

Braided Cobwebs: Cautionary Tales for Dynamic Retail Pricing in End-to-End Power Systems

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Abstract—This study investigates the effects of dynamic-price retail contracting on end-to-end power system operations. Performance is evaluated by means of carefully defined metrics for system stability, market efficiency, and market participant welfare. The study is carried out for an Integrated Retail and Wholesale (IRW) Test Case for which households have smart (price-responsive) air-conditioning (A/C) systems. A simplified version of the IRW Test Case with a directly postulated linear household demand curve is first used to derive, analytically, a set of necessary and sufficient conditions for system stability under dynamic-price retail contracting. A key finding is that dynamic-price retail contracts induce braided cobweb dynamics consisting of two interwoven cycles for power and price outcomes that can exhibit point convergence, limit-cycle convergence, or divergence depending on a small set of structural parameters. Outcomes are then reported for a dynamic welfare sensitivity study undertaken using the full IRW Test Case with smart household A/C systems. One surprising finding is that dynamic-price retail contracts with a positive price mark-up result in worse welfare outcomes for generators and household residents than flat-rate retail contracts for treatments exhibiting convergent cobweb dynamics.

Index Terms—Electric power system, end-to-end operations, retail dynamic pricing, cobweb dynamics, system stability, market efficiency, market participant welfare

NOMENCLATURE

Parameters:

a	GenCo cost coefficient (\$/MWh)
b	GenCo cost coefficient (\$/(MW) ² h)
c	Household demand coefficient (\$/MWh)
$c(m)$	$c(m) = c/[1 + m]$
d	Household demand coefficient (\$/(MW) ² h)
$d(m)$	$d(m) = d/[1 + m]$
m	Mark-up factor (scalar) for retail price
α	Household parameter (Utils/\$) reflecting comfort-cost trade-off preferences

Variables and Functions:

$E^{w*}(m)$	Market equilibrium point for wholesale power and price outcomes as a function of m
$E^{r*}(m)$	Market equilibrium point for retail power and price outcomes as a function of m
FD	LSE fixed demand bid (MW)
p	Generic symbol for a power level (MW)
p^{DA}	Power dispatch (MW) determined in the day-ahead market

p^{RT}	Power dispatch (MW) determined in the real-time market
p^{RET}	Retail power usage (MW)
π	Generic symbol for a power price (\$/MWh)
π^{DA}	Locational marginal price (\$/MWh) determined in the day-ahead market
π^{RT}	Locational marginal price (\$/MWh) determined in the real-time market
π^{RET}	Retail power price (\$/MWh)
R	Administratively-set flat rate (\$/MWh)
$\mathbf{r}(t)$	Retail power and price vector at time t

I. INTRODUCTION

THIS study investigates the effects of dynamic-price retail contracting on end-to-end power system operations. Performance is evaluated by means of carefully defined metrics for system stability, market efficiency, and market participant welfare separately calculated for wholesale generators, load-serving entities, and retail households.

The study is carried out using an *Integrated Retail and Wholesale (IRW) Test Case* for which 500 households have air-conditioning (A/C) systems with price-responsive controllers. This test case permits the integrated study of retail and wholesale electric power markets operating during successive days over linked distribution and transmission grids.

A simplified version of the IRW Test Case with a postulated linear aggregate household demand curve is first used to derive, analytically, a set of necessary and sufficient conditions for power system stability under dynamic-price retail contracting. A complete analysis of short-run welfare and market efficiency outcomes under both dynamic-price and flat-rate retail contracting is also provided. A key finding is that the use of dynamic-price retail contracts induces braided cobweb dynamics consisting of two interwoven cobweb cycles for power and price outcomes. These braided cobweb cycles can exhibit either point convergence, limit cycle convergence, or divergence depending on a small set of structural parameters characterizing power supply and demand conditions.

A dynamic welfare sensitivity study is then conducted for the full IRW Test Case with smart household A/C systems, implemented by means of the *Integrated Retail and Wholesale (IRW) Test Bed* [41]. The IRW Test Bed consists of four key components: (i) a wholesale power sector, implemented by means of the AMES Wholesale Power Market Test Bed [2]; (ii) a retail power sector, implemented in part by GridLAB-D [13]; (iii) a C++ modeling of price-responsive A/C loads [44];

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and (iv) a MySQL database server to facilitate data storage and transfer among the components (i)-(iii).

The three treatment factors for this sensitivity study are: (i) the form of retail contracts, either flat-rate or dynamic-price; (ii) a mark-up factor m that determines the percentage by which retail prices are marked up over wholesale prices in the case of dynamic-price retail contracts; and (iii) the preference parameter α^h for each household h that determines h 's willingness to trade off comfort against energy cost savings.

Welfare outcomes and energy costs are reported for a range of treatments. A key finding for the case of convergent cobweb dynamics is that dynamic-price retail contracts with a positive mark-up m result in *worse* welfare outcomes for the GenCo and household residents than flat-rate retail contracts.

The organization of this study is as follows. Section II provides background motivation, including a review of the existing literature on dynamic retail pricing. Section III discusses the key features and capabilities of the IRW Test Bed. The IRW Test Case, implemented by means of the IRW Test Bed, is carefully presented and explained in Section IV.

Section V undertakes a careful analytical study of the braided cobweb dynamics arising in a simplified version of the IRW Test Case with a postulated linear downward-sloping aggregate demand curve for households. Welfare and market efficiency outcomes for this simplified IRW Test Case are analytically derived and graphically depicted in Section VI.

A welfare sensitivity study for the full IRW Test Case is outlined in Section VII, and key findings for this study are reported in Section VIII. Concluding discussion is provided in Section IX. Technical materials regarding the approximately linear form of the aggregate household demand curve for the full IRW Test Case are relegated to an appendix.

II. BACKGROUND MOTIVATION

In traditional U.S. power systems based on vertically integrated utilities, the power usage of residential, commercial, and industrial customers was assumed to be highly unresponsive to price changes. Utilities typically charged their customers a flat hourly rate for power usage, plus additional fixed charges, on an extended (e.g., monthly) basis.

A critical utility task, referred to as *load following*, was then to ensure the continual balance of real-time customer power usage with real-time power generation, whatever form this power usage took. Customers thus became accustomed to extracting power from the grid without any consideration of its actual production cost or environmental impact.

This traditional conception of customer power usage as externally determined load in need of balancing has been carried forward into restructured U.S. wholesale electric power systems. Although, in principle, LSEs participating in day-ahead markets are permitted to submit hourly demand bids for the next-day power needs of their customers in two parts – a price-responsive demand schedule and a fixed power amount – most LSE hourly demand bids take a fixed form.

As far back as 2002, power economists have forcefully argued the need for participants on both sides of a power

market, buyers and sellers, to be able to express their reservation values¹ for power in order to achieve an efficient pricing of power; see, e.g., [38, Chapter 1-1], [21], [32], and [39]. However, given the relatively primitive state of metering technology, it was not feasible for power customers to adjust their power usage in real time in response either to system operator commands or to automated signals. Consequently, power customers continued to play a largely passive role in power system operations.

This traditional relationship between power buyers and sellers is now rapidly changing. Recent breakthroughs in metering technology, referred to as *Advanced Metering Infrastructure (AMI)*, have radically improved the potential for more active customer participation [17]. AMI broadly refers to an integrated system of meters, communication links (wired or wireless), and data management processes that permits rapid two-way communication between power customers and the agencies (e.g., utilities) that manage their power supplies.

AMI enables the implementation of *demand-response (DR)* initiatives designed to encourage fuller demand-side participation in power system operations. Power system researchers are currently exploring three basic types of DR initiatives:²

- (i) *Incentive-Based Load Control*: Down/up adjustments in the power usage of business and household devices are undertaken, either in response to direct requests from designated parties,³ or via device switches under the remote control of designated parties, with compensation at administratively set rates.
- (ii) *Dynamic Pricing*: Down/up power usage adjustments are undertaken by businesses and/or households in response to changes in power prices communicated to them by designated parties.
- (iii) *Transactive Energy*: Demands and supplies for power and ancillary services by businesses and households are determined by decentralized bid/offer-based transactions.

The implementation of these DR initiatives can result in curtailments (or increases) in total power withdrawal from the grid, or in shifts in the timing of power withdrawals from the grid with no significant change in total power withdrawal. In some cases, DR resources might be willing and able to offset curtailments (or increases) in their power withdrawals from the grid by resorting to local “behind the meter” generation and storage facilities, such as an on-site wind turbine or a small-scale battery system with no grid connection.

A key goal of type-(i) DR initiatives is to permit ancillary services to be extracted from demand-side resources in support of system *reliability*. A key goal of type-(ii) DR initiatives

¹As detailed in [39], a *buyer's reservation value* for a good or service at a particular point in time is defined to be the buyer's maximum willingness to pay for the purchase of an additional unit of this good or service at that time. A *seller's reservation value* for a good or service at a particular point in time is defined to be the minimum payment that the seller is willing to receive for the sale of an additional unit of this good or service at that time.

²For surveys covering all three types of DR research, see [16], [30], [33], [36]. For demand-response deployment in the U.S., see [11], [12].

³These designated parties can be ISOs/RTOs or utilities. They can also be intermediaries who manage collections of customer-owned *distributed energy resources (DERs)* in accordance with the operational requirements of ISOs/RTOs or utilities.

is to enhance system *efficiency* by permitting business and household customers to express their reservation values for power at different times and locations. A key goal of type-(iii) DR initiatives is to enhance the reliability *and* efficiency of system operations by enabling a balancing of demands and supplies for power and ancillary services across an entire electrical infrastructure on the basis of business and household reservation values.

Researchers focusing on type-(i) and type-(ii) DR initiatives have primarily stressed metering, control, and planning aspects for system operators and power customers. For example, refs. [4, Chapters 2-3] and [28] investigate the ability of type-(i) DR programs to provide reserve services for system operators. Ref. [46] studies various forms of control strategies for type-(i) initiatives designed to maximize the net benefits of building residents subject to constraints.

Ref. [44] proposes a type-(ii) intelligent air-conditioning system controller able to determine optimal next-day power usage for a household based on the household's comfort/cost trade-off preferences, conditional on price signals for next day retail power usage and a forecast for next-day environmental conditions. Ref. [45] develops a two-stage co-optimisation framework for a customer whose stage-1 decision is to install a battery energy storage system and whose stage-2 decision is to join one of several offered type-(i) and type-(ii) DR programs.

However, some work has explored the effects of type-(ii) DR initiatives on power system operations over time. For example, ref. [47] develops an agent-based computational platform to study the effects of price-responsive power demand by commercial buildings, modeled as autonomous agents with reinforcement learning capabilities. These building agents compete to offer DR services into a wholesale power market operating over a transmission grid. Also, ref. [43] reports results from a preliminary investigation of the effects of price-responsive household demands for power on integrated T&D system operations over time.

A *Transactive Energy System (TES)* is a set of economic and control mechanisms that permits the balancing of demands and supplies for power across an entire electric power system while still maintaining system reliability [14, p. 11]. Researchers focusing on type-(iii) DR initiatives are interested in understanding the potential effects of proposed TES designs on the end-to-end operations of an entire electric power system.

As detailed in ref. [40], the study of TES designs is still relatively new, and the source code developed for most of these studies is not publicly available. A key exception is work based on GridLAB-D [5], [13], an open-source agent-based computational platform developed by researchers with the U.S. Department of Energy at Pacific Northwest National Laboratory (PNNL). GridLAB-D permits the customized simulation of a distribution system populated with residential, commercial, and industrial customers that own a wide range of electrical devices.

III. THE IRW TEST BED

As illustrated in Fig. 1, the *Integrated Retail and Wholesale (IRW) Test Bed* [41] is an agent-based computational platform

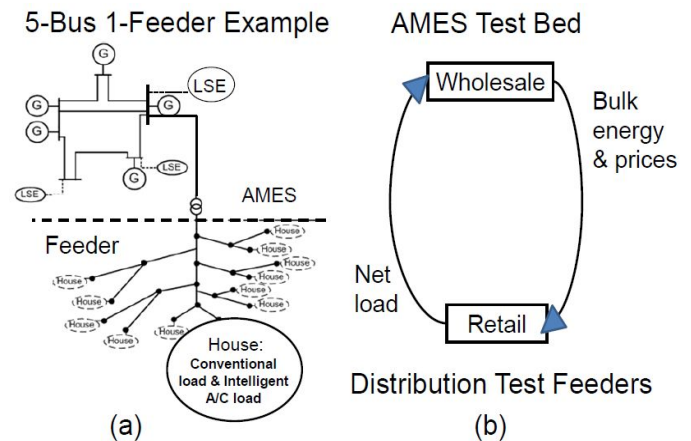


Fig. 1. (a) Simple illustration of the Integrated Retail and Wholesale (IRW) Test Bed; and (b) the basic IRW feedback loop.

permitting researchers to study the integrated grid-constrained operations of retail and wholesale power sectors over time. The IRW Test Bed has four key components:

- *C1*: Wholesale power sector, implemented by means of the AMES Wholesale Power Market Test Bed [2].
 - AMES⁴ is an open-source agent-based computational platform that permits the simulation over successive 24-hour days of a wholesale power market adhering to standard practices in U.S. ISO/RTO-managed wholesale power markets.
- *C2*: Retail power sector, implemented in part by GridLAB-D [13].
 - GridLAB-D is an open-source agent-based computational platform for the study of distribution systems that provides detailed physically-based models for a wide variety of appliances and devices owned by residential, commercial, and industrial customers.
- *C3*: C++ modeling of price-responsive loads
 - Price-responsive loads take the form of household A/C systems managed by smart price-responsive controllers [44].
- *C4*: MySQL database server.
 - Data storage and transfer among components C1–C3 is facilitated by a MySQL database server.

