Analysis of Ultrasonic Scattering from Simply Shaped Objects

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Analysis of Ultrasonic Scattering from Simply Shaped Objects

Abstract
The simple shapes referred to in my title are actually restricted to spheres and spheroids, that is ellipsoids of revolution. Actually, I will not even say too much about the spheroids except to the extent that I will concentrate on the sphere as the limiting case and as the jumping-off point for the complete analysis of the spheroid.

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The simple shapes referred to in my title are actually restricted to spheres and spheroids, that is ellipsoids of revolution. Actually, I will not even say too much about the spheroids except to the extent that I will concentrate on the sphere as the limiting case and as the jumping-off point for the complete analysis of the spheroid.

To begin with, I would like to justify the interest in the solution of the elastic wave equation by showing the difference in the calculated scattering cross section for a cavity under two different assumptions. Consider a plane compressional wave incident on a spherical cavity. Figure 1 is a comparison of (1) the solution of the scalar wave equation which results from assuming that the scattering medium is a fluid which cannot sustain shear waves, and (2) the solution of the vector wave equation which allows mode conversion and the generation of shear waves when the incident compressional wave interacts with the cavity. Shown in Fig. 1 are the reduced cross sections, \( c/2\pi a^2 \) (\( a \) = radius of the cavity), for the energy scattered into unit solid angle. The scattering predicted by the scalar wave approximation shows little structure in comparison with the more structured pattern of the scattered compressional wave of the elastic approximation. There is also a surprising reduction in the overall magnitude of the scattering cross section. Furthermore, the mode converted shear wave is relatively strong and sharply lobed.

We begin the analysis therefore with the reasonably well-known equation of motion of elastic waves written in terms of the Lame constants, \( \lambda \) and \( \mu \), and the displacement, \( s \),

\[
(\lambda + \mu) \nabla \cdot \mathbf{s} + \mu \nabla^2 \mathbf{s} + \rho \omega^2 \mathbf{s} = 0 \tag{1}
\]

where we have assumed that the time dependence is harmonic so that Eq. (1) may be considered as the equation for the Fourier component of frequency \( \omega \). If we consider a purely longitudinal displacement, that is, one for which the curl vanishes, we may write the displacement in terms of the gradient of a scalar potential:

\[
\nabla \times \mathbf{s} = 0, \quad \mathbf{s} = -\nabla \psi : \quad \nabla \left( (\lambda + 2\mu) \nabla^2 \psi + \rho \omega^2 \psi \right) = 0 \tag{1.a}
\]

On the other hand, if we consider the transverse component of the displacement, one whose divergence vanishes, and which therefore may be expressed as the curl of a vector potential we obtain
Fig. 1. Scattering Patterns for a Spherical Cavity in an Elastic Medium.

The incident wave is a plane compressional wave with propagation vector $k$ coming from the left. Upper patterns, $ka = 0.1$, lower patterns, $ka = 1.0$. The small circles at the center of the patterns indicate the position of the cavity. The patterns marked "no shear, $\nu = 0"$ refer to solutions of the scalar wave equation (fluid approximation). The other two contours for each case are the solutions of the elastic equations corresponding to the ratio of Lame constants, $\mu/\lambda = 0.56$, (ratio of shear to compression wave velocities, 0.51 corresponding to titanium). All patterns are to the same scale.
\[ \nabla \cdot \mathbf{s}_t = 0, \quad \mathbf{s}_t = \nabla A : \quad \nabla \left( -\mu \nabla \times \mathbf{A} + \rho \omega^2 \mathbf{A} \right) = 0 \quad (1.b) \]

It is then possible to express the vector potential in terms of two scalar functions,

\[ A = F\Phi - F\times \Psi \quad (2) \]

and each of these functions satisfies a scalar wave equation:

\[ \mu \nabla^2 \Phi + \rho \omega^2 \Phi = 0; \quad \mu \nabla^2 \Psi + \rho \omega^2 \Psi = 0 \quad (2.a) \]

We shall find in spherical geometry that \( \psi \), the scalar potential for the compressional wave, and the component \( \Phi \) of the transverse wave are coupled, whereas the component \( \psi \) is decoupled.

Breaking up the displacement into its spherical components we obtain

\[ r_s = -r \frac{\partial \Phi}{\partial r} - \Omega \psi \quad (3.a) \]

\[ s_\theta = \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \theta} + \frac{\partial U}{\partial \theta} \quad (3.b) \]

\[ s_\phi = -\frac{\sin \theta}{\sin \phi} \frac{\partial U}{\partial \phi} \quad (3.c) \]

where \( \Omega = \frac{1}{\sin \theta} \frac{\partial^2}{\partial \theta^2} \sin \theta + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \) is the angular part of the Laplacian (or the two-dimensional Laplacian on a unit sphere) and

\[ U = \frac{1}{r} \frac{\partial}{\partial r} \left( r \mathbf{v} \right) - \psi \quad (4) \]

If \( s_\theta \) and \( s_\phi \) are continuous across a spherical boundary then also \( \Phi \) and \( U \) must be continuous.

