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Theoretical and numerical studies of some ill-posed problems in partial differential equations

Richard Allan Smith
Iowa State University

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THEORETICAL AND NUMERICAL STUDIES OF SOME ILL-POSED PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

Iowa State University

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Theoretical and numerical studies of some ill-posed problems in partial differential equations

by

Richard Allan Smith

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CHAPTER 1. INTRODUCTION

Consider the problem of obtaining a solution \( u = u(x_1, \ldots, x_m) \) of a partial differential equation

\[
P(u, u_{x_1}, \ldots, u_{x_m}, u_{x_1 x_1}, \ldots) = 0
\]
given data \( f \). Special cases include the Cauchy (or initial-value) problem, where the data are values of \( u \) prescribed on a hypersurface in \( \mathbb{R}^m \) (or \( C^m \)); and the initial-boundary value (or mixed) problem, where \( u \) is prescribed for \( x_m = 0 \) on a domain \( D \) in \( \mathbb{R}^{m-1} \) (or \( C^{m-1} \)), and for \( x_m > 0 \) on the boundary \( D \). Such problems are said to be well-posed in the sense of Hadamard provided

(a) a solution \( u \) exists for given data \( f \);
(b) \( u \) is determined uniquely by \( f \); and
(c) \( u \) depends continuously on \( f \).

Prototypical well-posed problems include the Cauchy problem

\[
\frac{dy}{dt} = y, \ t > 0; \ y(0) = a;
\]

and the Dirichlet problem for Laplace's equation (John [10, p. 155]).

A problem is ill-posed if solutions fail to satisfy one or more of (a), (b), and (c). Examples for which (a) fails include the Cauchy problem
\[
\frac{dy}{dt} = y^2, \quad t > 0; \quad y(0) = 1,
\]
whose solution \( y(t) = (1-t)^{-1} \) cannot be continued past \( t = 1 \); and the general Cauchy problem for the Laplace equation ([10, p. 98]). An extreme example is furnished by Hans Lewy's linear partial differential equation with no solutions, regardless of the type or form of data prescribed ([24]). Examples for which (b) or (c) fails abound (see [10, pp. 155-156], and Smoller [34]).

One of the major advances in the study of ill-posed problems has been the development of a concept of weak or distribution solution. The notions of uniqueness, continuity, differentiability, and satisfaction of differential equation and side conditions are sufficiently generalized, so that in the new context a problem may become well-posed. Functions with weak derivatives were first used by Friedrichs [7,8] and spaces \( W^{k,p} \) of such functions by Sobolev [35].

Weak solutions are especially useful for studying nonlinear problems, which can model physical phenomena in such fields as hydrodynamics, chemical kinetics, and biophysics ([34]). Solutions of such problems often exhibit behavior, such as the development of gradient catastrophes or shock waves, unknown to solutions of linear problems.

In this work three nonlinear initial-boundary value problems are considered. Each is already known to be ill-posed under certain conditions on initial data. Two basic questions confront the would-be solver of these problems. In what space of functions should one look for a solution? Under what conditions on the data can solutions be
continued for all time, and under what conditions is this impossible?

These questions are considered in the next three chapters.
CHAPTER 2. A POTENTIAL WELL THEORY FOR THE WAVE EQUATION
WITH A NONLINEAR BOUNDARY CONDITION

Introduction

Let $D$ be an open, bounded, connected subset of $\mathbb{R}^n$ with a Lipschitz boundary $\partial D$. Let $\partial D$ be the union of two disjoint $(n-1)$-dimensional submanifolds $\sigma, \Sigma$ (each of positive Lebesgue measure) and their Lipschitz confluence. Consider the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta u \quad \text{in } D \times (0, T)$$

$$u(x, 0) = U(x), \quad \frac{\partial u}{\partial t} (x, 0) = V(x) \quad \text{in } D$$

(W)

$$u(x, t) = 0 \quad \text{on } \sigma \times (0, T)$$

$$\frac{\partial u}{\partial n} = f(u(x, t)) \quad \text{on } \Sigma \times (0, T),$$

where $\Delta$ denotes the n-dimensional Laplacian, and $\frac{\partial}{\partial n}$ the outward normal derivative. The kinetic and potential energy functionals associated with (W) are given by

$$K(u) = \frac{1}{2} \int_D \left| \frac{\partial u}{\partial t} \right|^2 \, dx, \quad (2.1)$$

$$J(u) = \frac{1}{2} \sum_{i=1}^{n} \int_D \left| \frac{\partial u}{\partial x_i} \right|^2 \, dx - \int_\Sigma F(u) \, dS, \quad (2.2)$$
respectively, where \( F(u) = \int_0^u f(s)ds \). The total energy \( E(t) \) of a solution of \((W)\) at time \( t \) is defined to be the sum

\[
E(t) \equiv K(u(\cdot, t)) + J(u(\cdot, t)) .
\]  

(2.3)

For a class of nonlinearities \( f \) (which includes the example \( f(u) = |u|^p \) with \( p > 1 \)), it has been shown [21] that solutions of \((W)\) cannot exist for all time whenever \( E(0) < 0 \). Generalizations of this nonexistence result to a wider class of problems are given in [22], [12]. In this chapter, the existence of global solutions of \((W)\) by means of potential well arguments, and the nonexistence of global solutions for some data with \( E(0) > 0 \), is established.

The arguments herein are adapted from those in [28] and [32], with the following important exceptions:

(a) the presence of a nonlinearity in a boundary condition necessitates certain trace and imbedding results, to be presented in the following section;

(b) global solutions in [32] are approximated by expansions in the Dirichlet eigenfunctions of the Laplacian on the domain of interest. In this work, global solutions are approximated by a double expansion in these eigenfunctions, and the eigenfunctions for the modified Steklov problem
\[ \Delta \psi = 0 \quad \text{in} \quad D \]
\[ \psi = 0 \quad \text{on} \quad \sigma \]
\[ \frac{\partial \psi}{\partial n} = \mu \psi \quad \text{on} \quad \Sigma , \]

again because of the nonlinear boundary condition;

(c) global existence of solutions is proved in [32] for \( n = 1, 2, \) and 3 space variables, and for higher dimensions when the potential well is infinitely deep. The proof of global existence in Lemma 2.13 of this chapter is valid for all dimensions \( n \) and all potential wells of positive depth, and can be modified to prove global existence for all dimensions and potential wells considered in [32].

(d) the authors of [28] have noted (see [18]) that the proof of Lemma 2.7 in [29] is in error for certain nonlinearities \( f \). The lemma is a key step in establishing nonexistence. In this chapter a corresponding result, Lemma 2.15, is proved under slightly stronger hypotheses on \( f \) (see (2.71)). The proof of Lemma 2.15 can supplant that of Lemma 2.7 in [28] when the hypotheses there on \( f \) are correspondingly strengthened.

Preliminaries

This section presents the notation, Sobolev spaces, and compact imbedding results required in the sequel.

For a domain \( G \) in \( \mathbb{R}^m \) and a (possibly empty) subset \( \Gamma \) of \( \partial G \) denote by \( C^k(G \cup \Gamma) \) the set of real-valued functions \( g \) such that \( g \) and all its partial derivatives of orders \( \leq k \) are continuous on \( G \),
and can be extended to be continuous on $G \cup \Gamma$. The set $C^k_0(G \cup \Gamma)$
consists of those $g \in C^k(G \cup \Gamma)$ with compact support, $\text{supp } g$, such
that $(\text{supp } g) \cap (G \cup \Gamma)$ is compact. Set $C^\infty(G \cup \Gamma) = \cap_{k=0}^\infty C^k(G \cup \Gamma)$,
with a similar definition for $C^\infty_0(G \cup \Gamma)$.

The volume element of integration in $\mathbb{R}^m$ is denoted by
$dx = dx_1 \ldots dx_m$. The symbol $\nabla u$ denotes the gradient,
$\text{grad } u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_m}\right)$, of $u$. Define
\begin{align*}
\|u\|_{q,G} &= \left[ \int_G |u|^q \, dx \right]^{1/q}, \quad 1 < q < \infty \tag{2.4} \\
\|u\|_G &= \left[ \int_G |\nabla u|^2 \, dx \right]^{1/2} \tag{2.5} \\
\|u\|_{1,2,G} &= \left[ \|u\|_{2,G}^2 + \|u\|_G^2 \right]^{1/2} \tag{2.6}
\end{align*}

where $u$ is any function on $G$ for which the right-hand side makes
sense. $L_q(G), \quad 1 < q < \infty$, denotes the Banach space of all measurable
functions $u$ on $G$ for which the norm (2.4) is finite.

Let $u,^i$ or $\frac{\partial u}{\partial x_i}$ denote the weak first partial derivative of $u$
with respect to $x_i$, satisfying
\begin{equation}
\int_G u \frac{\partial n}{\partial x_i} \, dx = - \int_G u,^i \, n \, dx \tag{2.7}
\end{equation}

for all $n \in C^1_0(G)$. The Sobolev space $W^{1,2}(G)$ consists of those
$u \in L_2(G)$ having weak first partial derivatives $u,^1, \ldots, u,^m$ which
are also in $L_2(G)$. Alternatively, $W^{1,2}(G)$ can be viewed as the
completion of \( \{ u \in C^1(G) : \| u \|_{1,2,G} < \infty \} \) with respect to the norm 
\[ \| \cdot \|_{1,2,G} \] ([2]).

The following compact imbedding theorem of Sobolev will be required.

**Lemma 2.1:** If \( G \) is a bounded region in \( \mathbb{R}^m \) possessing the cone property [2], then bounded sets in \( W^{1,2}(G) \) are precompact as subsets of 
\( L_q(G) \) for \( 1 < q < \frac{2m}{m-2} \) if \( m > 2 \), and \( 1 < q < \infty \) if \( m = 2 \).

A proof of Lemma 2.1 may be found in [2].

A one-to-one transformation \( \phi \) of a domain \( G_1 \subseteq \mathbb{R}^m \) onto a domain 
\( G_2 \subseteq \mathbb{R}^m \) is called **Lipschitz** if each component of both \( \phi \) and \( \phi^{-1} \) is 
Lipschitz continuous. An \((m-1)\)-dimensional submanifold \( \Gamma \) of \( G \) (with 
boundary \( \partial_{m-1}\Gamma \) in the manifold \( \partial G \)) is said to be **Lipschitz** if there 
exist an open cover \( \{ \Theta_1, \ldots, \Theta_j \} \) of \( \Gamma \) and corresponding Lipschitz 
transformations \( \phi_1, \ldots, \phi_j \) satisfying

(i) \( \phi_j \) maps \( \Theta_j \) onto the unit ball \( B \) in \( \mathbb{R}^m \);

(ii) If \( \Theta_j \) contains no points of \( \partial_{m-1}\Gamma \), then

\[
\phi_j(G \cap \Theta_j) = B^+ = \{ y \in B : y_m > 0 \} \quad \text{and} \quad \phi_j(\Gamma \cap \Theta_j) = B_0 = \{ y \in B : y_m = 0 \};
\]

(iii) If \( \Theta_j \) contains points of \( \partial_{m-1}\Gamma \) then \( \phi_j(G \cap \Theta_j) \) 
(respectively \( \phi_j(\Gamma \cap \Theta_j) \)) is a connected subset of \( B^+ \) (resp. \( B_0 \)).

If a function \( g \) has support in \( \Gamma \cap \Theta_j \) with \( \Gamma \) Lipschitz, one may 
define the surface integral of \( g \) over \( \Gamma \) as
\[ \int \mathbf{g}(x) \, dS = \int_{\Phi_j(\Gamma \cap \mathcal{E}_j)} g \circ \Phi_j^{-1}(y',0) J_j(y') \, dy' \]

where \( y' = (y_1, \ldots, y_{m-1}) \), \( x = \Phi_j^{-1}(y) \), and

\[ J_j(y') = \left[ \sum_{i=1}^{m} \left( \frac{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)}{\partial(y_1, \ldots, y_{m-1})} \right)^2 \right]^{1/2} \bigg|_{y_n=0} \]

For arbitrary functions \( \mathbf{g} \) defined on \( \Gamma \) one may select a \( C^\infty \) partition of unity \( \{w_j\} \) subordinate to \( \{\mathcal{E}_j\} \) and set

\[ \int \mathbf{g}(x) \, dS = \sum_{j=1}^{J} \int_{\Gamma} \mathbf{g}(x) w_j(x) \, dS . \quad (2.9) \]

\( L_q(\Gamma), \ 1 < q < \infty, \) denotes the Banach space of functions \( \mathbf{g} \) for which the norm

\[ \| \mathbf{g} \|_{L_q(\Gamma)} = \left[ \int_{\Gamma} |\mathbf{g}|^q \, dS \right]^{1/q} \quad (2.10) \]

is defined and finite.

The following result makes precise what is meant by the values of a function \( u \in W^{1,2}(\Omega) \) on \( \Gamma \).

**Lemma 2.2:** Let \( \Omega \) be bounded and have a Lipschitz boundary. Each \( u \in W^{1,2}(\Omega) \) is the limit in \( W^{1,2}(\Omega) \) of a sequence \( \{u_n\} \subseteq C(\Omega) \). For \( 2 < q < \frac{2(n-1)}{n-2} \) if \( n > 2 \), and for \( 2 < q < \infty \) if \( n = 2 \), the sequence \( \{u_n|_{\Gamma}\} \) of restrictions of \( u_1, u_2, \ldots \) to \( \Gamma \) converges in \( L_q(\Gamma) \) to a function \( B_{\Gamma}u \) satisfying
where $K_q > 0$ is independent of $u$. Moreover, $B_\Gamma : W^{1,2}(G) \to L_q(\Gamma)$ is a well-defined, compact linear operator. $B_\Gamma u$ is called the trace of $u$ on $\Gamma$. Lemma 2.2 is a consequence of the fact that for bounded domains, having a Lipschitz boundary is equivalent to possessing the uniform cone property (see [9]); Calderón's Extension Theorem; the hypothesis that $\Gamma$ can be flattened locally by a finite number of Lipschitz transformations; and the Rellich-Kondrachov Theorem. See [2, pp. 91, 144].

$H^1_0(G)$ denotes the completion of $C^1_0(G)$ under the norm (2.6). For $\gamma = \partial G \cap \Gamma^c$ (where $\Gamma^c$ denotes the complement of $\Gamma$ in $\mathbb{R}^m$), let $H^1_{0\gamma}(G)$ denote the completion of $C^1_0(G \cup \Gamma)$ under the norm (2.6). Functions in the closed subspace $H^1_{0\gamma}(G)$ of $W^{1,2}(G)$ vanish on $\gamma$ in the weak sense, and their images under $B_\Gamma$ are in $L^2(\Gamma)$ by (2.11).

**Lemma 2.3:** If $G$ is bounded, open, connected, and has a Lipschitz boundary, and if $\gamma$ is an $(m-1)$-dimensional submanifold of $\partial G$ of positive measure with a Lipschitz boundary in the manifold $\partial G$, then Poincaré's inequality

$$\|B_\Gamma u\|_{L^q(\Gamma)} \leq K_q \|u\|_{W^{1,2}(G)},$$

is valid for all $u \in H^1_{0\gamma}(G)$, where $C > 0$ is independent of $u$. 

\[ 1 \]
Thus $H^1_{0\gamma}(G)$ becomes a Hilbert space when endowed with inner product
\[(u,v)_{G,\Gamma} = \int_G \nabla u \cdot \nabla v \, dx + \int_\Gamma (Bu)(Bv) \, dS, \quad (2.13)\]
since $\| \cdot \|_G$ and the norm
\[\|u\|_{H^1_{0\gamma}(G)} = \sqrt{(u,u)} \quad (2.14)\]
induced by (2.13) are each equivalent to $\| \cdot \|_{1,2,G}^1$ by Lemmas 2.3 and 2.2.

Note that the conclusions of Lemmas 2.1, 2.2, and 2.3 hold with $G = D, \gamma = \sigma, \Gamma = \Sigma, \text{ and } m = n$.

For any set $A$ and any $T > 0$, let $A_T = A \times (0,T)$, $A_\infty = A \times (0,\infty)$. Then Lemmas 2.1, 2.2, 2.3 also hold with $G = D_T, \gamma = \sigma_T, \Gamma = \Sigma_T, \text{ and } m = n+1$. In particular, Lemma 2.2 can be applied to yield compact operators $B_{E} : W^{1,2}(D) \rightarrow L^q(\Sigma)$ and $B_{E_\Gamma} : \tilde{W}^{1,2}(D_T) \rightarrow L^q(\Sigma_T)$. These will both be denoted $B$ when no confusion can result.

The Potential Well

In this section, the functional $J(u) = \frac{1}{2} \|u\|_D^2 - \int_\Sigma F(Bu) \, dS$ of (2.2) is shown to determine a potential well with local minimum at the origin in the function space $H^1_{0\sigma}(D)$.

In this and the following section, the assumptions on the functions, $f, F$ are similar to those in [28]:

...
(i) $f \in C^1(\mathbb{R})$ and $f(0) = f'(0) = 0$; $f$ does not vanish identically in a neighborhood of the origin;

(ii) Either (a) $f(s)$ is monotone increasing, and convex for $s > 0$, concave for $s < 0$; or

(b) $f$ is convex; and

(iii) $(p+1)F(s) < sf(s)$, and

$$|sf(s)| < \gamma |F(s)|,$$

for all $s \in \mathbb{R}$, where $2 < p+1 < \gamma < \frac{2(n-1)}{n-2}$.

**Lemma 2.4:** Let $f, F$ satisfy (i)-(iii). Then as $s \rightarrow +\infty$

$$|F(s)| = o(|s|^{\gamma}),$$

$$|f(s)| = o(|s|^{\gamma-1}),$$

$$|f'(s)| = o(|s|^{\gamma-2}).$$

Moreover

$$F(s) = O(|s|^{p+1}) \text{ as } s \rightarrow 0^+;$$

and in case $f$ satisfies (ii)(a), then (2.20) also holds as $s \rightarrow 0^-.$

**Proof:** Multiplication of both sides of (2.15) by $s^{-(p+2)}$ for $s > 0$ yields $\frac{d}{ds} (s^{-(p+1)}F(s)) > 0$. Hence $F(s) = s^{p+1}I_1(s)$ for $s > 0$, where $I_1 \in C^2((0,+)\infty))$ is positive and monotone increasing. This establishes (2.20).
For $s > 0$ (2.16) yields $\frac{d}{ds} (s^{-\gamma} F(s)) < 0$, so $F(s) = s^\gamma D_1(s)$ where $D_1 \in \mathcal{C}^2((0, +\infty))$ is positive and monotone decreasing. Thus $F(s) = O(s^\gamma)$ as $s \to +\infty$.

Suppose $f$ satisfies (ii)(a). Then (2.16) yields $F(s) = |s|^\gamma I_2(s)$ for $s < 0$, where $I_2 \in \mathcal{C}^2((-\infty, 0))$ is positive and monotone increasing; so $F(s) = O(|s|^\gamma)$ as $s \to -\infty$ and (2.17) is verified in this case. Inequality (2.15) can be used to show (2.20) holds as $s \to 0^-$.

One may also verify that if $f$ satisfies (ii)(b) then (2.16) also implies (2.17).

The growth restriction (2.18) follows from (2.17) and (2.16).

To obtain (2.19) note that for $0 < s < s_0$, 
$$f(s_0) > \int_{s_0}^{s_0} f'(n)dn > f'(s_0)(s_0-s).$$
In particular, when $s_0 = 2s > 0$, 
$$0 < sf'(s) < f(2s).$$
Similar considerations when $s_0 < s < 0$ in either of the cases (ii)(a), (ii)(b) show that $|sf'(s)| < |f(2s)|$. The growth restriction (2.19) then follows from (2.18).

The order conditions (2.17) and (2.20) are the best obtainable when $\gamma > p+1$, as may be seen by taking

$$F(s) = \begin{cases} 
   s^{p+1} + s^\gamma, & s > 0 \\
   |s|^{p+1}, & s < 0.
\end{cases}$$

The behavior of the functional $J$ along rays emanating from the origin in $H^1_{00}(D)$ may now be considered.
Lemma 2.5: For \( f, F \) satisfying (i)-(iii) and for fixed \( u \in H^1_{00}(D) \),\nthe function \( j_u(\lambda) = J(\lambda u) \) is in \( C^2(\mathbb{R}) \) and satisfies\n\[
 j'_u(\lambda) = \lambda u u D - \frac{1}{\Sigma} \int_{\Sigma} (Bu)f(\lambda Bu)dS, \tag{2.21} \]
\[
 j''_u(\lambda) = u u D - \frac{1}{\Sigma} \int_{\Sigma} (Bu)^2 f'(\lambda Bu)dS. \tag{2.22} \]

Proof: If suffices to show \( g(\lambda) = \int_{\Sigma} F(\lambda Bu)dS \) is in \( C^2(\mathbb{R}) \) and\npossesses the appropriate expressions for its derivatives. For \( h_m \neq 0 \) one may write\n\[
 g(\lambda + h_m) - g(\lambda) \quad \frac{1}{h_m} \int_{\Sigma} f((\lambda + \eta h_m)Bu)(Bu)d\eta dS. \tag{2.23} \]

By (2.18) and Lemma 2.2, for \( \lambda \) fixed, \( 0 < \eta < 1 \), and a sequence\n\( \{h_m\} \) tending to zero, the integrand in (3.11) is dominated in absolute\nvalue by a fixed integrable function \( (A_1 + A_2 |Bu|^\gamma) |Bu| \), where \( A_1, A_2, C \) are positive constants and \( C > |\lambda + \eta h_m|^\gamma \) for all \( m \). By the\nLebesgue dominated convergence theorem the integral in (2.23) approaches\n\[
 \int_{\Sigma} \frac{1}{\Sigma} \int_{\Sigma} f(\lambda Bu)(Bu)d\eta dS = \int_{\Sigma} f(\lambda Bu)(Bu)dS \tag{2.24} \]
as \( m \to +\infty \). Therefore \( g'(\lambda) \) exists and is given by (2.24) for each \( \lambda \).

Similar proofs employing (2.19) and Lemma 2.2 establish the\nexistence and continuity of, and the integral expression for, \( g'' \).
Now \( j_u(0) = j_u'(0) = 0 \) and \( j_u''(0) = \| u \|^2_D > 0 \) for \( u \) nonzero, so \( j_u(\lambda) \) is a convex function of \( \lambda \) for small \( \lambda \).

**Lemma 2.6:** Let \( f, F \) satisfy conditions (i)-(iii), and let \( u \) be a function in \( \mathbb{H}_{00}^1(D) \) with \( Bu \) nonzero.

When \( f \) satisfies (ii)(a),

1. \( \lim_{\lambda \to +\infty} j_u(\lambda) = -\infty \);
2. there exists a unique positive root \( \lambda^* = \lambda^*(u) \) of \( j_u'(\lambda) = 0 \); and
3. \( j_u''(\lambda^*) < 0 \).

In case \( f \) satisfies (ii)(b), then

4. \( \lim_{\lambda \to +\infty} j_u(\lambda) = \pm \infty \);
5. if \( Bu > 0 \) a.e., then (1), (2), and (3) hold.

**Proof:** Let \( g(\lambda) = \int F(\lambda Bu) dS \). Then from (2.15),

\[
(p+1)g(\lambda) < \lambda \int \int_{\Sigma} (Bu)f(\lambda Bu)dS = \lambda g'(\lambda).
\]

Hence \( g(\lambda) = \lambda^{p+1}I(\lambda) \) where \( I \in C^2(0, +\infty) \) is monotone increasing, and

\[
j_u(\lambda) = \frac{1}{2} \| u \|^2_D \lambda^2 + \lambda^{p+1}(-I(\lambda)).
\]

Let \( L = \lim_{\lambda \to +\infty} (-I(\lambda)) \). Then \( L > 0 \) implies \( j_u(\lambda) + \infty \), and \( L < 0 \) or \( L = -\infty \) implies \( j_u(\lambda) - \infty \), as \( \lambda \to +\infty \). In particular, (4) is verified.

Suppose now that either \( f \) satisfies (ii)(a), or \( f \) satisfies
(ii)(b) and $\mathcal{B}u > 0$ is nonzero. Then $L$ is negative or $-\infty$, so (1) holds. The existence of a positive $\lambda^*$ satisfying $j_u'(\lambda^*) = 0$ is then guaranteed by the convexity of $j_u(\lambda)$ in a neighborhood of the origin. One has

$$j_u^-(\lambda^*) = \max D - \frac{1}{2} \int_S (\mathcal{B}u)^2 f'(\lambda^* \mathcal{B}u) dS$$

$$= (\lambda^*)^{-2} \int_S (\mathcal{B}u)[f(\mathcal{B}u) - f(\mathcal{B}u)] dS.$$  \hspace{1cm} (2.25)

Note that $f(s) - sf'(s)$ is the y-intercept of the tangent line to the graph of $f$ at the point $(s, f(s))$, so that

$$s(f(s) - sf'(s)) < 0$$  \hspace{1cm} (2.26)

for $s \neq 0$ in case (ii)(a), and for $s > 0$ in case (ii)(b). Thus the integrand in (2.25) is negative and $j_u^-(\lambda^*) < 0$. By (1), $\lambda^*$ must be the only positive critical value of $j_u$, so (2), (3), and (5) are verified. \hfill \blacksquare

When $f$ satisfies (ii)(a) let $\mathcal{C}$ consist of all $u \in H^1_{0\sigma}(D)$ with nonzero trace $\mathcal{B}u$ on $\Sigma$. Lemma 2.6 shows that for $u \in \mathcal{C}$, $j_u(\lambda^*)$ is the maximum value of $j_u$ achieved when leaving the origin in $H^1_{0\sigma}(D)$ along a ray in the direction $u$. Let

$$d = \inf_{u \in \mathcal{C}} j_u(\lambda^*).$$  \hspace{1cm} (2.27)
Clearly $0 < d < \infty$, and in Lemma 2.7 below it is shown that $d > 0$. A potential well $W$ in $H^1_{0}(D)$ of depth $d$ may then be defined by

$$W = \{ u \in H^1_{0}(D) : 0 < j_u(\lambda) < d \text{ for } 0 < \lambda < 1 \} . \quad (2.28)$$

Note that if $u \in C$ is always scaled so that $\lambda^* = 1$, then the variational problem (2.27) is equivalent to the problem

$$d = \inf_{u \in C} J(u) \quad (2.29)$$

subject to the constraint

$$Q(u) = \| u \|_D^2 - \int_{\Sigma} f(Bu)f(Bu) dS = 0 . \quad (2.30)$$

When nonzero $u \in H^1_{0}(D)$ has zero trace on $\Sigma$, $\lim_{\lambda \to +\infty} j_u(\lambda) = +\infty$ and the well is "infinitely deep" in the direction $u$.

