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Anticommutative derivation alternator rings

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Anticommutative derivation alternator rings

by

Steven Dale Nimmo

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I. INTRODUCTION

Definition 1. A nonassociative ring with characteristic $\neq 2$ is called a derivation alternator ring if it satisfies the identities

(1) $$(x, x, x) = 0,$$

(2) $$(yz, x, x) = y(z, x, x) + (y, x, x)z,$$

(3) $$(x, x, yz) = y(x, x, z) + (x, x, y)z,$$

where the associator $(x, y, z) = (xy)z - x(yz)$. These rings are a generalization of alternative rings which are rings that satisfy the identities $(x, x, y) = 0$ and $(y, x, x) = 0$. These derivation alternator rings were initially studied in [1] by Hentzel, Hogben, and Smith. They showed that derivation alternator Lie rings, which are anticommutative rings that also satisfy the Jacobi identity,

$$(xy)z + (yz)x + (zx)y = 0,$$

are solvable of index at most 2, meaning that the product $(xy)(zw) = 0$. In this same paper, they also showed that a simple flexible derivation alternator ring is either alternative or anticommutative.

In this dissertation, we shall investigate the structure of nonassociative, anticommutative rings that satisfy (1), (2), and (3).

II. PRELIMINARIES

Note. Throughout the remainder of this dissertation, all rings and algebras are assumed to be nonassociative and all scalars are complex numbers.

Definition 2. A ring is said to be flexible if $(x, y, x) = 0$.

Proposition 1. *If a ring is anticommutative, then it is flexible.*

PROOF: By anticommutativity,

$$\begin{aligned}(x, y, x) &= (xy)x - x(yx)\\ &= (xy)x + x(xy)\\ &= (xy)x - (xy)x\\ &= 0.\end{aligned}$$

Proposition 2. *In an anticommutative ring, equation (2) implies equation (3).*

PROOF: Let A be anticommutative. By Proposition 1, A is flexible. Using $(x, y, x) = 0$ linearized, (2), and $(x, y, x) = 0$ linearized again, we see that

$$\begin{aligned}(x, x, yz) &= -(yz, x, x)\\ &= -y(z, x, x) - (y, x, x)z\\ &= y(x, x, z) + (x, x, y)z\end{aligned}$$

which establishes (3).

In light of Proposition 2, an anticommutative, derivation alternator ring can be defined simply by identity (2) and by the identity

$$xy = -yx. \tag{5}$$

If we linearize equation (2), it becomes

$$(yz, x, w) + (yz, w, x) = y(z, x, w) + (y, x, w)z + y(z, w, x) + (y, w, x)z,$$

which, by (5), simplifies to

$$(yz \cdot x)w + (yz \cdot w)x = y(zx \cdot w) + (yx \cdot w)z + y(zw \cdot x) + (yw \cdot x)z \tag{6}$$

where juxtaposition is the priority operation.

Definition 3. We define an algebra A over a field F to be a vector space over F with a multiplication satisfying

i. $(x + y)z = xz + yz$,

ii. $z(x + y) = zx + zy$,

iii. $c(xy) = (cx)y = x(cy)$,

for all $c \in F$ and $x, y, z \in A$.

Let A be an algebra satisfying (5) and (6). For $a \in A$, let the operator of right multiplication by a be denoted by R_a. Via this operator, we decompose A into generalized eigenspaces. Henceforth, we shall assume that A is finite dimensional and we shall adopt the notation that $x_i^{(j)}$ is a generalized eigenvector of order j with the associated eigenvalue r_i, $i, j \in \mathbf{N} \cup \{0\}$. In this notation, $x_i^{(0)}$, a generalized eigenvector of order 0, is a pure eigenvector. Let the set

$$\{x_i^{(j)} : i = 0, 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots, N_i\}$$

be a basis for A.

Theorem 1. *Let A be an algebra that satisfies (5) and (6). Also let $y_p^{(k)}$ and $z_q^{(\ell)}$ be generalized eigenvectors with associated eigenvalues r_p and r_q, respectively, and let $\{r_i : i = 1, \ldots, n\}$ be the eigenvalues for R_a. If we define $I = \{i : r_i^2 = r_p^2 + r_q^2\}$ then $y_p^{(k)} z_q^{(\ell)} = \sum_{i,j} t_{i,j} x_i^{(j)}$ where $i \in I$, $r_i, t_{i,j} \in \mathbf{C}$, and $a \in A$.*

PROOF: We intend to show that $y_p^{(k)} z_q^{(\ell)} = 0$ unless there is some r_i such that $r_p^2 + r_q^2 = r_i^2$, and if such an r_i exists, then $y_p^{(k)} z_q^{(\ell)} = \sum_{i,j} t_{i,j} x_i^{(j)}$ where $i \in I$.

Since $y_p^{(0)} z_q^{(0)} = \sum_{i,j} t_{i,j} x_i^{(j)}$, then by (2)

$$\begin{aligned}\left(y_p^{(0)} z_q^{(0)}\right) a \cdot a &= \left(\sum_{i,j} t_{i,j} x_i^{(j)}\right) a \cdot a \\ &= \sum_{i,j} t_{i,j} \left(x_i^{(j)} a \cdot a\right) \\ &= \sum_{i,j} t_{i,j} \left(r_i^2 x_i^{(j)} + 2 r_i x_i^{(j-1)} + x_i^{(j-2)}\right)\end{aligned}$$

where we use the notation that the term $x^{(j)}$ is zero if $j < 0$. But also

$$\begin{aligned}\left(y_p^{(0)} z_q^{(0)}\right) a \cdot a &= \left(y_p^{(0)} a \cdot a\right) z_q^{(0)} + y_p^{(0)} \left(z_q^{(0)} a \cdot a\right) \\ &= \left(r_p^2 + r_q^2\right) y_p^{(0)} z_q^{(0)} \\ &= \left(r_p^2 + r_q^2\right) \sum_{i,j} t_{i,j} x_i^{(j)}.\end{aligned}$$

Equating the coefficients of $x_i^{(j)}$ yields

$$(r_p^2 + r_q^2) t_{i,j} = r_i^2 t_{i,j} + 2 r_i t_{i,j+1} + t_{i,j+2}$$

which implies

$$t_{i,j+2} = \left(r_p^2 + r_q^2 - r_i^2\right) t_{i,j} - 2 r_i t_{i,j+1}.$$

If we now set $j = N_i$, then $t_{i,j+1} = t_{i,j+2} = 0$ and so either $r_p^2 + r_1^2 - r_i^2 = 0$ or $t_{i,j} = 0$. If $r_p^2 + r_q^2 - r_i^2 \neq 0$ then $t_{i,N_i} = 0$ and by finite induction, $t_{i,j} = 0$ for all j. Therefore $y_p^{(0)} z_q^{(0)} = \sum_{i,j} t_{i,j} x_i^{(j)}$ where $i \in I$.

Now consider $y_p^{(1)} z_q^{(0)} = \sum_{i,j} t_{i,j} x_i^{(j)}$. Proceeding as we did above

$$\left(y_p^{(1)} z_q^{(0)}\right) a \cdot a = \sum_{i,j} t_{i,j} \left(r_i^2 x_i^{(j)} + 2r_i x_i^{(j-1)} + x_i^{(j-2)}\right)$$

and also

$$\begin{aligned}\left(y_p^{(1)} z_q^{(0)}\right) a \cdot a &= y_p^{(1)} \left(z_q^{(0)} a \cdot a\right) + \left(y_p^{(1)} a \cdot a\right) z_q^{(0)} \\ &= \left(r_p^2 + r_q^2\right) y_p^{(1)} z_q^{(0)} + 2r_p y_p^{(0)} z_q^{(0)} \\ &= \left(r_p^2 + r_q^2\right) \sum_{i,j} t_{i,j} x_i^{(j)} + 2r_p y_p^{(0)} z_q^{(0)}.\end{aligned}$$

Equating the coefficients of $x_i^{(j)}$ and using the previous case for $y_p^{(0)} z_q^{(0)}$ we obtain

$$\left(r_p^2 + r_q^2\right) t_{i,j} + 2r_p \left(\sum_{i \in I, j} s_{i,j}\right) = r_i^2 t_{i,j} + 2r_i t_{i,j+1} + t_{i,j+2}.$$

