of the gauge-boson mass ratio, but is nonetheless an informative model.

A more realistic model is the doublet-triplet model. The Higgs contribution to the Lagrangian is

$$L_H = |D_\mu \phi|^2 + |D_\mu \Delta|^2 + L_V .$$

Insertion of (2.10) and (2.30) gives

$$L_{\text{Mass}} = \frac{g^2}{4} \frac{1}{2} v^2 + \frac{1}{2} \kappa^2 W^+ W^- + g_Z^2 \frac{1}{8} v^2 + \frac{1}{2} \kappa^2 ZZ ,$$

and

$$M_W = \frac{g}{2} \sqrt{v^2 + 2\kappa^2} ,$$

$$M_Z = \frac{g_Z}{2} \sqrt{v^2 + 4\kappa^2} .$$
Figure 2.4. Couplings and strengths for the (1,2)-triplet model
Figure 2.4. (Continued)
\[ \rho = \frac{1 + 2\kappa^2/v^2}{1 + 4\kappa^2/v^2} . \]

Thus, as \( v \gg \kappa, \rho \approx 1. \)

The gauge boson-scalar terms are

\[ \text{ig}_W^{-} \partial_{\alpha} \left( \frac{-K\phi^+}{2} + \frac{v\phi^+}{2} \right) + \text{c.c.} + g_Z Z \partial \left( \kappa \chi + \frac{v}{2} \psi \right) , \quad (2.67) \]

so the Goldstone bosons are

\[ \xi^\pm = \left( \sqrt{2} \kappa \xi^\pm + v \phi^\pm \right)/\sqrt{v^2 + 2\kappa^2} , \quad (2.68) \]

\[ \xi^0 = \left( 2 \kappa \chi + v \psi \right)/\sqrt{v^2 + 4\kappa^2} . \]

We have physical Higgs particles \( H^\pm \) and \( H^0 \) (the fields orthogonal to (2.68)), \( \xi^{\pm \pm} \), \( \phi^0 \), and \( s^0 \). The interaction Lagrangian for the physical particles is

\[ L_{\text{Int}} = g^2 W^+ W^- (\kappa s^0 + \frac{v}{2} \phi^0) + g_Z^2 Z Z (\kappa s^0 + \frac{v}{4} \phi^0) \]

\[ - \frac{g g}{\sqrt{2}} \alpha Z W^- H^+ + \frac{g^2}{\sqrt{2}} Z W^+ s^- + \text{c.c.} \quad (2.69) \]

\[ + i \frac{g}{\sqrt{2}} W^+ \left[ \beta (\phi^0 a H^+ - H^+ a \phi^0) + \sqrt{2} \kappa (s^0 a H^+ - H^+ a s^0) \right] + \text{c.c.} \]

\[ + \frac{g}{\sqrt{2}} W^- (2 \sqrt{2} \beta \beta' + \sqrt{2} \alpha' \alpha') (H^0 a H^+ - H^+ a H^0) + \text{c.c.} \]

\[ - ig W^- a (H^- a s^{++} - s^{++} a H^-) + \text{c.c.} \]

\[ + i g \frac{g}{\sqrt{2}} Z \partial (2 \beta^2 - 2 x_W) (H^- a H^+ - H^+ a H^-) + 2 (1 - 2 x_W) (s^- a s^{++} - s^{++} a s^-) + \text{c.c.} \]
+ g_2 Z \beta' (\phi^0 e H^0 - H^0 e \phi^0) - \alpha' (\delta^0 e H^0 - H^0 e \delta^0) \\
+ i e A (H^- e H^+ - e H^-) + 2 (\delta^- e \delta^+ - \delta^+ e \delta^-) \\
+ g_2 W^+ W^- S^{++} S^{--} - \frac{1}{2} S^0 S^0 + \frac{1}{4} \phi^0 \phi^0 + \frac{1}{2} \alpha^2 + \beta^2 \delta^0 H^0 + (2 \alpha^2 + \beta^2) H^+ H^- \\
+ g_2 Z Z Z (1 - 2 x_W) S^{++} S^{--} + \frac{1}{2} S^0 S^0 + \frac{1}{8} \phi^0 \phi^0 + \frac{1}{2} \alpha' \beta + \frac{1}{2} \delta^0 H^0 + \frac{1}{2} \alpha^2 + \beta^2 H^+ H^- \\
+ e^2 A a e H^+ H^- + 4 S^{++} S^{--} \\
+ g_2 Z Z Z (1 - 2 x_W) S^{++} S^{--} + (2 \beta^2 - 2 x_W) H^+ H^- \\
+ \frac{\alpha^2}{\sqrt{2}} W^+ W^- \delta^{++} (\delta^0 + i \alpha' H^0) + c.c. \\
+ g W^+ (e a - g_2 x_W) \left( \frac{\sqrt{2}}{2} \delta^+ (\phi^0 - 2 i \beta' H^0) \right) - \frac{\sqrt{2}}{2} \alpha^+ (\delta^0 + i \alpha' H^0) \\
- (3 e + g_2 (1 - 3 x_W) Z \delta^{++} \alpha H^- + g_2 Z \frac{\sqrt{2}}{2} \alpha H^+ (\delta^0 + i \alpha' H^0) + c.c.,

where

\[ \alpha, \beta = \frac{\nu, \kappa}{\sqrt{\nu^2 + 2 \kappa^2}} \]

\[ \alpha', \beta' = \frac{\nu, \kappa}{\sqrt{\nu^2 + 4 \kappa^2}} \]
The couplings and strengths for the doublet-(1,2)-triplet model are summarized in Figure 2.5.

There is also another kind of triplet allowed. The (1,0)-triplet, or real triplet, is

$$\Delta(1,0) = \begin{pmatrix} \delta^+ \\ \delta^0 \\ \delta^- \end{pmatrix}. \quad (2.70)$$

We will take the triplet to be self-conjugate. This implies that $\delta^0$ is a real field. Note we only have three degrees of freedom. As there is no neutral imaginary field, the $Z$ boson will not gain a mass from the symmetry breaking. The mass term is

$$L_{\text{Mass}} = g^2 \kappa^2 W^+ W^-,$$  \quad (2.71)

and

$$M_W = g \kappa,$$

$$M_Z = 0,$$ \quad (2.72)

$$\rho \to \infty.$$  

The Goldstone bosons $\delta^\pm$ are eaten to give the $W$ mass. The remaining physical Higgs is the $\delta^0$. The interaction terms are

$$L_{\text{Int}} = 2gM_W W^+ W^- \delta^0 + g^2 W^+ W^- \delta^0 \delta^0.$$ \quad (2.73)

The couplings are given in Figure 2.6.
Figure 2.5. Couplings and strengths for the doublet-(1,2)-triplet model
Figure 2.5. (Continued)
Figure 2.5. (Continued)
\( \sqrt{2} g^2 \left[ \alpha \right] \)

\[
\begin{align*}
\text{w}^- & \rightarrow s^0, h^0 \\
\text{h}^- & \rightarrow s^{++}
\end{align*}
\]

\[
\begin{align*}
\text{z} & \rightarrow \phi^0, s^0, h^0, s^{++} \\
\text{h}^- & \rightarrow h^+, h^+, h^+, h^-
\end{align*}
\]

\[
\begin{align*}
\text{a} & \rightarrow \phi^0, s^0, h^0, s^{++} \\
\text{w}^- & \rightarrow h^+, h^+, h^+, h^-
\end{align*}
\]

\[
\frac{gg_{Z}}{\sqrt{2}} - \beta_\text{W}, (1 + x_\text{W})\alpha, 2\left(\alpha_\text{W} + \beta_\text{W}\right), -\sqrt{2}(1 - 3x_\text{W})\alpha\]

\[
\frac{ge_{Z}}{\sqrt{2}} [-\beta_\text{W}, x_\text{W}\alpha, i(2\alpha_\text{W} + \beta_\text{W}), -3\sqrt{2}\alpha]
\]

Figure 2.5. (Continued)
Figure 2.6. Couplings and strengths for the (1,0)-triplet model
We can also combine the (1,0)-triplet with a doublet. Then, from (2.64) with (2.70), we get

\[ L_{\text{Mass}} = \frac{1}{4} g^2 (v^2 + 4\kappa^2) W^+ W^- + \frac{1}{8} g_Z v^2 ZZ, \]  

(2.74)

and

\[ M_W = \frac{1}{2} g \sqrt{v^2 + 4\kappa^2}, \]

\[ M_Z = \frac{1}{2} g_Z v, \]  

(2.75)

\[ \rho = 1 + 4\kappa^2/v^2. \]

The gauge-boson–scalar-boson terms are

\[ igW^- \delta (\frac{v}{2} \phi^+ - \kappa \delta^+) + c.c. + g_Z Z \delta (\frac{v}{2} \psi). \]  

(2.76)

The Goldstone bosons are

\[ \xi^+ = \frac{1}{\sqrt{v^2 + 4\kappa^2}} (v \phi^+ - 2\kappa \delta^+), \]

(2.77)

\[ \xi^0 = \psi. \]

These are eaten to give the gauge-boson masses. The remaining scalar fields \( \phi^0, \delta^0, \) and \( H^\pm \) (the combination orthogonal to (2.77)) are the physical Higgs particles. The interaction Lagrangian is

\[ L_{\text{Int}} = g M_W W^+ W^- (c \phi^0 + 2 s \delta^0) + \frac{1}{2} g_Z M_Z Z \phi^0 - g_Z M_Z s c Z (H^- W^+ + H^+ W^-) \]

\[ + \frac{1}{2} g M_W [s (\phi^0 \delta H^+ - H^+ \delta \phi^0) - 2 c (\delta^0 \delta H^+ - H^+ \delta \delta^0) - c c] + c.c. \]
\[
+ \frac{1}{2} g_Z (c^2 + 1 - 2x_W) Z (H^- d H^+ - H^+ d H^-) + i e A (H^- d H^+ - H^+ d H^-)
\]
\[
+ \frac{1}{2} g^2 W^+ W^- (\phi^0 \phi^0 + 4 \delta^0 \delta^0) + \frac{1}{8} g_Z^2 Z \phi^0 \phi^0
\]
\[
+ e e^2 A A + e g_Z A Z (c^2 + 1 - 2x_W^\lambda) + \frac{1}{4} g_Z^2 Z Z (1 - 4x_W^\lambda + 4x_W^\lambda + 3c^2 - 4c^2 x_W)
\]
\[
+ \frac{1}{4} g^2 W^+ W^- (1 + c^2) H^+ H^- - \frac{1}{2} g^2 c^2 (W^+ W^- H^+ H^- + c.c.)
\]
\[
+ e A (H^+ W^- + H^- W^+) (\frac{S}{2} \phi^0 - c \delta^0)
\]
\[
- g g_Z Z (H^- W^+ + H^+ W^-) (x_W^\phi \phi^0 + (1 - x_W) c \delta^0)
\]

where

\[
c = \frac{v}{\sqrt{v^2 + 4k^2}}.
\]
\[
s = \frac{2k}{\sqrt{v^2 + 4k^2}}.
\]

The couplings are shown in Figure 2.7.

We may now use the Lagrangians and Feynman rules to calculate physical quantities for these models.
Figure 2.7. Couplings and strengths for the doublet-(1,0)-triplet model
Figure 2.7. (Continued)
$W^+ \rightarrow \phi^0, \delta^0, H^+
\frac{1}{4g^2} [2, 8, (1+c^2)]$

$W^- \rightarrow \phi^0, \delta^0, H^-
\frac{1}{4g^2} [2, (1-4x_W^+4x_W^2+3c^2-4c^2x_W)]$

$Z \rightarrow \phi^0, H^+
\frac{1}{4g^2} [2, (1-4x_W^+4x_W^2+3c^2-4c^2x_W)]$

$Z_\phi^0, H^-
\frac{1}{2} eg_Z (c^2+1-2x_W), 2e^2$

$Z \rightarrow \phi^0, \delta^0
\frac{1}{2} eg_Z [sx_W, 2(1-x_W)]$

$W^- \rightarrow \phi^0, \delta^0
\frac{1}{2} eg[s, -c]$

Figure 2.7. (Continued)
III. THE RENORMALIZATION PROCESS

A. Introduction

Quantum field theory has given us the ability to derive physical quantities from a theory. We can derive the Feynman rules, which are a prescription for doing calculations. Unfortunately, quantum field theories are highly non-linear. Since their formulation several decades ago, only free-field theories have been found to have exact closed-form solutions. The solutions to interacting field theories have escaped us so far. Therefore, we have to use approximation methods, i.e., perturbation, to find solutions. We expand in powers of some parameter, such as a coupling constant. A proper choice for the expansion parameter results in a convergent series, and we can make a meaningful approximation by taking a finite number of terms of the series.

For QED and the SM, we take the perturbation expansion parameter to be the fine structure constant, \( \alpha \). The lowest-order, or tree-level, approximation is straightforward and can be used to give meaningful predictions. This, however, is not sufficient. At some point, we must do higher-order calculations. First, the tree level is only an approximation. When experiments can be done with sufficient accuracy or at high enough interaction energies, we will need to take higher-
order terms of the expansion into account. More importantly, the higher-order terms give a test of the consistency of the theory, as they represent quantum fluctuations which are present in a quantum theory. Only if a theory allows calculation of the higher-order terms can it be regarded as a fundamental theory.

Thus, we deem it necessary to evaluate higher-order diagrams. This appears to be a straightforward task. It turns out to be highly non-trivial, however. Divergent expressions arise when the higher-order diagrams are evaluated. The renormalization program is the procedure by which these divergences are removed in a systematic manner, and meaningful quantities are obtained. The renormalization program consists of two steps: regularization and renormalization. The regularization is to define the divergent integrals in terms of finite ones; the renormalization is a procedure to eliminate all the potentially divergent quantities. The renormalization process is successful if finite quantities result.

As an example of divergences in higher-order amplitudes, we consider the QED process $e^+e^- \rightarrow \mu^+\mu^-$. The tree-level and first-order-correction diagrams are given in Figure 3.1. The
Figure 3.1. Diagrams for the QED process $e^+e^- \rightarrow \mu^+\mu^-$
lowest-order matrix amplitude, from diagram a, is

\[ T^{(0)} = \overline{e}(i\gamma^\mu)e\left(\frac{q_{\mu\nu}}{q^2}\right)\overline{\mu}(i\gamma^\nu)\mu \ . \quad (3.1) \]

Here, the spinors are represented by their symbols, e and \( \mu \).

We can consider a vertex correction, like diagram c. Using the momentum and Lorentz index notation shown in Figure 3.2.

![Figure 3.2. Momentum and index notation for vertex correction diagram](image)

the contribution from this diagram is

\[ T^{(1)}_c = \left[ \frac{d^4k}{(2\pi)^4i} \right] \overline{e}(i\gamma^\rho) \left( \frac{p_2+\mathbf{k}+m}{(p_2+\mathbf{k})^2-\mathbf{m}^2} \right) (i\gamma^\mu) \left( \frac{p_1-\mathbf{k}+m}{(p_1-\mathbf{k})^2-\mathbf{m}^2} \right) \]

\[ \left( \frac{q_{\rho\sigma}}{k^2} \right) (i\gamma^\sigma) e \left( \frac{q_{\mu\nu}}{q^2} \right) \overline{\mu}(i\gamma^\nu)\mu \ . \quad (3.2) \]
Here, $k$ is the loop momentum, and $\mathbf{p} \equiv p_\alpha \gamma^\alpha$. The $k$ integration is over all of 4-dimensional space-time. We note that the denominators of the fermion propagators look like $k^2 \pm 2k \cdot p$ as the external electron goes on-mass-shell. As $k \to \infty$, the integral goes like

$$\int d^4k \, k^{-4} = \int dk \, k^{-1} \to \infty \quad (\text{as } k \to \infty).$$

(3.3)

This is known as an ultraviolet (UV) divergence. As $k \to 0$, the integration also goes like

$$\int d^4k \, k^{-4} = \int dk \, k^{-1} \to \infty \quad (\text{as } k \to 0).$$

(3.4)

This is the infrared (IR) divergence. For this diagram, both types of divergences are logarithmic. We note that, in Figure 3.1, diagrams b, c, and d have UV divergences, while diagrams c, d, and e have IR divergences.

B. Regularization

The first step in the renormalization process is regularization. Regularization is the procedure by which the divergent integral is given meaning, such as the limit of a certain function. The divergence can then be handled in a consistent and mathematically meaningful way.

There are several regularization methods available. The Pauli-Villars method\textsuperscript{48} met with much success in QED. A massive spinor field is added to the theory. The theory is
then characterized by the mass $M$. The divergences manifest themselves as functions of $M$ which are finite for finite $M$. In the end, we must take the limit $M \to \infty$, and the spinor decouples from the real particles.

Another regularization method has gained enormous popularity since its introduction in the early 1970s. This is the dimensional regularization method. Both UV and IR divergences can be taken care of using dimensional regularization. The idea behind the method is the observation that the divergent Feynman integrals would be convergent in a smaller dimension. Thus, we compute the integrals in an arbitrary dimension $n$, and analytically continue it back to the physical space-time dimension $n = 4$. The divergences manifest themselves as poles at integer values of $n$. We want the physical limit $n \to 4$, so we look at the poles there. That is, our loop integrals become

\[
\left( \frac{d^4k}{(2\pi)^4} \right) \to \lim_{n \to 4} \left( \frac{d^nk}{(2\pi)^n} \right). \tag{3.5}
\]

The procedure then is this: we make the substitution (3.5) to $n$-dimensional space. Quantities are calculated, and in the end we take the limit $n \to 4$. Our final quantities must not have poles at $n = 4$, or they could not correspond to physical quantities. We will find how to remove these poles later.
A handy tool to use in dimensional regularization is the Feynman parameterization. Using identities, we are able to combine denominational factors. For instance,

\[ \frac{1}{AB} = \int_0^1 dx \left[ A x + B (1-x) \right]^{-2}. \tag{3.6} \]

Other identities are given in Table 3.1. The quantities A, B, ..., will be denominators of propagators. This enables us to write our loop integrals in the form

\[ \int \frac{d^4 k}{(2\pi)^4} \frac{1, k_\mu, k_\mu' \cdots}{\Gamma^2 - 2k \cdot p - q^2} \cdot \tag{3.7} \]

We now go to n-space, where the integrals are finite and can be performed. For instance,

\[ \int d^n k \frac{1}{\Gamma^2 - 2k \cdot p - q^2} = (-1)^{\alpha+2} \frac{\Gamma^{\alpha+2+\epsilon} \Gamma^{\alpha-2+\epsilon}}{\Gamma^{\alpha-2+\epsilon}} \tag{3.8} \]

where \( \epsilon \equiv (n-4)/2 \). Other integrals are given in Table 3.2. The \( \Gamma(x) \) in equation (3.8) is the gamma function. This is where the divergence is at. We can expand the gamma function:

\[ \lim_{\epsilon \to 0} \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma, \tag{3.9} \]

where \( \gamma = 0.577251... \) is Euler's constant. The divergence shows up explicitly as a pole at \( \epsilon = 0 \). We also observe that (3.8) is no longer dimensionally correct. Thus, we include a multiplicative factor of
Table 3.1. Feynman identities for use in combining denominator factors

\[
\frac{1}{AB} = \int_0^1 dx \left[ A x + B (1-x) \right]^{-2}
\]

\[
\frac{1}{ABC} = \int_0^1 dx \int_0^{1/2} dy \left[ A x + B (1-x) y + C (1-y) \right]^{-3}
\]

\[
\frac{1}{ABCD} = \int_0^1 dx \int_0^{1/2} dy \int_0^{1/3} dz \left( \left[ A x + B (1-x) y + C (1-y) z + D (1-z) \right]^{-4}
\]

\[
\frac{1}{ABCDE} = \int_0^1 dx \int_0^{1/2} dy \int_0^{1/3} dz \int_0^{1/4} \omega^3 \int_0 \left[ \left( A x + B (1-x) y + C (1-y) z + D (1-z) \right) \right]^{-5}
\]

etc.
Table 3.2. Feynman integral formulae in the dimensional regularization scheme \((\epsilon \equiv (n-4)/2)\)

\[
\int d^n k \frac{1}{(k^2 - 2k \cdot p - Q^2)^\alpha} = (-1)^{\alpha} \pi^2 \frac{\Gamma(\alpha - 2 + \epsilon)}{\Gamma(\alpha)} \frac{1}{(p^2 + Q^2)^{\alpha - 2 + \epsilon}}
\]

\[
\int d^n k \frac{k_\mu}{(k^2 - 2k \cdot p - Q^2)^\alpha} = (-1)^{\alpha} \pi^2 \frac{\Gamma(\alpha - 2 + \epsilon)}{\Gamma(\alpha)} \frac{P_\mu}{(p^2 + Q^2)^{\alpha - 2 + \epsilon}}
\]

\[
\int d^n k \frac{k_\mu k_\nu}{(k^2 - 2k \cdot p - Q^2)^\alpha} = (-1)^{\alpha} \pi^2 \frac{\Gamma(\alpha - 3 + \epsilon)}{2\Gamma(\alpha)} \frac{q_{\mu \nu}}{(p^2 + Q^2)^{\alpha - 3 + \epsilon}}
\]

\[+ \frac{\Gamma(\alpha - 2 + \epsilon)}{\Gamma(\alpha)} \frac{P_\mu P_\nu}{(p^2 + Q^2)^{\alpha - 2 + \epsilon}} \]

\[
\int d^n k \frac{k_\mu k_\nu k_\lambda}{(k^2 - 2k \cdot p - Q^2)^\alpha} = (-1)^{\alpha} \pi^2 \frac{\Gamma(\alpha - 3 + \epsilon)}{2\Gamma(\alpha)} \frac{p_\mu q_{\nu \lambda} + p_\nu q_{\mu \lambda} + p_\lambda q_{\mu \nu}}{(p^2 + Q^2)^{\alpha - 3 + \epsilon}}
\]

\[+ \frac{\Gamma(\alpha - 2 + \epsilon)}{\Gamma(\alpha)} \frac{P_\mu P_\nu P_\lambda}{(p^2 + Q^2)^{\alpha - 2 + \epsilon}} \]

etc.
where $\mu$ is some arbitrary parameter with dimension of mass, on the right-hand side. We then can expand the right-hand side of (3.8) using (3.9); we have things like

$$\frac{(4\pi\mu^2)^\varepsilon}{(p^2+q^2)^\varepsilon} = \frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \ln[\mu^2/(p^2+q^2)] \, .$$

We emphasize that regularization is a procedure by which we can manipulate the divergences. The divergences are still present when we take the $\varepsilon \to 0$ limit, in the $1/\varepsilon$ term. On the other hand, the regularization procedure is an integral part of a theory; without it the renormalization process is meaningless.