One must proceed more carefully when $f$ satisfies (ii)(b), since critical values of $j_u$, even when they exist, may not be unique. Let $C$ consist of all $u \in H^1_{0}(D)$ for which positive roots of $j_u'(\lambda) = 0$ exist, and denote by $\lambda^* = \lambda^*(u)$ the smallest such positive root. The well $W$ and depth $d$ are then defined as in (2.28), (2.27). (If nonzero $u \in H^1_{0}(D)$ is not in $C$, then by Lemma 2.6

$$\lim_{\lambda \to +\infty} j_u(\lambda) = +\infty.$$ Note that for $u \in C$ one has $J(\lambda u) < J(\lambda^* u)$ for $0 < \lambda < \lambda^*$. Also $J(\lambda|u|) < J(\lambda u)$ for all $\lambda > 0$, and
Q(λ^*|u|) < Q(λ^*u) = 0. By Lemma 2.6 (5) there exists a unique \( \widetilde{\lambda} \),
0 < \( \widetilde{\lambda} \) < \( \lambda^* \), satisfying Q(\( \widetilde{\lambda}|u| \)) = 0, and so J(\( \lambda^*u \)) > J(\( \widetilde{\lambda}u \)) > J(\( \lambda|u| \)).

Hence it suffices in (2.27) to minimize only over
u \( \in \mathbb{C}^+ \equiv \{u \in H^1_0(D) : Bu \text{ is nonzero and } Bu > 0 \text{ a.e.} \} \). The variational problem (2.27) is therefore equivalent to the problem

\[
d = \inf_{u \in \mathbb{C}^+} J(u) \tag{2.31}
\]

subject to the constraint (2.30).

**Lemma 2.7**: If \( f, F \) satisfy (i)-(iii), then \( d > 0 \).

**Proof**: Choose \( u \in \mathbb{C} \) when \( f \) satisfies (ii)(a), \( u \in \mathbb{C}^+ \) when \( f \) satisfies (ii)(b), and let \( u \) satisfy the constraint (2.30). (2.15) implies

\[
J(u) > \frac{1}{2} \|u\|^2_D - \frac{1}{p+1} \int_D (Bu)f(Bu)\,dS \tag{2.32}
\]

If one can show that in addition \( \|u\|^2_D \) is bounded below by some positive constant \( K \), then it follows that \( d > \frac{p-1}{2(p+1)} K^2 > 0 \).

By Lemma 2.4 there exist constants \( A_1, A_2 > 0 \) such that

\[
0 < F(s) < A_1|s|^{p+1} + A_2|s|^\gamma
\]
for all $s$ when $f$ satisfies (ii)(a), and for $s > 0$ when $f$ satisfies (ii)(b). By Lemma 2.2, the constraint (2.30), and the inequality (2.16),

$$
\| Bu \|_{Y, \Sigma}^2 \leq C^2 \| u \|_D^2 = C^2 \sum \int (Bu)f(Bu)\text{d}S
$$

$$
< C^2 \gamma \sum \int \text{F}(Bu)\text{d}S
$$

$$
< C^2 \gamma [A_1 \| Bu \|_{p+1, \Sigma}^{p+1} + A_2 \| Bu \|_{Y, \Sigma}^\gamma]
$$

$$
< A_3 \| Bu \|_{Y, \Sigma}^{p+1} [1 + \| Bu \|_{Y, \Sigma}^{\gamma-p-1}]
$$

(2.33)

where $C$, $A_3$ are positive constants. Now $\| Bu \|_{Y, \Sigma} \neq 0$, so (2.33) implies

$$
\| Bu \|_{Y, \Sigma} > \min \{ 1, \left( \frac{1}{2A_3} \right)^{p-1} \} \overset{\text{def}}{=} A_4.
$$

Therefore, $\| u \|_D > \frac{A_4}{C}$.

The following two lemmas will be used to construct a weak solution of problem (W).

**Lemma 2.8:** The potential well $W$ is precompact as a subset of $L_2(D)$, and $\{ Bu : u \in W \}$ is precompact as a subset of $L_2(\Sigma)$. 
Proof: For \( u \in W \), \( 0 < J(u) < d \) and \( Q(u) > 0 \), so (2.32) holds and
\[
\|u\|_D^2 < \frac{2(p+1)}{p-1} d . \tag{2.34}
\]

Thus \( W \) is a bounded set in \( H^1_{0\sigma}(D) \) by Lemma 2.3. The conclusions then follow from Lemmas 2.1 and 2.2. \( \blacksquare \)

**Lemma 2.9:** If \( f, F \) satisfy (i)-(iii), then the functionals \( J \) and \( Q \) are continuous on \( H^1_{0\sigma}(D) \).

Proof: One need only show \( \int F(Bu) \, dS, \int (Bu)f(Bu) \, dS \) are continuous for \( u \in H^1_{0\sigma}(D) \). A proof for the first is given; the proof for the second is similar.

For \( u, v \in H^1_{0\sigma}(D) \) write \( u_b = Bu, v_b = Bv \); then
\[
F(u_b) - F(v_b) = (u_b - v_b) \int_0^1 f((1-\tau)u_b + \tau v_b) \, d\tau .
\]

Then
\[
\left| \int_{\Sigma} [F(u_b) - F(v_b)] \, dS \right| < \left\{ \int_0^1 \int_{\Sigma} |u_b - v_b| |f((1-\tau)u_b + \tau v_b)| \, dS \right\} \, d\tau \tag{2.35}
\]
\[
< \|u_b - v_b\|_{\Sigma} \int_0^1 \int_{\Sigma} |f((1-\tau)u_b + \tau v_b)| \, dS \right\} \, d\tau
\]
by Hölder's inequality.

Now by (2.18) there exist constants \( C_1, C_2 > 0 \) such that
\[
|f(s)| < C_1 + C_2 |s|^{\gamma-1} \text{ for all } s \in \mathbb{R}.
\]
Thus
\[ \| (1-\tau)u_b + \tau v_b \|^{\gamma-1} \leq C_1 + C_2 \| (1-\tau)u_b + \tau v_b \|^{\gamma-1} \]

by the triangle inequality, where \( C_3 = C_1 \int_{\Sigma} \| \gamma \). Recall from Lemma 2.2 that

\[ \| u_b - v_b \|_\Sigma < K \| u - v \|_D. \]

Combining (2.35)-(2.37), one obtains

\[ \int_{\Sigma} |[F(u_b) - F(v_b)]| d\Sigma \]

\[ < K \| u - v \|_D \int_0^1 [C_3 + C_2 \| (1-\tau)u_b + \tau v_b \|^{\gamma-1}] d\tau \]

\[ < K (\| u_b \|_{\Sigma}, \| v_b \|_{\Sigma}) \| u - v \|_D, \]

where \( K \) depends only on \( \gamma, \| u_b \|_{\Sigma}, \| v_b \|_{\Sigma}, \) and is bounded for bounded \( \| u \|_D, \| v \|_D \). Hence \( J \) is in fact Lipschitz continuous on \( H^1_{0\sigma}(D) \).

**Global Solution**

One says that \( u \) is a weak solution of (W) on the interval \([0, T]\) provided
(1) \( u(t) : [0,T) + H^1_{0\sigma}(D), \ u_{\tau}(t) : [0,T) + L_2(D); \) \( \|u(t)\|_D \) and \( \|u_{\tau}(t)\|_{L_2(D)} \) are uniformly bounded on compact subsets of \([0,T);\)

(2) for each \( t, 0 < t < T, \) and every \( v \in L_2(D), \)

\[
    \int_D [u(x,t) - U(x)]v(x)dx = \int_0^t \int_D [u_{\tau}(x,\tau)v(x)d\tau; \quad (2.39)
\]

(3) for each \( t, 0 < t < T, \) and every \( \eta(t) : [0,T) + H^1_{0\sigma}(D) \) with the properties of \( u \) in (1) and (2) (with \( \eta(x,0) \) replacing \( U(x) \) in (2.39)),

\[
    \int_D [u_{\tau}(x,t)\eta(x,t) - \nu(x)\eta(x,0)]dx
\]

\[
    + \int_0^t \int_D [\nu u_{\tau}\eta - u_{\tau}\nu\eta]\] dxdt \quad (2.40)

\[
    - \int_0^t \int_\Sigma f(\nu u)(\nu\eta)dSd\tau = 0; \text{ and}
\]

(4) for each \( t, 0 < t < T, \)

\[
    E(t) \leq E(0), \quad (2.41)
\]

where \( E(0) \equiv \frac{1}{2} \int_D \nu^2 dx + \frac{1}{2} \|U\|_D^2 - \int_\Sigma F(\nu U)dS, \) and \( E(t) \) is given by (2.3).
The function \( u \) is called a global weak solution provided it satisfies (1)-(4) with \( T = +\infty \). In this section, the following result is established.

**Theorem 2.1:** Let \( W, d \) be as defined in (2.28-30), and let \( f, F \) satisfy (i)-(iii). Then provided \( U \in W, \ V \in H^1_{0\sigma}(D) \), and \( E(0) < d \), problem (W) has a global weak solution.

The weak solution in Theorem 2.1 is approximated by functions of the form

\[
 u_{MN}(x,t) = \sum_{i=1}^{M} q_i(t)\phi_i(x) + \sum_{k=1}^{N} p_k(t)\psi_k(x) , \tag{2.42}
\]

where the functions \( q_i, p_k \) are solutions of a nonlinear initial value problem, to be detailed below. The \( \phi_i, \psi_k \) are generalized eigenfunctions for a Dirichlet problem and a modified Steklov problem:

\[
 \Delta \phi + \lambda \phi = 0 \quad \text{in } D , \quad \text{and} \quad \Delta \psi = 0 \quad \text{in } D
\]

\[
 \phi = 0 \quad \text{on } \partial D \quad \psi = 0 \quad \text{on } \sigma \tag{2.43}
\]

\[
 \frac{\partial \psi}{\partial n} = \mu \psi \quad \text{on } \Sigma .
\]

**Lemma 2.10:** There exist generalized eigenfunctions \( \phi_i \in H^1_0(D) \), \( \psi_k \in H^1_{0\sigma}(D) \), and corresponding eigenvalues \( \lambda_i, \mu_k > 0 \), which solve the weak formulations of (2.43); i.e.
\[
\int_D \nabla \phi_1 \cdot \nabla v \, dx = \lambda_1 \int_D \phi_1 v \, dx,
\]
and
\[
\int_D \nabla \psi_k \cdot \nabla w \, dx = \mu_k \int_{\Sigma} (B\psi_k)(Bw) \, ds
\]
for all \(v \in H^1_0(D), \ w \in H^1_0(D), \) and \(i, k = 1, 2, \ldots.\) The eigenvalues \(\lambda_i, \mu_k\) each have finite multiplicity and satisfy \(\lambda_i \rightarrow + \infty, \mu_k \rightarrow + \infty\) as \(i, k \rightarrow + \infty.\) The collection \(\{\phi_1, \phi_2, \ldots, \psi_1, \psi_2, \ldots\}\) (assumed normalized with respect to the norm \(\|\cdot\|\) defined in (2.14)) forms a complete orthonormal set in the Hilbert space \(H^1_0(D)\) endowed with the inner product \((\cdot, \cdot)_{D, \Sigma}\) of (2.13).

Proof: The proofs of the existence of the \(\lambda_i\) and \(\phi_i,\) and the fact that \(\{\phi_i\}\) forms a complete orthonormal set in \(H^1_0(D)\) endowed with the Dirichlet inner product, are standard and are omitted.

For any \(h \in L^2(\Sigma),\) the linear functional \(\phi\) given by
\[
\phi(v) = \int_{\Sigma} h(Bv) \, ds
\]
is bounded as a map \(H^1_0(D) \rightarrow \mathbb{R}\) by the Cauchy-Schwarz inequality and the imbedding (2.11). By Riesz's representation theorem there exists a unique \(u \in H^1_0(D)\) such that
\[
\phi(v) = \int_D \nabla u \cdot \nabla v \, dx
\]
for all \(v \in H^1_0(D);\) in other words, \(u\) solves the weak formulation (2.46) of the problem

\[\Delta u = 0 \quad \text{in } D\]
\[ u = 0 \quad \text{on } \sigma \]
\[ \frac{\partial u}{\partial n} = h \quad \text{on } \Sigma . \]

Therefore there is a well-defined linear Green's transformation
\[ \mathcal{G} : L_2(\Sigma) \rightarrow H_{0\sigma}^1(\Omega) \]
given by\[ \mathcal{G} h = u , \]
where \( u \) solves (2.46). By Lemma 2.2, the linear map\[ B G : L_2(\Sigma) \rightarrow L_2(\Sigma) \]
is compact; and for any \( h \in L_2(\Sigma) \),
\[ \int_{\Omega} (B G h) d\sigma = \int_{D} |\nabla (G h)|^2 dx , \]
which is nonnegative and vanishes only when \( h \) is the zero function in\( L_2(\Sigma) \), so that \( B G \) is a strictly positive operator. Hence \( B G \)
possesses a countable spectrum of positive eigenvalues, each of finite multiplicity, which are written as\[ \frac{1}{\mu_1} > \frac{1}{\mu_2} > \ldots , \]
satisfying\[ \mu_k \rightarrow +\infty \text{ as } k \rightarrow +\infty ; \]
and corresponding eigenfunctions \( \eta_1, \eta_2, \ldots \)
in \( L_2(\Sigma) \). Moreover, the eigenfunctions \( \{ \eta_k \} \)
can be chosen to form a complete orthonormal set in \( L_2(\Sigma) \). For each \( k \), \( \tilde{\psi}_k = G \eta_k \)
is in \( H_{0\sigma}^1(\Omega) \) and satisfies \( \eta_k = u_k B \tilde{\psi}_k ; \) hence \( u_k, \tilde{\psi}_k \)
satisfy (2.45). Let \( \psi_k \) be \( \tilde{\psi}_k \) normalized with respect to the norm (2.14). The pairwise orthogonality of \( \phi_i, \psi_k \)
in \( H_{0\sigma}^1(\Omega) \) is a direct consequence of (2.44) and (2.45).
To show the completeness of the generalized eigenfunctions, choose any \( u \in H^1_{0\Sigma}(D) \), and set

\[
\alpha_k = \langle u, \psi_k \rangle_{D, \Sigma}, \quad \beta_k = \int_{\Sigma} B u \psi_k \, dS.
\]

Then \( \alpha_k = (\mu_k + 1)\beta_k \) by (2.45). Since the \( \mu_k \) are positive, one has

\[
\sum_{k=1}^{\infty} \beta_k^2 \leq \sum_{k=1}^{\infty} \frac{\alpha_k^2}{(1+\mu_k)^2} \leq \sum_{k=1}^{\infty} \frac{\alpha_k^2}{C^2 u^2_D} < \infty
\]

by Bessel's inequality and Lemma 2.3. Therefore

\[
v = \sum_{k=1}^{\infty} \beta_k \psi_k \in H^1_{0\Sigma}(D);
\]

and

\[
w = u - v \in H^1_0(D)
\]

since \( B u = B v \).

The existence proof for \( \{\eta_k\} \) above is adapted from Fichera ([5, pp. 108-110]). Regularity of the generalized eigenfunctions \( \{\psi_k\} \) is difficult to prove and requires conditions on the confluence and smoothness of the boundary submanifolds \( \sigma, \Sigma \) (see, e.g., [25, pp. 233-236], [27], and the references cited therein), but is not required for the purposes of this chapter.

To obtain the nonlinear equations satisfied by \( q_i, p_k \), substitute (2.42) into the kinetic and potential energy functionals \( K, J \) to yield
\[ K(y) \equiv K(u_{MN}) = \frac{1}{2} \sum_{j} y_{j}^{T} A_{MN} y_{j}, \quad (2.47) \]
\[ J(y) \equiv J(u_{MN}) = \frac{1}{2} \sum_{j} y_{j}^{T} B_{MN} y_{j} - \int_{\Sigma} F(u_{j}) dS, \quad (2.48) \]

where

\[ y = (q_{1}, \ldots, q_{M}, p_{1}, \ldots, p_{N})^{T}, \]

\[ A_{MN} = \int_{D} \zeta \zeta^{T} dx \quad \text{where} \quad \zeta = (\phi_{1}, \ldots, \phi_{M}, \psi_{1}, \ldots, \psi_{N})^{T}, \]

\[ B_{MN} = \text{diag}(\lambda_{1}, \ldots, \lambda_{M}, \mu_{1} \int_{\Sigma} \phi_{1}^{2} dS, \ldots, \mu_{N} \int_{\Sigma} \psi_{N}^{2} dS), \quad \text{and} \]

\[ u_{N}(x,t) = \sum_{k=1}^{N} p_{k}(t) \psi_{k}(x). \]

The Lagrangian for (2.47), (2.48) is the functional

\[ L(y,\dot{y}) = K(\dot{y}) - J(y). \quad \text{From Lagrange's equations} \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial y_{i}} \right) - \frac{\partial L}{\partial y_{i}} = 0 \quad \text{for} \quad i = 1, \ldots, M+N, \]

one obtains the system of ordinary differential equations

\[ A_{MN} \ddot{y} + B_{MN} y = H_{N}(y), \quad (2.49) \]

where

\[ H_{N}(y) \equiv [0, \ldots, 0, \int_{\Sigma} f(u_{N}) \psi_{1} dS, \ldots, \int_{\Sigma} f(u_{N}) \psi_{N} dS]^{T}. \]

By Lemma 2.10 the eigenfunctions \( \phi_{1}, \ldots, \phi_{M}, \psi_{1}, \ldots, \psi_{N} \) are linearly
independent on \( D \cup \Sigma \), so 
\[
\xi^T A_{MN} T = \int_\Sigma (\zeta^T \xi)^2 \, dx > 0
\]
for all nonzero \( \xi \in \mathbb{R}^{M+N} \). Therefore \( A_{MN} \) is positive definite and (2.49) is equivalent to the system

\[
y + B_{MN} y = \tilde{H}_N(y),
\]

where \( B_{MN} = A_{MN}^{-1} B_{MN} \), \( \tilde{H}_N(y) = A_{MN}^{-1} H_N(y) \).

Let \( U_{MN}, V_{MN} \) be the partial Fourier series

\[
U_{MN}(x) = \sum_{i=1}^M \alpha_{ili} \phi_i(x) + \sum_{k=1}^N \alpha_{2k} \psi_k(x),
\]

\[
V_{MN}(x) = \sum_{i=1}^M \beta_{ili} \phi_i(x) + \sum_{k=1}^N \beta_{2k} \psi_k(x),
\]

where \( \alpha_{ili} = (\phi_i, D, \Sigma) \), \( \alpha_{2k} = (\phi_k, D, \Sigma) \), \( \beta_{ili} = (\phi_i, D, \Sigma) \), \( \beta_{2k} = (\psi_k, D, \Sigma) \). By Lemmas 2.2, 2.3, and 2.10, for \( U \in W \) and \( V \in H_0^1(D) \) the norms \( \|U_{MN}\|_{1,2, D} \), \( \|B_{MN} U\|_{2,2, \Sigma} \), \( \|V_{MN}\|_{1,2, D} \), \( \|B_{MN} V\|_{2,2, \Sigma} \) all approach zero as \( M,N \rightarrow +\infty \). The appropriate initial conditions for (2.50) are therefore

\[
y(0) = a, \quad \dot{y}(0) = \beta,
\]

where \( a = (a_{11}^T, \ldots, a_{1M}^T, a_{21}^T, \ldots, a_{2N}^T)^T \), \( \beta = (\beta_{11}^T, \ldots, \beta_{1M}^T, \beta_{21}^T, \ldots, \beta_{2N}^T)^T \).

Let \( M_{MN}^+ \) be the subspace of \( H_0^1(D) \) spanned by the functions \( \phi_1^T, \ldots, \phi_M^T, \psi_1^T, \ldots, \psi_N^T \) and set \( M_{MN}^+ = \{ u: M_{MN} : u \neq 0 \text{ and } u > 0 \} \) on \( D \). For \( u \in M_{MN}^+ \) one may write \( u = y^T \zeta \) for some \( y \in \mathbb{R}^{M+N} \), and
as in (2.48) define \( J(y) = J(y^T \zeta) \). Let \( \lambda^* = \lambda^*(y) \) denote the smallest positive root of \( \frac{d}{d\lambda} J(\lambda y) = 0 \), when it exists, and set

\[
d_{MN} = \inf J(\lambda^* y)
\]  

(2.54)

where the infimum is taken over all \( y \in \mathbb{R}^{M+N} \) such that \( y^T \zeta \neq 0 \) in case \( f \) satisfies (ii)(a) (or such that \( y^T \zeta \in H^+ \) in case \( f \) satisfies (ii)(b)). Since the minimization in (2.54) is over only a subclass of the set \( \mathcal{C} \) (or \( \mathcal{C}^+ \)) of the previous section, it follows that \( d_{MN} > d \). A potential well \( W_{MN} \) in \( \mathbb{R}^{M+N} \) of depth \( d_{MN} \) may then be defined by

\[
W_{MN} = \{ y \in \mathbb{R}^{M+N} : 0 < J(\lambda y) < d_{MN} \text{ for } 0 < \lambda < 1 \}.
\]  

(2.55)

Note \( W \subseteq W_{MN} \) in the sense that \( W \subseteq \{ y^T \zeta : y \in W_{MN} \} \).

**Lemma 2.11:** Let \( f, F \) satisfy (i)-(iii) of the previous section. If \( \alpha \in W \) and

\[
K(\beta) + J(\alpha) < d < d_{MN},
\]  

(2.56)

for each \( M, N \), then the initial value problem (2.50), (2.53) has a unique global solution \( y = y(t) \) satisfying \( (y(t))^T \zeta \in W \) for all \( t > 0 \).
Proof: The energy relation

\[ K(\dot{y}(t)) + J(y(t)) = K(\dot{y}) + J(y) < d \quad , \tag{2.57} \]

valid for all \( t \) in the existence interval for \( y \), follows from the differential equations (2.50) and initial conditions (2.53).

For any \( \alpha_0, \beta_0 \in \mathbb{R}^{n+N} \) satisfying \( \alpha_0 \in W \) and \( K(\beta_0) + J(\alpha_0) < d \), there are positive \( a, b \) depending only on \( M, N, \) and \( d \) such that 
\[ |\alpha_0| < a, \quad |\beta_0| < b \] (here \(|\cdot|\) denotes Euclidean norm). Indeed, from Lemma 2.8 it is clear that \( W \) is a bounded set in \( \mathbb{R}^{n+N} \), so the bound for \(|\alpha_0|\) follows. Also \( \alpha_0 \in W \) implies \( J(\alpha_0) > 0 \), so that
\[ 0 < K(\beta_0) = \frac{1}{2} \beta_0^T A_{MN} \beta_0 < d . \] Since \( A_{MN} \) is positive definite, a uniform upper bound for \(|\beta_0|\) is obtainable.

The proof proceeds as in [32, p. 165]. Since \( f \in C^1(\mathbb{R}) \), an argument similar to the proof of Lemma 2.9 establishes that \( \overline{H}_N(y) \) is a Lipschitz continuous function of \( y \) on compact subsets of \( \mathbb{R}^{n+N} \). Therefore (2.50), (2.53) possess a unique local solution \( y(t) \) on some interval \( 0 < t < \delta \), where \( \delta \) depends on \( a, b, \) and the modulus of Lipschitz continuity of \( \overline{H}_N(y) \). If \( y \) leaves the potential well \( W \) at some time \( t_0 \), then \( J(y(t_0)) > d \) and (2.57) is violated. Therefore \( y(t) \in W \) for \( 0 < t < \delta \), and (2.57) holds when \( t = \delta \).

One has \(|y(\delta)| < a, \quad |\dot{y}(\delta)| < b, \) and the argument of the preceding paragraph can be employed to extend the solution \( y(t) \) uniquely to the interval \( 0 < t < 2\delta \). Continuing in this way one obtains a unique global solution \( y(t) \) which remains in \( W \) for all \( t > 0 \).
One may now obtain a candidate \( u = u(x,t) \) for a global weak solution to problem (W). Choose \( U \in W, V \in H^{1}_{0\sigma}(D) \) such that

\[
E(U,V) = \frac{1}{2} \int_{D} V^2 dx + \frac{1}{2} \|U\|_{D}^2 - \int_{\Sigma} \Sigma F(BU) dS < d.
\]

For simplicity in notation and argument relabel the orthogonal projections \( U_{MN}, V_{MN} \) of \( U, V \) given in (2.51), (2.52) as \( U_{k}, V_{k} \), where

\[
k + + \infty \text{ if and only if both } M + + \infty \text{ and } N + + \infty \text{ (using, e.g., Cantor's ordering for the rationals in } [0,1]).
\]

Clearly

\[
\lim_{k \to + \infty} K(V_{k}) = K(V), \text{ and by Lemma 2.9 } \lim_{k \to + \infty} J(U_{k}) = J(U);
\]

so for sufficiently large \( k \), \( E(U_{k},V_{k}) < d \).