The wholesale power sector for the IRW Test Bed, implemented using AMES [2], consists of a *Day-Ahead Market (DAM)* and a *Real-Time Market (RTM)* managed by an *Independent System Operator (ISO)*. The DAM and RTM operate through time over a high-voltage AC transmission grid. Congestion on the grid is handled by *locational marginal pricing (LMP)*, i.e., by the pricing of power in accordance with the location and time of its injection into, or withdrawal from, the transmission grid.

⁴AMES is an acronym for Agent-based Modeling of Electricity Systems. Pointers to down-loadable AMES software, manuals, and publications can be accessed at the AMES homepage [2]. The capabilities of the latest AMES release (V4.0) are discussed and demonstrated in [20], [25], which use AMES (V4.0) to develop an 8-zone test case based on structural attributes and data for ISO New England.

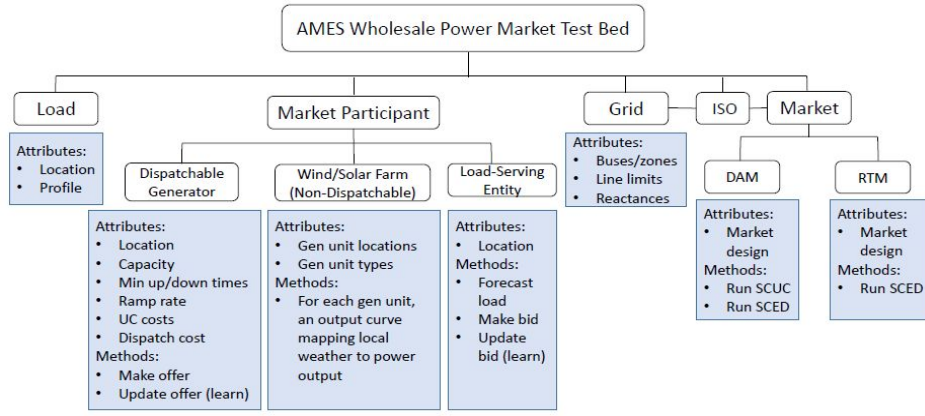


Fig. 2. Principal agents comprising the wholesale power sector for the Integrated Retail and Wholesale (IRW) Test Bed, implemented via AMES (V4.0).

The retail power sector for the IRW Test Bed consists of a distribution system populated by a collection of households with various appliances and electrical devices. GridLAB-D [13] is used to model *conventional (non-price-responsive)* household load. The A/C system for each household is simulated using a physically-based *Equivalent Thermal Parameter (ETP)* model. This A/C system is managed by the smart A/C controller developed in [44]. Each day the controller determines the household’s optimal next-day A/C power usage based on the household’s comfort/cost trade-off preferences, next-day retail power prices (announced today by the LSE), and forecasted next-day environmental conditions (outside air temperature, humidity, and solar radiation).

Figure 2 depicts the principal agents comprising the wholesale power sector for the IRW Test Bed. Each of these agents is characterized at any given time by its internal state (data, attributes, methods).⁵ These agents include: (i) a physical entity (the transmission grid); (ii) institutional entities (the DAM and RTM); and (iii) decision-making entities (the ISO and market participants). The market-participant agents consist of a collection \mathbb{G} of dispatchable and non-dispatchable *Generation Companies (GenCos)* and a collection \mathbb{L} of *Load-Serving Entities (LSEs)*. The LSEs are intermediary agents that purchase power in the DAM in order to service the power-usage demands of the retail power sector.

The GenCos and LSEs participate in the daily operations of the ISO-managed DAM and RTM; cf. Fig. 3. Roughly summarized, these daily operations are as follows.

During the morning of each day D the GenCos and LSEs submit into the DAM a collection of supply offers and demand bids, respectively, for all 24 hours H of day $D+1$. For each hour H , these offers and bids take the following general form:⁶

$$\text{GenCo Price-Responsive Supply: } \pi = a + 2bp \quad (1)$$

$$\text{LSE Price-Responsive Demand: } \pi = c - 2dp \quad (2)$$

$$\text{LSE Fixed Demand: } p = \text{FD}(H) \quad (3)$$

⁵For visual clarity, Fig. 2 only lists some of the more important attributes and methods for each agent, not the agent’s complete state.

⁶In current U.S. ISO-managed wholesale power markets, LSEs are permitted to submit hourly demand bids consisting of two parts: a price-responsive demand function; and a fixed (non-price-responsive) quantity demand.

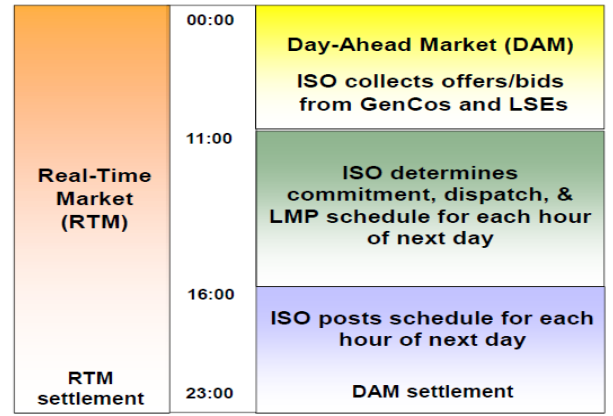


Fig. 3. Daily operation of the DAM/RTM in the Integrated Retail and Wholesale (IRW) Test Bed, as implemented via AMES (V4.0).

where: π (\$/MWh) denotes price, p (MW) denotes power, $\text{FD}(H)$ (MW) denotes a fixed (non-price-responsive) demand for power, and a (\$/MWh), b (\$/(MW)²h), c (\$/MWh), and d (\$/(MW)²h) are positive coefficients. The power levels in (1) through (3) represent constant power levels to be maintained during the entire hour H , either as injections into the grid (power supplies) or as withdrawals from the grid (power demands).

Given these offers and bids, the ISO solves *Security-Constrained Unit Commitment (SCUC)* and *Security-Constrained Economic Dispatch (SCED)* optimization problems subject to standard system⁷ and reserve requirement constraints in order to determine: (i) generation unit commitments; (ii) scheduled generation dispatch levels; and (iii) a locational marginal price $\pi^{DA}(B,H,D)$ (\$/MWh) at each transmission grid bus B for each hour H of day $D+1$. A generator located at bus B is paid $\pi^{DA}(B,H,D)$ for each MW it is scheduled to inject at B during hour H of day $D+1$, and an LSE located at a bus B must pay $\pi^{DA}(B,H,D)$ for each MW its household customers are scheduled to withdraw at

⁷As detailed in [25, Section V], the system constraints include: power balance constraints; line and generation capacity limits; down/up ramping restrictions; down/up ramping constraints; and minimum down/up-time constraints.

bus B during hour H of day D+1.

The RTM runs each hour of each day.⁸ At the start of the RTM for hour H on day D+1, the ISO is assumed to know the actual household power usage for hour H of day D+1. The ISO then solves a SCED optimization problem to resolve any discrepancies between the generation *scheduled* in the day-D DAM for dispatch during hour H of day D+1, which was based on day-D LSE demand bids, and the generation needed during hour H of day D+1 to balance *actual* household power usage. Any needed adjustments in the DAM-scheduled power supplies and demands at a bus B for hour H of day D+1 are settled at $\pi^{RT}(B,H,D+1)$, the RTM price at bus B for hour H of day D+1.

IV. IRW TEST CASE

A. Overview

This section describes a simplified version of the IRW Test Bed, hereafter referred to as the *IRW Test Case*. This test case will be used to explore how a switch from flat-rate to dynamic-price retail contracts for households affects IRW system operations over time under systematically varied parameter specifications. Particular attention will be paid to the effects of this switch on system volatility and market participant welfare.

B. IRW Test Case Assumptions

The IRW Test Case includes one GenCo and one LSE, each located at a single bus B; hence, transmission grid congestion is not an issue. The GenCo is automatically committed in the DAM each day without need for SCUC optimizations. The LSE serves as a wholesale power purchasing agent for 500 households populating the distribution system.

The power usage of each household arises solely from air conditioning (A/C). As in [44], the A/C load of each household is managed by a smart controller, responsive to both price and environmental conditions, that reflects the specific comfort/cost trade-off preferences of the household. The thermal dynamics of each household are represented by the physics-based *Equivalent Thermal Parameter (ETP)* model ([13], [37]). The ETP model supposes that the state of a house at any given time is described by inside air and mass temperatures whose movement over time can be expressed as a parameterized system of linear differential equations.

Line capacities for the distribution grid are large enough to avoid grid congestion. The distribution grid is connected to the distribution system at bus B, and the GenCo has sufficient capacity to meet the power needs of the H households.

Given these assumptions, the feedback loop determining retail prices and real-time loads over time for the IRW Test Case is as depicted in Fig. 4. A more detailed description of this loop will now be provided, starting from the lower right.

The supply offer (1) submitted by the GenCo to the DAM on day D for any specific hour H of day D+1 gives the GenCo's reservation value (marginal production cost) for each

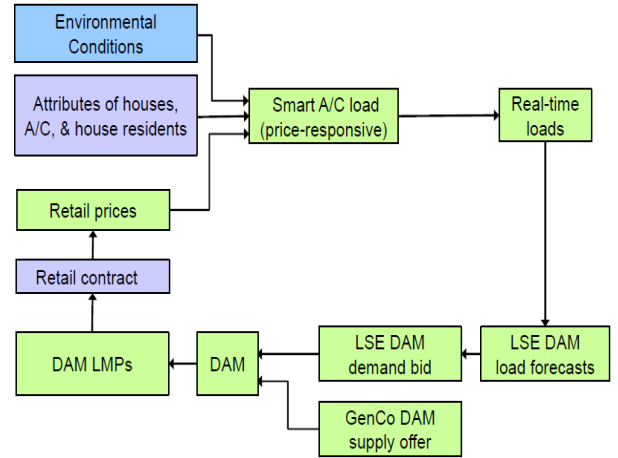


Fig. 4. Feedback loop for the Integrated Retail and Wholesale (IRW) Test Case. Since retail prices and real-time loads are not affected by Real-Time Market (RTM) imbalance adjustments, these adjustments are not depicted.

successive megawatt (MW) of power it might be required to generate during hour H of day D+1. Thus, (1) represents the *marginal cost function* of the GenCo.

The LSE's DAM demand bids take the fixed form (3). The demand bid (3) submitted by the LSE to the DAM on day D for any specific hour H of day D+1 is a forecast of household load for hour H of day D+1. This forecast is given by the actual household load observed for hour H on day D-1.

The retail contracts offered by the LSE to the 500 households take one of two possible forms: (i) a *flat-rate* retail contract with a flat rate R (\$/MWh) set to ensure the LSE breaks even over time; or (ii) a *dynamic-price* retail contract with one-way communication (LSE to households) in which DAM prices (marked up by a percentage $m \geq 0$) are passed through to households as next-day retail prices. More precisely, under dynamic-price retail contracts, the *retail price* charged by the LSE to each household withdrawing power for usage during hour H of day D is given by

$$\pi^{RET}(H,D) = [1 + m]\pi^{DA}(H,D-1) \quad (4)$$

where $\pi^{DA}(H,D-1)$ (\$/MWh) is the price determined in the day-(D-1) DAM for hour H of day D.⁹

As detailed in [44], the *net benefit (benefit minus cost)* attained by household h from the purchase and use of electric power during any hour H of any day D is given by

$$NB^h(H,D) = \text{Comfort}^h(H,D) - \alpha^h \text{EnergyCost}^h(H,D) \quad (5)$$

In (5), the comfort (Utils) attained by household h depends on the interior thermal conditions experienced by h during hour H of day D. Also, the energy cost (\$) charged to h depends on two factors: (i) h 's power usage during hour H of day D; and (ii) the form of h 's retail contract, either flat-rate or dynamic-price.

The key parameter α^h (Utils/\$) in (5) is a preference parameter measuring the degree to which household h is

⁸In actual U.S. ISO/RTO-managed wholesale power markets the RTM is conducted at least once every five minutes.