C. Infrared Divergences

Consider the QED process $e^+e^- \to \mu^+\mu^-$ (Figure 3.1) again. Diagrams of the type b, d, and e exhibit IR divergences as the loop integral variable $k \to 0$. Now the cross-section is the amplitude squared:

$$\sigma_{e^+e^- \to \mu^+\mu^-} = |M|^2 = |M^{(0)} + M^{(1)} + \ldots|^2 \, ,$$

where $M^{(0)} =$ lowest-order diagrams, $M^{(1)} =$ first-order corrections, etc. Then, using diagram notation, our cross-section of order $\alpha^3$ is
\[ \sigma^3 \alpha = \left( \begin{array}{c} \vdots \\ \end{array} \right) + \left( \begin{array}{c} \vdots \\ \end{array} \right) \]

which has IR divergent pieces, from section A,

\[ \sigma^3_{IR} = \left( \begin{array}{c} \vdots \\ \end{array} \right) \left( \begin{array}{c} \vdots \\ \end{array} \right) \]

Now the physical (measured) cross-section is actually

\[ \sigma_{PHYS} = \sigma_{e^+e^-\mu^+\mu^-} + \sigma_{e^+e^-\mu^+\mu^-\gamma(soft)} \]

where \( \gamma \) is a soft (low-energy) photon. That is, all detectors have a finite energy resolutions, say \( \Delta E \). Photons which have an energy less than \( \Delta E \) can not be detected. Hence, what is really a soft-photon process is being interpreted as an elastic process, and we must include the soft-photon contribution thru eq. (3.15).

The lowest-order diagrams for the inelastic process \( e^+e^-\rightarrow\mu^+\mu^-\gamma \) are given in Figure 3.3. To order \( \alpha^3 \), the cross-
Figure 3.3. Diagrams for the QED process $e^+e^- \rightarrow \mu^+\mu^- \gamma$

section is given by

$$\sigma_{\text{soft}} = \left\{ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{array} \right\}^2
\tag{3.16}$$

which has divergences as $k \to 0$ also. When we calculate the physical cross-section, eq. (3.15), we find that the divergences from

$$\begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4}
\end{array}$$

cancel the divergences from
This should not amaze us. We can think of it in the following way. The incoming electron emits a photon. In the IR limit, we don't know if the photon escapes or is reabsorbed. When we include both, there is no divergence. Likewise, divergences from

\[
\begin{align*}
\left( \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram1}} \\
\text{\includegraphics[width=1cm]{diagram2}}
\end{array} \right) \\
\end{align*}
\]

cancel the divergences from

\[
\begin{align*}
\left( \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram3}} \\
\text{\includegraphics[width=1cm]{diagram4}}
\end{array} \right) \\
\end{align*}
\]

the divergences from

\[
\begin{align*}
\left( \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram5}} \\
\text{\includegraphics[width=1cm]{diagram6}}
\end{array} \right) \\
\end{align*}
\]

cancel the divergences from

\[
\begin{align*}
\left( \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram7}} \\
\text{\includegraphics[width=1cm]{diagram8}}
\end{array} \right) + \left( \begin{array}{c}
\text{\includegraphics[width=1cm]{diagram9}} \\
\text{\includegraphics[width=1cm]{diagram10}}
\end{array} \right) \\
\end{align*}
\]

and so on.
The infrared divergence is, in fact, a divergence appearing in degenerate perturbation theory. The inclusion of soft photons is to take into account the degenerate states. In the limit of zero photon energy, the processes $e^+e^- \rightarrow e^+e^-$ and $e^+e^- \rightarrow e^+e^-(n\gamma) \ (n = 1, \ldots)$ are degenerate. Artificially breaking the problem into two-body and three-body problems led to the divergences.

In the physical cross-section, (3.15), all the divergent pieces have canceled. However, $\sigma_{\text{soft}}$ has some non-divergent pieces left. These pieces will contribute to our final answer. There will be a cut-off energy $\Delta E$ dependence in this contribution, where $\Delta E$ defines what we mean by a soft photon. The actual value for $\Delta E$ is detector-dependent.

The extension of the above analysis to electroweak models is straightforward. We have no problem with Z or W radiation, as they are massive. Thus, there is no IR divergence difficulty associated with the W or Z. The only corrections involving the IR divergences are the electromagnetic corrections.

Experimental results for low-energy processes are generally reported with electromagnetic radiative effects removed. The reasons for this are both traditional, relating to the Fermi current-current theory, and practical, as the corrections are detector-dependent. That is, the measured
data are meaningful only after the detector effects have been eliminated. Our input data have been corrected for the IR divergences, and we need not consider the contributions from the soft-photon cross-sections.

D. Ultraviolet Divergences

We now consider the renormalization process, by which we are able to remove the UV divergences. We assume a suitable regularization scheme, Section B, has been used to make the divergent quantities meaningful. We now need to remove the divergences to get finite quantities.

Consider a theory with one free parameter $g$. By a free parameter we mean that $g$ is not fixed by the theory. We can calculate a physical process (as a function of $g$) and compare to experiments to evaluate $g$. The predictive power of the theory is that it is the same $g$ for any process we evaluate. If we evaluate $n$ processes, then we have $n-1$ predictions, which can be used to test the theory.

An important point is that $g$ does not necessarily have any physical significance. If we like, we can transform to another parameter

$$g' = f(g) .$$

(3.17)

The physical quantities we calculate may now look quite
different, but they contain the same information. The physics is the same.

This is what we do in the renormalization process. We calculate quantities in terms of our (bare) parameter $g_0$. At the tree level, we may give $g_0$ some physical meaning by associating it with some measurement. But when we calculate the first-order corrections, we get divergences. However, we can make a transformation, as in (3.17), to another parameter $g$. This rescaling introduces renormalization constants. The divergent terms can be absorbed into these constants, leaving a theory, based on the parameter $g$, which gives finite predictions. Renormalization is the procedure of consistently rearranging terms in such a way that only finite terms are associated with physical quantities.

It is now $g$ which we give physical meaning by associating it with some measurement. In general, $g_0$ will not have physical significance, and will depend upon our regularization scheme parameters. However, if our perturbation expansion makes sense, $g$ will differ from the lowest-order $g_0$ by terms the size of the expansion parameter.

We illustrate this with an example from QED. Let us consider the photon inverse propagator,

$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu \right) \left[1 + \pi(q^2)\right]$, \hspace{1cm} (3.18)

where $\pi(q^2)$ is the self-energy, Figure 3.4. Evaluation of the
loop diagram shown, using the dimensional regularization tools from above, gives

\[ \tau_B(q^2) = \frac{\alpha_B}{3\pi \varepsilon} - \gamma - F(q^2/m^2), \quad (3.19) \]

where

\[ F(t) = 6\int_0^1 dx \frac{x(1-x)}{1-x(1-x)t}. \quad (3.20) \]

Here, \( m \) is mass of the loop fermion. We have used the notation \( \alpha_B \) to represent the bare parameter (which in the lowest order we associate with the physical charge). This is obviously divergent.

We now introduce the wavefunction renormalization constant \( Z_3 \):

\[ A_\mu = Z_3^{1/2} A_\mu^{\text{REN}}, \quad (3.21) \]

where \( A_\mu \) is the photon field. The subscript REN is for the renormalized field. From the definition of the propagator

\[ \Delta^{\mu\nu}(x_1-x_2) \equiv \langle T(A_\mu(x_1)A_\nu(x_2)) \rangle. \quad (3.22) \]
where $T$ is the time-ordered product, we find, by inserting eq. (3.21),

$$\Delta^{\mu\nu}_{\text{REN}}(x_1-x_2) \equiv \langle T(A^\mu_{\text{REN}}(x_1)A^\nu_{\text{REN}}(x_2)) \rangle$$

$$= Z_3^{-1} \Delta^{\mu\nu}(x_1-x_2).$$

(3.23)

(3.24)

The renormalized inverse propagator is then

$$1 + \pi(q^2) = Z_3 [1 + \pi_B(q^2)].$$

(3.25)

The charge is renormalized as

$$\alpha = Z_3 \alpha_B (4\pi\mu^2)^{-\varepsilon}.$$  

(3.26)

We have introduced the mass parameter $\mu$ in eq. (3.26) through

$$(4\pi\mu^2)^{-\varepsilon}$$

(3.27)

rather than in (3.11) as before, in order to maintain the renormalized fine structure constant dimensionless.

The divergences in $\pi_B(q^2)$ may be absorbed into $Z_3$ by requiring that the renormalized inverse propagator (3.25) be finite order by order. Equation (3.25) gives a constraint on $Z_3$; however, $Z_3$ is arbitrary as for its specific form. If we are only concerned about the renormalizability of the theory, this is sufficient. But, if one wishes to relate the parameters of the renormalized theory to physical quantities, this arbitrariness must be eliminated. This is done by
imposing a set of renormalization conditions, which define the renormalization scheme.

We emphasize that while the form of our theory may be radically different under different renormalization schemes, the physics is the same. Predictions for physical quantities should be independent of the scheme used. This is not entirely true, however, due to the fact that we must use approximation methods (perturbation theory) to solve our theory. While measurable quantities are independent of the scheme, approximations of measurable quantities will not be. Different schemes may lead to different results, differing in the next order of the expansion parameter. 50

As an example, we return to the QED photon inverse propagator. The renormalization constant $Z_3$ is to absorb the divergences in $\pi_B(q^2)$. The simplest form for $Z_3$ which accomplishes this is

$$Z_3 = 1 - \frac{\alpha}{3\pi \epsilon} .$$

(3.28)

When $\pi_B$, eq. (3.19), is inserted into the definition of the renormalized $\pi$, eq. (3.25), we find

$$\pi(q^2) = \frac{\alpha}{3\pi} \left[ -\gamma + \ln(4\pi) + \ln(\mu^2/m^2) - F(q^2/m^2) \right] ,$$

(3.29)

to order $\alpha$. We see that the renormalized self-energy is indeed finite. This scheme is known as the minimal subtraction (MS) scheme. 51 Since $\pi(q^2)$ enters physical
quantities and must therefore be independent of the regularization parameter $\mu$, we see that $\alpha$ must not be independent of $\mu$ in this scheme: $\alpha = \alpha(\mu)$.

An important feature of the MS scheme is that the renormalized coupling $\alpha(\mu)$ is not the physical (measured) charge of the electron, $\alpha_{\text{phys}} = 1/137$. We may set the coupling equal to the physical charge with a particular choice of $\mu$ (see eq. (3.29)):

$$\alpha(\mu) = \alpha_{\text{phys}} \quad \text{for} \quad \mu^2 = m^2 \frac{e^\gamma}{4\pi}.$$  \hspace{1cm} (3.30)

However, other renormalized parameters (such as the fermion mass) will also depend on $\mu$. These parameters are not in general equal to their physical values. There is no choice of $\mu$ such that any two parameters, such as the mass and electric charge of the electron, are equal to their physical values simultaneously.

We may also remark on a variation of the MS scheme. The parameter $\mu$ was introduced in eq. (3.10) solely on dimensional grounds. We could be more general by replacing (3.10) with

$$(4\pi \lambda \mu^2 \varepsilon)^2,$$  \hspace{1cm} (3.31)

where $\lambda$ is a dimensionless parameter. Thus, there is a family of MS schemes, corresponding to choices of $\lambda$. In particular, the value
defines the modified minimal subtraction (\(\overline{\text{MS}}\)) scheme. This is desirable in that the constants \(-\gamma + \ln(4\pi)\), which arise as an artifact of our dimensional regularization process, are eliminated in the above equations.

We emphasize that in the MS and \(\overline{\text{MS}}\) schemes the renormalized parameters in general do not correspond to the physical parameters. This is often inconvenient, especially when the physical parameters are directly measurable. The scheme in which the renormalized parameters correspond to the physical ones is the on-shell renormalization scheme. For instance, we can have \(\alpha = \alpha_{\text{phys}}\) in the above example by requiring

\[
1 + \pi(0) = 1 . \tag{3.33}
\]

This condition (together with corresponding ones for the electron) defines the on-shell scheme. Equation (3.33) can be satisfied by choosing

\[
\pi(q^2) = \pi_B(q^2) - \pi_B(0) . \tag{3.34}
\]

In this scheme,

\[
Z_3 = 1 - \frac{\alpha}{3\pi} \left[ \frac{1}{\varepsilon} - \gamma + \ln(4\pi) + \ln(\mu^2/m^2) \right] , \tag{3.35}
\]

\[
\pi(q^2) = -\frac{\alpha}{3\pi} F(q^2/m^2) . \tag{3.36}
\]
The renormalized coupling is a constant, equal to the physical coupling. We note that upon inserting eq. (3.35) into eq. (3.26), we find

\[ \alpha_g = \alpha \left[ 1 + \frac{\alpha}{3\pi} \left( \frac{1}{\varepsilon} - \gamma \right) \right], \tag{3.37} \]

and hence \( \alpha \) is independent of the parameter \( \mu \) (a necessity if it corresponds to a physical observable). We will discuss the on-shell scheme in more detail in the next chapter.
IV. ON-SHELL RENORMALIZATION IN ELECTROWEAK THEORY

A. The On-Shell Technique

A principal feature of the Weinberg-Salam model of electroweak theory is its renormalizability. Radiative corrections to tree-level processes can be calculated in a prescribed manner. These corrections offer an excellent laboratory for testing the SM and possibly differentiating between models. Yet, due to the diversity of renormalization prescriptions, the length and complexity of the calculations, and the lack of notation convention, there has not been much consensus on the status of the one-loop predictions. Fortunately, this situation has started to change. The subject has suddenly acquired renewed interest due to the discovery of the W and Z bosons. With this, the on-shell renormalization scheme has emerged as the preferred way to perform electroweak renormalization.\textsuperscript{53}

In the on-shell renormalization technique, the quantities $e$, $M_W$, and $M_Z$ are chosen as the free parameters of the theory. Boundary conditions are imposed during renormalization which guarantee that these parameters are the physical charge and masses. This choice of parameters offers several advantages. The parameters are well-defined, eliminating the ambiguities which plague other schemes. The procedure is a natural
extension of the renormalization in QED, and is convenient because the parameters are physical observables. Accurate measurements of the W and Z masses may be available in the near future, offering an excellent test of the radiative effects in electroweak models.

We will describe the on-shell renormalization procedure, developing the tools we need to find the gauge boson mass shifts for the various models presented in Chapter II. A complete review of the on-shell procedure is given by Aoki et al.\textsuperscript{54} The procedure is summarized in Table 4.1.

The first step is to choose the free parameters that describe our theory. As presented in Chapter II, the parameters in the bare Lagrangian before SSB are $g$ and $g'$ (the SU(2) and U(1) gauge couplings), $\mu^2$ and $\lambda$ (parameters in the Higgs potential), and $G_n$ (the Yukawa coupling constants of the fermions, with $n$ representing each fermion). The SSB introduces one other important parameter, $v$ (the Higgs vev), which is not independent because it must be the minimum of the Higgs potential and hence is related to $\mu$ and $\lambda$ (eq. (2.11)). However, this set

$$g, g', \mu^2, \lambda, G_n$$

appearing in the original Lagrangian, is not very convenient. We generally transfer to another set of parameters which are more physically meaningful. Various choices include:
Table 4.1. Flow chart for renormalization procedure

Choose a set of independent parameters.
\((e_0, M_W, M_Z, m_n, m_H)\)

Separate the bare parameters and fields into renormalized parts and counter terms.
\((e_0 = Y_e, \text{ etc.}, W_0 = Z_W^{1/2}, \text{ etc.})\)

Choose a subtraction scheme to fix the counter terms.

Calculate relevant amplitudes with a suitable regularization scheme.

Eliminate all divergences, using counter terms and subtraction conditions.

NOW HAVE S-MATRIX ELEMENTS AS FUNCTIONS OF PARAMETERS, BUT NUMERICAL VALUES ARE NOT YET FIXED.

Choose input data to fix values of renormalized parameters (muon decay, ev scattering, etc.)

CAN NOW MAKE PREDICTIONS.
where the quantities have been defined in Chapter II. There exist simple relationships between these sets (see, for example, eqs. (2.19)-(2.21), (2.25), and (2.27)) at the tree level. However, these relations become very complicated when higher-order effects are taken into account. In addition, there is no a priori definition of $x^w$. Rather, it may be defined in several differing ways, and its usage is inconvenient when considering higher-order effects.

As noted before, we choose set (3) in (4.2). Thus, we rewrite the Lagrangian (2.33)-(2.36) in terms of (3). We make use of the relations

\[ q = \frac{eM_Z}{\sqrt{M_Z^2 - M_W^2}} \]

\[ q' = \frac{eM_Z}{M_W} \]

\[ v = \frac{2M_W}{2M_Z^2 - M_W^2} \]

\[ \lambda = \frac{e^2 M_Z^2 M_H^2}{8M_W^2 (M_Z^2 - M_W^2)} \]
\[ \mu^2 = M_H^2/2, \]
\[ G_n = \sqrt{2} m_n/v, \]

which hold to all orders in the perturbation expansion. The Lagrangian now is in terms of the parameters (3) of (4.2). We will use the parameters \( g, g', \) and \( z_n \) only as shorthand notation for the expressions in (4.3).