Suppose that for all sufficiently large \( k \), \( U_{k} \notin W \). Then there is a subsequence \( \{\lambda_{k}\} \) satisfying \( 0 < \lambda_{k} < 1 \) and \( J(\lambda_{k} U_{k}) > d \). Some subsequence \( \{\lambda_{k'}\} \) of \( \{\lambda_{k}\} \) converges to a limit \( \lambda_{0} \in [0,1] \).

Then \( \|\lambda_{k'} U_{k'} - \lambda_{0} U_{1,2,D} \| \to 0 \) as \( k' + + \infty \), so

\[
\lim_{k' \to + \infty} J(\lambda_{k'} U_{k'}) = J(\lambda_{0} U) < d, \text{ a contradiction. Therefore there is an infinite subsequence } \{U_{k_{1}}\} \text{ such that } U_{k_{1}} \in W. \text{ One may assume without loss of generality that } E(U_{k_{1}},V_{k_{1}}) < d \text{ for all } k_{1}.
\]

Using Lemma 2.11 and (2.42) one may obtain corresponding approximating functions \( \{u_{k_{1}}(x,t)\} \) defined for all \( t > 0 \) and satisfying

\[
u_{k_{1}} = U_{k_{1}}, \quad u_{k_{1}}(x,t) = V_{k_{1}} \text{ at } t = 0, \quad (2.58)
\]

\[
u_{k_{1}} \in W, \quad E(u_{k_{1}},u_{k_{1}}) < d \text{ for all } t > 0. \quad (2.59)
\]
Thus for every $T > 0$, the sequence $\{u_{k_1}\}$ is bounded in norm in $W^{1,2}(D_T)$. Applying Lemma 2.1, one obtains a subsequence $\{u_{k_2}\}$ of the $u_{k_1}$'s such that $u_{k_2}$ converges strongly in $L_2(D_T)$ to a limit $u \in L_2(D_T)$ for every $T > 0$. Bounded sets in $W^{1,2}(D_T)$ are also conditionally compact with respect to weak convergence in $W^{1,2}(D_T)$ (see, e.g., [26, p. 70]). Thus one may arrange (by taking subsequences if necessary) that the derivatives $u_{k_2,1}, \ldots, u_{k_2,n}, u_{k_2,t}$ converge weakly in $L_2(D_T)$ to limits $u_1, \ldots, u_n, u_{n+1} \in L_2(D_T)$, respectively, for all $T > 0$. It is easily verified that $u \in W^{1,2}(D_T)$ and that $u_{k_2,t} = u_{n+1}, u_{k_2,i} = u_i$ for $i = 1, \ldots, n$. Applying Lemma 2.2 and an elementary property of compact operators (see [37]), $Bu_{k_2}$ converges strongly in $L_2(\Sigma_T)$ to $Bu$ for every $T > 0$. By taking subsequences again if necessary, we may arrange that $u_{k_2} + u$ a.e. on $D_T$, and $Bu_{k_2} + Bu$ a.e. on $\Sigma_T$, for all $T > 0$.

**Lemma 2.12:** \[ \lim_{k_2 \to +\infty} \int \int_{\Sigma_T} f(Bu_{k_2}) \eta \, ds \, dt = \int \int_{\Sigma_T} f(Bu) \eta \, ds \, dt \] for every bounded measurable function $\eta$ and each $T > 0$.

The proof is omitted since it is entirely analogous to that of Lemma 4.2 in [32].

**Lemma 2.13:** The limit function $u$ satisfies (1)-(4) for every $T > 0$.

**Proof:** The weak convergence of $u_{k_2,i}$ to $u_i$ and of $u_{k_2,t}$ to $u_t$
implies

\[ \int_A \|u_i\|_{2,D}^2 \, dt < \int_A \|u_{k_2,i}\|_{2,D}^2 \, dt \quad (2.60) \]

for each \( k_2 \), each \( i = 1, \ldots, n \), and every measurable subset \( A \) of \([0,\infty)\) of finite measure, and a similar inequality involving \( u, u_{k_2} \). Therefore

\[ \|u\|_D^2 < \|u_{k_2}\|_D^2, \quad \|u_t\|_{2,D}^2 < \|u_{k_2,t}\|_{2,D}^2. \quad (2.61) \]

almost everywhere on \([0,\infty)\) for each \( k_2 \). If necessary, \( u \) and its first partial derivatives may be modified on a set of measure zero in \([0,\infty)\) so that (2.61) holds everywhere, and \( u(t) \in H^1_{00}(D) \), \( u_{k_2}(t) \in L^2(D) \) for all \( t > 0 \). This establishes (1) for each \( T > 0 \).

For each \( v \in L^2_D \) and each \( t \in [0,\infty) \),

\[ \int_D \left[ u_{k_2}(x,t) - u_{k_2}(x) \right] v(x) \, dx = \int_0^t \int_D u_{k_2,t}(x,\tau) v(x) \, dx \, d\tau. \]

Letting \( k_2 \to \infty \) and using the facts that \( u_{k_2} \) tends strongly to \( u \) in \( L^2(D) \) for almost all \( t > 0 \), \( u_{k_2} \) tends strongly to \( U \) in \( L^2(D) \), and \( u_{k_2,t} \) tends weakly to \( u_t \) in \( L^2(D_T) \) for each \( T > 0 \), one obtains (2.39) for almost all \( t > 0 \). Again, if necessary, one may redefine \( u \) on a subset of \([0,\infty)\) of measure zero, so that (2.39) holds for every \( t > 0 \).

To verify (3), let \( \rho_{\xi}(x,t) = C_{\xi}(t) \psi_\xi(x) \) where \( C_{\xi} \in C^1([0,\infty)) \).

Choose \( k_2 \) so large that \( u_{k_2} = u_{MN} \) where \( N > \xi \), multiply the
(M+ε)-th equation of (2.49) by $C_\varepsilon(t)$ and integrate over $[0,t)$ to obtain

$$0 = \sum_{i=1}^{M} \left( \int \phi_i \psi \, dx \right) q_i + \sum_{k=1}^{N} \left( \int \psi_k \psi \, dx \right) p_k$$

$$+ \int_{\varepsilon} (\sum \psi_k^2 \, dS) p_k - \int_{\varepsilon} f(Bu^N) \psi \, dS \, C_\varepsilon \, dt$$

$$= \sum_{i=1}^{M} \int_{\varepsilon} \alpha_i \phi_i \, dx \, dt$$

$$+ \sum_{k=1}^{N} \left( \int \psi_k \psi \, dx \right) p_k$$

$$+ \int_{\varepsilon} \int_{\varepsilon} f(Bu^N) \rho \, dS \, dt$$

Integration by parts yields

$$0 = \phi(u^N, v^N, n) ,$$

where $\phi(u,v,n)$ denotes the left-hand side of (2.40). Letting $k_2 \to \infty$ in (2.62), and using $\mathbf{W}_{k_2} \to \mathbf{W}_{2,D} \to 0$, $u_{k_2,t} \to u_t$ and $v_{k_2,u} \to v_{u}$ weakly in $L_2(D_1)$, $u_{k_2,t} \to u_t$ strongly in $L_2(D)$ and
almost everywhere on \([0, \infty)\), and Lemma 2.12, one obtains \(\phi(u, V, \rho) = 0\) for almost all \(t > 0\).

Similar considerations show \(\phi(u, V, V_j) = 0\) for almost all \(t > 0\) and all \(v_j(x, t) = B_j(t)\phi_j(x)\), where \(B_j \in C^1([0, \infty))\). From the linearity of \(\phi\) in its third argument it is evident that \(\phi(u, V, \eta_{MN}) = 0\) for almost all \(t > 0\) and all functions \(\eta_{MN}\) of the form

\[
\eta_{MN} = \sum_{j=1}^{M} B_j \phi_j + \sum_{\ell=1}^{N} C_\ell \phi_\ell .
\]

Now, for any \(\eta = \eta(x, t)\) which is \(C^1\) in \(t\) on \([0, \infty)\), and in \(H^1_0(D)\) for each \(t > 0\),

\[
|\phi(u, V, \eta)| = |\phi(u, V, \eta - \eta_{MN})|<
\]

\[
\leq \|u\|_{2, D} \|\eta - \eta_{MN}\|_{2, D} + \|V\|_{2, D} \|\eta(x, 0) - \eta_{MN}(x, 0)\|_{2, D} + \|u\|_{\infty, D} \|\eta - \eta_{MN}\|_{\infty, D} + \|u\|_{2, D} \|\eta - \eta_{MN}\|_{2, D} + \|f(Bu)\|_{s, \infty, \Sigma} \|\eta - \eta_{MN}\|_{\gamma, \Sigma}
\]

by H"older's inequality, where \(s = \frac{\gamma}{\gamma - 1}\). (Note that by (2.18) and Lemma 2.2, \(f(Bu) \in L_s(\Sigma_\ell)\)). Using Lemma 2.10 one may choose a sequence \(\{\eta_{MN}\}\) of functions of the form (2.63) (where \(B_j, C_\ell\) are the Fourier coefficients of \(\eta\)) such that for each \(t > 0\),

\[
\|\eta(x, t) - \eta_{MN}(x, t)\|_{D} \to 0
\]

(2.65)
as \( M,N \to +\infty \). Now \( \{ \eta_{MN}^n \} \) is a monotone increasing sequence of functions continuous on \( [0,\infty) \), which converge pointwise on \( [0,\infty) \) to \( \eta_D^M \). By the monotone convergence theorem,

\[
\int_0^t \eta_{MN}^n d\tau + \int_0^t \eta_D^m d\tau ,
\]

and

\[
\| \eta - \eta_{MN}^n \|_D^t = \int_0^t \| \eta - \eta_{MN}^n \|_D d\tau + 0 \]

as \( M,N \to +\infty \), for each \( t > 0 \).

Since \( \| \eta_{x,t} \|_D \) is uniformly bounded on compact subsets of \( [0,\infty) \), one may obtain from (2.65) and Bessel's inequality that

\[
\| \eta_{x,t} - \eta_{MN}^n(x,t) \|_D \to 0 \quad \text{for} \quad t > 0.
\]

By the argument of the preceding paragraph one may conclude that \( \| \eta_{x,t} - \eta_{MN}^n \|_D \to 0 \) for each \( t > 0 \). Then, by Lemma 2.3, for each \( t > 0 \)

\[
\| \eta_{x,t} - \eta_{MN}^n \|_{2,D} \to 0
\]

as \( M,N \to +\infty \). By (2.65) and Lemma 2.2,

\[
\| \eta(x,0) - \eta_{MN}(x,0) \|_{2,D} + 0 ;
\]

and by (2.67) and Lemma 2.2, for each \( t > 0 \)

\[
\| \eta - \eta_{MN}^n \|_{x,t} \to 0 .
\]
Let $M, N \to +\infty$ in (2.64) and use (2.65-70) to obtain $\Phi(u, V, \eta) = 0$ for all $t > 0$. If $\eta : [0, \infty) \to H^1_{0\sigma}(D)$ has the properties of $u$ in (1) and (2), simply choose $\{\eta_k\}$ which are each $C^1$ in $t$ on $[0, \infty)$ and in $H^1(\Omega)$ for each $t > 0$, such that $\eta_k \to \eta$ strongly in $H^1_{0\sigma_T}(D_T)$ for each $T > 0$, and $\eta_k \to \eta$ strongly in $L_2(D)$ for almost all $t > 0$, to obtain $\Phi(u, V, \eta) = 0$ almost everywhere. By redefining $u$ on a set of measure zero one may arrange that (2) holds for all $T > 0$.

For all measurable subsets $A$ of $[0, \infty)$ of finite measure one may establish

\[
\lim_{k_2 \to \infty} \int_A \int F(Bu_k) dS dt = \int_A \int F(Bu) dS dt
\]

by a proof similar to that of Lemma 4.2 of [32]. Thus by (2.61)

\[
\int_A E(t) dt < \lim_{k_2 \to \infty} \int_A \int E(u_{k_2}, u_{k_2}, t) dt
\]

\[
= \lim_{k_2 \to \infty} \int_A \int E(U_{k_2}, V_{k_2}) dt
\]

\[
= \int_A E(0) dt .
\]

Consequently

\[
\int_A [E(0) - E(t)] dt > 0
\]
for all such \( A \), so that \( E(t) < E(0) \) for almost all \( t > 0 \). The function \( u \) and its weak derivatives may be altered on a set of measure zero in \([0, \infty)\) to make (2.41) hold everywhere. □

Thus (1)-(4) hold with \( T = + \infty \), and \( u \) is the global weak solution of Theorem 2.1.

**Nonexistence**

In this section, the following result is established.

**Theorem 2.2:** Let \( f \) satisfy the conditions below, and let \( u \) be a weak solution to \((W)\) in the sense of (1)-(4) of the previous section. There is a region \( \widetilde{\mathcal{E}} \), exterior to the potential well \( W \) and characterized by

\[
\mathcal{E} = \{ v \in H^1_0(D) : J(v) < d \text{ and } Q(v) < 0 \},
\]

such that if \( U \in \mathcal{E} \) and \( E(0) = E(U, V) < d \), then \( u \) can only exist on a set \( D \times [0, T_0) \) with \( T_0 < \infty \).

In this section, it is assumed that either \( f \) satisfies (i), (ii)(a), and (iii); or else \( f \) satisfies (i), (ii)(b), (2.16), and the condition

\[
psf(s) < s^2 f'(s)
\]  

(2.71)

for all \( s \in \mathbb{R} \). Note that (2.15) can be derived from (2.71) by integration, but the two conditions are not equivalent.
Lemma 2.14: Let $f$ satisfy the above conditions, and let $u$ be any nonzero function in $H^1_{0\sigma}(\Omega)$. Then either

(a) $j_u'(\lambda) = 0$ has a unique positive root $\lambda^* = \lambda^*(u)$, $j_u'(\lambda) < 0$ and $j_u''(\lambda) < 0$ for $\lambda > \lambda^*$, and $\lim_{\lambda \to +\infty} j_u(\lambda) = -\infty$; or

(b) $j_u(\lambda)$ is strictly increasing for $\lambda > 0$, and

$\lim_{\lambda \to +\infty} j_u(\lambda) = +\infty$.

Proof: Suppose first that $f$ satisfies (i), (ii)(a), and (iii). By Lemma 2.6 (1)-(3), whenever $\mathcal{B} u > 0$ is nonzero one has $j_u'(\lambda) < 0$ for $\lambda > \lambda^*$, so

$$j_u''(\lambda) < \frac{1}{\lambda^2} \int_{\Sigma} (\lambda \mathcal{B} u)[f(\lambda \mathcal{B} u) - (\lambda \mathcal{B} u)f'(\lambda \mathcal{B} u)]dS < 0$$

for $\lambda > \lambda^*$, by (2.26).

Suppose now that $f$ satisfies (i), (ii)(b), (2.71), and (2.16). By (2.71),

$$j_u''(\lambda) < \|u\|^2_D - \frac{p}{\lambda} \int_{\Sigma} (\mathcal{B} u)f(\lambda \mathcal{B} u)dS$$

$$= (1-p)\|u\|^2_D + \frac{p}{\lambda} j_u'(\lambda)$$

(2.72)

for all $\lambda > 0$. If $j_u'(\lambda) = 0$ has a smallest positive root $\lambda^*$, multiply both sides of (2.72) by $\lambda^{-p} > 0$ and integrate from
\[ j_u'(\lambda) < \frac{2}{\lambda^p D^2} \lambda^{1-p}(\lambda^*)^{1-p} \cdot \]

Since \( p > 1 \), \( j_u'(\lambda) < 0 \) for \( \lambda > \lambda^* \); which in turn implies \( j_u''(\lambda) < 0 \) for \( \lambda > \lambda^* \) by (2.72). In particular, \( \lambda^* \) must be the only positive root of \( j_u'(\lambda) = 0 \), and \( \lim_{\lambda \to +\infty} j_u(\lambda) = -\infty \) by Lemma 2.6 (4).

Therefore (a) is established. The result in (b) follows from the convexity of \( j_u \) in a neighborhood of the origin, and Lemma 2.6.

The following lemma will play a central role in the proof of Theorem 2.2.

**Lemma 2.15**: Suppose \( \{u_n^\lambda\} \) is a sequence in \( H^1_0(D) \) satisfying

\[ \|u_n\|_D \neq 0 \quad \text{and} \quad Q(u_n) < 0 \quad \text{for all} \quad n, \quad \text{and} \quad Q(u_n) \to 0 \quad \text{as} \quad n \to +\infty, \]

where \( Q \) is as defined in (2.30). Then

\[ \lim_{n \to +\infty} J(u_n) > d. \]

**Proof**: Suppose to the contrary that there exists an \( \varepsilon > 0 \) and a subsequence of \( \{u_n^\lambda\} \) (still denoted \( \{u_n^\lambda\} \)) such that \( J(u_n^\lambda) < d - \varepsilon \) for all \( n \).

Define \( j_n^\lambda(\lambda) = J(\lambda u_n^\lambda) \). Since \( Q(u_n^\lambda) < 0 \), by Lemma 2.14 for each \( n \) there exists a unique positive root \( \lambda_n^\lambda \) of \( j_n^\lambda(\lambda) = 0 \), and \( 0 < \lambda_n^\lambda < 1 \). Also by Lemma 2.14, \( j_n^\lambda(\lambda) < 0 \) for \( \lambda_n^\lambda < \lambda < 1 \), so
This is a contradiction, since \( j_n'(1) < d - \varepsilon \) for all \( n \), and 
\( j_n'(\lambda_n) > d \) for all \( n \) by the variational definition of \( d \).

The proof of Theorem 2.2 uses a well-known concavity technique ([15],[19],[21],[22]), and is very similar to that of [28]. It is included here for the convenience of the reader. Suppose \( u \) is a global weak solution of (W) in the sense of the previous section, and suppose \( U \in \mathcal{E} \) and \( E(0) < d \). Define
\[
M(t) = \int \frac{u^2(x,t)}{D} dx.
\]
It will be shown that \( M \to +\infty \) in finite time, contradicting the assumption that \( u \) is a global solution of (W).

Define \( P(t,s) = \int_D u(x,t)u(x,s)dx \). By (2.39) with \( v(x) = u(x,s) \),
\[
P(t,s) = P(0,s) + \int_0^t \int_D u(x,t)u(x,s)dx \, dt.
\]
for \( t,s > 0 \). Since \( \int_D u_{,t}(x,t)u(x,s)dx \) is an integrable function of \( t \) on compact subsets of \( [0,\infty) \) for each \( s > 0 \),
\[
\frac{\partial}{\partial \tau} P(t,s) = \int_D u_{,t}(x,t)u(x,s)dx.
\]
for $s > 0$ and almost all $t$ in $[0, \infty)$. Since $P$ is symmetric in its arguments,

$$\hat{M}(t) = \left[ \frac{\partial}{\partial t} P(t,s) + \frac{\partial}{\partial s} P(t,s) \right]|_{s=t}$$

$$= 2 \int_D u_t(x,t)u(x,t)dx \quad (2.73)$$

almost everywhere on $[0, \infty)$. If necessary, $u$ and $u_t$ may be modified on a set of measure zero in $D_{\infty}$ so that (2.73) holds for all $t > 0$.

Now, taking $\eta = u$ in (2.40), one obtains

$$\dot{M}(t_2) - \dot{M}(t_1) = 2 \int_{t_1}^{t_2} \int_D \left[ |u_t|^2 - |\nu u|^2 \right]dx dt$$

$$+ 2 \int_{t_1}^{t_2} \int_\Sigma (Bu)f(Bu)dS dt$$

for $t_2 > t_1 > 0$. Since $\|u_t\|_{2,D}$, $\|u\|_{D}$, and $\int_\Sigma (Bu)f(Bu)dS$ are uniformly bounded on compact subsets of $[0, \infty)$, $\dot{M}$ is Lipschitz continuous on such sets. Hence $\ddot{M}$ exists a.e. in $[0, \infty)$ and is given by

$$\ddot{M}(t) = 2[\|u_t\|_{2,D} - \|u\|_{D}^2 + \int_\Sigma (Bu)f(Bu)dS]. \quad (2.74)$$

One may show that $u(t) \in \mathcal{E}$ for all $t > 0$. Indeed, if $u$ leaves $\mathcal{E}$ at some smallest time $t = t_0 > 0$, then

$$Q(u(t_0)) < \lim_{n \to \infty} Q(u(t_n)) < 0$$
where \( t_n \to t_0^- \). If \( Q(u(t_0)) < 0 \), then \( u(t_0) \in \mathcal{E} \). If \( Q(u(t_0)) = 0 \), then \( J(u(t_0)) > d \) by the variational definition of \( d \); but this contradicts the energy inequality (2.41), which requires that \( J(u(t)) < E(0) < d \) for all \( t > 0 \).

Therefore

\[
Q(u) = \frac{1}{2}u^2 - \int_{\Sigma} (Bu)f(Bu)dS < 0 \quad \text{on } [0,\infty),
\]

and

\[
M(t) > 0 \quad \text{for all } t > 0.
\]

One may show \( \dot{M}(t_1) > 0 \) for some \( t_1 > 0 \). Suppose to the contrary that \( \dot{M}(t) < 0 \) for all \( t > 0 \). Then, since \( M > 0 \) and is convex, \( L = \lim_{t \to +\infty} M(t) \) exists and is finite. Furthermore, \( L > 0 \) since \( u \) remains in the region \( \mathcal{E} \) exterior to the potential well \( \mathcal{W} \) for all positive time. Choose \( t_n \to +\infty \) such that \( M(t_n) + L, \dot{M}(t_n) + 0, \) and \( \ddot{M}(t_n) + 0 \). Since \( Q(u(t_n)) < 0 \) for all \( n \), one sees from (2.74) that

\[
\left\| u_n(x,t_n) \right\|_{L^2}^2 + 0 \quad (2.75)
\]

and

\[
Q(u(t_n)) + 0
\]

as \( n \to \infty \). Applying Lemma 2.15,

\[
\lim_{n \to \infty} J(u(t_n)) > d > E(0).
\]
But from (2.75) and the energy inequality (2.41),

$$\lim_{n \to \infty} J(u(t_n)) < E(0),$$

a contradiction.

Using (2.15), (2.41), and Lemma 2.3,

\[ M(t) > 2 \| u_t \|_{L^2(D)}^2 - \| u \|_{L^2(D)}^2 + (p+1) \int F(Bu) dS \]

\[ > 2 \| u_t \|_{L^2(D)}^2 - 2 \| u \|_{L^2(D)}^2 \]

\[ + 2(p+1)[\frac{1}{2} \| u_t \|_{L^2(D)}^2 + \frac{1}{2} \| u \|_{L^2(D)}^2 - E(0)] \]

\[ = (p+3) \| u_t \|_{L^2(D)}^2 + (p-1) \| u \|_{L^2(D)}^2 - 2(p+1)E(0) \]

\[ > (p+3) \| u_t \|_{L^2(D)}^2 + C(p-1)M(t) - 2(p+1)E(0) \]

for \( t > 0 \), where \( C > 0 \) is a constant. Since \( M \) is convex and \( \dot{M}(t_1) > 0 \) by the argument of the previous paragraph, one may conclude that \( M(t) \) is increasing for all \( t > t_1 \), and that the quantity

\[ C(p-1)M(t) - 2(p+1)E(0) \]

must eventually become positive, and remain positive thereafter. Therefore

\[ \ddot{M}(t) > (p+3) \| u_t \|_{L^2(D)}^2 \]
for all sufficiently large $t$. Let $\alpha = \frac{p-1}{4} > 0$. Then for $t$ sufficiently large,

$$(M^{-\alpha})'' = \frac{\alpha}{M^{\alpha+2}} [(a+1)\dot{M}^2 - \ddot{M}]$$

$$< \frac{\alpha}{M^{\alpha+2}} [(a+1)\|u\|^2_{L^2(D)} - (p+3)\|u\|_{L^2(D)}^2 \|u\|_{L^2(D)}^2$$

$$= \frac{\alpha(p+3)}{M^{\alpha+2}} \left[ \frac{1}{4} \|u\|^2_{L^2(D)} - \|u\|_{L^2(D)}^2 \|u\|_{L^2(D)}^2 \right] < 0.$$ 

Therefore there exists a finite time $T_0 > 0$ for which

$$\lim_{t \to T_0^-} M^{-\alpha}(t) = 0; \text{ and consequently}$$

$$\lim_{t \to T_0^-} M(t) = +\infty.$$ 

This completes the proof of Theorem 2.2.
CHAPTER 3. THE POTENTIAL WELL THEORY APPLIED TO THE HEAT EQUATION

WITH A NONLINEAR BOUNDARY CONDITION

Introduction

In this chapter, the following problem for the linear heat equation is considered:

\[ \begin{align*}
    \mathbf{u}_t &= \Delta \mathbf{u} & \text{in } D \times (0,T) \\
    \mathbf{u} &= 0 & \text{on } \sigma \times (0,T) \\
    \frac{\partial \mathbf{u}}{\partial n} &= f(\mathbf{u}) & \text{on } \Sigma \times (0,T) \\
    \mathbf{u}(x,0) &= U(x) & \text{in } D.
\end{align*} \] (H)

In general, methods employed to study solutions of hyperbolic problems cannot be employed to study solutions of parabolic problems and conversely. Nevertheless the arguments in Chapter 2 used to study problem (W) can be modified to obtain corresponding results for problem (H) and can be used to prove Sattinger's assertions about the parabolic problems he mentions in [32]. The key results of this chapter, being of a function analytic nature, are similar to those in Chapter 2. However, there are several important differences in their proofs which are emphasized here.

