ANALYSIS AND DESIGN OF EDDY-CURRENT MEASUREMENT SYSTEMS

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INTRODUCTION

Calculations of the change in probe impedance produced by a flaw in or beneath the surface of a conducting material are necessary for the analysis of eddy-current flaw-detection systems. However, probe sensitivity alone does not completely determine the flaw-detection capability of such a system; the effect of noise and clutter in the system must also be considered. The objective of this work is to develop a statistical detection model for an eddy-current system, and to use this model to calculate probability of detection as a function of probability of false rejection (false alarm) for the purpose of determining the optimality of a particular system. The target flaw chosen for these calculations is a 0.015-in.-long, 0.007-in.-deep surface crack in a bolt hold in a turbine disk made of material such as IN-100, which has a conductivity of about $10^6$ mho/m.

The methodology involved in developing such a statistical model is illustrated here for a specific type of probe, lift-off motion, and noise distribution. Calculations made using this specific model illustrate certain general features of eddy-current flaw detection. In particular, they suggest the definition of two figures of merit that should prove useful for comparing eddy-current probes.

BACKGROUND

A typical eddy-current system is shown in Fig. 1. The probe impedance changes as it is passed over a flaw. This impedance change produces a change in the output voltage of the bridge, which is
subsequently amplified, filtered, and detected in a phase-sensitive detector. The reference phase for the detector is adjusted so that the change in the bridge's output voltage produced by any initial motion of the probe away from the work piece (lift-off) appears primarily in the in-phase (I) channel (lift-off motion includes both translation and rotation (tilt)). Lift-off discrimination is achieved in this system by observing the flaw signal in the quadrature (Q) channel and taking advantage of the fact that the phase of the flaw signal differs from that of the lift-off signal for most flaws.

The change in the output voltage of the system produced by a change in probe impedance, \( \Delta Z \), can be written as

\[
V_I = \frac{1}{2} V_{\text{inc}} F_M K \left| \frac{\Delta Z}{Z_{TP}} \right| \cos (\phi - \theta_R) \quad 1(\text{a})
\]

and

\[
V_Q = \frac{1}{2} V_{\text{inc}} F_M K \left| \frac{\Delta Z}{Z_{TP}} \right| \sin (\phi - \theta_R) \quad 1(\text{b})
\]

where \( V_{\text{inc}} \) is proportional to the generator voltage, \( F_M \) is the transfer function of the bridge, \( K \) is the transfer function of the amplifier and filter, \( Z_{TP} \) is the probe impedance at its rest position, \( \phi \) is the phase angle of \( V_{\text{inc}} F_M K (\Delta Z/Z_{TP}) \), and \( \theta_R \) is a reference phase angle. Although we have been referring specifically to an eddy-current system that uses an absolute probe and a bridge, the equations for a system that uses a differential probe or another method of measuring the change in probe voltage have the same form as Eqs. 1(a) and 1(b).
In the development that follows, it will be convenient to use a normalized voltage, $v_Q$, where

$$v_Q = \left| \frac{\Delta Z}{Z_{TP}} \right| \sin(\phi - \theta_R) \quad .$$  

(2)

Note that, in general, the normalizing factor is a function of frequency.

Considerable progress has been made in analyzing the interaction between an eddy-current probe and a flaw in a metallic specimen.\textsuperscript{2,3} For example, Kincaid\textsuperscript{2} has derived the following expression for the change in probe impedance produced by a surface-breaking elliptical crack:

$$\Delta Z_F = \left( \frac{H_0}{I} \right)^2 \cdot \sigma \delta Z_s^2 \cdot \frac{4ac}{3} \cdot \left\{ \left( \frac{a}{\delta} \right) - \frac{5}{8} \left( \frac{a}{\delta} \right)^2 - j \left[ \frac{5}{8} \left( \frac{a}{\delta} \right)^2 - \frac{4}{15} \left( \frac{a}{\delta} \right)^3 \right] \right\} \quad ,$$  