We also use the subscript 0 for the bare parameters which appear in the Lagrangian:
\[ L = \mathcal{L}(e_0^0, M_{W0}^0, M_{Z0}^0, m_0^0, M_{H0}^0). \]

(4.4)

When we do a tree-level calculation, the bare parameters are the physical parameters. But when we include first-order corrections, we encounter divergences. The UV divergences can be absorbed into renormalization constants, as seen in Chapter III. We introduce wavefunction renormalization constants
\[ W_0 = (Z_W^{1/2}) W, \]
\[ \left( \begin{array}{c} Z_0 \\ A_0 \end{array} \right) = \left( \begin{array}{c} Z^{1/2} \\ A \end{array} \right), \]

(4.5)
\[ \psi_{0R,L} = (Z_{R,L}^{1/2}) \psi_{R,L}, \]
\[ \phi_0 = (Z_{\phi}^{1/2}) \phi. \]
Similarly, we transform to a set of (renormalized) parameters

\[ M_{W0}^2 = M_W^2 + \delta M_W^2, \]

\[ M_{Z0}^2 = M_Z^2 + \delta M_Z^2, \]

\[ M_{H0}^2 = M_H^2 + \delta M_H^2, \]  

(4.6)

\[ m_{n0} = m_n + \delta m_n, \]

\[ e_0 = Ye. \]

Thus, we have renormalization constants

\[ \delta M_W^2, \delta M_Z^2, \delta M_H^2, \delta m_n, Y, \text{ and } Z. \]  

(4.7)

where Z represents all the wavefunction renormalization constants. Note that the constants are additive for the masses but multiplicative for the coupling constant and wavefunctions. Their form is as yet unspecified. We will impose boundary conditions that will require that \( M_W, M_Z, M_H, \)
\( m_n, \) and \( e \) be the physical quantities; this will in turn specify what the renormalization constants are. For instance, the constants \( \delta M_W^2 \) and \( Z_W \) are set such that the renormalized \( W \)-boson self-energy goes to zero when the momentum transfer \( q^2 \to M_W^2 \). With this condition, the renormalized mass \( M_W \) is identical to the pole of the \( W \) propagator, i.e., the physical
mass. For this reason, our procedure is known as the on-shell scheme.

We write the Lagrangian

\[ L = L(e_0, M^2_{W0}, Z_0, M^2_{Wn0}, M^2_{H0}, \psi_0) \]

\[ = L(Ye, M^2_W + SM^2_Z, M^2_W + SM^2_Z, m^2_n + SM^2_n, M^2_H + SM^2_H, Z\psi) , \tag{4.8} \]

where \( \psi \) represents all the field variables. We break this into two parts:

\[ L = L_0(e, M^2_W, M^2_Z, m^2_n, M^2_H, \psi) \]

\[ + L_C(e, M^2_W, M^2_Z, m^2_n, M^2_H, \psi, Z) \tag{4.9} \]

We define \( L_0 \) to have the same functional form as our original Lagrangian, with all the bare parameters replaced by renormalized ones. We also have a new contribution, \( L_C \), which gives rise to what are known as counter terms. The counter terms will depend on both the renormalized parameters and the renormalization constants. The success of renormalization theory is that the divergences in the one-loop diagrams (from \( L_0 \)) are cancelled by divergences in the counter terms (from \( L_C \)).

We can use (4.9) to actually find the form of the counter terms. Unfortunately, there is a counter term for each propagator and vertex in \( L \), and thus the list is quite long.
We will only present the ones we need; the complete set can be found in reference 54.

We can narrow our list of renormalization constants and counter terms considerably with several observations. First, we are only interested in first-order corrections. This simplifies the counter terms in that we can reduce the form to being linear in the renormalization constants (higher-power terms being higher-order corrections). Second, we will limit ourselves to doing corrections to leptonic processes, such as muon decay, $\nu e$ scattering, etc. In each case, we have a four-fermion interaction. This eliminates the need for many of the renormalization constants. Finally, we note that the Higgs-boson-fermion coupling is small (for light fermions), and will be ignored.

The types of necessary corrections are then given diagramatically in Figures 4.1. Here, the "blob" represents all the subdiagrams possible to the required order. Diagrams a-d are the gauge-boson self-energies, diagrams e and f are the lepton self-energies, and diagrams g-k are the vertex corrections. The one-loop diagrams which comprise these corrections in the MSM are shown in Figures 4.2-4.12. All but the last diagram in each figure comes from the Lagrangian $L_0$ of eq. (4.9). The last diagram of each figure is the counter term, which comes from $L_c$. 
Figure 4.1. Necessary corrections for leptonic electroweak processes
Figure 4.2. One-loop correction diagrams for the $W$ self-energy in the MSM
Figure 4.3. One-loop correction diagrams for the Z self-energy in the MSM

Figure 4.4. One-loop correction diagrams for the Z-A transition in the MSM
Figure 4.5. One-loop correction diagrams for the photon self-energy in the MSM

Figure 4.6. One-loop correction diagrams for the charged lepton self-energy in the MSM
Figure 4.7. One-loop correction diagrams for the neutrino self-energy in the MSM

Figure 4.8. One-loop correction diagrams for the $\ell\ell\gamma$ vertex in the MSM
Figure 4.9. One-loop correction diagrams for the $\ell\ell Z$ vertex in the MSM

Figure 4.10. One-loop correction diagrams for the $\nu\nu Z$ vertex in the MSM
Figure 4.11. One-loop correction diagrams for the $\nu\nu\gamma$ vertex in the MSM.

Figure 4.12. One-loop correction diagrams for the $\nu\nu W$ vertex in the MSM.
The next step is to find the form of the counter terms. Evaluation of (4.9), with the MSM Lagrangian of Chapter II and the simplifications mentioned, gives the necessary terms. First, we consider the vector bosons. For convenience, we break the self-energies into transverse and longitudinal parts:

\[ (q_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2})A(k^2) + \frac{k_{\mu}k_{\nu}}{k^2}B(k^2) . \]  

(4.10)

Then, Figure 4.2 can be written as

\[ A_R^W(q^2) = \sum_n A_n^W(q^2) + A_C^W(q^2) . \]  

(4.11)

Here, \( A_R^W \) represents the transverse part of the corrected propagator (diagram a of Figure 4.1, or the left-hand side of Figure 4.2), while \( A_n^W \) is the transverse part of each one-loop diagram of Figure 4.2, with the sum over all the diagrams a-h, and \( A_C^W \) is the transverse part of the counter term, diagram i. We define

\[ A_n^W(q^2) \equiv \sum_n A_n^W(q^2) . \]  

(4.12)

Then, (4.11) becomes

\[ A_R^W(q^2) = A(q^2) + A_C^W(q^2) . \]  

(4.13)

and likewise with the longitudinal part,

\[ B_R^W(q^2) = B(q^2) + B_C^W(q^2) . \]  

(4.14)
We find that with this notation the W self-energy counter term is

\[ A^W_C(q^2) = 8M_W^2 + Z_W(M_W^2 - q^2) , \]  

(4.15)

\[ B^W_C(q^2) = 8M_W^2 + Z_WM_W^2 . \]

We now proceed to the Z self-energy, Figure 4.3. We have the same notation:

\[ A^Z_R(q^2) = A^Z(q^2) + A^Z_C(q^2) , \]  

(4.16)

\[ B^Z_R(q^2) = B^Z(q^2) + B^Z_C(q^2) , \]

where \( A^Z \) and \( B^Z \) represent the sum over diagrams a-e. The Z self-energy counter term, diagram f of Figure 4.3, is

\[ A^Z_C(q^2) = 8M_Z^2 + 2Z_{ZZ}^{1/2}(M_Z^2 - q^2) , \]  

(4.17)

\[ B^Z_C(q^2) = 8M_Z^2 + Z_{ZZ}M_Z^2 . \]

The Z-A mixing channel, Figure 4.4, has a counter term (diagram d) of the form

\[ A^{ZA}_C(q^2) = 2^{1/2}Z_A(M_Z^2 - q^2) - q^{2}Z_A^{1/2} , \]  

(4.18)

\[ B^{ZA}_C(q^2) = Z_A^{1/2}M_Z^2 . \]

The photon self-energy, Figure 4.5, has a counter term (diagram d)
For the charged-lepton self-energy, Figure 4.6, we introduce the notation
\[ \Sigma^\nu = \Sigma_1^\nu + \Sigma_2^\nu q^\mu \gamma^\mu + \Sigma_3^\nu q^\mu \gamma^\mu \gamma^5, \] (4.20)

where
\[ \Sigma^\nu(q^2) \equiv \Sigma_n^\nu(q^2), \Sigma_R^\nu(q^2), \Sigma_C^\nu \]
are the self-energies from \( L_0 \), \( L \), and \( L_C \), respectively.

Again,
\[ \Sigma^\nu(q^2) \equiv \Sigma_n^\nu(q^2), \] (4.21)

where the summation is over the diagrams a-c, and \( \Sigma_C^\nu \) is the counter term, diagram d. Of course,
\[ \Sigma_R^\nu(q^2) = \Sigma_C^\nu(q^2) + \Sigma_C^\nu. \] (4.22)

With this notation, the self-energy counter term for the charged lepton is
\[ \Sigma_{1_C}^\nu = -\delta m^\nu - \frac{1}{2}(Z_L + Z_R) m^\nu, \]
\[ \Sigma_{2_C}^\nu = \frac{1}{2}(Z_L + Z_R), \] (4.23)
\[ \Sigma_{5C}^\nu = \frac{1}{2}(Z_L - Z_R). \]
For the neutrino self-energy, Figure 4.7, the counter term (diagram c) is

$$\Sigma_C^\nu(q) = -Z_L^\nu q_\mu \gamma^\mu (1 - \gamma^5) .$$  \hfill (4.24)

The vertex functions, Figures 4.8-4.12, are denoted by

$$\Gamma_\alpha(p, p', q) ,$$  \hfill (4.25)

where $p$ and $p'$ are the momenta of the fermion legs and $q$ is the boson momentum. We suppress these arguments. Again, we write

$$\Gamma_{R\alpha} = \Gamma_\alpha + \Gamma_{C\alpha} .$$  \hfill (4.26)

where $\Gamma_\alpha$ represents the sum of the vertex functions of the one-loop diagrams (for instance, diagrams a-c of Figure 4.8). We consider the charged-lepton-photon vertex, Figure 4.8, first. We find that the counter term, diagram d, for the $\nu\nu\nu$ vertex is

$$\Gamma_{C\nu} = -e\gamma_\alpha (Y + Z_{AA}^{1/2} + \frac{1}{2}(Z_L^\nu + Z_R^\nu) - \frac{1}{2}(Z_L^\nu - Z_R^\nu) \gamma_5)$$

$$+ q_2 Z_{Z\nu}^{1/2} \gamma_\alpha (\frac{1}{2} (M_Z^2 - M_W^2) + \gamma_5) .$$  \hfill (4.27)

The $\nu\nu\nu$ vertex, Figure 4.9, has counter term (diagram e)

$$\Gamma_{C\nu} = -eZ_{\nu\nu}^{1/2} \gamma_\alpha$$

$$- q_2 \gamma_\alpha + \frac{M_Z^2 - 2M_W^2}{2(M_Z^2 - M_W^2)} \left( \frac{S M_Z^2}{M_Z^2} - \frac{S M_W^2}{M_W^2} \right) .$$
For the $v\nu Z$ vertex, Figure 4.11, the counter term, diagram d, is

$$
\Gamma_{C\alpha} = - \frac{g}{g'} \frac{e^2}{2} \left( \frac{Z_{\nu Z}^2}{M_Z^2} - \frac{Z_{\nu W}^2}{M_W^2} \right) + Z_{ZZ}^{1/2} + Z_{L}^{2} + Z_{R}^{2} - \frac{1}{2} (Z_{L}^{2} - Z_{R}^{2}) \gamma_{5}. \tag{4.29}
$$

The $v\nu Z$ vertex, Figure 4.11, has counter term

$$
\Gamma_{C\alpha} = \frac{1}{4} Z_{ZA}^{1/2} \gamma_{\alpha}(1-\gamma_{5}). \tag{4.30}
$$

Finally, the $v\nu W$ vertex, Figure 4.12, has counter term

$$
\Gamma_{C\alpha} = \frac{g}{4} \frac{e^2}{2} \left( \frac{Z_{\nu Z}^2}{M_Z^2} - \frac{Z_{\nu W}^2}{M_W^2} \right) + Z_{ZL}^{2} + Z_{ZR}^{2} + Z_{W}^{2} \gamma_{\alpha}(1-\gamma_{5}). \tag{4.31}
$$

These are the necessary counter terms from $L_{C}$. They involve the as-yet unspecified renormalization constants.
We now specify the boundary conditions. These conditions will be used to determine the renormalization constants. This, in turn, gives us the counter terms (eqs. (4.15)-(4.31)). Then, by calculating the unrenormalized corrections and adding the counter terms, we may evaluate processes including the first-order corrections.

We determine the constants $\delta M^2_W$ and $Z_W$ with the requirement that the transverse part of the renormalized $W$ self-energy, Figure 4.2, behaves as

$$\delta M^2_W = 0 ,$$

$$Z_W = 0 ,$$

where

$$A'(M^2) \equiv \lim_{\substack{q^2 \to M^2}} \frac{d}{d(q^2)} A(q^2) .$$

With this condition, the renormalized mass $M_W$ is identical to the pole of the $W$ propagator, i.e., the physical mass. That is why this is called the on-shell renormalization scheme. The renormalized self-energy part $A_R^W(q^2)$ is then finite. We note that determination of the renormalization constants
through the conditions (4.33) also make $B^W(q^2)$ finite, and the renormalized propagator is the physical mass propagator.

Applying conditions (4.33) to (4.13), with the counter term given by (4.15), the renormalization constants are determined:

$$8M_W^2 = - A^W(M_W^2),$$

$$Z_W = A^W(M_W^2).$$

Likewise the renormalization conditions for the $Z$ propagator, Figure 4.3, which make $M_Z$ the physical $Z$ mass, are

$$A^{Z}(M_Z^2) = 0,$$

$$A^{Z^*}(M_Z^2) = 0.$$  \hspace{1cm} (4.35)

With the counter term (4.17), these conditions give the renormalization constants:

$$8M_Z^2 = - A^Z(M_Z^2),$$

$$Z_{ZZ}^{1/2} = \frac{1}{2} A^{Z^*}(M_Z^2).$$

For the $A-Z$ mixing channel, Figure 4.4, the renormalization conditions are

$$A^{ZA}_{R}(M_Z^2) = 0,$$

$$A^{ZA}_{R}(0) = 0.$$  \hspace{1cm} (4.37)
With the counter term (4.18), the renormalization constants are
\[ Z_{AZ}^{1/2} = A^Z \left( M_Z^2 \right) / M_Z^2. \]  
(4.38)
\[ Z_{ZA}^{1/2} = - A^Z \left( 0 \right) / M_Z^2. \]

For the photon self-energy, Figure 4.5, the on-shell renormalization conditions are
\[ A_R^A(0) = 0, \]  
(4.39)
\[ A_R^{A'}(0) = 0. \]

However, the first condition is identical to the second condition of (4.37). This is a result of the unbroken U(1) symmetry which guarantees the existence of the massless pole in the A-Z channel. The second condition, with the counter term (4.19), gives
\[ Z_{AA}^{1/2} = \frac{1}{2} A^{A'}(0). \]  
(4.40)

Thus, all the gauge-boson renormalization constants have been determined.

We now find the charged lepton self-energy (Figure 4.6) renormalization conditions. The lepton mass \( m_\ell \) will correspond to the pole of the propagator provided
\[ \Sigma_{1R}^{\ell}(m_\ell^2) + m_\ell \Sigma_{YR}^{\ell}(m_\ell^2) = 0. \]
\begin{equation}
\Sigma_5^{\nu}(m_\xi^2) = 0 , \tag{4.41}
\end{equation}

\begin{equation}
2m_\xi \Gamma \Sigma_{1R}^\xi(m_\xi^2) + m_\xi \Sigma_{YR}^\xi(m_\xi^2) \mathbb{I} - \Sigma_{YR}^\xi(m_\xi^2) = 0 . \tag{4.42}
\end{equation}

where we have used the notation introduced in (4.20).

Applying these conditions to (4.22), with the counter term given by (4.23), the renormalization constants are

\begin{equation}
\delta m_\xi = \Sigma_1^\xi(m_\xi^2) + m_\xi \Sigma_{Y}^\xi(m_\xi^2) , \tag{4.43}
\end{equation}

\begin{equation}
Z_R^\xi = Z_{YC}^\xi + \Sigma_{5C}^\xi , \tag{4.44}
\end{equation}

\begin{equation}
Z_L^\xi = Z_{YC}^\xi - \Sigma_{5C}^\xi , \tag{4.45}
\end{equation}

where

\begin{equation}
\Sigma_{YC}^\xi = - \Sigma_{Y}^\xi(m_\xi^2) - 2m_\xi \Gamma \Sigma_{1R}^\xi(m_\xi^2) + m_\xi \Sigma_{YR}^\xi(m_\xi^2) \mathbb{I} , \tag{4.46}
\end{equation}

\begin{equation}
\Sigma_{5C}^\xi = - \Sigma_{5}^\xi(m_\xi^2) . \tag{4.47}
\end{equation}

For the neutrino, Figure 4.7, the on-shell condition is

\begin{equation}
\Sigma_5^{\nu}(0) = 0 . \tag{4.48}
\end{equation}

This determines the renormalization constant

\begin{equation}
Z^{\nu} = -2\Sigma_{Y}^{\nu}(0) \tag{4.49}
\end{equation}

\begin{equation}
= 2\Sigma_{5}^{\nu}(0) . \tag{4.50}
\end{equation}
The final renormalization constant, $Y$, can be determined through the $eeA$ vertex, Figure 4.8. We require that the renormalized vertex be the QED vertex at zero momentum transfer:

$$\bar{u}(m_e)\Gamma_{eeA}^\text{ren} u(m_e) \bigg|_{k^\mu=0} = 0. \quad (4.46)$$

From the counter term (4.26) and using the previous renormalization constants, we determine $Y$.

Thus, all the necessary renormalization constants (4.32) which appear in the counter terms (4.15)-(4.31) have been determined (in terms of the unrenormalized self-energies and vertices). In the next chapter, we will consider the calculation of the one-loop diagrams, hence determining the counter terms. We are then prepared to examine the radiative effects in leptonic electroweak interactions, and evaluate the gauge-boson mass shifts.

B. Survey of Electroweak Radiative Correction Methods

In this section, we will review some other methods for calculating radiative corrections in electroweak models. We can divide the methods into two categories: methods based on conventional perturbation in powers of a fixed coupling constant, and methods based on the renormalization group method.
Most studies have followed a conventional perturbation theory similar to the on-shell method. The differences between the various studies are in the choice of independent parameters. All choices are, in principle, equivalent, and should lead to identical physical quantities. However, some choices are more convenient than others in that the definition of the parameters is unambiguous to all orders in perturbation theory, and possess a clear physical meaning. The best example is the Weinberg angle, $\theta_W$. This parameter is widely used to analyze neutral current data. At the tree level, eqs. (2.20) and (2.40) hold and $\theta_W$ is well-defined. At higher levels, however, these relations are modified, and no unique and proper definition of $\theta_W$ exists. Thus, it is an inconvenient parameter to use. The method we have outlined avoids this problem by not using $\theta_W$, except as a bookkeeping device. We elaborate on several of these other procedures as an illustration.