Note that the potential energy functional associated with problem (H) is the same as that associated with problem (W):

\[ J(\mathbf{u}) = \frac{1}{2} \int_D |\nabla \mathbf{u}|^2 dx - \int_\Sigma F(Bu) dS. \] (3.1)
Suppose that $f, F$ satisfy the following conditions (i)-(iii) of Chapter 2. Then as in Chapter 2, there is a potential well $W$ of positive depth $d$ in the function space $H^1_{00}(D)$ characterized by

$$W = \{ u \in H^1_{00}(D) : 0 < J(\lambda u) < d \text{ for } 0 < \lambda < 1 \}, \quad (3.2)$$

and

$$d = \inf J(u) \quad (3.3)$$

where $u$ is subject to the constraint

$$Q(u) \equiv \|u\|_D^2 - \int_\Sigma (Bu)f(Bu) \, dS = 0. \quad (3.4)$$

The infimum in (3.3) is taken over all $u \in H^1_{00}(D)$ with nonzero trace $Bu$ on $\Sigma$ in case $f$ satisfies (ii)(a); and may be taken over all $u \in H^1_{00}(D)$ with $Bu > 0$ a.e. in case $f$ satisfies (ii)(b).

Under conditions on the nonlinearity $f$ weaker than (i)-(iii) it has been shown ([21]) that solutions of problem (H) cannot exist for all time whenever $J(U) < 0$. Generalizations of this result to a wider class of initial-boundary value problems are given in [12], [22]. In this chapter, the following theorem is proved, which is the analogue of the main results, Theorems 2.1 and 2.2, of Chapter 2:
Theorem 3.1: Let $W, d$ be as defined in (3.2)-(3.4).

(a) Suppose $f$ satisfies (i)-(iii). Then if $U \in W$, problem (H) has a global weak solution in the sense of (1)-(4) of the following section.

(b) Suppose that either $f$ satisfies (i), (ii)(a), and (iii), or else $f$ satisfies (i), (ii)(b), (2.16), and the condition (2.71). Let $u$ be a weak solution of problem (H). Then there is a region $E$, exterior to $W$ and characterized by

$$E = \{ v \in H^1_0(D) : J(v) < d \text{ and } Q(v) < 0 \},$$

such that if $U \in E$, then $u$ can only exist on a set $D \times [0, T_0)$ with $T_0 < \infty$.

The proof of Theorem 3.1 will be outlined giving particular attention to how it differs from the proofs of Theorems 2.1 and 2.2.

Existence

In this section, part (a) of Theorem 3.1 is established. Recall the following results:

(A) the potential well $W$ is bounded as a subset of $W^{1,2}(D)$;

(B) bounded sets in $W^{1,2}(D)$ (respectively, $W^{1,2}(D_T)$) are pre-compact as subsets of $L_2(D)$ ($L_2(D_T)$), and are conditionally
compact with respect to weak convergence in $W^{1,2}(D)$ ($W^{1,2}(D_T)$);

(C) the trace operator $\mathcal{B}$ is compact as a linear mapping from $W^{1,2}(D)$ into $L_q(\Gamma)$ for $2 < q < \frac{2(n-1)}{n-2}$ if $n > 2$, and for $2 < q < \infty$ if $n = 2$; and

(D) Poincaré's inequality

$$\|u\|_{2,D} \leq C\|u\|_D$$

holds for all $u \in H^1_0(D)$, where $C > 0$ is independent of $u$.

The kinetic and total energy functionals associated with problem (H) are given by

$$K(u) = \int_0^T \int_D u_t(x,t)^2 \, dx \, dt,$$

and

$$E(t) = K(u(\cdot,t)) + J(u(\cdot,t)),$$

respectively. A function $u = u(x,t)$ is called a weak solution of problem (H) on $D_T$ provided

1. $u(t) : [0,T) \to H^1_0(D), \; u_t(t) : [0,T) \to L_2(D)$;

2. $\|u(t)\|_D$ and $\|u_t(t)\|_{L_2,D}$ are uniformly bounded on compact subsets of $[0,T)$;
(2) for each \( t, \quad 0 < t < T \), and every \( \eta \in L^2(D) \),

\[
\int_D [u(x,t) - U(x)]\eta(x)dx = \int_0^t \int_D u_{\tau}(x,\tau)\eta(x)dx\ d\tau; \quad (3.6)
\]

(3) for each \( t, \quad 0 < t < T \), and every \( \eta(t) : [0,T) \to H^1_0(D) \) with the properties of \( u \) in (1) and (2),

\[
\int_D u_{\tau} \eta dx + \int_D V u \nabla \eta \ dx - \int_{\Sigma} \overline{f(Bu)}(B\eta)dS = 0; \quad (3.7)
\]

and

(4) for each \( t, \quad 0 < t < T \),

\[
E(t) < E(0) = J(U). \quad (3.8)
\]

The function \( u \) is a global weak solution of problem (H) provided \( u \) satisfies (1)-(4) with \( T = +\infty \). The global weak solution of Theorem 3.1(a) is approximated by functions of the form

\[
u_{MN}(x,t) = \sum_{i=1}^M q_i(t)\phi_i(x) + \sum_{k=1}^N p_k(t)\psi_k(x), \quad (3.9)
\]

where \( \phi_1, \phi_2, \ldots \in H^1_0(D) \), with corresponding eigenvalues \( \lambda_1, \lambda_2, \ldots \), and \( \psi_1, \psi_2, \ldots \in H^1_0(D) \), with corresponding eigenvalues \( \nu_1, \nu_2, \ldots \), are as in Chapter 2. The \( q_i \)'s and \( p_k \)'s are chosen to satisfy a nonlinear system of ordinary differential equations. Let

\[
y = (q_1, \ldots, q_M, p_1, \ldots, p_N)^T,
\]

\[
\zeta = (\phi_1, \ldots, \phi_M, \psi_1, \ldots, \psi_N)^T,
\]
and substitute (3.9) into the expressions (3.5), (3.1) to obtain

\[ K(\dot{y}) = K(u_{MN}) = \int_0^t (\dot{y}(\tau))^T A_{MN} \dot{y}(\tau) d\tau, \]

\[ J(y) = J(u_{MN}) = \frac{1}{2} y^T B_{MN} y - \int_{\Sigma} F(u) dS, \]

where

\[ A_{MN} = \int_D \zeta \zeta^T d\mathbf{x}, \]

\[ B_{MN} = \text{diag}(\lambda_1, \ldots, \lambda_M, \mu_1 \int \psi_1^2 dS, \ldots, \mu_N \int \psi_N^2 dS), \]

\[ u_N = \sum_{k=1}^N p_k \psi_k. \]

Form the Lagrangian \( L(\dot{y}, y) = K(\dot{y}) - J(y). \) From Lagrange's equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}_i} \right) - \frac{\partial L}{\partial y_i} = 0, \quad i = 1, \ldots, M+N, \]

one obtains the system of \( M+N \) ordinary differential equations

\[ A_{MN} \ddot{y} + B_{MN} y = H_N(y), \quad (3.10) \]

where

\[ H_N(y) = [0, \ldots, 0, \int_{\Sigma} f(u_1)\psi_1 dS, \ldots, \int_{\Sigma} f(u_N)\psi_N dS]^T. \]

In Chapter 2, it was shown that \( A_{MN} \) is a real symmetric, positive definite matrix. Thus (3.10) is equivalent to the system
\[ \dot{y} + \bar{B}_{MN} y = \bar{H}_N(y), \]  
\[ \text{(3.11)} \]

where \( \bar{B}_{MN} = A_{MN}^{-1} B_{MN}, \) \( \bar{H}_N = A_{MN}^{-1} H_N. \) Expand \( U \in W \) in the partial Fourier series

\[ U_{MN}(x) = \sum_{i=1}^{M} \alpha_{1i} \phi_i(x) + \sum_{k=1}^{N} \alpha_{2k} \psi_k(x), \]  
\[ \text{(3.12)} \]

where \( \alpha_{1i} = (U, \phi_i)_{D, \Sigma}, \) \( \alpha_{2k} = (U, \psi_k)_{D, \Sigma}, \) to obtain the appropriate initial conditions for (3.11):

\[ y(0) = \alpha, \]  
\[ \text{(3.13)} \]

where \( \alpha = (\alpha_{11}, \ldots, \alpha_{1M}, \alpha_{21}, \ldots, \alpha_{2N})^T. \) The energy equality

\[ K(\dot{y}(t)) + J(y(t)) = J(\alpha), \]  
\[ \text{(3.14)} \]

valid for all \( t \) in the existence interval for \( y, \) follows directly from (3.11) and (3.13).

One may then prove the following analogue of Lemma 2.11.

**Lemma 3.1:** Provided \( U_{MN} = \alpha_T \xi \in W, \) the initial-value problem (3.11), (3.13) has a unique global solution \( y \) satisfying \( (y(t))^T \xi \in W \) for all \( t > 0. \)

The proof uses the boundedness of \( \{ \alpha_0 \in \mathbb{R}^{M+N} : \alpha_0^T \xi \in W \} \) in \( \mathbb{R}^{M+N}, \) and the Lipschitz continuity of \( \bar{H}_N(y) \) on compact subsets of \( \mathbb{R}^{M+N}, \) both established in Chapter 2.
Just as in Chapter 2, one may choose a doubly infinite subsequence 
\{U_{M,N}\} of functions of the form (3.12) satisfying \( U_{M,N} \in W \) for all 
\( M, N \). Using Lemma 3.1 and (3.9), one obtains corresponding approximating functions \( \{u_{M,N}(x,t)\} \) defined for all \( t > 0 \) and satisfying

\[
u_{M,N}(x,0) = U_{M,N}(x),
\]
\( u_{M,N} \in W \) and \( K(u_{M,N}) + J(u_{M,N}) = J(U_{M,N}) \) for all \( t > 0 \).

Using (A), (B), and (C), choose a doubly infinite subsequence \( \{u_{M,N}\} \)
of approximating functions, and a function \( u \in W^{1,2}(D_T) \), such that for all \( T > 0 \), as \( M, N \to \infty \) the sequence \( u_{M,N} \) converges to \( u \)
strongly in \( L_2(D_T) \) and a.e. on \( D_T \); \( u_{M,N}, u_{M,N}^t, u_{M,N}^{t_1}, \ldots, u_{M,N}^{t_n} \)
converges to \( u_t, u_1, \ldots, u_n \) weakly in \( L_2(D_T) \); and \( Bu_{M,N} \)
converges to \( Bu \) strongly in \( L_2(\Sigma_T) \) and a.e. on \( \Sigma_T \). By Lemma 2.12,

\[
\lim_{M,N \to \infty} \int_0^T \int_{\Sigma_T} f(Bu_{M,N}) \eta \, dS \, dt = \int_0^T \int_{\Sigma_T} f(Bu) \eta \, dS \, dt \quad (3.15)
\]

for every bounded measurable function \( \eta \) on \( \Sigma_T \) and each \( T > 0 \); and

\[
\lim_{M,N \to \infty} \int_G F(Bu_{M,N}) \, dS \, dt = \int_G F(Bu) \, dS \, dt \quad (3.16)
\]

for each measurable subset \( G \) of \([0,T]\) and each \( T > 0 \).
Conditions (1) and (2) on a weak solution of (H) are identical to conditions in Chapter 2 on a weak solution of problem (W). Consequently the limit function \( u \) of the preceding paragraph satisfies (1) and (2) for each \( T > 0 \) and all \( 0 < t < T \) by the same arguments as in Chapter 2.

To establish (3), let \( \Psi(u, \eta) \) denote the left-hand side of (3.7). Choose any \( T > 0 \), and for each integer \( j > 1 \) let \( C^j = C^j(t) \) be any continuously differentiable function on \([0, T]\). For fixed \( j \) choose \( N_j > j \), multiply both sides of the \((M_j + m)\)-th equation in (3.10) by \( C^j \), integrate both sides with respect to \( t \) over any measurable subset \( G \) of \([0, T]\), and use the orthogonality properties of the \( \psi_k \)'s to obtain

\[
0 = \int \left[ \sum_{i=1}^{M_j} \left( \int \phi_i \psi_{\ell}^j \, dx \right) q_i + \sum_{k=1}^{N_j} \left( \int \psi_k \psi_{\ell}^j \, dx \right) p_k \right] \psi_{\ell}^j \, dt \\
+ u_{\ell} \left( \int \psi_{\ell}^2 \, ds \right) p_{\ell} - \int f(Bu) \psi_{\ell}^j \, ds \right] C^j \, dt \quad (3.17)
\]

Let \( M_j, N_j + \infty \) in (3.17) and use (3.15) and the facts that

\[
\int \Psi(u, \eta) \, dt = 0 \quad \text{for each} \quad j > 1 . \quad \text{In a similar manner one may show}
\]

that \( \int \Psi(u, B_j \phi_j) \, dt = 0 \) for each \( j > 1 \), where \( B_j = B_j(t) \) is any
continuously differentiable function on \([0,T]\). Since \(\psi(u,n)\) is linear in \(n\), it follows that

\[
\int_G \psi(u, n_{MN}) \, dt = 0 \tag{3.18}
\]

for all functions \(n_{MN}\) of the form

\[
n_{MN} = \sum_{j=1}^{M} B_j \phi_j + \sum_{k=1}^{N} C_k \psi_k. \tag{3.19}
\]

Since (3.18) holds for all measurable subsets \(G\) of \([0,T)\), it follows that \(\psi(u, n_{MN}) = 0\) a.e. on \([0,T)\). If necessary, \(u\) may be modified on a set of measure zero in \([0,T)\) so that \(\psi(u, n_{MN}) = 0\) everywhere.

For any \(n(t) : [0,T) \rightarrow H^1_{0\sigma}(D)\) satisfying the same conditions as \(u\) in (1) and (2), we may choose a sequence \(\{n_{MN}\}\) of functions of the form (3.19) such that for each \(t, 0 < t < T,\)

\[
\|n(t) - n_{MN}(t)\|_D \rightarrow 0 \tag{3.20}
\]

as \(M,N \rightarrow \infty\). By Hölder's inequality,

\[
|\psi(u,n)| = |\psi(u,n-n_{MN})| \\
\leq \|u\|_{L^2_D} \|n-n_{MN}\|_{L^2_D} + \|u\|_D \|n-n_{MN}\|_D \tag{3.21}
\]

+ \|f(Bu)\|_{s,\Sigma} \|Bn-Bn_{MN}\|_{\gamma,\Sigma}, \]
where $s = \frac{\gamma}{\gamma - 1}$ and $\|v\|_{q, \Sigma} = \left[\int_{\Sigma} |v|^q \, dS\right]^{1/q}$. (Using the uniform boundedness of $1u(t)1_D$ on compact subsets of $[0,T]$ from (1), the order condition $|f(u)| = (6|u|^\gamma - 1)$ as $|u| \to +\infty$ proved in Chapter 2, and results (C) and (D), $f(Bu) \in L_s(\Sigma)$ for each $t$, $0 < t < T$). By (3.20), (C), and (D), the right-hand side of (3.21) approaches zero as $M, N \to \infty$. Therefore $\Psi(u, \eta) = 0$ for each $t$, $0 < t < T$, and (3) is established for each $T > 0$.

To establish (4), note that since $u_{M_2, N_2, t} + u_t$ and $\nabla u_{M_2, N_2} + \nabla u$ weakly in $L_2(D, T)$, one has

$$K(u(t)) < K(u_{M_2, N_2}(t))$$

for each $0 < t < T$ and each $M_2, N_2$, and

$$\int \int |\nabla u|^2 \, dx \, dt < \int \int |\nabla u_{M_2, N_2}|^2 \, dx \, dt$$

for each measurable subset $G$ of $[0,T)$ and each $M_2, N_2$. Thus

$$\int E(t) \, dt = \int [K(u) + J(u)] \, dt \quad G$$

$$< \lim_{M_2, N_2 \to \infty} \int [K(u_{M_2, N_2}) + J(u_{M_2, N_2})] \, dt$$

$$= \lim_{M_2, N_2 \to \infty} \int J(u_{M_2, N_2}) \, dt = \int J(u) \, dt \quad G$$
using the energy equality for the approximating solutions $u_{M^2N^2}$, the
continuity of $J$ on $H^1_{\partial D}$ (proved in Lemma 2.9), and the fact that
$u_{M^2N^2} + U$ strongly in $H^1_{\partial D}$. Therefore $E(t) < J(U)$ for almost all
t $\in [0,T)$. If necessary, $u$ may be redefined on a set of measure
zero so that (4) holds everywhere on $[0,T)$ for each $T > 0$.

Nonexistence

Suppose $f$ satisfies the conditions of part (b) of Theorem 3.1, suppose $U \in E$, and suppose $u$ is a global weak solution to problem
(H). Then one can show that

$$M(t) = \int_0^T \int_D u^2(x,t)dxdt$$

approaches infinity in finite time, which contradicts (1) of the previous
section and establishes Theorem 3.1(b). The proof is similar to that in
Section 5 of [28] and uses a well-known concavity technique ([15], [19]);
an outline is given here.

Since $\|u(t)\|_{L^2,D}$ is integrable on compact subsets of $[0,\infty)$,

$$\dot{M}(t) = \int_D u^2(x,t)dx = \|u\|_{L^2,D}^2 + 2 \int_0^T \int_D u(x,\tau)u_\tau(x,\tau)dxd\tau \quad (3.22)$$

a.e. on $[0,\infty)$. Since $\|u(t)\|_{L^2,D}$, $\|u_\tau(t)\|_{L^2,D}$ are uniformly bounded on
compact subsets of $[0,\infty)$, one may use (3.22) and (possibly) redefine
$u$ on a set of measure zero to show that $\dot{M}$ is Lipschitz continuous on
compact subsets of $[0,\infty)$. Therefore $\dot{M}$ exists a.e. on $[0,\infty)$, and
using (3.22) and (3.6),
\[ M(t) = 2 \int_D u_t(x,t)u(x,t)dx \]  
\[ = 2\left[ \int_{\mathbb{R}} (Bu)f(Bu)dS - \frac{2}{D} \right] = -2Q(u) \]  
a.e. in \([0,\infty)\), where \( Q \) is as defined in (3.4).

Suppose \( u \) leaves \( E \) at some smallest time \( t = t_0 > 0 \). Then

\[ Q(u(t_0)) < \lim_{n \to \infty} Q(u(t_n)) < 0, \]

where \( t_n \to t_0^- \). If \( Q(u(t_0)) < 0 \), then \( u(t_0) \notin E \), a contradiction.

If \( Q(u(t_0)) > d \), then \( J(u(t_0)) > d \) by the variational definition (3.3), (3.4) of \( d \); but this contradicts (3.8), which requires that \( J(u(t)) < J(U) < d \) for all \( t > 0 \). Therefore \( u(t) \notin E \), and

\[ Q(u(t)) < 0, \text{ for all } t > 0. \]

If there exists a sequence \( t_n \to \infty \) such that \( Q(u(t_n)) \to 0^- \), then one may obtain the contradiction

\[ \lim_{n \to \infty} J(u(t_n)) > d > J(U) \]

of (3.8), as a consequence of Lemma 2.15.

Therefore one has in fact that \( Q(u(t)) \) is negative and bounded away from zero for \( 0 < t < \infty \). Using (3.23) one may easily show that both \( M(t) \) and \( \tilde{M}(t) \) approach \( +\infty \) as \( t \to \infty \).

By (3.23) and the energy inequality (3.8),

\[ \tilde{M}(t) > 2(p+1)K(u) + (p-1)C^2 - 2(p+1)J(U) \]  
\[ \text{ (3.24) } \]
for all \( t > 0 \), where \( C \) is as in (D) of the previous section. If one sets \( \alpha = \frac{p-1}{2} > 0 \), then using (3.24) and the expression (3.22) for \( \dot{M} \), one obtains

\[
(M^{-\alpha})'' = - \frac{\alpha}{M^{\alpha+2}} (MM - \frac{p+1}{2} M^2) \\
< - \frac{\alpha}{M^{\alpha+2}} \left\{ 2(p+1) [K(u)M - (\int_0^t \int_0^D uu, dx dx)^2] \\
+ (p-1)C^2 M \right\} \dot{M} \\
- 2(p+1)J(U)M + \frac{(p+1)}{2} \|U\|_{2,D}^4 \dot{M}.
\] (3.25)

The term in square brackets in (3.25) is nonnegative by Schwarz's inequality, and \((p-1)C^2 M \dot{M}\) will eventually become and stay larger than the remaining three terms in (3.25). Therefore \((M^{-\alpha})'' < 0\) for all sufficiently large \( t \), so that \( M^{-\alpha} \to 0^+ \), and \( M \to +\infty \), in finite time. This is the desired contradiction.
CHAPTER 4. A HYPERBOLIC QUENCHING PROBLEM
IN SEVERAL DIMENSIONS

Introduction

Let $D$ be an open, bounded subset of $\mathbb{R}^n$ with boundary $\partial D$. Let $\phi : (-\infty, M) \rightarrow (0, \infty)$, $M > 0$, be continuously differentiable, monotone increasing, convex, and satisfy $\lim_{u \to M^-} \phi(u) = \infty$; and let $\varepsilon > 0$. In this chapter the following initial-boundary value problem is considered:

$$u_{tt} = \Delta_n u + \varepsilon \phi(u) \quad \text{in} \quad D \times (0, T)$$

$$u = 0 \quad \text{on} \quad \partial D \times (0, T)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x) \quad \text{in} \quad D,$$

where $\Delta_n$ denotes the $n$-dimensional Laplacian.

Let (Al) denote problem (An) with $D = (0, 1)$. When $\phi(u) = \frac{1}{M-u}$, a solution of (Al) has a physical interpretation as describing the motion of a wire composed of a magnetic material and carrying an electric current, in the presence of another current-carrying wire ([17]). Chang and Levine [4] showed that for suitably regular initial data, problem (Al) has a unique local piecewise $C^2$ solution $u$ which can be continued as long as $u < M$. They also established the existence of numbers $\varepsilon_1 > \varepsilon_0 > 0$ such that
(a) if $\varepsilon > \varepsilon_1$, then for some finite $T > 0$,

$$\lim_{t \rightarrow T} - \left[ \sup_{0 < x < 1} u(x,t) \right] = M.$$ 

Hence one of $u_{tt}, u_{xx}$ becomes infinite on $[0,1] \times [0, T)$;

(b) if $0 < \varepsilon < \varepsilon_0$, and the initial data $u_0, v_0$ are appropriately restricted, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|u(x,t)| < M - \delta \text{ on } [0,1] \times [0, \infty).$$

Note that by applying to (Al) the change of scale $x' = Lx$, $t' = Lt$, $\varepsilon = L^2$, one obtains (Al) with $\varepsilon$ replaced by 1 and $x'$ varying between 0 and L. Results (a) and (b) assert, therefore, that global solutions do not exist for long wires, but do exist for short wires.

If $u$ behaves as in (a), it is said to quench in finite time.

Speaking loosely, one says that a solution of some evolutionary problem quenches in finite (or infinite) time $T$ if some norm of the solution remains bounded, while some norm of one of its derivatives becomes unbounded, on the interval $[0,T)$ ([16]).

For space dimensions $n > 2$ and $\varepsilon$ sufficiently large, solutions of problem (An) also quench in finite time ([4]). However, the proof of (b) in [4] relies strongly on the inequality

$$4 u^2(x,t) < \int_0^1 |u_x(x,t)|^2 \, dx,$$  \hspace{1cm} (4.1)
which guarantees the imbedding of $H^1_0(0,1)$ into $C([0,1])$. In general, no such imbedding $H^1_0(D) \rightarrow C(D)$, or even $H^1_0(D) \rightarrow L^\infty(D)$, is possible in higher dimensions, and the question of existence of global solutions of (An) for $n > 2$ remains open.

If instead of (An) one considers the abstract problem

$$\frac{d^2u}{dt^2} + Au = \varepsilon \phi(u) \quad 0 < t < \infty$$

(B)

$$u(0) = u_0 \in V, \quad u'(0) = v_0 \in L^2(D),$$

where $V \subseteq L^2(D)$ imbeds in $L^\infty(D)$ and $A$ is an operator of elliptic type mapping $V$ into its dual, then a global existence theorem for $\varepsilon$ sufficiently small may be proved; see Levine and Smiley [23]. Their results apply, for example, to solutions of (B) when $D$ is the interior of a rectangle in $\mathbb{R}^2$, $u(x,t) = \Delta_2u(x,t) = 0$ on $\partial D \times [0,\infty)$, $A$ is the biharmonic operator $\Delta_2^2$, and $V = H^2(D) \cap H^1_0(D)$.

Acker and Walter [1] have proved a higher-dimensional global existence theorem for small $\varepsilon$ for solutions of

$$u_t = \Delta_n u + \varepsilon \phi(u) \quad \text{in } D \times (0,T)$$

(C)

$$u = 0 \quad \text{on } \partial D \times (0,T)$$

$$u(x,0) = u_0(x) \quad \text{in } D.$$

Their proof relies on consequences of the maximum principle for parabolic problems, which are available only in much weaker forms for hyperbolic problems. Hyperbolic problems in which the driving term
\[ e^* \] appears in a boundary condition instead of in the differential equation have also been studied ([17]), but the question of global existence of solutions in space dimensions higher than one also remains unanswered. For a comprehensive survey of the literature on quenching see Levine [16].

In this chapter, problem (An) is shown to have a unique local continuous solution \( u \) in low dimensions for small \( \varepsilon \) under appropriate assumptions on \( \phi \), \( u_0 \), \( v_0 \), and \( \mathcal{D} \), which can be continued as long as \( u < M \). It is also shown that for any \( \varepsilon > 0 \), there exists no potential well about any equilibrium solution of (An), so that a proof of global existence along the lines of Sattinger [32] is not possible. Nevertheless, an a priori inequality for solutions of (An) similar to (4.1) is shown, via energy considerations, to guarantee global existence. Numerical evidence is given which suggests that such an a priori inequality is sometimes satisfied by solutions of (An).

Theoretical Considerations

Local continuous solutions of (An) for \( n = 2, 3 \) are obtained by applying the abstract theory of Reed [30] to an appropriately modified problem.