⁹Retail prices are typically reported in cents/kWh, not in \$/MWh, which would require a conversion factor be given on the right-hand side of (4). This price conversion is ignored in the current paper for ease of exposition.

willing to trade off comfort against cost. The higher the value of α^h , the greater the weight that h places on energy cost savings relative to thermal comfort.¹⁰

V. ANALYSIS OF BRAIDED COBWEB DYNAMICS FOR THE IRW TEST CASE UNDER DYNAMIC PRICING

A. Overview

Suppose retail contracts for the IRW Test Case take a dynamic-price form. In this case the LSE uses two simple adaptive methods, one to determine its wholesale demand bids on behalf of its household customers, and one to set retail prices for these household customers.

First, during the afternoon of each day D-1, after the closing of the day-(D-1) DAM, the LSE engages in *adaptive advance retail pricing*. Specifically, as depicted in (4), the LSE announces to its household customers a retail price $\pi^{RET}(H,D)$ for power usage during hour H of day D that is an m -percent mark-up of the price $\pi^{DA}(H,D-1)$ determined in the day D-1 DAM for hour H of day D. Note that $\pi^{DA}(H,D-1)$ is the price the LSE itself must pay for each MW of power it purchases in the day-(D-1) DAM for its retail customers during hour H of day D.

Second, during the morning of each day D the LSE engages in *adaptive load forecasting*. Specifically, the LSE participates in the day-D DAM by submitting a demand bid $p^{DA}(H,D)$ consisting of a forecast for retail load during hour H of day D+1. The day-D DAM takes place before the realization of retail loads for day D. Consequently, the LSE sets $p^{DA}(H,D)$ equal to the retail load realized during hour H of day D-1.

These two adaptive LSE methods, together with other features of the IRW Test Case, give rise to dynamic market cycling behavior known in the economics literature as *cobweb dynamics* [8]. However, a unique aspect of this cobweb dynamics for the IRW Test Case is that the cycling is *braided*.

More precisely, as depicted in Fig. 5, this braided cobweb dynamics consists of two intertwined cycles determining retail and wholesale outcomes on alternative days. For example, starting on any day D-1, one cycle determines retail outcomes for days D-1, D+1, D+3, . . . and the other cycle determines retail outcomes for days D, D+2, D+4, . . .

This section provides a comprehensive analysis of the braided cobweb dynamics arising for the IRW Test Case when retail contracts take a dynamic-price form. Section V-B presents a summary explanation of these findings through a series of graphical depictions. Section V-C provides a detailed analytical derivation of these findings.

Seven key assumptions characterizing the IRW Test Case with dynamic-price retail contracts, explained with care in

¹⁰More precisely, α^h measures the benefit to h of an additional dollar of income. It permits costs measured in dollars to be expressed in benefit units (Utils), so that comfort/cost trade-offs can be calculated. The precise sense in which α^h quantifies the trade-off between comfort satisfaction and energy cost for h is explained in some detail in [44, Appendix]. Roughly, it is shown that α^h can be derived as the shadow price for h 's budget constraint in a more fully articulated constrained benefit maximization problem: namely, the maximization of h 's benefit from consumption of multiple goods/services (including thermal comfort) subject to a budget constraint. Thus, α^h measures h 's *marginal benefit of income* at the optimization point, i.e., the drop in the maximized value of h 's benefit that would result if h had one less dollar of income to spend (e.g., due to a higher energy price).

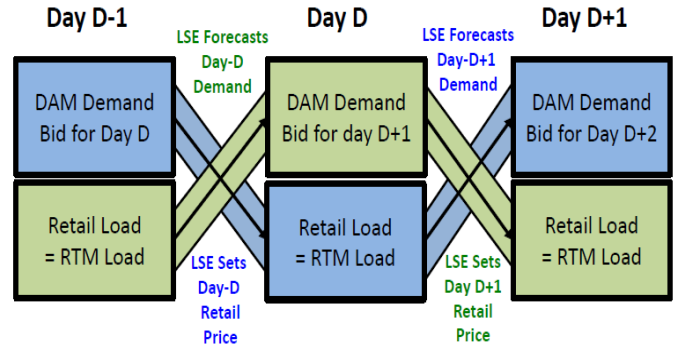


Fig. 5. Braided cobweb dynamics resulting for the Integrated Retail and Wholesale (IRW) Test Case under dynamic-price retail contracts.

Section IV, are listed below in summary form for easy later reference. These assumptions reflect the basic simplifying IRW Test Case assumption that no grid congestion arises, hence power and price outcomes do not have to be differentiated by location.

Summary of Key IRW Test Case Assumptions:

- (A1) RTM load $p^{RT}(H,D)$ is equal to retail load $p^{RET}(H,D)$ for each hour H of each day D.
- (A2) The GenCo's RTM supply offer for each hour H of each day D takes form (1).
- (A3) The ISO ensures RTM market clearing for each hour H of each day D.
- (A4) The LSE's DAM demand bids take the fixed-demand form (3). Specifically, on each day D the LSE sets its day-D DAM demand bid $p^{DA}(H,D)$ for each hour H of day D+1 equal to the retail load $p^{RET}(H,D-1)$ previously realized during hour H of day D-1.
- (A5) The GenCo's DAM supply offer for each hour H of each day D takes form (1).
- (A6) The ISO ensures DAM market clearing for each hour H of each day D.
- (A7) The LSE makes use of a dynamic-price retail contract with mark-up $m \geq 0$. That is, on each day D-1 the LSE sets the retail price $\pi^{RET}(H,D)$ for hour H of day D equal to $[1 + m]\pi^{DA}(H,D-1)$, the DAM price determined on day D-1 for hour H of day D with an m -percent mark-up.

In addition, for the purposes of this analytical section, the following auxiliary assumption is imposed:

- (A8) The retail aggregate demand function determined by household price-responsive A/C controllers for any hour H of any day D can be approximated by a linear function taking form

$$\pi^{RET} = c - 2dp^{RET}, \quad (6)$$

where $c > a$.¹¹

Support for the linear approximation assumption (6) is provided in an appendix.

¹¹The restrictions $a, b, c, d > 0$ and $c > a$ ensure that the plots of the supply and demand functions (1) and (6) in the power-price plane intersect each other at a positive power-price point (p^*, π^*) .

Given (A7) and (A8), the actual retail power usage $p^{RET}(H, D)$ of households during any hour H of any day D is determined from

$$[1 + m]\pi^{DA}(H, D-1) = c - 2dp^{RET}(H, D) \quad (7)$$

where $\pi^{DA}(H, D-1)$ denotes the price of power previously determined in the day-(D-1) DAM for hour H of day D. For later purposes, note that the determination of $p^{RET}(H, D)$ via (7) is equivalent to determining $p^{RET}(H, D)$ by means of the modified demand relationship

$$\pi^{DA}(H, D-1) = c(m) - 2d(m)p^{RET}(H, D) \quad (8)$$

where $c(m) = c/[1 + m]$ and $d(m) = d/[1 + m]$.

B. Graphical Preview of Analytical Findings

Figure 6 depicts supply and demand relationships for the IRW Test Case, assuming retail contracts take a dynamic-price form with a positive mark-up ($m > 0$). The supply curve, a plot of (1), represents the GenCo's marginal production costs in the wholesale power market. The demand curve is an approximate representation for the power demand arising in the retail sector from household price-responsive A/C systems, expressed in two forms: (i) with original coefficients c and d , using the retail price, as in (6); and (ii) with m -modified coefficients $c(m)$ and $d(m)$, using the DAM price, as in (8).

The LSE is the connecting link between the wholesale and retail power sectors. The LSE purchases bulk power at wholesale prices in the DAM and the RTM and then resells this power to households at retail prices.

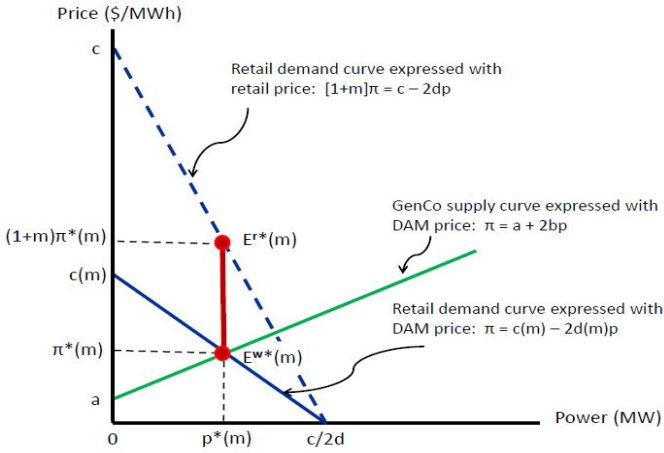


Fig. 6. A market equilibrium configuration for the IRW Test Case for any fixed hour over successive days, given dynamic-price retail contracts with a positive mark-up ($m > 0$). The system endlessly cycles between $E^{w*}(m)$ and $E^{r*}(m)$.

More precisely, Fig. 6 depicts a market equilibrium configuration for any fixed hour H over successive days. To see this, suppose the LSE's fixed demand bid submitted into the day-(D-1) DAM for hour H of day D induces a vertical demand curve at the power level $p^*(m)$. To balance this fixed demand, the GenCo must be scheduled to produce $p^*(m)$ during hour H of day D. Consequently, the day-(D-1) DAM price for hour H of day D must be set at $\pi^*(m)$, as determined from the

GenCo's supply curve. The resulting DAM market clearing point $E^{w*}(m)$, depicted in Fig. 6, takes the following specific numerical form:

$$E^{w*}(m) = \begin{bmatrix} p^*(m) \\ \pi^*(m) \end{bmatrix} = \begin{bmatrix} \frac{[c(m)-a]}{2[b+d(m)]} \\ \frac{[ad(m)+bc(m)]}{[b+d(m)]} \end{bmatrix} \quad (9)$$

The LSE then sets the retail price for hour H of day D equal to $[1 + m]\pi^*(m)$. This retail price induces a retail power usage (load), determined from the retail demand function. As depicted in Fig. 6, it follows by straightforward calculation that this retail power usage is equal to $p^*(m)$ in (9). Let

$$E^{r*}(m) = \begin{bmatrix} p^*(m) \\ [1 + m]\pi^*(m) \end{bmatrix} \quad (10)$$

denote these retail power and price outcomes. The LSE then submits $p^*(m)$ as its fixed-demand bid for the day-D DAM for hour H of day D+1, and the entire process repeats.

Thus, the market equilibrium configuration in Fig. 6 is a 2-point cycle that alternates between the vector $E^{w*}(m)$ of wholesale power and price outcomes and the vector $E^{r*}(m)$ of retail price and power outcomes. Note that RTM operations are not depicted in Fig. 6. The reason for this is that the RTM is an imbalance adjustment mechanism designed to handle discrepancies between DAM scheduled generation and real-time loads. However, no such discrepancies arise in the market equilibrium configuration depicted in Fig. 6.

A key issue is whether the market participants in the IRW Test Case are able to attain the market equilibrium configuration depicted in Fig. 6. Comparing the form of $E^{w*}(m)$ in (9) to the form of $E^{r*}(m)$ in (10), it is clear that hour-H wholesale power and price outcomes ($p^{DA}(H, D-1), \pi^{DA}(H, D-1)$) converge to $E^{w*}(m)$ over successive days D if and only if hour-H retail power and price outcomes ($p^{RET}(H, D-1), \pi^{RET}(H, D-1)$) converge to $E^{r*}(m)$ over successive days D.

These two intertwined wholesale and retail processes constitute the braided cobweb dynamics for the IRW Test Case. Given any start-day D-1, the study of the wholesale process is most naturally undertaken by choosing a *wholesale-start point*, i.e., a start-point W_0 on the plot of the GenCo's supply function (1); and the study of the retail process is most naturally undertaken by choosing a *retail-start point*, i.e., a start-point R_0 on the plot of the retail demand function (6).

Figure 7 depicts these two braided cobweb cycles. The wholesale-start cycle is depicted as $W_0 \rightarrow A_1 \rightarrow A_2 \dots$, and the retail-start cycle is depicted as $R_0 \rightarrow B_1 \rightarrow B_2 \dots$.

As will be established analytically in Section V-C, below, the type of cobweb dynamics exhibited by each of these braided cobweb cycles is entirely determined by the sign of $[b - d(m)]$, where b is the slope coefficient for the GenCo's supply offer (1) and $d(m)$ is the slope coefficient for the m -modified retail demand function (8). In particular, assuming the start point W_0 (R_0) does not coincide with $E^{w*}(m)$ ($E^{r*}(m)$), a necessary and sufficient condition for wholesale (retail) power and price outcomes to converge to $E^{w*}(m)$ ($E^{r*}(m)$) is $b < d(m)$.