(3)

where $H_0$ is the applied magnetic field (assumed to be uniform over the crack), $I$ is the current in the probe, $a$ is the crack depth, $2c$ is the crack length, $\sigma$ is the conductivity of the metal, $\delta$ is the skin depth, and $Z_s$ is the ratio of tangential electric field to tangential magnetic field at the work-piece surface in the uniform-field region. This relation was derived assuming $a/\delta < 1$. On the other hand, if $a/\delta \gg 1$, Auld et al\textsuperscript{3} have shown that the change in impedance produced by a uniformly illuminated circular slot of depth $a$, length $2c$, and width $\Delta u$ is

$$\Delta Z_F = \left( \frac{H_0}{I} \right)^2 \cdot (1-j)Z_s \cdot 2c^2 \cdot \left\{ \left[ \frac{a}{c} - \frac{1}{n} \cdot \frac{\Delta u}{c} \right] + j \left[ \frac{a}{c} \left( 1 + L \cdot \frac{\Delta u}{c} \right) - \frac{1}{n} \cdot \frac{\Delta u}{c} \right] \right\} \quad ,$$  

(4)

where

$$n = 2 + \frac{4}{\pi} \tan^{-1} \left( \frac{a}{c} \right)$$  

(5)

$$\frac{a}{c} = \int_{-1}^{1} F(x,n) \, dx$$  

(6)
\[ F(x,n) = \frac{(1/n) \sin(2\pi/n)}{\cosh \left[ \frac{(2/n) \ln \left( \frac{1 + x}{1 - x} \right)}{1 - x} \right] - \cos(2\pi/n)} \]  
\[ L = \sqrt{2}(c/\delta) \]  

It can be seen from Eqs. 3 and 4 that a knowledge of the excitation fields produced by a probe is needed to compute \( \Delta Z_F \) because of the presence of \( H_0/I \) and \( Z_s \) in the expressions for \( \Delta Z_F \). In addition, an analytic model for the probe is needed to compute lift-off effects. At present, an analytic probe model is available only for an air-core coil whose axis is perpendicular to the surface of a planar work piece. Hence, we will use this model to illustrate the methodology involved in computing the probability of detection for an eddy-current system.

The geometry of the air-core coil we wish to consider is shown in Fig. 2. All the dimensions shown in the figure have been normalized to the average radius of the coil, \( \bar{r} \). The relevant analytic expressions that describe a coil with \( N \) turns are as follows:

1. **Coil impedance at normalized height, \( h \):**

\[ Z_{TP} = j \frac{\pi \bar{r}N^2}{(\bar{r} - \Delta r)^2} \cdot \frac{2}{\sigma \delta} \cdot I_{TP}(h) \]  

where

\[ I_{TP}(h) = \int_0^\infty \frac{1^2(\alpha \cdot \Delta r)}{\alpha^5} \left[ 2t + \frac{1}{\alpha} \left( 2(e^{-\alpha t} - 1) + e^{-2\alpha h(e^{-\alpha t} - 1)} \right) \right] \, d\alpha \]  

\[ I(\alpha \cdot \Delta r) = \int_{\alpha(1-\Delta r/2)}^{\alpha(1+\Delta r/2)} uJ_1(u) \, du \]  

\[ J_1 = \text{a Bessel function of the first kind and first order} \]  
\[ \alpha_1 = \sqrt{\frac{2}{\alpha} + j2(\bar{r}/\delta)^2} \]
(2) Radial magnetic field, $H_0$, at radial position $r$ on the conducting surface:

$$
\frac{H_0}{r} = \frac{-N}{r(t \cdot \Delta r)} I_F(h)
$$

where

$$
I_F(h) = \int_0^\infty \frac{I(\alpha \cdot \Delta r)}{\alpha^2} J_1(\alpha) e^{-\alpha h} \left(1 - e^{-\alpha t}\right) \left(\frac{\alpha l}{\alpha + \alpha_1}\right) d\alpha
$$

$I = $ the driving current in the coil.