In this section \( \phi \) will be assumed to be bounded and uniformly Lipschitz continuous on intervals of the form \((- \infty, M-d]\), \( d > 0 \). For \( 0 < d < M \) define
Then by suitably defining $\phi_\delta$ on the interval $(M - \delta, M - \delta/2)$, one may arrange that $\phi_\delta \in C^1(\mathbb{R})$ and $\phi'$ is bounded and uniformly Lipschitz continuous on $\mathbb{R}$. Let $(An, \delta)$ represent problem $(An)$ with $\phi$ replaced by $\phi_\delta$.

It is assumed that problem $(An)$ has a stationary solution $f \in C^2(D)$, which is analytic in $D$ and satisfies

$$\Delta f + \epsilon \phi(f) = 0 \quad \text{in } D,$$

$$f = 0 \quad \text{on } \partial D.$$

Such is indeed the case when, e.g., $D$ is a ball in $\mathbb{R}^n$ and

$$\phi(u) = (M + au)^\beta, \quad \alpha, \beta < 0; \quad \text{see Joseph and Lundgren [11].}$$

Applying the transformation $\tilde{u} = u - f$ to problem $(An, \delta)$, one obtains the problem

$$\tilde{u}_{tt} = \Delta_n \tilde{u} + \epsilon \psi_\delta(x, \tilde{u}) \quad \text{in } D \times (0, T),$$

$$\tilde{u} = 0 \quad \text{on } \partial D \times (0, T) \quad (4.2)$$

$$\tilde{u}(x, 0) = u_0(x) - f(x), \quad \tilde{u}_t(x, 0) = v_0(x) \quad \text{in } D,$$

where $\psi_\delta(x, u) = \phi_\delta(u + f(x)) - \phi_\delta(f(x))$. By setting
\[ \tilde{v} = \tilde{u}_t, \quad \eta = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} u_0 - f \\ v_0 \end{pmatrix}, \]

\[ F(\eta) = \begin{pmatrix} 0 \\ \epsilon \phi_\delta(x, \tilde{u}) \end{pmatrix}, \quad \text{and} \quad A = -\begin{pmatrix} 0 & I \\ \Lambda & 0 \end{pmatrix}, \]

one may write (4.2) as the equivalent system

\[ \eta'(t) = -A\eta(t) + F(\eta(t)), \quad 0 < t < T \]

\[ \eta(0) = \eta_0. \quad (4.3) \]

Let \( H \) denote the Hilbert space of real-valued functions

\( H_0^1(D) \oplus L_2(D) \), with inner product

\[ \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle = \int_D vu \cdot vw \, dx + \int_D vz \, dx. \quad (4.4) \]

Provided \( D \) is of class \( C^2 \), \( A \) is a closed skew-adjoint operator on \( H \) with domain \( \text{Dom}(A) \equiv [H^2(D) \cap H_0^1(D)] \oplus H_0^1(D) \), and generates on \( H \) the continuous one-parameter group \( W(t) = e^{-tA} \). Therefore (4.3) can be reformulated as the integral equation problem

\[ \eta(t) = e^{-tA}\eta_0 + \int_0^t e^{-(t-s)A}F(\eta(s)) \, ds, \quad (4.5) \]

which may then be solved by the contraction mapping principle.
The following theorem summarizes results proved in [30].

**Theorem 4.1:** Let \( \mathcal{D} \) be of class \( C^2 \), and for a fixed \( m > 1 \) let \( \eta_0 \) be in \( \text{Dom}(A^m) \). Let \( \| \cdot \| \) denote the norm on \( H \) induced by (4.4). Suppose that for all \( 1 < j < m, \)

\[
\| A^j F(\eta) \| < C(\| \eta \|, \ldots, \| A^{j-1} \eta \|)\| A^j \eta \|, \tag{4.6}
\]

and

\[
\| A^j (F(\eta) - F(\nu)) \| < \tag{4.7}
\]

\[
C(\| \eta \|, \| \nu \|, \ldots, \| A^j \eta \|, \| A^j \nu \|)\| A^j (\eta - \nu) \|
\]

for all \( \eta, \nu \) in \( \text{Dom}(A^j) \), where the constants \( C \) are nondecreasing, everywhere finite functions of all their variables. Then there is a \( T > 0 \) such that (4.3) has a unique continuously differentiable solution \( \eta(t) \) for \( 0 < t < T \), with \( \phi(t) \) in \( \text{Dom}(A^m) \) for all \( 0 < t < T \).

If in addition \( \| \eta(t) \| \) is bounded on any finite interval of existence of \( \eta(t) \), then \( \eta(t) \) exists globally in time.

Let \( \| \cdot \|_p \) denote the norm in \( L_p(D) \), and let \( K_1, K_1, \ldots \) denote positive constants. If \( \eta \in \text{Dom}(A) \) has first component \( u \), then

\[
\| AF(\eta) \|^2 = \varepsilon^2 \int_D |\nabla \phi(\eta(x,u))|^2 \, dx \]

\[
< K_1 (\| \phi(x,u+f) \|_2^2 + \| \phi(x,u+f) - \phi(x,f) \|_2^2] \]

< \ K_2 (\| u \|_2^2 + \| u \|_2^2] < K_3 \| u \|_2^2 < K_4 \| \eta \|_2^2 ,
\]
where use was made of the boundedness and uniform Lipschitz continuity of \( \phi'_0 \) on \( \mathbb{R} \), the boundedness of \( |\nabla f| \) on \( \overline{D} \), and the Poincaré inequalities

\[
\|u\|_2 < K\|\nabla u\|_2 < K^2\|\Delta_n u\|_2, \quad (4.8)
\]

valid for \( u \in H^2(D) \cap H^1_0(D) \). This establishes (4.6) with \( j = 1 \).

If \( \eta, \nu \in \text{Dom}(A) \) have respective first components \( u, w \), then by applying Hölder's inequality, the Sobolev inequality

\[
\|u\|_p < C \|\nabla u\|_2, \quad 1 < p < \frac{2n}{n-2} \quad (4.9)
\]

valid for \( u \in H^1(D) \), and (4.8), one obtains

\[
\|A(F(\eta) - F(\nu))\| \\
< K_1 [\|u-w\|_2^2 + \|\nabla(u-w)\|_2^2 + \|\nabla u\|_4^2 \|\nabla w\|_4^2] \\
< K_2 \|\Delta_n (u-w)\|_2^2 + K_3 \|\Delta_n u\|_2^2 \|\nabla(u-w)\|_2^2 \\
< K_4 (1 + \|\Delta u\|_2^2) \|A(\eta-u)\|_2^2.
\]

This establishes (4.7) with \( j = 1 \) for \( 1 < n < 4 \).

From the integral equation (4.5), from the fact that

\[ |\phi'_0(x,u)| < C|u| \]

for some constant \( C > 0 \) for all \( x \in \overline{D} \) and \( u \in \mathbb{R} \), and from (4.8), one may obtain
for all $t$ in the existence interval for $\eta(t)$.

The above arguments and Theorem 4.1 together yield the following

**Corollary 4.1:** Let $D \subseteq \mathbb{R}^n$ with $1 < n < 4$ have a $C^2$ boundary, and suppose that $\eta_0 \in \text{Dom}(A) = [H^2(D) \cap H^1_0(D)] \oplus H^1(D)$. Then for all $0 < \delta < M$ and for all sufficiently small $\varepsilon > 0$, problem (4.3) has a unique continuously differentiable solution $\eta(t)$ which is global in time and remains in $\text{Dom}(A)$ for all $t > 0$.

Four remarks are in order. Note that the proof of Corollary 4.1 does not require $\phi$ to be convex. Theorem 4.1 cannot be used to obtain greater regularity of solutions of (4.3) due to the lack of a Poincaré inequality of the form

$$
\|\nabla^j u\|_2 \leq C\|\nabla^{j+1} u\|_2, \quad u \in H^{j+1}(D) \cap H^1_0(D)
$$

for $j > 2$. The first component $\vec{u}$ of the solution $\eta = \begin{pmatrix} u \\ v_1 \\ \vdots \\ v_{n-1} \\ \bar{u} \end{pmatrix}$ in
Corollary 4.1 with \( n = 2 \) or \( 3 \) is continuous on \( \overline{D} \times [0, \infty) \) in view of the imbedding inequality

\[
\max_{x \in \overline{D}} |\tilde{u}(x,t)| < C \Delta_n \tilde{u}(\cdot,t)_{L^2}
\]

valid for \( \tilde{u}(\cdot,t) \in H^2(D) \cap H^1_0(D), \ 1 < n < 3, \) and the continuity in time of \( \tilde{u} \) in the norm on \( H^2(D) \cap H^1_0(D). \) Since \( H^2(D) \) imbeds in \( C(\overline{D}) \) only for \( n = 1, 2, 3 \) ([6, p. 30]), any extension of Corollary 4.1 to cases \( n > 5 \) would not be useful for the purposes of this paper.

Now suppose that the solution \( n \) in Corollary 4.1 with \( n = 2 \) or \( 3 \) begins with \( \max_{X \in D} [u_0(x)] < M - \delta. \) If \( \max_{x \in \overline{D}} [\tilde{u}(x,t) + f(x)] < M - \delta \) for all \( t > 0, \) then \( u = \tilde{u} + f \) is a global solution of problem (An). Otherwise there is a first time \( T > 0 \) at which \( \max_{x \in \overline{D}} [\tilde{u}(x,T) + f(x)] = M - \delta; \) by choosing \( 0 < \delta_1 < \delta \) and applying Corollary 4.1 to problem (4.3) with \( \delta \) replaced by \( \delta_1, \) one may extend \( \tilde{u} + f \) uniquely to an interval \( [0, T_1) \) with \( T_1 > T \) such that \( \max_{x \in \overline{D}} [\tilde{u}(x,t) + f(x)] < M - \delta_1 \) for \( 0 < t < T_1, \) and \( u = \tilde{u} + f \) solves (An) on \( \overline{D} \times [0, T_1). \) This argument establishes the following

**Corollary 4.2:** Let \( D \subseteq \mathbb{R}^n \) with \( n = 2 \) or \( 3 \) have a \( C^2 \) boundary, and suppose that \( \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \text{Dom}(A) \) with \( \max_{x \in \overline{D}} u_0(x) < M. \) Then for all sufficiently small \( \varepsilon > 0, \) the system form of problem (An)
\[
\begin{align*}
\begin{pmatrix} u \\ u_t \end{pmatrix}' &= -A \begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon \phi(u) \end{pmatrix}, \quad 0 < t < T \\
\begin{pmatrix} u \\ u_t \end{pmatrix} \bigg|_{t=0} &= \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}
\end{align*}
\]

has a unique solution \( \begin{pmatrix} u \\ u_t \end{pmatrix} \in \text{Dom}(A) \) on some time interval \([0,T)\) which is continuously differentiable in time in the norm on \( H \). The solution can be continued as long as \( \max_{x \in \overline{D}} u(x,t) < M \).

Define \( \mathfrak{u}(u) = \int_0^u \phi(s) \, ds \). A solution \( u \) of (An) with the regularity properties given in Corollary 4.2 satisfies the energy equality

\[
E(t) = \frac{1}{2} \int_D |u_t|^2 \, dx + J(u) = E(0),
\]

where

\[
J(u) = \frac{1}{2} \int_D |\nabla u|^2 \, dx - \varepsilon \int_D \phi(u) \, dx
\]

represents the potential energy of \( u \) at time \( t \).

By defining \( j(\lambda) = J(f + \lambda u) \) for \( \lambda > 0 \), one may examine the behavior of \( J \) along rays emanating from the stationary solution \( f \) of problem (An) in the function space \( H_0^1(D) \). Note
\[ j'(0) = \int_D \nabla f \cdot \nabla u \, dx - \epsilon \int_D \phi(f) u \, dx \]
\[ = \int_D \nabla f \cdot \nabla u \, dx + \int_D (\Delta f) u \, dx = 0. \]

Let \( f_\infty = \max_{x \in D} f(x) \); then by (4.8),
\[ j''(0) = \int_D |\nabla u|^2 \, dx - \epsilon \int_D \phi'(f) u^2 \, dx \]
\[ > \int_D |\nabla u|^2 \, dx - \epsilon \phi'(f_\infty) \int_D u^2 \, dx \]
\[ > (1 - \epsilon K^2 \phi'(f_\infty)) \int_D |\nabla u|^2 \, dx. \]

Suppose \( f \) satisfies \( f_\infty + 0 \) as \( \epsilon \to 0^+ \). (Equilibrium solutions with this property do exist; see [11]). Then for all sufficiently small \( \epsilon \) one has \( 1 - \epsilon K^2 \phi'(f_\infty) > 0 \), and hence \( j''(0) > 0 \); i.e., \( J \) is convex along rays emanating from \( f \). This is a necessary condition that \( f \) be a local minimum of \( J \) in \( H_0^1(D) \).

Nevertheless, as the following results show, there exists no potential well in the function space \( H_0^1(D) \) about any stationary solution of (An) for any \( \epsilon > 0 \) and any \( n > 2 \).

**Lemma 4.1**: Let \( \epsilon > 0 \) be fixed, and let \( f \) be an equilibrium solution of (An), \( n > 2 \). Choose \( x_0 \in D \) such that \( f(x_0) = f_\infty \). Then for any \( \delta > 0 \) one may find a ball \( B_\delta \subseteq D \) with center \( x_0 \), and functions \( \{ w_\lambda : f_\infty < \lambda < M \} \subseteq C_0^\infty(D) \), such that \( w_\lambda = \lambda - f \) on \( B_\delta \) and
\[ \int_D |\nabla w_\lambda|^2 \, dx < \delta \] for all \( f_\infty < \lambda < M \).
Proof: Let \( B(x_0, a) \), \( B(x_0, b) \) denote concentric open balls in \( \mathbb{R}^n \) with center \( x_0 \) and respective radii \( 0 < a < b \). By Friedman ([6, p. 9]) there exists a \( \zeta \in C^\infty(\mathbb{R}^n) \) such that \( \zeta = 1 \) in \( B(x_0, a) \), \( 0 < \zeta < 1 \) in \( B(x_0, b) - B(x_0, a) \), and \( \zeta = 0 \) outside \( B(x_0, b) \). The function \( \zeta \) will be called a \( C^\infty \) cutoff from \( B(x_0, a) \) to \( B(x_0, b) \). Note that for \( 0 < \rho < 1 \), \( \zeta_\rho(x) \equiv \zeta(\frac{x}{\rho}) \) is a \( C^\infty \) cutoff from \( B(x_0, a_\rho) \) to \( B(x_0, b_\rho) \) which satisfies \( |\nabla \zeta_\rho(x)| < \frac{K_0}{\rho} \) for all \( x \in \mathbb{R}^n \), where \( K_0 > 0 \) depends on \( a, b \), but is independent of \( \rho \).

The proof for \( n > 3 \) proceeds as follows. Choose \( 0 < a < b \), and for each \( f_\infty < \lambda < M \) and each \( \rho > 0 \) define

\[
\omega_{\lambda, \rho} = \zeta_\rho(\lambda - f).
\]

Then for all \( \rho \) sufficiently small \( \omega_{\lambda, \rho} \in C^\infty_0(D) \), and \( \omega_{\lambda, \rho} = \lambda - f \) on \( B(x_0, a_\rho) \). Now \( f \) is nonnegative on \( \overline{D} \) by the maximum principle, so for \( f_\infty < \lambda < M \)

\[
\int_D |\nabla \omega_{\lambda, \rho}|^2 \, dx = \int_{B(x_0, b_\rho)} [|\nabla \zeta_\rho|^2|\lambda-f|^2 + |\zeta_\rho|^2|\nabla f|^2] \, dx < 2V_n(K_0^2 \lambda^2 + \rho^2 \max_D |\nabla f|^2) \rho^{n-2},
\]

where \( V_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). By taking \( \rho = \rho_0 > 0 \) sufficiently small, one may arrange that

\[
\int_D |\nabla \omega_{\lambda, \rho_0}|^2 \, dx < \delta \text{ for all } f_\infty < \lambda < M.
\]
One may then take $B_\lambda$ to be $B(x_0,\lambda p_0)$, $w_\lambda$ to be $w_\lambda p_0$ for $f_\infty < \lambda < M$.

When $n = 2$, choose fixed $b > a > 0$ so small that $B(x_0,b) \subseteq D$. Let $r = |x - x_0|$ denote the distance from $x_0$ to $x$ in $\mathbb{R}^2$. For $f_\infty < \lambda < M$ choose $B \equiv \lambda - f_\infty > A > 0$, and choose $0 < 2p < a$. For $\alpha < 0$ let

$$C = \frac{B - A}{\lambda - b}, \quad D = \frac{A\alpha - B\alpha}{\lambda - b},$$

so that $C\alpha + D = B$ and $C\alpha + D = A$. Let $\zeta$ denote a $C^\infty$ cutoff from $B(x_0,1)$ to $B(x_0,2)$, and define $\zeta_\alpha(r) = \zeta(\frac{r}{\lambda})$. Let $\zeta_{a,b}$ be a $C^\infty$ cutoff from $B(x_0,a)$ to $B(x_0,b)$. For $f_\infty < \lambda < M$ define

$$w_{\lambda,\rho}(x) = \zeta_{a,b}(r)[\zeta_\rho(r)(\lambda - f(x) - Cr^\alpha - D) + Cr^\alpha + D].$$

Then setting

$$I_1 = \int_{B(x_0,2\rho)} |\nabla \psi|^2 \, dx, \quad I_2 = \int_{B(x_0,2\rho) - B(x_0,\rho)} |\nabla \rho|\, |\lambda - f(x) - Cr^\alpha - D|^2 \, dx$$

$$I_3 = \int_\rho^b |\nabla(Cr^\alpha)|^2 \, r \, dr$$

$$I_4 = \int_a^b |\nabla \zeta_{a,b}|^2 |Cr^\alpha + D|^2 \, r \, dr.$$
one may obtain that
\[ \int_D |\nabla \chi_{\lambda, \rho}|^2 \, dx < K_1 \sum_{j=1}^{4} I_j \]
for an absolute constant \( K_1 > 0 \).

Since \(|\nabla f|\) is bounded on \( B(x_0, b) \), one may choose a \( \rho_0 > 0 \) independent of \( A, \alpha, \) and \( \lambda \) such that \( I_4 < \delta / (4K_1) \) for all \( 0 < \rho < \rho_0 \).

Using the facts that \( C^{\alpha} + D \) is positive and decreasing for \( 0 < r < b \), and that \( |\nabla \chi_{\rho}| < \frac{K_0}{\rho} \) for some absolute constant \( K_0 > 0 \), one may show that
\[ I_2 < 6\pi K_0^2 \left[ c^2(1-2\alpha)^2 \rho^{2\alpha} + \max_{B(x_0, 2\rho)} |f(x) - f_\infty|^2 \right], \]
\[ I_3 = \frac{a\, c^2}{2} (b^{2\alpha} - \rho^{2\alpha}), \]
\[ I_4 < K_2 [A^2 + c^2(a^\alpha - b^\alpha)^2], \]
for some positive absolute constant \( K_2 \). It is a simple matter to show that the expressions \( c^2(1-2\alpha)^2 \rho^{2\alpha} \), \( \frac{ac^2}{2} (b^{2\alpha} - \rho^{2\alpha}) \), \( c^2(a^\alpha - b^\alpha)^2 \) can be made arbitrarily small independent of \( \lambda, f_\infty < \lambda < M \), by taking \( |a| > 0, \rho > 0 \) sufficiently small. Hence by choosing \( A = A_0, \rho = \rho_1 < \rho_0, \alpha = \alpha_0 \) all sufficiently close to zero, one may arrange that \( I_j < \frac{\delta}{4K_1} \) for \( 1 < j < 4 \) for all \( f_\infty < \lambda < M \). Set \( B_M = B(x_0, \rho_1) \), and \( \chi_\lambda = \chi_{\lambda, \rho_1} \) with \( A = A_0, \alpha = \alpha_0 \). Then
\[ \int_D |\nabla \chi_{\lambda}|^2 \, dx < \delta \] for all \( f_\infty < \lambda < M \).
In particular, for any $\varepsilon > 0$, $n > 2$, there are functions
\[ v_\varepsilon = w_\varepsilon + f \]
with essential supremum equal to $M$ in any neighborhood of $f$ in $H^1_0(D)$.

**Lemma 4.2:** Let $y(t)$ denote the solution of the ordinary initial-value problem

\[
\begin{align*}
  y &= \varepsilon \phi(y) & t > 0 \\
  y(0) &= y_0, \quad \dot{y}(0) = 0,
\end{align*}
\]

(4.11)

where $y_0 < M$. Then there is a finite $T_\varepsilon > 0$ such that
\[ \lim_{t \to T_\varepsilon} y(t) = M; \ 	ext{i.e., } y \text{ quenches in finite time.} \]

As $y_0 \to M^-$,
\[ T_\varepsilon \to 0^+. \]

**Proof:** The uniform Lipschitz continuity of $\phi$ on intervals of the form $(-\infty, M-\delta]$ for $\delta > 0$, guarantees that (4.11) has a unique local $C^2$ solution $y(t)$ which can be continued as long as $y(t) < M$. On the existence interval $[0, T_\varepsilon)$ for $y$, $\dot{y} > 0$ and hence $\ddot{y}(t)$ is strictly increasing in $t$. Since $\ddot{y}(0) = 0$, $\dot{y}(t) > 0$ and hence $y(t) > y_0$ for $0 < t < T_\varepsilon$. Since $\phi$ is strictly increasing,

\[ y(t) > \varepsilon \phi(y_0), \]

so that

\[ y(t) - y_0 > \varepsilon \phi(y_0) t^2, \]
for $0 < t < T_q$. Hence

$$\frac{\sqrt{M - y_0}}{\epsilon \Phi(y_0)} > T_q.$$ 

Clearly $T_q \to 0^+$ as $y_0 \to M^-$. 

**Theorem 4.2:** Let $\epsilon, \delta, T_0$ be any fixed positive numbers, and let $k$ be any nonnegative integer. Let $f \in C^k(\overline{D})$ be an equilibrium solution of (An) with $n > 2$. Then there exists $u_0 \in C^k(\overline{D})$ with $u_0 = 0$ on $\partial D$, $\max_{\overline{D}} u_0 < M$, and $\int_D |\nabla (u_0 - f)|^2 \, dx < \delta$, such that the solution $u$ of problem (An) with $v_0 = 0$ quenches in finite time $T < T_0$.

**Proof:** Let $w_\lambda \in C^\infty(\overline{D})$ be the functions satisfying $w_\lambda = \lambda - f$ on $B_M$, $\int_D |\nabla w_\lambda|^2 \, dx < \delta$ for all $f_\infty < \lambda < M$, whose existence is guaranteed by Lemma 4.1. Let $\rho > 0$ denote the radius of $B_M$. By Lemma 4.2, one may choose $y_0 < M$ so close to $M$ that the solution $y = y(t,y_0)$ of (4.11) quenches in time $T_q < \min\{T_0, \rho\}$.

Define $u_0 = w_\lambda + f$; then $u_0 \in C^k(\overline{D})$, $u_0 = 0$ on $\partial D$, $u_0 = y_0$ on $B_{M'}$ and $\int_D |\nabla (u_0 - f)|^2 \, dx < \delta$. If $u$ denotes the solution of problem (An) with this $u_0$ and with $v_0 = 0$, one has

$$u(x,t) = y(t,y_0)$$

for all $(x,t)$ in the retrograde characteristic cone with vertex $(x_0, \rho)$ and base $B_M \times \{0\}$. Hence $u$ must quench in time $T < T_q$. 


The idea of comparing solutions inside retrograde characteristic cones in the half-space $t > 0$ was first used by Keller [13] to show pointwise blow-up in finite time of solutions of $u_{tt} = c^2 \Delta u + f(u)$ for certain $f \in C^2(\mathbb{R})$.

Suppose $u$ is a continuous solution of (An) with sufficient regularity to satisfy energy equality (4.10) (or the inequality $E(t) < E(0)$) for all $t$ in its existence interval. Define

$$u_\infty^+ = u_\infty^+(t) \equiv \max_{x \in D} (u(x,t),0), \quad \gamma = \gamma(t) \equiv (u_\infty^+)^{-2} \int_D |\nabla u|^2 \, dx.$$ Then

$$\gamma < 2(u_\infty^+)^{-2} [E(0) + \varepsilon \mu(D)\Phi(u_\infty^+)] \equiv g(u_\infty^+), \quad (4.12)$$

where $\mu(D)$ denotes the $n$-dimensional Lebesgue measure of $D$.

When $E(0) > 0$, $g(s)$ achieves a positive absolute minimum $g_m = g(s_0)$ on the interval $(0,M]$. Note that $g_0^2$, and hence $g_m$ itself, can be made arbitrarily small by taking both $\varepsilon > 0$ and $E(0) > 0$ sufficiently small.

If $u$ satisfies an a priori inequality of the form

$$\gamma(t) > g_m, \quad t > 0 \quad (4.13)$$

and begins in the region $R$ depicted in Figure 1 (i.e. with $u_0^+ < s_0$), then $u_\infty^+$ remains bounded away from $M$ for all time, and $u$ will be a global solution of (An).

When $n = 1$ one has $\gamma(t) > 4$ for all $t > 0$ by (4.1), and these observations underlie the proof of the result (b) of the previous section in [4].
Figure 1. An invariant region for solutions of \((An)\) satisfying \((4.13)\).