This analysis is previewed here using graphical depictions for illustrative simulation runs. For concreteness, attention is

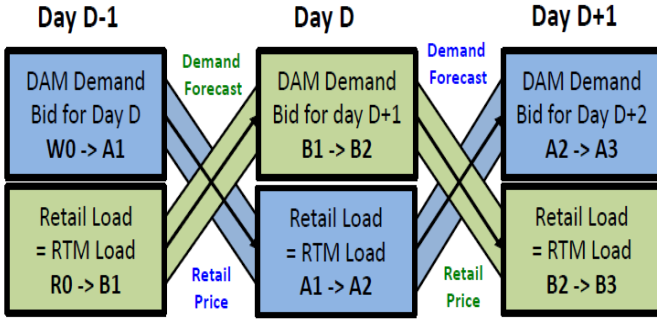


Fig. 7. Depiction of the two braided cobweb cycles for the IRW Test Case under dynamic-price retail contracting, differentiated by start point. The wholesale-start cycle starts at W_0 and the retail-start cycle starts at R_0 .

focused on the cobweb dynamics exhibited by the retail-start cycle $R_0 \rightarrow B_1 \rightarrow B_2 \dots$ for a particular hour H on successive days D , commencing on day $D-1$ from a start-point R_0 on the plot of the retail demand function (6). For increased graphical clarity, the mark-up m for the dynamic-price retail contracts is set to zero, which implies $E^{w*}(0) = E^{r*}(0) \equiv E^*$.

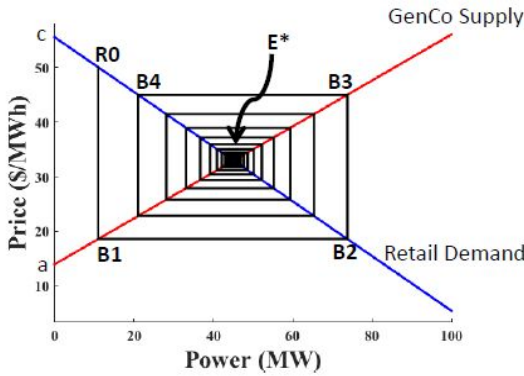


Fig. 8. Illustration of convergent cobweb dynamics ($b < d(m)$) for the IRW Test Case with dynamic-price retail contracts. Parameter settings: $m=0$, $a=14$, $b=0.2$, $c=55.5$, $d=0.25$, and start point $R_0=(11,50)$.

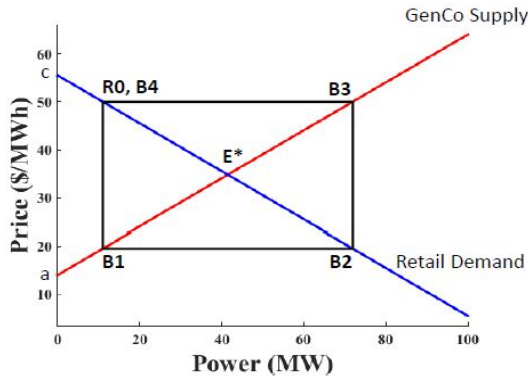


Fig. 9. Illustration of a fixed cobweb cycle ($b = d(m)$) for the IRW Test Case with dynamic-price retail contracts. Parameter settings: $m=0$, $a=14$, $b=0.25$, $c=55.5$, $d=0.25$, and start point $R_0=(11,50)$.

Figures 8–10 illustrate the three possible types of cobweb dynamics that can be exhibited by the retail-start cycle, con-

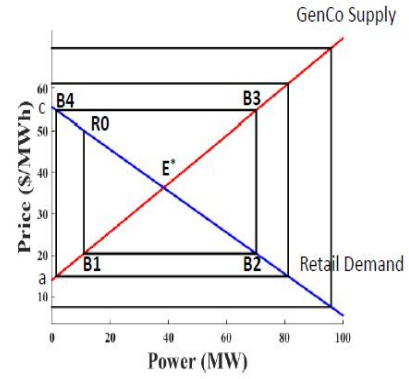


Fig. 10. Illustration of divergent cobweb dynamics ($b > d(m)$) for the IRW Test Case with dynamic-price retail contracts. Parameter settings: $m=0$, $a=14$, $b=0.3$, $c=55.5$, $d=0.25$, and start point $R_0=(11,50)$.

ditional on the magnitude of $[b - d(0)]$, assuming a retail-start point R_0 that is not coincident with E^* . In each case the power system repeatedly cycles through four market phases. The first occurrences of these four market phases correspond to the movements $R_0 \rightarrow B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow B_4$. The second occurrence of the first market phase then corresponds to the movement $B_4 \rightarrow B_5$.

Specifically, at the start point R_0 , values are determined for retail power usage (load) and the retail price for day $D-1$, which then determines RTM load for day $D-1$ by assumption (A1). The RTM load on day $D-1$ determines the RTM price for day $D-1$, by assumptions (A2) and (A3), which moves the system from R_0 to B_1 . The LSE then uses RTM load on day $D-1$ as its day- D DAM demand bid (in fixed demand form) by assumption (A4), which determines the day- D DAM price through DAM market clearing by assumptions (A5) and (A6). The LSE then sets the day $D+1$ retail price equal to $[1 + m]$ times the day- D DAM price, by assumption (A7), which determines retail power usage for day $D+1$ by assumption (A8); this moves the system from B_1 to B_2 .

At B_2 , the retail load for day $D+1$ determines the RTM load for day $D+1$, by (A1), which then determines the RTM price for day $D+1$ by assumptions (A2) and (A3); this moves the system from B_2 to B_3 . The LSE uses the RTM load for day $D+1$ to determine its day- $(D+2)$ DAM demand bid, by assumption (A4), which then determines the day- $(D+2)$ DAM price by assumptions (A5) and (A6). The LSE then uses this day- $(D+2)$ DAM price, multiplied by $[1 + m]$, to set the retail price for day- $(D+3)$, by assumption (A7), which determines retail power usage by assumption (A8). This moves the system from B_3 to B_4 , and the entire process then repeats.

C. Analytical Derivation of Findings

As discussed in Section V-B, and illustrated in Fig. 7, assumptions (A1) through (A8) for the IRW Test Case induce market dynamics having a braided cobweb form. That is, two cycles are intertwined with each other, a wholesale-start cycle and a retail-start cycle. In particular, starting on any day $D-1$, the wholesale-start cycle determines retail outcomes on days D , $D+2$, $D+3, \dots$, and the retail-start cycle determines retail outcomes on days $D-1$, $D+1$, $D+3, \dots$.

For concreteness, this section focuses on the retail-start cycle. Note that a wholesale-start cycle starting on any arbitrarily selected day $D-1$ can be decomposed into a wholesale start on day $D-1$ followed by a retail-start cycle starting on day D . As will be established below in Proposition 2, the long-run dynamical behavior of the retail-start cycle is entirely determined by the relative size of two exogenously given parameters, regardless of the start day. Consequently, the wholesale-start cycle always exhibits the same long-run dynamical behavior as the retail-start cycle.

Consider an IRW Test Case satisfying assumptions (A1) through (A8). Let H denote any particular hour of a 24-hour day, hereafter suppressed in all notation for easier readability. Let $D-1$ denote any given day, and let

$$R_0 = (p^{RET}(D-1), \pi^{RET}(D-1))^T \quad (11)$$

denote an arbitrarily selected point on the plot of the retail demand function (6) for hour H of day $D-1$. For each $t = 0, 1, 2, \dots$, define

$$s(t) = ([D-1] + 4t) ; \quad (12)$$

$$\mathbf{r}(t) = (p^{RET}(s(t)), \pi^{RET}(s(t)))^T . \quad (13)$$

By construction, the $\mathbf{r}(t)$ process represents retail power and price outcomes for the IRW Test Case during hour H on every fourth day, starting from R_0 on day $D-1$.

PROPOSITION 1: Let R_0 and $\mathbf{r}(t)$ be defined as in (11) and (13). Then the dynamical behavior of the $\mathbf{r}(t)$ process is determined by the following system of linear difference equations:

$$\mathbf{r}(t) = \mathbf{q}(m) + \mathbf{J}(m)\mathbf{r}(t-1), \quad \forall t = 1, 2, \dots ; \quad (14)$$

$$\mathbf{r}(0) = R_0, \quad (15)$$

where:

$$\mathbf{q}(m) = [\mathbf{I} + \mathbf{M}(m)]\mathbf{v}(m); \quad (16)$$

$$\mathbf{J}(m) = [\mathbf{M}(m)]^2; \quad (17)$$

$$\mathbf{v}(m) = \begin{bmatrix} \frac{[c(m)-a]}{2d(m)} \\ \frac{(1+m)[ad(m)+bc(m)]}{d(m)} \end{bmatrix}; \quad (18)$$

$$\mathbf{M}(m) = (-b/d(m)) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (19)$$

$$\begin{bmatrix} c(m) \\ d(m) \end{bmatrix} = \begin{bmatrix} \frac{c}{1+m} \\ \frac{d}{1+m} \end{bmatrix}. \quad (20)$$

Proof of Proposition 1: By assumption, R_0 in (11) satisfies the retail demand function (6) for hour H of day $D-1$, implying

$$p^{RET}(D-1) = \frac{[c - \pi^{RET}(D-1)]}{2d}. \quad (21)$$

The following additional relationships can be verified for IRW Test Case outcomes over the next four days by successive

application of assumptions (A1) through (A8).

$$\text{By (A1): } p^{RT}(D-1) = p^{RET}(D-1) \quad (22)$$

$$\text{By (A2),(A3): } \pi^{RT}(D-1) = a + 2bp^{RT}(D-1) \quad (23)$$

$$\text{By (A4): } p^{DA}(D) = p^{RT}(D-1) \quad (24)$$

$$\text{By (A5),(A6): } \pi^{DA}(D) = a + 2bp^{DA}(D) \quad (25)$$

$$\text{By (A7): } \pi^{RET}(D+1) = [1 + m]\pi^{DA}(D) \quad (26)$$

$$\text{By (A8): } p^{RET}(D+1) = [c - \pi^{RET}(D+1)]/2d \quad (27)$$

For each $n = 0, 1, 2, \dots$, define

$$k(n) = [D-1] + 2n \quad (28)$$

$$\mathbf{y}(n) = (p^{RET}(k(n)), \pi^{RET}(k(n)))^T. \quad (29)$$

Note that relationship (27) for day $D+1$ has the same form as relationship (21) for day $D-1$. By a simple induction argument, it follows that relationships (22) through (27) can be recursed forward indefinitely for days $k(n-1)$ to $k(n)$ for each $n = 1, 2, \dots$. Consequently, it follows by straightforward calculation that

$$\mathbf{y}(n) = \mathbf{v}(m) + \mathbf{M}(m)\mathbf{y}(n-1), \quad n = 1, 2, \dots \quad (30)$$

Finally, for each $t = 1, 2, \dots$, note that $s(t) = k(2t)$ and $\mathbf{r}(t) = \mathbf{y}(2t)$. Thus, for each $t = 1, 2, \dots$, one has

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{y}(2t) \\ &= \mathbf{v}(m) + \mathbf{M}(m)\mathbf{y}(2t-1) \\ &= \mathbf{v}(m) + \mathbf{M}(m)[\mathbf{v}(m) + \mathbf{M}(m)\mathbf{y}(2t-2)] \\ &= [\mathbf{I} + \mathbf{M}(m)]\mathbf{v}(m) + [\mathbf{M}(m)]^2\mathbf{y}(2t-2) \\ &= [\mathbf{I} + \mathbf{M}(m)]\mathbf{v}(m) + [\mathbf{M}(m)]^2\mathbf{r}(t-1) \\ &= \mathbf{q}(m) + \mathbf{J}(m)\mathbf{r}(t-1). \end{aligned} \quad (31)$$

QED

COROLLARY 1: Let R_0 , $\mathbf{r}(t)$, and $\mathbf{J}(m)$ be defined as in Proposition 1. For each $t = 0, 1, 2, \dots$, define

$$\mathbf{z}(t) = \mathbf{r}(t) - E^{r^*}(m), \quad (32)$$

where $E^{r^*}(m)$ is the retail market equilibrium point given by (10). Then

$$\mathbf{z}(t) = \mathbf{J}(m)\mathbf{z}(t-1), \quad \forall t = 1, 2, \dots ; \quad (33)$$

$$\mathbf{z}(0) = R_0 - E^{r^*}(m). \quad (34)$$

Proof of Corollary 1: By simple recursion and manipulation of terms,

$$\begin{aligned} \mathbf{z}(t) &= \mathbf{q}(m) + \mathbf{J}(m)\mathbf{x}(t-1) - E^{r^*}(m) \\ &= \mathbf{q}(m) + \mathbf{J}(m)[\mathbf{x}(t-1) - E^{r^*}(m)] \\ &\quad + \mathbf{J}(m)E^{r^*}(m) - E^{r^*}(m) \\ &= \mathbf{q}(m) + \mathbf{J}(m)\mathbf{z}(t-1) + [\mathbf{J}(m) - \mathbf{I}]E^{r^*}(m) \\ &= \mathbf{J}(m)\mathbf{z}(t-1) + (\mathbf{q}(m) + [(b/d(m))^2 - 1]E^{r^*}(m)) \\ &= \mathbf{J}(m)\mathbf{z}(t-1) + \mathbf{u}(m). \end{aligned} \quad (35)$$

It then follows by straightforward calculation and definition (10) for $E^{r^*}(m)$ that $\mathbf{u}(m) = \mathbf{0}$. QED

If the start-point R_0 for the $\mathbf{r}(t)$ process in Proposition 1 coincides with the retail market equilibrium point $E^{r^*}(m)$, it follows immediately from Corollary 1 that $\mathbf{r}(t) = E^{r^*}(m)$ for all $t = 0, 1, \dots$; hence, no dynamics are generated. Consequently, the following proposition focuses on cases for which R_0 does not coincide with $E^{r^*}(m)$.