A set of parameters $e$, $g$, and $M_W$ was first employed by Appelquist et al. Here, $e$ and $M_W$ are the electric charge and the mass of the W boson. The parameter $g$ is the SU(2) coupling constant, defined on-mass-shell at the $W_{\mu\nu}$ vertex. The Weinberg angle is given by $\theta_W = e^2/g^2$. The Z-boson mass is not an independent parameter and is defined as the pole of the Z propagator. As a result, the relation $M_W = M_Z \cos \theta_W$ is
no longer true beyond the lowest order of the perturbation expansion. This scheme is also inconvenient in that the $W_{\mu \nu}$ vertex function is infrared divergent. This divergence can, of course, be cancelled, but then $g$ is cutoff-dependent.

Another scheme, employed by Sirlin,\textsuperscript{56} has $g$, $g'$, and $v$ as independent parameters. Other parameters such as $x_W$, $M_W$, and $M_Z$ are defined as functions of $g$, $g'$, and $v$. In this method, $M_W = M_Z \cos \theta_W$ is set to hold to all orders, but $v$ is no longer the minimum of the Higgs potential beyond the tree level. In addition, there are no field renormalization constants. This simplifies the form of the counter terms, but results in unrenormalized Green's functions. Of course, $S$-matrices and hence physical quantities are still made finite by the renormalization procedure.

Another common scheme uses $g$, $M_W$, and $x_W$ as parameters.\textsuperscript{57,58} Here, $x_W$ is defined as the ratio of elastic $\bar{\nu}_\mu e$ to $\nu_\mu e$ scattering. Neither the relationships $e = g \sin \theta_W$ nor $M_W = M_Z \cos \theta_W$ hold to second-order or higher any longer.

Nonetheless, these and other similar methods\textsuperscript{59} are essentially the same as the on-shell technique we have presented. The differences are generally that of convention, with results being comparable once relationships between parameters in the different schemes have been worked out.
An approach based on conventional perturbation theory but structured quite differently was formed by Llewellyn Smith and Wheater. They employed the modified minimal subtraction (MS) scheme, which was mentioned in Chapter III. The set of parameters used is

$$\alpha(\mu), m_W(\mu), m_Z(\mu), m_H(\mu), m_f(\mu),$$ \hspace{1cm} (4.47)

where $\mu$ is the mass scale introduced in the dimensional regularization process. They are finite, well-defined parameters, but they are in general not equal to the corresponding physical ones. Calculation of renormalized quantities in terms of the set (4.47) is particularly simple. However, we must relate the MS parameters to the physical parameters, and the total amount of work in obtaining a given physical quantity remains the same.

We note that the MS method most closely mimics the QCD renormalization procedure, whereas the on-shell technique mimics QED. The "running coupling constant" idea in QCD, where $\mu$ is taken as the relevant mass scale, has no meaning in the on-shell scheme. The only coupling constant is the physical one, which relates to a directly measurable coupling strength.

An alternative to conventional perturbation theory has been formulated by means of the renormalization group approach. For example, the perturbative expansion of a
theory may give rise to logarithmic terms of the form
\((\alpha \ln(\mu'/\mu))^n\), where \(\mu\) and \(\mu'\) are two different mass scales of
the theory. In cases such as QCD, in which the coupling
constant is not so small, the series must be summed to all
orders when \(\mu' \gg \mu\). The renormalization group method is used
to deal in part with this problem. The renormalization group
equation sums up to all orders the leading log contributions.
In electroweak theory, we also have two mass scales: the gauge
boson masses, and low-energy variables like the momentum
transfer of a process. The logarithmic terms are not as
dominant, though, due to the small coupling constant.
Nonetheless, we can use renormalization group equations to sum
these terms. This was originally formulated for electroweak
theory by Marciano,\(^62\) and expanded by the Rome group\(^63\) and the
Harvard group.\(^64\) For weak corrections, effects of the sum of
the leading-log terms to all orders is small in comparison to
the first-order non-log term. Thus, this method is
complementary to but cannot replace conventional perturbation
theory. However, summation of the leading-log contributions
is essential if one considers the QCD corrections to weak
processes involving hadrons. As we confine ourselves to
reactions involving leptons as external particles, the QCD
corrections enter only at the two-loop level. These
corrections are negligible at the experimental accuracy
achieved presently.
V. MASS SHIFTS IN THE MINIMAL AND EXTENDED MODELS

A. Mass Shift Formulas

Thus far, we have developed the formalism to calculate radiative effects in the SM. We will now use this formalism to calculate the gauge-boson mass shifts. We are particularly interested in the contributions of the Higgs sector, in both the minimal and extended variations of the standard model.

In Chapters II and IV we presented the SM in terms of the independent parameters $e$, $M_W$, and $M_Z$ (we will suppress the parameters $m_t$ and $M_H$). The theory is of no use, however, until we establish these parameters by comparing predictions to experimental results. The parameter $e$ is known to great precision from Thompson scattering (without recourse to perturbation), thus we need to determine $M_W$ and $M_Z$. The values we get depend on the order of perturbation we have considered; the differences between successive orders are called the mass shifts. The mass shifts due to the one-loop corrections to the tree-level values are the most important. We will examine this mass shift for the Minimal Standard Model and models with extended Higgs sectors.

Let us consider two low-energy processes, called R and S, which we will use as input data points. The actual choice for R and S will be discussed later. We calculate R and S, to
lowest order, as functions of the parameters $M_W$ and $M_Z$, then compare to the experimental values:

$$R^{(0)}(M_W, M_Z) = R^{\text{exp}}.$$  

(5.1)

$$S^{(0)}(M_W, M_Z) = S^{\text{exp}}.$$  

Here, $R^{(0)}$ and $S^{(0)}$ represent the tree-level calculations, $R^{\text{exp}}$ and $S^{\text{exp}}$ their experimental values. The solutions to (5.1), which we designate as $M_W^{(0)}$ and $M_Z^{(0)}$, are the lowest-order predictions for the boson masses. When we include the first-order radiative corrections, we have

$$R^{(1)}(M_W, M_Z) = R^{\text{exp}}.$$  

(5.2)

$$S^{(1)}(M_W, M_Z) = S^{\text{exp}}.$$  

The solutions to this, $M_W^{(1)}$ and $M_Z^{(1)}$, are the physical masses (except for terms of higher order in $\alpha$). The differences

$$\Delta M_W = M_W^{(1)} - M_W^{(0)}$$

(5.3)

$$\Delta M_Z = M_Z^{(1)} - M_Z^{(0)}$$

are the mass shifts. The tree-level expressions are readily calculable; the problem is to solve (5.2). In general, the functional dependence of $R^{(1)}$ and $S^{(1)}$ on $M_W$ and $M_Z$ is complicated, and the equation must be solved numerically.
This presents a problem in that we wish to examine the contributions to $\Delta M_W$ and $\Delta M_Z$ due to the Higgs boson, requiring explicit expressions for $\Delta M_W$ and $\Delta M_Z$.

Consider a process, $R$, dependent on a single variable, $M$. Then, as in (5.1) and (5.2),

$$R_{\exp} = R(0)(M(0)) = R(1)(M(1)) .$$

(5.4)

We can write

$$R(1)(M(1)) = R(0)(M(1)) + \Delta R(M(1))$$

(5.5)

$$= R(0)(M(0) + \Delta M) + \Delta R(M(0)) ,$$

(5.6)

where $\Delta R$ contains all the second-order corrections. In going from (5.5) to (5.6) we have noted that $\Delta R(M(1)) = \Delta R(M(0))$ to this order. Making a Taylor series expansion and keeping only the first-order term in $\Delta M$:

$$R(0)(M(0) + \Delta M) = R(0)(M(0)) + \Delta M \frac{\partial R(0)}{\partial M} \bigg|_{M=0} .$$

(5.7)

Inserting into (5.6), with (5.4), gives

$$\Delta M = - \frac{\Delta R(M(0))}{R'(M(0))} ,$$

(5.8)

where

$$R'(M(0)) \equiv \frac{\partial R(m)}{\partial M} \bigg|_{M=0} ,$$

(5.9)

which we assume to be non-vanishing. Thus, we have an explicit expression for $\Delta M$, depending only on the lowest-order mass.
We can extend this to two processes and two variables. We find

\[ \Delta M_W = \frac{R_2 \Delta S - S_2 \Delta R}{R_1 S_2 - S_1 R_2} \]

\[ \Delta M_Z = \frac{S_1 \Delta R - R_1 \Delta S}{R_1 S_2 - S_1 R_2} \]  

(5.10)

where

\[ R_i \equiv \frac{\partial R^{(0)}}{\partial m_i}(m_1, m_2) \bigg|_{m_1 = M_W^{(0)}, m_2 = M_Z^{(0)}} \]

(5.11)

for \( i = 1, 2 \), and

\[ \Delta R \equiv \Delta R(M_W^{(0)}, M_Z^{(0)}) \]

(5.12)

and similarly for \( S_i \) and \( \Delta S \). Thus, we have explicit expressions for the mass shifts. We may find the contributions from the Higgs boson (or any other type of contribution) by finding the corresponding contribution to \( \Delta R \) and \( \Delta S \). Note that the only masses which appear are the lowest-order ones, because we have substituted \( M_W^{(0)} \) and \( M_Z^{(0)} \) for \( M_W \) and \( M_Z \), to this order. We point out that while the input data points no longer explicitly appear, their numerical values manifest themselves in the determination of \( M_W^{(0)} \) and \( M_Z^{(0)} \).
B. Calculation of Low-Energy Processes

The next step is to choose the low-energy processes $R$ and $S$ which we wish to use as input data. As mentioned, $e$ can be determined from Thompson scattering. The fermion masses $m_f$ are known from other considerations, while the Higgs mass $M_H$ is undetermined by the theory and will be allowed to vary within certain bounds. Thus, we are looking for two processes which will determine $M_W$ and $M_Z$. The obvious choice for one is the muon decay rate, which is one of the best measurable numbers in weak processes. For the other, we will take the ratio of $\bar{\nu}_e$ to $\nu_e$ elastic scattering cross-sections. This is theoretically much easier to calculate than other processes, such as $eD$ scattering, atomic parity violation, and so on, which involve quark structure functions and QCD corrections. However, the precision of the $\nu_e$ experiments is not good due to poor statistics. Given the difficulties in doing such experiments and the small rates involved, the accuracy of the $\nu_e$ experiments are not likely to improve greatly in the near future.

We then have two low-energy processes which we use to give the lowest-order mass predictions and the predicted mass shifts. Comparison to the physical $W$ and $Z$ masses then offers a test of the radiative corrections. For instance, the existing values of $M_W$ and $M_Z$ from the CERN collider already
offer a preliminary check, showing that the mass shifts are at
least approximately of the sizes predicted by the SM.

An interesting alternative to the above method would be to
take one of the masses as an experimentally determined value
instead of the neutral current data. We give up one check
of our theory in return for more accuracy. It is expected
that the mass of the Z boson will be measured accurately (to
within 0.1 GeV) in the near future at SLC and LEP. We take
\( M_Z^{\text{exp}} \) as input, as well as the muon decay rate. The mass
shift of the \( M_W \) can be found from (5.5):

\[
\Delta M_W = - \frac{\Delta R(M_W^{(0)}, M_Z^{\text{exp}})}{\frac{\partial R(0)}{\partial m}(m, M_Z^{\text{exp}})|_{m=M_W^{(0)}}}.
\]

The shift for the \( W \) mass can be obtained to good accuracy, and
a measurement of the physical \( W \) mass may be used to test our
theory. We will call this the 1-variable method.

First, we look at muon decay. The diagrams describing the
process to first-order are given in Figure 5.1. The general
matrix element can be written as

\[
T = \frac{G_F}{\sqrt{2}} \bar{\nu}_\mu \gamma^5 \nu_e (1-\gamma^5) \bar{\nu}_e \gamma^5 \nu_e.
\]

where \( G_F \) is the Fermi constant (Chapter I). Here, we have
represented the spinor wavefunctions by their symbols, \( u = u_\mu \),
\( \bar{\nu} = \nu_e \), and so on. We also use \( e \) to denote the electric
Figure 5.1. Muon decay diagrams, including one-loop corrections.
charge; however, no confusion should arise from this notation. The matrix element (5.14) gives the decay width

\[ \Gamma = \frac{G_F^2 m^5}{192 \pi^3} . \]  

(5.15)

The experimental results are traditionally expressed for $G_F$ rather than $\Gamma$. Thus, for our process $R$ we take the coefficient of the matrix element (5.14). That is,

\[ R^{\text{exp}} = G_F/\sqrt{2} . \]  

(5.16)

Taking this coefficient as our process $R$, rather than the actual decay rate, is more convenient. From diagram a of Figure 5.1, we find that the tree-level matrix element in the SM is

\[ M^{(0)} = \frac{e^2 M_Z^2}{8(M_Z^2 - M_W^2)} \mu \gamma (1-\gamma^5) \nu e (1-\gamma^5) \nu e \frac{1}{M_W^2} , \]  

(5.17)

where we have taken the low-momentum-transfer limit. The process (to lowest order) as a function of the parameters is then

\[ R^{(0)} = \frac{e^2 M_Z^2}{8(M_Z^2 - M_W^2) M_W^2} . \]  

(5.18)

We now include the first-order corrections, diagrams b-d of Figure 5.1. The total matrix element is

\[ T^{(1)} = T^{(0)} + T_D + T_C + T_d , \]  

(5.19)
where $T_b^\ell$, $T_c^\ell$, and $T_d^\ell$ correspond to diagrams b, c, and d in Figure 5.1, respectively. We find that

\begin{align}
   T_b^\ell &= \frac{g}{2\sqrt{2}} \bar{\nu}_\mu \Gamma^\nu W_{\mu R} e (1-\gamma^5) \nu e \frac{1}{M_W^2}, \\
   T_c^\ell &= \frac{g}{2\sqrt{2}} \bar{\nu}_\mu (1-\gamma^5) \nu e (1-\gamma^5) \nu e \frac{1}{M_W^2}. \\
   T_d^\ell &= \frac{g^2}{8} \bar{\nu}_\mu (1-\gamma^5) \nu e (1-\gamma^5) \nu e \frac{1}{M_W^2} \frac{A_R^W(0)}{M_W^2}.
\end{align}

We have used the notation introduced in Chapter IV, where the "blobs" represent the one-loop corrections, including the counter terms. The renormalized vertex functions are given by $\Gamma_R^\alpha$ and $A_R^W(0)$ is the transverse part of the $W$-boson self-energy, evaluated at $q^2 \to 0$. The fermion self-energy contributions (diagrams e-h of Figure 5.1) are cancelled in the renormalization process, and for convenience we will not consider them. The box diagrams (diagrams i-l) are small, and hence are ignored. It should be noted that the $g$ appearing in equations (5.20)-(5.22) and the rest of this chapter is only a shorthand notation defined by equation (4.3). As long as $\Gamma_R^\nu W_{\mu R}$ has a V-A form, we can define $\Gamma_R^\nu W_{\mu R}$ by

\begin{equation}
   \Gamma_R^\nu W_{\mu R} = \Gamma_R^\nu W_{\mu R} (1-\gamma^5) \\
   \Gamma_R^\nu W_{\mu R} = \Gamma_R^\nu W_{\mu R} (1-\gamma^5) \nu e (1-\gamma^5) \nu e \frac{1}{M_W^2} \frac{A_R^W(0)}{M_W^2}.
\end{equation}
Then, (5.19) can be put in the form of (5.14), and the coefficient of the matrix element (5.19) is

\[ R(1) = \frac{e^2}{8m^2_W} [1 + \frac{2\sqrt{2}}{g} (\Gamma^{\tilde{v}eW} + \Gamma^{\nu\mu W}) + \frac{A_R^W(0)}{m^2_W}] . \]  

(5.24)

The total weak correction is then

\[ \Delta R = \frac{e}{2\sqrt{2}} \frac{(\Gamma^{\tilde{v}eW} + \Gamma^{\nu\mu W})}{m^2_W} + \frac{e^2}{8m^2_W} \frac{A_R^W(0)}{m^2_W} . \]  

(5.25)

The second process we choose is the ratio of muon-neutrino-electron elastic scattering ($\nu_\mu e \rightarrow \nu_\mu e$) to muon-antineutrino-electron elastic scattering ($\bar{\nu}_\mu e \rightarrow \bar{\nu}_\mu e$). The neutral current reaction $\nu_\mu e \rightarrow \nu_\mu e$ has contributions through the one-loop level from the diagrams in Figure 5.2. The tree-level process, diagram a of Figure 5.2, has a corresponding matrix element

\[ t^{(0)} = \frac{g_Z^2}{16M^2_Z} \bar{\nu}_\mu \gamma(1-\gamma^5)\nu_\mu \bar{e}^\gamma X_0 (a_0 - b_0 \gamma^5) e \]  

(5.26)

Again we have taken the low-momentum-transfer limit. Here $a_0$ and $b_0$ are the vector and axial-vector couplings, respectively, of the charged lepton to the Z boson:

\[ a_0 = -1 + 4 \frac{m_Z^2 - m_W^2}{m_Z^2} \]  

(5.27)
Figure 5.2. Diagrams for $\nu\mu$ elastic scattering, including one-loop corrections
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\[ b_0 = -1 \ . \]

The matrix element for \( \bar{\nu}_\mu e \to \bar{\nu}_\mu e \) is just (5.26) with \( b_0 \to -b_0 \). These amplitudes in turn give a cross-section ratio of

\[
\frac{\sigma(\bar{\nu}_e e \to \bar{\nu}_e e)}{\sigma(\nu e \to \nu e)} = \frac{\xi^2 - \xi + 1}{\xi^2 + \xi + 1} , \tag{5.28}
\]

where

\[
\xi = \frac{a_0}{b_0} . \tag{5.29}
\]

Note that in the conventional notation

\[
\xi = 1 - 4x_W , \tag{5.30}
\]

where \( x_W \) is the commonly quoted experimental value for \( \sin^2 \theta_W \), where \( \theta_W \) is the Weinberg angle. Following the common procedure, we take as our process \( S \) the ratio of the vector to the axial-vector couplings (times a factor of \(-1/4\) for convenience). The tree level calculation for \( S \), from (5.27), is

\[
S^{(0)} = - \frac{1}{4} \frac{a_0}{b_0} = - \frac{1}{4} \left( 1 - \frac{M_Z^2 - M_W^2}{M_Z^2} \right) . \tag{5.31}
\]

We take (5.28) as the definition of \( \xi \). It is then the ratio of vector to axial-vector current strength. This is an
experimentally determined number, and compares to the process \( S \) by

\[
S_{\text{exp}} = -\frac{1}{4} \xi .
\]  

(5.32)

Radiative corrections to \( S \) arise when we take into account one-loop diagrams, such as diagrams \( b-m \) of Figure 5.2. Again, the fermion self-energy diagrams, \( g-j \), give no finite contribution, and the box diagrams, \( k-m \), are small and can be ignored. The total matrix element, including one-loop corrections, is then

\[
T^{(1)} = T^{(0)} + T_b + T_c + T_d + T_e + T_f .
\]  

(5.33)

Now any of the one-loop corrections to the amplitude can be written as

\[
T_J = \frac{g^2}{16 M_Z^2} \frac{1}{4} \bar{\nu} \gamma^\alpha (1-\gamma^5) \nu \bar{e} \gamma^\beta (a_J - b_J \gamma^5) e,
\]  

(5.34)

where the index \( J \) represents a particular diagram. Then, the total one-loop-corrected vector and axial-vector couplings, \( A \) and \( B \) respectively, are

\[
A = a_0 + \sum J a_J
\]  

\[ B = b_0 + \sum J b_J , \]

where the summation runs over all diagrams \( b-f \). The one-loop-corrected expression for \( S \) is
The one-loop corrections to the process $S$ are then

$$
\Delta S = -\frac{1}{4} \frac{a_0}{b_0} \left[ \frac{\Sigma a_J}{a_0} - \frac{\Sigma b_J}{b_0} \right].
$$

We need to evaluate each diagram to find $a_J$ and $b_J$ to determine $\Delta S$.