**Numerical Results**

An explicit finite-difference scheme was used to approximate solutions \(u = u(r,t)\) of

\[
\begin{align*}
    u_{tt} &= u_{rr} + \frac{n-1}{r} u_r + \varepsilon \phi(u), \\
    u_r(0,t) &= u(l,t) = 0, \\
    u(t,0) &= u_0(r), \\
    u_t(r,0) &= v_0(r),
\end{align*}
\]

which are radial solutions of \((An)\) when \(r = |x|\) and \(D\) is the unit ball centered at the origin in \(\mathbb{R}^n\). (Note that to ensure compatibility, the initial data must satisfy \(u_{0,r}(0) = v_{0,r}(0) = u_0(l) = v_0(l) = 0\).)
The difference scheme used is adapted from John [10, pp. 172-174]. Divide the interval \([0,1]\) into \(N\) subintervals of equal length \(h = \frac{1}{N}\), and let \(k\) denote the stepsize in time, with

\[
\lambda \equiv \frac{k}{h} < 1. \tag{4.17}
\]

For \(1 < i < N+1, \ j > 0\), \(w = w(r,t)\) define

\[
w_{ij} = w((i-1)h, jk).
\]

Let \(\delta_t\) denote the forward divided difference operator

\[
\delta_t w_{ij} = \frac{1}{k} \left[ w_{i,j+1} - \frac{1}{2} \left( w_{i+1,j} + w_{i-1,j} \right) \right],
\]

with space averaging in the lower step; and let \(\delta_r\) denote the central divided difference operator

\[
\delta_r w_{ij} = \frac{1}{2h} \left[ w_{i+1,j} - w_{i-1,j} \right].
\]

For \(2 < i < N-1\) and \(j > 0\), (4.14) was replaced by the difference equation

\[
\delta_t^2 w_{ij} = \delta_r^2 w_{ij} + \frac{n-1}{(i-1)h} \delta_r w_{ij} + \varepsilon \phi(w_{ij}). \tag{4.18}
\]
(Values \( w_{0,j}, w_{-1,j}, \ldots \) are interpreted by extending \( w \) as an even function of \( r \) through \( r = 0 \)). When \( i = 1 \), (4.14) was replaced by

\[
\delta_t^2 w_{1j} = n \delta_r^2 w_{1j} + \varepsilon \phi(w_{1j}). \quad (4.19)
\]

When \( i = N \), backward differences in \( r \) must be used, and space averaging abandoned, wherever necessary to avoid going past \( r = 1 \); consequently (4.14) was replaced by

\[
\frac{1}{k} \delta_t (w_{N,j+1} - w_{N,j}) = \frac{1}{h} \delta_r (w_{N,j} - w_{N-1,j}) + \frac{n-1}{(i-1)h} \delta_r w_{N,j} + \varepsilon \phi(w_{N,j}). \quad (4.20)
\]

Of course, when \( i = N + 1 \) the boundary condition \( u(1,t) = 0 \) of (4.15) translates to

\[
w_{N+1,j+2} = 0. \quad (4.21)
\]

Since each of (4.18-21) can be solved for \( w_{i,j+2} \) in terms of \( w_{i0,j0} \), with \( j_0 < j+2 \), the scheme is explicit. The Taylor series approximation

\[
w_{i1} = (u_0)_i + k(u_0)_i + \frac{1}{2} k^2 \delta_r^2 (u_0)_i + \frac{n-1}{(i-1)h} \delta_r (u_0)_i + \varepsilon \phi(u_0)_i + \ldots
\]
was used to obtain the values \( w_{i,1} \) (with the obvious modification when \( i = 1 \)).

The difference scheme used is stable, consistent, and convergent when applied to the pure initial-value problem obtained by linearizing (4.14) about a stationary solution \( f \). Grave difficulties are encountered, however, in attempts to prove consistency and convergence for (4.18-21), due to the boundary conditions and the presence of the nonlinearity \( \phi \). In particular, one is unable to derive useful upper bounds for higher difference quotients of \( w \). This is analogous to the difficulties encountered with the abstract approach to the differential problem in the previous section. Therefore for the numerical tests the following checks and safeguards were implemented:

(a) the Courant-Friedrichs-Lewy condition (4.17), which is a necessary condition for stability, was ensured to be satisfied by taking \( \lambda = \frac{1}{4} \) in all tests;

(b) stationary solutions \( f \) of (Rn) were approximated by a procedure described below. The difference scheme (4.18-21) was then applied with \( u_0 = f, \ v_0 = 0 \) as a check of the computer code. Since the approximate stationary solutions are not exact, these checks (as well as checks with \( v_0 = 0, \ u_0 = \text{small perturbation of } f \)) served as empirical evidence of the scheme's stability;

(c) the convergence of the scheme (4.18-21) was checked empirically for several examples by letting \( h, k \to 0 \) while keeping \( \lambda = \frac{1}{4} \); and
(d) the energy equality (4.10) was checked for the difference scheme at selected time steps using Simpson's rule to approximate the integrals involved.

Double-precision arithmetic was used for all computations. The experiments were performed on a National Advanced Systems AS/9160 computer with MVS/SP operating system.

To isolate the effects of the term $\varepsilon \phi(u)$, solutions of (4.18-21) with $u_0 = v_0 = 0$ were computed in dimensions $n = 2, 3, \ldots$ and 7. The behavior of such solutions agrees qualitatively with behavior reported in [4] for solutions in the case $n = 1$. In particular, in each dimension $n$ considered there appears to be an $\varepsilon_n^* > 0$ such that solutions quench in finite time when $\varepsilon > \varepsilon_n^*$, and do not quench (even in infinite time) when $\varepsilon < \varepsilon_n^*$. For $\varepsilon < \varepsilon_n^*$ the solution displays a sequence of relative maxima which occur along the line $r = 0$, $t > 0$; the first such relative maximum appears to be an absolute maximum, which approaches 1 from below as $\varepsilon$ approaches $\varepsilon_n^*$ from below.

The following table lists values of $\varepsilon_n$ obtained when $\phi(u) = (1-u)^{-1}$; the value of $\varepsilon_1$ is taken from [4]. Figures 2 and 3 contrast the behavior of a solution $w$ of (4.18-21) for values of $\varepsilon$ respectively greater than or less than $\varepsilon_n$. Both figures were generated using $\phi(u) = (1-u)^{-1}$, $n=2$, and $h = 1/200$. In Figure 2, $\varepsilon = 1.5 > \varepsilon_2^*$, and the solution quenches in time $T = 1.01$; while in Figure 3, $\varepsilon = 0.9 < \varepsilon_2^*$, and the solution is displayed for $0 < t < 8$. 
Table. Values of $\varepsilon_n$, $\varepsilon_*$ for $\phi(u) = (1-u)^{-1}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\varepsilon_n$</th>
<th>$\varepsilon_*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.341</td>
<td>0.383</td>
</tr>
<tr>
<td>2</td>
<td>1.017</td>
<td>1.309</td>
</tr>
<tr>
<td>3</td>
<td>1.520</td>
<td>2.139</td>
</tr>
<tr>
<td>7</td>
<td>2.563</td>
<td>6.000</td>
</tr>
</tbody>
</table>

Figure 2. A solution of (4.18-21) with $\varepsilon > \varepsilon_n$. 
Figure 3. A solution of (4.18-21) with $\varepsilon < \varepsilon_n$. 
The stability of the solution \( f \) of the stationary problem

\[
f''(r) + \frac{n-1}{r} f'(r) + \varepsilon \psi(f(r)) = 0, \quad 0 < r < 1 \quad (4.22)
\]

\[f'(0) = f(1) = 0 \quad (4.23)\]

satisfying \( f \to 0^+ \) as \( \varepsilon \to 0^+ \), was also investigated. The solutions \( f \) were obtained via the shooting method; i.e., the boundary conditions (4.23) are replaced with "initial" conditions

\[f'(0) = 0, \quad f(0) = A, \quad (4.24)\]

and \( A \) is chosen so that the solution \( f(r,A) \) of (4.22), (4.24) (obtained via the classical fourth-order Runge-Kutta method) satisfies \( f(1,A) = 0 \). Since \( A \) is a root of the nonlinear equation \( f(1,A) = 0 \), it may be found using an iterative procedure such as Newton's method. The secant method was used for the numerical experiments since it does not require the extra calculation of \( \frac{\partial f}{\partial A} \).

Care must be taken in the choice of \( A = A_0 \) to begin the shooting procedure, since solutions of (4.22),(4.24) are not always unique.

Indeed, Joseph and Lundgren showed in [11] that for \( \psi(u) = (1+au)^\beta \) with \( \alpha, \beta < 0, \quad \tau = \frac{2}{\beta-1}, \quad \varepsilon = \frac{1}{\alpha} (n-2-\tau), \) and \( f(\beta) = 2\beta + 2(2\varepsilon \tau)^{1/2} \),

(a) there is an \( \varepsilon_* > 0 \) such that positive solutions of (4.22), (4.23) do not exist when \( \varepsilon > \varepsilon_* \);
(b) for $3 < n < 2 + f(\beta)$ and $\varepsilon_* > \bar{\varepsilon}$, there are a large but finite number of positive solutions when $\varepsilon < \varepsilon$ is close to $\bar{\varepsilon}$, and a countably infinite number of solutions when $\varepsilon = \bar{\varepsilon}$; and

(c) for $n > 2 + f(\beta)$ and $\varepsilon = \varepsilon_*$, there is exactly one positive solution when $\varepsilon < \varepsilon_*$.

See the previous table for values of $\varepsilon_*$ when $\phi(u) = (1-u)^{-1}$ and $n = 1, 2, 3, 7$.

Bifurcation diagrams plotting $\varepsilon$ as a function of $f_\infty$ were generated using the procedure described in [11]. It could then be checked that the shooting method above converged to the stationary solution with smallest maximum, $f_\infty$. Figure 4 contains bifurcation diagrams for space dimensions $n = 2, 3, 9$ when $\alpha = -1, \beta = -3$.

Figure 4. Bifurcation diagram for positive solutions of (4.22), (4.23) when $\phi(u) = (1-u)^{-1}$. 
Perturbations $p$ of $f$ used as initial data $u_0$ were of the form

$$p = vf \text{ for } 0 < v < \frac{1}{f_{\infty}}; \text{ or of the form}$$

$$p(r) = \begin{cases} v & 0 < r < r_0, \\ f(r) & r_1 < r < 1 \end{cases}$$

where $0 < r_0 < r_1 < 1$, $f_{\infty} < v < 1$, and $p$ is defined on $r_0 < r < r_1$ to be strictly decreasing and $C^2$ on $[0,1]$. Numerical experiments with initial data $u_0 = p$, $v_0$ chosen so that $E(0) > 0$, indicate that whenever $p$ is sufficiently close to $f$ in sup norm and $E(0)$ is small, the a priori inequality (4.13) is satisfied for all time by solutions of (4.18-21). The following diagrams, Figures 5 and 6, contrast the trajectories of $\gamma(t)$ in quenching and nonquenching cases. Each figure was generated with $n = 3$, $\phi(u) = (1-u)^{-1}$, $u_0 = vf$, $v_0(r) = \mu(1-r^2)$ where $\mu$ is chosen so that $E(0) > 0$, $\epsilon = 0.5$, $h = 1/100$.

Figure 5. Quenching trajectory for $\gamma$ when $v = 0.5$, $E(0) = 2.0$. 
Figure 6. Nonquenching trajectory for $\gamma$ when $v = 0.5$, $E(0) = 0$. 
CHAPTER 5. DISCUSSION

One can more generally require that the initial velocity $V$ of problem (W) belong to $L_2(D)$, at the expense of losing uniqueness of the Fourier expansion of $V$ in terms of the eigenfunctions $\phi$, $\psi$.

The global solutions obtained in [32] and in Chapters 2 and 3 are not proved to be unique. In addition, it remains to be shown whether global solutions have greater regularity under more restrictive assumptions on initial data and geometry; and whether global solutions depend continuously, in some sense, on initial data.

Ball [3] was the first to note that concavity arguments like those of Chapters 2 and 3 do not prove that nonexistence actually occurs by "blow-up" of some norm of the solution. It is an open question whether nonexistence of global solutions of problems (W), (H), and of problems considered in [28], is indeed caused by such blow-up.

The definition of a weak solution of problem (H) given in (1)-(4) of Chapter 3 is nonstandard. One might ask whether the global solution obtained in Chapter 3 satisfies the conventional definition of weak solution (see, e.g., [14, pp. 418-419]).

Potential well arguments similar to those of Chapters 2 and 3 are also applicable to problems of the form
\[ u_{tt} = \Delta u + g(u) \quad \text{in } D_T, \]
\[ u = 0 \quad \text{on } \sigma_T, \]
\[ \frac{3u}{\partial n} = f(u) \quad \text{on } \Sigma_T, \]
\[ u(x,0) = U(x), \quad u_t(x,0) = V(x) \quad \text{in } D, \]

where \( g \) satisfies (i)-(iii) of Chapter 2 with \( 1 < p' + 1 < \gamma' < \frac{2m}{m-2} \).

Here four cases need be considered, depending on whether each of \( f, g \) satisfies (ii)(a) or (ii)(b). When \( f \) satisfies (ii)(a), and \( g \) (ii)(b), one must require that \( f(-s) > -f(s) \) for all \( s < 0 \), in order to ensure that the depth of the potential well is determined by nonzero functions \( u \in H^1_{0\sigma}(D) \) with \( u > 0 \) a.e. (This condition on \( f \) holds when, for example, \( f \) is symmetric through the origin). Similarly, when \( f \) satisfies (ii)(b), and \( g \) (ii)(a), one must require \( g(-s) > -g(s) \) for all \( s < 0 \). When \( f, g \) both satisfy (ii)(a), or both satisfy (ii)(b), no additional restrictions on \( f \) or \( g \) are necessary. The positive depth of the potential well is established by showing that

\[ a = \max \{ \| u \|_{\gamma, D}, \| B^u \|_{\gamma, \Sigma} \} \]

is uniformly bounded below for all appropriately restricted \( u \in H^1_{0\sigma}(D) \) satisfying

\[ Q_0(u) \equiv \| u \|^2_D - \int_D u g(u) \, dx - \int_{\Sigma} f(Bu) \, dS = 0. \]
The extraction of a suitable convergent subsequence in the proof of
global existence requires the compact imbedding of $H^1_0(D)$ into
$L^q(D)$ for $2 < q < \frac{2n}{n-2}$.

In [28] Payne and Sattinger considered the stability properties of
the positive solution (ground state) of $\Delta u + g(u) = 0$ with Dirichlet
boundary conditions. In a pending work Sternberg [36] demonstrates the
instability of higher modes of $u_{tt} = \Delta u + g(u)$. It appears that
problems (H),(W) do not possess higher modes, at least in one dimension.
The question of existence and stability properties of higher modes of
problems (H),(W) in several dimensions is being studied.

Sacks [31] derived decay estimates and described the asymptotic
behavior of solutions of the problem

$$u_t = \Delta (|u|^{m-1} u) + \lambda |u|^{p-1} u \quad \text{in } D_T$$

$$u = 0 \quad \text{on } \partial D_T$$

$$u(x,0) = u_0(x) \quad \text{in } D.$$ 

A similar analysis may be possible when the nonlinearity appears in a
boundary condition:

$$u_t = \Delta (|u|^{m-1} u) \quad \text{in } D_T$$

$$u = 0 \quad \text{on } \sigma_T$$

$$\frac{\partial u}{\partial n} = \lambda |u|^{p-1} u \quad \text{on } \Sigma_T$$

$$u(x,0) = u_0(x) \quad \text{in } D,$$
and when nonlinearities appear in both equation of motion and boundary condition.

The phenomenon of barrier penetration, i.e., of a quantum-mechanical particle's "tunneling" to a region which is classically inaccessible, is well known (see, e.g., [29]). One might consider an initial-boundary value problem for Schrödinger's equation with a nonlinear boundary condition of the form

\[
\frac{1}{i} \frac{\partial u}{\partial t} = c^2 \Delta u \quad \text{in } D_T \\
u = 0 \quad \text{on } \sigma_T \\
\frac{\partial u}{\partial n} = f(u) \quad \text{on } \Sigma_T \\
u(x,0) = U(x), \quad u_t(x,0) = V(x) \quad \text{in } D,
\]

where the complex-valued function \( u \) of space and time represents the wave function of a particle. It may be possible for a solution of \((S)\) which begins in a potential well of finite positive depth \( d \), and with total initial energy less than \( d \), to tunnel out of the well in finite time.

Much more research needs to be done on the difficult problem \((An)\) of Chapter 4, particularly on the existence of local continuous solutions in dimensions \( n > 4 \), and on the obtainment of greater regularity of solutions for \( n > 2 \). Due to the limited success of the abstract theory, progress on these fronts must rely heavily on the particular form of the differential equation and side conditions. Future numerical research
might be directed toward determining whether there are initial data \( u_0 \)
in each neighborhood in \( H^1_0(D) \) of each equilibrium solution for which
global solutions of (An) appear to exist.
REFERENCES


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Many thanks are due to Jan Nyhus for her excellent and patient typing of this manuscript through its many versions.

Without the love and staunch support of my parents, Allan E. and Mary L. Smith, none of this would have been possible.

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APPENDIX. FORTRAN CODE FOR THE NUMERICAL EXPERIMENTS

C PROGRAM TO SOLVE
C
C DTTU = LAPLACIAN(U) + EPSL*PHI(U)
C U = 0,  R = |X1 = 1, T > 0
C U(R,0) = U0(R),  0 < R < 1 (*)
C DTU(R,0) = VO(R),  0 < R < 1
C VIA FINITE DIFFERENCES, WHERE
C U IS A RADIAL FUNCTION OF DIM > 1 SPACE VARIABLES
C X1,...,XDIM AND TIME T. DIM NEED NOT BE AN INTEGER;
C EPSL IS A POSITIVE GEOMETRIC FACTOR EQUAL TO THE SQUARE
C OF THE RADIUS OF THE ORIGINAL DOMAIN OF SOLUTION;
C PHI(S) IS CONTINUOUSLY DIFFERENTIABLE, STRICTLY
C INCREASING, CONVEX, AND APPROACHES INFINITY AS S
C APPROACHES EM > 0 FROM THE LEFT.
C INITIAL DATA U0,VO ARE DETERMINED BY SUBROUTINE INIDAT.
C
C LATEST VERSION CODED BY R.A. SMITH, IOWA STATE UNIVERSITY,
C
C ALL COMPUTATIONS USE DOUBLE-PRECISION ARITHMETIC.
C
C IMPPLICIT REAL*8 (A-H,O-Z)
C
C W CONTAINS THE FINITE DIFFERENCE APPROXIMATION OF U. W(.,1),
C W(.,2),W(.,3) CONTAIN VALUES OF W AT THREE SUCCESSIVE
C TIME STEPS.
C R(.,1) CONTAINS VALUES OF THE RADIAL VARIABLE R. R(.,2) CONTAINS
C VALUES OF R**(DIM-1), WHICH ARE USED REPEATEDLY IN ENERGY
C COMPUTATIONS.
C
C DIMENSION W(1000,3), R(1000,2)
C COMMON EM, EPSL, DIM, DIM1, H, N, N1, NMIN1, NRSKIP, R,
C 1 OMEGA, FAC1, FAC2, FAC3
C
C FUNCTION DEFINITIONS OF PHI,DPHI,CAPPHI FOLLOW. DPHI IS THE
C DERIVATIVE OF PHI AND IS USED IN THE TAYLOR SERIES
C APPROXIMATION BELOW. CAPPHI(S) IS THE INTEGRAL FROM 0 TO S
C OF PHI AND IS USED IN ENERGY COMPUTATIONS.
C
C PHI(S) = 1.DO/(1.DO - S)
C DPHI(S) = 1.DO/(1.DO - S)**2
C CAPPHI(S) = -DLOG(1.DO - S)
C
C EM = 1.DO
C
C NOTE: CHANGING PHI REQUIRES CHANGING THE FUNCTION DEFINITIONS
C OF DPHI,CAPPHI, AND THE VALUE OF EM (THE BLOW-UP POINT OF
C PHI). MUST ALSO CHANGE LITERAL OUTPUT DEFINING PHI IN
C FORMAT STATEMENT 20.

C READ IN PARAMETERS FOR (*) AND THE DIFFERENCE SCHEME.

C XLAMDA = (DELTA T)/(DELTA R). MUST BE POSITIVE, AND LESS
C THAN 1 TO SATISFY THE COURANT-FRIEDRICHLS-LEWY CONDITION.
C N = NUMBER OF SUBINTERVALS INTO WHICH THE INTERVAL (0,1) FOR
C THE RADIAL VARIABLE R IS DIVIDED. MUST BE EVEN,
C GREATER THAN 3, AND LESS THAN 1000.
C KEND = NUMBER OF TIME STEPS AT WHICH W IS COMPUTED. TIME
C INTERVAL OF SOLUTION IS (0, KEND*XLAMDA/N). MUST BE
C POSITIVE.
C NRSKIP: AT A GIVEN TIME STEP, VALUES OF W ARE OUTPUT EVERY
C NRSKIP/N UNITS IN THE R VARIABLE. MUST BE POSITIVE AND
C CANNOT EXCEED N.
C NTSKIP: CHOSEN VALUES OF W ARE OUTPUT EVERY NTSKIP*XLAMDA/N
C UNITS IN THE T VARIABLE. MUST BE POSITIVE AND CANNOT
C EXCEED KEND.

C READ(5,10) EPSL, DIM, XLAMDA, N, KEND, NRSKIP, NTSKIP
10 FORMAT(3(D23.16/),3(I4/),I4)
WRITE(6,20) EPSL, DIM
20 FORMAT(' SOLUTION OF DTTU = LAPLACIAN(U) + EPSL*','
1 'PHI(U)',15X,'R.A.SMITH'/,
2 ' EPSILON = SQUARE OF RADIUS = ',D13.6/,'
3 ' DIM = NUMBER OF SPACE VARIABLES = ',D13.6)
WRITE(6,30) XLAMDA, N, KEND, NRSKIP, NTSKIP
30 FORMAT(' XLAMDA = (DELTA T)/(DELTA R) = ',D13.6/,
1 ' N = NUMBER OF R PARTITION POINTS = ',I4/,
2 ' KEND = NUMBER OF TIME STEPS = ',I4/,
3 ' NRSKIP = ',I4/, ' NTSKIP = ',I4/)

C CHECKS THAT INPUT PARAMETERS ARE ACCEPTABLE.

C IF (EPSL .LE. 0.D0) GO TO 270
IF (DIM .LE. 1.D0) GO TO 270
IF ( (XLAMDA .LE. 0.D0) .OR. (XLAMDA .GT. 1.D0) ) GO TO 270
IF ( (N .LE. 3) .OR. (N .GE. 1000) ) GO TO 270
IF (KEND .LE. 0) GO TO 270
IF ( (NRSKIP .LT. 1) .OR. (NRSKIP .GT. N) ) GO TO 270
IF ( (NTSKIP .LT. 1) .OR. (NTSKIP .GT. KEND) ) GO TO 270

C ASSIGN PRIMARY VARIABLES USED BY DIFFERENCE SCHEME.

C H = DELTA R = STEPSIZE IN R VARIABLE.
C AK = DELTA T = STEPSIZE IN T VARIABLE.