PROPOSITION 2: Let R_0 and $\mathbf{r}(t)$ be defined as in Proposition 1. Suppose $R_0 \neq E^{r^*}(m)$. Then, ignoring non-negativity restrictions on price and power levels, the long-run dynamical behavior of the $\mathbf{r}(t)$ process is entirely determined by the sign of $[b - d(m)]$, as follows:

Case 1 (Cobweb Convergence): If $b < d(m)$, $\mathbf{r}(t)$ converges to $E^{r^*}(m)$ as $t \rightarrow \infty$.

Case 2 (Cobweb Fixed Cycle): If $b = d(m)$, $\mathbf{r}(t) = R_0$ for all $t = 0, 1, \dots$

Case 3 (Cobweb Divergence): If $b > d(m)$, the components of $\mathbf{r}(t)$ diverge to plus or minus ∞ .

Proof of Proposition 2: From Corollary 1,

$$\begin{aligned} \mathbf{z}(t) &= \mathbf{J}(m)\mathbf{z}(t-1) \\ &= [\mathbf{J}(m)]^2\mathbf{z}(t-2) \\ &= \dots = [\mathbf{J}(m)]^t\mathbf{z}(0), \end{aligned} \quad (36)$$

where $\mathbf{z}(t) = [\mathbf{x}(t) - E^{r^*}(m)]$ and $[\mathbf{J}(m)]^t = (b/d(m))^{2t}\mathbf{I}$. If $b < d(m)$, it follows from (36) that $\mathbf{z}(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, $\mathbf{r}(t) \rightarrow E^{r^*}(m)$ as $t \rightarrow \infty$. This establishes Case 1.

Suppose, instead, that $b = d(m)$. It then follows from (36) that $\mathbf{z}(t) = \mathbf{z}(0)$ for all $t = 0, 1, \dots$, which implies $\mathbf{r}(t) = \mathbf{r}(0) = R_0$ for all $t = 0, 1, \dots$. This establishes Case 2.

Finally, suppose $b > d(m)$. Consider, first, what happens if R_0 lies to the *left* of $E^{r^*}(m)$ along the plot of the retail demand function (6); see Fig. 6. This implies that the components of the vector $\mathbf{z}(0) = (z_1(0), z_2(0))^T$ satisfy

$$z_1(0) < 0; \quad (37)$$

$$z_2(0) > 0. \quad (38)$$

It then follows from (36), (37), and (38) that $z_1(t)$ diverges to $-\infty$ as $t \rightarrow \infty$ and $z_2(t)$ diverges to $+\infty$ as $t \rightarrow \infty$. The reverse divergence directions for $z_1(t)$ and $z_2(t)$ can easily be shown to hold if R_0 instead lies to the *right* of $E^{r^*}(m)$ along the plot of the retail demand function (6). This establishes Case 3. QED

Consider once again Figs. 8–10. These three figures depict the three possible types of cobweb dynamics that can be exhibited by power and price outcomes for the IRW Test Case, given dynamic-price retail contracts with a zero mark-up ($m = 0$) and a start point R_0 on the plot of the retail demand function (6) that is not coincident with the retail market equilibrium point $E^* \equiv E^{r^*}(0)$. Note that the long-run dynamical behavior of power and price outcomes depicted in these figures can in fact be inferred from the relative positions of R_0 , $\mathbf{r}(1) \equiv B_4$, and E^* along the depicted demand function.

The following corollary establishes that this finding holds more generally for any IRW Test Case with dynamic-price retail contracts regardless of the magnitude of the mark-up $m \geq 0$; see Fig. 11.

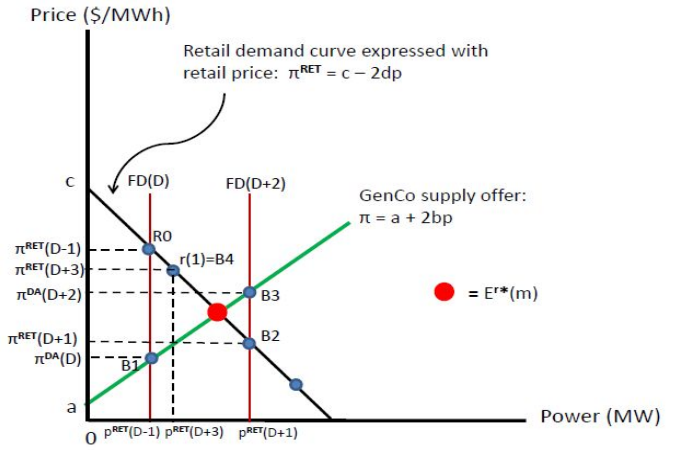


Fig. 11. Illustration of Corollary 2 depicting how the form of cobweb dynamics for IRW Test Case outcomes under dynamic-price retail contracts with mark-up $m \geq 0$ is determined by the relative positions of R_0 , $\mathbf{r}(1)$, and $E^{r^*}(m)$ along the retail demand curve.

COROLLARY 2: Let R_0 and $\mathbf{r}(t)$ be defined as in Proposition 1. Then the form of cobweb dynamics exhibited by IRW Test Case outcomes under dynamic-price retail contracts with mark-up $m \geq 0$ can be determined from the positions of R_0 , $\mathbf{r}(1)$, and $E^{r^*}(m)$ along the plot of the retail demand function (6), as follows:

- (i) *Stationary Equilibrium:* R_0 coincides with $E^{r^*}(m)$, implying $\mathbf{r}(t) = E^{r^*}(m)$ for all $t \geq 0$;
- (ii) *Cobweb Convergence:* $R_0 \neq E^{r^*}(m)$ and $\mathbf{r}(1)$ lies strictly between R_0 and $E^{r^*}(m)$ along the plot of (6).
- (iii) *Cobweb Fixed Cycle:* $R_0 \neq E^{r^*}(m)$ and $\mathbf{r}(1)$ is coincident with R_0 ;
- (iv) *Cobweb Divergence:* $R_0 \neq E^{r^*}(m)$ and the distance between $\mathbf{r}(1)$ and $E^{r^*}(m)$ is greater than the distance between R_0 and $E^{r^*}(m)$ along the plot of (6).

Proof of Corollary 2: As established by Corollary 1 and Proposition 2, the long-run dynamical behavior of the $\mathbf{z}(t)$ process (33), hence the $\mathbf{r}(t)$ process (14), is determined by the properties of the matrix $\mathbf{J}(m) = (b/d(m))^2\mathbf{I}$, in particular by whether $(b/d(m))^2$ is greater than, equal to, or less than 1. If $\mathbf{z}(0) \neq 0$, the magnitude of $(b/d(m))^2$ can be determined by comparing $\mathbf{z}(1) = (b/d(m))^2\mathbf{z}(0)$ to $\mathbf{z}(0)$.

Thus, there are only four possible long-run dynamical behaviors that can be exhibited by the $\mathbf{r}(t)$ process, each identifiable as a particular relationship among R_0 , $\mathbf{r}(1)$, and $E^{r^*}(m)$ as follows:

- (a) *Stationary Equilibrium:* R_0 coincides with $E^{r^*}(m)$, implying $\mathbf{r}(t) = E^{r^*}(m)$ for all times $t \geq 0$;
- (b) *Cobweb Convergence:* $R_0 \neq E^{r^*}(m)$, and $\mathbf{r}(1)$ lies strictly between R_0 and $E^{r^*}(m)$ along the plot of (6). This implies the magnitude of $\mathbf{z}(1)$ is strictly less than the magnitude of $\mathbf{z}(0)$, hence $b < d(m)$;
- (c) *Cobweb Fixed Cycle:* $R_0 \neq E^{r^*}(m)$, and $\mathbf{r}(1)$ is coincident with R_0 ; this implies $\mathbf{z}(1) = \mathbf{z}(0)$, hence $b = d(m)$;
- (d) *Cobweb Divergence:* $R_0 \neq E^{r^*}(m)$, and the distance between $\mathbf{r}(1)$ and $E^{r^*}(m)$ is greater than the distance

between R_0 and $E^{r^*}(m)$ along the plot of (6). This implies that the magnitude of $\mathbf{z}(1)$ exceeds the magnitude of $\mathbf{z}(0)$, hence $b > d(m)$.

QED

VI. WELFARE AND MARKET EFFICIENCY ANALYSIS FOR THE IRW TEST CASE

A. Overview

This section uses standard applied economic analysis [39] to derive short-run (hourly) welfare and market efficiency outcomes for the IRW Test Case under both dynamic-price and flat-rate retail contracting. To facilitate the analysis, the auxiliary assumption (A8) in Section V is imposed. This assumption postulates that the hourly aggregate demand function arising from household price-responsive A/C controllers can be approximated by a linear function taking form (6).¹²

For increased clarity, a short-run welfare analysis is first undertaken in Section VI-B for a simple market example with a single Seller and a single Buyer. A detailed short-run welfare analysis for the IRW Test Case under dynamic-price and flat-rate retail contracts is then given in Sections VI-C and VI-D, respectively.

B. Welfare and Efficiency Analysis for a Simple Market

Consider a market for power for some hour H . The only participants in this market are a Seller that submits a price-responsive supply offer (1) and a Buyer that submits a price-responsive demand bid (2). There is no grid congestion, and the Seller's capacity is sufficient to meet the Buyer's power demands.

Assume the Seller's supply offer (1) represents the operational (production) cost for each increment of power that the Seller produces during hour H ; that is, assume (1) represents the *marginal cost function* of the Seller. Assume, also, that the Buyer's demand bid (2) represents the maximum willingness of the Buyer to pay for each increment of power during hour H ; that is, assume (2) represents the *marginal benefit function* of the Buyer.

Suppose the plots of this supply offer and demand bid are as depicted in Fig. 12. Then the unique *market equilibrium* for this market is the point $E^* = (p^*, \pi^*)$ in the non-negative orthant of the p - π plane where (1) and (2) intersect. Suppose all settlements for hour H are based on E^* .

At E^* the Seller is paid the same price π^* for every MW of power it supplies from $p = 0$ to $p = p^*$. Hence, the Seller's benefit at E^* , measured in terms of its revenues, is given by $\pi^* p^*$. On the other hand, the Seller's *total operating cost* at E^* is measured by the area underneath the plot of its marginal cost function (1) in Fig. 12 from $p = 0$ to $p = p^*$; that is, it is given by the integration of the Seller's marginal cost function from $p = 0$ to $p = p^*$. Let this area be denoted by $SCost^*$. It follows that *Seller welfare* at E^* , measured in term of the Seller's *net benefit* (i.e., benefit minus cost), is given by

$$SWelfare(E^*) = \pi^* p^* - SCost^*. \quad (39)$$

¹²As detailed in the Appendix, simulation testing conducted with the IRW Test Case provides support for (A8).

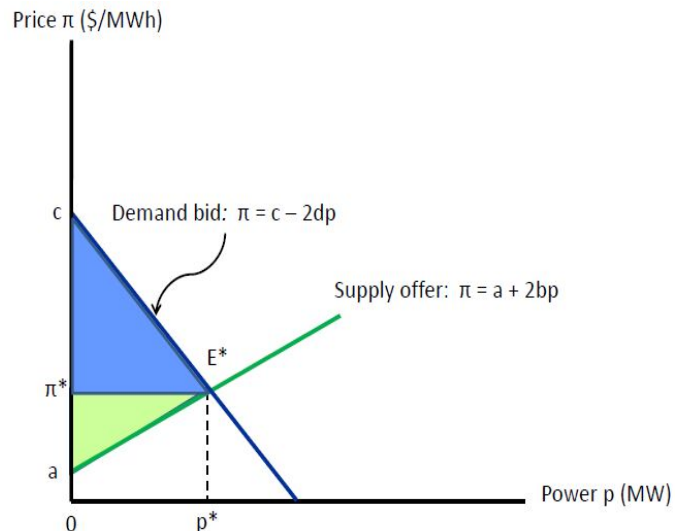


Fig. 12. Market clearing and welfare outcomes for a seller and buyer with price-responsive supply and demand curves.

The Seller's welfare (39) appears in Fig. 12 as the lighter (green) shaded area.

The Buyer's benefit at E^* , measured in terms of its willingness to pay, is measured by the area underneath the plot of its marginal benefit function (2) in Fig. 12 from $p = 0$ to $p = p^*$; that is, it is given by the integration of the Buyer's marginal benefit function from $p = 0$ to $p = p^*$. Let this area be denoted by $BBenefit^*$. On the other hand, the Buyer pays the same price π^* for every MW of power it purchases from $p = 0$ to $p = p^*$. Hence, the Buyer's total purchase cost at E^* is given by $\pi^* p^*$. It follows that *Buyer Welfare* at E^* , measured in terms of the Buyer's net benefit, is given by

$$BWelfare(E^*) = BBenefit^* - \pi^* p^*. \quad (40)$$

The Buyer's welfare (40) appears in Fig. 12 as the darker (blue) shaded area.