(3) Surface impedance, $Z_s$, at radial position $r$ on the conducting surface:

$$
Z_s = j2\pi R_s \left(\frac{r}{\delta}\right) \frac{I_E(h)}{I_F(h)}
$$

where

$$
I_E(h) = \int_0^\infty \frac{I(\alpha \cdot \Delta r)}{\alpha^2} J_1(\alpha) e^{-\alpha h} \left(1 - e^{-\alpha t}\right) \frac{d\alpha}{\alpha + \alpha_1}
$$

$$
R_s = \frac{1}{\sigma \delta}
$$
It is interesting to consider how the surface impedance at \( \vec{r} \)
directly beneath the coil windings varies as a function of skin depth
(frequency). Since the field in this region is fairly uniform, one
might expect the surface impedance to be nearly equal to the surface
impedance for a plane wave, i.e.,

\[
Z_s(\text{plane wave}) = R_s(1 + j) \quad . \tag{18}
\]

However, the curves in Fig. 3 (calculated for \( \Delta r = t \) and \( h = 0.1 \))
show that this is not true unless \( \vec{r}/\delta \) is large. For small coils or
large skin depths the surface impedance produced by a coil is largely
inductive. This behavior of the surface impedance plays an important
role in determining the lift-off discrimination that can be achieved
with a coil probe.

Besides the change in probe impedance produced by a flaw, another
important impedance change is that produced by lift-off. There are
two components to this effect: one is the change in probe impedance,
\( Z_{TP} \), and the other is the change in flaw impedance, \( \Delta Z_F \). These
changes are given by

\[
\Delta Z_{LO} = Z_{TP}(h) - Z_{TP}(h_0) \quad \tag{19}
\]

and

\[
\Delta Z_{LF} = \Delta Z_F(h) - \Delta Z_F(h_0) \quad , \tag{20}
\]

where \( h_0 \) is the reference height (rest position) of the probe. To
use these equations we must assume, of course, that only lift-off
motion perpendicular to the work piece is allowed.

The normalized voltages in the Q channel that correspond to these
various impedance changes can all be written in the form given by
Eq. 2:

\[
v_F = \left| \frac{\Delta Z_F}{Z_{TP}} \right| \sin (\phi_F - \phi_{LO}) \quad , \tag{21}
\]

\[
v_{LO} = \left| \frac{\Delta Z_{LO}}{Z_{TP}} \right| \sin (\phi_{LO} - \phi_{LO}) \quad , \tag{22}
\]

\[
v_{LF} = \left| \frac{\Delta Z_{LF}}{Z_{TP}} \right| \sin (\phi_{LF} - \phi_{LO}) \quad . \tag{23}
\]
Fig. 3. Variation with Skin Depth of the Skin-Effect Impedance for a Coil

Here, $\phi_F$, $\phi_{LO}$, and $\phi_{LF}$ are the phase angles of each corresponding $\Delta Z$. The phase angle, $\phi_{LO}^i$, is the phase angle of $\Delta Z_{LO}$ for very small changes in $h$. Hence, it is given by

$$\phi_{LO}^i = \tan^{-1} \left[ \frac{\text{Im} \ I_{TP}'}{\text{Re} \ I_{TP}'} \right] + \pi/2 \quad ,$$  \hspace{1cm} (24)

where

$$I_{TP}' = \left. \frac{dI_{TP}}{dh} \right|_{h_0} \quad .$$  \hspace{1cm} (25)
This reference phase is compared with $\phi_F$ in Fig. 4. It is interesting to note that these phases are equal at a low value of $a/\delta$. Hence, there is no lift-off discrimination at this point.

STATISTICAL DETECTION MODEL

The eddy-current crack detection problem in its simplest form is a classical binary decision problem, namely, whether or not a crack is present. The probability that a crack is present is determined by the individual probability density functions for these two situations, such as those shown schematically in Fig. 5. The density functions are functions of the measured random variable, $z$, which is equal to $v_Q$ for the eddy-current system. By establishing a threshold, $\gamma$, one can decide whether a crack is present by observing whether $|v_Q|$ exceeds $|\gamma|$. There are several strategies for selecting $\gamma$, which can be either fixed or variable. In the following discussion we will assume that $\gamma$ is fixed at a value determined by a given probability of false rejection, $P_{FR}$. The probability of detection (POD) is equal to 1 minus the probability of false acceptance, $P_{FA}$. The probabilities $P_{FR}$ and $P_{FA}$ are given by the shaded areas shown in Fig. 5. It is clear that for the best performance the probability density functions should overlap as little as possible.