Solving for $t_{i,j+2}$ we have

$$t_{i,j+2} = \left(r_p^2 + r_q^2 - r_i^2\right) t_{i,j} - 2r_i t_{i,j+1} + 2r_p \left(\sum_{i \in I, j} s_{i,j}\right).$$

Setting $j = N_i$ we get

$$0 = \left(r_p^2 + r_q^2 - r_i^2\right) t_{i,j} + 2r_p \left(\sum_{i \in I, j} s_{i,j}\right).$$

Thus if $i \notin I$, then $t_{i,N_i} = 0$, and by induction we see that $t_{i,j} = 0$ for all j. Hence,

$$y_p^{(1)} z_q^{(0)} = \sum_{i,j} t_{i,j} x_i^{(j)} \qquad \text{where} \quad i \in I.$$

Now we proceed by induction on $m = k + \ell$. We will assume that

$$y_p^{(k)} z_q^{(\ell)} = \sum_{i,j} s_{i,j} x_i^{(j)} \qquad \text{where} \quad i \in I \quad \text{for} \quad k + \ell < m.$$

Consider the product

$$y_p^{(k)} z_q^{(\ell)} = \sum_{i,j} t_{i,j} x_i^{(j)} \qquad \text{where} \quad k + \ell = m.$$

Then by (2),

$$\begin{aligned}
\left(y_p^{(k)} z_q^{(\ell)}\right) a \cdot a &= \left(\sum_{i,j} t_{i,j} x_i^{(j)}\right) a \cdot a \\
&= \sum_{i,j} t_{i,j} \left(x_i^{(j)} a \cdot a\right) \\
&= \sum_{i,j} t_{i,j} \left(r_i^2 x_i^{(j)} + 2 r_i x_i^{(j-1)} + x_i^{(j-2)}\right).
\end{aligned}$$

Also

$$\begin{aligned}
\left(y_p^{(k)} z_q^{(\ell)}\right) a \cdot a &= y_p^{(k)} \left(z_q^{(\ell)} a \cdot a\right) + \left(y_p^{(k)} a \cdot a\right) z_q^{(\ell)} \\
&= \left(r_p^2 + r_q^2\right) y_p^{(k)} z_q^{(\ell)} + 2 \left(r_p y_p^{(k-1)} z^{(\ell)} + r_q y_p^{(k)} z_q^{(\ell-1)}\right) \\
&\quad + y_p^{(k-2)} z_q^{(\ell)} + y_p^{(k)} z_q^{(\ell-2)} \\
&= \left(r_p^2 + r_q^2\right) \sum_{i,j} t_{i,j} x_i^{(j)} + \left(\sum_{i \in I, j} s_{i,j} x_i^{(j)}\right)
\end{aligned}$$

by the induction assumption. Equating coefficients of $x_i^{(j)}$ and solving for $t_{i,j+2}$, we obtain

$$t_{i,j+2} = \left(r_p^2 + r_q^2 - r_i^2\right) t_{i,j} - 2 r_i t_{i,j+1} + \left(\sum_{i \in I, j} s_{i,j}\right).$$

Setting $j = N_i$ yields

$$0 = \left(r_p^2 + r_q^2 - r_i^2\right) t_{i,j} + \left(\sum_{i \in I, j} s_{i,j}\right).$$

Thus if $i \notin I$ then $t_{i,N_i} = 0$ and so by induction $t_{i,j} = 0$ for all j. Therefore

$$y_p^{(k)} z_q^{(\ell)} = \sum_{i,j} t_{i,j} x_i^{(j)} \qquad \text{where} \quad i \in I.$$

Theorem 2. *Let A be an algebra that satisfies (5) and (6), and let $a \in A$. If b and c are pure eigenvectors for R_a with associated eigenvalues λ and μ, respectively, then bc is a sum of two pure eigenvectors or $\lambda^2 + \mu^2 = 0$.*

PROOF: Suppose $\lambda^2 + \mu^2 = t^2 \neq 0$ and that

$$bc = k_j x_t^{(j)} + \cdots + k_0 x_t^{(0)}.$$

Invoking the same technique we used in Theorem 1,

$$\begin{aligned}(bc \cdot a)a &= b(ca \cdot a) + (ba \cdot a)c \\ &= t^2 bc \\ &= t^2 \left(k_j x_t^{(j)} + \cdots + k_0 x_t^{(0)}\right)\end{aligned}$$

and also

$$\begin{aligned}(bc \cdot a)a &= \left(k_j x_t^{(j)} + k_{j-1} x_t^{(j-1)} + \cdots + k_0 x_t^{(0)}\right) a \cdot a \\ &= k_j t^2 x_t^{(j)} + \left(2k_j t + k_{j-1} t^2\right) x_t^{(j-1)} + \ldots .\end{aligned}$$

Equating the coefficients of $x_t^{(j-1)}$ we obtain

$$k_{j-1} t^2 = 2k_j t + k_{j-1} t^2$$

which implies that $2k_j t = 0$. Thus $k_j = 0$ or $t = 0$, but both of these possibilities lead to a contradiction. Hence, bc is a pure eigenvector.

In the case that bc lives in the sum of the generalized eigenspaces corresponding to t and $-t$,

$$bc = k_j x_t^{(j)} + \cdots + k_0 x_t^{(0)} + \ell_j y_{-t}^{(j)} + \cdots + \ell_0 y_{-t}^{(0)},$$

then an analogous argument shows that bc must be a sum of two pure eigenvectors.

Theorem 2 does not guarantee that bc is a pure eigenvector if $t = 0$. The following example shows that if $t = 0$, then bc does not have to be a pure eigenvector.

Example 1.

	a	b	c	$d^{(1)}$	d
a	0	0	0	$-d$	0
b	0	0	$d^{(1)}$	0	0
c	0	$-d^{(1)}$	0	0	0
$d^{(1)}$	d	0	0	0	0
d	0	0	0	0	0

As one can see, the only nonzero products are $d^{(1)}a$, bc, and $bc \cdot a$.

Since none of these products contain four elements, equation (6) holds trivially. From the table, it is clear that (5) holds. Hence, this example shows that if $t = 0$, then bc does not have to be a pure eigenvector even though b and c are pure eigenvectors.

III. THE PURE EIGENVECTOR CASE

Note. In this section, we shall assume that the algebra A satisfies identities (5) and (6). Also, for $a \in A$, we shall assume that the operator R_a decomposes A into pure eigenspaces.

Let b, c, and d be eigenvectors with eigenvalues λ, μ, and η, respectively. Consider the products $(bc)d$, $(cd)b$, and $(db)c$. Earlier we saw that these products must individually lie in the sum of the generalized eigenspaces corresponding to $\pm\Delta$ where $\Delta^2 = \lambda^2 + \mu^2 + \eta^2$. By Theorem 2,

$$k_1(bc)d + k_2(cd)b + k_3(db)c = k_1(x_1 + y_1) + k_2(x_2 + y_2) + k_3(x_3 + y_3)$$

where $x_i a = \Delta x_i$, $y_i a = -\Delta y_i$. If

$$k_1(bc)d + k_2(cd)b + k_3(db)c = 0,$$

then we obtain the two equations

$$k_1 x_1 + k_2 x_2 + k_3 x_3 = 0$$

and

$$k_1 y_1 + k_2 y_2 + k_3 y_3 = 0$$

by the linear independence of the x_i's from the y_i's. For this reason, whenever an equation only involves the products $(bc)d$, $(cd)b$, and $(db)c$, we can separate the equation into two equations, one involving Δ and the other involving $-\Delta$.