We first consider the $Z$ self-energy graph, diagram $b$ of Figure 5.2. The matrix element is

$$
T_b = \frac{g_Z^2}{16} \bar{v}_\gamma(1-\gamma^5)v_\alpha(-1+4\frac{M_Z^2-M_W^2}{M_Z^2}+\gamma^5)e_\frac{Z_R(0)}{M_Z^4}.
$$

The renormalized $Z$ self-energy, $Z_R$, has been defined in Chapter IV. Putting (5.38) into the form of (5.34), we find the vector and axial-vector couplings

$$
a = (-1 + 4\frac{M_Z^2-M_W^2}{M_Z^2}) \frac{Z_R(0)}{\frac{M_Z^2}{M_Z^2}},
$$

$$
b = (-1) \frac{Z_R(0)}{M_Z^2},
$$

so
\[ \frac{a}{a_0} - \frac{b}{b_0} = 0 \quad . \quad (5.40) \]

and the contribution from this graph is zero.

The Z-A exchange graph, diagram c of Figure 5.2, has matrix element

\[ T_c = \frac{1}{4} e g_Z \gamma^\alpha (1 - \gamma^5) \gamma^\alpha (\gamma - 1) e \frac{A_R(q^2)}{M_Z^2 q^2} \bigg| q^2 = 0 . \quad (5.41) \]

The couplings are

\[ a = -4 e \frac{A_R(q^2)}{g_Z} \bigg| q^2 = 0 . \quad (5.42) \]

\[ b = 0 . \]

This gives

\[ \frac{a}{a_0} - \frac{b}{b_0} = \frac{-4 e A_R(q^2)}{g_Z(-1 + 4(M_Z^2 - M_W^2)/M_Z^2) q^2} \bigg| q^2 = 0 . \quad (5.43) \]

Next we consider the \( \nu \nu Z \) vertex, diagram d of Figure 5.2. The matrix element is

\[ T_d = \frac{1}{4} g \Gamma_R \nu \nu Z \nu e^{(-1 + 4M_Z^2 - M_W^2 - \gamma^5) e} \frac{1}{M_Z^2} . \quad (5.44) \]

The vertex function \( \Gamma_R \) has been introduced in Chapter IV. We make the definition

\[ \Gamma_R^{\nu \nu Z} = a^{\nu \nu Z} - b^{\nu \nu Z} \gamma^5 . \quad (5.45) \]
We then put (5.44) in the form of (5.34), getting
\[
a = \frac{4}{g} (-1 + 4 \frac{M_W^2 - M_N^2}{M_Z^2}) a_{\nu\nu Z}
\]
(5.46)
\[
b = -\frac{4}{g} b_{\nu\nu Z}.
\]
This gives
\[
\frac{a}{a_0} - \frac{b}{b_0} = \frac{4}{g} (a_{\nu\nu Z} - b_{\nu\nu Z}).
\]
(5.47)

The $\nu\nu A$ vertex graph is shown in diagram e of Figure 5.2. The matrix element for it is
\[
T_e = -e^2 \nu\nu A \nu\nu Y \frac{a}{q^2} \left| q^2 = 0 \right.
\]
(5.48)

Making a definition similar to (5.45), we find
\[
a = -\frac{16e^2 M_Z^2}{q_Z^2 q^2} a_{\nu\nu A}
\]
(5.49)
\[
b = 0,
\]
with the limit $q^2 \to 0$ understood. The contribution to (5.37) is then
\[
\frac{a}{a_0} - \frac{b}{b_0} = -\frac{16e^2 M_Z^2}{q_Z^2 q^2 (-1 + 4 (M_Z^2 - M_W^2)/M_Z^2)} a_{\nu\nu A}.
\]
(5.50)

The final graph is the $\nu\nu Z$ vertex, diagram f of Figure 5.2. The matrix element is
\[ T_f = \frac{g_Z}{4} \gamma_{\alpha}(1-\gamma^5)\bar{\nu}e\nu eZ e \frac{1}{M_Z^2}, \]  

(5.51)

giving

\[ a = \frac{4}{g_Z} a_{eeZ}, \]  

(5.52)

\[ b = \frac{4}{g_Z} b_{eeZ}. \]

The contribution to (5.37) is then

\[ \frac{a}{a_0} - \frac{b}{b_0} = \frac{4}{g_Z} \frac{a_{eeZ}}{(-1+4(M_Z^2-M_W^2)/M_Z^2)} + b^{eeZ}. \]  

(5.53)

The total weak corrections to the process S are then

\[ \Delta S = -\frac{1}{4}(1-4\frac{M_Z^2-M_W^2}{M_Z^2}) \frac{4e^2 A^2(q^2)}{g_Z(-1+4(M_Z^2-M_W^2)/M_Z^2)} + \frac{4}{g}(a^{\nu\nu Z}-b^{\nu\nu Z}) \]

\[ -\frac{16e^2 M_Z^2}{g_Z^2 q^2 (-1+4(M_Z^2-M_W^2)/M_Z^2)} a^{\nu\nu A} \]  

(5.54)

\[ +\frac{4}{g_Z} \frac{a_{eeZ}}{(-1+4(M_Z^2-M_W^2)/M_Z^2)} + b^{eeZ}]. \]

To summarize, we have examined two low-energy processes with the goal of determining the gauge-boson masses. The processes are muon decay (R) and the neutrino-electron elastic
scattering cross-section \( S \). To the lowest order, as functions of the parameters, they are

\[
R^{(0)} = \frac{e^2 M_Z^2}{8(M_Z^2 - M_W^2)} \frac{M_Z^2}{M_W^2} 
\]

(5.55)

\[
S^{(0)} = -\frac{1}{4} \left( 1 - 4 \frac{M_Z^2 - M_W^2}{M_Z^2} \right)
\]

The first-order-correction expressions are (5.25) and (5.54).

Experimentally, the processes are

\[
R^{\text{exp}} = G_F/\sqrt{2} 
\]

(5.56)

\[
S^{\text{exp}} = -\frac{1}{4} \xi
\]

We note that (5.55) gives

\[
R_1 = -\frac{e^2 M_Z^2 (M_Z^2 - 2M_W^2)}{4M_W^2 (M_Z^2 - M_W^2)^2}
\]

\[
R_2 = -\frac{e^2 M_Z^2}{4(M_Z^2 - M_W^2)^2}
\]

(5.57)

\[
S_1 = -2 \frac{M_W^2}{M_Z^2}
\]

\[
S_2 = 2 \frac{M_W^2}{M_Z^2}
\]
We are now ready to calculate the explicit expressions for (5.25) and (5.54), and hence the mass shifts, for the MSM.

C. Dominant Terms of the Mass Shifts in the Minimal Model

The last step in our procedure is to finally calculate the "blobs" - the one-loop diagrams in Figures 4.2-4.12. From Chapter IV, we may then calculate the renormalization constants and the counter terms. This gives the renormalized self-energies and vertices, which we use to find $\Delta R$ and $\Delta S$, as shown in the previous section. Using the results of section A will then give us the mass shifts.

The calculation of all the one-loop diagrams of Figures 4.2-4.12 is a monumental task, generally left to algebraic manipulation computer programs. The results are presented explicitly in reference 54; we will not repeat them here. We will develop the Higgs-dependent terms in the next section as a prelude to working with models with extended Higgs sectors.

The gauge-boson mass shifts for the MSM have been presented by many authors. They have found that the radiative corrections are important in the prediction for the $W$ and $Z$ masses. Using $G_F$ and $\xi$ as experimental input data, the tree-level predictions are

$$M_W \equiv M_W^{(0)} \sim 77 \text{ GeV}$$

(5.58)
\[ M_Z = M_{Z}^{(0)} \approx 88 \text{ GeV} \, . \]

However, the one-loop-corrected predictions differ by
\[ \Delta M_{W} = +2.6 \text{ GeV} \, . \]
\[ \Delta M_{Z} = +3.0 \text{ GeV} \, . \]

Thus, we expect to find the gauge bosons, not at 77 and 88 GeV, but at 79.6 and 91 GeV.

This large mass shift is almost entirely due to fermion loops in the boson self-energies, Figure 5.3. Corrections

\[ \Delta M \propto \alpha \ln \left( \frac{m^2}{M^2} \right) \, . \]  

where \( M \) is the mass of the \( W \) or \( Z \) and \( m \) is a fermion mass. Obviously, for the light fermions, this type of correction is large. For the first generation of fermions, the actual
shifts are

\[ \Delta M_W = -\frac{\alpha^2 M_W}{2\pi^2} \left( \frac{5}{36} + \frac{1}{48} \ln\left(\frac{m_u^2}{M_W^2}\right) + \ln\left(\frac{m_d^2}{M_W^2}\right) + \ln\left(\frac{m_e^2}{M_W^2}\right) \right) \]

\[ + \frac{1}{\sqrt{2}} \left( \frac{2c^4 - 1}{4} \ln\left(\frac{m_d^2}{M_Z^2}\right) + \frac{1}{48} \left(4c^4 - 6c^2 + 3\right) \ln\left(\frac{m_e^2}{M_Z^2}\right) \right) \]

\[ (5.61) \]

\[ \Delta M_Z = -\frac{\alpha^2 M_Z}{2\pi^2} \left( \frac{1}{108} \left(40c^4 - 50c^2 + 25\right) + \frac{1}{72} \left(8c^4 - 10c^2 + 5\right) \ln\left(\frac{m_u^2}{M_Z^2}\right) \right) \]

where \( m_u, m_d, \) and \( m_e \) are the up quark, down quark, and electron masses, respectively. Here \( c = \cos \theta_W = M_W/M_Z \). There are similar expressions for the other generations.

Contributions from other diagrams of Figures 4.2-4.12, including Higgs particle diagrams and vertex diagrams, total to about .1 GeV. The result is not very sensitive to the Higgs mass.

We reflect that while these corrections are rather large, the current uncertainty, due to the neutrino data, in the \( W \) and \( Z \) masses is as large. Thus, we are not yet to the point where the radiative corrections are numerically meaningful.

In addition, measurements of the \( W \) and \( Z \) masses at CERN are not precise enough to compare to the theoretical predictions for the radiative corrections. As mentioned before, this situation will be improved in the next few years.
D. Higgs Terms of the Mass Shifts in the Minimal Model

As part of the total mass shift calculation discussed in section C, we have some terms which involve the Higgs boson. The minimal model has a single neutral Higgs; the mass shifts can be written as a function of the Higgs-boson mass. In this section, we present the details of the calculation of the Higgs-mass-dependent terms. 67

First, we examine the $W$-boson self-energy. The diagrams involving the Higgs particle are shown in Figure 5.4. The loop integrals have been worked out in Appendix A. The contributions to the $W$ self-energy from diagrams a and b of Figure 5.4, respectively, are

---

![Diagram](image.png)

Figure 5.4. Higgs-dependent contributions to the $W$ self-energy in the MSM
\[ \text{diag } a = \frac{q^2 M^2_W}{(2\pi)^4 i} \left[ B_0(M^2_W, M^2_H, q^2) - \frac{1}{M^2_W} B_{22}(M^2_W, M^2_H, q^2) \right] \]

\[ \text{diag } b = -\frac{1}{4} \frac{q^2}{(2\pi)^4 i} A_0(M^2_H) , \]

where the functions \( B_0, B_{22}, \) and \( A_0 \) are defined and evaluated in Appendix A. The Higgs-dependent part of the \( W \) self-energy is then the sum of (5.62a) and (5.62b):

\[ \frac{g^2}{(2\pi)^4 i} M^2_W B_0(M^2_W, M^2_H, q^2) - B_{22}(M^2_W, M^2_H, q^2) \]

By (4.34), the Higgs-dependent parts of the renormalization constants \( S_{W} \) and \( Z_{W} \) are

\[ \frac{g^2}{(2\pi)^4 i} M^2_W B_0(M^2_W, M^2_H, M^2) - B_{22}(M^2_W, M^2_H, M^2) \]

\[ + \frac{1}{4} A_0(M^2_H) ] . \]

and

\[ Z_{W} = A_{W}(M^2_W) \]
For the case of large Higgs mass, $M_H \gg M_W$, we can expand the functions $B_0$ and $B_{22}$ in powers of $M_W^2/M_H^2$. The details of the expansion are given in Appendix B. Using eq. (B.13), the Higgs dependence of the unrenormalized self-energy, (5.63), becomes

\[
\text{[} A_{W}(0)\text{]}_H = \frac{g^2}{16\pi^2} \left[ \frac{1}{8} M_H^2 + 3 M_W^2 \ln \left( \frac{M_H^2}{M_W^2} \right) \right],
\]

(5.66)

where we have dropped terms of order $M_W^2$ and smaller. Using equation (B.14), the Higgs-dependent part of the renormalization constant $\delta M_W^2$, (5.64), becomes

\[
\text{[} \delta M_W^2 \text{]}_H = \frac{g^2}{16\pi^2} \left[ - \frac{1}{8} M_H^2 + \frac{5}{6} M_W^2 \ln \left( \frac{M_H^2}{M_W^2} \right) \right].
\]

(5.67)

The leading term is of order $M_H^2$. The Higgs contribution to the wavefunction renormalization constant is

\[
\text{[} Z_{W} \text{]}_H = 0,
\]

(5.68)

due to (B.22). The renormalized $W$ self-energy, (4.15), is the sum of (5.66) and (5.67), in the limit $M_H \gg M_Z$:

\[
A_W(0) = A_W(0) + A_{W}(0)
\]

\[
= A_W(0) + \delta M_W^2 + M_W^2 Z_W
\]
Next, we calculate the Z-boson self-energy. The relevant diagrams are shown in Figure 5.5. The calculation is exactly like the W-boson self-energy, except with $g \rightarrow g_Z$, $M_W \rightarrow M_Z$.

Thus, the Higgs contributions to the mass counter term and the renormalized Z self-energy, in the large Higgs-mass limit, are

$$[\delta m_Z^2]_H = \frac{g_Z^2}{16\pi^2} \left[ -\frac{1}{8} M_W^2 - \frac{5}{6} M_Z^2 \ln(\frac{M_H^2}{M_Z^2}) \right]$$

(5.70)

and

$$[A_R(0)]_H = \frac{g_Z^2}{16\pi^2} \left[ -\frac{1}{12} M_Z^2 \ln(\frac{M_H^2}{M_Z^2}) \right].$$

(5.71)
The Higgs particle does not couple directly to the photon and we are taking the coupling to the light fermions as zero. Thus, all the other renormalization constants — Y, $Z_{ZA}$, $Z_{AZ}$, $Z_{A}$, and $\delta m_f$ — have no Higgs dependence in them, to the one-loop order. In addition, the unrenormalized vertex functions have no diagrams with Higgs particles in them, to the one-loop order. The only Higgs dependence in the vertex functions is in the counter terms, through the renormalization constants $\delta M_W^2$ and $\delta M_Z^2$. Using the counter term from (4.26), the Higgs-dependent portion of the renormalized $eeZ$ vertex is

$$ [\Gamma_{eeZ}]_H = \frac{g^2(q_Z^2 - 2g^2)}{8e^2} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) \gamma_\alpha (1 - \gamma^5) $$

(5.72)

$$ + \frac{g_Z}{e} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) \gamma_\alpha $$

We see that it has vector and axial-vector couplings, (5.45).

$$ a_{eeZ} = - \frac{g^2(q_Z^2 - 2g^2) + 8g_Z e}{8e^2} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) $$

(5.73)

$$ b_{eeZ} = - \frac{g^2(q_Z^2 - 2g^2)}{8e^2} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right). $$

The renormalized $\nu\nuZ$ vertex, using the counter term (4.28), has Higgs-dependent terms.
\[ \Gamma_{R\alpha}^{\nu\nu} |_H = \frac{g_Z (g'^2 - g^2)}{8g^2} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) \gamma_\alpha (1 - \gamma^5) , \quad (5.74) \]

giving vector and axial-vector couplings

\[ a^{\nu \nu} = b^{\nu \nu} = \frac{g_Z (g'^2 - g^2)}{8g^2} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) . \quad (5.75) \]

For the \( \nu \nu W \) vertex, the Higgs-dependent contributions to the renormalized vertex come from the counter term (4.30):

\[ \Gamma_{R\alpha}^{\nu \nu W} |_H = \frac{g}{4\sqrt{2}} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) \gamma_\alpha (1 - \gamma^5) , \quad (5.76) \]

leading to couplings

\[ a^{\nu \nu W} = b^{\nu \nu W} = \frac{g}{4\sqrt{2}} \left( \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \right) . \quad (5.77) \]

For the \( e e A \) vertex and the \( \nu \nu A \) vertex, the counter terms (4.26) and (4.29) have no dependence on \( \delta M_W^2 \) and \( \delta M_Z^2 \), and hence

\[ \Gamma_{R\alpha}^{eeA} |_H = 0 , \quad (5.78) \]

and

\[ \Gamma_{R\alpha}^{\nu \nu A} |_H = 0 . \quad (5.79) \]

Finally, we note that the renormalized lepton self-energy has
no Higgs-dependent terms, as the Higgs-fermion coupling is small (of order \( m_f/M_W \)).

Inserting the above equations into the expressions for \( \Delta R \) and \( \Delta S \), (5.25) and (5.54), and then using the mass shift formulas (5.10), we find the Higgs-dependent portions of the mass shifts to be

\[
[\Delta M^W_W]_H = -\frac{1}{2} \frac{[\Delta W(0)]_H}{M_W^2} + \frac{[\delta M^2_W]_H}{M_W^2},
\]

\[
[\Delta M^Z_Z]_H = -\frac{1}{2} \frac{[\Delta Z(0)]_H}{M_Z^2} + \frac{[\delta M^2_Z]_H}{M_Z^2}.
\]

Finally, using the expansions for \( \Delta W(0) \), \( \delta M^2_W \), and \( \delta M^2_Z \), (eqs. (5.66), (5.67), and (5.70)), we have the Higgs contributions to the gauge-boson mass shifts (in the limit \( M_H >> M_W \)):

\[
[\Delta M^W_W]_H = \frac{g^2}{384 \pi^2} M_W \ln(M_H^2/M_W^2),
\]

\[
[\Delta M^Z_Z]_H = \frac{g^2}{384 \pi^2} (1 + 10 s^2/c^2) M_Z \ln(M_H^2/M_Z^2).
\]

Here, \( s^2 = 1 - c^2 = 1 - \cos^2 \theta \).

We see that the only Higgs-mass dependence is in the logarithm. Thus, the contribution is not be large, for Higgs masses within the bounds discussed in Chapter 2 (\( M_H < 1 \) TeV).

For instance, a Higgs mass of 1 TeV gives a mass shift of
Figure 5.6. Higgs-dependent contributions to the $W$ self-energy in the 2-doublet model
Figure 5.7. Higgs-dependent contributions to the Z self-energy in the Z-doublet model
Figure 5.8. Higgs-dependent contributions to the $Z-\gamma$ transition in the 2-doublet model
and the only Higgs dependence for the vertex functions will be through the renormalization constants in the counter terms. We note that, as in the MSM, the wavefunction renormalization constants will not give any Higgs-dependent contributions. This is because they involve derivatives of the self-energies. Then, by the expansions of Appendix B, the leading terms will only be of order \( \frac{M_W^2 \ln(M_H^2/M_W^2)}{M_W^2} \) or smaller, and can be ignored. Finally, we note the Z-A mixing self-energy, \( A_{ZA}^A \), and the photon self-energy, \( A^A \), are always zero to this order. This is because a particle can only couple to itself through the electromagnetic interaction.