Finally, *total net benefit* for a market is defined to be the summation of all net benefit extracted from the market by market participants. Thus, for the market example at hand, total net benefit is given by the sum of (39) and (40), which results in

$$TotalNetBenefit^* = BBenefit^* - SCost^*. \quad (41)$$

The total net benefit (41) is represented in Fig. 12 by the sum of the two shaded areas. It is easily shown that the total net benefit (41) is the maximum possible net benefit that can be extracted from this market. Any further sale of power beyond p^* would have the property that the Seller's cost for producing this power exceeds the willingness of the Buyer to pay for this power, implying a decrement to total net benefit.

C. Welfare and Market Efficiency Analysis for the IRW Test Case with Dynamic-Price Retail Contracts

Consider an IRW Test Case with dynamic-price retail contracting that satisfies the auxiliary assumption (A8) in Section V-A. The market participants consist of a GenCo

that produces bulk (wholesale) power, a collection of retail households that consume power, and an LSE that operates as an intermediary between the wholesale and retail sectors. Specifically, the LSE purchases bulk power in wholesale markets on behalf of the retail households and then sells this power to the households in accordance with a dynamic-price retail contract with a mark-up $m \geq 0$.

Let H refer to a particular operating hour during a 24-hour day. Suppose the current day is $D-1$ and the IRW Test Case is at a point $R0$ on the retail demand curve. How can welfare outcomes be measured and compared for the GenCo, the households, and the LSE for hour H of day $D+1$?

Three complicating factors must be kept in mind in the calculation of these welfare outcomes. First, the LSE does not know next-day retail loads. Rather, on any given day D the LSE's forecast for hour- H retail load on day $D+1$ is given by the retail load realized for hour H on day $D-1$; the LSE uses this forecast as its day- D DAM demand bid for hour H of day $D+1$. Second, the LSE uses current-day DAM prices to set next-day retail prices. Third, the DAM is a forward financial market used each day by the ISO to determine a *planned* next-day GenCo dispatch schedule. The GenCo does not incur any operational costs until it engages in actual real-time production.

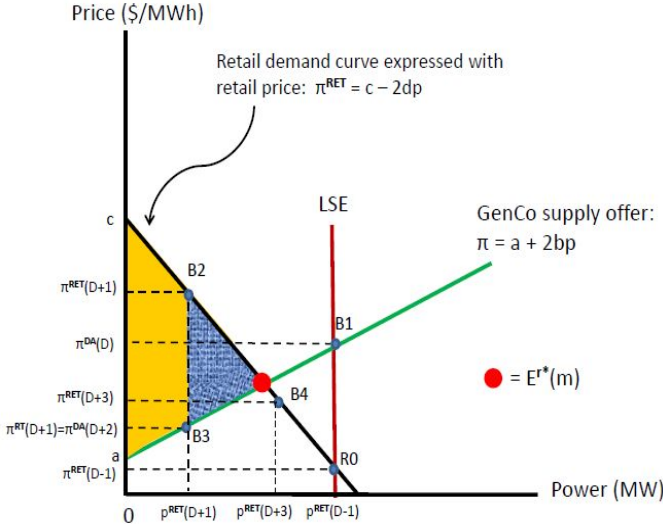


Fig. 13. Illustrative depiction of power, price, and welfare outcomes for the IRW Test Case with dynamic-price retail contracts for a particular hour H on day $D+1$, assuming a positive retail-price mark-up ($m > 0$).

Figure 13 illustrates possible power, price, and welfare outcomes for the IRW Test Case for a particular hour H on a particular day $D+1$, assuming that the dynamic-price retail contract has a positive mark-up $m > 0$. For clarity of exposition, dependence on H has been suppressed in the notation.

The start point depicted in Fig. 13 is an arbitrary point $R0 = (p^{RET}(D-1), \pi^{RET}(D-1))$ on the retail demand curve for hour H of day $D-1$. The LSE's day- D DAM demand bid for hour H of day $D+1$ is then given by $p^{RET}(D-1)$, implying a vertical day- D DAM demand curve at this power level. Thus, the market clearing point for the day- D DAM is given by $B1$

$$= (p^{RET}(D-1), \pi^{DA}(D)).$$

Given the assumed dynamic-price form of retail contracts, the LSE then sets the retail price $\pi^{RET}(D+1)$ for hour H of day $D+1$ equal to $[1+m]$ times the day- D DAM price $\pi^{DA}(D)$, resulting in a retail power demand $p^{RET}(D+1)$ for hour H of day $D+1$ that is strictly *less* than $p^{RET}(D-1)$. This moves the system to point $B2$. At $B2$, the retail power usage $p^{RET}(D+1)$ for day $D+1$ is used by the LSE as its demand bid for the day- $(D+2)$ DAM. The day- $(D+2)$ DAM is then cleared at a price $\pi^{DA}(D+2)$ for hour H of day $D+3$, which moves the system to point $B3$. The LSE then sets the retail price $\pi^{RET}(D+3)$ for hour H of day $D+3$ equal to $[1+m]$ times $\pi^{DA}(D+2)$. This retail price results in a retail power usage in amount $p^{RET}(D+3)$, which moves the system to $B4$.

The GenCo receives compensation in amount $\pi^{DA}(D)p^{RET}(D-1)$ in the day- D DAM for its scheduled power dispatch level $p^{RET}(D-1)$ for hour H of day $D+1$. However, actual household power usage during hour H of day $D+1$ is $p^{RET}(D+1)$, which is strictly less than $p^{RET}(D-1)$. In the day- $(D+1)$ RTM for hour H the GenCo must now in effect “buy back” power in amount $[p^{RET}(D-1) - p^{RET}(D+1)]$ at the RTM price. This RTM price, denoted by $\pi^{RT}(D+1)$, is determined by the intersection of the GenCo's supply curve with the vertical RTM demand curve at power level $p^{RT}(D+1) = p^{RET}(D+1)$. For later purposes, note from Fig. 13 that $\pi^{RT}(D+1) = \pi^{DA}(D+2)$.

The GenCo's total operating cost for the production of $p^{RET}(D+1)$ during hour H of day $D+1$ is given by the area under its supply offer (marginal cost curve) from $p = 0$ to $p = p^{RET}(D+1)$. Let $GCost(D+1)$ denote this area. *GenCo welfare* for hour H of day $D+1$, measured in terms of net benefit (i.e., revenues minus costs), is then given by¹³

$$\begin{aligned} GWelfare(D+1) &= \pi^{DA}(D)p^{RET}(D-1) \\ &\quad - \pi^{RT}(D+1)[p^{RET}(D-1) - p^{RET}(D+1)] \\ &\quad - GCost(D+1) \\ &= \pi^{DA}(D)p^{RET}(D-1) \\ &\quad - \pi^{DA}(D+2)[p^{RET}(D-1) - p^{RET}(D+1)] \\ &\quad - GCost(D+1) \end{aligned} \quad (42)$$

The LSE pays $\pi^{DA}(D)p^{RET}(D-1)$ in the day- D DAM for its forecasted fixed demand $p^{RET}(D-1)$ for hour H of day $D+1$. However, actual household power usage during hour H of day $D+1$ is only $p^{RET}(D+1)$. Consequently, the LSE receives compensation at rate $\pi^{RT}(D+1)$ in the RTM for hour H of day $D+1$ for this over-payment. On the other hand, the amount the LSE receives in payment from households for their actual power usage during hour H of day $D+1$ is given by $\pi^{RET}(D+1)p^{RET}(D+1) = [1+m]\pi^{DA}(D)p^{RET}(D+1)$.

Thus, *LSE welfare* for hour H of day $D+1$, measured in

¹³In the opposite case in which the actual retail power demand $p^{RET}(D+1)$ for hour H of day $D+1$ is strictly *greater* than the power level $p^{RET}(D-1)$ scheduled in the day- D DAM for hour H of day $D+1$, the ISO must secure additional power from the GenCo in the RTM for hour H of day $D+1$. This additional power is compensated at $\pi^{RT}(D+1)$. It is easily shown that the GenCo's welfare for hour H of day $D+1$ is once again given by formula (42).

terms of net benefit (i.e., revenues minus costs), is given by¹⁴

$$\begin{aligned}
\text{LWelfare}(D+1) &= \pi^{RET}(D+1)p^{RET}(D+1) \\
&+ \pi^{RT}(D+1)[p^{RET}(D-1) - p^{RET}(D+1)] \\
&- \pi^{DA}(D)p^{RET}(D-1) \\
&= [1 + m]\pi^{DA}(D)p^{RET}(D+1) \\
&+ \pi^{RT}(D+1)[p^{RET}(D-1) - p^{RET}(D+1)] \\
&- \pi^{DA}(D)p^{RET}(D-1) \\
&= m\pi^{DA}(D)p^{RET}(D+1) \\
&+ [\pi^{DA}(D+2) - \pi^{DA}(D)][p^{RET}(D-1) - p^{RET}(D+1)]. \tag{43}
\end{aligned}$$

The sign of (43) depends on exact magnitudes. However, since the product of the bracketed terms is always non-positive, a necessary condition for (43) to be positive is that the mark-up m be set sufficiently large.

Suppose household welfare is measured as benefit (willingness to pay) minus cost (actual payment). The willingness of households to pay for the power usage level $p^{RET}(D+1)$ during hour H of day D+1 is given by the area under the retail demand curve from $p = 0$ to $p = p^{RET}(D+1)$; let this area be denoted by $\text{HBen}(D+1)$. The actual household payment for this power usage is $\pi^{RET}(D+1)p^{RET}(D+1)$. Consequently, *household welfare* for hour H of day D+1, measured as household net benefit (i.e., benefits minus costs), is given by

$$\text{HWelfare}(D+1) = \text{HBen}(D+1) - \pi^{RET}(D+1)p^{RET}(D+1). \tag{44}$$

If the three welfare (net benefit) measures (42), (43), and (44) are added together, one obtains the *total net benefit* extracted by market participants for hour H of day D+1: namely,

$$\text{TotalNetBenefit}(D+1) = \text{HBen}(D+1) - \text{GCost}(D+1). \tag{45}$$

The total net benefit (45) is given by the lighter (gold) shaded area in Fig. 13.¹⁵

The total net benefit (45) is less than the *maximum possible* total net benefit that could be extracted for hour H of day D+1. The “missing” net benefit is the darker (blue) triangular shaded region in Fig. 13. This missing net benefit, which economists refer to as *deadweight loss*, measures the degree to which market outcomes are *inefficient* in the following “wasted resource” sense: some units of good fail to be produced even though the willingness to pay for these units exceeds their cost of production. In Fig. 13, these “wasted” units of good are the

¹⁴It is straightforward to show that (43) still correctly represents LSE welfare in the opposite case in which the actual retail power demand $p^{RET}(D+1)$ for hour H of day D+1 is strictly *greater* than the power level $p^{RET}(D-1)$ scheduled in the day-D DAM for hour H of day D+1.

¹⁵Recall that a basic simplifying assumption for the IRW Test Case is that no grid congestion arises. Thus, only a single LMP is determined in the DAM/RTM for any given hour H of any given day D; i.e., the LMPs do not separate across the transmission grid buses due to congestion. Consequently, each MW of power transacted in the DAM/RTM has the property that its purchase price (collected from the LSE) equals its sale price (paid out to the GenCo). Hence the ISO, who collects all payments and pays out all compensation in the DAM/RTM, has no net revenues left over from these transactions; i.e., the ISO has no “congestion rent” that it extracts from the market. See [24] for a careful discussion of welfare and congestion-rent calculations for the case in which transmission grid congestion can arise.

MWs of power from $p^{RET}(D+1)$ to the power level $p^*(m)$ at the market equilibrium point $E^{r^*}(m)$.

This deadweight loss arising for the IRW Test Case under dynamic-price retail contracts is caused by two factors: (i) the LSE uses an adaptive forecast for retail demand in the DAM in place of the true retail demand function (6); and (ii) the LSE is a private profit-seeking entity that uses adaptive advance retail pricing with a positive mark-up m in an attempt to ensure for itself a positive net benefit (revenues minus costs).

Suppose, instead, that the LSE is a fully-informed non-profit entity with a fiduciary responsibility for household welfare. In this case the LSE would submit a demand bid into the day-D DAM that correctly represents the true household demand function (6) for hour H of day D+1, and it would charge households a retail price $\pi^{RET}(D+1)$ for power usage during hour H of day D+1 that is equal to its true purchase price $\pi^{DA}(D)$, without imposing a price mark-up $m > 0$.

In this idealized case the retail power and price outcomes for hour H of day D+1 would be at $E^{r^*}(m)$ in Fig. 13, with a zero deadweight loss, and the net earnings (revenues minus costs) of the LSE would be zero. In effect, households would be direct active participants in wholesale power market operations.