C DIMM1 = DIM - 1.D0
XN = N
H = 1.D0/XN
\[ AK = \Xi \lambda^*H \]
\[ AKSQ = AK^*AK \]
\[ N1 = N+1 \]

C ASSIGN VARIABLES USED IN ENERGY COMPUTATIONS.
C OMEGA = SURFACE AREA OF UNIT SPHERE IN DIM DIMENSIONS.
C FAC1, FAC2, FAC3 ARE FACTORS FOR ENERGY EXPRESSIONS.
C
\[ PI = 4.0^*\text{D}A\tan(1.0) \]
\[ OMEGA = 2.0^*\text{P}I^{*}(\text{DIM}/2.0)/\text{DGAMMA}(\text{DIM}/2.0) \]
C
\[ FAC1 = OMEGA/(24.0^*H) \]
\[ FAC2 = OMEGA^*H/(6.0^*AKSQ) \]
\[ FAC3 = EPSL^*OMEGA^*H/3.0 \]
\[ NMIN1 = N-1 \]

C ASSIGN VALUES OF R IN R(.,1), AND VALUES OF R^*DIMM1 IN R(.,2).
C
\[ R(1,1) = 0.0 \]
\[ R(1,2) = 0.0 \]
DO 40 I = 2,N
    TEMP = I-1
    R(I,1) = TEMP^*H
    R(I,2) = R(I,1)^*DIMM1
40 CONTINUE
\[ R(N1,1) = 1.0 \]
\[ R(N1,2) = 1.0 \]

C COMPUTE INITIAL POSITION DATA UO AND STORE IN W(.,1);
C MAXIMUM VALUE UOMAX OF UO AND VALUE ROMAX OF R AT WHICH
C IT OCCURS; INITIAL VELOCITY DATA VO AND TEMPORARILY STORE
C IN W(.,3); INITIAL GAMMA=GAMO; AND INITIAL ENERGIES. SEE
C SUBROUTINE INIDAT FOR DETAILS.
C
\[ \text{CALL INIDAT(W,UOMAX,ROMAX,GAMO,IERR)} \]
\[ \text{IF (IERR .NE. 0) STOP} \]
C ASSIGN VALUES OF W(I,2) = W(R,DELTA T) USING THE PDE AND THE
C TAYLOR EXPANSION OF U IN DELTA T ABOUT T=0.
C ALSO CHECK TO SEE IF W(.,2) EXCEEDS EM (I.E., IF QUENCHING
C OCCURS).
C
\[ FC1 = 0.5^*AKSQ \]
\[ FC2 = 1.0^*/(H^*H) \]
\[ FC3 = DIMM1/(2.0^*H) \]
\[ FC4 = AKSQ^*AK/6.0 \]
C
\[ W(1,2) = W(1,1) + AK^*W(1,3) + \]
\[ FC1^*(2.0^*DIM^*FC2^*(W(2,1)-W(1,1))+EPSL^*\text{PHI}(W(1,1))) + \]


```plaintext
2     FC4*( 2.D0*DIM*FC2*( W(2,3)-W(1,3) ) +
3     EPSL*DPHI(W(1,1))*W(1,3) )
IF (W(1,2) .LT. EM) GO TO 45
I = 1
TLAST = AK
GO TO 260
C
45 DO 50 I = 2,N
   W(I,2) = W(I,1) + AK*W(I,3) +
     1 FC1*( FC2*( W(I+1,1)-2.D0*W(I,1)+W(I-1,1) ) ) +
     2 FC3*( W(I+1,1)-W(I-1,1) )/R(I,1) +
     3 EPSL*PHI(W(I,1))
   W(I,2) = W(I,2) +
     1 FC4*( FC2*( W(I+1,3)-2.D0*W(I,3)+W(I-1,3) ) ) +
     2 FC3*( W(I+1,3)-W(I-1,3) )/R(I,1) +
     3 EPSL*DPHI(W(I,1))*W(I,3) )
IF (W(I,2) .LT. EM) GO TO 50
TLAST = AK
GO TO 260
50 CONTINUE
C
W(N1,2) = 0.DO
C
C ASSIGN SECONDARY VARIABLES USED IN THE DIFFERENCE SCHEME.
C
AKSQEP = AKSQ*EPSL
XLAMSQ = XLAMDA*XLAMDA
ALL1 = 0.25D0*(XLAMSQ - 1.DO)
ALL2 = 0.50D0*(XLAMSQ + 1.DO)
FACTOl = 2.D0*XLAMSQ*DIMM1
FACT02 = AKSQ*DIMM1/(2.DO*H)
FACT03 = 0.25D0*(3.DO*XLAMSQ + 1.DO)
C
C COMPUTE SOLUTION, ENERGIES AT LATER TIME STEPS.
C
C THE FINITE DIFFERENCE SCHEME USES CENTRAL R DIFFERENCES, AND
C BACKWARD TIME DIFFERENCES WITH SPACE AVERAGING IN THE LOWER
C STEP. SEE FRITZ JOHN, PARTIAL DIFFERENTIAL EQUATIONS
C (REFERENCE 10). THIS SCHEME IS STABLE.
C
FOR EACH FIXED T, U IS EXTENDED AS AN EVEN FUNCTION OF R THROUGH
C R = 0. FOR REGULARITY IT IS ASSUMED THAT DRU(0,T) = 0; HENCE
C VALUES OF W(0,T) ARE OBTAINED BY REPLACING THE DRU(R,T)/R
C EXPRESSION IN A TERM OF LAPLACIAN(U) BY A DIFFERENCE
C APPROXIMATION TO DRU(0,T). VALUES OF W NEAR OR AT R = 1
C ARE OBTAINED USING THE BOUNDARY CONDITIONS U(1,T) =
C DTU(1,T) = 0, AND BACKWARD R DIFFERENCES WHERE NECESSARY TO
C AVOID GOING PAST R = 1.
C
WMAX = GLOBAL MAXIMUM VALUE OF W ACHIEVED.
```
C RMAX = VALUE OF R AT WHICH WMAX OCCURS.
C TMAX = VALUE OF T AT WHICH WMAX OCCURS.
C GAMOLD, GAMNEW ARE USED TO COMPUTE THE TIME DERIVATIVE OF GAMMA.
C
C GAMMA(T) IS TWICE THE LINEAR POTENTIAL ENERGY OF W(T),
C DIVIDED BY WMAX(T)**2.
C
WMAX = UOMAX
RMAX = ROMAX
TMAX = 0.DO
GAMOLD = GAMO
TSKIP = NTSKIP
C
DO 230 J = 1,KEND
C
C W(.,3) CONTAINS THE VALUES OF W COMPUTED AT EACH NEW TIME STEP.
C W(1,3),W(2,3),W(N,3),W(N1,3) MUST BE COMPUTED INDIVIDUALLY.
C EACH W(I,3) IS CHECKED TO SEE IF IT EXCEEDS EM (I.E., TO SEE
C IF QUENCHING HAS OCCURRED). THE LOCAL MAXIMUM VALUE WMAXJ OF
C W AT EACH TIME STEP IS DETERMINED, ALONG WITH THE VALUE RMAXJ
C OF R AT WHICH WMAXJ OCCURS.
C
W(1,3) = 2.DO*( W(2,2)+ALL1*W(3,1) ) - ALL2*W(1,1) +
1 FACT01*( W(2,1)-W(1,1) ) + AKSQQP*PHI(W(1,1))
IF (W(1,3) .LT. EM) GO TO 60
I = 1
GO TO 130
60 WMAXJ = W(1,3)
RMAXJ = 0.DO
C
W(2,3) = W(3,2) + W(1,2) + ALL1*( W(4,1)+W(2,1) ) -
1 ALL2*W(2,1) + FACT02*( W(3,1)-W(1,1))/H +
2 AKSQQP*PHI(W(2,1))
IF (W(2,3) .LT. EM) GO TO 70
I = 2
GO TO 130
70 IF (W(2,3) .LT. WMAXJ) GO TO 80
WMAXJ = W(2,3)
RMAXJ = R(2,1)
C
DO 90 I = 3,NMIN1
W(I,3) = W(I+1,2) + W(I-1,2) + ALL1*( W(I+2,1)+W(I-2,1) ) -
1 ALL2*W(I,1) + FACT02*( W(I+1,1)-W(I-1,1))/R(I,1) +
2 AKSQQP*PHI(W(I,1))
IF (W(I,3) .GE. EM) GO TO 130
IF (W(I,3) .LE. WMAXJ) GO TO 90
WMAXJ = W(I,3)
RMAXJ = R(I,1)
90 CONTINUE
C
\[ W(N,3) = W(N-1,2) - \text{FACT03} \times W(N,1) + \text{ALL1} \times W(N-2,1) - \]
\[ \text{FACT02} \times W(N-1,1)/R(N,1) + \text{AKSQEP}^2 \phi(W(N,1)) \]

\text{IF} (W(N,3) \text{ .LT. EM}) \text{ GO TO 100}
\text{I = N}
\text{GO TO 130)
100 \text{ IF} (W(N,3) \text{ .LE. WMAXJ}) \text{ GO TO 110}
WMAXJ = W(N,1)
RMAXJ = R(N,1)

\text{C}
110 \text{ W(N1,3) = 0.DO)

\text{C IF WMAXJ EXCEEDS THE GLOBAL MAXIMUM WMAX, REPLACE THE PREVIOUS}
\text{VALUE STORED IN WMAX WITH WMAXJ.)}
\text{C}
\text{IF} (WMAXJ \text{ .LE. WMAX}) \text{ GO TO 120}
WMAX = WMAXJ
RMAX = RMAXJ
TMAX = J

\text{C IF QUENCHING HAS NOT OCCURRED (JFLAG=0), DETERMINE WHETHER W,}
\text{ENERGIES ARE TO BE OUTPUT AT THIS TIME STEP (I.E., WHETHER}
\text{J+1 IS AN INTEGRAL MULTIPLE OF NTSKIP).}
\text{C}
120 \text{ JFLAG = 0}
JLIM = J+1
\text{IF} ((JLIM - (JLIM/NTSKIP) \times NTSKIP) \text{ .NE. 0}) \text{ GO TO 210}
XJ = J
T = (XJ + 1.DO) \times AK
\text{GO TO 160)

\text{C IF QUENCHING HAS OCCURRED (JFLAG=1), OUTPUT W,ENERGIES AT THE}
\text{TIME STEP JUST PRIOR TO QUENCHING. MUST FIRST CHECK THAT THESE}
\text{VALUES HAVE NOT ALREADY BEEN OUTPUT. MUST ALSO RESTORE W TO ITS}
\text{STATE BEFORE QUENCHING.)}
\text{C}
130 \text{ JFLAG = 1}
\text{IF} ((J - (J/NTSKIP) \times NTSKIP) \text{ .EQ. 0}) \text{ GO TO 250)
DO 140 K = 1,I
W(I,3) = W(I,2)
140 \text{ CONTINUE)
DO 150 K = 1,N1
W(I,2) = W(I,1)
150 \text{ CONTINUE)
XJ = J
T = XJ \times AK

\text{C}
160 \text{ WRITE(6,165) T, (W(K,3), K=1,N1,NTSKIP)
165 \text{ FORMAT(/' SOLUTION AND ENERGIES AT TIME LEVEL T = ',}
1 \text{ D13.6//,50(' ',7(D13.6,3X),/)})
C EVALUATE ENERGIES USING SIMPSON'S RULE.
C SI = LINEAR POTENTIAL ENERGY = 0.5D0*OMEGA*
C (THE INTEGRAL FROM 0 TO 1 OF (R**DIMM1)*(DRW**2)
C WITH RESPECT TO R).
C S2 = KINETIC ENERGY = 0.5D0*OMEGA*(THE INTEGRAL
C FROM 0 TO 1 OF (R**DIMM1)*(DTW**2) WITH RESPECT
C TO R).
C S3 = MINUS NONLINEAR POTENTIAL ENERGY =
C EPSL*OMEGA*(THE INTEGRAL FROM 0 TO 1 OF
C (R**DIMM1)*CAPPHI(W) WITH RESPECT TO R).
C DERIVATIVES IN THE INTEGRANDS ARE APPROXIMATED BY BACKWARD
C TIME DIFFERENCES AND CENTRAL R DIFFERENCES.
C THE EXACT SOLUTION OF (*) SATISFIES THE ENERGY EQUALITY
C S2 + SI - S3 = E0, WHERE E0 IS THE TOTAL INITIAL ENERGY.
C
S1 = 0.D0
S2 = R(1,2)* ( W(1,3)-W(2,2) )**2
S3 = R(1,2)*CAPPHI(W(1,3))
DO 170 K = 2,NMIN1,2
   TEMP1 = 4.D0*R(K,2)
   TEMP2 = 2.D0*R(K+1,2)
   S1 = S1 + TEMP1*( W(K+1,3)-W(K-1,3) )**2 +
   TEMP2*( W(K+2,3)-W(K,3) )**2
1  S2 = S2 + TEMP1*( W(K,3)-0.5D0*( W(K+1,2)+W(K-1,2) ) )**2
   + TEMP2*( W(K+1,3)-0.5D0*( W(K+2,2)+W(K,2) ) )**2
   S3 = S3 + TEMP1*CAPPHI(W(K,3)) +
   TEMP2*CAPPHI(W(K+1,3))
170 CONTINUE
S1 = FAC1*(S1 + 4.D0*R(N,2)*W(N-1,3)**2 + W(N,3)**2))
S2 = FAC2*(S2 + 4.D0*R(N,2)* ( W(N,3)-0.5D0*W(N-1,2) )**2)
S3 = FAC3*(S3 + 4.D0*R(N,2)*CAPPHI(W(N,3)))
C
C T3 = NONLINEAR POTENTIAL ENERGY
C PE = TOTAL POTENTIAL ENERGY
C TE = TOTAL ENERGY
C
T3 = -S3
PE = S1 - S3
TE = PE + S2
WRITE(6,180) S1, T3, PE, S2, TE
180 FORMAT(/' LPE=',D13.6,' , NPE=',D13.6,' , PE=',D13.6,
      1 ', KE=',D13.6,5X,'E(T) = ',D15.8)
C
C OUTPUT LOCAL MAXIMUM WMAXJ AND VALUE GAMNEW OF GAMMA.
C
GAMNEW = 2.D0*S1/WMAXJ**2
WRITE(6,190) WMAXJ, GAMNEW
190 FORMAT(' MAXIMUM W(T) = ',D13.6/, ' GAMMA(T) = ',D13.6)
IF (JFLAG .EQ. 1) GO TO 250

DTGAM = TIME DERIVATIVE OF GAMMA, COMPUTED USING A BACKWARD
DIFFERENCE BETWEEN TWO SUCCESSIVE OUTPUT VALUES OF GAMMA.

\[ DTGAM = \frac{(GAMNEW - GAMOLD)}{(AK \times TSKIP)} \]

WRITE (6, 200) DTGAM

200 FORMAT (' DTGAMMA(T) = ', D13.6)

GAMOLD = GAMNEW

SHIFT VECTORS TO BEGIN COMPUTATIONS AT THE NEXT TIME STEP.

210 DO 220 I = 1, N1
   W(I,1) = W(I,2)
   W(I,2) = W(I,3)
220 CONTINUE

230 CONTINUE

IF W NEVER QUENCHED, OUTPUT ITS GLOBAL MAXIMUM WMAX
AND THE R VALUE RMAX AND T VALUE TMAX AT WHICH IT OCCURRED.

IF (TMAX .EQ. 0.0) GO TO 240

\[ TMAX = (TMAX + 1.0) \times AK \]

240 WRITE (6, 245) RMAX, TMAX, WMAX

245 FORMAT (/ 'JOB COMPLETED WITH NO QUENCHING/',
   1 'MAXIMUM VALUE OF U ACHIEVED = U(', D13.6, ', ',
   2 D13.6, ') = ', D13.6)

STOP

IF W QUENCHED, OUTPUT WHERE AND WHEN.

250 TLAST = (XJ + 2.0) \times AK
260 WRITE (6, 265) R(I,1), TLAST

265 FORMAT (' SOLUTION EQUALLED OR EXCEEDED A AT R = ',
   1 D13.6, 'AT TIME TLAST = ', D13.6)

STOP

IF AN INPUT VARIABLE EXCEEDS ACCEPTABLE PARAMETERS:

270 WRITE (6, 275)

275 FORMAT (' **ERROR: INPUT VARIABLE IN MAIN PROGRAM EXCEEDS',
   1 ' ACCEPTABLE PARAMETERS', ' COMputation DISCONTINUED.')

STOP

END

SUBROUTINE INIDAT(W, U0MAX, R0MAX, GAMO, IERR)
SUBROUTINE TO GENERATE INITIAL DATA FOR THE MAIN PROGRAM.

A. DETERMINES U₀ AS A PERTURBATION OF AN EQUILIBRIUM SOLUTION F OF (*) AND STORES U₀ IN W(:,1). SEE INIUO FOR DETAILS.
B. COMPUTES THE MAXIMUM VALUE U₀MAX OF U₀, AND THE VALUE R₀MAX OF R AT WHICH U₀MAX OCCURS.
C. DETERMINES V₀(R) = XMU*(1-R**2), AND STORES V₀ IN W(:,3). XMU IS CHOSEN SO THAT THE TOTAL INITIAL ENERGY OF THE SOLUTION OF (*) IS E₀. NOTE: E₀ MUST EQUAL OR EXCEED THE TOTAL POTENTIAL ENERGY OF U₀.
D. COMPUTES THE INITIAL ENERGIES OF THE SOLUTION OF (*).
E. DETERMINES THE INITIAL VALUE GAM₀ OF GAMMA.
F. DETERMINES WHETHER GAMMA BEGINS IN THE REGION R OF FIGURE 1. SEE DISSERTATION AND SUBROUTINE FINMIN FOR DETAILS.

IMPLICIT REAL*8(A-H,O-Z)

DIMENSION W(1000,3), R(1000,2), F(1000,2)
COMMON EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R, 1, OMEGA, FAC1, FAC2, FAC3

C

CAPPHI(S) = -DLOG(1.DO - S)

C

READ IN INITIAL TOTAL ENERGY E₀, AND PARAMETER SEED FOR SUBROUTINE EQSOLN. SEED IS CHOSEN USING THE BIFURCATION DIAGRAM PLOTTING EPSL AS A FUNCTION OF FMAX.

READ(5,400) E₀, SEED
400 FORMAT(D23.16/D23.16)
WRITE(6,405) SEED
405 FORMAT('/' EQUILIBRIUM SOLUTION SEED = ',D13.6/)

C

CHECK THAT SEED IS WITHIN ACCEPTABLE PARAMETERS.

IF ( (SEED .LE. 0.DO) .OR. (SEED .GE. EM) ) GO TO 500

C

COMPUTE AN EQUILIBRIUM SOLUTION F OF (*) WITH FMAX CLOSEST TO SEED. F(:,1) CONTAINS VALUES OF F; F(:,2) CONTAINS VALUES OF DF. SEE SUBROUTINE EQSOLN FOR DETAILS.

CALL EQSOLN(F,SEED,NRITS,JERR)
IF (JERR .NE. 0) GO TO 510

C

OUTPUT F.

WRITE(6,410) NRITS,(F(1,1), 1=1,N1,NRSKIP)
410 FORMAT('/' EQUILIBRIUM SOLUTION F ('',I3,
1 ' ITERATIONS REQUIRED')//,50(' ',7(D13.6,3X),/))
C COMPUTE UO AND STORE IN W(.,1). SEE SUBROUTINE INIUO.
C
CALL INIUO(W,F,UOMAX,ROMAX,JERR)
IF (JERR .NE. 0) GO TO 510
C
C OUTPUT UO.
C
WRITE(6,420) (W(I,1), I=1,N1,NRSKIP)
420 FORMAT(//' INITIAL VALUES OF U'//,50(' ',7(D13.6,3X),/))
C
C COMPUTE
C SIO = INITIAL LINEAR POTENTIAL ENERGY;
C S30 = INITIAL MINUS NONLINEAR POTENTIAL ENERGY.
C SEE MAIN PROGRAM FOR DESCRIPTION OF HOW ENERGIES ARE COMPUTED.
C
SIO = 0.D0
S30 = R(1,2)*CAPPHI(W(1,1))
DO 430 I = 2,NMIN1,2
 TEMP1 = 4.D0*R(I,2)
 TEMP2 = 2.D0*R(I+1,2)
 SIO = SIO + TEMP1*( W(I+1,1)-W(I-1,1) )**2 +
 1 TEMP2*( W(I+2,1)-W(I,1) )**2
S30 = S30 + TEMP1*CAPPHI(W(I,1)) + TEMP2*CAPPHI(W(I+1,1))
430 CONTINUE
SIO = FAC1*(S10 + 4.DO*(R(N,2)**W(N-1,1)**2 + W(N,1)**2))
S30 = FAC3*(S30 + 4.DO*R(N,2)*CAPPHI(W(N,1)))
C
C COMPUTE VO(R) = XMU**2, WHERE XMU IS CHOSEN SO THAT
C THE TOTAL INITIAL ENERGY IS EO. STORE VO IN W(.,3). S20=
C INITIAL KINETIC ENERGY.
C
S20 = EO + S30 - S10
IF (S20 .LT. 0.D0) GO TO 520
XMU = DSQRT(DIM*(DIM+2.D0)*(DIM+4.D0)*S20/(4.D0*OMEGA))
DO 440 I = 1,N1
 W(I,3) = XMU*(1.D0-R(I,1)**2)
440 CONTINUE
C
C OUTPUT VO.
C
WRITE(6,450) (W(I,3), I=1,N1,NRSKIP)
450 FORMAT(//' INITIAL VALUES OF DTU'//,50(' ',7(D13.6,3X),/))
C
C OUTPUT INITIAL ENERGIES. SEE MAIN PROGRAM FOR EXPLANATION OF
C T3,PE.
C
T30 = -S30
PE0 = S10 + T30
WRITE(6,460) S10, T30, PEO, S20, E0
460 FORMAT(/' LPE0=' ,D13.6,' , NPE0=' ,D13.6,' , PE0=' ,D13.6,
1 ' , KE0=' ,D13.6/,' INITIAL TOTAL ENERGY E0 = ' ,D13.6/)

C
C COMPUTE INITIAL VALUE GAMO OF GAMMA.
C
GAMO = 2.DO*S10/UOMAX**2
WRITE(6,470) ROMAX, UOMAX, GAMO
470 FORMAT(' MAXIMUM VALUE OF INITIAL U = UOMAX = ',
1 'U0(',D13.6,') = ',D13.6/,' INITIAL GAMMA = GAMO = ',D13.6/)

C
C DETERMINE WHETHER INITIAL GAMMA LIES IN REGION R OF FIGURE 1.
C
CALL FINMIN(EO,SO,GMIN,JERR)
IF (JERR .NE. 0) GO TO 510
IERR = 0
IF ( (UOMAX .GT. 80) .OR. (GAMO .LT. GMIN) ) GO TO 490
WRITE(6,480)
480 FORMAT(/' INITIAL GAMMA LIES IN REGION R'/)
RETURN

490 WRITE(6,495)
495 FORMAT(/' INITIAL GAMMA IS EXTERIOR TO REGION R'/)
RETURN

C
C ERROR-HANDLING SECTION.
C
C IF SEED WAS OUTSIDE INTERVAL (0,EM):
C
500 WRITE(6,505)
505 FORMAT( ' -^"ERROR: SEED EXCEEDS ACCEPTABLE PARAMETERS'/)
IERR = 1
RETURN

C
C IF ONE OF SUBROUTINES EQSOLN,RUNKUT,INIUO,FINT-IN failed
C (ERROR MESSAGE WAS OUTPUT BY FAILED ROUTINE):
C
510 IERR = 1
RETURN

C IF INITIAL KINETIC ENERGY WAS NEGATIVE:
C
520 PEO = S10 - S30
WRITE(6,525) PEO, E0, S20
525 FORMAT(' ^ERROR: PEO = ',D13.6,' EXCEEDS E0 = ',D13.6/,
1 ' YIELDING NEGATIVE INITIAL KINETIC ENERGY S20 = ',D13.6/)
IERR = 1
RETURN
END
SUBROUTINE EQSOLN(F, SEED, NRITS, JERR)

SUBROUTINE TO OBTAIN AN EQUILIBRIUM SOLUTION F OF THE MAIN PROGRAM'S PROBLEM (*), VIA THE SHOOTING METHOD.

LET F(R, A) DENOTE THE SOLUTION OF

\[ DRF + (\text{DIM}-1) \cdot DRF/R + EPSL \cdot \phi(F) = 0 \]

\[ F(0) = A \quad (*) \]

\[ DRF(0) = 0, \]

WHERE A IS A PARAMETER. THE SECANT METHOD IS USED TO FIND THE A FOR WHICH F(1, A) = 0, SO THAT F(R, A) IS THE DESIRED STATIONARY SOLUTION OF (*).

IMPLICIT REAL*8(A-H, O-Z)
DIMENSION F(1000, 2), R(1000, 2)
COMMON EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R, \[ \text{1 OMEGA, FAC1, FAC2, FAC3} \]

A1, A2 ARE INITIAL ESTIMATES FOR A, AND ARE USED TO START THE SECANT METHOD.

A1 = SEED
A2 = 0.95D0 * SEED
NRITS = 0

COMPUTE F(R, A1)

CALL RUNKUT(F, A1, KERR)
IF (KERR .NE. 0) GO TO 720
F0 = F(N1, 1)

LOOP TO CALCULATE A. SECANT METHOD IS ITERATED UNTIL THE RELATIVE ERROR BETWEEN SUCCESSIVE A1, A2 IS LESS THAN 5.D-12.