In order to complete the boundary conditions required to specify the scattering of elastic waves we must also consider the stresses generated in the scattering object and in the external medium. The stresses across the bounding surface of the spherical scatterer must be continuous; these stresses are \( \sigma_{rr} \), \( \sigma_{r\theta} \), and \( \sigma_{r\phi} \). We may write these stress components in terms of the potentials in the form:

\[ \sigma_{rr} = \rho \omega^2 \psi + \frac{2}{r} \frac{\partial \psi}{\partial r} - 2r \frac{\partial \Phi}{\partial r} + \Omega \psi - r \frac{\partial^2 \psi}{\partial r^2} + \psi \quad (5.a) \]

\[ r\sigma_{r\theta} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \phi} \quad (5.b) \]

\[ r\sigma_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} - \frac{\partial \psi}{\partial \phi} \quad (5.c) \]
where

\[ V = -\rho \omega^2 r^2 \psi - 2\mu r^2 \phi + (\nu+1)\psi r \frac{\partial \psi}{\partial r} - \psi \]  

(5.d)

and

\[ \dot{W} = \mu r^2 \frac{\partial}{\partial r} \left( \frac{\phi}{r} \right) \]  

(5.e)

The continuity of the displacement and stresses across the boundary of the scattering sphere is thus reduced to the continuity of \( s_r, U \) and \( \Phi \), and \( \sigma_{rr} \).

For the case of the scattering of an incident plane longitudinal wave we specify the displacement of the incident wave to be

\[ \tilde{\psi} = \hat{u}_z e^{i(\omega t-kz)} ; \quad \psi = -\frac{i}{k} e^{i(\omega t-kz)} \]  

(6)

where \( \hat{u}_z \) is a unit vector in the z-direction, and \( K \) is the propagation constant for compressional waves, \( K = \omega \sqrt{\mu/(\lambda + 2\mu)} \).

For the transverse wave we choose the x-direction to lie in the plane of polarization of the incident shear waves which are propagating in the z-direction

\[ \tilde{\psi} = \hat{u}_x e^{i(\omega t-kz)} = \nabla x A^i = \nabla x (\vec{r} \Phi^i - \vec{r} x \nabla \psi) \]  

(7)

where \( K \) is the propagation constant for shear waves, \( K = \omega \sqrt{\rho/\mu} \). When we integrate Eq. (7) however, we have a complication which was not present in the longitudinal case, since \( A^i \) is not yet uniquely specified but contains an as yet undefined gradient of a scalar function. Therefore, we find

\[ A^i = -\frac{i}{K} \hat{u}_y e^{-ikz+i\omega t} + \nu G = -\vec{r} \Phi^i - \vec{r} x \nabla \psi \]  

(8)

In order to determine the potentials for the incident waves we make use of the expansion of a plane wave in terms of a superposition of spherical waves:

\[ e^{-ikz} = \sum (2n+1)(-i)^n j_n(kr) P_n(cos \theta) ; \quad z = rcos \theta. \]  

(9)

Inserting this expression into Eq. (6) gives us immediately the form of \( \Phi^i \) for the incident compressional wave, and we can write the total potential by adding additional terms which represent the scattered waves. These must have the same angular behavior as the incident wave, \( P_n(cos \theta) \), and in particular, by symmetry, have no \( \phi \)-dependence. Furthermore in this case \( \Phi \) may also be dropped since there must be no \( \phi \)-component of the displacement. The scattered components of \( \psi \) and \( \Psi \) in the external medium are therefore constructed from products of \( P_n(cos \theta) \) and spherical Bessel function of the third kind and order \( n \), since these represent outgoing waves. Inside the scattering medium the waves excited by the incident wave must be finite at the origin and hence are constructed from spherical Bessel functions of the
first kind. These potentials are inserted into Eqs. (3a,b), (4), and (5a,b,d) to yield expressions for the displacements and stresses at the boundary \( r = a \). The continuity of these quantities across the boundary determines the expansion coefficients and defines the solution.