In an eddy-current measurement, $v_Q$ is composed of a flaw signal contaminated by signals produced by lift-off (including the effect

![Fig. 4. Comparison Between Phase of Flaw Signal and Reference Phase](image-url)
of surface roughness), material variations, external pickup, and electronic noise. To develop a statistical detection model, one needs a priori statistical information about these contaminating signals, as well as about the distribution of flaw size, orientation, and location beneath the surface. Much of this information either is not available or varies considerably with the type of application. Thus, in order to proceed, we will make the following simplifying assumptions:

- The surface is smooth.
- There are no material variations.
- There is no external pickup.
- The electronic noise in the Q channel is normally distributed (Gaussian).
- The flaw is a single half-penny surface crack aligned along the radius of the coil.
- Lift-off motion occurs along a direction perpendicular to the surface from \( h_0 = 0 \) to \( h = h_{\text{max}} \).
- The radius of curvature of the bolt hole is large compared to the probe size so that the planar theory applies.

Although these assumptions eliminate some important practical features of eddy-current testing, the model should still provide insight into the relative importance of various probe parameters in crack detection.

Under these assumptions, the output voltage is composed of at most four terms:
where \( v_N \) is the voltage due to Gaussian noise, and \( v_F \) is the flaw voltage produced when the coil height is \( h_0 \). For convenience, we will call

\[ v_C = v_{LO} + v_{LF} \]  \hspace{1cm} (27)

the clutter voltage.

The statistics of \( v_C \) are determined by the statistics of the lift-off motion that produces signals within the output bandwidth of the system. These latter statistics are not known. Hence, in the calculations that follow, we will assume that \( h \) is distributed uniformly over the range \([0, h_{max}]\). The corresponding probability density function for \( h \) is

\[ p_h(h) = \frac{1}{h_{max}} \quad 0 \leq h \leq h_{max} \]  \hspace{1cm} (28)

One can make a plausability argument for this choice as follows. We imagine that the measurement statistics are determined by scanning past a single crack a large number of times. If we assume that minimum and maximum lift-off occur at least once during each scan at completely random positions along the scan, then the lift-off observed each time the probe is over a crack will tend toward a uniform distribution as the number of scans increases without limit.

We note from Eq. 27 that, in the absence of a flaw, the clutter voltage becomes equal to the lift-off voltage, \( v_{LO} \). An example of how the presence of a crack affects the clutter voltage is shown in Fig. 6. The magnitude of the clutter voltage is actually less than the magnitude of the lift-off voltage because the decrease in flaw signal is compensated for by an increase in lift-off signal as \( h \) increases. Also, as might be expected, the lift-off voltage is closely approximated by a quadratic function of \( h \) for small \( h \).

Based on the measurement model described by Eq. 26, the general formulas for computing the desired probability density functions are found to be

\[ p(v_Q|nc) = \int p_h(h)p_N\left[v_Q - v_{LO}(h)\right]dh \]  \hspace{1cm} (29)

and

\[ p(v_Q|c) = \iint p_F(v_F)p_h(h)p_N\left[v_Q - v_F - v_C(h)\right]dhdv_F \]  \hspace{1cm} (30)
If we now take
\[ p_N(v_N) = \frac{1}{\sqrt{2\pi} \sigma_N} e^{-v_N^2 / 2\sigma_N^2}, \] (31)

where \( \sigma_N^2 \) is the average noise power at the output of the Q channel, and assume there is only one surface crack with a predetermined orientation, we find that Eqs. 29 and 30 become
\[
p(v_Q|nc) = \frac{1}{\sqrt{2\pi} \sigma_N h_{\text{max}}} \int_{0}^{h_{\text{max}}} e^{-\left(v_Q - v_{LO}\right)^2 / 2\sigma_N^2} dh, \] (32)

and
\[
p(v_Q|c) = \frac{1}{\sqrt{2\pi} \sigma_N h_{\text{max}}} \int_{0}^{h_{\text{max}}} e^{-\left(v_Q - v_{F0} - v_{C}\right)^2 / 2\sigma_N^2} dh, \] (33)
respectively. Here, $v_{F0}$ is the flaw voltage for the given crack. Since $v_Q$ is mostly negative in our example, we have that

$$P_{FR} = \int_{-\infty}^{\gamma} p(v_Q|nc) \, dv_Q .$$

(34)