An important caveat regarding the above welfare analysis is as follows. The metric used for household welfare – willingness to pay minus actual payment – provides only an approximate measure of household welfare. As detailed in Section IV-B, in the IRW Test Case the true welfare attained by a household in any hour H for any day D is measured by comfort minus energy cost; see (5).

D. Welfare and Market Efficiency Analysis for the IRW Test Case with Flat-Rate Retail Contracts

Consider, instead, the IRW Test Case with flat-rate retail contracting, still augmented with auxiliary assumption (A8) that directly postulates an hourly linear household aggregate demand function (6). Thus, the retail price charged to households for their power usage is set at an administratively determined rate R (\$/MWh) that is held constant over an extended time interval T .

To guarantee the LSE can stay in business over T , the flat rate R must be at a *break-even* level, i.e., at a level that ensures the LSE’s revenues are at least as great as the LSE’s costs over T . For each hour H of each day D, these revenues and costs are determined by the following four factors:

- The amount *charged* to the LSE in the day-(D-1) DAM for its demand bid (load forecast) for hour H of day D;
- The amount *charged* to the LSE in the RTM for hour H of day D if its day-(D-1) DAM demand bid (load forecast) for hour H of day D is *smaller than* actual retail power usage during hour H of day D.
- The payment *received* by the LSE in the RTM for hour H of day D to compensate the LSE if its day-(D-1) DAM demand bid (load forecast) for hour H of day D is *larger than* actual retail power usage during hour H of day D;
- The payment *received* by the LSE from households during hour H of day D for actual household power usage.

It is difficult to set R in advance at a level ensuring the LSE breaks even over time since each of the four factors listed above is affected by the setting of R . For example, it might seem reasonable simply to set the flat rate R equal to the average DAM price over time, arguing that the RTM price will fluctuate randomly around the DAM price. However, analysis suggests that this would be a disastrous policy for the LSE, resulting in strictly negative LSE welfare outcomes.

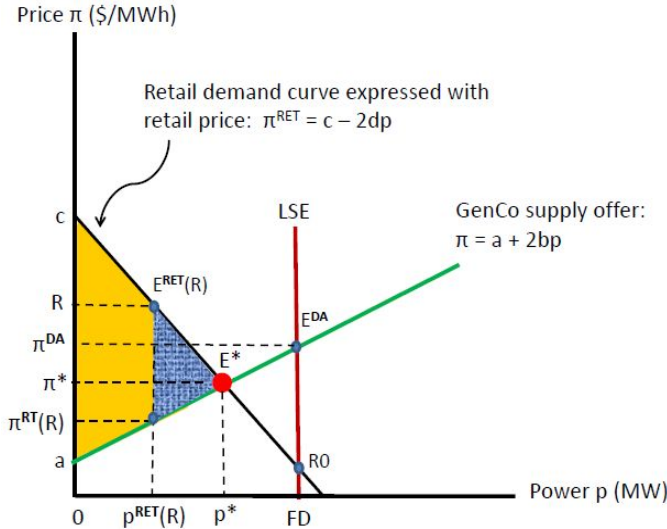


Fig. 14. Illustration of power, price, and welfare outcomes during an arbitrary hour H of an arbitrary day D+1 for the IRW Test Case with flat-rate retail contracts, assuming the flat rate R is at least as large as the DAM price π^{DA} for hour H of day D+1.

To see the difficulty, consider the IRW Test Case market processes depicted in Fig. (14) for a particular hour H of a particular day D+1 under the assumption that retail contracts take a flat-rate form. As in Section VI-C, the start point depicted in Fig. 13 is an arbitrary point $R0 = (p^{RET}(D-1), \pi^{RET}(D-1))$ on the retail demand curve for hour H of day D-1. The LSE's day-D DAM demand bid for hour H of day D+1 is then given by $FD = p^{RET}(D-1)$, implying a vertical day-D DAM demand curve at this power level. Thus, the market clearing point for the day-D DAM is given by $E^{DA} = (FD, \pi^{DA}(D))$.

Suppose the flat rate R charged to households for their retail power usage during hour H of day D+1 is at least as large as π^{DAM} . The actual power usage of households during hour H of day D+1, denoted by $p^{RET}(R)$, will then be strictly less than FD , the LSE's day-D DAM forecast for household power usage during hour H of day D+1.

The GenCo receives compensation in amount $\pi^{DA}FD$ in the day-D DAM for its scheduled power dispatch level FD during hour H of day D+1. In the day-(D+1) RTM for hour H the GenCo must now in effect "buy back" power in amount $[FD - p^{RET}(R)]$ at the RTM price. This RTM price, denoted by $\pi^{RT}(R)$, is determined by the intersection of the GenCo's supply curve with the induced vertical RTM demand curve at power level $p^{RET}(R)$.

The GenCo's total operating cost for the production of $p^{RET}(R)$ during hour H of day D+1 is given by the area under its supply curve (marginal cost function) from $p = 0$ to

$p = p^{RET}(R)$. Let $GCost(R)$ denote this area. It then follows that *GenCo welfare* for hour H of day D+1, measured in terms of net benefit (i.e., revenues minus costs), is given by¹⁶

$$\begin{aligned} GWelfare(R) &= \pi^{DA}FD \\ &\quad - \pi^{RT}(R)[FD - p^{RET}(R)] \\ &\quad - GCost(R) \end{aligned} \quad (46)$$

The LSE pays $\pi^{DA}FD$ in the day-D DAM for its forecasted fixed demand FD for hour H of day D+1. However, the actual household power usage during hour H of day D+1, i.e., $p^{RET}(R)$, is strictly less than FD .

Consequently, the LSE receives compensation at rate $\pi^{RT}(R)$ in the RTM for hour H of day D+1 for this overpayment. In addition, the LSE receives payments from households for their actual power usage during hour H of day D+1, given by $Rp^{RET}(R)$. Thus, *LSE welfare* for hour H of day D+1, measured in terms of net benefit (i.e., revenues minus costs), is given by¹⁷

$$\begin{aligned} LSEWelfare(R) &= Rp^{RET}(R) \\ &\quad + \pi^{RT}(R)[FD - p^{RET}(R)] \\ &\quad - \pi^{DA}FD \\ &= [\pi^{RT}(R) - \pi^{DA}][FD - p^{RET}(R)] \\ &\quad + [R - \pi^{DA}]p^{RET}(R). \end{aligned} \quad (47)$$

As long as the LSE's vertical demand curve at FD does not intersect the GenCo's supply curve at E^{r*} , the bracketed terms in (47) will have opposite signs; see Fig. 14. Consequently, a necessary condition for LSE welfare to be positive for hour H of day D+1 is that R be sufficiently greater than π^{DA} .

Finally, suppose household welfare is measured as benefit (willingness to pay) minus cost (actual payment). The willingness of households to pay for the power usage level $p^{RET}(R)$ during hour H of day D+1 is given by the area under the retail demand curve from $p = 0$ to $p = p^{RET}(R)$; let this area be denoted by *HouseBen*(R). The actual household payment for this power usage during hour H of day D+1 is $Rp^{RET}(R)$. Consequently, *household welfare* for hour H of day D+1, measured as household net benefit (i.e., benefits minus costs), is given by

$$HWelfare(R) = HouseBen(R) - Rp^{RET}(R). \quad (48)$$

Finally, if one now adds together the three welfare (net benefit) calculations (46), (47), and (48), one obtains the *Total Net Benefit* extracted by market participants for hour H of day D+1, given by

$$TotalNetBenefit(R) = HouseBen(R) - GCost(R) \quad (49)$$

The total net benefit (49) is given by the lighter (gold) shaded area in Fig. 14.

¹⁶It is straightforward to show that (46) still correctly represents the GenCo's welfare in the opposite case in which the retail power demand $p^{RET}(R)$ for hour H of day D+1 is strictly greater than the power level FD scheduled in the day-D DAM for hour H of day D+1.

¹⁷It is straightforward to show that (47) still correctly represents the LSE's welfare in the opposite case in which the retail power demand $p^{RET}(R)$ for hour H of day D+1 is strictly greater than the power level FD scheduled in the day-D DAM for hour H of day D+1.

The total net benefit (49) is less than the *maximum possible* total net benefit that could be extracted for hour H of day D. The “missing” net benefit is the darker (blue) triangular shaded region in Fig. 14. As before, this missing net benefit (deadweight loss) measures the degree to which market outcomes in the IRW Test Case are inefficient in the sense that some units of good fail to be produced even though the willingness to pay for these units exceeds their cost of production. In Fig. 14, these “wasted” units of good are the MWs of power from $p^{RET}(R)$ to the power level p^* at the market equilibrium point E^* .

As seen in Fig. 14, market efficiency increases (i.e., the deadweight loss decreases) as the distance between the flat-rate R and the market equilibrium price π^* is decreased. Suppose the administrators of the flat-rate retail contract program are somehow able to learn the equilibrium point E^* and they then set R equal to π^* for hour H for all days *subsequent* to D-1. Starting from an arbitrary point R_0 on the retail demand curve during hour H of day D-1, what welfare outcomes result for the GenCO, LSE, and households during hour H for all days *subsequent* to D-1?

First note that the *long-run* welfare outcomes for the GenCo, the LSE, and the households at E^* are as depicted in Fig. 12: namely, the Seller (GenCo) and the Buyer (the Households) split the extracted net benefit among themselves, and the LSE’s net benefit is zero. Call these welfare outcomes the *equilibrium welfare outcomes*.

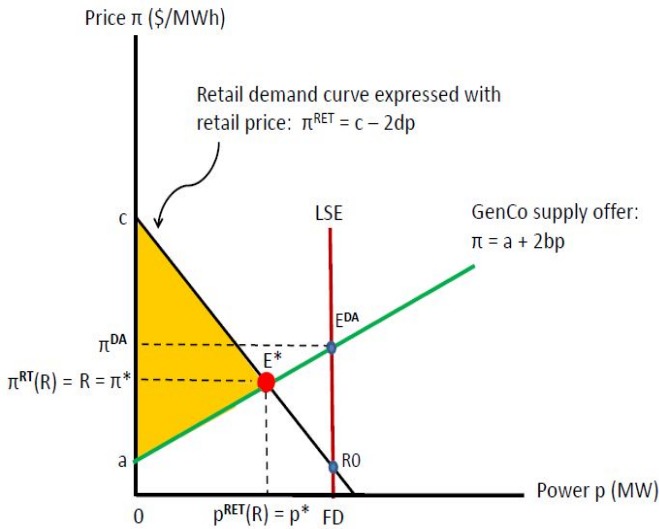


Fig. 15. Illustration of power, price, and welfare outcomes during some hour H of an arbitrary day D+1 for the IRW Test Case with flat-rate retail contracts, assuming the flat rate R is set equal to the market equilibrium price π^* .

This optimal- R case is illustrated in Fig. 15 for the specific case in which the start-point R_0 for hour H of day D-1 lies to the *right* of E^* along the retail demand curve. In this case π^{DA} *exceeds* π^* , and the following implications are easily seen.

First, GenCo welfare for hour H of day D+1 strictly *exceeds* GenCo equilibrium welfare by an amount $[\pi^{DA} - \pi^*]FD$. Second, LSE welfare for hour H of day D+1 is strictly *smaller* than the break-even LSE equilibrium welfare level (0); specifically, LSE welfare is $[\pi^* - \pi^{DA}]FD < 0$. Third, household

welfare for hour H of day D+1 equals household equilibrium welfare. Fourth, during hour H on all days *subsequent* to D+1, welfare outcomes for all market participants coincide with equilibrium welfare outcomes.

For the opposite case in which the start-point R_0 for hour H of D-1 lies to the *left* of E^* along the retail demand curve, π^{DA} is strictly *smaller* than π^* and the reverse situation holds for GenCo and LSE welfare. Specifically, GenCo welfare for hour H of day D+1 is strictly *lower* than GenCo equilibrium welfare by an amount $[\pi^* - \pi^{DA}]FD$. On the other hand, LSE welfare for hour H of day D+1 strictly *exceeds* the break-even LSE equilibrium welfare level (zero) by an amount $[\pi^* - \pi^{DA}]FD$.

VII. A DYNAMIC WELFARE SENSITIVITY DESIGN FOR THE IRW TEST CASE

This section explains the design of a dynamic welfare sensitivity study undertaken for the IRW Test Case described in Section IV.¹⁸ In contrast to the analytical studies carried out in Sections V and VI, this sensitivity study does not rely on the auxiliary assumption (A8) postulating a specific linear form for the hourly household aggregate demand function. Instead, the hourly power demands for each of the 500 households in the IRW Test Case are determined by a price-responsive A/C controller, as in [44].

The three treatment factors for this study are: (i) the form of retail contracts, either flat-rate or dynamic-price; (ii) the mark-up m in (4) that determines the percentage by which retail prices are marked up over wholesale prices in the case of dynamic-price retail contracts; and (iii) the household comfort-cost trade-off parameters α^h in (5). Four values are tested for m : 0.0, 0.2, 0.4, and 0.6. Also, four values are tested for α : 0, 1000 (“Low”), 2000 (“Medium”), and 3000 (“High”). The four α values were selected by trial and error to correspond to a price-elasticity of aggregate household demand ranging from zero to high.