For the case of an incident shear wave the introduction of Eq. (9) into Eq. (8) leads to equations for the three partial deviations of the unknown function \( G \). By taking appropriate cross-derivatives of these expressions and equating \( \partial^2 G/\partial r \partial \theta = \partial^2 G/\partial \theta \partial r \), etc., we ultimately eliminate \( G \) and obtain expressions for the form of the incident potentials \( \Phi \) and \( \Psi \).

\[
\phi^i = \sin \phi \sum \frac{2n+1}{n(n+1)} (-i)^n j_n(Kr)p_n^\dagger(\cos \theta) \tag{10.a}
\]

and

\[
\psi^i = -\frac{\cos \phi}{K} \sum \frac{2n+1}{n(n+1)} (-i)^{n+1} j_n(Kr)p_{n+1}^\dagger(\cos \theta) \tag{10.b}
\]

Again the scattered wave must also contain the corresponding angular dependences of the incident wave with the appropriate spherical Bessel functions for the radial dependence. The amplitudes of these partial waves are again determined by imposing the conditions of continuity of displacement, Eq. (3), and stress, Eq. (5), across the boundary. In the far field the components of displacement may then be determined to be

\[
rs_r = i \cos \phi e^{-i Kr} \sum \frac{2n+1}{n(n+1)} E_n p_n^\dagger(\cos \theta) \tag{11.a}
\]

\[
rs_\theta = \frac{i \cos \phi}{K} e^{-i Kr} \sum \frac{2n+1}{n(n+1)} \left[ F_n \frac{d p_n^\dagger}{d \theta} + K_n p_n^\dagger \right], \quad r >> a \tag{11.b}
\]

\[
rs_\phi = -\frac{i \sin \phi}{K} e^{-i Kr} \sum \frac{2n+1}{n(n+1)} \left[ F_n \frac{p_n^\dagger}{\sin \theta} + K_n \frac{d p_n^\dagger}{d \theta} \right], \tag{11.c}
\]

the constants \( E_n, F_n, \) and \( K_n \) having been determined by imposing the boundary conditions. The radial displacement is obviously the compressional wave with propagation constant \( k \); the transverse displacements are the shear wave with propagation constant \( K \). The scattered shear wave is in general elliptically polarized.

From a consideration of the energy flow (the elastic Poynting vector) we may define the scattering cross section. The energy scattered per unit solid angle with unit incident energy flow per unit area is the scattering function; this is given by
\[ S(\theta, \phi) = a^2 \frac{K}{K^2} \cos^2 \phi \left| \sum \frac{2n+1}{n(n+1)} \frac{P_n}{\sin \theta} \right|^2 + \frac{\cos^2 \phi}{K^2} \left| \sum \frac{2n+1}{n(n+1)} \left( \frac{F_n dP_n}{n \sin \theta} + \frac{K_n dP_n}{n \sin \theta} \right) \right|^2 + \frac{\sin^2 \phi}{K^2} \left| \sum \frac{2n+1}{n(n+1)} \left( \frac{F_n dP_n}{n \sin \theta} + \frac{K_n dP_n}{n \sin \theta} \right) \right|^2 \] (12)

The total scattering cross section is the integral of \( S(\theta, \phi) \) over all directions:

\[ a = \int \int S(\theta, \phi) \sin \theta d\theta = 2\pi a^2 [g_L + g_T], \] (13)

where \( g_L \) is the reduced scattering cross section for longitudinal (mode converted) waves

\[ g_L = \frac{K}{K} \sum \frac{2n+1}{n(n+1)} E^*E, \] (13.a)

and \( g_T \) is that for transverse waves

\[ g_T = \frac{1}{(K a)^2} \sum (2n+1)[F_n F_n + K_n K_n]. \] (13.b)

If we consider a spheroidal obstacle and hence use spheroidal coordinates, the expressions are more complex. We shall discuss here only the prolate (cigar-shaped) coordinates for which the transformations to cartesian coordinates are

\[ x = b \sinh n \sin \theta \cos \phi = (r^2-b^2)^{1/2} \sin \theta \cos \phi \]
\[ y = b \sinh n \sin \theta \sin \phi = (r^2-b^2)^{1/2} \sin \theta \sin \phi \]
\[ z = b \cosh n \cos \theta = r \cos \theta \] (14)

for which the coordinate surfaces \( n = \) constant are ellipses of revolution rotated about the major axis and \( b \) is one half of the interfocal distance of the ellipse. For \( n \to \infty \) with \( b \sinh n \to b \cosh n = r \) the ellipsoids become spheres. Thus in the limit as the semi-interfocal distance \( b \) goes to zero the spheroidal coordinates become spherical coordinates. The scalar wave equation in spheroidal coordinates is
\[
\frac{1}{\sinh \eta} \frac{\partial}{\partial \eta} \left( \sinh \eta \frac{\partial \psi}{\partial \eta} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left( \sin \theta \frac{\partial \psi}{\partial \phi} \right) + \left( \frac{1}{\sinh^2 \eta} + \frac{1}{\sin^2 \theta} \right) \frac{\partial^2 \psi}{\partial \phi^2} \\
+ k^2 b^2 (\cosh^2 \eta - \cos^2 \theta) \psi = 0
\]