Theorem 3. *Let b, c, and d be eigenvectors with eigenvalues λ, μ, and η, respectively. Then*

$$(\lambda - \Delta)(bc \cdot d) + (\eta - \mu)(cd \cdot b) + (\Delta - \lambda)(db \cdot c) = 0 \tag{7}$$

$$(\Delta - \mu)(bc \cdot d) + (\mu - \Delta)(cd \cdot b) + (\lambda - \eta)(db \cdot c) = 0 \tag{8}$$

$$\left(\sqrt{\lambda^2 + \mu^2} + \Delta\right)(bc \cdot d) + \left(\sqrt{\mu^2 + \eta^2} + \mu\right)(cd \cdot b) + \left(\sqrt{\lambda^2 + \eta^2} + \lambda\right)(db \cdot c) = 0 \tag{9}$$

where $\Delta^2 = \lambda^2 + \mu^2 + \eta^2$ *and* $b, c, d \in A$.

PROOF: By (6) we have

$$(ab \cdot c)d + (ab \cdot d)c = a(bc \cdot d) + (ac \cdot d)b + a(bd \cdot c) + (ad \cdot c)b$$

which implies

$$\lambda(bc \cdot d) + \lambda(bd \cdot c) = \Delta(bc \cdot d) + \mu(cd \cdot b) + \Delta(bd \cdot c) + \eta(dc \cdot b).$$

Using (5) and bringing all terms to the left hand side, we obtain (7).

To establish (8), we proceed as above starting with

$$(ac \cdot b)d + (ac \cdot d)b = a(cb \cdot d) + (ab \cdot d)c + a(cd \cdot b) + (ad \cdot b)c$$

which comes from (6).

Finally, equation (6) implies

$$(bc \cdot d)a + (bc \cdot a)d = b(cd \cdot a) + (bd \cdot a)c + b(ca \cdot d) + (ba \cdot d)c$$

which simplifies to

$$\Delta(bc \cdot d) + \sqrt{\lambda^2 + \mu^2}(bc \cdot d) = \sqrt{\mu^2 + \eta^2}(b \cdot cd) + \sqrt{\lambda^2 + \eta^2}(bd \cdot c) + \mu(b \cdot cd) + \lambda(bd \cdot c).$$

Using (5) and moving all of the terms to the left hand side establishes (9).

Corollary 4. *Let b, c, and d be eigenvectors with eigenvalues λ, μ, and η, respectively. Then*

$$\frac{bc \cdot d}{(\Delta - \eta)(\Delta + \eta - \lambda - \mu)} = \frac{cd \cdot b}{(\Delta - \lambda)(\Delta + \lambda - \mu - \eta)} = \frac{db \cdot c}{(\Delta - \mu)(\Delta + \mu - \lambda - \eta)}$$

assuming that each denominator is nonzero and $b, c, d \in A$. *If exactly one of the denominators is zero, then the relationship still holds between the other two terms.*

PROOF: By Theorem 3, we know that (7) and (8) are true. Dividing (7) by $(\lambda - \Delta)$ and dividing (8) by $(\Delta - \mu)$, we can solve each of the resulting equations for $bc \cdot d$. Thus we obtain

$$bc \cdot d = cd \cdot b - \left(\frac{\lambda - \eta}{\Delta - \mu}\right) db \cdot c = -\left(\frac{\mu - \eta}{\Delta - \lambda}\right) cd \cdot b + db \cdot c.$$

Combining like terms gives us

$$\left(\frac{\Delta - \lambda + \mu - \eta}{\Delta - \lambda}\right) cd \cdot b = \left(\frac{\Delta + \lambda - \mu - \eta}{\Delta - \mu}\right) db \cdot c$$

which implies

$$\frac{cd \cdot b}{(\Delta - \lambda)(\Delta + \lambda - \mu - \eta)} = \frac{db \cdot c}{(\Delta - \mu)(\Delta + \mu - \lambda - \eta)}.$$

To obtain the other relation, we add (7) and (8) to get

$$(10) \qquad (\lambda - \mu)(bc \cdot d) + (\eta - \Delta)(cd \cdot b) + (\Delta - \eta)(db \cdot c) = 0.$$

If we divide (10) by $(\Delta - \eta)$ and divide (7) by $(\Delta - \lambda)$, we can solve each of the resulting equations for $db \cdot c$. Proceeding as we did above yields

$$\frac{bc \cdot d}{(\Delta - \eta)(\Delta + \eta - \lambda - \mu)} = \frac{cd \cdot b}{(\Delta - \lambda)(\Delta + \lambda - \mu - \eta)}.$$

Corollary 5. *Let b and c be eigenvectors with eigenvalues λ and μ, respectively. Then* $b \cdot bc = 0$ *or* $\lambda = 0$ *where* $b, c \in A$.

PROOF: We first note that setting $x = y$ in (5) yields $x^2 = -x^2$ which implies $2x^2 = 0$. Since we assumed characteristic $\neq 2$, we have $x^2 = 0$.

Now we set $d = b$ and $\eta = \lambda$ in equation (7) of Theorem 3. Thus

$$(\lambda - \Delta)(bc \cdot b) + (\lambda - \mu)(cb \cdot b) + (\Delta - \lambda)(bb \cdot c) = 0.$$

By (5) and our note, this simplifies to

$$(\Delta - \mu)(b \cdot bc) = 0.$$

Therefore $b \cdot bc = 0$ or $\Delta = \mu$. If $\Delta = \mu$ then $\Delta^2 = 2\lambda^2 + \mu^2 = \mu^2$ which implies $\lambda = 0$. Hence, $b \cdot bc = 0$ or $\lambda = 0$.

Theorem 6. *Let b, c, and d be eigenvectors with eigenvalues λ, μ, and η, respectively. Then $bc \cdot d = cd \cdot b = db \cdot c = 0$ unless the determinant of the matrix*

$$M = \begin{pmatrix} \lambda - \Delta & \eta - \mu & \Delta - \lambda \\ \Delta - \mu & \mu - \Delta & \lambda - \eta \\ \sqrt{\lambda^2 + \mu^2} + \Delta & \sqrt{\mu^2 + \eta^2} + \mu & \sqrt{\lambda^2 + \eta^2} + \lambda \end{pmatrix}$$

is zero where $b, c, d \in A$ and $\Delta^2 = \lambda^2 + \mu^2 + \eta^2$.

PROOF: By Theorem 3, equations (7), (8), and (9) hold. We can represent these three equations in the following matrix equation

$$\begin{pmatrix} \lambda - \Delta & \eta - \mu & \Delta - \lambda \\ \Delta - \mu & \mu - \Delta & \lambda - \eta \\ \sqrt{\lambda^2 + \mu^2} + \Delta & \sqrt{\mu^2 + \eta^2} + \mu & \sqrt{\lambda^2 + \eta^2} + \lambda \end{pmatrix} \begin{pmatrix} bc \cdot d \\ cd \cdot b \\ db \cdot c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, if M is nonsingular, the only solution of this matrix equation is

$$bc \cdot d = cd \cdot b = db \cdot c = 0.$$

If we expand the determinant of the matrix M appearing in Theorem 6, we obtain

$$\sqrt{\lambda^2 + \mu^2}(\Delta - \eta)(\Delta + \eta - \lambda - \mu) + \sqrt{\lambda^2 + \eta^2}(\Delta - \mu)(\Delta + \mu - \lambda - \eta) \\ + \sqrt{\mu^2 + \eta^2}(\Delta - \lambda)(\Delta + \lambda - \mu - \eta) + 2\lambda\mu\eta.$$

If we set this determinant equal to zero and solve for $\sqrt{\lambda^2+\mu^2}(\Delta-\eta)(\Delta+\eta-\lambda-\mu)+2\lambda\mu\eta$, then we can square both sides to obtain an equation of the form

$$K_1 + K_2\sqrt{\lambda^2+\mu^2} = K_3 + K_4\sqrt{(\lambda^2+\eta^2)(\mu^2+\eta^2)}.$$