Substituting Eq. 32 into Eq. 34 and interchanging the order of integration gives the result

$$P_{FR} = \frac{1}{2} + \frac{1}{2h_{max}} \int_{0}^{h_{max}} \text{erf} \left( \frac{\gamma - v_{LO}}{\sqrt{2} \sigma_N} \right) \, dh ,$$

(35)

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^2} \, du .$$

(36)

Similarly,

$$P_{FA} = \frac{1}{2} - \frac{1}{2h_{max}} \int_{0}^{h_{max}} \text{erf} \left( \frac{\gamma - v_{F0} - v_C}{\sqrt{2} \sigma_N} \right) \, dh .$$

(37)

Before proceeding to a discussion of POD calculations, let us consider in more detail the probability density functions given by Eqs. 32 and 33. Examples of these functions are shown plotted in Fig. 7 for $a/\delta = 0.7, 1.0, \text{and } 8.0$. The values $a/\delta = 0.7 \text{ and } 8.0$ give the same value of flaw signal, assuming the coil size is fixed at $2F = 4a$ and $\Delta r = \Delta t = 0.535$. It was also assumed in these calculations that the crack length $2c = 2.14a$. The calculations for $a/\delta = 1.0$ were made using Eq. 3 for $\Delta Z_F$.

The effect of electronic noise on the density functions is also illustrated in Fig. 7. When electronic noise is small (which is often the case in practice), the density functions exhibit a sharp peak that corresponds to the reference lift-off position $h_0$ in the case of no crack. This peak results from the fact that the initial effects of lift-off have been filtered out of the Q channel. The widths of the density functions correspond to the maximum lift-off excursion, and the density functions for the case when a crack is present are essentially the no-crack functions shifted by an amount equal to the flaw signal. The widths of the density functions for
the crack and no-crack cases are slightly different because of the difference between $v_C$ and $v_{LO}$. 

Fig. 7. Probability Density Functions
When electric noise becomes significant, the density functions become more Gaussian-like and the sharp peaks are lost. As discussed earlier, the most important characteristic of these density functions is the degree of overlap between the crack and no-crack cases. This overlap is determined by the separation between the peaks in these functions and the width of the no-crack density function. The usual way of describing this relation between the density functions is by a parameter called signal-to-noise ratio (SNR), which we define by

\[
\text{SNR} = \frac{2\nu_{F0}^2}{\sigma_N^2 + \sigma_{LO}^2}
\]

(38)

where

\[
\sigma_{LO}^2 = \int_{-\infty}^{\infty} p_{LO}(\nu_{LO}) (\nu_{LO} - \bar{\nu}_{LO})^2 \, d\nu_{LO}
\]

and \(\bar{\nu}_{LO}\) is the mean value of \(\nu_{LO}\). Some examples of SNR as a function of \(a/\delta\) are shown in Fig. 8. We see that SNR maximizes in the neighborhood of \(a/\delta = 1.0\), and that a smaller coil generally produces a larger SNR than does a larger coil. The discontinuities in the curves at \(a/\delta = 1.0\) are a result of using the \(\Delta Z_F\) formulas (Eqs. 3 and 4) beyond their range of validity and should not be considered meaningful.

When electronic noise is small the SNR remains large for \(a/\delta > 1.0\), and thus it would appear that operation at large \(a/\delta\) is acceptable. However, we shall see that this is not necessarily true because a relatively small decrease in SNR (in decibels) in this range results in a relatively large decrease in POD.

Despite this sensitivity of POD to small changes in SNR, it is still meaningful to use SNR as a basis for defining a figure of merit for an eddy-current probe. However, to be really useful, such a figure of merit should not depend on a knowledge of the statistical properties of the system. Hence, as suggested by Kincaid,\(^5\) we propose the following two deterministic figures of merit for an eddy-current probe:
Fig. 8. SNR as a Function of $a/\delta$

(1) electronic noise dominant:

$$S = \frac{|v_{F0}|}{\sigma_N}$$

(40)

(2) lift-off dominant:

$$D = \frac{|v_{F0}|}{|v_{LO(h_{ref})}|}$$

(41)

In Eq. 40, $\sigma_N$ can be taken to be an arbitrary normalizing constant; in Eq. 41, $v_{LO(h_{ref})}$ is the lift-off voltage measured in the Q channel when the probe is at a specific value of lift-off, $h_{ref}$.