With the above observations noted, the only remaining renormalization constants are \( \delta M_W^2 \) and \( \delta M_Z^2 \). Then, equations (5.78) and (5.79) become

\[
[\Delta M_W^2]_H = -\frac{1}{2} M_W^2 \left[ \frac{A_W(0)}{M_W^2} - \frac{A_W(M_W^2)}{M_W^2} \right] \\
[\Delta M_Z^2]_H = -\frac{1}{2} M_Z^2 \left[ \frac{A_W(0)}{M_W^2} - \frac{A_Z(M_Z^2)}{M_Z^2} \right]
\]

(5.82) (5.83)

When we make expansions for the functions appearing in \( A_W(q^2) \) and \( A_Z(q^2) \) for large \( M_H^2 \), we are dropping terms smaller than \( M_H^2 \) in the self-energies. Since the argument \( q^2 \) appears as \( q^2/M_H^2 \) for a leading term, it is always less than the desired order, and
\[ A^W(0) = A^W(M_W^2) \equiv A^W. \] (5.84)

\[ A^Z(0) = A^Z(M_Z^2) \equiv A^Z. \]

Thus we see, to this order,

\[ [\Delta M_W] = 0, \]

\[ [\Delta M_Z]_H = -\frac{1}{2} M_W \left[ \frac{A^W}{M_W^2} - \frac{A^Z}{M_Z^2} \right]. \] (5.85)

The Higgs part of the W-boson mass shift is always zero.

Similarly, we find for the 1-variable method, (5.13),

\[ [\Delta M_W]_H = -\frac{1}{2} M_W \frac{M_W^2}{M_W^2 - 2 M_Z^2} \left[ \frac{A^W}{M_W^2} - \frac{A^Z}{M_Z^2} \right]. \] (5.86)

We rely on (5.85) and (5.86) to give us the mass shifts. The remaining task is to calculate the Higgs-dependent contributions to \( A^W \) and \( A^Z \) from the one-loop diagrams.

Using the Feynman rules of Chapter II and the integrals of Appendix A, we write down the Higgs-dependent terms to the W and Z self-energies. The Higgs-dependent contributions to \( A^W \) from the diagrams of Figure 5.6 are

\[ \text{diag } a = \frac{q^2 c^2}{16 \pi^4} \left[ \frac{1}{2} \mathcal{B}_0(M_W^2, M_Z^2, q^2) - \mathcal{B}_{22}(M_W^2, M_Z^2, q^2) \right] \] (5.87a)
For the Z self-energy (Figure 5.7), the contributions from the individual diagrams are

\[ \text{diag } a = \frac{g^2 c^2}{16\pi i} \left[ \mathcal{M}_Z^2 B_0(\phi_1^2, M_Z^2, q^2) - B_{22}(\phi_1^2, M_Z^2, q^2) \right] \]

\[ \text{diag } b = \frac{g^2 s^2}{16\pi i} \left[ \mathcal{M}_W^2 B_0(\Phi_2^2, M_W^2, q^2) - B_{22}(\Phi_2^2, M_W^2, q^2) \right] \]

\[ \text{diag } c = \frac{g^2 c^2}{16\pi i} \left[ -B_{22}(\Phi_1^2, M_{H^\pm}^2, q^2) \right] \]

\[ \text{diag } d = \frac{g^2 c^2}{16\pi i} \left[ -B_{22}(\Phi_2^2, M_{H^\pm}^2, q^2) \right] \]

\[ \text{diag } e = \frac{g^2}{16\pi i} \left[ \frac{1}{2} A_0(\Phi_1^2) \right] \]

\[ \text{diag } f = \frac{g^2}{16\pi i} \left[ \frac{1}{2} A_0(\Phi_2^2) \right] \]

\[ \text{diag } g = \frac{g^2}{16\pi i} \left[ \frac{1}{4} A_0(\Phi_0^2) \right] \]

\[ \text{diag } h = \frac{g^2}{16\pi i} \left[ \frac{1}{2} A_0(\Phi_0^2) \right] \]

\[ \text{diag } i = \frac{g^2}{16\pi i} \left[ \frac{1}{2} A_0(\Phi_0^2) \right] \]
In the limit that \( M_{H^+} \) is much larger than the other Higgs masses and the gauge-boson masses, the diagrams c, e, and i of Figure 5.6 and diagrams e and i of Figure 5.7 become dominant. Adding (5.87c), (5.87e), and (5.87i), we find

\[
\text{diag } b = \frac{g^2 s^2}{16\pi^4 i} \left[ M_Z^2 B_0 (M_{\phi_2}^2, M_{H^0}^2, q^2) - B_{22} (M_{\phi_2}^2, M_{H^0}^2, q^2) \right] \hspace{0.5cm} (5.88b)
\]

\[
\text{diag } c = \frac{g^2 s^2}{16\pi^4 i} \left[ -B_{22} (M_{\phi_1}^2, M_{H^0}^2, q^2) \right] \hspace{0.5cm} (5.88c)
\]

\[
\text{diag } d = \frac{g^2 c^2}{16\pi^4 i} \left[ -B_{22} (M_{\phi_1}^2, M_{H^0}^2, q^2) \right] \hspace{0.5cm} (5.88d)
\]

\[
\text{diag } e = \frac{g^2 (1-2x_W)^2}{16\pi^4 i} \left[ -B_{22} (M_{\phi_1}^2, M_{H^0}^2, q^2) \right] \hspace{0.5cm} (5.88e)
\]

\[
\text{diag } f = \frac{g^2}{16\pi^4 i} \left[ \frac{1}{4} A_0 (M_{\phi_1}^2) \right] \hspace{0.5cm} (5.88f)
\]

\[
\text{diag } g = \frac{g^2}{16\pi^4 i} \left[ \frac{1}{4} A_0 (M_{\phi_2}^2) \right] \hspace{0.5cm} (5.88g)
\]

\[
\text{diag } h = \frac{g^2}{16\pi^4 i} \left[ \frac{1}{4} A_0 (M_{H^0}^2) \right] \hspace{0.5cm} (5.88h)
\]

\[
\text{diag } i = \frac{g^2 (1-2x_W)^2}{16\pi^4 i} \left[ \frac{1}{2} A_0 (M_{H^+}^2) \right] \hspace{0.5cm} (5.88i)
\]
Adding \((5.88e)\) and \((5.88i)\), we find

\[
[A^Z_H] = \frac{g^2}{16\pi^2} (1 - 2x_\phi) 2[A_0 - 2B_{22}] \tag{5.91}
\]

We have used the notation from Appendix B, with expansions (10.16) and (10.21). From (5.85), we find that the Z-boson mass shift is

\[
[A^Z_H] = -\frac{g^2}{128\pi^2} M_Z^2 \left( M_H^2 / M_Z^2 \right) . \tag{5.92}
\]

This is potentially quite large. The 2-doublet model is significantly different from the minimal standard model. A heavy charged Higgs boson will give rise to a significant Higgs-dependent correction to the Z mass.

If we take the heavy Higgs boson to be the \(H^0\), the important diagrams are e and h of Figure 5.6, and c, d, and h of Figure 5.7. The dominant Higgs terms in the self-energies are
Then, by (5.85), the Higgs-dependent $Z$-boson mass shift is

$$[\Delta M_Z]_H = 0$$

(5.94)

to this order of the mass expansion. The mass shift has no strong Higgs dependence; the dominant terms are of order $\ln(M_H^2/M_W^2)$, as in the MSM.

When $\phi_1$ or $\phi_2$ is the heavy Higgs, the dominant diagrams are a, c, and f, or b, d, and g, respectively, of Figures 5.6 and 5.7. The self-energy Higgs terms are

$$[\Delta W]_H = \frac{g^2}{16\pi^2} \frac{1}{4}[A_0 - 4B_{22}]$$

(5.95)

and
From (5.85), the Higgs-dependence of the mass shift, to this order in the mass expansion, is again zero.

We have found that for the 2-doublet model, the Z-boson mass shift has a strong dependence on the charged Higgs. The dependence on the neutral Higgs is weak, as in the MSM. The W-boson mass shift has only a weak dependence on any of the Higgs. This strong Higgs dependence is illustrated in Figure 5.9. We have plotted the total Z mass shift as a function of \( M_H \), as \( M_H = M_{H^\pm} \) gets large. The W mass shift and the MSM Z mass shift are shown for comparison. We see that as the Higgs mass gets large, its contribution gets large in magnitude, and, being negative, cancels the other contributions. The 2-doublet model with a heavy charged Higgs may well be discernible from the MSM by an accurate measurement of the gauge-boson masses.

We may also use the 1-variable method, as outlined in section A. Here, we take the Z mass as a measured quantity, along with \( R^{\text{exp}} \). Then, by (5.86), we see the same sort of effect. The W-boson mass shift shows a strong dependence on the mass of the charged Higgs:

\[
[A_Z]_H = \frac{\frac{1}{2}}{16\pi} \frac{1}{4} [A_0 - 4B_{22}] (5.96)
\]
Figure 5.9. Plot of total Z mass shift vs. charged-Higgs mass
as $M_H = M_{H^\pm}$ gets larger than the gauge-boson and other Higgs masses. Here, $M_Z$ is the experimentally determined value for the Z mass, while $M_W^\text{exp}$ is the lowest-order prediction for the W mass. In Figure 5.10, we plot the first-order-corrected $M_W$ vs. $M_W^\text{exp}$, for various values of the charged-Higgs mass. The MSM values for $M_W$ are also shown. In Figure 5.11, we have chosen a value of $M_Z^\text{exp}$ as an example. We show the functional dependence of the W mass on the charged-Higgs mass for $M_Z^\text{exp} = 93.5$ GeV. Accurate measurements of the W and Z mass will enable us to compare the 2-doublet model with the MSM, and possibly put bounds on the charged-Higgs mass.

We note that the present experimental values for the W and Z masses from CERN have an accuracy of about ±3.0 GeV. In comparing to Figures 5.9-5.11, we see that this is too large of an uncertainty to discern between the models through the mass shifts. However, the accuracy of ~1% to 1% which is expected in future accelerators will enable us to differentiate between models.

In this analysis, we have taken the mass of one Higgs to be large. By using the appropriate expansion, one may take two or more Higgs' masses to be large. The Z mass shift is then a complicated function of the large masses. In this way,
Figure 5.10. Plot of $W$ mass vs. $M_Z^{exp}$ for various values for the charged-Higgs mass in the 2-doublet model
$M_Z^{\text{exp}} = 93.5 \text{ GeV}$

Figure 5.11. Plot of $W$ mass vs. the charged-Higgs mass for a selected value of $M_Z^{\text{exp}}$ in the 2-doublet model
the shift may gain a dependence on a neutral-Higgs mass. However, the shift is largest in the limit of the charged Higgs being more massive than the others.

We may extend the procedure to include n Higgs doublets. The n-doublet model is presented in Chapter II. The n-doublet model has n neutral scalar Higgs $\phi_k^0$, n-1 neutral pseudoscalar Higgs $H_k^0$, and n-1 charged pairs $H_k^\pm$. Typical diagrams for the Higgs contributions to the $W$ and $Z$ self-energies are shown in Figures 5.12 and 5.13. The contributions to the $W$ self-energy, Figure 5.12, can be found using the couplings of (2.57) and the integrals of Appendix A:

$$\text{diag } a = \frac{g^2}{16\pi^2 i} \left( \sum_{j,j'} V_{j,n} V_{j',n} T_{jk} T_{j'k} \right) \left( M_W^2 B_0 \left( M_{\phi_k^0}^2, M_W^2, q^2 \right) - B_{22} \left( M_{\phi_k^0}^2, M_W^2, q^2 \right) \right)$$

(5.98a)

$$\text{diag } b = \frac{g^2}{16\pi^2 i} \left( \sum_{j,j'} T_{jk} T_{j'k} U_{j,l} U_{j',l} \right) \left( - B_{22} \left( M_{H_k^0}^2, M_W^2, q^2 \right) \right)$$

(5.98b)

$$\text{diag } c = \frac{g^2}{16\pi^2 i} \left( \sum_{j,j'} V_{j,k} V_{j',k} U_{j,l} U_{j',l} \right) \left( - B_{22} \left( M_{H_k^0}^2, M_{H_1^0}^2, q^2 \right) \right)$$

(5.98c)

$$\text{diag } d = \frac{g^2}{16\pi^2 i} \left( \sum_{j,j'} T_{jk} T_{j'k} \right) \left( \frac{1}{4} A_0 \left( M_{\phi_k^0}^2 \right) \right)$$

(5.98d)
Figure 5.12. Higgs-dependent contributions to the $W$ self-energy in the $n$-doublet model
Figure 5.13. Higgs-dependent contributions to the $Z$ self-energy in the $n$-doublet model
The contributions to the $Z$ self-energy, Figure 5.12, are

\[ \text{diag } a = \frac{g_Z^2}{16\pi^4 i} \left( \Sigma_{jj'} V_j V_{j'} T_{jj'} T_{jj'} \right) [M_Z^2 B_0(M_{oZ}^2, M_{Z}^2, q^2) - B_{22}(M_{oZ}^2, M_{Z}^2, q^2)] \]  

(5.99a)

\[ \text{diag } b = \frac{g_Z^2}{16\pi^4 i} \left( \Sigma_{jj'} T_{jj'} T_{jj'} T_{jj'} V_{j} V_{j'} \right) [- B_{22}(M_{oZ}^2, M_{H_1}^2, q^2)] \]  

(5.99b)

\[ \text{diag } c = \frac{g_Z^2}{16\pi^4 i} \left( 1 - 2x_W \right)^2 [- B_{22}(M_{H_1}^2, M_{H_1}^2, q^2)] \]  

(5.99c)

\[ \text{diag } d = \frac{g_Z^2}{16\pi^4 i} \left[ \frac{1}{4} A_0(M_{oZ}^2) \right] \]  

(5.99d)

\[ \text{diag } e = \frac{g_Z^2}{16\pi^4 i} \left[ \frac{1}{4} A_0(M_{H_1}^2) \right] \]  

(5.99e)

\[ \text{diag } f = \frac{g^2}{16\pi^4 i} \left( 1 - 2x_W \right)^2 [\frac{1}{2} A_0(M_{H_1}^2)] \]  

(5.99f)
For the limit that $H_k^\pm$, for a fixed $k$, is much heavier than the gauge bosons and other Higgs, diagrams b, c, and f of Figure 5.12 and diagrams c and f of Figure 5.13 become important. We remember to sum the index 1 to include all the diagrams. Using the unitarity of the rotation matrices $T$, $U$, and $V$ to add (5.98b), (5.98c), and (5.98f), we find

$$\left[\Delta \tilde{W}\right]_H = \frac{g^2}{16\pi^2} \frac{1}{4} \hat{A}_0 - 4B_{22}$$

$$= \frac{g^2}{16\pi^2} \left(- \frac{1}{4} M^2\right) \quad (5.100)$$

Adding (5.99c) and (5.99f), we find

$$\left[\Delta Z\right]_H = \frac{g_Z^2}{16\pi^2} \frac{1}{2} (1 - 2x_W)^2 \hat{A}_0 - B_{22}$$

$$= 0 \quad (5.101)$$

This gives a mass shift

$$\left[\Delta M_Z\right]_H = - \frac{g_Z^2}{128\pi^2} M_Z (M_H^2 / M_Z^2) \quad (5.102)$$

as in the 2-doublet model.

We can also take one of the neutrals to be heavy. For $\phi_k^0$, the relevant diagrams are a, b, and d of Figures 5.12 and 5.13. For $H_k^0$, the relevant diagrams are c and e of Figure 5.12 and b and e of Figure 5.12. In both cases, we find
and hence, by (5.85), the mass shift is zero.

In summary, the n-doublet model is similar to the 2-doublet model. The Z mass shift has a strong dependence on the charged Higgs but only a weak dependence on the neutral Higgs.

F. Mass Shifts in the Triplet and Doublet-Triplet Models

Next, we consider the triplet and doublet-triplet models of section E of Chapter 2. The triplet models are not realistic because they do not have ρ ≠ 1. The doublet-triplet models can have ρ ≠ 1 provided v >> κ, where v and κ are the doublet and triplet vevs, respectively.

The (2,1)-triplet has three physical Higgs - a doubly-charged pair $\tilde{S}^{±±}$ and a neutral $\tilde{S}^0$. The corresponding contributions to the gauge-boson self-energies are given in Figures 5.14 and 5.15. Using the coupling constants from (2.63) and the integrals of Appendix A, we can evaluate these diagrams. The Higgs-dependent terms of $A^\tilde{W}$, Figure 5.14, are
Figure 5.14. Higgs-dependent contributions to the W self-energy in the (2,1)-triplet model
Figure 5.15. Higgs-dependent contributions to the Z self-energy in the (2,1)-triplet model.
\[
\text{diag a} = \frac{g^2}{16\pi^4} 4[M^2_W B_0(M_{0^\pm}, M_{0^\pm}, q^2) - B_{22}(M_{0^\pm}, M_{0^\pm}, q^2)]
\]

\[
(5.105a)
\]

\[
\text{diag b} = \frac{g^2}{16\pi^4} 2[M^2_W B_0(M_{0^\pm}, M_{0^\pm}, q^2) - B_{22}(M_{0^\pm}, M_{0^\pm}, q^2)]
\]

\[
(5.105b)
\]

\[
\text{diag c} = \frac{g^2}{16\pi^4} A_0(M_{0^\pm})
\]

\[
(5.105c)
\]

\[
\text{diag d} = \frac{g^2}{16\pi^4} \frac{1}{2} A_0(M_{0^\pm})
\]

\[
(5.105d)
\]

The Z self-energy contributions, Figure 5.14, are

\[
\text{diag a} = \frac{g_Z^2}{16\pi^4} 4(1-2x_W)^2 [B_{22}(M_{0^\pm}, M_{0^\pm}, q^2)]
\]

\[
(5.106a)
\]

\[
\text{diag b} = \frac{g_Z^2}{16\pi^4} 4[M^2_Z B_0(M_{0^\pm}, M_{0^\pm}, q^2) - B_{22}(M_{0^\pm}, M_{0^\pm}, q^2)]
\]

\[
(5.106b)
\]

\[
\text{diag c} = \frac{g_Z^2}{16\pi^4} 2(1-2x_W)^2 A_0(M_{0^\pm})
\]

\[
(5.106c)
\]

\[
\text{diag d} = \frac{g_Z^2}{16\pi^4} A_0(M_{0^\pm})
\]

\[
(5.106d)
\]

For \(0^\pm\) much heavier than the gauge bosons, the relevant diagrams are a and c of Figures 5.14 and 5.15. The
contributions to the self-energy are then

\[
[A^W]_H = \frac{g^2}{16\pi^2 i} \left[ A_0 - 4B_{22} \right] \tag{5.107}
\]

and

\[
[A^Z]_H = \frac{g_Z^2}{16\pi^2 i} 2(1-2x_W)^2 [A_0 - 2B_{22}^S] \tag{5.108}
\]

giving

\[
[\Delta M^Z]_H = -\frac{g_Z^2}{64\pi^2} M_Z (M_H^2/M_Z^2) \tag{5.109}
\]

Thus, the mass shift has a strong dependence on the doubly-charged Higgs.

For the neutral Higgs, the important diagrams are b and d of Figures 5.14 and 5.15. The contributions are

\[
[A^W]_H = \frac{g^2}{16\pi^2 i} \frac{1}{2} [A_0 - 4B_{22}^S] \tag{5.110}
\]

and

\[
[A^Z]_H = \frac{g_Z^2}{16\pi^2 i} [A_0 - 4B_{22}^S] \tag{5.111}
\]

giving

\[
[\Delta M^Z]_H = +\frac{g_Z^2}{128\pi^2} M_Z (M_H^2/M_Z^2) \tag{5.112}
\]
Unlike the doublet models, there is a strong dependence on the neutral Higgs also. Note that the sign of the shift contribution is positive, the opposite of that for the charged particles.