From here, we solve for $K_4\sqrt{(\lambda^2+\eta^2)(\mu^2+\eta^2)}$ and again square both sides. The resulting equation has the form

$$K_5 = K_6\sqrt{\lambda^2+\mu^2}.$$

Squaring this equation will give us an equation in λ, μ, η, and Δ. Next, we put all of the terms with odd exponents for Δ on one side of the equation and all terms with even exponents for Δ on the other side. Now we square both sides and move all of the terms to one side. The resulting equation is of the form

$$f(\lambda,\mu,\eta) = 0$$

where f is symmetric in the variables λ, μ, η. (Actually, the original determinant is symmetric in λ, μ, and η, and this symmetry is preserved under the operation we have performed.) If we make the change of variables

$$S = \lambda+\mu+\eta, \qquad P = \lambda\mu\eta, \qquad Q = \lambda\mu+\mu\eta+\lambda\eta,$$

then $f(\lambda,\mu,\eta)$ can be rewritten as

$$\begin{aligned}
4096P^6\,\big[&2048S^{15}P^5 + 22784S^{14}Q^2P^4 - 20992S^{13}Q^4P^3 - 108032S^{13}QP^5 \\
&+ 6096S^{12}Q^6P^2 - 159104S^{12}Q^3P^4 + 83200S^{12}P^6 - 576S^{11}Q^8P \\
&+ 173984S^{11}Q^5P^3 + 925952S^{11}Q^2P^5 - 45016S^{10}Q^7P^2 + 288224S^{10}Q^4P^4 \\
&- 969216S^{10}QP^6 + 2136S^9Q^9P - 546704S^9Q^6P^3 - 3222144S^9Q^3P^5 \\
&+ 178176S^9P^7 + 288S^8Q^{11} + 114425S^8Q^8P^2 + 281568S^8Q^5P^4 \\
&+ 4158464S^8Q^2P^6 + 6528S^7Q^{10}P + 834168S^7Q^7P^3 + 5257984S^7Q^4P^5 \\
&- 1343488S^7QP^7 - 2304S^6Q^{12} - 103160S^6Q^9P^2 - 1363760S^6Q^6P^4 \\
&- 8192640S^6Q^3P^6 + 112128S^6P^8 - 44352S^5Q^{11}P - 770960S^5Q^8P^3
\end{aligned}$$

$$- 3881120S^5Q^5P^5 + 3493632S^5Q^2P^7 + 6912S^4Q^{13} + 11576S^4Q^{10}P^2$$
$$+ 1297760S^4Q^7P^4 + 7307616S^4Q^4P^6 - 511488S^4QP^8 + 76800S^3Q^{12}P$$
$$+ 680096S^3Q^9P^3 + 1288448S^3Q^6P^5 - 3513216S^3Q^3P^7 + 30720S^3P^9$$
$$- 9216S^2Q^{14} - 39136S^2Q^{11}P^2 - 553536S^2Q^8P^4 - 2448000S^2Q^5P^6$$
$$+ 623872S^2Q^2P^8 - 44160SQ^{13}P - 426432SQ^{10}P^3 - 540544SQ^7P^5$$
$$+ 947456SQ^4P^7 - 69120SQP^9 + 4608Q^{15} + 76816Q^{12}P^2$$
$$+ 364160Q^9P^4 + 361600Q^6P^6 - 91648Q^3P^8 + 6400P^{10}\big].$$

The products $bc \cdot d$, $cd \cdot b$, and $db \cdot c$ will be zero unless the determinant of the matrix M of Theorem 6 is zero. To find these exceptional cases, we first check to see if this polynomial evaluates to zero. For zeros of this polynomial, we then check the actual determinant to see which of the eight sign choices for the square roots will lead to a possibility for a nonzero product.

Since P^3 is a factor of this polynomial, it is evident that if one or more of λ, μ, and η is zero, then the polynomial is zero. (If $\lambda = 0$ and we let $\sqrt{\mu^2} = \mu$, $\sqrt{\eta^2} = \eta$, $\Delta = \sqrt{\mu^2 + \eta^2}$ be our choice of signs for the square roots, then the determinant is zero as well.)

In the next theorem, we shall examine, in detail, what happens if $\lambda = 0$.

<u>Theorem 7</u>. *Let b, c, and d be eigenvectors with eigenvalues λ, μ, and η, respectively. If $\lambda = 0$, then one of the following must happen:*

(i) $\Delta = 0$, $cd \cdot b = 0$, *and* $bc \cdot d = \pm i(db \cdot c)$,

(ii) $\Delta = 0$, $\mu = 0$, *and* $\eta = 0$,

(iii) $\mu = 0$, $\Delta = -\eta$, $bc \cdot d = 0$, *and* $cd \cdot b = db \cdot c$,

(iv) $\mu = 0$, $\Delta = \eta$, *and* $bc \cdot d = cd \cdot b + db \cdot c$,

(v) $\eta = 0$, $\Delta = -\mu$, $db \cdot c = 0$, *and* $bc \cdot d = cd \cdot b$,

(vi) $\eta = 0$, $\Delta = \mu$, *and* $db \cdot c = bc \cdot d + cd \cdot b$,

(vii) $\dfrac{bc\cdot d}{(\Delta-\eta)(\Delta+\eta-\mu)}=\dfrac{cd\cdot b}{\Delta(\Delta-\mu-\eta)}=\dfrac{db\cdot c}{(\Delta-\mu)(\Delta+\mu-\eta)}$, *where* $\Delta^2=\mu^2+\eta^2$.

PROOF: By Corollary 4, we know that (vii) holds unless the denominators are zero. Therefore, we need to see what happens when (vii) does not hold, i.e., when any one of the following expressions is zero:

$$\Delta$$
$$\Delta-\mu$$
$$\Delta-\eta$$
$$\Delta-\mu-\eta$$
$$\Delta-\mu+\eta$$
$$\Delta+\mu-\eta.$$

If $\Delta=\lambda=0$, then $\Delta^2=\mu^2+\eta^2$ implies that $\mu=\pm\eta\neq 0$ or $\lambda=\mu=\eta=0$.

If $\Delta-\mu=0$, then $\Delta^2=\mu^2+\eta^2$ implies that $\eta=0$ and $\Delta=\mu$. Similarly, if $\Delta-\eta=0$, then $\mu=0$ and $\Delta=\eta$.

For the possibilities $\Delta\pm\mu\pm\eta=0$, we can solve for Δ and upon squaring both sides we obtain

$$u^2+\eta^2=\Delta^2=\mu^2+\eta^2\pm 2\mu\eta$$

which implies that $\mu=0$ and $\Delta=\pm\eta$ or $\eta=0$ and $\Delta=\pm\mu$.

Now by Theorem 3,

$$-\Delta(bc\cdot d)+(\eta-\mu)(cd\cdot b)+\Delta(db\cdot c)=0, \tag{11}$$

$$(\Delta-\mu)(bc\cdot d)+(\mu-\Delta)(cd\cdot b)-\eta(db\cdot c)=0. \tag{12}$$

If $\Delta=\lambda=0$ and $\mu=\pm\eta\neq 0$, then from (11) we obtain

$$cd\cdot b=0,$$

and from (12) we obtain

$$bc \cdot d = \pm i(db \cdot c).$$

Now assume that $\lambda = \mu = 0$. Under this assumption, we can add (11) and (12) to get the equation

$$(\eta - \Delta)(cd \cdot b) + (\Delta - \eta)(db \cdot c) = 0.$$

From this we see that either $cd \cdot b = db \cdot c$ or $\Delta = \eta$. If $\Delta = -\eta$, then (11) implies

$$\eta(bc \cdot d) + \eta(cd \cdot b) = \eta(db \cdot c).$$

Hence, either $\lambda = \mu = \eta = 0$ or $bc \cdot d + cd \cdot b = db \cdot c$, which means that $\lambda = \mu = \eta = 0$ or

$$bc \cdot d = 0 \qquad \text{and} \qquad cd \cdot b = db \cdot c.$$

If $\Delta = \eta$, then (11) becomes

$$-\eta(bc \cdot d) + \eta(cd \cdot b) + \eta(db \cdot c) = 0.$$

Thus, $\lambda = \mu = \eta = 0$ or $bc \cdot d = cd \cdot b + db \cdot c$.

If $\lambda = \eta = 0$, then by symmetry we arrive at possibilities (v) and (vi).