These figures of merit are illustrated in Fig. 9 as functions of $a/\delta$. The quantity $\sigma_N$ was taken to be $10^{-3}$, and $h_{ref}$ was 0.1 in the case where $\bar{r} = 2a$ and 0.05 when $\bar{r} = 4a$. This choice for $h_{ref}$ made the absolute value of lift-off the same for both probe sizes. We see that $S$ and $D$ as functions of $a/\delta$ have the same general shape as the SNR curves in Fig. 8, and thus should provide a meaningful basis for selecting a probe.

Using Eqs. 35 and 37, POD and $P_{PR}$ were computed as functions of $a/\delta$ and threshold voltage, $\gamma$. Figure 10 shows these quantities plotted as functions of $a/\delta$, with $\gamma$ as a parameter. When $\gamma$ is a small
negative number the POD is essentially unity over a wide range of \(a/\delta\), but this result is obtained at the expense of \(P_{FR}\) being large over this range. Making \(\gamma\) a larger negative number reduces the range over which \(POD \approx 1\), but also decreases \(P_{FR}\). Note that the POD decreases at large \(a/\delta\) even though the SNR remains large.

Perhaps of more interest is the value of POD for a given value of \(P_{FR}\). Curves showing POD as a function of \(P_{FR}\) with \(a/\delta\) as parameter are called system operating characteristics; these are shown in Fig. 11. The most desirable operating characteristics are portrayed by those curves located closest to the upper-left-hand corner of the graph. These curves are meant only to be illustrative; because of the simplifications made in the model, they do not represent any particular real eddy-current system.

**SUMMARY**

The work described here illustrates the methodology involved in computing the statistical detection characteristics of an eddy-current system. In general, this computation requires a knowledge of the following quantities:
Fig. 10. Probability of Detection and of False Rejection as Functions of $a/\delta$
Fig. 11. Eddy-Current System Operating Characteristics
• The flaw distribution in size, orientation, and location relative to the surface of the work piece.
• The flaw voltage as a function of flaw size, orientation, and location.
• The probability distribution for probe motion (lift-off).
• The clutter voltage as a function of probe motion.
• The probability distributions for surface roughness, material variations, and electronic noise.

The statistical calculations reported here also have suggested the definition of two deterministic figures of merit, S and D, for an eddy-current probe that should be useful in comparing probes. These figures of merit are measures of the SNR associated with a given probe and flaw. The results also indicate that best performance is achieved when $a/\delta = 1$.

The statistical detection model presented thus far is fairly rudimentary. A number of improvements are being considered for incorporation into the model. These include the following effects:

• Flaw distributions
• Surface roughness
• Material variations
• Different probe motions
• Filtering
• Multi-frequency measurements.

In addition, the possibility of using a variable threshold and multiple measurements, and the effect of the measurement statistics on inversion, are of interest.

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DISCUSSION

F. Muennemann (Stanford University): One of your givens for calculating the probability of detection was the statistics of probe motion. It seems that probe motion will reflect motion of the entire assembly including its "base". Have you considered putting that in by adding to your statistics some semideterministic motion for the probe?

A.J. Bahr (SRI International): Adding in the deterministic motion?

F. Muennemann: Well, if you were to plot the probe motion as a function of time, it would presumably start at one point.

A.J. Bahr: No. I assume that anything like that is filtered out. A periodic lift-off, for example, which you might have in a rotating probe, would have been filtered out. There are many reasons that you might want to choose a different distribution for lift-off and that should be based on experiments. This was more of a heuristic choice.

R. Chance (Grumman Aerospace): Are you contemplating incorporating in your model the analysis of the surface condition? That is a major contributor to problems in the reliability of eddy currents. As you increase your frequency, you are gaining sensitivity for small flaw detection, but you also increase your sensitivity to these conditions.

A.J. Bahr: We will, but we haven't had the inputs for the model up to now. We will definitely consider doing that, and see if we can get that information.