A more realistic model is the doublet-(1,2)-triplet model. It has seven physical Higgs - a doubly-charged pair $\delta^{\pm\pm}$, a charged pair $H^\pm$, and three neutrals $\phi^0$, $\delta^0$, and $H^0$. The relevant diagrams are given in Figures 5.16 and 5.17. From the Lagrangian (2.69), we find the Higgs-dependent contributions to $A^W$. Figure 5.16, are

\[
\text{diag a} = \frac{g^2}{16\pi^4 i} \frac{2\kappa^2}{M_W^2} \left[ M_W^2 B_0 (M_{\phi^0}, M_W^2, q^2) - B_{22} (M_{\phi^0}, M_W^2, q^2) \right] \]

\[(5.113a)\]

\[
\text{diag b} = \frac{g^2}{16\pi^4 i} \frac{1}{4} \frac{\nu^2}{M_W^2} \left[ M_W^2 B_0 (M_{\phi^0}, M_W^2, q^2) - B_{22} (M_{\phi^0}, M_W^2, q^2) \right] \]

\[(5.113b)\]

\[
\text{diag c} = \frac{g^2}{16\pi^4 i} \frac{\kappa^2}{M_W^2} \left[ M_W^2 B_0 (M_{\phi^0}, M_W^2, q^2) - B_{22} (M_{\phi^0}, M_W^2, q^2) \right] \]

\[(5.113c)\]

\[
\text{diag d} = \frac{g^2}{16\pi^4 i} \frac{2\kappa^2}{\nu^2 + 2\kappa^2} \left[ - B_{22} (M_{\phi^0}, M_{H^\pm}, q^2) \right] \]

\[(5.113d)\]

\[
\text{diag e} = \frac{g^2}{16\pi^4 i} \frac{2(\nu^2 + 2\kappa^2)}{\nu^2 + 4\kappa^2} \left[ - B_{22} (M_{\phi^0}, M_{H^\pm}, q^2) \right] \]

\[(5.113e)\]
Figure 5.16. Higgs-dependent contributions to the $W$ self-energy in the doublet-(1,2)-triplet model
Figure 5.17. Higgs-dependent contributions to the $Z$ self-energy in the doublet-(1,2)-triplet model
\[ \text{diag } f = \frac{q^2}{16\pi^4 i} \left( \frac{2v^2}{v^2 + 2\kappa} \right) \mathcal{B}_{22}(M_0^2, M_+^2, q^2) \]  
(5.113f)

\[ \text{diag } g = \frac{q^2}{16\pi^4 i} \left( \frac{4v^2}{v^2 + 2\kappa} \right) \mathcal{B}_{22}(M_0^2, M_+^2, q^2) \]  
(5.113g)

\[ \text{diag } h = \frac{q^2}{16\pi^4 i} \left( \frac{\kappa^2 v^2}{2M_0^2(v^2 + 2\kappa^2)} \right) \mathcal{B}_0(M_0^2, M_Z^2, q^2) \]  
\[ \mathcal{B}_{22}(M_+^2, M_Z^2, q^2) \]  
(5.113h)

\[ \text{diag } i = \frac{q^2}{16\pi^4 i} \left( \frac{1}{4} A_0(M_0^2) \right) \]  
(5.113i)

\[ \text{diag } j = \frac{q^2}{16\pi^4 i} \left( \frac{1}{2} A_0(M_0^2) \right) \]  
(5.113j)

\[ \text{diag } k = \frac{q^2}{16\pi^4 i} \left( \frac{v^2 + 2\kappa^2}{2(v^2 + 4\kappa^2)} \right) A_0(M_0^2) \]  
(5.113k)

\[ \text{diag } l = \frac{q^2}{16\pi^4 i} \left( \frac{2v^2 + \kappa^2}{v^2 + 2\kappa^2} \right) A_0(M_0^2) \]  
(5.113l)

\[ \text{diag } m = \frac{q^2}{16\pi^4 i} A_0(M_0^2) \]  
(5.113m)

The vevs \( v \) and \( \kappa \) can be written in terms of the masses; we leave them in for convenience. The \( Z \) self-energy
contributions. Figure 5.17, are

\[
\text{diag a} = \frac{g_{Z}^{2}}{16\pi i} \frac{4\kappa^{2}}{M_{Z}^{2}} B_{0}(M_{0}^{2}, M_{Z}^{2}, q^{2}) - B_{22}(M_{0}^{2}, M_{Z}^{2}, q^{2}) \]
\[
\text{diag b} = \frac{g_{Z}^{2}}{16\pi i} \frac{4\kappa^{2}}{M_{Z}^{2}} B_{0}(M_{0}^{2}, M_{Z}^{2}, q^{2}) - B_{22}(M_{0}^{2}, M_{Z}^{2}, q^{2}) \]
\[
\text{diag c} = \frac{g_{Z}^{2}}{16\pi i} \frac{4\kappa^{2}}{v^{2}+4\kappa^{2}} B_{22}(M_{0}^{2}, M_{H}^{2}, q^{2}) \]
\[
\text{diag d} = \frac{g_{Z}^{2}}{16\pi i} \frac{4\kappa^{2}}{v^{2}+4\kappa^{2}} B_{22}(M_{0}^{2}, M_{H}^{2}, q^{2}) \]
\[
\text{diag e} = \frac{g_{Z}^{2}}{16\pi i} \frac{4(\kappa^{2}-x_{W}^{2})^{2}}{(v^{2}+4\kappa^{2})^{2}} B_{22}(M_{H}^{2}, M_{H}^{2}, q^{2}) \]
\[
\text{diag f} = \frac{g_{Z}^{2}}{16\pi i} \frac{4(1-2x_{W}^{2})^{2}}{v^{2}+4\kappa^{2}} B_{22}(M_{S}^{2}, M_{S}^{2}, q^{2}) \]
\[
\text{diag g} = \frac{g_{Z}^{2}}{16\pi i} \frac{\kappa^{2} v^{2}}{2M_{W}^{2}(v^{2}+2\kappa^{2})} B_{0}(M_{H}^{2}, M_{W}^{2}, q^{2}) - B_{22}(M_{H}^{2}, M_{W}^{2}, q^{2}) \]
\[
\text{diag h} = \frac{g_{Z}^{2}}{16\pi i} \frac{1}{4A_{0}(M_{0}^{2})} \]
diag $i = \frac{g_Z^2}{16\pi^4 i} A_0(M_\phi^0)^2 \quad (5.114i)$

diag $j = \frac{g_Z^2}{16\pi^4 i} \frac{v^2 + \kappa^2}{(v^2 + 4\kappa^2)} A_0(M_{H^0}^2) \quad (5.114j)$

diag $k = \frac{g_Z^2}{16\pi^4 i} \frac{\kappa^2 (1-2x_W)^2 + 2x_W v^2}{v^2 + 2\kappa^2} A_0(M_{H^\pm}^2) \quad (5.114k)$

diag $l = \frac{g_Z^2}{16\pi^4 i} (1-2x_W)^2 A_0(M_{S^{\pm}}^2) \quad (5.114l)$

For $S^{\pm}$ much heavier than the gauge bosons and other Higgs, we have contributions from diagrams a, g, and m of Figure 5.16 and diagrams f and l of figure 5.17. Adding these, the contributions to the self-energies are

$$[A^W]_H = \frac{g^2}{16\pi^4 i} [A_0 - 4B_{22}] \quad (5.115)$$

and

$$[A^Z]_H = \frac{g_Z^2}{16\pi^4 i} (1-2x_W)^2 [A_0 - 4B_{22}] \quad (5.116)$$

giving

$$[\Delta M_{Z}]_H = -\frac{g_Z^2}{64\pi^2} [1 - (1-2x_W)^2] \frac{M_Z}{M_{H^0}^2} \frac{M_{H^0}^2}{M_{Z^0}^2} \quad (5.117)$$
For $H^+$ much heavier than the other Higgs and the gauge bosons, the dominant contributions come from diagrams d, e, f, g, h, and l of Figure 5.16 and diagrams e, g, and k of Figure 5.17. Adding, we find

$$\begin{align*}
\left[ \mathcal{A}_W \right]_H &= \frac{g^2}{16\pi^2} \frac{2\nu^2 + \kappa^2}{\nu^2 + 2\kappa^2} \left[ \mathcal{A}_0 - 4\mathcal{B}_{22} \right] \\
\text{and}
\left[ \mathcal{A}_Z \right]_H &= \frac{g_\nu^2}{16\pi^2} \left[ \frac{1}{2} \left( \frac{\kappa^2}{\nu^2 + 2\kappa^2} \right) - \frac{1}{2} \frac{\kappa^2 \nu^2}{(\nu^2 + 2\kappa^2)^2} \right] \left[ \mathcal{A}_0 - 4\mathcal{B}_{22} \right],
\end{align*}$$

(5.118)

and

$$\begin{align*}
\left[ \Delta M^2 \right]_H &= -\frac{g_\nu^2}{64\pi^2} \left( \frac{\kappa^2}{\nu^2 + 2\kappa^2} \right)^2 \left( M_H^2 / M_Z^2 \right).
\end{align*}$$

(5.119)

giving

$$\begin{align*}
\left[ \Delta M^2 \right]_H &= -\frac{g_\nu^2}{64\pi^2} \left( \frac{\kappa^2}{\nu^2 + 2\kappa^2} \right)^2 \left( M_H^2 / M_Z^2 \right).
\end{align*}$$

(5.120)

For $H^0$ much heavier, the dominant contributions are from diagrams e and k of Figure 5.16 and e, d, and j of Figure 5.17. The contributions to the self-energies are

$$\begin{align*}
\left[ \mathcal{A}_W \right]_H &= \frac{g^2}{16\pi^2} \frac{\nu^2 + \kappa^2}{2(\nu^2 + 4\kappa^2)} \left[ \mathcal{A}_0 - 4\mathcal{B}_{22} \right] \\
\text{and}
\left[ \mathcal{A}_Z \right]_H &= \frac{g_\nu^2}{16\pi^2} \frac{\nu^2 + \kappa^2}{\nu^2 + 2\kappa^2} \left[ \mathcal{A}_W - 4\mathcal{B}_{22} \right],
\end{align*}$$

(5.121)

and

$$\begin{align*}
\left[ \Delta M^2 \right]_H &= -\frac{g_\nu^2}{64\pi^2} \left( \frac{\nu^2 + \kappa^2}{\nu^2 + 2\kappa^2} \right)^2 \left( M_H^2 / M_Z^2 \right).
\end{align*}$$

(5.122)

giving
For the $\phi^0$ neutral, the relevant diagrams are $b$, $d$, and $i$ of Figure 5.16 and $b$, $c$, and $h$ of Figure 5.17. Adding these contributions, we find

$$[\Delta M_Z]_H = \frac{g_Z^2}{128\pi^2} \frac{v^2}{v^2 + 4\kappa^2 M_Z^2} (M_H^2 / M_Z^2) \quad (5.123)$$

$g_2^2 = 5 \frac{\alpha}{12} \quad (5.124)$

and

$$[\Delta Z]_H = \frac{g_Z^2}{16\pi^4} \frac{1}{4} [A_0 - 4B_{22}] \quad (5.125)$$

giving

$$[\Delta M_Z]_H = 0 \quad (5.126)$$

For the $\phi^0$ Higgs, which came from the Higgs doublet, there is no shift contribution (as in the MSM).

For the $\delta^0$ neutral heavy, the dominant contributions are from diagrams $c$, $f$, and $j$ of Figure 5.16 and diagrams $a$, $d$, and $i$ of Figure 5.17. Adding, we find

$$[\Delta W]_H = \frac{g_Z^2}{16\pi^4} \frac{1}{2} [A_0 - 4B_{22}] \quad (5.127)$$

and
\[ [A^2]_H = \frac{g_2^2}{16\pi^2} [A_0 - 4B_{22}] \]  
\hspace{1cm} (5.128)

giving

\[ [\Delta M_Z]_H = + \frac{g_2^2}{128\pi^2} M_Z (M_{H^+}^2 / M_Z^2) \]  
\hspace{1cm} (5.129)

There is, as in the triplet model, a shift contribution from the \( \delta^0 \) Higgs.

In general, the actual physical mass eigenstates will be combinations of \( \phi^0 \) and \( \delta^0 \). Then, we will have shift contributions from both neutrals. However, in a realistic model with \( v \gg \kappa \), the mixing between the states tends to be suppressed, and the states are approximately mass eigenstates. In addition, if we require that the couplings of each neutral to the fermions be of the same order, then \( \phi^0 \) is much heavier than \( \delta^0 \). The ratio of the \( \phi^0 \) mass to the \( \delta^0 \) mass will be of the order \( v/\kappa \). Given the upper bounds on the Higgs masses, we do not expect the \( \delta^0 \) mass to be more than a few times the \( W \) and \( Z \) masses. In this scenario, we do not expect the neutrals to give a major effect on the mass shifts.

We may also calculate the mass shift in the 1-variable method, by use of eq. (5.86). Using \( A_W \) and \( A_Z \) from (5.115) and (5.116), we find that for \( \delta^{\pm\pm} \) much heavier than the others,
\[ \Delta M_{W}^{\pm} = \frac{g^2}{64\pi^2} \frac{M_{W}^2}{2M_{W}^2 - M_{Z}^2} \left[ 1 - (1 - 2x_{W})^2 \right] M_{W}(m_{H}^2/M_{W}^2). \] (5.130)

Likewise, for \( H^+ \) heavy

\[ \Delta M_{W}^{\pm} = \frac{g^2}{64\pi^2} \frac{M_{W}^2}{2M_{W}^2 - M_{Z}^2} \left[ \frac{2(v^2 + \kappa^2)^2}{(v^2 + 2\kappa^2)^2} \right] M_{W}(m_{H}^2/M_{W}^2). \] (5.131)

for \( H^0 \) heavy

\[ \Delta M_{W}^{\pm} = \frac{g^2}{16\pi^2} \frac{M_{W}^2}{2M_{W}^2 - M_{Z}^2} \left[ \frac{v^2}{v^2 + 4\kappa^2} \right] M_{W}(m_{H}^2/M_{W}^2). \] (5.132)

and for \( \phi^0 \) heavy

\[ \Delta M_{W}^{\pm} = \frac{g^2}{16\pi^2} \frac{M_{W}^2}{2M_{W}^2 - M_{Z}^2} M_{W}(m_{H}^2/M_{W}^2). \] (5.133)

For \( \phi^0 \) heavy, as in the minimal model, there is no strong dependence on the mass.

We may also consider the \((1,0)\)-triplet model. It has one physical Higgs, the \( \phi^0 \). This Higgs does not directly couple to the \( Z \) boson. However, it does have contributions to the \( W \) self-energy, as shown in Figure 5.18. These contributions are calculated to be

\[ \text{diag } a = \frac{g^2}{16\pi^2 i} 4[M_{Z}^2 B_{0}(M_{\phi}^2, M_{W}^2, q^2) - B_{22}(M_{\phi}^2, M_{W}^2, q^2)]. \] (5.134a)
Figure 5.18. Higgs-dependent contributions to the W self-energy in the (1,0)-triplet model

and

\[ \text{diag } b = \frac{g^2}{16\pi^2} A_0(M_0^2) \]  \hspace{1cm} (5.134b)

As the Higgs mass gets larger than the gauge-boson masses, we find

\[ [A^W]_H = \frac{g^2}{16\pi^2} [A_0 - 4B_{22}] \]  \hspace{1cm} (5.135)

and

\[ [A^Z]_H = 0 . \]  \hspace{1cm} (5.136)

giving

\[ [\Delta M_Z]_H = + \frac{g_Z^2}{64\pi^2} M_Z(M_H^2/M_Z^2) \]  \hspace{1cm} (5.137)
The mass shift does have a strong dependence on the Higgs boson mass in this model.

Finally, we consider the doublet–(1,0)–triplet model. The physical Higgs particles are $H^\pm$, $\phi^0$, and $s^0$. The relevant diagrams are shown in Figures 5.19 and 5.20. The contributions to $A^W$, Figure 5.19, are

\[
\text{diag } a = \frac{g^2}{16\pi^4 i} b \left[ M^2_\text{W} B_0 (M_{\phi^0}^2, M^2_\text{W}, q^2) - B_{22} (M_{\phi^0}^2, M^2_\text{W}, q^2) \right] \\
(5.138a)
\]

\[
\text{diag } b = \frac{g^2}{16\pi^4 i} 4s^2 [M^2_\text{W} B_0 (M_{s^0}^2, M^2_\text{W}, q^2) - B_{22} (M_{s^0}^2, M^2_\text{W}, q^2)] \\
(5.138b)
\]

\[
\text{diag } c = \frac{g^2}{16\pi^4 i} s^2 [-B_{22} (M_{\phi^0}^2, M_{H^\pm}^2, q^2)] \\
(5.138c)
\]

\[
\text{diag } d = \frac{g^2}{16\pi^4 i} 4c^2 [-B_{22} (M_{s^0}^2, M_{H^\pm}^2, q^2)] \\
(5.138d)
\]

\[
\text{diag } e = \frac{g^2}{16\pi^4 i} \frac{M^2_\text{Z} M^2_\text{W}}{M^2_\text{Z}} b \left[ M^2_\text{Z} B_0 (M_{H^\pm}^2, M^2_\text{Z}, q^2) - B_{22} (M_{H^\pm}^2, M^2_\text{Z}, q^2) \right] \\
(5.138e)
\]

\[
\text{diag } f = \frac{g^2}{16\pi^4 i} \frac{1}{4} A_0 (M_{\phi^0}^2) \\
(5.138f)
\]
Figure 5.19. Higgs-dependent contributions to the W self-energy in the doublet-(1,0)-triplet model.
Figure 5.20. Higgs-dependent contributions to the Z self-energy in the doublet-(1,0)-triplet model
The $Z$ self-energy contributions, Figure 5.20, are

\[
\text{diag } a = \frac{g_Z}{16\pi^2 i} \left[ M_Z^2 B_0(M_\phi^0, M_Z^2, q^2) - B_{22}(M_\phi^0, M_Z^2, q^2) \right] 
\]  
\[ (5.139a) \]

\[
\text{diag } b = \frac{g_Z}{16\pi^2 i} \left( c^2 + 1 - 2x_W \right)^2 \left[ -B_2(M_{H^\pm}^2, M_{H^\pm}^2, q^2) \right] 
\]  
\[ (5.139b) \]

\[
\text{diag } c = \frac{g_Z}{16\pi^2 i} \left[ 2s^2c^2[M_W^2 B_0(M_{H^\pm}^2, M_W^2, q^2) - B_{22}(M_{H^\pm}^2, M_W^2, q^2)] \right] 
\]  
\[ (5.139c) \]

\[
\text{diag } d = \frac{g_Z}{16\pi^2 i} A_0(M_\phi^0) 
\]  
\[ (5.139d) \]

\[
\text{diag } e = \frac{g_Z}{16\pi^2 i} \frac{1}{2} (1 + c^2) A_0(M_{H^\pm}^2) 
\]  
\[ (5.139e) \]

As the charged Higgs mass gets larger than the gauge boson and other Higgs masses, the dominant diagrams are $c$, $d$, $e$, and $h$ of Figure 5.19 and $b$, $c$, and $e$ of Figure 5.20. The contributions to the gauge-boson self-energies are
\[
[\Delta W^+_{\mu}]_H = \frac{g^2}{16\pi^2 i} \frac{1}{2} (1+c^2) \left[ A_0 - 4B_{22} \right] \quad (5.140)
\]
and
\[
[\Delta Z]_H = \frac{g_Z^2}{16\pi^2} \left[ - (1+c^2-2x_W^2) \right] \left[ A_0 - 2B_{22} \right] + \frac{1}{2} s^2 c^2 \left[ A_0 - 4B_{22} \right] \right) , \quad (5.141)
\]
giving
\[
[\Delta M^2_{\mu}]_H = - \frac{g_Z^2}{128\pi^2} (1+c^4) M_Z^2 \left( M_H^2 / M_Z^2 \right) \quad (5.142)
\]
For the limit \( \phi^0 \) getting heavy, the relevant diagrams are a, c, and f of Figure 5.19 and a and d of Figure 5.20. The contributions are
\[
[\Delta W^+_{\mu}]_H = \frac{g^2}{16\pi^2 i} \frac{1}{4} \left[ A_0 - 4B_{22} \right] \quad (5.143)
\]
and
\[
[\Delta Z]_H = \frac{g_Z^2}{16\pi^2} \frac{1}{4} \left[ A_0 - 4B_{22} \right] , \quad (5.144)
\]
giving
\[
[\Delta M^2_{\mu}]_H = 0 . \quad (5.145)
\]
As in the MSM, there is no strong contribution from the \( \phi^0 \).