Theorem 8. *Let b, c, and d be eigenvectors with eigenvalues λ, μ, and η, respectively. If $\Delta = 0$, then one of the following must happen:*

(i) $\lambda = 0$, $cd \cdot b = 0$, $bc \cdot d = \pm i(db \cdot c)$,

(ii) $\mu = 0$, $db \cdot c = 0$, $bc \cdot d = \pm i(cd \cdot b)$,

(iii) $\eta = 0$, $bc \cdot d = 0$, $cd \cdot b = \pm i(db \cdot c)$,

(iv) $\eta = \lambda + \mu$, $bc \cdot d = 0$, $cd \cdot b = db \cdot c$,

(v) $\lambda = \mu + \eta$, $cd \cdot b = 0$, $bc \cdot d = db \cdot c$,

(vi) $\mu = \lambda + \eta$, $db \cdot c = 0$, $bc \cdot d = cd \cdot b$,

(vii) $\lambda = \mu = \eta = 0$,

(viii) $\dfrac{bc \cdot d}{\eta(\eta - \lambda - \mu)} = \dfrac{cd \cdot b}{\lambda(\lambda - \mu - \eta)} = \dfrac{db \cdot c}{\mu(\mu - \lambda - \eta)}$

where $\Delta^2 = \lambda^2 + \mu^2 + \eta^2$ and $b, c, d \in A$.

<u>PROOF</u>: If $\Delta = \lambda = 0$, then as we showed in Theorem 7, $cd \cdot b = 0$ and $bc \cdot d = \pm i(db \cdot c)$ or $\lambda = \mu = \eta = 0$. By symmetry, we obtain (ii) and (iii).

Now by Corollary 4, (viii) holds unless one of the denominators is zero. If λ, μ, or η is zero, then (i), (ii), or (iii) holds, respectively. Thus we need only to consider the cases in which $\eta - \lambda - \mu$, $\lambda - \mu - \eta$, or $\mu - \lambda - \eta$ are individually zero . Again, by symmetry, we shall only consider $\eta - \lambda - \mu = 0$.

With $\Delta = 0$ and $\eta = \lambda + \mu$, equations (7) and (8) of Theorem 3 become

$$\lambda(bc \cdot d) + \lambda(cd \cdot b) - \lambda(db \cdot c) = 0$$

and

$$-\mu(bc \cdot d) + \mu(cd \cdot b) - \mu(db \cdot c) = 0.$$

Since we have already considered the possibilities of λ and μ being zero, we may assume that λ and μ are nonzero. Hence, our two equations imply

$$bc \cdot d + cd \cdot b - db \cdot c = 0$$

and

$$bc \cdot d - cd \cdot b + db \cdot c = 0.$$

Therefore,

$$bc \cdot d = 0 \qquad \text{and} \qquad cd \cdot b = db \cdot c.$$

Another result stemming from Theorem 3 is the following:

Theorem 9. *Let b, c, and $\tilde{c}$ be eigenvectors in A with eigenvalues λ, μ, and μ, respectively, $\mu \neq 0$. Then $bc \cdot \tilde{c} + b\tilde{c} \cdot c = 0$, i.e., if c and $\tilde{c}$ are in the same eigenspace, then $R_c R_{\tilde{c}} + R_{\tilde{c}} R_c = 0$.*

PROOF: By (7) of Theorem 3 with $d = \tilde{c}$ and $\eta = \mu$ we obtain

$$(\lambda - \Delta) bc \cdot \tilde{c} + (\Delta - \lambda)\tilde{c}b \cdot c = 0$$

where $\Delta^2 = \lambda^2 + 2\mu^2$. Since $\mu \neq 0$, then $\Delta \neq \lambda$ and so by (5), this equation becomes

$$bc \cdot \tilde{c} + b\tilde{c} \cdot c = 0.$$

Since $f(\lambda, \mu, \eta)$ is homogeneous, then $f(\lambda, \mu, \eta) = \mu^{48} f\left(\frac{\lambda}{\mu}, 1, \frac{\eta}{\mu}\right)$. Hence, for $\mu \neq 0$, we can assume that $\mu = 1$ in Theorem 9. Then by Theorem 6 and the discussion that follows it,

$$bc \cdot \tilde{c} = c\tilde{c} \cdot b = \tilde{c}b \cdot c = 0$$

unless $f(\lambda, 1, 1) = 0$. Substituting these values into f gives us the equation

$$\begin{aligned} f(\lambda, 1, 1) = K \big[&3136\lambda^{16} + 10304\lambda^{15} + 25712\lambda^{14} + 65296\lambda^{13} \\ &+ 82168\lambda^{12} + 140208\lambda^{11} + 17791\lambda^{10} + 90720\lambda^{9} \\ &+ 247512\lambda^{8} - 12448\lambda^{7} + 68968\lambda^{6} + 104768\lambda^{5} \\ &+ 194976\lambda^{4} + 168320\lambda^{3} - 127120\lambda^{2} + 42240\lambda - 4608\big] \\ = 0, \end{aligned}$$

where K is a constant, which has the following approximate solutions for λ:

$$\begin{array}{ll} 0.25604 \pm 0.00196i, & 0.22518 \pm 0.70320i, \\ -1.21259 \pm 0.34785i, & 0.79121 \pm 0.79387i, \\ -0.14284 \pm 0.84681i, & -0.94891 \pm 1.80837i, \\ 0.42936 \pm 2.11726i, & 0.54548 \\ -2.62609, & \text{and} \quad 0. \end{array}$$

Therefore, $bc \cdot \tilde{c} = c\tilde{c} \cdot b = \tilde{c}b \cdot c = 0$ unless $\frac{\lambda}{\mu}$ is equal to one of these values.

Along the same vein, let us consider what happens if the eigenvectors b, c, and d have corresponding eigenvalues λ, 1, and i. Then $\Delta^2 = \lambda^2$ and by Theorem 6 and the discussion that follows it,

$$bc \cdot d = cd \cdot b = db \cdot c = 0$$

or $f(\lambda, 1, i) = 0$. If $f(\lambda, 1, i) = 0$, then λ is a root of the following polynomial:

$$\begin{aligned} x^6 \big[&3200ix^{14} + 960(i-1)x^{13} - 2064x^{12} + 2272(i+1)x^{11} \\ &+ 2400ix^{10} - 2544(i-1)x^9 + 4088x^8 + 4152(i+1)x^7 \\ &- 1248ix^6 - 1528(i-1)x^5 + 279x^4 - 864(i+1)x^3 \\ &-2672ix^2 - 768(i-1)x + 576\big]. \end{aligned}$$

The roots of this polynomial are approximately:

$$\begin{array}{ll} -0.57923 + 0.27801i, & 0.27801 - 0.57923i, \\ -0.63884 - 0.63884i, & 0.29428 + 0.95572i, \\ 0.95572 + 0.29428i, & 0.48830 - 0.94824i, \\ -0.94824 + 0.48830i, \text{ and } 0. & \end{array}$$

Let b, c and d be eigenvectors with corresponding eigenvalues λ, μ, and η, respectively. With $\Delta^2 = \lambda^2+\mu^2+\eta^2$, we saw in Theorem 8 what happens if $\Delta = 0$. We can actually say a little more. If $\Delta = 0$, then $f(\lambda,\mu,\eta)$ can be written in terms of $S = \lambda+\mu+\eta$ and $P = \lambda\mu\eta$ since $\Delta^2 = S^2 - 2Q$. The polynomial becomes $8192P^6[40P^2 - 12PS^3 + S^6]$. Hence, the polynomial is zero when $P = 0$, $20P = (3+i)S^3$ or $20P = (3-i)S^3$. Therefore, $bc \cdot d = cd \cdot b = db \cdot c = 0$ unless $\lambda\mu\eta = 0$, $20\lambda\mu\eta = (3+i)(\lambda+\mu+\eta)^3$ or $20\lambda\mu\eta = (3-i)(\lambda+\mu+\eta)^3$.

Next we shall see what happens if $Q = 0$.

If $Q = 0$, then the polynomial becomes

$$65536P^4(5P+8S^3).$$

Hence, $bc \cdot d = cd \cdot b = db \cdot c = 0$ unless $P = \lambda\mu\eta = 0$ or $P = -\frac{8}{5}S^3$, i.e., $\lambda\mu\eta = -\frac{8}{5}(\lambda+\mu+\eta)^3$.