700 NRITS = NRITS + 1
CALL RUNKUT(F, A2, KERR)
IF (KERR .EQ. 0) GO TO 720
F1 = F(N1, 1)
TEMP = F1 - F0
IF (TEMP .EQ. 0.D0) GO TO 730
TEMP = F1 * (A2 - A1) / TEMP
IF (DABS(TEMP) .GE. (5.D-12 * DABS(A2))) GO TO 710
JERR = 0
RETURN

710 A1 = A2
A2 = A1 - TEMP
F0 = F1
IF (NRITS .GT. 10) GO TO 740
GO TO 700
C
C ERROR-HANDLING SECTION.
C
C IF A CALL TO RUNKUT FAILS:
C
720 WRITE(6,725) NRITS
725 FORMAT(/' DURING CALL ',I3,' TO SUBROUTINE RUNKUT')
JERR = 1
RETURN
C
C IF SECANT METHOD FAILS DUE TO DIVISION BY ZERO:
C
730 WRITE(6,735) NRITS
735 FORMAT(/' ERROR: DIVISION BY ZERO IN THE CALCULATION',
1 ' OF A2 AT STEP ',I2,' OF EQSOLN')
C
C IF SECANT METHOD FAILS TO CONVERGE:
C
740 WRITE(6,745) F1
745 FORMAT(/' ERROR: SECANT METHOD DOES NOT CONVERGE',
1 ' AFTER 10 ITERATIONS IN SUBROUTINE EQSOLN',/,
2 ' LAST COMPUTED F(1,A2) = ',D13.6)
JERR = 1
RETURN
END
C
C SUBROUTINE RUNKUT(F,ALPHA,KERR)
C
SUBROUTINE TO SOLVE (***) IN EQSOLN VIA THE CLASSICAL FOURTH-ORDER RUNGE-KUTTA METHOD.
C
IMPLICIT REAL*8(A-H,0-Z)
DIMENSION F(1000,2), R(1000,2), TK(4,2)
COMMON EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
1 OMEGA, FAC1, FAC2, FAC3
C
PHI(S) = 1 DO/(1.DO - S)
C
F(1,1) = ALPHA
F(1,2) = 0.DO
J = 1
TK(1,1) = F(1,2)
IF (F(1,1) .GE. EM) GO TO 780
TK(1,2) = -EPSL*PHI(F(1,1))/DIM
COEF = 0.5DO**H
760 TEMP = F(J,1) + COEF*TK(1,1)
   IF (TEMP .GE. EM) GO TO 780
   TK(2,1) = F(J,2) + COEF*TK(1,2)
   QUANT = R(J,1) + COEF
   TK(2,2) = -(DIMM1*TK(2,1)/QUANT + EPSL*PHI(TEMP))
   TEMP = F(J,1) + COEF*TK(2,1)
   IF (TEMP .GE. EM) GO TO 780
   TK(3,1) = F(J,2) + COEF*TK(2,2)
   TK(3,2) = -(DIMM1*TK(3,1)/QUANT + EPSL*PHI(TEMP))
   TEMP = F(J,1) + H*TK(3,1)
   IF (TEMP .GE. EM) GO TO 780
   TK(4,1) = F(J,2) + H*TK(3,2)
   JTEMP = J
   J = J+1
   TK(4,2) = -(DIMM1*TK(4,1)/R(J,1) + EPSL*PHI(TEMP))
   F(J,1) = F(JTEMP,1) + H*(TK(1,1) + 2.DO*TK(2,1) +
   1 2.DO*TK(3,1) + TK(4,1))/6.DO
   F(J,2) = F(JTEMP,2) + H*(TK(1,2) + 2.DO*TK(2,2) +
   1 2.DO*TK(3,2) + TK(4,2))/6.DO
   IF (J .GE. N1) GO TO 770
   TEMP = F(J,1)
   IF (TEMP .GE. EM) GO TO 780
   TK(1,1) = F(J,2)
   TK(1,2) = -(DIMM1*TK(1,1)/R(J,1) + EPSL*PHI(TEMP))
   GO TO 760

C IF QUENCHING OCCURS IN RUNKUT:
C
770 KERR = 0
RETURN
780 WRITE(6,785) R(J,1)
785 FORMAT(/' ERROR: ARGUMENT OF PHI EQUALED OR','
     1 ' EXCEEDED EM AT R = ',D13.6)
   KERR = 1
RETURN
END

SUBROUTINE INIUO(W,F,UOMAX,ROMAX,JERR)

C THIS VERSION CREATES INITIAL DATA UO FOR THE MAIN
C PROGRAM SUCH THAT:
C  A. UO IS TWICE CONTINUOUSLY DIFFERENTIABLE ON (0,1);
C  B. UO(R) = UO FOR 0 <= R <= RO;
C  C. UO(R) IS STRICTLY DECREASING ON RO <= R <= R1;
C  D. UO(R) = F(R) FOR R1<= R<= 1, WHERE F IS COMPUTED BY
C     SUBROUTINE EQSOLN.
C THE VALUES OF UO ON (RO,R1) ARE OBTAINED BY ADDING AN
C  APPROPRIATE FOURTH- OR FIFTH-ORDER POLYNOMIAL IN R TO F.
C  VALUES OF U0 ARE STORED IN W(.,1).
C
C IMPLICIT REAL*8(A-H,O-Z)
DIMENSION W(1000,3), R(1000,2), F(1000,2)
COMMON EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R, 1
  OMEGA, FAC1, FAC2, FAC3
C
C READ IN PARAMETERS FOR CREATING U0.
C
READ(5,800) UOMAX,RO,R1
800 FORMAT(2(D23.16/),D23.16)
WRITE(6,805) UOMAX, RO, R1
805 FORMAT(/' MAXIMUM VALUE OF INITIAL U = UOMAX = ',D13.6/, 1
  1 ' RO = ',D13.6,5X,'R1 = ',D13.6/)
C
C CHECK THAT PARAMETERS ARE ACCEPTABLE.
C
IF ( (UOMAX .LT. F(1,1)) .OR. (UOMAX .GT. EM) ) GO TO 850
IF ( (RO .LT. O.DO) .OR. (RO .GE. R1) .OR. 1
  (R1 .GE. 1.DO) ) GO TO 850
C
C CREATE U0.
C
ROMAX = 0.DO
W(1,1) = UOMAX
NRO = RO/H
NR0 = NRO+1
NR1 = R1/H
NR1 = NR1+1
IF (NRO .EQ. NR1) GO TO 860
IF (NRO .GT. 1) GO TO 815
G4 = 3.DO*(F(1,1)-UOMAX)/R1**4
G3 = 4.DO*(F(1,1)-UOMAX)/R1**3
DO 810 I = 2,NR1
     W(I,1) = ((R(I,1) - R1)**3) * 1
     (G4*(R(I,1)-R1)+G3) + F(I,1)
810 CONTINUE
GO TO 830
815 DO 820 I = 2,NR0
     W(I,1) = UOMAX
820 CONTINUE
NROPL1 = NRO+1
AL1 = R1-R0
AL1SQ = AL1*AL1
AL1CU = AL1*AL1SQ
A1 = (F(NRO,1)-UOMAX)/AL1CU
DRFRO = (F(NROPL1,1)-F(NRO-1,1))/(2.DO*H)
A2 = (3.DO*AL1SQ*A1 + DRFRO)/AL1CU
DRRFRO = (F(NROPL1,1)-2.DO*F(NRO,1)+F(NRO-1,1))/H**2
$A_3 = \frac{(D_{RRF0} - 6.0 \cdot A_1 \cdot A_1 + 6.0 \cdot A_1 \cdot A_1 \cdot A_1 \cdot A_1)}{(2.0 \cdot A_1 \cdot C_1)}$ 

DO 825 I = NR0PL1, NR1
   $W(I,1) = A_3 \cdot (R(I,1) \cdot R_0) + A_2$
   $W(I,1) = ((R(I,1) - R_0) + W(I,1) + A_1) \cdot (R(I,1) - R_1) \cdot 3 + F(I,1)$
825 CONTINUE

830 IF (NR1 .GE. N1) GO TO 840
   NR1PL1 = NR1 + 1
   DO 835 I = NR1PL1, N1
      $W(I,1) = F(I,1)$
835 CONTINUE

840 JERR = 0
RETURN

C ERROR-HANDLING SECTION.

C IF INPUT VARIABLE UNACCEPTABLE:

850 WRITE(6,855)
855 FORMAT(/' **ERROR: INPUT VARIABLE IN SUBROUTINE INIUO', 1 ' EXCEEDS ACCEPTABLE PARAMETERS'/)
   JERR = 1
   RETURN

C IF R0,R1 TOO CLOSE FOR COMFORT:

860 WRITE(6,865) NR0
865 FORMAT(/' **ERROR: R0,R1 BOTH YIELD PARTITION', 1 ' VALUE ',15,' IN SUBROUTINE INIDAT AND ARE', 2 ' INDISTINGUISHABLE'/)
   JERR = 1
   RETURN

END

SUBROUTINE FINMIN(E0, S0, GMIN, JERR)

SUBROUTINE TO DETERMINE THE ABSOLUTE MINIMUM POINT (S0,GMIN) OF
THE FUNCTION $G(S) = (E_0 + \epsilon_1 \cdot VOL \cdot \psi(S))/S^{\alpha_2}$, WHERE
VOL = THE LEBESGUE MEASURE OF THE DIM-DIMENSIONAL UNIT
SPHERE;
G(UMAX) IS AN UPPER BOUND FOR GAMMA AS DETAILED IN CHAPTER
FOUR OF DISSERTATION.

IMPORTANT: THIS SUBROUTINE ASSUMES E0 >= 0.

IMPLICIT REAL*8(A-H,0-Z)
DIMENSION R(1000,2)
COMMON EM, EPSL, DIM, DIMM1, H, N, N1, NMIN1, NRSKIP, R,
1 OMEGA, FAC1, FAC2, FAC3

C CAPPHI(S) = -DLOG(1.DO - S)
FZERO(S) = 1.DO - S - DEXP(CONST - S/(2.DO*(1.DO-S)))/1
DFZERO(S) = DEXP(CONST-S/(2.DO*(1.DO-S)))/(2.DO*(1.DO-S)**2)

C FZERO IS CHOSEN SO THAT SOLVING FZERO(S)=0 FOR S IS EQUIVALENT
C TO SOLVING DG(S)=0.
C SOLUTION OF FZERO(S)=0 IS ACCOMPLISHED BY APPLYING THE BISECTION
C ALGORITHM TO LOCATE S0 TO WITHIN 2**(-10), AND THEN APPLYING
C NEWTON'S METHOD.
C
C NOTE: IF THE FUNCTION DEFINITION OF PHI AND VALUE OF EM ARE
C CHANGED IN THE MAIN PROGRAM, ONE NEED ONLY CHANGE THE FUNCTION
C DEFINITIONS OF CAPPHI,FZERO,DFZERO IN THIS SUBROUTINE.
C
VOLUM = OMEGA/DIM
CONST = EO/(EPSL*VOLUM)

C APPLY THE BISECTION ALGORITHM. BOUND1,BOUND2 ARE RESPECTIVELY
C LOWER,UPPER BOUNDS FOR S0. THE ALGORITHM IS WRITTEN USING
C THE FACTS THAT S0 IS UNIQUE, S0 IS LOCATED BETWEEN 0 AND EM,
C AND DFZERO(S0)>0.
C
BOUND1 = 0.DO
BOUND2 = EM
NRITS = 0
910 SO = (BOUND1 + BOUND2)/2.DO
NRITS = NRITS + 1
IF (NRITS .GT. 11) GO TO 940
IF (FZERO(SO)) 920, 960, 930
920 BOUND1 = SO
   GO TO 910
930 BOUND2 = SO
   GO TO 910
940 NRITS = 0
C
C APPLY NEWTON'S METHOD UNTIL THE RELATIVE ERROR BETWEEN TWO
C SUCCESSIVE COMPUTED S0 IS LESS THAN 5.D-14.
C
C THE METHOD IS DISCONTINUED, AND AN ERROR MESSAGE ISSUED, IF MORE
C THAN 10 ITERATIONS ARE REQUIRED, IF DFZERO(S0) 'VANISHES' AT
C SOME ITERATION, OR IF A COMPUTED S0 ESCAPES THE INTERVAL
C (BOUND1,BOUND2) DETERMINED BY THE BISECTION ALGORITHM ABOVE.
C
950 IF (NRITS .GT. 10) GO TO 970
NRITS = NRITS + 1
TEMP = DFZERO(SO)
IF (DABS(TEMP) .LT. 5.D-16) GO TO 980
TEMP = FZERO(SO)/TEMP
SO = SO - TEMP
IF ( (SO .LT. BOUND1) .OR. (SO .GT. BOUND2) ) GO TO 990
IF (DABS(TEMP) .GE. (5.D-14*DABS(SO))) GO TO 950
C
C IF NEWTON'S METHOD WAS SUCCESSFUL, OUTPUT SO,GMIN,NRITS.
C
960  GMIN = (E0 + EPSL*VOLUM*CAPPHI(SO))/SO**2
WRITE(6,965) GMIN, SO, NRITS
965  FORMAT(' G ASSUMES ABSOLUTE MINIMUM OF GMIN = ',D13.6,
1    ' AT SO = ',D13.6/,',' ',I2,' ITERATIONS OF NEWTON''S',
2    ' METHOD REQUIRED'/)
JERR = 0
RETURN
C
C IF NEWTON'S METHOD WAS UNSUCCESSFUL, OUTPUT APPROPRIATE ERROR
C MESSAGE.
C
970  WRITE(6,975) SO
975  FORMAT(' **ERROR: 11 ITERATIONS OF NEWTON''S METHOD',
1    ' REQUIRED BY FINMIN/', LAST COMPUTED SO = ',D13.6/)
JERR = 1
RETURN
C
980  WRITE(6,985) SO, TEMP, NRITS
985  FORMAT(' **ERROR: DFZERO(',D13.6,') = ',D13.6,
1    ' IS CONSIDERED TO VANISH/', AFTER ',',I2,
2    ' ITERATIONS OF NEWTON''S METHOD IN FINMIN'/)
JERR = 1
RETURN
C
990  WRITE(6,995) SO, NRITS, BOUND1, BOUND2
995  FORMAT(' **ERROR: SO = ',D13.6,' COMputed AT ITERATION ',I2,
1    ' OF NEWTON''S METHOD IN FINMIN/', 'Escaped INTERVAL (',
2    D13.6,', ',D13.6,') DETERMINED BY BISECTION ALGORITHM'/)
JERR = 1
RETURN
END
C
C SAMPLE LIST OF INPUT VARIABLES.
C
1.DO  EPSL
2.DO  DIM
0.25DO  XLAMDA
100  N
400  KEND
10  NRSkip
40  NTskip
1.DO  EO
0.334589D0  SEED
PROGRAM TO OBTAIN THE BIFURCATION DIAGRAM OF EPSL VS. FMAX, WHERE F = F(R) IS A POSITIVE SOLUTION OF THE STATIONARY PROBLEM

\[ \text{D} \text{R} \text{R} \text{F} + (\text{D} \text{I} \text{M} - 1) \text{D} \text{R} \text{F} / R + \text{EPSL} \text{P} \text{H} \text{I}(F) = 0 \quad (*) \]

\[ F(1) = \text{DRF}(0) = 0 \]

VIA EMDEN'S METHOD, WHERE \( \text{PHI}(S) = (1 + \text{ALPHA} \times S)^{\text{BETA}} \).

FOR A DESCRIPTION OF EMDEN'S SOLUTION TECHNIQUE SEE JOSEPH AND LUNDGREN, "QUASILINEAR DIRICHLET PROBLEMS DRIVEN BY POSITIVE SOURCES" (REFERENCE 11).

LATEST VERSION CODED BY RICHARD A. SMITH, IOWA STATE UNIVERSITY, NOVEMBER 27, 1985.

ALL COMPUTATIONS USE DOUBLE-PRECISION.

IMPLICIT REAL*8 (A-H,O-Z)

V(.,1) CONTAINS VALUES V OF EMDEN'S SOLUTION. V(.,2) CONTAINS VALUES OF DXV.

X CONTAINS VALUES OF THE ARGUMENT OF V.

EPSL CONTAINS VALUES OF EPSL AS A FUNCTION OF FMAX.

VDIF: V(X) IS ASYMPTOTIC TO \( X^{(\beta - 1)} \), WHERE \( \tau = \frac{2}{\beta - 1} \).

G: THE VECTOR (V(.,1),V(.,2)) SOLVES EMDEN'S ORDINARY DIFFERENTIAL EQUATION \( DX(V(.,1),V(.,2)) = (G(.,1),G(.,2)) \).

DIMENSION V(10001,2), X(10001), EPSL(1000), FMAX(1000),

1 VDIF(10001),G(10001,2)

READ IN PARAMETERS.

W = THE HOMOLOGY CONSTANT OF THE SOLUTION OF THE EMDEN PROBLEM. MUST BE POSITIVE;

N = THE NUMBER OF SUBINTERVALS INTO WHICH THE INTERVAL (0,1) FOR X IS TO BE DIVIDED. MUST BE POSITIVE, AND CANNOT EXCEED 10000;

DIM = THE NUMBER OF SPACE VARIABLES IN (\#). MUST BE GREATER THAN OR EQUAL TO 1, BUT NEED NOT BE AN INTEGER.

NXSKIP SELECTS VALUES OF X AT WHICH TO OUTPUT V. MUST BE POSITIVE, AND CANNOT EXCEED 10000.

FMAXIN = THE INTERVAL WIDTH FOR FMAX. MUST BE POSITIVE, AND
C LESS THAN THE BLOW-UP POINT 1/DABS(ALPHA) OF PHI.
C NFMAX = THE NUMBER OF SUBINTERVALS INTO WHICH THE INTERVAL
C (0,FMAXIN) FOR FMAX IS TO BE DIVIDED. MUST BE POSITIVE.
C ALPHA,BETA MUST BOTH BE NEGATIVE.
C DELTA = ACCURACY WITHIN WHICH GPRED MUST PREDICT GCRC IN THE
C PREDICTOR-CORRECTOR METHOD BELOW.

C READ (5,10) W,N,DIM,NXSKIP,FMAXIN,NFMAX,ALPHA,BETA,DELTA
WRITE(6,20) W,N,DIM,NXSKIP,FMAXIN,NFMAX,ALPHA,BETA,DELTA
20 FORMAT(' EPSL VS. FMAX VIA EMDEN'S METHOD'//,' W = ','
1 D13.6/,' N = ',15/,' DIM = ',D13.6/,' NXSKIP = ',
2 I4/,' FMAXIN = ',D13.6/,' NFMAX = ',I4/,
3 ' ALPHA = ',D13.6/,' BETA = ',D13.6/,' DELTA = ',D13.6///,
4 ' EMDEN''S SOLUTION WITH HOMOLOGY CONSTANT W'/)
C CHECKS THAT INPUT VARIABLES ARE ACCEPTABLE.
C IF(W .LE. 0.DO) GO TO 240
IF ( (N .LE. 0) .OR. (N .GT. 10000) ) GO TO 240
IF (DIM .LT. l.DO) GO TO 240
IF ( (NXSKIP .LT. 1) .OR. (NXSKIP .GT. 10000) ) GO TO 240
IF ( (FMAXIN .LE. O.DO) .OR. (FMAXIN .GE. (1.DO/DABS(ALPHA))) ) GO TO 240
IF (FMAXIN .LE. 0.DO) GO TO 240
IF ((ALPHA .GE. O.DO) .OR. (BETA .GE. O.DO)) GO TO 240
IF (DELTA .LE. 0.DO) GO TO 240
C SET UP INTERNAL VARIABLES FOR DIFFERENCE SCHEME.
C H IS THE STEPSIZE IN X VARIABLE.
C
XN = N
N1 = N+1
H = 1.DO/XN
DIMM1 = DIM - 1.DO
C
C INITIALIZE X.
C DO 30 I = 1,N1
  TEMP = I-1
  X(I) = TEMP*H
30 CONTINUE

C COMPUTE EMDEN'S SOLUTION V USING SIXTH ORDER ADAMS-BASHFORTH
C PREDICTOR WITH ADAMS-MOULTON CORRECTOR.
C THE FIRST FIVE STARTING VALUES ARE GENERATED USING THE
C CLASSICAL FOURTH-ORDER RUNGE-KUTTA METHOD.
C
TAU = 2.DO/(BETA-1.DO)
CONST = TAU*(DIM-2.DO-TAU)
V(1,1) = 1.DO/W**(-TAU)
V(1,2) = 0.DO
TK11 = V(1,2)
TK12 = -CONST/(DIM**V(1,1)**(-BETA))
G(1,1) = TK11
G(1,2) = TK12
J = 1
COEF = 0.5DO**H
40 TEMP = V(J,1) + COEF*TK11
  IF (TEMP .LT. 1.D-15) GO TO 260
  TK21 = V(J,2) + COEF*TK12
  QUANT = X(J) + COEF
  TK22 = -DIMM1*TK21/QUANT - CONST/TEMP**(-BETA)
  TEMP = V(J,1) + COEF*TK21
  IF (TEMP .LT. 1.D-15) GO TO 260
  TK31 = V(J,2) + COEF*TK22
  TK32 = -DIMM1*TK31/QUANT - CONST/TEMP**(-BETA)
  TEMP = V(J,1) + H*TK31
  IF (TEMP .LT. 1.D-15) GO TO 260
  TK41 = V(J,2) + H*TK32
  JTEMP = J
  J = J+1
  TK42 = -DIMM1*TK41/X(J) - CONST/TEMP**(-BETA)
  V(J,1) = V(JTEMP,1) + H*(TK11 + 2.DO*TK21 +
               1     2.DO*TK31 + TK41)/6.DO
  V(J,2) = V(JTEMP,2) + H*(TK12 + 2.DO*TK22 +
               1     2.DO*TK32 + TK42)/6.DO
  G(J,1) = V(J,1)
  G(J,2) = -DIMM1*V(J,2)/X(J) - CONST/V(J,1)**(-BETA)
  IF (J .GE. 5) GO TO 50
  TEMP = V(J,1)
  IF (TEMP .LT. 1.D-15) GO TO 260
  TK11 = V(J,2)
  TK21 = -DIMM1*TK11/X(J) - CONST/TEMP**(-BETA)
  GO TO 40

C
C INITIALIZE CONSTANTS FOR PREDICTOR-CORRECTOR SCHEME.
C
50   D1 = 720.DO
    B0 = 1901.DO/D1
    B1 = -2774.DO/D1
    B2 = 2616.DO/D1
    B3 = -1274.DO/D1
    B4 = 251.DO/D1
    D2 = 1440.DO
    C0 = 475.DO/D2
    C1 = 1427.DO/D2
    C2 = -798.DO/D2
    C3 = 482.DO/D2
C4 = -173.0/D2
C5 = 27.0/D2

C APPLY PREDICTOR-CORRECTOR METHOD.

VPRED1, VPRED2 CONTAIN PREDICTED VALUES OF V.
VCRC1, VCRC2 CONTAIN CORRECTED VALUES OF V.

A maximum of 8 corrections is allowed. For details, see
Henrici, Discrete Variable Methods in Ordinary Differential
Equations, page 200.

DO 80 J = 6, N1

VPRED1 = V(J-1,1) + H*(B4*G(J-5,1) + B3*G(J-4,1)
1 + B2*G(J-3,1) + B1*G(J-2,1) + B0*G(J-1,1))

VPRED2 = V(J-1,2) + H*(B4*G(J-5,2) + B3*G(J-4,2)
1 + B2*G(J-3,2) + B1*G(J-2,2) + B0*G(J-1,2))

GPRED1 = VPRED2

GPRED2 = -DIMM1*VPRED2/X(J) - CONST/VPRED1*(-BETA)

CON1 = V(J-1,1) + H*(C5*G(J-5,1) + C4*G(J-4,1) +
1 C3*G(J-3,1) + C2*G(J-2,1) + C1*G(J-1,1))

CON2 = V(J-1,2) + H*(C5*G(J-5,2) + C4*G(J-4,2) +
1 C3*G(J-3,2) + C2*G(J-2,2) + C1*G(J-1,2))

ITER = 1

60 VCRC1 = CON1 + H*C0*GPRED1
VCRC2 = CON2 + H*C0*GPRED2

GCRC1 = VCRC1
GCRC2 = -DIMM1*VCRC2/X(J) - CONST/VCRC1*(-BETA)

TEMP = DABS(GCRC1-GPRED1) + DABS(GCRC2-GPRED2)
IF (TEMP .LT. DELTA) GO TO 70

IF (ITER .GT. 8) GO TO 320

GPRED1 = GCRC1
GPRED2 = GCRC2
ITER = ITER + 1
GO TO 60

70 G(J,1) = GCRC1
G(J,2) = GCRC2
V(J,1) = VCRC1
V(J,2) = VCRC2

80 CONTINUE

C OUTPUT V AND CHECK ITS ASYMPTOTIC BEHAVIOR USING VDIF.

LINES = N1/(7*NXSKIP)
IF ((LINES*7*NXSKIP) .NE. N1) LINES = LINES+1
INCR = 7*NXSKIP
JLOW = 1-INCR
JHIGH = 1-NXSKIP
DO 130 I = 1, LINES

    JLOW = JLOW + INCR
JHIGH = JHIGH + INCR
IF (JHIGH .GT. N1) JHIGH = N1
WRITE(6,90) (X(J),J=JLOW,JHIGH,NXSKIP)
90 FORMAT(' X: ',7(D13.6,2X))
WRITE(6,100) (V(J,1),J=JLOW,JHIGH,NXSKIP)
100 FORMAT(' V: ',7(D13.6,2X))
DO 110 J = JLOW,JHIGH,NXSKIP
  VDIFF(J) = V(J,1) - X(J)**(-TAU)
110 CONTINUE
WRITE(6,120)(VDIFF(J),J=JLOW,JHIGH,NXSKIP)
120 FORMAT(' VDIF: ',7(D13.6,2X)/)
130 CONTINUE
C CHECK FOR STRICTLY INCREASING V.
C
DO 140 I = 1,N
  IF (V(I,1) .GE. V(I+1,1)) GO TO 300
140 CONTINUE
C USING V, DETERMINE THE UNIQUE EPSL CORRESPONDING TO EACH
C FMAX. THIS IS ACCOMPLISHED USING THE OUTER BOUNDARY
C CONDITION WHICH V SATISFIES; SEE JOSEPH AND LUNDGREN.
C
XFMAX = NFMAX
DELFMX = -FMAXIN/(XFMAX*ALPHA)
JSTART = 1
NUMPI = NFMAX + 1
DO 180 I = 2,NUMPI
  TEMP = I-1
  FMAX(I) = TEMP*DELFMX
  VVAL = V(1,1)/(1.0D0 + ALPHA*FMAX(I))
  DO 150 J = JSTART,N1
    IF (V(J,1) .GE. VVAL) GO TO 160
 150 CONTINUE
  EPSL(I) = (CONST/ALPHA)*((XTEMP**W-2)**(1.0D0+BETA)
180 CONTINUE
C OUTPUT EPSL VS. FMAX.
C
190 IF (NFMAX .LE. 0) GO TO 280
WRITE(6,200)
200 FORMAT(' EPSL VS. FMAX'/)
LINES = NUMPI/7
IF ((LINES*7) .NE. NUMPl) LINES = LINES+1
JLOW = -5
JHIGH = 1
DO 230 I = 1,LINES
   JLOW = JLOW + 7
   JHIGH = JHIGH + 7
   IF (JHIGH .GT. NUMPl) JHIGH = NUMPl
   WRITE(6,210) (FMAX(J),J=JLOW,JHIGH)
210 FORMAT(• FMAX: ',7(D13.6,2X))
   WRITE(6,220) (EPSL(J),J=JLOW,JHIGH)
220 FORMAT(• EPSL: ',7(D13.6,2X)/)
230 CONTINUE
STOP
C
C ERROR-HANDLING SECTION.
C
240 WRITE(6,250)
250 FORMAT(/• "ERROR: INPUT VARIABLE EXCEEDS ACCEPTABLE •
    1 'PARAMETERS'/)
STOP
C
260 WRITE(6,270) J
270 FORMAT(/• "ERROR: R-K SOLUTION FOR V FAILS AT STEP •
    1 •I4/)
STOP
C
280 WRITE(6,290)
290 FORMAT(/• NO VALUES OF FMAX, EPSL TO OUTPUT'/)
STOP
C
300 WRITE(6,310)
310 FORMAT(/• "ERROR: V NOT STRICTLY INCREASING'/)
STOP
C
320 WRITE(6,330) X(J)
330 FORMAT(• "ERROR: NINE ITERATIONS REQUIRED IN PREDICTOR-CORRECTOR AT X = •
    1 \D13.6/, 'COMPUTATION DISCONTINUED')
STOP
END
C
C SAMPLE LIST OF INPUT VARIABLES.
C
1000.DO W
10000 DO N
3.DO DIM
100 NXSKEP
0.99DO FMAXIN
99 NFMAX
-1.DO ALPHA
-1.DO BETA
0.00001DO DELTA