(15)

If we generalize the operator \( \mathcal{L}(\eta, \phi) \), which we previously defined, by adding an additional term and expressed it in terms of \( \mu = \cos^{-1} \theta \):

\[
\Omega(\mu, \phi; \rho) = \frac{3}{\mu} (1 - \mu^2) - \frac{\partial^2}{\partial \rho^2} - \mu^2
\]

(16)

we may write the Helmholtz equation in the form

\[
[\mathcal{L}(\cosh^{-1} \eta, \phi; k_b) - \mathcal{L}(\cos^{-1} \eta, \phi; \rho)] \psi = 0.
\]

(17)

Thus the solutions of both the "radial" and "angular" parts of the wave equation (since both components actually contain the azimuthal dependence, the term "radial" is more figurative than accurate) are constructed from solutions of what we can call a generalized Legendre function. Incidentally, this also demonstrates a connection between spherical Bessel functions and Legendre functions as limiting cases of the same general set of functions.

We have not yet explored this solution of the spheroidal geometry problem further, but we see an obvious generalization emerging and, in the limit of \( k_b \ll 1 \) an indication of what will be, hopefully, a basis for the expansion of the solution as a series in \( k_b \).
DISCUSSION

DR. BRUCE THOMPSON (Rockwell International Science Center): I guess we have time for one or two questions.

DR. YIH PAO (Cornell University): I'd like to make two remarks. One is about the solution for spheres. But I'm from Cornell and we have learned that any solution seems to be reproduced every few years or so. This exact thing happened for scattering by spheres. The first solution was produced by Hermite in 1868. Lamb reproduced the same solution around 1910 and Nishimura in Japan did the whole thing in 1919. The solution for a sphere (scattering by an elastic sphere in a solid matrix) was documented by Morse and Feshbach which appeared in 1963 and also was documented in the report by Pao and Mow of the Rand Corporation in 1972. Now, the difficulty here is not formulating a mathematical equation, the difficulty is to calculate from those functions, as Dr. Cohen has shown, the numerical results you want, which, indeed, is very difficult. Now, my next question is about your remark about spheroidal inclusion. Now, spheroidal inclusion also has been solved. The cavity was solved by Boninski around 1961. While he produced a scattered field by a spheroidal cavity in elastic solid, the scattering by a spheroidal of a rigid solid was solved by me and my students around 1965. There I don't think that I used generalized Legendre functions or generalized Bessel functions to solve your equation. You must use spheroidal wave functions. There we encountered a tremendous numerical difficulty in programming these functions. We dropped the whole thing because I came up with a dead end as far as a few analytical numerical solutions are concerned.

DR. COHEN: Thank you. In terms of the spheroidal solutions, I was using "generalized Legendre" as synonymous with the solutions of the spheroidal wave equation. So, I agree with you as to what functions are needed to be used. I was not aware that the ellipsoidal problem had been solved as completely as you indicated. I was aware, of course, of the solutions of the spherical scattering. What I've done here is a somewhat, I won't say independent derivation, but an independent re-development as the basis for the generalization to the spheroidal case, and also to independently check the results of Einspruch, Witterholt and Truell, in which there seem to be some errors. I think Jim Krumhansl said he would make some comments on that. I have not made detailed numerical comparisons between my results and the results that already exist in the literature, but they seem to be in agreement. On the other hand, I've got to be careful, because as Jim points out there are some obvious errors in the existing solutions. Jim, do you want to--

PROF. JIM KRUMHANSL (Cornell University): No, I'll cover it in my talk.

DR. COHEN: Okay
DR. BRUCE THOMPSON: I guess we have time for a short question.

DR. SY FRIEDMAN (NRDC): It wasn't quite clear to me that you have assumed that the form of the solution of the wave equation developed was that of an incident plane wave plus a scattered wave.

DR. COHEN: Yes.

DR. FRIEDMAN: Is that what you had in practice assumed?

DR. COHEN: Yes.

DR. FRIEDMAN: Secondly, did you assume a mixture of longitudinal and transverse waves incident on it, or did you take these separately?

DR. COHEN: You take these separately for the incident wave, because they do separate. On the other hand, there is mode conversion.

DR. FRIEDMAN: Okay. I see, yes. Thirdly, of course, just the wave equation was for a forceless medium?

DR. COHEN: Yes.

DR. FRIEDMAN: Okay. Thank you.