In Corollary 5, we saw that $x^2 = 0$ for all $x \in A$. In the next theorem, we shall show that the product of three eigenvectors from the same eigenspace is zero unless the eigenvalue is zero.

Theorem 10. *Let b, c, and d be eigenvectors all from the same eigenspace with eigenvalue $\lambda \neq 0$. Then $bc \cdot d = cd \cdot b = db \cdot c = 0$ where $b, c, d \in A$.*

PROOF: $f(\lambda,\lambda,\lambda) = -888191\lambda^{30}$.

Therefore, by Theorem 6 and the fact that $\lambda \neq 0$, it must happen that $bc \cdot d = cd \cdot b = db \cdot c = 0$.

Theorem 11. *Let b, c, and d be eigenvectors with distinct eigenvalues λ, μ, and η, respectively. If one of the denominators in Corollary 4 is zero, then the numerator is zero, e.g., if $(\Delta-\lambda)(\Delta+\lambda-\mu-\eta) = 0$ then $cd \cdot b = 0$.*

PROOF: By symmetry, we only need to show that if $(\Delta - \lambda)(\Delta + \lambda - \mu - \eta) = 0$ then $cd \cdot b = 0$.

If $\Delta - \lambda = 0$, then by equation (7) of Theorem 3,

$$(\eta - \mu)(cd \cdot b) = 0.$$

Since we assumed that μ and η were distinct, this implies that $cd \cdot b = 0$.

If $\Delta + \lambda - \mu - \eta = 0$, then equation (8) of Theorem 3 becomes

$$(\eta - \lambda)(bc \cdot d) + (\lambda - \eta)(cd \cdot b) + (\lambda - \eta)(db \cdot c) = 0.$$

Upon adding (7) and (8) and substituting $\Delta = -\lambda + \mu + \eta$ we also obtain

$$(\lambda - \mu)(bc \cdot d) + (\lambda - \mu)(cd \cdot b) + (\mu - \lambda)(db \cdot c) = 0.$$

Since λ, μ, and η are assumed to be distinct, these two equation become

$$-bc \cdot d + cd \cdot b + db \cdot c = 0$$

and

$$bc \cdot d + cd \cdot b - db \cdot c = 0.$$

Adding these two equations and knowing that the characteristic $\neq 2$, we find that $cd \cdot b = 0$.

Our next two results will allow us to prove that if A is simple, then the only eigenspace of A relative to R_a corresponds to the eigenvalue zero. Thus, R_a is nilpotent for any $a \in A$.

Lemma 12. *Let b, $\bar{b}$, c, and $\bar{c}$ be eigenvectors with corresponding eigenvalues λ, $i\lambda$, μ and $i\mu$, respectively, $\lambda \neq 0$. Then $(b\bar{b})(c\bar{c}) = 0$ where $b, \bar{b}, c, \bar{c} \in A$.*

<u>PROOF</u>: By equation (6),

$$(a \cdot c\bar{c})\bar{b} \cdot b + (a \cdot c\bar{c})b \cdot \bar{b}$$
$$= a \cdot (c\bar{c} \cdot \bar{b})b + (a\bar{b} \cdot b) \cdot c\bar{c} + a \cdot (c\bar{c} \cdot b)\bar{b} + (ab \cdot \bar{b}) \cdot c\bar{c}$$

which implies that

$$0 = i\lambda(\bar{b}b \cdot c\bar{c}) + \lambda(b\bar{b} \cdot c\bar{c}).$$

Thus $\lambda(1-i)(b\bar{b} \cdot c\bar{c}) = 0$ and since $\lambda \neq 0$, it must happen that $(b\bar{b})(c\bar{c}) = 0$.

<u>Lemma 13</u>. *Let b_j and $\bar{b}_j$ be eigenvectors with eigenvalues λ_j and $i\lambda_j$, respectively, $\lambda_j \neq 0$, for $j = 1, \ldots, n$. Then $a \neq \sum_{j=1}^{n} b_j\bar{b}_j$.*

<u>PROOF</u>: Suppose $a = \sum_{j=1}^{n} b_j\bar{b}_j$. Then

$$\begin{aligned} 0 \neq -\lambda b_1\bar{b}_1 &= ab_1 \cdot \bar{b}_1 \\ &= \left(\sum_{j=1}^{n} b_j\bar{b}_j\right) b_1 \cdot \bar{b}_1 \\ &= \sum_{j=1}^{n} (b_j\bar{b}_j)b_1 \cdot \bar{b}_1. \end{aligned}$$

If we can show that $(b_j\bar{b}_j)b_1 \cdot \bar{b}_1 = 0$ for all j, then we are done. As we saw in the proof of Theorem 9, we may assume that $\lambda_j = 1$. Then since $(\Delta - i\lambda_1)(\Delta + i\lambda_1 - 0 - \lambda_1) = -i\lambda_1(i\lambda_1 - \lambda_1) \neq 0$ and $(\Delta - \lambda_1)(\Delta + \lambda_1 - 0 - i\lambda_1) = -\lambda_1(\lambda_1 - i\lambda_1) \neq 0$, Corollary 4 allows us to say that

$$(b_j\bar{b}_j)b_1 \cdot \bar{b}_1 = K\left[(b_j\bar{b}_j)\bar{b}_1 \cdot b_1\right] \tag{13}$$

where K is nonzero. Therefore, it suffices to show that either side of (13) is zero.

If $(b_j\bar{b}_j)b_1 \neq 0$, then $f(1, i, \lambda_1) = 0$, and if $(b_j\bar{b}_j)\bar{b}_1 \neq 0$, then $f(1, i, i\lambda_1) = 0$. But earlier we found all values of x such that $f(1, i, x) = 0$, and no pair of these values has the property that $x_1 = ix_2$. This means that either $f(1, i, \lambda_1) \neq 0$ or $f(1, i, i\lambda_1) \neq 0$. Therefore one side or the other of equation (13) is zero.

Theorem 14. *Let $A^{\#}$ be the linear span of all of the nonzero eigenspaces and let A^0 be the linear span of the kernel of R_a. If A is simple, then $A = A^0$, and hence, R_a is nilpotent for any $a \in A$.*

PROOF: Consider the ideal, I, generated by $A^{\#}$. Then $I = A^{\#} + Z$ where $Z =$ linear span$\{b\bar{b} : ab = \lambda b, \quad a\bar{b} = i\lambda\bar{b}, \quad \lambda \neq 0\}$. Since A is simple, $I = 0$ or $I = A$. Now by Lemmas 12 and 13, $a \neq I$. Hence, $I = 0$ which contradicts the existence of $A^{\#}$. Therefore, $A = A^0$.

IV. THE GENERALIZED EIGENVECTOR CASE

Note. In this section, we shall assume that the algebra A satisfies identities (5) and (6). Also we shall assume that the operator R_a decomposes A into generalized eigenspaces where $a \in A$.

Theorem 15. *Let $x^{(k)}$ and $y^{(\ell)}$ be generalized eigenvectors with eigenvalues λ and μ, respectively. Then*

$$2\Delta\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)+\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^2=2\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)$$
$$+\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right)$$

where $\Delta^2=\lambda^2+\mu^2$ and $x^{(k)}, y^{(\ell)} \in A$.