The metrics used to measure welfare (net benefit) for the GenCo and LSE under dynamic-price and flat-rate retail contracts are as derived in Sections VI-C and VI-D, respectively. Specifically, given dynamic-price retail contracting, hourly GenCo welfare is measured as in (42), and hourly LSE welfare is measured as in (43). Given flat-rate retail contracting, hourly GenCo welfare is measured as in (46), and hourly LSE welfare is measured as in (47).

For all treatment configurations, hourly household welfare (net benefit) is measured as a trade-off between comfort and energy cost, as in (5). The degree to which a household h is willing to sacrifice comfort for energy cost savings is measured by α^h in (5).

As noted in Section IV, the thermal dynamics governing each of the 500 households in the IRW Test Case are represented by an ETP model. As detailed in [44], the basic ETP model is a linear differential system with two state variables, T^a and T^m , denoting inside air temperature and inside mass temperature. To reduce computational time, the current study

¹⁸Complete source code and data files for this sensitivity study, including external forcing terms (e.g., outside air temperature), maintained parameter values, and functional forms, can be accessed at [42].

uses the following simplified single-state version of the ETP model for each household h ,

$$\frac{dT^a(t)}{dt} = z_1^h [T^o(t) - T^a(t)] + z_2^h, \quad (50)$$

where $T^o(t)$ denotes outside air temperature at time t . Values were randomly assigned to z_1^h and z_2^h for each household h at the beginning of the sensitivity study and then maintained throughout all subsequent simulation runs.

Additional attributes maintained at fixed settings for all simulation runs are as follows:

- A/C rating for each household h ;
- Initial inside air temperature for each household h ;
- Bliss point temperature (70° F) for each household h ;
- Daily outside air temperature profile (same for each h on each simulated day);
- Initial retail prices communicated by the LSE to all households during the first two simulated days.

For the flat-rate case, the retail price charged to households in each simulation run was set at a fixed *break-even* rate R (\$/MWh) for the entire simulation, i.e., at a fixed rate R for which hourly LSE net earnings are at least zero and are as close to zero as possible. The break-even value for R was found by trial and error for each flat-rate simulation run.¹⁹

Below are the flat rates R determined for different values of α for the flat-rate retail contracting outcomes reported in Section VIII.

- $R = 27.9$ (\$/MWh) for $\alpha = 0$;
- $R = 27.7$ (\$/MWh) for $\alpha = 1000$ (low);
- $R = 27.5$ (\$/MWh) for $\alpha = 2000$ (medium);
- $R = 27.2$ (\$/MWh) for $\alpha = 3000$ (high).

Note these R values are a *decreasing* function of α .

Finally, each simulation run in the sensitivity study consisted of 20 simulated days. Hourly electricity costs and hourly GenCo, LSE, and household welfare levels were calculated during simulated days 11 to 20 and then divided by ten to obtain average daily values.

VIII. WELFARE SENSITIVITY FINDINGS FOR THE IRW TEST CASE

Figure 16 reports illustrative welfare outcomes for the LSE, the GenCo, and the 500 households for the IRW Test Case under a range of treatments exhibiting point-convergent cobweb dynamics. Only treatments with α and m values commonly set across all households are shown.

The following regularities are seen in the welfare sensitivity effects reported in Fig. 16. All of these regularities are subject to an “all else equal” qualifier.

- *Overall Welfare Findings:*
 - Suppose households are concerned about energy cost ($\alpha > 0$). Then a dynamic-price retail contract with a *positive* mark-up m results in a *better* welfare

¹⁹As explained in Section VI-D, LSE net earnings are zero for the IRW Test Case under flat-rate contracts if the IRW Test Case is initialized at the market equilibrium point (p^*, π^*) and the flat rate R is set equal to π^* ; cf. Fig. 15. However, none of the simulation runs for the welfare sensitivity study were initialized at this equilibrium point.

	m	Alpha = Zero		Alpha = Low		Alpha = Medium		Alpha = High	
		Flat	Dynamic	Flat	Dynamic	Flat	Dynamic	Flat	Dynamic
Avg. Daily LSE Welfare (\$)	0.00	0.00	0.00	0.04	0.00	0.08	0.00	0.04	0.00
	0.20		9.11		7.67		6.17		4.79
	0.40		18.21		14.84		11.40		8.35
	0.60		27.32		21.51		15.69		10.95
Avg. Daily GenCo Welfare (\$)	0.00	12.59	12.59	10.96	10.99	9.24	9.30	7.59	7.64
	0.20		12.59		10.66		8.62		6.73
	0.40		12.59		10.32		7.98		5.89
	0.60		12.59		9.97		7.33		5.16
Avg. Daily Household Welfare (Utils)	0.00	10446.29	10446.29	8581.52	8587.73	6707.30	6733.29	4879.86	4913.59
	0.20		10446.29		8216.86		5996.66		3870.66
	0.40		10446.29		7845.11		5275.68		2882.13
	0.60		10446.29		7473.19		4563.41		1965.76
Avg. Daily Energy Cost (\$)	0.00	1.82	1.82	1.58	1.58	1.33	1.33	1.09	1.09
	0.20		2.19		1.84		1.48		1.15
	0.40		2.55		2.08		1.60		1.17
	0.60		2.92		2.29		1.67		1.17

Fig. 16. LSE, GenCo, and household welfare outcomes for a range of IRW Test Case treatments exhibiting point-convergent cobweb dynamics.

outcome for the LSE and *worse* welfare outcomes for the GenCo and households than a flat-rate retail contract.

- Suppose households are *not* concerned about energy cost ($\alpha = 0$). Then GenCo and household welfare are independent of the retail price of power, hence independent of the form of the retail contract; but LSE welfare is higher under a dynamic-price retail contract than under a flat-rate retail contract for any $m > 0$.
- *LSE Welfare:*
 - Given dynamic-price retail contracting, LSE welfare *increases* as m *increases*.
 - Given a dynamic-price retail contract with $m > 0$, LSE welfare *decreases* as α *increases*.
- *GenCo and Household Welfare:*
 - Given $\alpha > 0$ and dynamic-price retail contracting, GenCo and household welfare *decrease* as m *increases*.
 - Given $\alpha > 0$, GenCo and household welfare are *higher* under a dynamic-price retail contract with a *zero* mark-up m than under a flat-rate retail contract.
 - GenCo and household welfare *decrease* as α *increases*.
- *Electricity Cost:*
 - Given dynamic-price retail contracting, average daily energy cost *increases* as m *increases*.
 - Average daily energy cost *decreases* as α *increases*.
 - The same average daily energy cost is realized under a dynamic-price retail contract with zero mark-up ($m = 0$) and a flat-rate retail contract.

IX. CONCLUDING DISCUSSION

Economists have known for decades that possibly divergent cycles can arise for prices and quantities in “cobweb” market

models for which a lag exists between the decision to produce a nonstorable good and its actual production. Economic research on this topic remains active; see, e.g., [3], [8], [15]. Power engineers have raised similar concerns for real-time electric power markets; see, e.g., [6], [26], [35].

For example, [35] analyze the global properties of a system of nonlinear differential equations derived for an ISO-managed real-time power market. The authors make various simplifying assumptions (e.g., non-binding capacity constraints for generation and transmission grid lines) that reduce the ISO's optimization problem in each successive period to a straightforward economic dispatch problem in which expected load (power consumption) is balanced by scheduled generation (planned power supply).

Power supplies and demands are specified as parameterized functional forms interpreted as the optimal solutions for myopic price-taking utility-maximizing producers and consumers. The discrepancy between scheduled generation and subsequent actual power consumption (hence actual power supply) then results in a form of cobweb cycling for market prices. Given a sufficiently large "Maximal Relative Price Elasticity," roughly defined to be demand price elasticity in ratio to supply price elasticity, the authors prove that prices can become increasingly volatile over time.

All of these previous cobweb studies highlight a common cautionary concern for demand-response researchers: namely, initiatives designed to encourage the more active participation of retail customers in power system operations must be designed with care in order to avoid adverse unintended consequences for power system operations. In what way, then, does the IRW Test Bed provide additional capabilities for demand-response researchers in general, and for TES researchers in particular, to address this concern?

The previous studies of cobweb cycle effects within electric power systems have largely been conceptual studies. In contrast, the IRW Test Bed is an agent-based computational platform. Its modular extensible architecture permits systematic studies of alternative demand-response initiatives in a plug-and-play mode. In scope, it covers the entire range of wholesale/transmission and retail/distribution operations, and it permits these operations to play out over time as an open-ended dynamic process.

Moreover, since analytical tractability is not an issue, the user's initial specifications for physical conditions, institutional arrangements, and the decision-making processes of human participants (including learning processes) can be as strongly grounded in empirical reality as warranted by the user's purpose. Last but not least, the IRW Test Bed is open source software, thus permitting later researchers to build directly and systematically upon previous findings.

By exploiting the capabilities of the IRW Test Bed to model increasingly larger systems with increasingly greater degrees of empirical verisimilitude, these studies could become the foundation for an increasingly sophisticated sequence of demand-response studies that bridge the gap from conceptual studies to real-world implementations.

APPENDIX

A fundamental assumption supporting the analytical findings reported in Sections V and VI is that the hourly aggregate demand function for the 500 households in the IRW Test Case can be approximated by a linear function taking form (6). Simulation tests provide strong support for this presumption.

Specifically, multiple simulation runs r were conducted for the IRW Test Case, where each run r was initialized at a distinct point in the retail power-price plane. (The initial retail prices were increased in successive steps across the runs r .) All other exogenous aspects were set identically across each run r . In particular, a single daily outside temperature profile was used from one simulated day to the next, the same for each run r ; and the form of retail contracting (dynamic-price or flat-rate) was set the same for each run r .

For each hour H during each simulated day D of each simulation run r , the resulting aggregate household power usage $p(H, D, r)$ and retail price $\pi(H, D, r)$ were calculated and plotted in superimposed fashion in the power-price plane, resulting in a scatter-plot of (p, π) points. These scatter-plots indicate that the hourly aggregate household demand function for the IRW Test Case can be well approximated by a linear function with negative slope over a broad range of power levels.

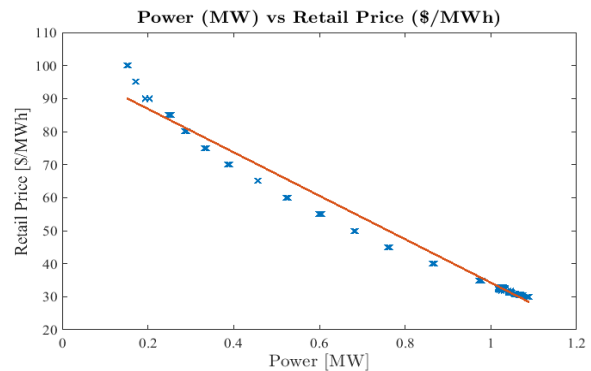


Fig. 17. Simulation-generated scatter plot for retail price and corresponding aggregate household power demand for a particular hour over twenty successive simulated days, with corresponding trend line.

An example of such a scatter-plot is depicted in Fig. 17. In this example, the selected hour of the day is 18:00, the GenCo cost coefficient b was set at 0.00001, and the form of retail contracting is dynamic-price contracting with a zero markup m . As can be seen more clearly in Fig. 18, the resulting household aggregate demand function is approximately linear with a negative slope over a broad interior range.

Moreover, the estimated slope coefficient \hat{d} for this simulated aggregate household demand function over this broad interior range is approximately 0.0004. Consequently, $b < \hat{d}/[1 + m] = \hat{d}$, indicating point-convergent cobweb dynamics in this range. Indeed, for each of the runs used for Figs. 17 and 18, the resulting retail price sequence over 20 successive simulated days was observed to converge to approximately 30 (\$/MWh).

The scatter plot depicted in Fig. 17 also indicates that the simulated household aggregate demand curve for the

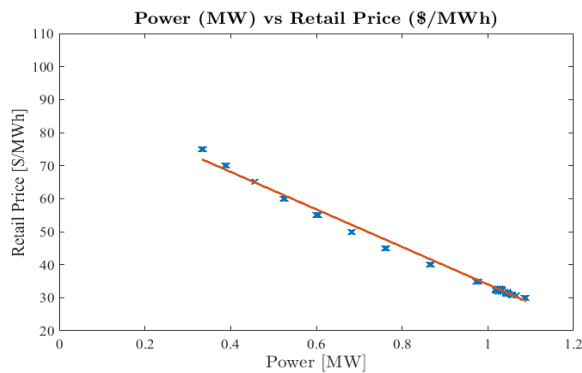


Fig. 18. Trend line drawn for the linear portion of the scatter plot in Fig. 17.

IRW Test Case becomes exponential as power approaches zero. Consequently, the linear approximation for the aggregate household demand function does not hold at the boundary where power goes to zero. This is not surprising, given that household power demands are for a critical purpose: namely, maintaining inside air temperature at a desired level. In such cases one would expect that households will have to be faced with very high retail prices in order to reduce their power usage to near-zero levels.

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