For the case of \( \delta^0 \) much heavier than the other bosons, we have diagrams b, d, and f of Figure 5.19. Adding, we have
The $\delta^0$ does not interact with the $Z$, and there are no contributions to $A^Z$. Then,

$$\left[ A^W \right]_H = \frac{g^2}{16\pi^2} \left[ A_{0-4B_{22}} \right]. \quad (5.146)$$

We can also find the $W$ mass shift in the 1-variable case for the doublet-$(1,0)$-triplet model, by use of eq. (5.86). For the case where $H^+$ is much heavier than the other Higgs and the gauge bosons, then, by eqs. (5.140) and (5.141),

$$\left[ \Delta M_Z \right]_H = - \frac{g^2}{64\pi^2} M_Z (M_H^2/M_Z^2). \quad (5.147)$$

$$\left[ \Delta M_W \right]_H = + \frac{g^2}{128\pi^2} \frac{M_W^2}{2M_Z^2 - 2M_W^2} (1+c^2) M_W (M_H^2/M_W^2). \quad (5.148)$$

For $\delta^0$ much heavier, eqs. (5.145) and (5.146) give

$$\left[ \Delta M_W \right]_H = \frac{g^2}{64\pi^2} \frac{M_W^2}{2M_W^2 - M_Z^2} M_W (M_H^2/M_W^2). \quad (5.149)$$

As in the MSM, the mass shift does not have a strong dependence on the mass of the $\phi^0$.

We have found that, unlike the minimal model, models with extended Higgs sectors have potentially dominant Higgs-dependent terms for the $Z$-boson mass shift in the 2-variable case, or for the $W$-boson mass shift in the 1-variable case.
VI. CONCLUSIONS

A proposed physical theory can be made meaningful only when it is compared to experiment. In this way, the parameters can be determined and the theory becomes realistic in its ability to make predictions. Of particular interest in a gauge-theory description of electroweak interactions are the gauge-boson masses. Low-energy data can be used to give predictions for these masses. As we calculate processes by use of perturbation theory, we get different predictions for the masses depending upon to what order we make the calculations. The difference in the mass predictions between the tree-level and higher-order radiative calculations is known as the mass shift. Comparison of the corrected mass and the measured physical mass offers a test of the higher-order corrections of the theory, verifying the theory as a physically relevant quantum theory.

We choose to use the on-shell renormalization scheme to enable us to calculate radiative corrections. This scheme makes sense in that the parameters are physical quantities. Thus, they may directly be compared to experimental measurements. Appropriate counter terms render higher-order corrections finite.

The processes of muon decay and $e\nu$ ($e\bar{\nu}$) elastic scattering are used as the input data. One-loop corrected
expressions for these processes can be derived, and the gauge-

boson masses $M_W$ and $M_Z$ can then be predicted. Unfortunately,
this method suffers in that the neutrino elastic scattering

data lacks the accuracy needed for comparison to radiative
corrections. An alternative scheme is to take muon decay,
which has been accurately measured, and the Z mass as input
data. While part of the test of the theory is sacrificed,
this method will become preferred as the mass of the Z becomes
known accurately.

The mass shifts for the $W$ and $Z$ are expected to be

significant. In the minimal standard model with a single
Higgs doublet, the mass shift is on the order of 4%. This
shift is almost entirely due to light-fermion loops in the
gauge-boson self-energies. The Higgs-boson dependent terms
are on the order of $\ln(M^2_{H^2}/M^2)$, where $M$ is the mass of the $W$
or $Z$, which is small even if the Higgs boson is heavy. Thus,
little can be learned about the Higgs sector from the mass
shifts for the minimal model.

We have found, however, the situation is quite different
for models with more complex Higgs structures. In the 2-

doublet model, the $Z$ mass shift shows a strong dependence on
the charged Higgs boson. The Higgs contribution is negative
and of order $M^2_{H^2}/M^2_Z$, in the limit that the mass of the
charged Higgs is large. This contribution becomes important,
and may even cancel the light-fermion contributions. Such a large contribution should be noticeable. On the other hand, the $W$ mass shift exhibits a only a weak dependence on the charged Higgs, of the same order as in the minimal model. In addition, both mass shifts have only a weak dependence on the neutral Higgs particles, and it will be difficult to learn anything about the neutral Higgs in this manner.

When the measured $Z$ mass is used as input, along with muon decay, to determine the parameters of the theory, we have the same type of effect. The $W$ mass shift has a strong dependence on the charged Higgs. Again, a heavy Higgs will result in a smaller mass shift. An accurate measurement of the $Z$ mass will be able to predict the $W$ mass shift to good accuracy, and comparison to the measured $W$ mass will allow us to gain information on the charged-Higgs structure. The contributions from the neutral Higgs are again small. These results hold for the general $n$-doublet model as well.

For models with a triplet Higgs representation, the same sort of behavior is exhibited. The $Z$ mass shift has a strong dependence on both doubly-charged (if present) and singly-charged Higgs. In addition, there is also a strong dependence on the neutral Higgs, although this tends to be suppressed in realistic models. The $W$ mass shift has no such dependence on the Higgs masses. As in the $2$-doublet model, when the $Z$ mass
is used as input, the W mass shift exhibits the same sort of dependence.

Thus, we see that accurate measurements of the W and Z masses will shed light on the character of the Higgs sector. The determination of the mass shifts may put strong limits on Higgs masses in extended models. We look forward to the next few years as a chance to differentiate between the minimal and extended models, and gain information about the Higgs sector.
VII. LITERATURE CITED


19. For a review of the experimental verification of the Standard Model, see:


53. For a review of the current state of electroweak radiative corrections and the on-shell scheme, see: Workshop on Radiative Corrections in SU(2)LxU(1), ed. by B. W. Lynn and J. F. Wheater (World Scientific, Singapore, 1984).


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IX. APPENDIX A: ONE-LOOP FEYNMAN DIAGRAMS

We present the evaluation of the one-loop diagrams which occur in the gauge-boson self-energies of Chapter V. We use the Feynman rules of Chapter II and the dimensional regularization technique of Chapter III.

In the dimensional regularization procedure, we introduce the dimension $n$. We define

$$\epsilon \equiv \frac{1}{2}(n-4) . \quad (9.1)$$

The ultraviolet divergences manifest themselves as poles at $\epsilon=0$. We express the ultraviolet divergent part as

$$\Delta \equiv \frac{1}{\epsilon} - \gamma + \ln(4\pi) . \quad (9.2)$$

For the convergent parts, we take the limit $n \to 4$ at the onset.

The types of diagrams we need in finding the Higgs contributions to the gauge-boson self-energies are shown in Figure 9.1. In finding the two-point functions, we use the convention of Figure 9.2. Here, $k$ is the loop momentum, $q$ is

![Diagram](image)

Figure 9.1. Typical Higgs-dependent diagrams involved in the gauge-boson self-energy calculation.
Figure 9.2. Conventions used in evaluating the two-point functions.

the external momentum, and $m_1$, $m_2$, and $m$ are the masses of the particles of the internal lines.

All of the diagrams we need are of one of these three types; the diagrams differ only in the couplings. For the diagram involving the gauge-boson-Higgs-boson couplings, diagram a of Figure 9.1, the contribution to the self-energy is

$$\text{diag a} \propto \left[ \frac{d^4k}{(2\pi)^4 i} - \frac{\sigma_{\alpha\beta}^{(k-q)\alpha(k-q)\beta/m_2^2}}{(k-q)^2 - m_2^2} \frac{1}{k^2 - m_1^2} \right]$$

$$= \frac{-1}{(2\pi)^4 i} \Sigma_{0}^{\alpha\beta}(m_1^2, m_2^2, q^2) - \frac{1}{m_2^2} \Sigma_{22}^{\alpha\beta}(m_1^2, m_2^2, q^2) \Sigma_{\alpha\beta}^{\alpha\beta}$$

$$+ q_{\alpha} q_{\beta} \text{ terms}$$

Diagram b, involving the Higgs-Higgs derivative couplings, gives a contribution

$$\text{diag b} \propto \left[ \frac{d^4k}{(2\pi)^4 i} \frac{(k-q) - (k)}{\alpha} \frac{(q-k) - k}{\beta} \frac{1}{(k-q)^2 - m_2^2} \frac{1}{k^2 - m_1^2} \right]$$

$$= \frac{-1}{(2\pi)^4 i} \Sigma_{0}^{\alpha\beta}(m_1^2, m_2^2, q^2) - \frac{1}{m_2^2} \Sigma_{22}^{\alpha\beta}(m_1^2, m_2^2, q^2) \Sigma_{\alpha\beta}^{\alpha\beta} + q_{\alpha} q_{\beta} \text{ terms}.$$
Diagram c, involving the quartic Higgs coupling, has contribution

\[
\text{diag c} = \frac{1}{(2\pi)^4 i} \int \frac{d^4k}{k^2 - m^2} \frac{\sigma_{\alpha\beta}}{\alpha^2} \text{, (9.7)}
\]

where

\[
\sigma_{\alpha\beta} = \frac{-1}{(2\pi)^4 i} A_0(m^2) \sigma_{\alpha\beta} \text{. (9.8)}
\]

We have introduced the functions

\[
B_0(m_1^2, m_2^2, q^2) = \int d^4k \frac{1}{(k-q)^2 - m_2^2} \frac{1}{k^2 - m_1^2} \text{. (9.9)}
\]

\[
B_{2\alpha\beta}(m_1^2, m_2^2, q^2) = \int d^4k \frac{k_\alpha k_\beta}{(k-q)^2 - m_2^2} \frac{1}{k^2 - m_1^2} \text{. (9.10)}
\]

\[
= B_{21}(m_1^2, m_2^2, q^2) \sigma_{\alpha\beta} + B_{22}(m_1^2, m_2^2, q^2) \sigma_{\alpha\beta}
\]

and

\[
A_0(m^2) = \int d^4k \frac{1}{k^2 - m^2} \text{. (9.11)}
\]

We use the Feynman identity (3.6):

\[
\frac{1}{(k-q)^2 - m_2^2} \frac{1}{k^2 - m_1^2} = \int_0^1 dx \frac{1}{(k-q)^2 - m_2^2} x + \frac{1}{k^2 - m_1^2} (1-x) J^{-2} \text{. (9.12)}
\]

\[
= \int_0^1 dx [k^2 - 2k.p - q^2] J^{-2} \text{.}
\]
where
\[ p_x = x q_x \]
\[ Q^2 = -q^2 x + m_2^2 x + m_1^2 (1-x) \]

We define
\[ D = p^2 + Q^2 = m_1^2 (1-x) + m_2^2 x - q^2 x (1-x) \quad (9.13) \]

Then,
\[ B_0 = \int d^4 k \int_0^1 dx \, [k^2 - 2k.p - Q^2]^2 \]
\[ = i \pi^2 \int_0^1 dx \frac{\Gamma(\epsilon)}{\Gamma(2)} D^{-\epsilon} \quad (9.15) \]

where we have used (3.8). We expand (9.15) about the pole \( \epsilon = 0 \), and find
\[ B_0 = i \Delta - i \pi^2 \int_0^1 dx \ln(D) \quad (9.16) \]

Likewise
\[ B_2^{\alpha \beta} = \int d^4 k \int_0^1 dx \, k^\alpha k^\beta [k^2 - 2k.p - Q^2]^2 \]
\[ = i \pi^2 \int_0^1 dx \frac{1}{2} \Gamma(\epsilon-1) D^{-(\epsilon-1)} q^{\alpha \beta} + \Gamma(\epsilon) D^{-\epsilon} q^{\alpha q^{\beta}} \quad (9.18) \]
\[ = \left[ \frac{1}{2} i \Delta \pi^2 \right] \int_0^1 dx D - \frac{1}{2} i \pi^2 \int_0^1 dx \ln(D) \, q^{\alpha \beta} + q^{\alpha q^{\beta}} \text{ terms.} \quad (9.19) \]

We introduce the notation
\[ F_n (m_1^2, m_2^2, q^2) = \int_0^1 dx \, x^n \ln[1 - x + m_2^2 x - q^2 x (1-x)] \quad (9.20) \]
Then, we have

$$B_0(m_1^2, m_2^2, q^2) = i\Delta - i\pi^2 F_0(m_1^2, m_2^2, q^2), \quad (9.21)$$

and

$$B_{22}(m_1^2, m_2^2, q^2) = i(\Delta + \pi^2)(\frac{1}{4}m_1^2 + \frac{1}{4}m_2^2 - \frac{1}{12}q^2)
- \frac{1}{2}i\pi^2 \text{Im}_1^2 F_0(m_1^2, m_2^2, q^2) \quad (9.22)$$

$$+ (m_2^2 - m_1^2 - q^2) F_1(m_1^2, m_2^2, q^2) + q^2 F_2(m_1^2, m_2^2, q^2).$$

Note that the functions $F_n$ are symmetric in the first two arguments. The integrals in the $F_n$ functions can be done explicitly; however, the results are functions which are highly singular in various limits. Hence, it is more convenient to derive various expansions for the $F_n$. We will give some of these expansions in Appendix B.

Finally, we evaluate the function from (9.11). Using (3.11), we find

$$A_0(m^2) = \int d^4k \frac{1}{k^2 - m^2}$$

$$= -i\pi^2 \Gamma(\epsilon - 1)(m^2)^{-(\epsilon - 1)} \quad (9.23)$$

$$= i m^2 \Delta + i\pi^2 [m^2 - m^2 \ln(m^2)]. \quad (9.24)$$

Thus, the Feynman diagrams of Figure 9.1 are given by (9.4), (9.6), and (9.8), respectively, with functions defined in (9.22) and (9.24).
As seen in Appendix A, evaluation of the Feynman diagrams for the gauge-boson self-energies gives us the functions

\begin{align}
B_0(m_1^2, m_2^2, q^2) &= i\Delta - i\pi^2 F_0(m_1^2, m_2^2, q^2), \\
B_{22}(m_1^2, m_2^2, q^2) &= i(\Delta + \pi^2)(\frac{1}{4}m_1^2 + \frac{1}{4}m_2^2 - \frac{1}{12}q^2) \\
&- \frac{1}{2}i\pi^2\ln(m_1^2 F_0(m_1^2, m_2^2, q^2)) \\
&+ (m_2^2 - m_1^2 - q^2)F_1(m_1^2, m_2^2, q^2) + q^2 F_2(m_1^2, m_2^2, q^2),
\end{align}

where

\begin{align}
F_n(m_1^2, m_2^2, q^2) &= \int_0^1 dx x^n \ln(D), \\
D &= m_1^2 (1-x) + m_2^2 x - q^2 x (1-x).
\end{align}

We are interested in the expansions of the functions as \( m_1^2 \gg m_2^2, q_2 \). Letting \( M \) represent the large mass (corresponding to a Higgs-boson mass) and \( m \) the small mass (corresponding to a gauge-boson or other Higgs mass), we can expand:

\begin{align}
D &= M^2 (1-x) + m^2 x - q^2 x (1-x) \\
&= [M^2 (1-x) + m^2 x] (1+x),
\end{align}

where
We then expand $\ln(D)$ in powers of $f$. The expansion is uniformly convergent; we can integrate term-by-term:

\[
\int_0^1 dx \, x^n \ln(D) = \int_0^1 dx \, x^n \ln\left(M^2(1-x) + m^2x\right) + \int_0^1 dx \, x^n \frac{x(1-x)q^2}{M^2(1-x)+m^2x} \\
+ \int_0^1 dx \, x^n \frac{x^2(1-x)^2q^4}{2M^2(1-x)+m^2x^2} + \ldots
\]  

(10.6)

Explicitly, we find

\[
F_0(M^2,m^2,q^2) = \ln(M^2) - 1 - \frac{1}{M^2} \left[\frac{1}{2}q^2 + m^2\ln(m^2/M^2)\right] ,
\]  

(10.7)

\[
F_1(M^2,m^2,q^2) = \frac{1}{2}\ln(M^2) - \frac{3}{4} - \frac{1}{3M^2} \left[\frac{1}{2}q^2 + \frac{1}{2}m^2 + m^2\ln(m^2/M^2)\right] ,
\]  

(10.8)

\[
F_2(M^2,m^2,q^2) = \frac{1}{3}\ln(M^2) - \frac{11}{18} - \frac{1}{6M^2} \left[\frac{1}{4}q^2 + \frac{5}{6}m^2 + m^2\ln(m^2/M^2)\right] ,
\]  

(10.9)

dropping terms of order $m^2/M^2$, $q^2/M^2$, and smaller. We may use these expansions in the $B$ functions. From Chapter V, we see that we are interested in combinations of $m^2B_0$ and $B_{22}$.

Dropping terms of order $m^2$, $q^2$, and smaller, we have

\[
m^2B_0(M^2,m^2,q^2) = i\pi^2 m^2\ln(M^2)
\]  

(10.10)
We also have

\[ A_0(M^2) = iM^2\Delta + i\pi^2 M^2 - i\pi^2 M^2 \ln(M^2) \]  

(10.12)

from Appendix A. We look at combinations which occur in Chapter V:

\[ m^2 B_0(M^2, m^2, 0) - B_{22}(M^2, m^2, 0) + \frac{1}{4} A_0(M^2) = - \frac{1}{8} i\pi^2 M^2 + \frac{3}{4} i\pi^2 m^2 \ln(M^2) . \]  

(10.13)

\[ m^2 B_0(M^2, m^2, m^2) - B_{22}(M^2, m^2, m^2) + \frac{1}{4} A_0(M^2) = - \frac{1}{8} i\pi^2 M^2 + \frac{5}{6} i\pi^2 m^2 \ln(M^2) . \]  

(10.14)

In working with extended models, we will only keep terms of order \( M^2 \). In this limit, the cases \( q^2 = 0 \) and \( q^2 = m^2 \) are the same. In addition, we get no contribution from \( B_0 \). Using the notation

\[ B_{22} \equiv \lim_{M \to m, q} B_{22}(M^2, m^2, q^2) . \]  

(10.15)

we have

\[ \frac{1}{4} A_0 - B_{22} = - \frac{1}{8} i\pi^2 M^2 . \]  

(10.16)
The other scenario we have is when $m_1^2 = m_2^2 \gg q^2$. Then, we expand

$$D = M^2 - q^2 x(1-x) ,$$

(10.17)

giving

$$F_n(M^2, M^2, q^2) = \frac{1}{n+1} \ln(M^2)$$

(10.18)

to order $M^2$. Then,

$$B_{22}(M^2, M^2, q^2) = \frac{1}{2} i M^2 \Delta + \frac{1}{2} i \pi^2 M^2 - \frac{1}{2} i \pi^2 M^2 \ln(M^2) ,$$

(10.19)

and, defining

$$B_{22}^S = \lim_{M \gg q} B_{22}(M^2, M^2, q^2) ,$$

(10.20)

we have

$$B_{22}^S - \frac{1}{2} A_0 = 0 \quad \text{(to order } M^2) .$$

(10.21)

Finally, we note that, since the leading term in $q^2$ is of order $q^2$, the first derivative with respect to $q^2$ is zero, to this order:

$$B_{22}^S = B_{22}^S' = A_0' = 0 .$$

(10.22)

These are the expansions necessary to find the mass shift contributions when a Higgs mass is much larger than the gauge-boson masses.