Proof: By Theorem 1, $(x^{(k)}y^{(\ell)})$ is a generalized eigenvector with eigenvalue Δ. Thus,

$$\left(x^{(k)}y^{(\ell)}\right)a\cdot a=\Delta^2\left(x^{(k)}y^{(\ell)}\right)+2\Delta\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)+\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^2.$$

But also

$$\begin{aligned}\left(x^{(k)}y^{(\ell)}\right)a\cdot a&=x^{(k)}\left(y^{(\ell)}a\cdot a\right)+\left(x^{(k)}a\cdot a\right)y^{(\ell)}\\&=\left(\lambda^2+\mu^2\right)\left(x^{(k)}y^{(\ell)}\right)+2\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)\\&\quad+\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right).\end{aligned}$$

Equating these two equations and noting that $\left(\lambda^2+\mu^2\right)\left(x^{(k)}y^{(\ell)}\right)=\Delta^2\left(x^{(k)}y^{(\ell)}\right)$ we obtain

$$2\Delta\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)+\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^2=2\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)$$
$$+\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right).$$

Theorem 16. *Let $x^{(k)}$ and $y^{(\ell)}$ be generalized eigenvectors with eigenvalues λ and μ, respectively. Then*

$$\left(x^{(k)}y^{(\ell)}\right)(a-\Delta) \subseteq \sum_{i=0}^{k}\sum_{j=0}^{\ell} s_{i,j}\left(x^{(i)}y^{(j)}\right)(a-\Delta)^n \subseteq \sum_{i=0}^{k}\sum_{j=0}^{\ell} t_{i,j}x^{(i)}y^{(j)}$$

where $i+j<k+\ell, \Delta^2=\lambda^2+\mu^2$, and $x^{(k)}, y^{(\ell)} \in A$.

<u>Proof</u>: By Theorem 15,

$$\begin{aligned}\left(x^{(k)}y^{(\ell)}\right)(a-\Delta) = &-\frac{1}{2\Delta}\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^2+\frac{1}{\Delta}\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)\\ &+\frac{1}{2\Delta}\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right).\end{aligned}$$

But rewriting $\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^2$ as $\left[\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)\right](a-\Delta)$, we can define $\left(x^{(k)}y^{(\ell)}\right)(a-\dot{\Delta})$ recursively to obtain

$$\begin{aligned}\left(x^{(k)}y^{(\ell)}\right)(a-\Delta) &= -\frac{1}{2\Delta}\left[\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)\right](a-\Delta)+\frac{1}{\Delta}\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)\\ &\quad+\frac{1}{2\Delta}\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right).\\ &= -\frac{1}{2\Delta}\Bigg[-\frac{1}{2\Delta}\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^2+\frac{1}{\Delta}\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)\\ &\quad+\frac{1}{2\Delta}\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right)\Bigg](a-\Delta)\\ &\quad+\frac{1}{\Delta}\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)\\ &\quad+\frac{1}{2\Delta}\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right)\\ &= \frac{1}{4\Delta^2}\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^3-\frac{1}{2\Delta^2}\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)(a-\Delta)\\ &\quad-\frac{1}{4\Delta^2}\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right)(a-\Delta)\\ &\quad+\frac{1}{\Delta}\left(\lambda x^{(k-1)}y^{(\ell)}+\mu x^{(k)}y^{(\ell-1)}\right)\\ &\quad+\frac{1}{2\Delta}\left(x^{(k-2)}y^{(\ell)}+x^{(k)}y^{(\ell-2)}\right).\end{aligned}$$

∎

By rewriting $\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^q$ as $\left[\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)\right](a-\Delta)^{q-1}$ and proceeding as we did above we obtain

$$\left(x^{(k)}y^{(\ell)}\right)(a-\Delta) = p_m\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^m + \sum_{r\geq 0}\sum_{i=0}^{k}\sum_{j=0}^{\ell} s_{i,j}\left(x^{(i)}y^{(j)}\right)(a-\Delta)^{n_r}$$

where $i+j<k+\ell$. Since chains of the form $\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)$, $\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^2$, $\left(x^{(k)}y^{(\ell)}\right)(a-\Delta)^3$, …, must eventually be zero, we obtain

$$\left(x^{(k)}y^{(\ell)}\right)(a-\Delta) = \sum_{r\geq 0}\sum_{i=0}^{k}\sum_{j=0}^{\ell} s_{i,j}\left(x^{(i)}y^{(j)}\right)(a-\Delta)^{n_r}, i+j<k+\ell \tag{14}$$

From here we proceed by induction on $k+\ell$. If $k+\ell=0$, then the right hand side of (14) is zero and our result holds trivially. Now suppose that

$$\left(x^{(u)}y^{(v)}\right)(a-\Delta) = \sum_{i=0}^{u}\sum_{j=0}^{v} t_{i,j}x^{(i)}y^{(j)}, i+j<u+v,$$

for all u and v such that $u+v<k+\ell$. Applying the induction hypothesis repeatedly to the right hand side of (14), we obtain

$$\left(x^{(k)}y^{(\ell)}\right)(a-\Delta) = \sum_{i=0}^{k}\sum_{j=0}^{\ell} t_{i,j}\left(x^{(i)}y^{(j)}\right) \text{ where } i+j<k+\ell.$$

Notice that we say nothing about the order of the products $x^{(i)}y^{(j)}$. For all we know, they may be larger or smaller than the order of $x^{(k)}y^{(\ell)}$. In subsequent induction proofs, it will be enough to know that $i+j<k+\ell$ regardless of the order of $x^{(i)}y^{(j)}$.

In the next theorem, we find it easier to use the notation $x^{(m-1)} = x^{(m)}(a-\Delta)$.

Theorem 17. *Let $x^{(\ell)}$, $y^{(m)}$, and $z^{(n)}$ be generalized eigenvectors with eigenvalues λ, μ, and η, respectively. Then $\left(x^{(\ell)}y^{(m)}\cdot z^{(n)}\right)^{(p-1)}$*

$= \sum_{i=0}^{\ell} \sum_{j=0}^{m} \sum_{k=0}^{n} t_{i,j,k} \left(x^{(i)}y^{(j)} \cdot z^{(k)}\right)$ *if* $\Delta^2 = \lambda^2 + \mu^2 + \eta^2$ *and* $\lambda^2 + \mu^2$ *are nonzero where* $x^{(\ell)}, y^{(m)}, z^{(n)} \in A$ *and* $i + j + k < \ell + m + n$.

<u>PROOF</u>: Since $x^{(\ell)}y^{(m)}$ is a generalized eigenvector, say of order q, and $\lambda^2+\mu^2+\eta^2 \neq 0$, then by Theorem 16,

$$\left(x^{(\ell)}y^{(m)} \cdot z^{(n)}\right)^{(p-1)} = \sum_{r=0}^{q} \sum_{k=0}^{n} s_{r,k} \left(x^{(\ell)}y^{(m)}\right)^{(r)} z^{(k)}$$

and by Theorem 16 again, we break the sum into two parts depending on whether $r < q$ or $r = q$.

$$\begin{aligned}\left(x^{(\ell)}y^{(m)} \cdot z^{(n)}\right)^{(p-1)} &= \sum_{k=0}^{n} \left(\sum_{i=0}^{\ell} \sum_{j=0}^{m} u_{i,j} \left(x^{(i)}y^{(j)}\right) \right) z^{(k)} \\ &\quad + \sum_{k=0}^{n-1} v_k x^{(\ell)}y^{(m)} \cdot z^{(k)} \\ &= \sum_{i=0}^{\ell} \sum_{j=0}^{m} \sum_{k=0}^{n} t_{i,j,k} \left(x^{(i)}y^{(j)} \cdot z^{(k)}\right).\end{aligned}$$

With Theorem 17, we can now prove a result analogous to Theorem 3 for generalized eigenvectors.

<u>Theorem 18</u>. *Let* $b^{(\ell)}$, $c^{(m)}$, *and* $d^{(n)}$ *be generalized eigenvectors with eigenvalues* λ, μ, *and* η, *respectively. If* $\Delta^2 = \lambda^2 + \mu^2 + \eta^2$, $\lambda^2 + \mu^2$, $\lambda^2 + \eta^2$, $\mu^2 + \eta^2$ *are all nonzero and* $f(\lambda, \mu, \eta) \neq 0$, *then* $b^{(\ell)}c^{(m)} \cdot d^{(n)} = c^{(m)}d^{(n)} \cdot b^{(\ell)} = d^{(n)}b^{(\ell)} \cdot c^{(m)} = 0$ *where* $b^{(\ell)}, c^{(m)}, d^{(n)} \in A$.

<u>PROOF</u>: We will proceed by induction on $\ell + m + n$.

Theorem 3 shows that the equations hold for $\ell + m + n = 0$. Suppose (7), (8), and (9) hold for $i + j + k < p$, and assume that $\ell + m + n = p$. By equation (6),

$$\left(ab^{(\ell)}\cdot c^{(m)}\right)d^{(n)}+\left(ab^{(\ell)}\cdot d^{(n)}\right)c^{(m)}=a\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)+\left(ac^{(m)}\cdot d^{(n)}\right)b^{(\ell)}$$
$$+a\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)+\left(ad^{(n)}\cdot c^{(m)}\right)b^{(\ell)}$$

expanding both sides and knowing that each of the products are of some order, we gave them the names q, r, s. Note that r is used twice because the terms are the negatives of each other by anticommutivity.

$$\begin{aligned}
&\lambda\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q)} &&=\Delta\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q)}+\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q-1)}\\
&+\left(b^{(\ell-1)}c^{(m)}\cdot d^{(n)}\right) &&+\mu\left(c^{(m)}d^{(n)}\cdot b^{(\ell)}\right)^{(r)}+\left(c^{(m-1)}d^{(n)}\cdot b^{(\ell)}\right)\\
&+\lambda\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s)} &&+\Delta\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s)}+\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s-1)}\\
&+\left(b^{(\ell-1)}d^{(n)}\cdot c^{(m)}\right) &&+\eta\left(d^{(n)}c^{(m)}\cdot b^{(\ell)}\right)^{(r)}+\left(d^{(n-1)}c^{(m)}\cdot b^{(\ell)}\right).
\end{aligned}$$

By our induction assumption and the hypothesis $f(\lambda,\mu,\eta)\neq 0$, we obtain

$$\begin{aligned}
&\lambda\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q)} &&=\Delta\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q)}+\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q-1)}\\
&+\lambda\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s)} &&+\mu\left(c^{(m)}d^{(n)}\cdot b^{(\ell)}\right)^{(r)}+\Delta\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s)}\\
& &&+\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s-1)}+\eta\left(d^{(n)}c^{(m)}\cdot b^{(\ell)}\right)^{(r)}.
\end{aligned}$$

Now we use Theorem 17 and our induction hypothesis again to simplify this to the following equation:

$$\begin{aligned}
&\lambda\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q)} &&=\Delta\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q)}+\mu\left(c^{(m)}d^{(n)}\cdot b^{(\ell)}\right)^{(r)}\\
&+\lambda\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s)} &&+\Delta\left(b^{(\ell)}d^{(n)}\cdot c^{(m)}\right)^{(s)}+\eta\left(d^{(n)}c^{(m)}\cdot b^{(\ell)}\right)^{(r)}.
\end{aligned}$$

which by (5) becomes

$$\begin{aligned}
&(\lambda-\Delta)\left(b^{(\ell)}c^{(m)}\cdot d^{(n)}\right)^{(q)}+(\eta-\mu)\left(c^{(m)}d^{(n)}\cdot b^{(\ell)}\right)^{(r)}\\
&\quad+(\Delta-\lambda)\left(d^{(n)}b^{(\ell)}\cdot c^{(m)}\right)^{(s)}=0.
\end{aligned}$$

This is equation (7) for generalized eigenvectors.

Proceeding in a similar manner, we can establish (8) and (9).

Now since (7), (8), and (9) hold by the above and $f(\lambda, \mu, \eta) \neq 0$, then Theorem 6 is true for the generalized eigenvectors $b^{(\ell)}$, $c^{(m)}$, and $d^{(n)}$.

Corollary 19. *Let $b^{(\ell)}$, $c^{(m)}$, and $d^{(n)}$ be generalized eigenvectors from the same eigenspace with eigenvalue $\lambda \neq 0$. Then*

$$b^{(\ell)}c^{(m)} \cdot d^{(n)} = c^{(m)}d^{(n)} \cdot b^{(\ell)} = d^{(n)}b^{(\ell)} \cdot c^{(m)} = 0.$$

PROOF: Since $\lambda \neq 0$, then $2\lambda^2$ and $3\lambda^2$ are nonzero and Theorem 18 is true. Therefore, $f(\lambda, \lambda, \lambda) = -888191\lambda^{30}$ implies the result.

V. EXAMPLES

We shall conclude this dissertation with some examples of algebras that satisfy equations (5) and (6). It will be apparent from the multiplication tables that (5) holds so we shall only verify (2), (3), or (6) for each example.

Example 2. Let A consist of the elements a, b, ab, $a \cdot ab$, $a(a \cdot ab)$, ...

	a	b	ab	$a \cdot ab$	$a(a \cdot ab)$	...
a	0	ab	$a \cdot ab$	$a(a \cdot ab)$	$a \cdot a(a \cdot ab)$	
b	$-ab$	0	0	0	0	
ab	$-a \cdot ab$	0	0	0	0	
$a \cdot ab$	$-a(a \cdot ab)$	0	0	0	0	
$a(a \cdot ab)$	$-a \cdot a(a \cdot ab)$	0	0	0	0	
$\vdots$						

Here we shall verify (3). Since (5) holds, (3) simplifies to

$$x(x \cdot yz) = y(x \cdot xz) + (x \cdot xy)z. \tag{15}$$

By assumption, any product containing more than one b is zero. Hence, $x = a$ or (14) holds trivially. For the same reason, only one of the two elements y and z can contain a b. In view of (5), we may assume that $y = a$. If we use the notation that $a(a \cdots b)$ stands for stacking a's onto b from the left, we need only show that

$$a\,(a \cdot a\,[a(a \cdots b)]) = a\,(a \cdot a\,[a(a \cdot \cdot\cdot)]) + (a \cdot aa)\,[a(a \cdots b)]$$

But $aa = 0$ by (5) and this equation holds. This shows the existence of a derivation alternator ring that is finitely generated and is not nilpotent.

Example 3. Let A be generated by the elements a and b where we define $ab = \lambda b$, $\lambda \neq 0$. Then the only products that are nonzero are $ab, a \cdot ab, a(a \cdot ab), \ldots$. As in example 1, equation (3) holds. This shows the existence of a finite dimensional derivation alternator algebra that is nil but where R_a is not nilpotent.

Example 4. Let $a, b, c, d, \cdots \in A$ with the following multiplication table:

	a	b	c	d	$\ldots$
a	0	λb	μc	ηd	
b	$-\lambda b$	0	0	0	
c	$-\mu c$	0	0	0	
d	$-\eta d$	0	0	0	
$\vdots$					

The only product of the form $(xy \cdot z)w$ that is nonzero is $(xa \cdot a)a$ and since

$$(xa \cdot a)a = x(aa \cdot a) + (xa)a \cdot a$$
$$= (xa)a \cdot a$$

we see that (2) holds. This is an example where the operator R_a has an infinite number of distinct eigenspaces.

Example 5. Let $a, b^{(1)}, b^{(0)}, c^{(1)}, c^{(0)}, d^{(0)} \in A$ with the following multiplication table:

	a	$b^{(1)}$	$b^{(0)}$	$c^{(1)}$	$c^{(0)}$	$d^{(0)}$
a	0	$b^{(0)}$	0	$c^{(0)}$	0	0
$b^{(1)}$	$-b^{(0)}$	0	0	0	0	0
$b^{(0)}$	0	0	0	0	$d^{(0)}$	0
$c^{(1)}$	$-c^{(0)}$	0	0	0	0	0
$c^{(0)}$	0	0	$-d^{(0)}$	0	0	0
d	0	0	0	0	0	0

The only nonzero products are $ab^{(1)} = b^{(0)}$, $ac^{(1)} = c^{(0)}$, $b^{(0)}c^{(0)} = d^{(0)}$, $ab^{(1)} \cdot c^{(0)} = d^{(0)}$, and $ac^{(1)} \cdot b^{(0)} = -d^{(0)}$. Since none of the products contain four elements, all products of the form $(xy \cdot z)w = 0$. Therefore, equation (6) holds trivially.

From this example, we see that

$$\left(ab^{(1)}\right)\left(ac^{(1)}\right) = b^{(0)}c^{(0)} = d^{(0)} \neq 0$$

and so products of the form $(xy)(zw)$ are not necessarily zero. This shows that that derivation alternator rings are not necessarily solvable of index 2. This contrasts the result in [1] that derivation alternator Lie rings are solvable of index at most 2.

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